On Equilibrium Positions in the Problem of the Motion of a System of Two Bodies in a Uniform Gravity Field

Valentin Irtegov and Tatiana Titorenko(✉)

Institute for System Dynamics and Control Theory SB RAS,
134, Lermontov Street, Irkutsk 664033, Russia
{irteg,titor}@icc.ru

Abstract. In the problem of motion of a system of two rigid bodies connected by a spherical hinge in a uniform gravity field, the conditions for the existence of two- and one-dimensional invariant manifolds are presented, and the manifolds themselves are found with the use of computer algebra tools. From a mechanical point of view, these solutions correspond to equilibrium positions of the system. Their instability in the first approximation is proved.

1 Introduction

This work continues the study [5]. The rotation of the system of two connected rigid bodies S_1 and S_2 (see Fig. 1) in a uniform gravity field is considered. The first body has a fixed point O_1. The bodies are connected by an ideal spherical hinge O_2.

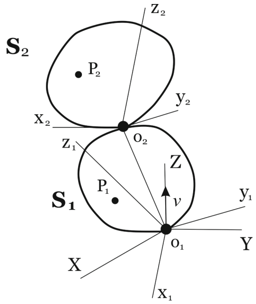

Fig. 1. .

To describe the motion of the mechanical system, the following coordinate systems are introduced: the inertial O_1XYZ (its Z axis with the unit vector $\boldsymbol{\nu}$ is directed vertically upwards), the moving frames $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$ attached rigidly to the bodies S_1 and S_2, respectively. The x_i, y_i, z_i $(i = 1, 2)$ axes are directed along the principal inertia axes of the bodies. The positions of

F. Boulier et al. (Eds.): CASC 2022, LNCS 13366, pp. 165–184, 2022.
https://doi.org/10.1007/978-3-031-14788-3_10

$O_1x_1y_1z_1$ with respect to O_1XYZ and $O_2x_2y_2z_2$ with respect to $O_1x_1y_1z_1$ are defined by Euler's angles $\psi_1, \theta_1, \varphi_1$ and $\psi_2, \theta_2, \varphi_2$.

The mechanical system studied in [5] is characterized as follows: the distribution of mass in the bodies is arbitrary, the connection point O_2 does not lie on the principal axes of inertia of the body S_1, but the centers of masses of the bodies P_1 and P_2 belong to their principal axes of inertia. The equations of motion for the system have been derived with the help of the software package [1] written in the language of computer algebra system (CAS) "Mathematica". First, according to a geometric description of the mechanical system, its characteristic function (the Lagrange function) in symbolic form has been constructed, then, using this function as a starting point, the equations of motion have been obtained. The problem of their qualitative analysis was stated. Within the framework of solving this problem, solutions of the equations corresponding to permanent rotations of the system have been found, and the sufficient conditions of their stability in the sense of Lyapunov have been derived.

In the present work, the above mechanical system is studied in a more general case. We assume that the centers of masses of the bodies do not lie on their principal axes of inertia. The equations of motion of the system are derived analogously to the previous case. The problem of qualitative analysis of the equations is stated. In the present paper, we restrict ourselves by considering equilibrium solutions of the equations. Two techniques are used to find them: from the stationary conditions for the family of the first integrals of the problem, and, directly, from the equations of motion. The stability of the solutions is analyzed on the base of Lyapunov's stability theorems in the first approximation. All computations are performed with the aid of CAS "Mathematica".

As was mentioned in [5], similar problems arise in many applications, e.g., in modelling and the study of dynamical properties of various technical devices and instruments. Such problems are considered, e.g., in [2,3].

The paper is organized as follows. In Sect. 2, the Lagrange function and the equations of motion with their first integrals for the mechanical system in question are given. In Sect. 3, we seek equilibrium positions of the system, using the stationary conditions of the first integrals. In Sect. 3.2, the same problem is solved with the help of the equations of motion. In Sect. 4, the stability of the solutions is analyzed. In Sect. 5, we give a conclusion.

