Optimal LQR Control for Longitudinal Vibrations of an Elastic Rod Actuated by Distributed and Boundary Inputs

Alexander Gavrikov[1,2] and Georgy Kostin[2(✉)]

[1] Pennsylvania State University, University Park, PA 16802, USA
Alexander.Gavrikov@psu.edu
[2] Ishlinsky Institute for Problems in Mechanics RAS, 119526 Moscow, Russia
kostin@ipmnet.ru

Abstract. We consider a vibrating system consisting of a thin rectilinear elastic rod actuated by external loads applied at the ends as well as by a normal force, which is distributed piecewise constantly in space. Such a force may be implemented by piezoelectric actuators. The intervals of constancy of this normal force are equal in length, and the force value on each of these sections is considered as an independent control input. We study the longitudinal motions of the rod and the means of control optimization. Based on the eigenmode decomposition, it is shown in the case of uniform rod that the original continuous system is split into several infinite vibrating subsystems each of which is controlled by a certain linearly independent combination of control inputs. It follows that if any of these combinations is taken equal to zero, then the corresponding subsystem becomes uncontrollable. Next, an optimal control problem on a finite time horizon is considered, where the terminal mechanical energy of the rod and energy losses in the control circuit are minimized with some weighting coefficients. We show that for a fixed number of actuators distributed along the rod, approximation of the problem is reduced to the design of linear-quadratic regulators. An example of a uniform rod is presented where finite expressions for the optimal control functions are obtained. Amplitudes of controlled and affected but not minimized modes are derived for approximated suboptimal control.

Keywords: Optimal control · Longitudinal vibrations · LQR · Distributed control

1 Introduction

The classical problem of active suppression of vibrations has many practical applications and has been studied for years, mainly as vibration control of discrete structures. Nowadays, continuous systems, e.g. elastic, are a substantial

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field of ongoing research. However, practical implementation of any control strategy implies reduction of either the vibrating system or its control inputs to finite-dimensional objects [1]. In this paper, we assume from the beginning that the input is finite-dimensional in space. Such an assumption allows for splitting the original continuous system into a finite number of subsystems each actuated by its own combination of the original inputs. We consider an elastic rod for which inputs, piecewise constant in space, may be implemented by means of piezo elements [2,3] attached to the rod's sides. The piezoelectric actuators are used in a wide range of engineering applications from precision positioning on nano-scale to motion control and vibration suppression on large scale in the aerospace field [4,5]. In our study, the subsystems, in which the original continuous system is split, are groups of vibrational modes. Since these groups are still infinite-dimensional, we most likely have to approximate them. Here, we use a simple approach by optimizing the motion of some lowest modes while higher modes are actuated by derived optimal inputs. We also estimate the influence of proposed strategy on all the modes by considering a suboptimal control law. This allows us to obtain exact expressions for amplitudes of both controlled and actuated modes.

In Sect. 2 we introduce the mechanical system and describe how it splits into subsystems (groups of modes) and state the optimal control problem. In Sect. 3, the original problem is reformulated in terms of these groups. We construct an LQ-optimal control strategy for the lowest mode in each group in Sect. 4 and propose an asymptotic approximation of the optimal feedback law. In Sect. 5 numerical results on the LQ-optimal and suboptimal strategies are presented.

2 Mechanical System

We consider a thin elastic rod that undergoes longitudinal vibrations. It is assumed that the rod is actuated by boundary forces $f^{\pm}(t)$ and a distributed force $f(t,x)$. The force f acts in normal direction to the cross-section and is distributed along the rod's length, so that it has N space intervals of constancy with equal length λ. Such a force may be implemented via a series of identical piezoelements placed symmetrically on the side surface of the rod. We suppose that the rod has a length of $2L$ and its center is at the point $x = 0$. Denoting the intervals of constancy of the distributed force f as $I_k := (x_{k-1}, x_{k+1})$, $k \in J_s$, we introduce auxiliary functions $f_k(t)$ (see also Fig. 1) such that

$$\begin{gathered} f_k(t) := f(t,x), \quad x \in I_k, \quad k \in J_s, \quad f_{\pm N \pm 1}(t) := f^{\pm}(t), \\ x_n = \tfrac{n\lambda}{2}, \quad \lambda = \tfrac{2L}{N}, \quad n \in J_x, \quad x_{\pm N} = \pm L, \\ J_s = \{1-N, 3-N, \ldots, N-1\}, \quad J_x := \{-N, -N-2, \ldots, N\}. \end{gathered} \tag{1}$$

