Visualizing Polymorphisms and Counter-Polymorphisms in S5 Modal Logic

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Abstract. We present a graphical representation that allows us to easily determine if a certain modal function is or is not a polymorphism of a given relation. While doing so, we provide a comparison between two ways (a calculative and a diagrammatic one) to analyze a claim about the Sheferness criterion in the theory of clones of (S5) modal functions.

Keywords: Modal logic · Clone theory · Sheffer functions

1 Introduction

We exhibit a rather complex logical/mathematical problem involving calculations that are pretty laborious when done by ordinary means, but which can be readily seen using diagrams.

In his excellent paper 'On functional completeness in the modal logic S5' [8] the Moldavian logician M. F. Ratsa commits a slight imprecision: he claims that a certain formula (f_{21}) is an example of an exclusive polymorphism (in a sense to be defined precisely) of a certain relation (R_{21}). We use an extension of the technique presented in an earlier paper [5] in order to show that his claim is incorrect (the technique is not necessary but, as we expect to show, useful), and we provide an alternative formula.

We start by giving an interpretation of S5 formulas as operations on n-dimensional cubes (we will focus on $n \leq 4$); then we define the *relation expressed* by a formula. Next, we define the notion of *polymorphism* of a relation, after giving a list of relations whose polymorphisms are maximal clones of modal operations. All these notions and results can be found in [8].

We then proceed to the elaboration and refutation of the claim about f_{21}, and we finish our paper presenting the above-mentioned alternative formula. We try to keep this material self-contained, but acquaintance with [5] can be helpful while interpreting the diagrams presented here.

2 Modal Formulas as Operations on Cubes

Following Ratsa, we will associate formulas of propositional S5 to operations on the structures A_1, A_2, and A_3 (cf. Fig. 1). We can think of each A_n as an

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n-dimensional cube or (using the familiar notion of a *proposition* as a set of possible worlds) as the set of all propositions in a model with n possible worlds.

An interesting way to interpret the structures A_n is thinking of them as the set of all bitstrings (i.e. sequences of 0's and 1's) of length n (cf. [2]). In fact, the bitstrings of length n can be seen as a sort of *characteristic function* of the propositions in the models with n possible worlds; e.g. in the model with two possible worlds the necessary proposition will be characterized as **11**, the contingent propositions as **10** and **01**, and the impossible proposition as **00**.

One advantage of thinking of propositions as bitstrings is that it is simple to define how boolean (and modal) operations behave on bitstrings, and if we wish we can translate these definitions back into the more philosophical realm of propositional operations.

The boolean operations on bitstrings can be defined in terms of bitwise (usual) boolean operations. Let $B = b_1, ..., b_n$ and $S = s_1, ..., s_n$ be bitstrings of length n. We define the *bitstring negation* $\neg B$ as the bitstring whose terms are, respectively, $\neg b_1, ..., \neg b_n$; and we define the *bitstring conjunction* $B \wedge S$ as the bitstring whose terms are, respectively, $b_1 \wedge s_1$, ... , $b_n \wedge s_n$. The modal operator $\Box$ has the rule: $\Box B = B$ if B has 1 in every bit, otherwise $\Box B =$ the bitstring of same length as B which has 0 in every bit.

Since we are here dealing with Ratsa's results, we also present the names he uses to refer to the elements of the structures A_1, A_2, and A_3. The elements of A_1 he calls simply 1 and 0. As for the elements of A_2: 1 stands for **11**, ρ stands for **10**, σ stands for **01**, and 0 stands for **00**. For A_3: 1 stands for **111**, ω stands for **110**, ν stands for **101**, σ stands for **011**, ρ stands for **100**, μ stands for **010**, ε stands for **001**, and 0 stands for **000**. Ratsa also has names for the elements of A_4 but, since we will not enter into details about A_4 here, we will omit them.

On top of all that, we decided (cf. Fig. 1) to give *colors* to the elements of each structure! The choice of colors is quite arbitrary, but we tried to organize them. The colorful colors in $A3$ are arranged almost like a rainbow, going from infrared to ultraviolet. The choice of black for 0 was suggested by the fact that the RGB code for black is (0, 0, 0). This use of colors allows us to represent operations over these structures (cf. Figs. 2 and 3).

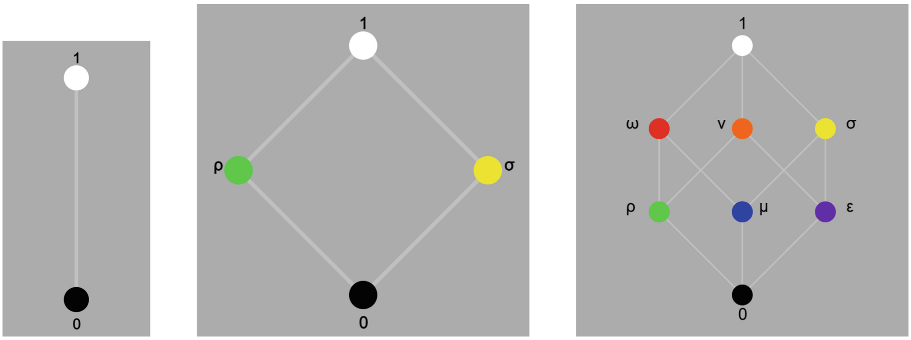

Fig. 1. Structures A1, A2, and A3

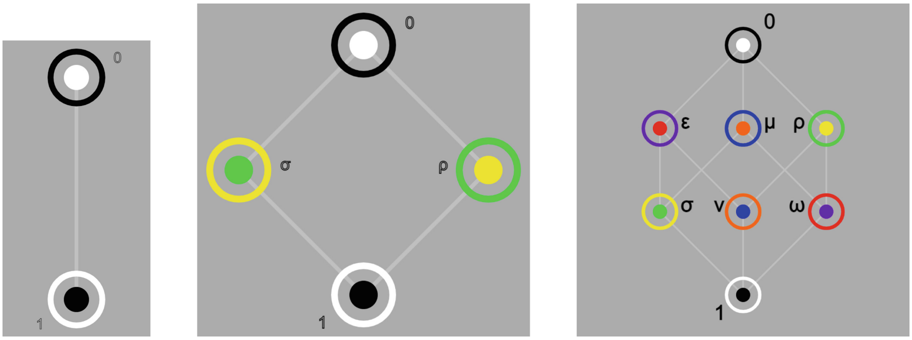

Fig. 2. A1, A2, and A3 under the effect of ¬. It is helpful to notice that the complementary elements of each structure (other than 0 and 1) have names that are either graphically similar (ω, ε / σ, ρ) or phonetically similar (ν, μ).

