On the Vibrations of the Bowed String Instruments

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Abstract. The present note considers the bowed string instruments of the violin family and focuses on the string-soundbox dynamical coupling in the low-frequency range, carrying out a numerical and an analytical modal approach in parallel and aiming at a simple theoretical model of the sound production. The numerical results show just slight aperiodic fluctuations of the amplitude and very slow phase shifts in comparison with the simple analytical solutions, suggesting the latter as a fairly realistic description of the instrument performances.

Keywords: String instrument · Modal approach · Numerical solution · Analytical approximation

1 Foreword

The vibration analysis of the bowed string instruments started with the early studies of Helmholtz and Raman [1, 2] but the scientific interest began to grow greatly only from the middle of the last century up until today, as testified by many papers (see [3–9], just as a few examples). Here, we focus on the string-soundbox coupling in the low-frequency range, i.e. the range of the signature modes, more or less, carrying out a numerical and an analytical modal approach in parallel, to identify the coupling effects on the dynamical response of the instrument and to work out an acceptable approximate model of the sound production. The analysis takes into account proper functional relations between the force and displacement at the string-bridge contact and those at the bridge feet, for symmetric, antisymmetric and general modes. The characteristic equation of the coupled system is formulated and, assuming realistic values for the soundbox own frequencies, the coupling frequency spectrum is identified. The motion equations are then solved in the time domain numerically and compared with the analytical results obtainable assuming the pure Helmholtz motion as the string exciting motion.

2 Theoretical Model

Let us consider a single bowed string, for example, the string A4 of a violin as in Fig. 1 (440 Hz), and refer to a frame *Oxyz* with the origin *O* on the nut, the *x*-axis along the

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string and the y-axis along the bow motion direction. The string displacement may be expanded in a series of sinusoidal eigenfunctions $e_i(x) = a_i \sin(\omega_i x/v_w)$, multiplied by generalized coordinates $q_i(t)$, that is $y(x, t) = \sum_i q_i(t)e_i(x)$, where ω_i are the natural frequencies, $v_w = \sqrt{T/(\mu_s S_s)}$ is the propagation speed, T is the pre-tensioning, μ_s and S_s are the mass density and the cross-section area of the string. The eigenfunctions $e_i(x)$ are not orthogonal to each other in general, nor are the frequencies in arithmetic progression as for the string with fixed-fixed ends, because the extreme on the bridge vibrates together with the soundbox.

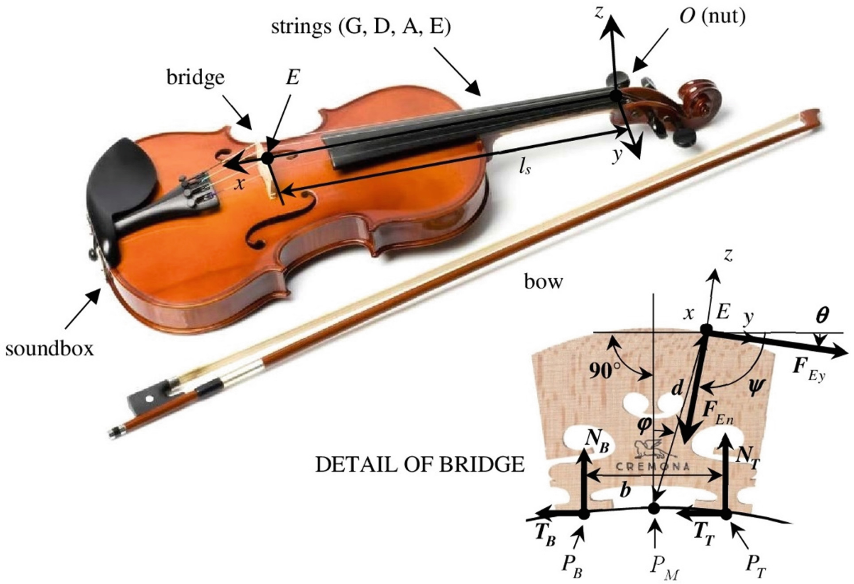

Fig. 1. Violin and reference frame. The detail shows the forces acting on the bridge: 1) $\boldsymbol{F}_{Ey}$, force applied by the string due to the bow thrust. 2) $\boldsymbol{F}_{En}$, force applied by the string due to the string tensioning. 3) $\boldsymbol{N}_B$, $\boldsymbol{N}_T$, $\boldsymbol{T}_B$, $\boldsymbol{T}_T$, normal and tangent forces applied by the soundbox top plate at the bass and treble feet of the bridge base, P_B and P_T, respectively.