2 The Lagrange Function and the Equations of Motion

The Lagrange function of the mechanical system under consideration derived by the technique [5] has the form: $L = T + U$, where

$$\begin{aligned}2T &= A_1p_1^2 + B_1q_1^2 + C_1r_1^2 + A_2(b_{11}p_1 + p_2 + b_{12}q_1 + b_{13}r_1)^2 \\ &+B_2(b_{21}p_1 + b_{22}q_1 + b_{23}r_1 + q_2)^2 \\ &+C_2(b_{31}p_1 + b_{32}q_1 + b_{33}r_1 + r_2)^2 + m_2(d_1^2 + d_2^2 + d_3^2) \\ &+2m_2\Big[a_2\left[(b_{31}p_1 + b_{32}q_1 + b_{33}r_1 + r_2)(b_{23}d_1 + b_{22}d_2 + b_{21}d_3)\right.\end{aligned}$$

$$
\begin{aligned}
&-(b_{21}p_1 + b_{22}q_1 + b_{23}r_1 + q_2)(b_{33}d_1 + b_{32}d_2 + b_{31}d_3)] \\
&-b_2[(b_{31}p_1 + b_{32}q_1 + b_{33}r_1 + r_2)(b_{13}d_1 + b_{12}d_2 + b_{11}d_3) \\
&-(b_{11}p_1 + b_{12}q_1 + b_{13}r_1 + p_2)(b_{33}d_1 + b_{32}d_2 + b_{31}d_3)] \\
&+c_2\,[(b_{21}p_1 + b_{22}q_1 + b_{23}r_1 + q_2)(b_{13}d_1 + b_{12}d_2 + b_{11}d_3) \\
&-(b_{11}p_1 + b_{12}q_1 + b_{13}r_1 + p_2)(b_{23}d_1 + b_{22}d_2 + b_{21}d_3)]\Big],
\end{aligned}
$$

$$
\begin{aligned}
U = -g\Big[&m_1(a_1a_{13} + b_1a_{23} + c_1a_{33}) + m_2\Big(a_{13}(a_2b_{11} + b_2b_{21} + c_2b_{31} + s_1) \\
&+a_{23}(a_2b_{12} + b_2b_{22} + c_2b_{32} + s_2) + a_{33}(a_2b_{13} + b_2b_{23} + c_2b_{33} + s_3)\Big)\Big]
\end{aligned}
$$

are the kinetic energy and the force function of the system, respectively.

Here $d_1 = p_1s_2 - q_1s_1,\ d_2 = r_1s_1 - p_1s_3,\ d_3 = q_1s_3 - r_1s_2$; $a_i, b_i, c_i\ (i = 1, 2)$ are the coordinates of the centers of masses of the bodies; s_1, s_2, s_3 are the coordinates of the connection point O_2; $A_i, B_i, C_i\ (i = 1, 2)$ are the principal moments of inertia of the bodies; m_1, m_2 are the masses of the bodies; g is the acceleration due to gravity; $p_i = \dot\psi_i \sin\varphi_i \sin\theta_i + \dot\theta_i \cos\varphi_i$, $q_i = \dot\psi_i \cos\varphi_i \sin\theta_i - \dot\theta_i \sin\varphi_i$, $r_i = \dot\varphi_i + \dot\psi_i \cos\theta_i$ are the projections of the vector of angular velocity of the body S_i onto the axes $O_ix_iy_iz_i$; $\alpha = \|a_{kl}\|$, $\beta = \|b_{kl}\|$ are the cosine matrices 3×3 of angles between the axes O_1XYZ and $O_1x_1y_1z_1$, and the axes $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$, respectively. Their elements are related to Euler's angles $\psi_i, \theta_i, \varphi_i$ as follows:

$$
\begin{aligned}
&a_{kl} = \zeta_{kl}^{(1)},\ b_{kl} = \zeta_{kl}^{(2)}\ (k, l = 1, 2, 3),\ \text{where} \\
&\zeta_{11}^{(i)} = \cos\varphi_i \cos\psi_i - \cos\theta_i \sin\varphi_i \sin\psi_i, \\
&\zeta_{12}^{(i)} = \cos\psi_i \cos\theta_i \sin\varphi_i + \cos\varphi_i \sin\psi_i,\ \zeta_{13}^{(i)} = \sin\varphi_i \sin\theta_i, \\
&\zeta_{21}^{(i)} = -\cos\psi_i \sin\varphi_i - \cos\varphi_i \cos\theta_i \sin\psi_i, \\
&\zeta_{22}^{(i)} = \cos\varphi_i \cos\psi_i \cos\theta_i - \sin\varphi_i \sin\psi_i, \\
&\zeta_{23}^{(i)} = \cos\varphi_i \sin\theta_i,\ \zeta_{31}^{(i)} = \sin\psi_i \sin\theta_i,\ \zeta_{32}^{(i)} = -\cos\psi_i \sin\theta_i, \\
&\zeta_{33}^{(i)} = \cos\theta_i\ (i = 1, 2).
\end{aligned}
\tag{1}
$$

The equations of motion produced by the package according to the formulae

$$
\frac{d}{dt}\Big(\frac{\partial L}{\partial \boldsymbol\omega^{(i)}}\Big) = \frac{\partial L}{\partial \boldsymbol\omega^{(i)}} \times \boldsymbol\omega^{(i)} + \frac{\partial L}{\partial \boldsymbol\alpha^{(i)}} \times \boldsymbol\alpha^{(i)} + \frac{\partial L}{\partial \boldsymbol\beta^{(i)}} \times \boldsymbol\beta^{(i)} + \frac{\partial L}{\partial \boldsymbol\gamma^{(i)}} \times \boldsymbol\gamma^{(i)},
$$

$$
\dot{\boldsymbol\alpha}^{(i)} = \boldsymbol\alpha^{(i)} \times \boldsymbol\omega^{(i)},\ \dot{\boldsymbol\beta}^{(i)} = \boldsymbol\beta^{(i)} \times \boldsymbol\omega^{(i)},\ \dot{\boldsymbol\gamma}^{(i)} = \boldsymbol\gamma^{(i)} \times \boldsymbol\omega^{(i)} \quad (i = 1, 2),
$$

where $\boldsymbol\omega^{(i)} = (p_i, q_i, r_i)$, $\boldsymbol\alpha^{T^{(1)}} = (a_{11}, a_{21}, a_{31})$, $\boldsymbol\beta^{T^{(1)}} = (a_{12}, a_{22}, a_{32})$, $\boldsymbol\gamma^{T^{(1)}} = (a_{13}, a_{23}, a_{33})$, $\boldsymbol\alpha^{T^{(2)}} = (b_{11}, b_{21}, b_{31})$, $\boldsymbol\beta^{T^{(2)}} = (b_{12}, b_{22}, b_{32})$, $\boldsymbol\gamma^{T^{(2)}} = (b_{13}, b_{23}, b_{33})$, are written as:

$$
\begin{aligned}
&[A_1 + A_2b_{11}^2 + B_2b_{21}^2 + b_{31}(C_2b_{31} + 2(a_2b_{23} - b_2b_{13})\,m_2s_2) \\
&\quad +m_2(s_2(2(b_2b_{11} - a_2b_{21})b_{33} + 2c_2(b_{13}b_{21} - b_{11}b_{23}) + s_2) \\
&\quad +2(b_2(b_{12}b_{31} - b_{11}b_{32}) + a_2(b_{21}b_{32} - b_{22}b_{31}) + c_2(b_{11}b_{22} - b_{12}b_{21}))s_3 + s_3^2)]\,\dot p_1 \\
&\quad +[A_2b_{11} + m_2((b_2b_{32} - c_2b_{23})s_2 + (c_2b_{22} - b_2b_{32})s_3)]\,\dot p_2
\end{aligned}
$$

