Piecewise Analytical Solution for Rub Interactions Between a Rotor and an Asymmetrically Supported Stator

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Abstract. Rotor-stator rubbing phenomenon is a very common fault in various engineering applications. Usually, the stator supports are assumed to be symmetric, however, in some conditions, the stator support stiffness is anisotropic in nature. In this paper, an approximate analytical solution for the rub interactions between a Jeffcott rotor and an asymmetrically supported stator is introduced. The present problem is a good example of piecewise nonlinear dynamics in which contact causes the system to switch between linear and nonlinear dynamic behavior. To evaluate an analytical solution to the contact equations, Taylor's expansion is used to linearize the contact forces. Then, an analytical solution is introduced based on this approximation. To investigate the validity of the present model, the proposed approximate analytical solution is compared with the numerical solution obtained using direct numerical integration for the same problem. The results show that the proposed piecewise analytical solution is efficient in evaluating the system response for both periodic and chaotic responses.

Keywords: Rotordynamics · Asymmetric stator · Jeffcott rotor · Analytical solution · Periodic · Chaotic · Piecewise nonlinear system

1 Introduction

Rub interactions between rotating and stationary parts of machinery have drastic effects on the reliability and safety of rotordynamic systems. Rotor-stator rubbing has been extensively investigated over the past decades as documented in several reviews [1–4]. However, the related phenomena need more investigation.

The dynamics of rotor-stator rubbing phenomenon is mostly explored using lumped parameter models. Owing to the inherent nonlinearity that characterizes rotor rubbing due to the discontinuous contact, the models are usually low-dimensional to keep the computational time reasonable. Most of the studies have solved the nonlinear differential

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equations of the system numerically using various techniques such as Runge-Kutta algorithm [5–7], Newmark-beta method as in [8] and finite element analysis as in [9]. The piecewise-smooth model is commonly used in most of the previous studies to describe the contact dynamics as a linear elastic stiffness that causes a normal restoring force [10–14].

The strong nonlinearity in contact systems typically generates different possible responses that include periodic, quasi-periodic and chaotic responses. Usually, a rich profile of subharmonics and super harmonics characterize the frequency response of rotor-stator rubbing systems [15–18]. The features of the rub response are greatly affected by changing the system parameters such as rotor speed, stator stiffness, friction coefficient, and system damping [15, 19, 20]. Varney and Green [12] studied the influence of support asymmetry on the response of a rotor-stator contact system. Direct stiffness asymmetry was shown to greatly influence the system response even for small values of stiffness asymmetry. However, rotor flexibility was not considered in the model. In [21], orbit asymmetry was observed experimentally during rubbing and this was attributed to lower bottom stator stiffness in comparison to that of the other sides. Several researchers have conducted experimental studies to validate the mathematical models. Different forms of rub response were observed experimentally [7, 16–18, 21–23] but more sophisticated behavior was exhibited compared to the mathematically obtained responses. Hence, it is inferred that an improvement of the mathematical models of rubbing is crucial to better represent the real phenomenon.

Very few analytical studies have been conducted owing to the strong non-linearity associated with the rubbing phenomenon. Childs [24] provided an explanation of the second (1/2 $\times$) and third (1/3 $\times$) order subharmonics based on parametric excitation analysis. Kim and Noah [25] employed the harmonic balance method (HBM) to obtain periodic solutions and conducted bifurcation analysis to predict chaotic motions. They also showed that increasing the cross-coupling stiffness term causes Hopf bifurcation that leads to quasi-periodic response. Lu et al. [26] used a semi-analytical approach to find periodic solutions of a rub-impact rotor system. Karpenko et al. [13] developed a piecewise approximate analytical solution to study the rotor-stator interactions using Taylor's series expansion. Most recently, El-Sayed et al.[27] extended the approximate analytical solution given in [13] to include Coulomb friction between the rotor and stator. The solution was applied to analyze whirling vibrations of drill-strings.