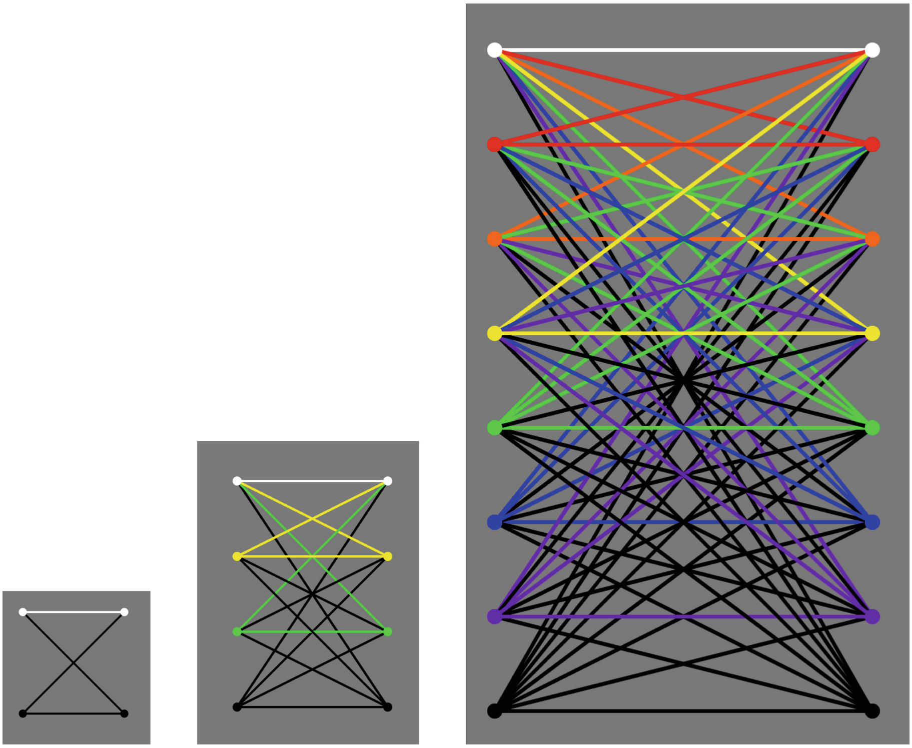

Fig. 3. The *graph representation of a binary operation on a structure* A is a function *from* the edges of the complete bipartite graph whose parts are copies of A *to* the elements of A. We 'abbreviate' this representation by giving colors to the elements of A and to the edges themselves. Here we see the action of $\wedge$ over $(A_1)^2, (A_2)^2$, and $(A_3)^2$. Much of the work done in this paper uses these graphs; most of the explanations on the graphs are given in the captions following their figures.

3 Modal Formulas as Relations on Cubes

For any formula $\phi(p_1, ..., p_n)$ of propositional S5, and any m-dimensional cube A_m, we can think of the A_m*-relation expressed by* ϕ as the set of n-tuples $\langle t_1, ..., t_n\rangle \in (A_m)^n$ such that $\phi(t_1, ..., t_n) = 1$. For instance, the formula $p \veebar q$ expresses the A_1-relation of *difference* $\{\langle 0, 1\rangle, \langle 1, 0\rangle\}$ (it also represents the relation of complementarity for every A – cf. Fig. 8). Also, the formula $p \rightarrow q$ expresses the A_1-relation *less than or equal to* $\{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle\}$. To give an example involving modality and a bigger cube, we note that the formula $p \leftrightarrow \Box q$ expresses the A_2-relation $\{\langle 0, 0\rangle, \langle 0, \rho\rangle, \langle 0, \sigma\rangle\ \langle 1, 1\rangle\}$. We give plenty of other examples in the next section.

4 Ratsa's Relations

The relations presented in this section (together with a pair of relations on A_4, omitted here for the sake of simplicity) constitute a *functional completeness criterion for sets of operations of propositional S5*. The proof of this fact is beyond the scope of this paper (details can be checked in [8] or in [4]), but some elaboration on it will be found in the next sections.

We start by considering some A_1-relations. Here $E^4(p, q, r, s)$ means: *there is an even number of truths among p, q, r, s* (a definition of E^4 in terms of the usual connectives is: $E^4(p, q, r, s) =_{df} (p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s))$. When defining a relation, we simply state a formula that expresses it. There is a correspondence between relations and matrices, to be clarified in the next section.

$$R_0 =_{df} \neg p, R_1 =_{df} p, R_2 =_{df} p \veebar q, R_3 =_{df} p \rightarrow q R_4 =_{df} E^4(p, q, r, s).$$

The corresponding A_1-matrices are:

$$M_0 = \begin{bmatrix} 0 \end{bmatrix}$$
$$M_1 = \begin{bmatrix} 1 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$M_4 = \begin{bmatrix} 0&0&0&0&1&1&1&1 \\ 0&0&1&1&0&0&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&1&1&0&1&0&0&1 \end{bmatrix}$$

We proceed to consider some A_2-relations. Here $\triangledown p$ reads 'it is contingent that p' and is defined as $\Diamond p \wedge \Diamond \neg p$; $\triangle p$ reads 'it is rigid that p' and is defined as $\neg \triangledown p$; $\triangledown^+ p$ reads 'it is contingently true that p' and is defined as $\triangledown p \wedge p$; $\triangledown^- p$

reads 'it is contingently false that p' and is defined as $\bigtriangledown p \wedge \neg p$; $\neg \bigtriangledown^{-} p$ reads 'it is not contingently false that p'; $\neg \bigtriangledown^{+} p$ reads 'it is not contingently true that p'. (Roderick Batchelor devised this notation for the more exotic unary modal functions. See [1].)

