The Structure of Configurations in One-Dimensional Majority Cellular Automata: From Cell Stability to Configuration Periodicity

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Abstract. We study the dynamics of (synchronous) one-dimensional cellular automata with cyclical boundary conditions that evolve according to the majority rule with radius r. We introduce a notion that we term *cell stability* with which we express the structure of the possible configurations that could emerge in this setting. Our main finding is that apart from the configurations of the form $(0^{r+1}0^* + 1^{r+1}1^*)^*$, which are always fixed-points, the other configurations that the automata could possibly converge to, which are known to be either fixed-points or 2-cycles, have a particular spatially periodic structure. Namely, each of these configurations is of the form s^* where s consists of $O(r^2)$ consecutive sequences of cells with the same state, each such sequence is of length at most r, and the total length of s is $O(r^2)$ as well. We show that an analogous result also holds for the minority rule.

1 Introduction

Dynamic processes that evolve according to the majority rule arise in various settings and as such have received wide attention in the past, primarily within the context of propagation of information or influence (e.g., [7,12,17]). Here we consider perhaps the most basic case, that of one-dimensional cellular automata, where our focus is on analyzing the structure of the configuration space. Specifically, we analyze the configuration space of one-dimensional cellular automata with cyclical boundary conditions that evolve according to the majority rule with radius r.

It is well-known [8,13] that these processes always converge to configurations that correspond to cycles either of length 1 (fixed-points) or of length 2 (period-2 cycles). In particular, it is easy to verify (see, e.g., [14]) that configurations in which each cell belongs to a consecutive sequence of at least $r + 1$ cells with the same state[1] are fixed-points. Not much is currently understood, however, about the structure of the other fixed-point configurations or of configurations that correspond to cycles of length 2.

[1] In this work, a state is a value in $\{0, 1\}$.

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The reason for this gap in understanding is largely due to the fact that most previous research has made assumptions about the mechanism producing the initial configuration. Namely, it is usually assumed that the state of each cell in the initial configuration is randomly chosen, independently from the other cells. See, for instance, the theoretical analysis in [14] and the experimental results in [15], both for one-dimensional majority cellular automata (and also the references within Sect. 5 for examples in other models). Under such assumptions, as shown in [14], these other configurations are indeed rarely encountered.

In this work, we tackle the problem of understanding the structure of the possible configurations without making assumptions about the mechanism behind the generation of the initial configuration. One of our main results (stated formally in Theorem 1) is that all period-2 configurations and all fixed-point configurations (other than those mentioned above) have a very special structure. Specifically, they have a "spatially" periodic structure with a period that is quadratic in the radius r. In the course of the proof of this result, we introduce several notions and prove several claims, which we believe are of interest in their own right as they shed light on the dynamics of the majority rule in cellular automata (and not only on the configurations they converge to).

1.1 Organization

In Sect. 2 we formally define the majority rule and other basic terms required for the formulation of our results. Then, in Sect. 3, we introduce the notion of cell stability and state Theorem 1, which is the main result of this paper. In Sect. 3.1, we illustrate Theorem 1 for the special cases of $r = 1, 2, 3$. In Sect. 4, we discuss some of the high-level ideas behind the proof of Theorem 1. Finally, in Sect. 5, we review related work.

2 Preliminaries

2.1 The Majority Rule with Radius *r*

In all that follows, when performing operations on cells $i \in \mathbb{Z}_n$, these operations are modulo n.

Definition 1 (cell interval). *For a pair of cells $i, j \in \mathbb{Z}_n$ we use $[i, j]$ to denote the sequence $i, i+1, \ldots, j$ (so that it is possible that $j < i$), which we refer to as a cell interval.*

For an integer n, we refer to a function $\sigma : \mathbb{Z}_n \to \{0, 1\}$ as a *configuration* and view σ as a (cyclic) binary string of length n.

Definition 2 (neighborhood). *For a cell $i \in \mathbb{Z}_n$ and an integer r, the r-neighborhood of i, denoted $\Gamma_r(i)$, is the cell interval $[i - r, i + r]$. For a set of cells $I \subseteq \mathbb{Z}_n$, we let $\Gamma_r(I)$ denote the set of cells in the union of cell intervals $[i - r, i + r]$ taken over all $i \in I$.*

Given a state $\beta \in \{0,1\}$, a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ and a cell interval $[i,j]$, we denote by $\#_\beta(\sigma[i,j])$ the number of cells $\ell \in [i,j]$ such that $\sigma(\ell) = \beta$.