In the dimensionless variables $x^* = x/L, t^* = t/\tau$ (the asterix is further omitted) the rod's motion is described by the following PDE system

$$\begin{gathered} \rho(x)\ddot{v}(t,x) = (\kappa(x)v'(t,x) + f(x,t))', \quad x \in (-1,1), \quad t \in (0,T), \\ \kappa(\pm 1)v'(t,\pm 1) = f^{\pm}(t), \quad v(0,x) = v^0(x), \quad \dot{v}(0,x) = \dot{v}^0(x). \end{gathered} \tag{2}$$

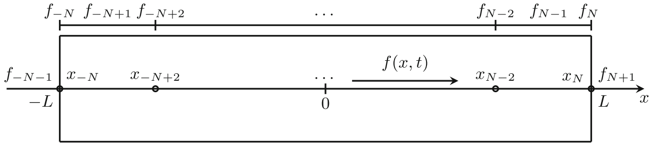

Fig. 1. Schematic representation of the elastic rod and applied forces.

Here, v is the longitudinal displacements of the cross-section, ρ and κ are dimensionless density and stiffness of the rod's material, τ is a characteristic time.

2.1 Control Problem

Let us formulate an optimal control problem of suppressing the rod's vibrations while minimizing the terminal internal mechanical energy E and energy losses F

$$E := \frac{1}{2}\int_{-1}^{1}(\rho\dot{v}^2 + \kappa(v')^2)|_{t=T}dx, \quad F := \frac{1}{T}\int_{0}^{T}\int_{-1}^{1} R\left(f_-, f_+, f\right) dxdt, \tag{3}$$

where R is a positive defined quadratic function w.r.t. forces $f^{\pm}, f$. That is, the aim is to minimize the cost function

$$\Phi := E + \gamma F \to \min_{f^{\pm}, f} \tag{4}$$

subject to constraints (2). In (4), $\gamma > 0$ is a weighting coefficient. We later assign different weights for forces $f_k(t)$ on each interval I_k in (3), but for simplicity of presentation we keep F in the form (3) in this section.

We decompose (2) by projecting this system in the weighted space $L_2(\rho(x), (-1,1))$ onto eigenfunctions $\{w_n(x)\}$, where w_n, $n = 0, 1, \dots$, are found by solving the eigenproblems

$$(\kappa(x)w_n'(x))' = \eta_n\rho(x)w_n(x), \quad \kappa(x)w_n'(\pm 1) = 0, \quad n = 0, 1, \dots . \tag{5}$$

As a result, the infinite-dimensional ODE system is obtained according to

$$\ddot{v}_n = -\eta_n v_n + \sum_{j\in J_x} w_n(x_j)f_j, \quad v_n(0) = v_n^0, \quad \dot{v}_n(0) = \dot{v}_n^0. \tag{6}$$

In (6), v_n, v_n^0 and $\dot{v}_n^0$ are components of projections of $v(t,x)$, $v^0(x)$ and $\dot{v}^0(x)$ onto $w_n(x)$, respectively. Additionally, we introduce in (6) the control functions

$$f_j(t) := f_{j+1}(t) - f_{j-1}(t), \quad j \in J_x. \tag{7}$$

That is, the rod is actuated only by the jumps of the forces f_k. Therefore, any forces $f(t,x), f^{\pm}(t)$ resulting in the same jumps yield the same motion. In what follows, we consider such forces equivalent. Moreover, the rod is uncontrollable if for at least one n we have

$$\sum_{j \in J_x} w_n(x_j) f_j(t) \equiv 0. \quad (8)$$

Therefore, the forces resulting in linear dependent jumps f_j s.t. (8) holds for some n do not actuate this specific mode. We exclude such forces from consideration.