The bridge translates along its axis in the symmetric modes of the plate and rotates around the base mid-point P_M in the antisymmetric modes.

Introduce the full symmetric matrix $[e_{ji}]$, where $e_{ji} = \mu_s S_s \int_0^{l_s} e_j(x)\, e_i(x) dx$, and impose that $\mu_s S_s \int_0^{l_s} e_i^2(x) dx = 1$, whence, indicating the string mass and length with m_s and l_s, one gets

$$a_i = \sqrt{\frac{2}{m_s\left[1 - \frac{v_w}{2\omega_i l_s} \sin\left(\frac{2\omega_i l_s}{v_w}\right)\right]}} \tag{1}$$

so, the physical dimensions of the eigenfunctions e_i are [kg$^{-1/2}$] and those of the generalized coordinates q_i are [kg$^{1/2}$ $\times$ m]. Moreover, introduce the one-dimensional

Dirac distribution $\delta(x - x_*)$ in order to manage generic concentrated forces F_*, so that $\int_0^{l_s} \delta(x - x_*)F_* e_j(x)dx = F_* e_j(x_*)$, and define the vectors $\{F_B e_j(x_B)\}$ and $\{F_{Ey} e_j(l_s)\}$, where x_B is the abscissa of the string-bow contact point B, F_B is the bow force and $F_{Ey} = -T \times \sum_i q_i (de_i/dx)_{x=l_s}$ is the bridge force (see Fig. 1). The motion equations of the string sub-system may be written in the matrix form

$$[e_{ji}]\left\{\frac{d^2 q_i}{dt^2} + 2\zeta_i \omega_i \frac{dq_i}{dt} + \omega_i^2 q_i\right\}^T = \{F_B e_j(x_B)\}^T - \{F_{Ey} e_j(l_s)\}^T \quad j = 1, 2, \ldots \tag{2}$$

where the damping effects are quite small and are here assumed uncoupled for the various modes. Observe that q_i must be considered in general as the sum of a variable part, $q_{i\sim}(t)$, which has the particular form $q_{i\sim}(t) = q_{i\mathrm{x}} \sin(\omega_i t)$ when considering the natural modes, and a constant part q_{i0}, which is the static part due to the mean bow force.

The violin string could not emit a vigorous and harmonious sound by itself but needs the dynamic cooperation of the soundbox. Using capital letters for the quantities of the soundbox sub-system, we introduce the natural frequencies Ω_I, the two-dimensional eigenfunctions E_I and the modal coordinates Q_I, where the subscripts I refer to the single characterizing modes of the soundbox alone. It is presumed that all these parameters are obtainable by experimental tests, e. g. by holographic interferometry or impulse hammer and accelerometers or laser Doppler vibrometry. Besides, other experiments should also permit evaluating the bandwidths of the singular modes and then the damping factors Z_I. The present analysis is just limited to the low modes (<~1500 Hz), which may be clearly identified and characterized in the frequency response. As well known, the high-frequency range coincides with the so-called “bridge hill”, where a large overlap of bandwidths occurs, the modal approach ceases to provide useful results and other methods should be applied (e. g. see [6]). The whole frequency response of the soundbox should be obtained by joining the two frequency ranges.