Definition 3 (the majority rule). *Denote by MAJ_r majority rule with radius r. That is, for a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$, $MAJ_r(\sigma)$ is the configuration σ' in which for each cell $i \in \mathbb{Z}_n$,*

$$\sigma'(i) = \begin{cases} 0 & \text{if } \#_0(\sigma[\Gamma_r(i)]) > \#_1(\sigma[\Gamma_r(i)]) \\ 1 & \text{otherwise} \end{cases}$$

For each $t \geq 0$, denote by $\mathrm{MAJ}_r^t(\sigma)$ the result of repeatedly applying the majority rule with radius r, starting from the configuration σ. In particular, $\mathrm{MAJ}_r^0(\sigma) = \sigma$ and $\mathrm{MAJ}_r^1(\sigma) = \mathrm{MAJ}_r(\sigma)$.

2.2 Temporal and Spatial Periodicity

Eventually, for every initial configuration, the majority rule, and, in fact, any rule, reaches a *cycle*: a periodic sequence of configurations. As mentioned earlier, in the case of the majority rule, that cycle is always either a *2-cycle* or a *fixed-point*.

Definition 4 (fixed-point). *We say that a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ is a fixed-point if $MAJ_r(\sigma) = \sigma$.*

Definition 5 (2-cycle). *We say that a pair of distinct configurations $\sigma, \sigma' : \mathbb{Z}_n \to \{0,1\}$ is a 2 cycle if $MAJ_r(\sigma) = \sigma'$ and $MAJ_r(\sigma') = \sigma$.*

We refer to the *configurations* that constitute a cycle as temporally periodic configurations. That is,

Definition 6 (temporally periodic). *We say that a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ is temporally periodic if $MAJ_r^2(\sigma) = \sigma$.*

Note that if a configuration σ is temporally periodic, then it is either the case that $\mathrm{MAJ}_r(\sigma) = \sigma$ (i.e., σ is a fixed-point), or $\mathrm{MAJ}_r(\sigma) = \sigma'$ for $\sigma' \neq \sigma$, in which case σ and σ' constitute a 2 cycle.

Definition 7 (transient). *If a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ is not temporally periodic, we say that σ is transient.*

Definitions 4–7 are all related to the notion of *temporal* periodicity, i.e., periodicity that occurs over time. In this paper, we relate temporal periodicity to *spatial* periodicity, i.e., periodic behavior exhibited within individual configurations. Formally,

Definition 8 (spatial period). *We say that a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ has spatial period p if p is the minimum positive integer such that for every cell $i \in \mathbb{Z}_n$, $\sigma(i+p) = \sigma(i)$.*

Definition 9 (spatially periodic). *We say that a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ is spatially periodic if its spatial period p satisfies $p < n$.*

3 Our Main Result and the Notion of Cell Stability

In this section we state our main result, Theorem 1, whose proof can be found in the full version of the paper [10] and some the proof's high level ideas appear in Sect. 4. In order to state Theorem 1, we introduce the notion of a cell's stability within a configuration via Definitions 10–12 (illustrated in Fig. 1).

Definition 10 (unstable). *We say that a cell $i \in \mathbb{Z}_n$ is* unstable *with respect to a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ if $\sigma(i) \neq \sigma''(i)$ where $\sigma'' = MAJ_r^2(\sigma)$.*

Recall that after a finite number of steps,[2] a one-dimensional cellular automaton that evolves according to the majority rule, reaches either a fixed-point or a 2 cycle. Thus, a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ is transient if and only if it contains unstable cells.

As for the "stable" cells, we define two variants: strongly stable and weakly stable.

Definition 11 (strongly stable). *We say that a cell $i \in \mathbb{Z}_n$ is* strongly stable *with respect to a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ if there exists a cell interval $[a,b]$ of length at least $r+1$ such that $i \in [a,b]$ and for each $j \in [a,b]$, $\sigma(i) = \sigma(j)$.*

Definition 12 (weakly stable). *We say that a cell $i \in \mathbb{Z}_n$ is* weakly stable *with respect to a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ if i is not strongly stable with respect to σ, but $\sigma(i) = \sigma''(i)$ where $\sigma'' = MAJ_r^2(\sigma)$.*

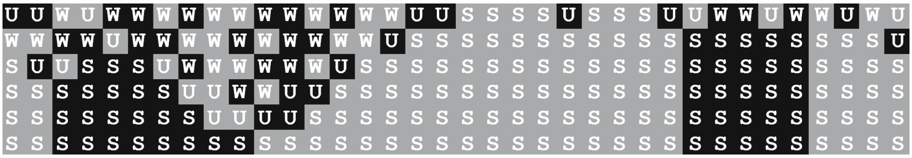

Fig. 1. The evolution under the majority rule with $r = 2$. Gray squares correspond to state-0 cells and dark squares correspond to state-1 cells. Each cell is labeled by a letter indicating the cell's *stability*, where S stands for *Strongly* stable, W for *Weakly* stable and U for *Unstable*.