2.2 Controlled Groups for a Uniform Rod

If the values of eigenfunctions are such that $w_i(x_k) = w_j(x_k)$ for some $i \neq j$ and all $k \in J_x$, the vibrational modes split in groups. It may happen if the rod consists of homogeneous pieces or, in the simplest case, both the density ρ and the rigidity κ are constant. Let $\rho = \kappa = 1$ in dimensionless units. Then

$$\begin{gathered} w_n(x) = \cos\left(\tfrac{\pi}{2} n(x+1)\right), \quad \eta_n = \tfrac{\pi^2 n^2}{4}, \quad n = 1, 2, \ldots, \\ w_0 = c_0 = \sqrt{2}/2, \quad \eta_0 = 0, \end{gathered} \quad (9)$$

and, for example, for $N = 2$ the system (6) takes the form

$$\ddot{v}_n = -\eta_n v_n + w_n(-1) f_{-2} + w_n(0) f_0 + w_n(1) f_2. \quad (10)$$

Since the functions $w_n(x_k) = \cos\left(\frac{\pi}{2} n \left(\frac{k}{N} + 1\right)\right)$ take only a finite number of values for fixed k and N, the number of possible linear combinations $\sum_{j \in J_x} w_n(x_j) f_j$ is also finite. We introduce the effective control functions

$$u_n = \sum_{j \in J_x} w_n(x_j) f_j. \quad (11)$$

Then all vibrational modes split into $N + 1$ independent groups such that each i-th group, $i = 0, \ldots, N$, with mode numbers $n = 2Nj \pm i \geq 0$, $j = 0, 1, \ldots$, is controlled by a specific u_i:

$$\ddot{v}_n = -\eta_n v_n + u_i. \quad (12)$$

For example, denoting for $N = 2$

$$\begin{gathered} u_0 = f_2 + f_0 + f_{-2} = f_3 - f_{-3}, \quad u_1 = -f_2 + f_{-2} = -f_3 + f_1 + f_{-1} - f_{-3}, \\ u_2 = f_2 - f_0 + f_{-2} = f_3 - 2f_1 + 2f_{-1} - f_{-3}, \end{gathered} \quad (13)$$

we obtain 3 groups, namely,

$$\begin{gathered} \ddot{v}_0 = c_0 u_0, \quad \ddot{v}_{4j+4} = -\eta_{4j+4} v_{4j+4} + u_0, \\ \ddot{v}_{2j+1} = -\eta_{2j+1} v_{2j+1} + u_1, \quad \ddot{v}_{4j+2} = -\eta_{4j+2} v_{4j+2} + u_2, \end{gathered} \quad (14)$$

where $j = 0, 1, \ldots$. Similarly, there are 4 groups for $N = 3$ with $n = 6j$, $6j \pm 1$, $6j \pm 2$, $6j \pm 3$ where $n > 0$. Likewise, the groups are formed for any N.

3 Transforming the Cost Function

Since the vibrational modes are split into independent groups, it is worth transforming the cost function Φ in (4) accordingly. By using the eigenfunction expansion of the displacements v, the internal mechanical energy E and the energy losses F in (3) take the form

$$E = \frac{1}{2}\sum_{n}\left(\dot{v}_n^2(T) + \eta_n v_n^2(T)\right), \quad F = \frac{1}{2T}\int_0^T \bar{f}^* R_f \bar{f} dt, \tag{15}$$

where the vector of control jumps $\bar{f} = (f_{1-N}, \ldots, f_{N-1}, f_{idle})^*$ is introduced, and $\bar{f}^*$ means the transpose of the vector $\bar{f}$; R_f is a positive weighting matrix. The component f_{idle} is equal to the sum of all original inputs f_k. Since the rod is effectively controlled by jumps of forces, any forces $f^{\pm}$, f having the same jumps yield the same control. Any "excessive" input does not change the state of the system. Minimizing $\Phi = E + \gamma F$, we imply that the function $f_{idle}(t)$ representing this "excessive" input is taken equal to zero: $f_{idle}(t) \equiv 0$.