In this paper, an approximate piecewise analytical solution is developed to study the rotor-stator dynamic interactions where the stator support is considered anisotropic. Direct and cross-coupled stiffness terms are included in the model. A two degree of freedom rotor model is employed taking into consideration the static offset between the rotor and stator centers. A linear elastic contact model is used to represent the contact dynamics and Taylor's expansion is used to linearize the nonlinear contact forces. After this introduction, the piecewise analytical solutions for the non-contact and contact regimes are presented in Sect. 2. Then, the results of the present model are given in Sect. 3. Finally, the main findings are summarized in the conclusions section.

2 Analytical Model

A two degree of freedom Jeffcott rotor is considered here to rotate within a stator which is asymmetrically supported. The Jeffcott rotor mass is negligible in comparison to the disk mass M. The stiffness and damping of the rotor are denoted by K and C respectively. The stator is mounted elastically on an anisotropic support with the following stiffness parameters K_{xx}, K_{yy}, K_{xy}, K_{yx} where x, y are the cartesian coordinates as shown in Fig. 1.b. The center of the stator is denoted by O_{s0} and the geometrical center of the rotor O_r with static equilibrium position at O_{r0}. At static equilibrium, the radial difference between O_{s0} and O_{r0} in x and y directions are denoted as ε_x, ε_y respectively. The rotor spins by an angular frequency Ω and is subjected to unbalance where the unbalance mass is m and the eccentricity is ρ. The rotor center position is described by x and y components. The diametral clearance between the rotor and the stator is $2\,\gamma$. When $R = \sqrt{(x-\varepsilon_x)^2 + \left(y-\varepsilon_y\right)^2} > \gamma$ the rotating disk comes in contact with the stator.

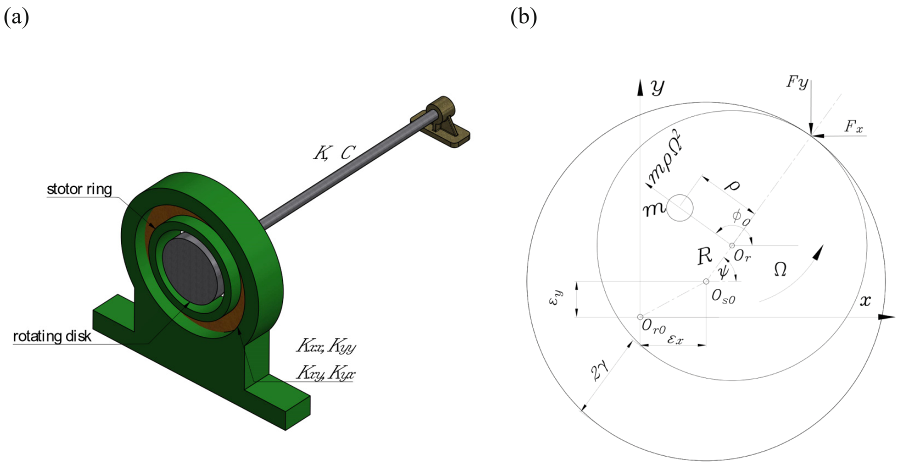

Figure 1 (a) Three-dimensional representation to the rotating disk and the stator ring mounted on anisotropic stiffness. (b) schematic drawing shows the coordinate systems and the contact forces. The rotor mass M is excited by the out of balance mass m. The out of balance force is $m\rho\Omega^2$. F_x, F_y are nonlinear contact force components which are in action when $R \geq \gamma$.