The crucial property of the strongly stable cells is that their states, unlike the states of the weakly stable cells, cannot change in later configurations. In that sense, their stability is "stronger" than that of the weakly stable cells. It is worth noting, though, that if a cell lies within a long cell interval of weakly stable cells, then that cell remains weakly stable, alternating between the same pair of states, for a number of steps that depends on the cell interval length.

[2] Which can be shown to be at most linear in n [10].

Accordingly, given a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$, we say that a cell interval $[i,j]$ is strongly stable, weakly stable or unstable if all the cells in that cell interval are, respectively, strongly stable, weakly stable or unstable.

Considering complete configurations, observe that all the configurations of the form $(0^{r+1}0^* + 1^{r+1}1^*)^*$ contain only strongly stable cells. As noted previously and explained in the characterization provided in [14], these configurations are always fixed-points, which means that they are, in particular, also temporally periodic (with a period of 1). However, there are more forms of temporally periodic configurations, both period-1 and period-2, that contain only weakly stable cells and are not addressed by [14]'s characterization, as the authors of [14] were only interested in "typical" configurations, which are not of that kind.[3]

Theorem 1 complements [14]'s characterization by additionally specifying the structure of the remaining temporally periodic configurations. In addition to temporally periodic configurations, Theorem 1 also includes a property of the transient configurations that is related to the dynamics by which they eventually converge.

Theorem 1. *For any configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$, exactly one of the following must hold:*

1. *The configuration σ is a temporally periodic configuration and it is either the case that:*
   (a) *all the cells in σ are strongly stable, in which case σ is of the form $(0^{r+1}0^* + 1^{r+1}1^*)^*$), or*
   (b) *all the cells in σ are weakly stable, in which case σ is spatially periodic with spatial period at most $2r(r+1)$.*
2. *The configuration σ is a transient configuration and the length of every unstable cell interval in σ is at most $2r$.*

In the full version [10] we show that an analog of Theorem 1 holds for the minority rule as well, with analogous variants of cell stability.

Under the assumption that r is a constant, Theorem 1 directly yields an output-sensitive algorithm that, given n, generates all the temporally periodic configurations of length n. The running-time of the algorithm is linear in the number of temporally periodic configurations.

Turning to transient configurations, recall that all transient configurations contain unstable cells, and the evolution of the transient configurations can be described using the notion of cell stability. Namely, the following can be shown regarding any transient configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$ (see proofs in the full version [10]). First, the configuration $\mathrm{MAJ}_r(\sigma)$ contains strictly fewer unstable cells

[3] Indeed, it is shown in [14] that the probability that a randomly selected configuration of length n being transient approaches 1 as $n \longrightarrow \infty$. As such, the additional temporally periodic configurations that we address in this work are, in a sense, not "typical". We, in contrast to [14], make no assumption about the distribution of the configuration space, and are therefore interested in understanding the structure of *all* configurations, not only the "typical" ones.

than σ. Second, if σ contains strongly stable cells, then $\mathrm{MAJ}_r(\sigma)$ contains even more strongly stable cells than σ, and the automaton eventually converges to a fixed-point of the form defined in Case (1a). Third, if there are no strongly stable cells in σ, then there are cases in which the automaton eventually converges to a fixed-point of the form defined in Case (1a)[4] and there are also cases in which it eventually converges to a fixed-point or to a 2 cycle of the form defined in Case (1b)[5].

3.1 Illustrating Theorem 1 for $r = 1, 2, 3$

To get a feel for the nature of the statement in Theorem 1, we demonstrate some of its aspects for $r = 1, 2, 3$.

1. For $r = 1$, the temporally periodic configurations are either
   (a) of the form $(000^* + 111^*)^*$, or
   (b) of the form $(01)^*$.[6]
2. For $r = 2$, the temporally periodic configurations are either
   (a) of the form $(0000^* + 1111^*)^*$, or
   (b) of one of the following forms: $(01)^*$, $(0011)^*$, $(001101)^*$, $(001011)^*$.
3. For $r = 3$, the temporally periodic configurations are either
   (a) of the form $(00000^* + 11111^*)^*$, or
   (b) of the form s^*, where s belongs to the set:[7]

$$\left\{\begin{array}{l} 01, \\ 0011, \\ 010011, 010110, 001110, \\ 01011001, 10100101, 10100110, 01011100, 10010011, 00011101, 10110001, \\ 0011001110, 1000111001 \end{array}\right\}$$

An interesting observation about the patterns in Case (1b) in our demonstration is that the number of zeros in each of them equals the number of ones. This, in fact, holds in general, as we prove in the full version [10].