The detail of the bridge in Fig. 1 shows the forces applied to the bridge by the string and the soundbox top plate. As the bridge own frequencies are higher than the examined range, the bridge may be presumed rigid, while its mass may be neglected. Hence, disregarding the moment of $\boldsymbol{T}_T + \boldsymbol{T}_B$ with respect to P_M, the calculation of the normal reaction forces at the bridge feet is quite straightforward:

$$\begin{aligned} N_T &= F_{Ey}\left[\frac{\sin\theta}{2} + \frac{d}{b}\cos(\varphi - \theta)\right] + F_{En}\left[\frac{\sin\psi}{2} + \frac{d}{b}\cos(\psi - \varphi)\right] \\ N_B &= F_{Ey}\left[\frac{\sin\theta}{2} - \frac{d}{b}\cos(\varphi - \theta)\right] + F_{En}\left[\frac{\sin\psi}{2} - \frac{d}{b}\cos(\psi - \varphi)\right] \end{aligned} \tag{3a, b}$$

As the force F_{En} is induced by the constant string tensioning, it yields only invariant deflections and is irrelevant in the analysis of the vibratory motion, so the second terms on the right sides of Eqs. (3a, b) may be ignored. The concentrated forces on the soundbox surface, $-\boldsymbol{N}_T$ and $-\boldsymbol{N}_B$, may be dealt with using two-dimensional Dirac distributions, $\delta(X - X_*, Y - Y_*)$, where X and Y are coordinates on this surface. Applying the usual modal separation technique to the soundbox own motions, these forces turn out to be multiplied by $E_I(X_{PT}, Y_{PT})$ and $E_I(X_{PB}, Y_{PB})$ for each mode I, and moreover, $E_I(X_{PT},$

$Y_{PT}) = \pm E_I(X_{PB}, Y_{PB})$ for the symmetric and antisymmetric mode shapes, respectively. Therefore, using P_T as a reference point, the overall effect is $h_{fI} F_{Ey} E_I(X_{PT}, Y_{PT})$, where the force factor h_{fI} is $\sin\theta$ or $2(d/b)\cos(\varphi - \theta)$ for the one or the other shape.

On the other hand, the soundbox vibration, even though very small compared with the string, implies small vibration components of the latter on the xz plane as well. Indicating the y-displacement of the string end on the bridge with y_E for each individual mode, the normal displacements of points P_T and P_B towards the box inside, identifiable by $Q_I E_I(X_{PT}, Y_{PT})$ and $Q_I E_I(X_{PB}, Y_{PB})$, are both equal to $y_E/\sin\theta$ for the symmetric modes, whereas they are opposite to each other and equal to $\pm y_E b/[2d\cos(\varphi - \theta)]$ for the antisymmetric ones. Hence, one may write $y_E = h_{dI} Q_I E_I(X_{PT}, Y_{PT})$, where the displacement factor h_{dI} is equal to $\sin\theta$ or $2(d/b)\cos(\varphi - \theta)$ for the former and latter modes and is then equal to the force factor h_{fI}, in perfect accordance with the virtual work principle.

It must be clarified that the introduction of the sound post and the bass bar in the inside of the harmonic box modifies its symmetry characteristic with respect to the preliminary artefact with no additional elements, so each mode shape turns out to be a sort of combination of a symmetric and an antisymmetric shape. Therefore, the above factors, h_{fI} and h_{dI}, must be combined by proper weighting coefficients, somehow guided by the results from the experimentation, and a similar operation must be applied to the point values of the eigenfunctions $E_I(X_P, Y_P)$. In practice, indicating with p_s and p_a the values of any parameter p for the symmetric and antisymmetric deformation, respectively, we will set $p_I = p_{sI} w_{sI} + p_{aI}(1 - w_{sI})$ (with $0 < w_{sI} < 1$), where w_{sI} is the "weight" of the symmetric shape in the specific mode I. Also, it will be assumed for simplicity in the following that $d/b = 1$ and $\varphi = \theta$.