The equations of motion for the present model can be presented in the dimensionless form as follows:

$$\text{Nocontact}\left(\hat{z} < 1\right): \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} + 2\nu\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} + \begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \eta_m\hat{\rho}\eta^2\begin{bmatrix}\cos(\eta\tau+\varphi_0)\\ \sin(\eta\tau+\varphi_0)\end{bmatrix}, \quad (1)$$

$$\text{Contact}\left(\hat{z} > 1\right): \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} + 2\nu\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} + \begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} - \begin{bmatrix}\widehat{F}_x\\ \widehat{F}_y\end{bmatrix} = \eta_m\hat{\rho}\eta^2\begin{bmatrix}\cos(\eta\tau+\varphi_0)\\ \sin(\eta\tau+\varphi_0)\end{bmatrix}. \quad (2)$$

The contact forces in dimensionless form can be given as:

$$\widehat{F}_x = \frac{F_x}{\gamma} = \widehat{K}_{xx}\left(1 - 1/\hat{z}\right)\left(\hat{x} - \hat{\varepsilon}_x\right) + \widehat{K}_{xy}\left(1 - 1/\hat{z}\right)\left(\hat{y} - \hat{\varepsilon}_y\right) \tag{3}$$

$$\widehat{F}_y = \frac{F_y}{\gamma} = \widehat{K}_{yx}\left(1 - 1/\hat{z}\right)\left(\hat{x} - \hat{\varepsilon}_x\right) + \widehat{K}_{yy}\left(1 - 1/\hat{z}\right)\left(\hat{y} - \hat{\varepsilon}_y\right) \tag{4}$$

The analytical solution for the linear noncontact case can be written as:

$$\hat{x} = e^{-\nu\tau}(c_1\cos(\beta\tau) + c_2\sin(\beta\tau)) + c_{11}\cos(\eta\tau + \varphi_0) + \tag{5}$$

$$c_{22}\sin(\eta\tau + \varphi_0), \tag{6}$$

$$\hat{y} = \mathrm{e}^{-\nu\tau}(c_3\cos(\beta\tau) + c_4\sin(\beta\tau)) + c_{33}\cos(\eta\tau + \varphi_0) + c_{44}\sin(\eta\tau + \varphi_0).$$

The analytical solution for the nonlinear contact equations can be obtained after linearizing the nonlinear contact forces which results in contact equations as follows:

$$\text{Contact}(\hat{z} > 1): \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\begin{bmatrix}\hat{\mathrm{x}}\\ \hat{\mathrm{y}}\end{bmatrix} + 2\nu\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix}\hat{\mathrm{x}}\\ \hat{\mathrm{y}}\end{bmatrix} + \begin{bmatrix}F_{a1} & F_{b1}\\ F_{a2} & F_{b2}\end{bmatrix}\begin{bmatrix}\hat{\mathrm{x}}\\ \hat{\mathrm{y}}\end{bmatrix} = -\begin{bmatrix}F_{d1}\\ F_{d2}\end{bmatrix} + \eta_m\hat{\rho}\eta^2\begin{bmatrix}\cos(\eta\tau + \varphi_0)\\ \sin(\eta\tau + \varphi_0)\end{bmatrix} \tag{7}$$

where

$$F_{a1} = \left(1 + \widehat{K}_{xx}A_1 + \widehat{K}_{xy}B_2\right), F_{b1} = \left(\widehat{K}_{xx}B_1 + \widehat{K}_{xy}A_2\right), F_{D1} = \left(\widehat{K}_{xx}D_1 + \widehat{K}_{xy}D_2\right),$$

$$F_{a2} = \left(1 + \widehat{K}_{yy}A_2 + \widehat{K}_{yx}B_1\right), F_{b2} = \left(\widehat{K}_{yy}B_2 + \widehat{K}_{yx}A_1\right), F_{D2} = \left(\widehat{K}_{yy}D_2 + \widehat{K}_{yx}D_1\right),$$

where

$$A_1 = \left(\alpha^{1.5} - \alpha + \left(\tilde{x}_0 - \hat{\varepsilon}_x\right)^2\right)/\alpha^{1.5},$$

$$B_1 = B_2 = \frac{\left(\tilde{x}_0 - \hat{\varepsilon}_x\right)\left(\tilde{y}_0 - \hat{\varepsilon}_y\right)}{\alpha^{1.5}},$$