[4] e.g., for $r = 3$, the transient configuration 001001001001001001 converges after one step to the fixed-point configuration $(0)^*$.

[5] e.g., for $r = 4$, the transient configuration 001011001011001011001011001011001011 converges after one step to the 2 cycle consisting of $(111000)^6$ and $(000111)^6$.

[6] Also $(10)^*$, but since the configurations are cyclic, the patterns $(01)^*$ and $(10)^*$ correspond to equivalent sets of configurations.

[7] The string s could also be the *mirror* or the *complement* of any of the specified patterns, which we omit for the sake of conciseness. For example, since we explicitly specified that s could be 010011, it means that s could also be 110010 (which is the mirror of 010011) or 101100 (which is the complement of 010011), even though these two are not explicitly specified.

4 The Alignment Mapping (High-Level Idea)

In proving Theorem 1, we define a number of notions and establish several claims, some of which we believe are valuable in and of themselves. We decided to focus in this section on a high-level description of only a few of the ideas underlying the proof of Theorem 1. The complete proof as well as the precise definitions of the notions we introduce in order to establish the proof can be found in the full version [10]. We have chosen to highlight the high-level idea behind one of the key tools we utilize, which is a *mapping* we introduce between *blocks* of consecutive configurations.

Given a configuration $\sigma : \mathbb{Z}_n \to \{0,1\}$, we say that a cell interval $[i,j]$ is a *maximal homogeneous block* in σ with value $\beta \in \{0,1\}$ if for every cell $\ell \in [i,j]$, $\sigma(\ell) = \beta$, and also $\sigma(i-1) = \sigma(j+1) \neq \beta$ if the length of $[i,j]$ is less than n.

We refer to this mapping, defined below (and illustrated in Fig. 2), as the alignment mapping. The alignment mapping, beyond being essential for the proof of Theorem 1, has several features that make it useful for reasoning about the dynamics of the majority rule, which is why we present its definition here.

Definition 13 (alignment mapping). *Let σ and σ' be a pair of configurations satisfying $MAJ_r(\sigma) = \sigma'$. Given a block $[i',j']$ in σ', let I be the block in σ that contains the cell $i+r$ and let J be the block in σ that contains the cell $j-r$. The alignment mapping maps the block $[i',j']$ (in σ') to the* middle[8] *block $[i,j]$ between I and J in σ.*

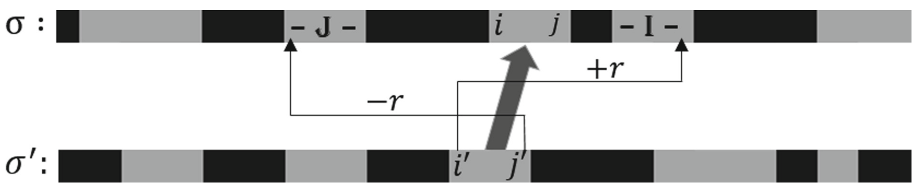

Fig. 2. *The alignment mapping.* The figure depicts a pair of configurations, σ and σ', where $\sigma' = \mathrm{MAJ}_r(\sigma)$, and also a pair of blocks, $[i,j]$ in σ and $[i',j']$ in σ', where $[i',j']$ is mapped to $[i,j]$ by the alignment mapping. The block I in σ is the one that contains the cell $i'+r$, and the block J in σ is the one that contains the cell $j'-r$. The block $[i,j]$ in σ is the one right in the middle of the interval of five blocks in σ whose left and right ends are I and J. Hence, by Definition 13, the alignment mapping maps $[i',j']$ to $[i,j]$.

We stress that the alignment mapping, as defined in Definition 13, is a *backward* mapping, in the sense that, given a configuration σ', it maps all blocks in σ' into those of the configuration σ that *precedes* σ'. This naturally suggests

[8] The middle block is well defined, as it is shown in the full version [10] that the number of blocks between I and J must be odd.

defining the notion of *the forward alignment mapping* as the *inverse* function of the backward alignment mapping that would map the blocks of the configuration σ to those of the configuration σ' that *follows* σ (for example, in Fig. 2, the forward alignment mapping maps $[i, j]$ in σ to $[i', j']$ in σ').

However, while it can be shown that the backward alignment mapping is always one-to-one, it is not necessarily *onto* (unless we apply it within a pair of temporally periodic configurations). Hence, under our definition of the forward alignment mapping, not all blocks will be mapped forward.