Assume normalized eigenfunctions, so that $\int_0^{S_P} \mu_p h_p E_R E_S dX dY = \delta_{RS}$ where μ_p, h_p and S_p are the mass density, the thickness and the surface area of the vibrating plates and δ_{RS} is the Kronecker delta. Then, the usual modal separation yields the motion equations of the soundbox sub-system when excited by the forces $-\boldsymbol{N}_T$ and $-\boldsymbol{N}_B$,

$$\frac{d^2 Q_J}{dt^2} + 2 Z_J \Omega_J \frac{dQ_J}{dt} + \Omega_J^2 Q_J = h_{fJ} E_J\left(X_{P_T}, Y_{P_T}\right) F_{Ey} \quad J = 1, 2, \ldots \tag{4}$$

3 Natural Modes

Looking for the natural modes, one has to ignore the damping terms in Eqs. (4), replace $F_{Ey} = -T \times \sum_i q_i (de_i/dx)_{x=l_s}$, consider only the time-varying terms and solve for the Q_J. Then, taking into account the equality $T = \mu_s S_s {v_w}^2$ and using the correlation formulae reported in the previous section for the forces and displacements, one has $y(t, l_s) = \sum_J h_{dJ} Q_J(t) E_J\left(X_{P_T}, Y_{P_T}\right)$, that is

$$\sum_i a_i \sin\left(\frac{\omega_i l_s}{v_w}\right) q_{ix} \sin(\omega_i t)$$
$$= \mu_s S_s l_s \sum_i a_i \left(\frac{\omega_i l_s}{v_w}\right) \cos\left(\frac{\omega_i l_s}{v_w}\right) q_{ix} \sin(\omega_i t) \left[\sum_J \frac{h_{dJ} h_{fJ} E_J^2\left(X_{P_T}, Y_{P_T}\right)}{\left(\frac{\omega_i l_s}{v_w}\right)^2 - \left(\frac{\Omega_J l_s}{v_w}\right)^2}\right] \tag{5}$$

Since Eq. (5) must hold instant by instant, the summation concerning i and the time functions $q_{ix}\sin(\omega_i t)$ may be dropped, obtaining the characteristic equation:

$$\tan\frac{\omega_i l_s}{v_w} = \mu_s S_s l_s \sum_J h_{dJ} h_{fJ} E_J^2(X_{P_T}, Y_{P_T}) \frac{\frac{\omega_i l_s}{v_w}}{\left(\frac{\omega_i l_s}{v_w}\right)^2 - \left(\frac{\Omega_J l_s}{v_w}\right)^2} \tag{6}$$

Since $\mu_s S_s l_s$ is the string mass and the order of magnitude of ${E_J}^2$ is the reciprocal of the vibrating mass of the soundbox, which is much greater than the string mass, the coupled frequencies ω_i turn out to be very close to the uncoupled ones, either to $\omega_i = i\pi v_w/l_s$, when $\tan(\omega_i l_s/v_w) \cong 0$, or to Ω_J, when $\omega_i \cong \Omega_J$. The sequence Ω_J is chosen in the following calculation using verisimilar soundbox frequencies and realistic values of the weighting coefficients w_{sJ}. Once fixing the frequencies Ω_J and all other parameters, the exact values of the natural frequencies ω_i of the full system may be calculated numerically using Eq. (6). It is remarkable that some of the Ω_J are well separated from the angular frequencies of the string with fixed-fixed ends, so the soundbox is feebly excited, whereas some are close to the string frequencies, so the soundbox is resonant and a vigorous sound level is emitted to the surrounding environment.

4 Numerical and Analytical Solutions in the Time Domain

The time solutions may be obtained by use of a Euler-Cauchy solver, considering a finite but sufficiently large number n of modes.