$$D_1 = \left(-\alpha\hat{\varepsilon}_x\left(\sqrt{\alpha} - 1\right) - \left(\tilde{x}_0 - \hat{\varepsilon}_x\right)\left(\tilde{x}_0\left(\tilde{x}_0 - \hat{\varepsilon}_x\right) + \tilde{y}_0\right) + \tilde{y}_0\left(\tilde{y}_0 - \hat{\varepsilon}_y\right)\right)/\alpha^{1.5}$$

$$A_2 = \left(\alpha^{1.5} - \alpha + \left(\tilde{y}_0 - \hat{\varepsilon}_y\right)^2\right)/\alpha^{1.5}$$

$$D_2 = \left(-\alpha\hat{\varepsilon}_y\left(\sqrt{\alpha} - 1\right) - \left(\tilde{y}_0 - \hat{\varepsilon}_y\right)\left(\tilde{x}_0\left(\tilde{x}_0 - \hat{\varepsilon}_x\right) + \tilde{y}_0\right) + \tilde{y}_0\left(\tilde{y}_0 - \hat{\varepsilon}_y\right)\right)/\alpha^{1.5}.$$

Then, the analytical solution for the contact case can be written as:

$$\hat{x} = e^{-\nu\tau}(c_1\cos(\gamma_1\tau) + c_2\sin(\gamma_1\tau) + c_3\cos(\gamma_2\tau) + c_4\sin(\gamma_2\tau)) + c_{11}\cos(\varphi_0 + \eta\tau) + c_{22}\sin(\eta\tau + \varphi_0) + c_{55}, \tag{8}$$

$$\hat{y} = e^{-\nu\tau}(s_1c_1\cos(\gamma_1\tau) + s_1c_2\sin(\gamma_1\tau) + s_2c_3\cos(\gamma_1\tau) + s_2c_4\sin(\gamma_1\tau)) + c_{33}\cos(\eta\tau + \varphi_0) + c_{44}\sin(\eta\tau + \varphi_0) + c_{66}, \tag{9}$$

where

$$s_1 = \frac{\gamma_1^2 + \nu^2 - F_{a1}}{F_{b1}}, s_2 = \frac{\gamma_2^2 + \nu^2 - F_{a1}}{F_{b1}}$$

To derive the linear and nonlinear constants the reader is referred to [13, 27]. After obtaining the solution constants, the analytical solutions for the individual piecewise equations are obtained. To connect the two solutions, the switching condition $\widehat{f}_z$ should be monitored precisely during the solution as:

$$\widehat{f}_z = \sqrt{\left(\hat{x}\left(\tau;\tau_0,\tilde{x}_0,\tilde{y}_0,\tilde{x}_0',\tilde{y}_0'\right) - \hat{\varepsilon}_x\right)^2 + \left(\hat{y}\left(\tau;\tau_0,\tilde{x}_0,\tilde{y}_0,\tilde{x}_0',\tilde{y}_0'\right) - \hat{\varepsilon}_y\right)^2} - 1 \tag{10}$$

When $\widehat{f}_z$ is greater than zero, the contact occurs and when $\widehat{f}_z$ is smaller than zero the noncontact condition applies. Bisection algorithm is used to precisely evaluate the switching times.

3 Results and Discussion

To prove the validity of the presented analytical solution a selected example parameters based on the rotor-snubber ring test rig in the center for Applied Dynamics Research (CADR)of Aberdeen university is selected [23, 28]. Then, the model is solved using both analytical and numerical methods. The input data for the model in dimensionless form is listed in Table 1. In this study, the stator was assumed to be asymmetrically supported by the following stiffnesses $\widehat{K}_{yy} = 40 and\ \widehat{K}_{xx} = \widehat{K}_{xy} = \widehat{K}_{yx} = 2$.