$R_5 =_{df} \bigtriangledown^{-} p$, $R_6 =_{df} \neg \bigtriangledown^{-} p$, $R_7 =_{df} \bigtriangledown p$, $R_8 =_{df} \Box(p \leftrightarrow \Box q)$, $R_9 =_{df} \Box(p \leftrightarrow \Diamond q)$, $R_{10} =_{df} (\Box p \wedge q) \vee (\neg\Diamond p \wedge \neg q)$, $R_{11} =_{df} \Box(p \leftrightarrow \Box q) \vee \Box(p \leftrightarrow \Diamond q)$, $R_{12} =_{df} \Box(p \leftrightarrow \Box q) \vee \Box(\neg p \leftrightarrow \Diamond q)$, $R_{13} =_{df} \Box(\neg p \leftrightarrow \Box q) \vee \Box(p \leftrightarrow \Diamond q)$, $R_{14} =_{df} \Box(\bigtriangledown^{+} p \leftrightarrow \bigtriangledown^{+} q)$, $R_{15} =_{df} \bigtriangleup p \leftrightarrow \bigtriangleup q$, $R_{16} =_{df} (p \leftrightarrow q) \vee (\bigtriangledown p \leftrightarrow \bigtriangledown q)$, $R_{17} =_{df} \bigtriangleup p \vee \bigtriangleup q$, $R_{18} =_{df} \bigtriangleup p \wedge \bigtriangleup r \wedge ((p \leftrightarrow r) \vee \bigtriangleup q)$, $R_{19} =_{df} \bigtriangleup p \wedge \bigtriangleup r \wedge ((p \leftrightarrow r) \vee \bigtriangledown q)$.

The corresponding A_2-matrices are:

$$M_5 = \begin{bmatrix} \rho \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 0 & \sigma & 1 \end{bmatrix}$$

$$M_7 = \begin{bmatrix} \rho & \sigma \end{bmatrix}$$

$$M_8 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \rho & \sigma & 1 \end{bmatrix}$$

$$M_9 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & \rho & \sigma & 1 \end{bmatrix}$$

$$M_{10} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & \rho & \sigma & 1 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & \rho & \sigma & \rho & \sigma & 1 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \rho & \sigma & 1 & 0 & 1 \end{bmatrix}$$

$$M_{13} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \rho & \sigma & 1 \end{bmatrix}$$

$$M_{14} = \begin{bmatrix} 0 & 0 & \rho & \sigma & 1 & 1 \\ 0 & 1 & \rho & \sigma & 0 & 1 \end{bmatrix}$$

$$M_{15} = \begin{bmatrix} 0 & 0 & \rho & \rho & \sigma & \sigma & \sigma & 1 & 1 \\ 0 & 1 & \rho & \sigma & \rho & \rho & \sigma & 0 & 1 \end{bmatrix}$$

$$M_{16} = \begin{bmatrix} 0 & \rho & \rho & \sigma & \sigma & 1 \\ 0 & \rho & \sigma & \rho & \sigma & 1 \end{bmatrix}$$

$$M_{17} = \begin{bmatrix} 0 & 0 & 0 & 0 & \rho & \rho & \sigma & \sigma & 1 & 1 & 1 & 1 \\ 0 & \rho & \sigma & 1 & 0 & 1 & 0 & 1 & 0 & \rho & \sigma & 1 \end{bmatrix}$$

$$M_{18} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \rho & \sigma & 1 & 1 & 0 & 0 & \rho & \sigma & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$M_{19} = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1 \\ 0\,\rho\,\rho\,\sigma\,\sigma\,1\,0\,\rho\,\rho\,\sigma\,\sigma\,1 \\ 0\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,1 \end{bmatrix}$$

Finally, we consider the A_3-relations corresponding to the following matrices. We do not have S5 formulas expressing these relations.

$$M_{20} = \begin{bmatrix} 0\ \rho\ \mu\ \nu\ \sigma\ 1 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 0\ \rho\ \mu\ \varepsilon\ \omega\ \nu\ \sigma\ 1 \\ 0\ \sigma\ \sigma\ \sigma\ \rho\ \rho\ \rho\ 1 \end{bmatrix}$$

$$M_{22} = \begin{bmatrix} 0\ \rho\ \mu\ \varepsilon\ \omega\ \nu\ \sigma\ 1 \\ 0\ \rho\ \nu\ \omega\ \varepsilon\ \mu\ \sigma\ 1 \end{bmatrix}$$

$$M_{23} = \begin{bmatrix} 0\ \rho\ \mu\ \varepsilon\ \omega\ \nu\ \sigma\ 1 \\ 0\ \sigma\ \omega\ \nu\ \mu\ \varepsilon\ \rho\ 1 \end{bmatrix}$$

5 Polymorphisms and Counter-Polymorphisms

We say that an n-ary operation $f(p_1, ..., p_n)$ *is a polymorphism of an m-ary A-relation* R if for every $\alpha_{ij} (i = 1, ..., m; j = 1, ..., n) \in A$:
if

$$R(\alpha_{11}, \alpha_{21}, ..., \alpha_{m1}) \wedge R(\alpha_{12}, \alpha_{22}, ..., \alpha_{m2}) \wedge ... \wedge R(\alpha_{1n}, \alpha_{2n}, ..., \alpha_{mn})$$

then

$$R(f(\alpha_{11}, \alpha_{12}, ..., \alpha_{1n}), f(\alpha_{21}, \alpha_{22}, ..., \alpha_{2n}), ... , f(\alpha_{m1}, \alpha_{m2}, ..., \alpha_{mn})).$$

In this definition, the relation R can be replaced by a matrix M whose columns are the m-sequences of elements of A satisfying R (say, arranged in the 'alphabetical' order induced by the order: $0, \rho, \mu, \varepsilon, \omega, \nu, \sigma, 1$). We say that *a matrix M' is a submatrix of a matrix M* if all columns of M' are columns of M. If M' is a submatrix of M we may write $M' \subseteq M$. Given an n-ary formula f and a matrix M with n columns, by $f(M)$ we mean the column generated applying f in each row of M. If c is a column of matrix M we may write $c \in M$ (or, if that is not the case, $c \notin M$). Using these notions, the above definition can be restated (equivalently, but perhaps more clearly) as follows:

A formula f is a counter-polymorphism of matrix M if there is an $M' \subseteq M$ such that $f(M') \notin M$.

A formula f is a polymorphism of matrix M if f is not a counter-polymorphism of M. If f is a polymorphism of M, we may write $f \in Pol(M)$.

Given a formula f and a family of relations $R^* = \langle R_1, ..., R_k \rangle$, the *polymorphic profile of f w.r.t. R^** is the k-tuple whose i-th term is 1 if $f \in Pol(R_i)$, and 0 otherwise. We say that f *is an exclusive polymorphism of R_i* (w.r.t. R^*) if the polymorphic profile of f has a single occurrence of 1, in its i-th place.

Let R^* be the family of the relations in Sect. 4 supplemented by the omitted A_4-relations R_{24} and R_{25}. Ratsa established that a set of modal operations F is *functionally complete* (i.e. sufficient to define every modal operation) if, for every relation $r \in R^*$ there is an operation $f \in F$ such that $f \notin Pol(r)$.