$$
\begin{aligned}
&+[A_2b_{11}b_{12}+B_2b_{21}b_{22}+C_2b_{31}b_{32}+(b_2b_{13}-a_2b_{23})b_{31}m_2s_1\\
&+(a_2b_{21}b_{32}-b_2b_{11}b_{32}+c_2(b_{11}b_{23}-b_{13}b_{21}))m_2s_1-m_2(b_2(b_{13}b_{32}-b_{12}b_{32})\\
&+a_2(b_{22}b_{32}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_2]\,\dot q_1\\
&+[B_2b_{21}+m_2((c_2b_{13}-a_2b_{32})s_2+(a_2b_{32}-c_2b_{12})s_3)]\,\dot q_2\\
&+[A_2b_{11}b_{13}+B_2b_{21}b_{23}+C_2b_{31}b_{32}+(a_2b_{22}b_{31}+b_2(b_{11}b_{32}-b_{12}b_{31}))\,m_2s_1\\
&-(a_2b_{21}b_{32}+c_2(b_{11}b_{22}-b_{12}b_{21}))m_2s_1-m_2(b_2(b_{13}b_{32}-b_{12}b_{32})\\
&+a_2(b_{22}b_{32}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_3]\,\dot r_1\\
&+[C_2b_{31}+m_2((a_2b_{23}-b_2b_{13})s_2+(b_2b_{12}-a_2b_{22})s_3)]\,\dot r_2+\Phi_1=0,\\
&[A_2b_{11}b_{12}+B_2b_{21}b_{22}+C_2b_{31}b_{32}+m_2s_1(b_2b_{13}-a_2b_{23})b_{31}\\
&\quad+m_2s_1\,(a_2b_{21}b_{33}-b_2b_{11}b_{33}+c_2(b_{11}b_{23}-b_{13}b_{21}))\\
&\quad-m_2s_2\,(b_2(b_{13}b_{32}-b_{12}b_{33})+a_2(b_{22}b_{33}-b_{23}b_{32})\\
&\quad+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)]\,\dot p_1\\
&\quad+[A_2b_{12}+m_2(s_1(c_2b_{23}-b_2b_{33})+s_3(b_2b_{31}-c_2b_{21}))]\,\dot p_2\\
&\quad+[B_1+A_2b_{12}^2+B_2b_{22}^2+b_{32}(C_2b_{32}+2m_2s_1\,(b_{13}b_2-a_2b_{23}))\\
&\quad+m_2(s_1(2(a_2b_{22}b_{33}-b_2b_{12}b_{33}+c_2(b_{12}b_{23}-b_{13}b_{22}))+s_1)\\
&\quad+2(b_2(b_{12}b_{31}-b_{11}b_{32})+a_2(b_{21}b_{32}-b_{22}b_{31})+c_2(b_{11}b_{22}-b_{12}b_{21}))s_3+s_3^2)]\,\dot q_1\\
&\quad+[B_2b_{22}+m_2(s_1(a_2b_{33}-c_2b_{13})+s_3(b_{11}c_2-a_2b_{31}))]\,\dot q_2\\
&\quad+[A_2b_{12}b_{13}+B_2b_{22}b_{23}+C_2b_{32}b_{33}+m_2s_2(b_2(b_{11}b_{32}-b_{12}b_{31})\\
&\quad+a_2(b_{22}b_{31}-b_{21}b_{32})+c_2(b_{12}b_{21}-b_{11}b_{22}))-m_2s_3(b_2(b_{11}b_{33}-b_{13}b_{31})\\
&\quad+a_2(b_{23}b_{31}-b_{21}b_{33})+c_2(b_{13}b_{21}-b_{11}b_{23})+s_2)]\,\dot r_1\\
&\quad+[C_2b_{32}+m_2(s_1(b_2b_{13}-a_2b_{23})+s_3(a_2b_{21}-b_2b_{11}))]\,\dot r_2+\Phi_2=0,\\
&[A_2b_{11}b_{13}+B_2b_{21}b_{23}+C_2b_{31}b_{33}+m_2s_1\,(a_2b_{22}b_{31}+b_2(b_{11}b_{32}-b_{12}b_{31}))\\
&\quad-m_2s_1\,(a_2b_{21}b_{32}+c_2(b_{11}b_{22}-b_{12}b_{21}))-m_2s_3\,(b_2(b_{13}b_{32}-b_{12}b_{33})\\