Formally, let $\sigma_0, ...\sigma_m$ be a sequence of configurations where $\text{MAJ}_r(\sigma_{t-1}) = \sigma_t$ for each $1 \leq t \leq m$. We define the step-t *forward alignment mapping*, denoted φ_t, as follows. Given a block $[i, j]$ in σ_t, if there is a block $[i', j']$ in σ_{t+1} such that the backward alignment mapping between the configuration pair σ_t, σ_{t+1} maps $[i', j']$ into $[i, j]$, then $\varphi_t([i, j]) = [i', j']$. Otherwise, $\varphi_t([i, j]) = \perp$. In the case in which $\varphi_t([i, j]) \neq \perp$, we also define $\varphi_t^2([i, j])$ as $\varphi_{t+1}(\varphi_t([i, j]))$.

One notable property of the forward alignment mapping is what we refer to as "identity preservation in stable intervals". Roughly speaking, consider any block $[i, j]$ residing in a sufficiently long weakly stable or strongly stable cell interval of σ_t. Then $\varphi_t([i, j]) \neq \perp$, and hence $\varphi_t^2([i, j])$ is defined and is equal to the same block $[i, j]$ we started with. In particular, for a pair of configurations comprising a 2 cycle, applying the forward alignment mapping *twice* essentially maps each block to itself.

In the proof of Theorem 1, we essentially use the forward alignment mapping and its properties to show that for a configuration in which all blocks are of length at most r, if the configuration is temporally periodic, then it is also spatially periodic. We achieve this through three steps.

In the first step, we employ the alignment mapping to express the length of each of the configuration's blocks in terms of the lengths of other $O(r)$ blocks in the preceding configuration. Specifically, given a pair of temporally periodic configurations σ_t and σ_{t+1}, we obtain a relationship between the length of each block $[i, j]$ in σ_t and the lengths of $O(r)$ consecutive blocks, belonging to a block sequence centered at the block $\varphi_t([i, j])$, in the configuration σ_{t+1}.

In the second step, we look at the *difference* between the length of each block $[i, j]$ and the lengths of the blocks at the two ends of the sequence mentioned above, and define *aligned difference vectors*, whose entries are these differences. We use the properties of the forward alignment mapping to establish that the aligned difference vectors (defined formally in the full version [10]) are spatially periodic with a spatial period that is *linear* in r.

In the third and final step, by applying the relationship between aligned difference vectors iteratively, we use the spatial periodicity of the aligned difference vectors to establish that the configurations themselves are spatially periodic as well, and that each configuration's spatial period must be quadratic in r.

5 Related Work

The main focus of most of the research on majority/minority (and more generally, threshold) cellular automata so far has been on the convergence time (e.g., [3,4,11]) and on the dominance problem[9] (e.g., [1,2,9]).

As mentioned earlier, most of the work on the problem of understanding the structure of the configuration space is based on the assumption that the initial configuration is random. For the one-dimensional case, the case with which the current paper is concerned, this includes the paper of Tosic and Agha [14]. In their paper, they distinguish between synchronous/sequential and finite/infinite majority cellular automata with radius r, and our work can be viewed as extending their result for the finite and synchronous case.

They show that whereas 2 cycles cannot emerge under the sequential model, in the synchronous model (the one we focus on in this paper), 2 cycles exist even for $r = 1$. They also show that a randomly picked configuration is a transient configuration (and, in particular, not a 2 cycle) with probability approaching 1 (both for finite and infinite configurations), and it can additionally be shown that the probability that such a random transient configuration eventually converges to a 2 cycle approaches 0. Finally, they characterize the "common" forms of fixed-point configurations (those that in our paper are described in Case (1a) of Theorem 1).

Their theoretical result is supplemented by a later experimental work [15], showing that in practice, convergence to these "common" fixed-point configurations occurs relatively quickly. Namely, the simulations in [15] demonstrate that convergence tends to occur in less than five steps for $n = 1000$ and $1 \leq r \leq 5$.

Additional work beyond the one-dimensional case includes [6] for two-dimensional majority cellular automata, [5] for majority in random regular graphs, [18] for majority in Erdos–Rényi graphs as well as expander graphs.

One notable work that does not rely on the assumption that the initial configuration is random is Turau's work [16] on characterizing all the temporally periodic configurations for majority and minority processes on trees. The characterization presented in [16] also yields an output-sensitive algorithm for generating these configurations.

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[9] In the dominance problem, one asks how many cells must initially be at a certain state so that eventually all cells have the same state.

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