In the reminder of this paper we only consider the family of relations presented in Sect. 4, so when we say *the polymorphic profile of* f we mean the polymorphic profile of f w.r.t. the family of relations in Sect. 4.

6 Diagrams for Polymorphisms and Counter-Polymorphisms on A_1

In this section we consider a simpler version of the diagrams that will be presented in the end of this paper. In Figs. 4 and 5 we consider the polymorphic profile of the functions $\wedge$ and $\downarrow$ w.r.t. the relations $R_0 - R_4$. These are the relations whose sets of polymorphisms are precisely the pre-complete systems of two-valued functions, determined by Emil Post in [7].

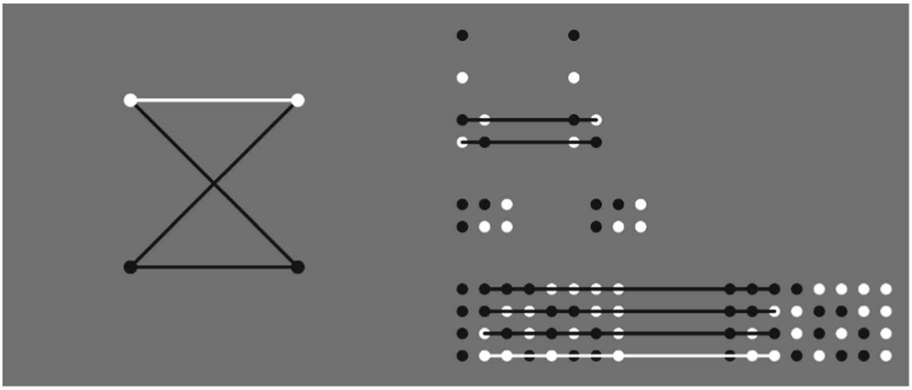

Fig. 4. The action of $\wedge$ on $(A_1)^2$ (on the left) and its polymorphic profile (on the right). We can see that $\wedge$ is a polymorphism of R_0 since the line connecting the black nodes in the left part is black; it is also a polymorphism of R_1 since the line connecting the white nodes in the left part is white; it is a counter-polymorphism of R_2 since, as the figure indicates $0 \wedge 1 = 0$ and $1 \wedge 0 = 0$, i.e. we can use arguments which are different to get values that are equal; it is a polymorphism of R_3, as the absence of lines connecting the copies of M_3 indicates, and is a counter-polymorphism of R_4, since (as indicated) with $\wedge$ we can construct, using arguments in M_4, a column of values that is not in M_4.

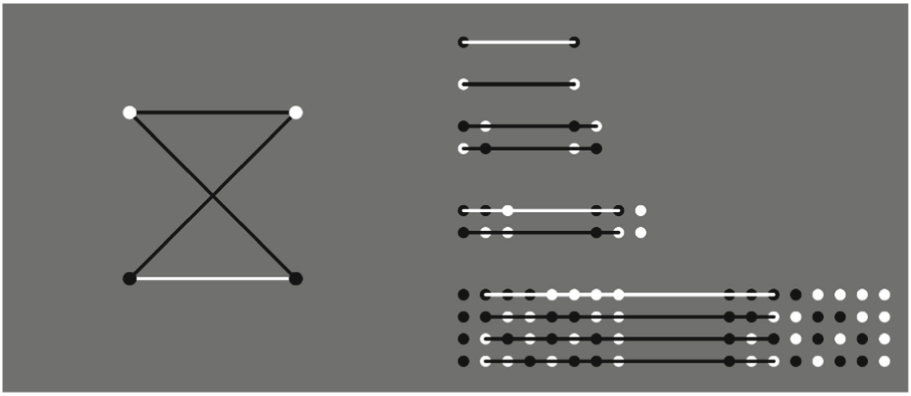

Fig. 5. $\downarrow$ and its polymorphic profile. It is well known that Peirce's arrow is a function in terms of which every other truth-function can be defined. This follows immediately from the fact that it is a counter-polymorphism of all relations R_0 – R_4, which characterize the maximal pre-complete systems of truth-functions.

7 Ratsa's Alleged Exclusive Polymorphism

Ratsa claims that a certain formula (which we call f_{21}) is an exclusive polymorphism of the relation R_{21} (or, what is the same, of the matrix M_{21}). He is interested in such a formula because it helps him to prove that his criterion for determining if a single function is functionally complete (i.e. if it is a Sheffer function for S5) is as good as it can be (cf. [8], p. 278).

To properly present Ratsa's formula, we introduce some preliminary notions (which are interesting in themselves). We start by defining the straightforward propositional relations of *independence*, *connection*, *compatibility*, and *incompatibility*:

$$Ind(p, q) =_{df} \Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \Diamond(\neg p \wedge \neg q).$$
$$Con(p, q) =_{df} \neg Ind(p, q).$$
$$Comp(p, q) =_{df} \Diamond(p \wedge q).$$
$$Incomp(p, q) =_{df} \neg\Diamond(p \wedge q).$$

It is interesting to notice that for $A_i (i \in \{1, 2, 3\})$ the A_i-relation expressed by $Ind(p, q) = \varnothing$. In order to find a pair of independent propositions, we need to resort to A_4 (this fact is noted w.r.t. bitstrings in [2], except that what we call independence is there called *unconnectedness*. In their terminology: 'unconnectedness requires bitstrings of length at least 4').

Since the compatibility relation will be significant in our next definition, we give an explicit characterization of its A_3-instances, from which the other instances may be derived. We start the characterization by listing some compatible elements of $(A_3)^2$: $\langle\mu, \omega\rangle$, $\langle\mu, \sigma\rangle$, $\langle\varepsilon, \nu\rangle$, $\langle\varepsilon, \sigma\rangle$, $\langle\nu, \omega\rangle$, $\langle\nu, \rho\rangle$, $\langle\nu, \sigma\rangle$, $\langle\omega, \rho\rangle$, $\langle\omega, \sigma\rangle$

and we finish it by noticing that everything different from 0 is compatible with 1 and with itself, and that compatibility is a symmetric relation.

The *modal profile* of a pair of propositions p, q is the 4-tuple $Modpro(p, q) =_{df} \langle Comp(p, q), Comp(p, \neg q), Comp(\neg p, q), Comp(\neg p, \neg q)\rangle$.