&\quad+a_2(b_{22}b_{33}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)]\,\dot p_1\\
&\quad+[A_2b_{13}+m_2(s_1(b_2b_{32}-c_2b_{22})+s_2(c_2b_{21}-b_2b_{31}))]\,\dot p_2\\
&\quad+[A_2b_{12}b_{13}+B_2b_{22}b_{23}+C_2b_{32}b_{33}+m_2s_2\,(b_2(b_{11}b_{32}-b_{12}b_{31})\\
&\quad+a_2(b_{22}b_{31}-b_{21}b_{32})+c_2(b_{12}b_{21}-b_{11}b_{22}))-m_2s_3\,(b_2(b_{11}b_{33}-b_{13}b_{31})\\
&\quad+a_2(b_{23}b_{31}-b_{21}b_{33})+c_2(b_{13}b_{21}-b_{11}b_{23})+s_2)]\,\dot q_1\\
&\quad+[B_2b_{23}+m_2(s_1(c_2b_{12}-a_2b_{32})+s_2(a_2b_{31}-c_2b_{11}))]\,\dot q_2\\
&\quad+[C_1+A_2b_{13}^2+B_2b_{23}^2+C_2b_{33}^2+m_2(s_1(2(b_2(b_{13}b_{32}-b_{12}b_{33})\\
&\quad+a_2(b_{22}b_{33}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22}))+s_1)+2(b_2(b_{11}b_{33}-b_{13}b_{31})\\
&\quad+a_2(b_{23}b_{31}-b_{21}b_{33})+c_2(b_{13}b_{21}-b_{11}b_{23}))s_2+s_2^2)]\,\dot r_1\\
&\quad+[C_2b_{33}+m_2(s_1(a_2b_{22}-b_2b_{12})+s_2(b_2b_{11}-a_2b_{21}))]\,\dot r_2+\Phi_3=0,\\
&[A_2b_{11}+m_2(s_2(b_2b_{33}-c_2b_{23})+s_3(c_2b_{22}-b_2b_{32}))]\,\dot p_1+A_2\dot p_2\\
&\quad+[A_2b_{12}+m_2(s_1(c_2b_{23}-b_2b_{33})+s_3(b_2b_{31}-c_2b_{21}))]\,\dot q_1\\
&\quad+[A_2b_{13}+m_2(s_1(b_2b_{32}-c_2b_{22})+s_2(c_2b_{21}-b_2b_{31}))]\,\dot r_1+\Phi_4=0,\\
&[B_2b_{21}+m_2(s_2(c_2b_{13}-a_2b_{33})+s_3(a_2b_{32}-c_2b_{12}))]\,\dot p_1\\
&\quad+[B_2b_{22}+m_2(s_1(a_2b_{33}-c_2b_{13})+s_3(c_2b_{11}-a_2b_{31}))]\,\dot q_1+B_2\dot q_2\\
&\quad+[B_2b_{23}+m_2(s_1(c_2b_{12}-a_2b_{32})+s_2(a_2b_{31}-c_2b_{11}))]\,\dot r_1+\Phi_5=0,\\
&[C_2b_{31}+m_2(s_2(a_2b_{23}-b_2b_{13})+s_3(b_2b_{12}-a_2b_{22}))]\,\dot p_1\\
&\quad+[C_2b_{32}+m_2(s_1(b_2b_{13}-a_2b_{23})+s_3(a_2b_{21}-b_2b_{11}))]\,\dot q_1\\
&\quad+[C_2b_{33}+m_2(s_1(a_2b_{22}-b_2b_{12})+s_2(b_2b_{11}-a_2b_{21}))]\,\dot r_1+C_2\dot r_2+\Phi_6=0;
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&\dot a_{11}=a_{21}r_1-a_{31}q_1,\ \dot a_{12}=a_{22}r_1-a_{32}q_1,\ \dot a_{13}=a_{23}r_1-a_{33}q_1,\\
&\dot a_{21}=a_{31}p_1-a_{11}r_1,\ \dot a_{22}=a_{32}p_1-a_{12}r_1,\ \dot a_{23}=a_{33}p_1-a_{13}r_1,\\
&\dot a_{31}=a_{11}q_1-a_{21}p_1,\ \dot a_{32}=a_{12}q_1-a_{22}p_1,\ \dot a_{33}=a_{13}q_1-a_{23}p_1,
\end{aligned}
\tag{3}
$$