Let us introduce the vector of effective control functions $\bar{u} = (u_0, \ldots, u_N)^*$. The energy losses F can be written as

$$F = \frac{1}{2T}\int_0^T \bar{f}^* R_f \bar{f} dt = \frac{1}{2T}\int_0^T \bar{u}^* R_u \bar{u} dt, \tag{16}$$

where the weighting matrix $R_u = C^* R_f C$ corresponds to linear transformation of $\bar{f}$ to $\bar{u}$: $\bar{f} = C\bar{u}$. For the following decomposition, we suppose further that R_u is equal to the identity matrix I since this can be achieved by assigning specific weights to each input f_j in (3).

As a result, the cost function Φ is split into $N+1$ independent terms

$$\begin{gathered} \Phi = \Phi_0 + \Phi_1 + \ldots + \Phi_N, \\ \Phi_0 = \frac{\dot{v}_0^2(T)}{2} + \frac{1}{2}\sum_{j=1}^{\infty}\left(\dot{v}_{2Nj}^2(T) + \eta_{2Nj} v_{2Nj}^2(T)\right) + \frac{\gamma}{2T}\int_0^T u_0^2(t)dt, \\ \Phi_k = \frac{1}{2}\sum_{j=0}^{\infty}\left(\dot{v}_{2Nj\pm k}^2(T) + \eta_{2Nj\pm k} v_{2Nj\pm k}^2(T)\right) + \frac{\gamma}{2T}\int_0^T u_k^2(t)dt, \end{gathered} \tag{17}$$

with $k \in J_1,\ J_1 := \{1, \ldots, N\}$. Therefore, the problem of vibration suppression (4) is transformed into $N+1$ independent control problems. Since each group is still infinite-dimensional, the direct solution of the corresponding problems is not straightforward, although an approximate solution is still possible [6]. In the next section we consider a finite-dimensional approximation of the optimal control problem.

4 Controlling a Finite-Dimensional Approximation

We consider the following approximation: let us minimize the cost function

$$\begin{gathered} \tilde{\Phi} := \tilde{\Phi}_0 + \tilde{\Phi}_1 + \ldots + \tilde{\Phi}_N \to \min_{u_k}, \quad \tilde{\Phi}_0 = \frac{\dot{v}_0^2(T)}{2} + \frac{\gamma}{2T}\int_0^T u_0^2(t)dt, \\ \tilde{\Phi}_k = \frac{1}{2}\left(\dot{v}_k^2(T) + \eta_k v_k^2(T)\right) + \frac{\gamma}{2T}\int_0^T u_k^2(t)dt, \quad k \in J_1. \end{gathered} \tag{18}$$

That is, we control only the lowest vibrational mode in each group, while the higher modes are actuated, but their motion is not optimized. For definiteness, we call the modes with $n \leq N$ the controlled modes, and the ones with $n > N$—the actuated modes. After denoting

$$\begin{gathered}
y_1^{(0)}(t) = v_0(t), \quad y_2^{(0)}(t) = \dot{v}_0(t), \quad y_1^{(0)}(0) = v_0^0, \quad y_2^{(0)}(0) = \dot{v}_0^0, \\
y_1^{(k)}(t) = \mu_k v_k(t),\ y_2^{(k)}(t) = \dot{v}_k(t),\ y_1^{(k)}(0) = \mu_k v_k^0,\ y_2^{(k)}(0) = \dot{v}_k^0 \\
\mu_k = \sqrt{\eta_k},\ y^{(0)} = (y_1^{(0)}, y_2^{(0)})^*,\ y^{(k)} = (y_1^{(k)}, y_2^{(k)})^*,\ k \in J_1,
\end{gathered} \tag{19}$$

the cost functions $\tilde{\Phi}_k$ are rewritten as follows

$$\begin{gathered}
\tilde{\Phi}_k = (y^{(k)}(T))^* Q^{(k)} y^{(k)}(T) + \int_0^T \bar{u}_k^*(t) R \bar{u}_k(t) dt, \quad \bar{u}_k = (0, u_k)^*, \\
R = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\gamma}{2T} \end{pmatrix}, \quad Q^{(0)} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q^{(j)} = \frac{1}{2} I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j \in J_1.
\end{gathered} \tag{20}$$

Therefore, instead of problem (4) we consider $N+1$ minimization problems:

$$\tilde{\Phi}_k \to \min_{u_k(t), t \in [0,T]}, \quad k \in J_0, \quad J_0 := \{0, \ldots, N\}, \tag{21}$$

subject to the constraints

$$\begin{gathered}
\dot{y}^{(k)} = A^{(k)} y^{(k)} + B \bar{u}_k, \quad k \in J_0, \quad A^{(0)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & c_0 \end{pmatrix}, \\
A^{(j)} = \begin{pmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{pmatrix}, \quad B^{(j)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad j \in J_1,
\end{gathered} \tag{22}$$

with initial conditions introduced in (19).