To present f_{21} we need to introduce some formulas used in its definition.

$$S(p, q) =_{df} \Box(p \vee q) \vee \Box(p \rightarrow q) \vee \Box(q \rightarrow p).$$
$$V(p, q) =_{df} S(p, q) \wedge S(p, \neg q) \wedge S(\neg p, q) \wedge S(\neg p, \neg q).$$

S 'says' that p and q are connected even if we disregard its (possible) incompatibility (or equivalently: there is at least one 0 in the last three entries of $Modpro(p, q)$), while V 'says' that p and q are *strongly connected*, i.e., either (at least) one of them is rigid, or they are both contingent but then either $\Box(p \leftrightarrow q)$ or $\Box(p \veebar q)$ (this is equivalent to say that sum of the terms of $Modpro(p, q)$ is less than 3).

Ratsa's formula is:

$$f_{21} = (V(p, q) \rightarrow ((p \rightarrow q) \wedge \neg\Box q)) \wedge (((p \leftrightarrow S(p, q)) \wedge (q \rightarrow S(p, q))) \vee V(p, q)).$$

To see that this is not an exclusive polymorphism of R_{21} it is enough to notice that it is not a polymorphism of R_{21}. This is obvious given that $\{\langle\rho, \sigma\rangle, \langle\mu, \sigma\rangle\} \subseteq R_{21}$ and $f_{21}(\rho, \mu) = \varepsilon$, $f_{21}(\sigma, \sigma) = 1$ and that $\langle\varepsilon, 1\rangle \notin R_{21}$. This last claim can perhaps be more easily checked by considering Fig. 10, where we present the action of f_{21} over A_1, A_2, and A_3, and its polymorphic profile.

8 Moody Truth-Functions

The definition in this section is essentially the same found in [3], p. 35. Recall the definition of $Modpro$, given in the last section.

The *moody truth-functional representation of a binary modal operation f* is a sequence of eight binary truth-functions $\langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\rangle$, together with the proviso:

if $Modpro(p, q) = \langle 1, 1, 1, 1\rangle$, apply f_1;
if $Modpro(p, q) = \langle 1, 1, 1, 0\rangle$ or $\langle 0, 0, 0, 1\rangle$, apply f_2;
if $Modpro(p, q) = \langle 1, 1, 0, 1\rangle$ or $\langle 0, 0, 1, 0\rangle$, apply f_3;
if $Modpro(p, q) = \langle 1, 0, 1, 1\rangle$ or $\langle 0, 1, 0, 0\rangle$, apply f_4;
if $Modpro(p, q) = \langle 0, 1, 1, 1\rangle$ or $\langle 1, 0, 0, 0\rangle$, apply f_5;
if $Modpro(p, q) = \langle 1, 1, 0, 0\rangle$ or $\langle 0, 0, 1, 1\rangle$, apply f_6;
if $Modpro(p, q) = \langle 1, 0, 1, 0\rangle$ or $\langle 0, 1, 0, 1\rangle$, apply f_7;
if $Modpro(p, q) = \langle 1, 0, 0, 1\rangle$ or $\langle 0, 1, 1, 0\rangle$, apply f_8.

Since we are here ignoring A_4, when using moody truth-functions, we will restrict ourselves to the 7-tuples corresponding to $f_2 - f_8$.

We claim that the operation expressed by $\langle\top, \wedge, \leftrightarrow, \uparrow, \wedge, \downarrow, \rightarrow\rangle$ is an exclusive polymorphism of R_{21}. We support our claim with Fig. 11 and with the captions of the figures preceding it.

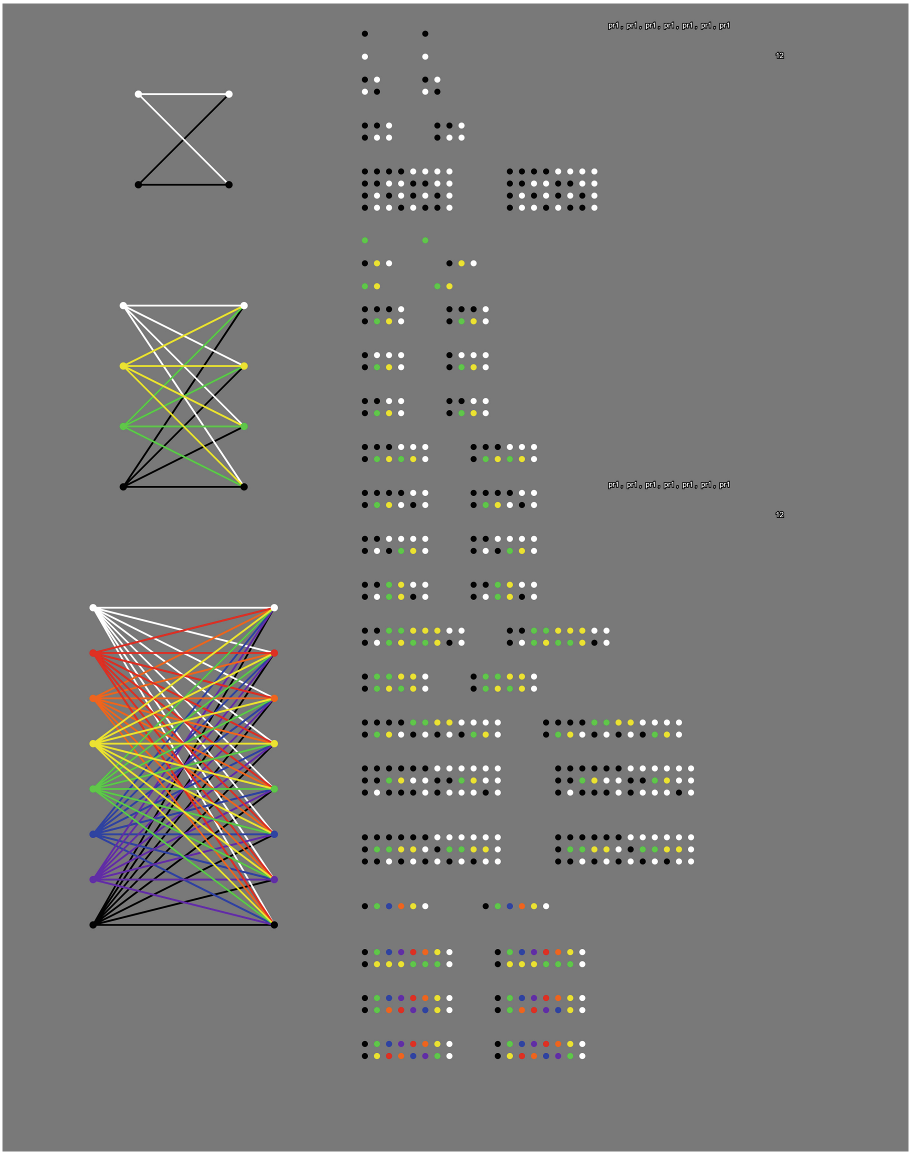

Fig. 6. The projection of the first argument (π_1^2) is a universal polymorphism. We take advantage of the space left by the absence of counter-polymorphisms of this operation to present the framework we are working with. On the right side of this figure you can see (pairs of) the translation into colors (following Fig. 1) of the relations presented in Sect. 6.