$$\begin{array}{l} \dot{b}_{11} = b_{21}r_1 - b_{31}q_1,\ \dot{b}_{12} = b_{22}r_1 - b_{32}q_1,\ \dot{b}_{13} = b_{23}r_1 - b_{33}q_1, \\ \dot{b}_{21} = b_{31}p_1 - b_{11}r_1,\ \dot{b}_{22} = b_{32}p_1 - b_{12}r_1,\ \dot{b}_{23} = b_{33}p_1 - b_{13}r_1, \\ \dot{b}_{31} = b_{11}q_1 - b_{21}p_1,\ \dot{b}_{32} = b_{12}q_1 - b_{22}p_1,\ \dot{b}_{33} = b_{13}q_1 - b_{23}p_1. \end{array} \quad (4)$$

Here Φ_i $(i = 1, \ldots, 6)$ are the quadratic polynomials of p_j, q_j, r_j $(j = 1, 2)$. These are rather cumbersome and are given in Appendix.

Equations (2)–(4) admit the following first integrals.

– The integrals of energy and kinetic moment:

$$H = T - U = h,\ V = \frac{\partial L}{\partial \boldsymbol{\omega}^{(1)}} \cdot \boldsymbol{\gamma}^{(1)} = c,$$

where h and c are some constants.
– The geometric integrals:

$$\begin{array}{ll} V_1 = a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, & V_7 = b_{11}^2 + b_{21}^2 + b_{31}^2 = 1, \\ V_2 = a_{12}^2 + a_{22}^2 + a_{32}^2 = 1, & V_8 = b_{12}^2 + b_{22}^2 + b_{32}^2 = 1, \\ V_3 = a_{13}^2 + a_{23}^2 + a_{33}^2 = 1, & V_9 = b_{13}^2 + b_{23}^2 + b_{33}^2 = 1, \\ V_4 = a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, & V_{10} = b_{11}b_{12} + b_{21}b_{22} + b_{31}b_{32} = 0, \\ V_5 = a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0, & V_{11} = b_{11}b_{13} + b_{21}b_{23} + b_{31}b_{33} = 0, \\ V_6 = a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0, & V_{12} = b_{12}b_{13} + b_{22}b_{23} + b_{32}b_{33} = 0. \end{array} \quad (5)$$

We pose the problem of finding the stationary solutions and invariant manifolds (IMs) [4] for Eqs. (2)–(4) and the investigation of their stability.

3 Finding Stationary Solutions and IMs

It is shown [4] that stationary solutions and IMs can be obtained from both the stationary conditions of the problem's first integrals and, directly, from the equations of motion. Let us use the first technique.