4.1 Feedback Optimal and Suboptimal Controls

Each subproblem (21), (22) can be solved in a standard way by means of the LQR theory [7]. The optimal feedback control functions are expressed as follows

$$\begin{gathered}
u_0(t) = -\frac{T c_0 y_2^{(0)}(t)}{c_0^2 T (T-t)^2 + \gamma}, \\
u_k(t) = \frac{2\mu_k \alpha_T y_1^{(k)}(t) + (\mu_k T^2 \sin(2\mu_k(T-t)) - 2\mu_k^2 \beta_T T) y_2^{(k)}(t)}{\alpha_T + \mu_k^2 \beta_T^2}, \\
\alpha_T = T^2(\cos^2(\mu_k(T-t)) - 1), \quad \beta_T = (T(T-t) + 2\gamma).
\end{gathered} \tag{23}$$

The behavior of the controlled modes is described by substituting (23) into (22) and integrating the result. For the zeroth mode, the solution is found explicitly

$$\begin{gathered}
y_1^{(0)}(t) = y_1^{(0)}(0) + \left(t - \frac{T c_0^2 t^2}{2(T^2 c_0^2 + \gamma)}\right) y_2^{(0)}(0), \\
y_2^{(0)}(t) = \left(1 - \frac{T c_0^2 t}{T^2 c_0^2 + \gamma}\right) y_2^{(0)}(0).
\end{gathered} \tag{24}$$

The other modes from the zeroth group with $k = 2Nj$, $j > 0$, are actuated by u_0 from (23) and expressed as

$$y_1^{(k)}(t) = \frac{T c_0 (\cos(\mu_k t) - 1)}{\mu_k (T^2 c_0^2 + \gamma)} y_2^{(0)}(0), \quad y_2^{(k)}(t) = -\frac{T c_0 \sin(\mu_k t)}{\mu_k (T^2 c_0^2 + \gamma)} y_2^{(0)}(0). \tag{25}$$

That is, their amplitudes decay as $\frac{1}{kT}$ for large k or T. It is worth mentioning that choosing T as a multiple of 4 in dimensionless units we obtain $y_1^{(k)}(T) = y_2^{(k)}(T) = 0$. Therefore, the proposed control completely suppresses excitation of higher modes with $n = 2Nj$ for such T at the terminal time instant.

Note that (23) holds for any $N > 0$. That is, this solution is applicable to any number N of piezoelements utilized. It is worth noting that it is possible to enhance approximation (18) by controlling several modes in each group. However, since we have to control several pendulums by means of one input in this case, we encounter a classical control problem when the number of inputs is much smaller than the number of controlled variables [8].

The behavior of actuated but not suppressed modes $y_{1,2}^n$, $n > N$, can be investigated numerically as done in the next section. However, under the assumption that N is large, and $k \gg 1$, we can estimate the modes amplitudes explicitly. Indeed, by introducing an approximate control

$$u_k^a(t) = -\frac{T}{\beta_T} y_2^k(t) \tag{26}$$

we obtain explicit expressions for amplitudes:

$$\begin{aligned} y_{1a}^{(k)}(t) &= \left(\tfrac{\beta_T y_2^k(0)}{T^2+2\gamma} + \tfrac{T(\mu_k(T^2+2\gamma)y_1^k(0)+Ty_2^k(0))}{\mu_k^2(T^2+2\gamma)^2}\right)\sin(\mu_k t) \\ &+ \left(\tfrac{\beta_T(\mu_k(T^2+2\gamma)y_1^k(0)+Ty_2^k(0)}{\mu_k(T^2+2\gamma)^2} - \tfrac{Ty_2^k(0)}{\mu_k(T^2+2\gamma)}\right)\cos(\mu_k t), \\ y_{2a}^{(k)}(t) &= -\mu_k\beta_T\left(\tfrac{(\mu_k(T^2+2\gamma)y_1^k(0)+Ty_2^k(0))\sin(\mu_k t)}{\mu_k^2(T^2+2\gamma)^2} - \tfrac{y_2^k(0)\cos(\mu_k t)}{\mu_k(T^2+2\gamma)}\right). \end{aligned} \tag{27}$$