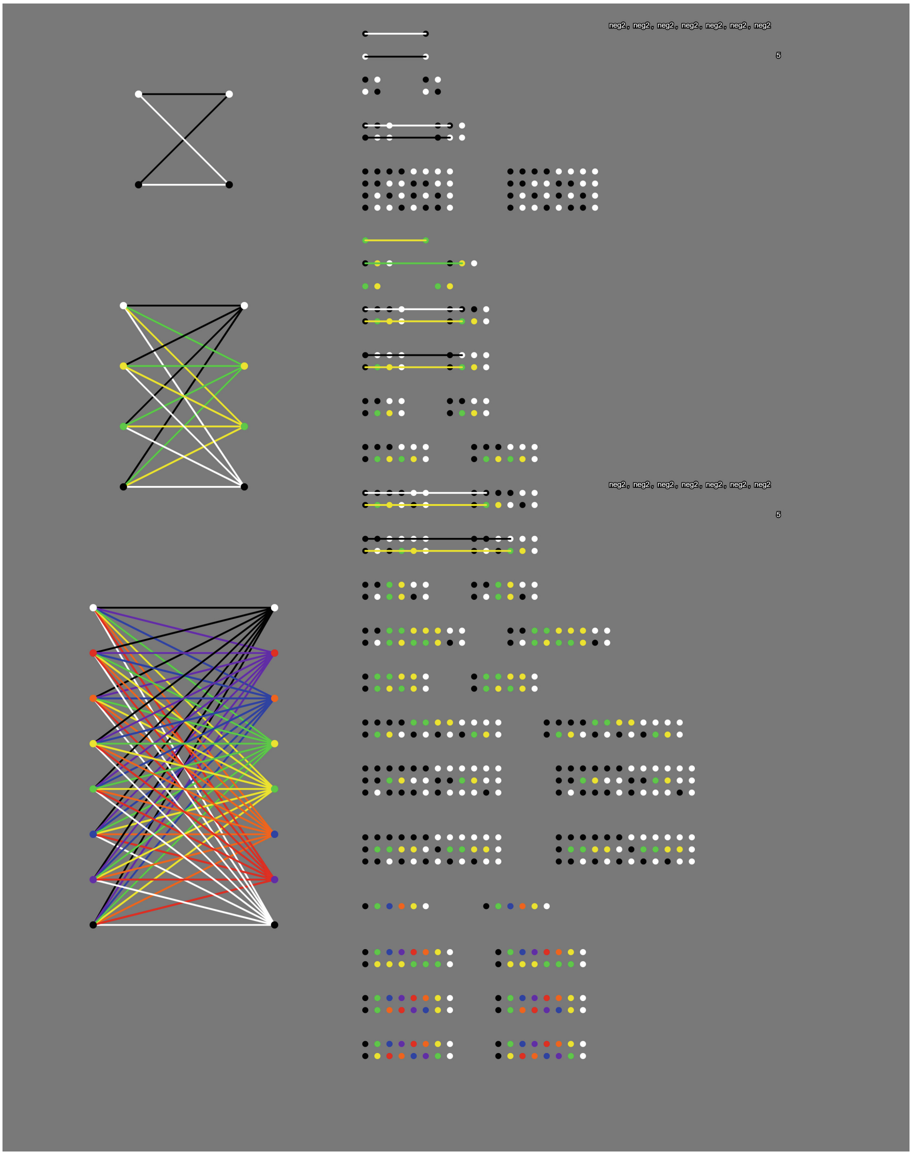

Fig. 7. The negation of the second argument $\neg\pi_2^2$ over A_1, A_2 and A_3 and its polymorphic profile. Notice that the counter-polymorphisms are indicated by horizontal lines connecting relevant columns of the matrices. Notice also that the polymorphic profile of $\neg\pi_2^2$ w.r.t. R_0 – R_4 (0, 0, 1, 0, 1) is complementary of that of $\wedge$ (1, 1, 0, 1, 0) (cf. Fig. 4).