Table 1. Dimensionless parameters of the system considered in the present analysis

Parameter	Value
Damping ratio, ζ	0.125
Mass ratio,η_m	0.005
Eccentricity ratio,$\widehat{\rho}$	70
Normalized static displacement in x-direction,$\hat{\varepsilon}_x$	1
Normalized static displacement in y-direction,$\hat{\varepsilon}_y$	0

For the present model, the analysis in this paper is based on evaluating the system orbits at several frequency ratios ranging from $\eta = 1$ to $\eta = 3.5$ as shown in Fig. 2 (a) to (f). The results are evaluated using the present analytical solution. In addition, direct integration numerical solution is evaluated using Runge-Kutta MATLAB function ODE45. In both numerical and analytical solutions, a special code is prepared to detect the switching points. The red circle in Fig. 2 indicates the commencing of contact state and the green circles indicates the start of the non-contact state.

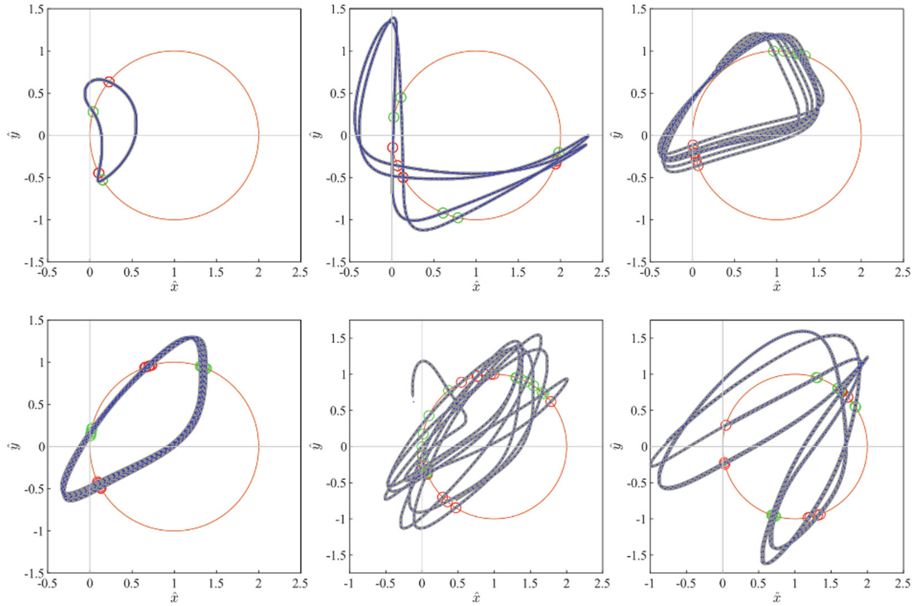

Fig. 2. Orbits for the nonlinear contact equations using numerical (ODE45) and analytical methods. $\widehat{K}_{yy} = 40$, $\widehat{K}_{xx} = \widehat{K}_{xy} = \widehat{K}_{yx} = 20$(a) $\eta = 1$, (b) $\eta = 1.5$, (c) $\eta = 2$, (d) $\eta = 2.5$, (e) $\eta = 3$, (f) $\eta = 3.5$, $[\tilde{x}_0 = 0.1, \tilde{x}_0' = 0, \tilde{y}_0 = 0.1, \tilde{y}_0' = 0]$ Analytical solution, Numerical solution indicates contact start indicates non-contact start.

Figure 2 shows that the rotor orbits are very sensitive to the frequency ratio. It can be realized that changing the rotational speed results in response changes as periodic, quasi periodic and chaotic. In the investigated cases, the results of the present analytical model are in good agreement with the numerical results as shown in Fig. 2.

4 Conclusions

In the present paper, an approximate piecewise analytical solution is introduced for the problem of rotor stator rubbing when the stator is mounted on an anisotropic support. The main novelty in this work is in including direct and cross-coupled stator stiffness asymmetry in the analytical solution of rotor-stator rubbing. The results show that the analytical solution is capable for evaluating the system response in the cases of periodic, quasi periodic and chaotic responses. The switching instances are calculated and compared with those captured by direct numerical integration method. The results of the analytical model are very promising and contribute towards more refined mathematical models of rotor-stator rubbing.

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