Note that $y_{2a}(T) = 0$ for T that are multiples of 4. The higher actuated modes, $n > N$, also can be derived analytically. We omit these lengthy expressions stating that their amplitudes behave as $\frac{1}{nT}$. In the next section, we show numerically that these approximations are close to the numerical (optimal) solution.

5 Example of Control

5.1 Numerical Results

In this section, we consider a numerical implementation of the strategy (23). Since the equations of motion (12) are linear, to estimate the influence of the control on the higher modes v_n with $n > N$, it is enough to consider zero initial conditions for such modes:

$$v_n^0 = \dot{v}_n^0 = 0 \text{ or } y_1^{(n)}(0) = y_2^{(n)}(0) = 0, \quad n > N. \tag{28}$$

For the lowest controlled modes, we choose initial conditions (ICs) such that energy of each mode is equal to 1 at $t = 0$:

$$\begin{gathered} E_0(0) = \tfrac{(\dot{v}_0^0)^2}{2} = \tfrac{(y_1^{(0)}(0))^2}{2} = 1, \\ E_k(0) = \tfrac{1}{2}\left((\dot{v}_k^0)^2 + \eta_k(v_k^0)^2\right) = \tfrac{1}{2}\left((y_2^{(k)}(0))^2 + (y_2^{(k)}(0))^2\right) = 1, \\ v_0^0 = \sqrt{2}, \quad \dot{v}_0^0 = 0, \quad v_k^0 = \tfrac{1}{\sqrt{\eta_k}} = \tfrac{1}{\mu_k}, \quad \dot{v}_k^0 = 1, \\ y_1^{(0)}(0) = \sqrt{2},\ y_2^{(n)}(0) = 0, \quad y_1^{(k)}(0) = y_2^{(k)}(0) = 1, \quad k \in J_1. \end{gathered} \tag{29}$$

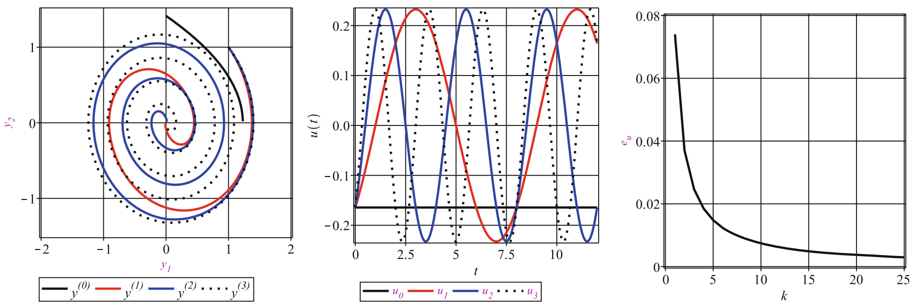

Fig. 2. The amplitudes $y^k_{1,2}$ of the controlled modes with $k = 0, 1, 2, 3$ (left). The optimal control inputs (middle). The relative error of the approximate control input (right).

Let us take $N = 3$. That is, we control the four lowest modes and the other modes are actuated but not optimized. The optimal control u_k acts on the modes with $n = 6j \pm k > 0$. We fix $\gamma = 1$, and $T = 12$, which corresponds to three periods of the first mode v_1. Fig. 2(left) shows the phase portrait of the controlled modes, and Fig. 2(middle) presents the controlled inputs. Here, the zeroth mode $y^{(0)}_{1,2}$ and the input u_0 are computed analytically according to (23), (24), while the amplitudes $y^{(k)}_{1,2}$, $k = 1, 2, 3$ are obtained numerically solving (22), with ICs (29) and u_k from (23). In Fig. 2(left), $y^{(0)}_1$ is scaled by 7 and at $t = T$ we have $y^{(0)}_1 \approx 8.6$, $y^{(0)}_2 \approx 0.02$, $y^{(1)}_1 \approx y^{(1)}_2 \approx -0.014$, $y^{(2)}_1 \approx y^{(1)}_2 \approx 0.014$, $y^{(3)}_1 \approx y^{(3)}_2 \approx -0.014$.