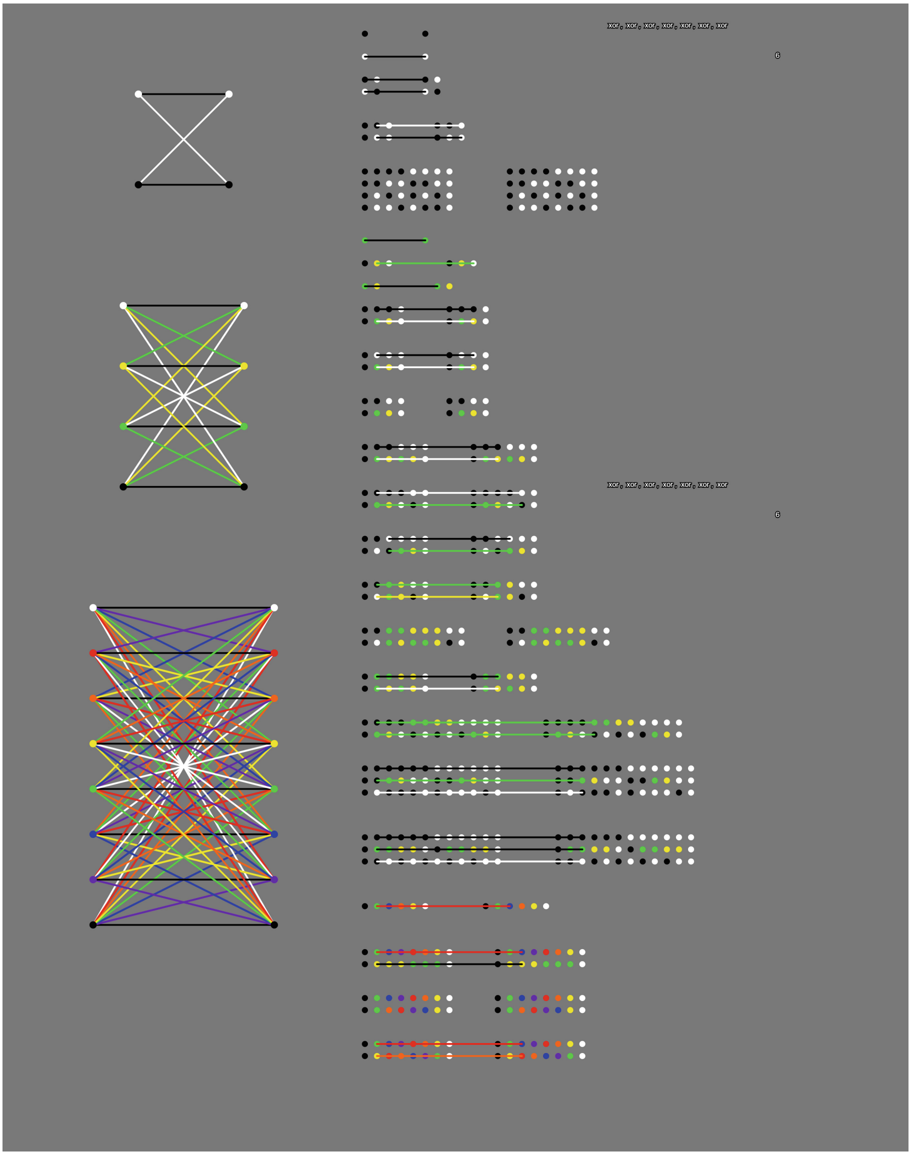

Fig. 8. This graph represents the action of $\veebar$. It is interesting that the white lines on the left side represent precisely the relation of complementarity in the structures A_1, A_2 and A_3.

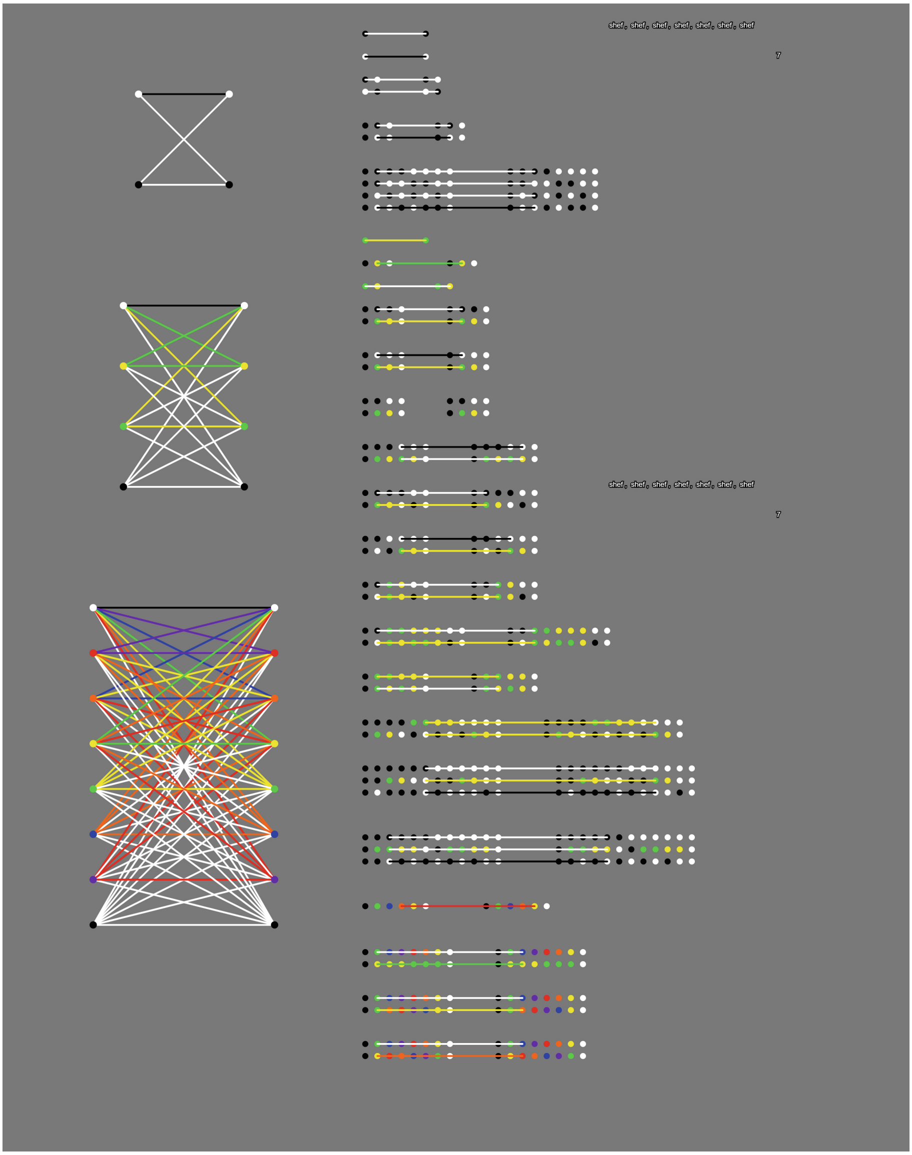

Fig. 9. This is the representation of the very well known (boolean) operation ↑ (the Sheffer stroke). Notice that it is an exclusive polymorphism of R_{10}.