Observe that Eqs. (2) and (4) refer in practice to the same modes, i. e. the common modes of the whole coupled system string + soundbox, but only the modes for which $e_j(l_s) \neq 0$ give their contribution to the vector $\{F_{Ey} e_j(l_s)\}^T$ at the right side of Eq. (2). These modes have nearly the same frequencies Ω of the soundbox and are characterized by $Q_I(t) = q_i(t)e_i(l_s)/[h_{dI} E_I(X_{PT}, Y_{PT})]$, according to Sect. 2. Therefore, it is possible to eliminate the force F_{Ey} between Eqs. (2) and Eqs. (4) obtaining

$$\left\{\frac{d^2 q_i}{dt^2} + 2\zeta_i\omega_i\frac{dq_i}{dt} + \omega_i^2 q_i\right\}^T = \left[e_{ji} + \frac{e_i^2(l_s)\delta_{ji}}{h_{dJ} h_{fJ} E_J^2\left(X_{P_T}, Y_{P_T}\right)}\right]^{-1} \left\{F_B e_j(x_B)\right\}^T \tag{7}$$

where δ_{ji} is the Kronecker delta and the small damping factors of the two sub-systems were equalized for simplicity.

The sequential procedure consists in solving Eqs. (7) for the q_i first, replacing the q_i into F_{Ey} in Eqs. (4) and then solving Eqs. (4) for the Q's. This can be considered an "experimental" result and tends to become all the more exact the more correct the input data are.

For the damping factors ζ_i, the following laws were used, assuming $r_i = \omega_i l_s/(\pi v_w)$:

$$\begin{aligned} 2\zeta_i\omega_i &\cong 5\pi\left(2.9 + 0.3 r_i^2\right)\ \mathrm{s}^{-1} && \text{for } r_i \leq 3 \\ 2\zeta_i\omega_i &\cong 5\pi\,\frac{[5.6(10-r_i)+23(r_i-3)]}{7}\ \mathrm{s}^{-1} && \text{for } r_i \geq 3 \end{aligned} \tag{8a, b}$$

As regards the bow force F_B, it depends on the state of slip or stick between the string and the bow. For the former state, it is possible to assume

$$F_B = F_{\text{slip}} = F_s \times \left[0.3 + 0.7 \times \exp(-1.25 \times v_{\text{rel.}})\right] \tag{9}$$

where $v_{\text{rel.}} = v_B - dy_B/dt$, v_B is the bow velocity and F_s is the maximum static friction force, a function of the normal force exerted by the violinist, who must control it in a very shrewd way. We here assume the formula $F_s = 0.036[l_s/(l_s - x_B)]^{1.43}$ N, which complies with Schelleng's diagram [3]. During the stick phase, on the other hand, one has $dy_B/dt = v_B =$ constant, whence $\sum_i e_i(x_B)d^2q_i/dt^2 = 0$ and Eq. (7) gives

$$F_B = F_{\text{stick}} = \frac{\sum_i e_i(x_B)\left[2\zeta_i\omega_i\frac{dq_i}{dt} + \omega_i^2(q_{i\sim} + q_{i0})\right]}{\sum_i e_i(x_B)\sum_j inv_{ji} \times e_j(x_B)} \tag{10}$$

where the coefficients inv_{ji} are those of the inverse matrix on the right side of Eq. (7). The mean temporal bow force F_m is found to be roughly equal to the constant slip force F_{slip}, so $q_{i0} \cong F_{\text{slip}}\sum_j \left[inv_{ji} \times e_j(x_B)\right]/\omega_i^2$ by Eq. (7),