Figure 3(left) shows the mechanical energy during the controlled motion. This figure presents the energy of the controlled modes with $k = 0, 1, 2, 3$ with the terminal value $E(T) = 7 \cdot 10^{-4}$, and the energy of several actuated modes with $n = 4, \ldots, 9$. Here, 4th and 8th modes are actuated by u_2, 5th and 7th—by u_1, 6th—by u_0, and 9th—by u_3. The terminal energy of these five actuated modes is $E(T) \approx 10^{-11}$. During the motion, the amplitudes of the actuated modes do not exceed 0.15 for $y^{(4)}_{1,2}$ and 0.06 for $y^{(n)}_{1,2}$, $n = 5, \ldots, 9$. The terminal value of the approximate cost functional (18) is $\tilde{\Phi} = 5.5 \cdot 10^{-2}$.

Next, the time horizon T is varied. In Fig. 3(middle), we present the terminal mechanical energy of the first ten modes $E(T)$ (the lowest four are controlled while the rest are actuated) depending on $T \in [2, 102]$ and the corresponding value of the energy loss F.

Further, we perform analysis of the approximate analytical solution. Figure 2 (right) shows the relative errors in $L_2(0, T)$ of the approximation (26) depending of the number k of controlled mode: $e_u = \frac{\|u^a_k - u_k\|_{L_2}}{\|u_k\|_{L_2}}$ for $T = 12$. Here, the estimate for a specific k is valid for any number of control elements $N \geq k + 1$. The relative errors $e_{yi} = \frac{\|y^{(k)}_{ia} - y^{(k)}_i\|_{L_2}}{\|y^{(k)}_i\|_{L_2}}$, $i = 1, 2$ of approximated mode amplitudes have the similar value and dependence on k.

In Fig 3(right), the approximate mechanical energy E_a and the approximate energy losses F_a are given depending on T for $N = 3$ for the first ten modes computed analytically. The zeroth mode is controlled by u^0 (23) and for $k = 1, 2, 3$ we use u_a^k (26). The other modes with numbers $n > 3$ are actuated by one of these inputs.

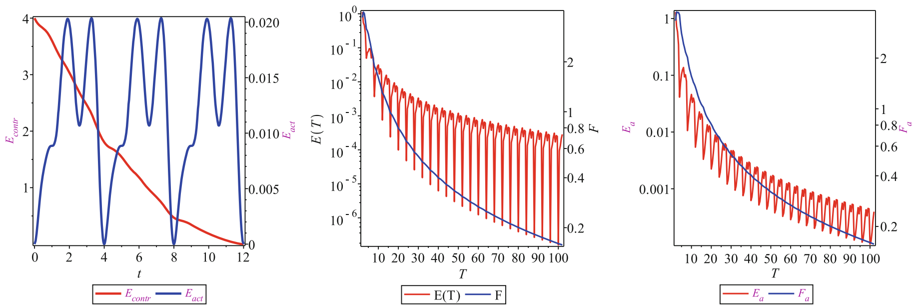

Fig. 3. The mechanical energy of the controlled and actuated modes during the motion (left). The terminal mechanical energy and the energy losses (middle). The approximate terminal mechanical energy and the energy losses (right)

5.2 Discussion

It follows from the presented results (see Figs. 2, 3) that the proposed control strategy allows for achieving a state relatively close to zero except for the zeroth mode. The velocity of this mode is rather small at $t = T$, but its amplitude, which corresponds to the motion of the rod as a rigid body, grows in time. This is due to the cost functional Φ (17) not including $v_0(T)$ since the change of the rod's position does not affect the mechanical energy of the rod. To minimize the displacement of the rod as a rigid body, the initial setup should be modified, e.g. by adding elastic springs at rod's ends. Then, all the modes will be described by similar pendulum equations.