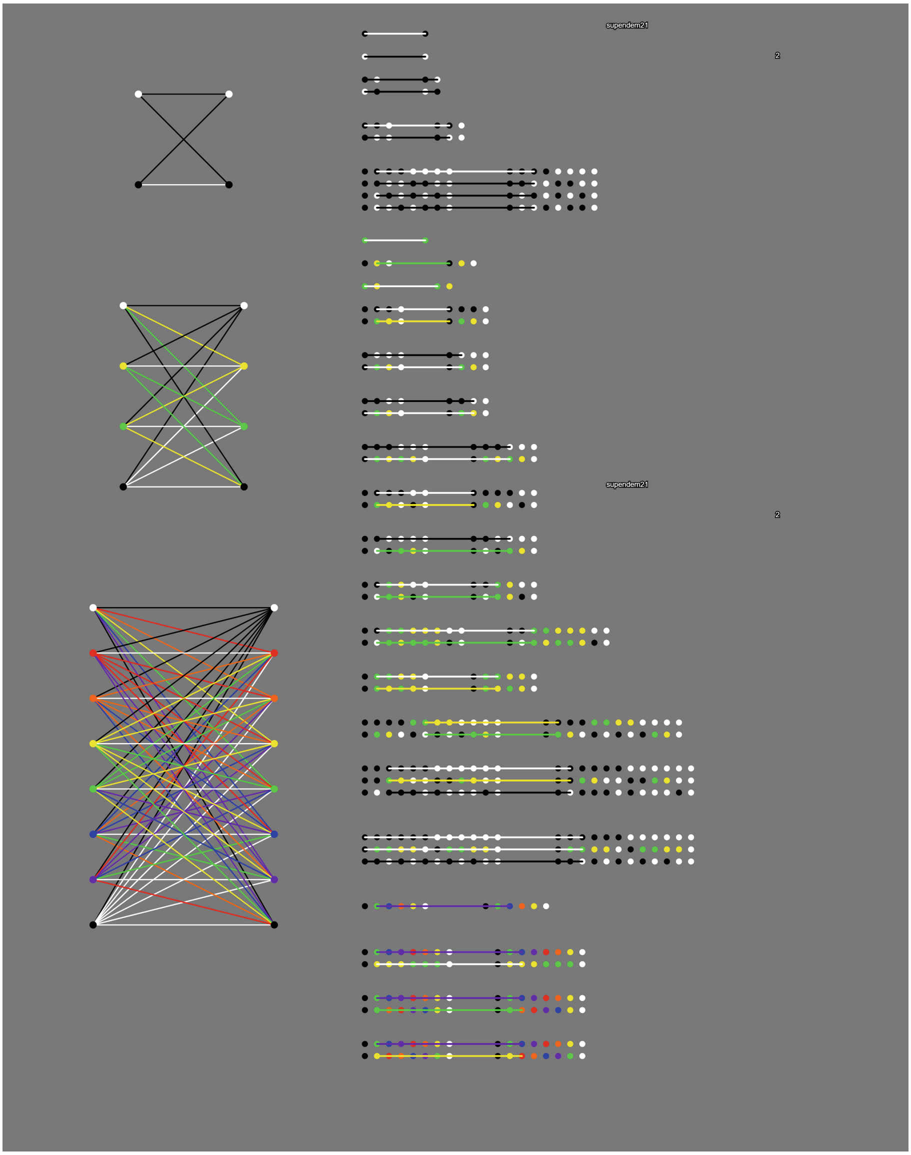

Fig. 10. f_{21} over A_1, A_2 and A_3 and its polymorphic profile.

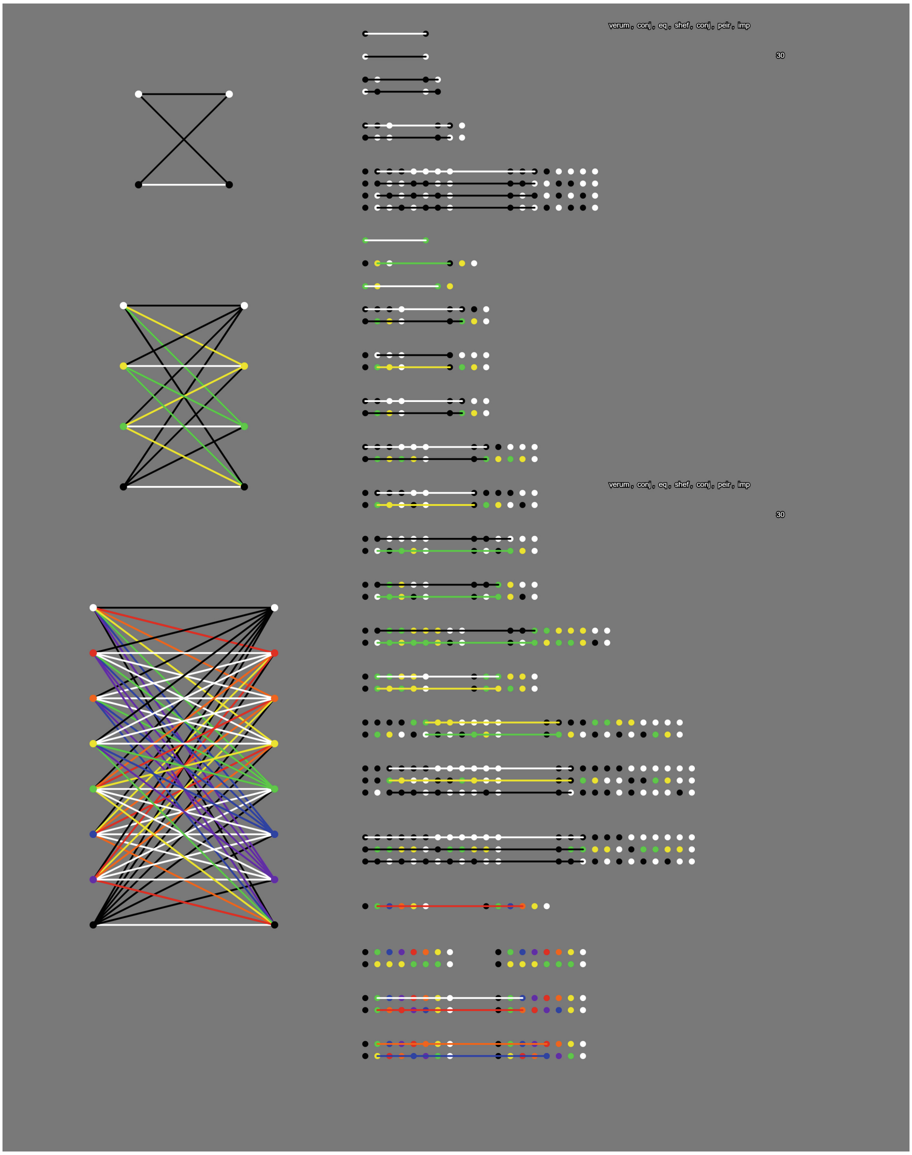

Fig. 11. $\langle \top, \wedge, \leftrightarrow, \uparrow, \wedge, \downarrow, \rightarrow \rangle$ is an exclusive polymorphism of M_{21}.

9 Conclusion

We are glad to give an exoteric presentation of a somewhat esoteric result, and we hope that this paper is not too enigmatic. We believe that the techniques presented here are also useful in the investigations on clones of k-valued functions (cf. [6]) and we expect to give some new results on this matter soon.

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