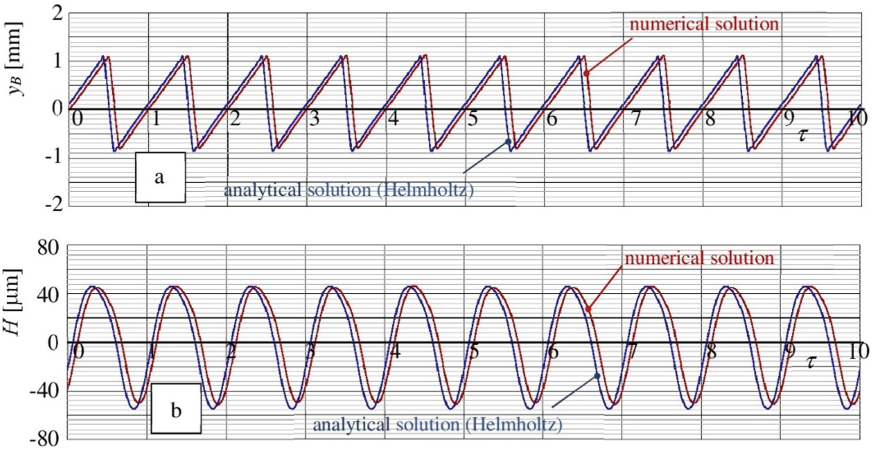

Fig. 2. String and soundbox response after a time $t_i \cong 0.1$ s. Time scale $\tau = (t - t_i)v_w/l_{sA4}$. (a) Stick-slip motion of bowed point B. (b) Harmonic table vibration, $H = \sum_J Q_J(t)E_J(X_{P_T}, Y_{P_T})$. Data: $n = 20$, $\mu_s S_s = 0.0008$ kg/m, $l_s = l_{sA4} = 325$ mm, $x_B = 285$ mm, $v_B = 1$ m/s, $T = 66$ N. Soundbox: frequencies = 275, 400, 450, 530, 620, 850, 980 [Hz]; quality factors = 50 (all modes).

In parallel, analytical approximations for the soundbox motion can be searched by expressing the $q_{i\sim}(t)$ by plausible functions, for example, referring to the Helmholtz motion and using the terms of its saw-tooth Fourier expansion,

$$q_{i\sim}(t) = \frac{2v_B}{\pi\omega a_i i^2} \times \frac{l_s}{(l_s - x_B)} \times \sin i\omega t \tag{11}$$

adding the static terms q_{i0}, replacing these quantities into the differential equations of the soundbox vibrations, Eqs. (4), and solving for the $Q_{J\sim}(t)$ and Q_{J0} by use of realistic bandwidths of the frequency response of the soundbox to calculate the Z_J.

Figure 2a, b shows the motion of the bowed point of the string and that of the reference soundbox point P_T, as they can be obtained by the numerical and the analytical procedure, for $l_s = 325$ mm (A4: 440 Hz). The former is quite similar to the Helmholtz motion (Fig. 2a) and points out that the very short run of the string endpoint does not affect so much the oscillations of the string itself. Due to the mutual incommensurability of the string and soundbox frequencies, the steady motions of the bowed point B and the soundbox point P_T, which occur after a transient period of one tenth of a second roughly, are not rigorously periodic but quasi-periodic with slight fluctuations of the amplitude and slow phase shifts. The analytical results show a good approximation anyway. Figure 3 shows results analogous to Fig. 2b when the string length is reduced by the violinist's finger to get the note C5 (523 Hz): there is a good agreement also in this case and a similar agreement may be found in many other test cases.

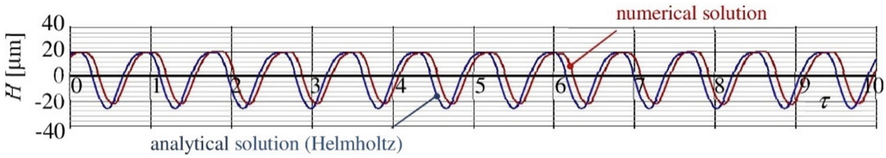

Fig. 3. Harmonic table vibration $H = \sum_J Q_J(t) E_J\left(X_{P_T}, Y_{P_T}\right)$ after a time $t_i \cong 0.1$ s, as a function of the dimensionless time $\tau = (t - t_i)v_w/l_{sA4}$ (same time scale as in Fig. 2). Same data as in Fig. 2, except $l_s = l_{sC5} = 273$ mm (C5).

5 Conclusion

The present report proposes a simple analytical approach to describe the low-frequency behaviour of the bowed string instruments. It proves sufficiently consistent with the more accurate numerical results and might be completed by adding the higher frequency response obtainable by other methods described in the literature. The whole frequency spectra of the various individual instruments are certainly different from each other, as all luthiers are well aware and careful experimental tests should be carried out to characterize their tone colour. Yet, the present methodology may provide a useful tool to analyse the influence of possible structural changes of the soundbox parts on the global performances of the instruments in the low-frequency range.

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