Choosing the time horizon T larger, we allow for more precise control, see Fig. 3(middle). As can be expected for a vibrational system, for some T, specifically, multiples of 4—the period of the first mode—the proposed strategy is distinctively more effective. Although the actuated modes are not excited significantly in general, for the above-mentioned T their terminal amplitudes are several orders smaller than the terminal perturbation of the controlled modes.

To estimate analytically the terminal amplitudes of both controlled and actuated modes, we introduce the asymptotic control u_a^k. As seen from Fig. 2(right), this control law provides an approximation that becomes more accurate as k grows, that is if we utilize more control elements (make N larger). The relative error of the energy losses F (see Fig. 3, right) achieves about 10^{-4} as T grows for fixed $N = 3$. However, the approximate control (26) does not contain feedback in position. Therefore, the terminal values of displacement amplitudes are not zeros: $y_{1a}^{(k)}(T) \sim \frac{1}{kT}$ while $y_{2a}^{(k)}(T) = 0$, providing larger values of the terminal energy (of the order $\sum_{k=1}^{N} \frac{1}{k^2T^2} + \sum_{n>N} \frac{1}{n^2T^2}$) than numerical (optimal) solution. It is worth mentioning that this estimate is stable: the approximate terminal energy is about 10 times larger than the exact one for all T except for multiples of 4, when the exact energy is much smaller.

6 Conclusions

In this paper, we study a vibrating system controlled by distributed and boundary inputs, where the distributed force is piecewise constant in space. We have shown that the continuum system is split into a finite number of subsystems each actuated by certain linear combination of inputs. We consider an optimal control problem of suppressing the vibrations by minimizing the terminal mechanical energy and energy losses. Taking an approximation of the original control problem, the motion of the lowest mode in each group is optimized and the optimal feedback control law is found explicitly by means of LQR theory. For the subsystem containing the zeroth mode, the amplitude of both controlled and actuated modes are obtained analytically. Amplitudes of the other subsystems are described numerically. To show that the amplitudes of the higher modes are not excited significantly, we have derived an approximate feedback control and the corresponding mode behavior. We present a numerical example and analyze the optimal solution and its analytical approximation.

We plan to derive rigorous estimates of quality of suboptimal control and combine the proposed feedback strategy with recently developed feedforward approach providing analytical solution [9]. Also, since the piezoelements may serve both as actuators and sensors, it is promising to add observes in the model.

References

1. Glowinski, R., Lions, J.L.: Exact and approximate controllability for distributed parameter systems. Acta Numer. **3**(269378), 269–378 (1994)
2. Preumont, A.: Vibration Control of Active Structures, 3rd edn. Springer, Berlin (2011). https://doi.org/10.1007/978-94-007-2033-6
3. Tzou, H.S.: Piezoelectric Shells: Sensing, Energy Harvesting, and Distributed Control. Springer, Dordrecht (2019). https://doi.org/10.1007/978-94-024-1258-1
4. Mohith, S., et al.: Recent trends in piezoelectric actuators for precision motion and their applications: a review. Smart Mater. Struct. **30**, 013002 (2021)
5. Shivashankar, P., Gopalakrishnan, S.: Review on the use of piezoelectric materials for active vibration, noise, and flow control. Smart Mater. Struct. **29**, 053001 (2020)
6. Akulenko, L.D.: Reduction of an elastic system to a prescribed state by use of a boundary force. J. Appl. Math. Mech. **45**(6), 827–833 (1981)
7. Sontag, E.: Mathematical Control Theory: Deterministic Finite Dimensional Systems. Springer, New York (1998). https://doi.org/10.1007/978-1-4612-0577-7

8. Ovseevich, A., Ananievski, I.: Robust feedback control for a linear chain of oscillators. J. Optim. Theory Appl. **188**, 307–316 (2021). https://doi.org/10.1007/s10957-020-01765-z
9. Kostin, G., Gavrikov, A.: Optimal motions of an elastic structure under finite-dimensional distributed control (submitted, 2022)