3.1 The Usage of Stationary Conditions

Compose the linear combination of the integrals

$$2\Omega = 2\lambda_0 H - \lambda_1 V_3 - \lambda_2 V_7 - \lambda_3 V_8 - \lambda_4 V_9 - 2(\lambda_5 V_{10} + \lambda_6 V_{11} + \lambda_7 V_{12})$$

and write down the necessary conditions of extremum for the integral Ω with respect to the variables $p_1, p_2, q_1, q_2,\ r_1, r_2, a_{13}, a_{23}, a_{33}, b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}$:

$$\begin{array}{l} \partial\Omega/\partial p_i = 0,\ \partial\Omega/\partial q_i = 0,\ \partial\Omega/\partial r_i = 0\ (i = 1, 2), \\ \partial\Omega/\partial a_{k3} = 0,\ \partial\Omega/\partial b_{kl} = 0\ (k, l = 1, 2, 3). \end{array} \quad (6)$$

The variables $a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}$ are sought with the help of the integrals V_1, V_2, V_j $(j = 4, 5, 6)$ under the corresponding values of $a_{13},\ a_{23}, a_{33}$.

Solutions of Eqs. (6) allow us to determine the stationary solutions and IMs of the differential equations (2)–(4) corresponding to the integral Ω. As was mentioned before, we are interested in equilibrium solutions. In order to find them, we put $p_i = q_i = r_i = 0$ $(i = 1, 2)$ in (6). The equations take the form:

$$\begin{aligned}
&\lambda_0 g\,(a_1 m_1 + m_2(a_2 b_{11} + b_2 b_{21} + c_2 b_{31} + s_1)) - \lambda_1 a_{13} = 0,\\
&\lambda_0 g\,(b_1 m_1 + m_2(a_2 b_{12} + b_2 b_{22} + c_2 b_{32} + s_2)) - \lambda_1 a_{23} = 0,\\
&\lambda_0 g(c_1 m_1 + m_2(a_2 b_{13} + b_2 b_{23} + c_2 b_{33} + s_3)) - \lambda_1 a_{33} = 0,\\
&\lambda_0 a_2 g\, m_2 a_{13} - \lambda_2 b_{11} - \lambda_5 b_{12} - \lambda_6 b_{13} = 0,\\
&\lambda_0 a_2 g\, m_2 a_{23} - \lambda_3 b_{12} - \lambda_5 b_{11} - \lambda_7 b_{13} = 0,\\
&\lambda_0 a_2 g\, m_2 a_{33} - \lambda_4 b_{13} - \lambda_6 b_{11} - \lambda_7 b_{12} = 0,\\
&\lambda_0 b_2 g\, m_2 a_{13} - \lambda_2 b_{21} - \lambda_5 b_{22} - \lambda_6 b_{23} = 0,\\
&\lambda_0 b_2 g\, m_2 a_{23} - \lambda_3 b_{22} - \lambda_5 b_{21} - \lambda_7 b_{23} = 0,\\
&\lambda_0 b_2 g\, m_2 a_{33} - \lambda_4 b_{23} - \lambda_6 b_{21} - \lambda_7 b_{22} = 0,\\
&\lambda_0 c_2 g\, m_2 a_{13} - \lambda_2 b_{31} - \lambda_5 b_{32} - \lambda_6 b_{33} = 0,\\
&\lambda_0 c_2 g\, m_2 a_{23} - \lambda_3 b_{32} - \lambda_5 b_{31} - \lambda_7 b_{33} = 0,\\
&\lambda_0 c_2 g\, m_2 a_{33} - \lambda_4 b_{33} - \lambda_6 b_{31} - \lambda_7 b_{32} = 0.
\end{aligned} \tag{7}$$

The resulting system is multiparametric (20 parameters) that leads to cumbersome expressions in the process of computation.

For the polynomials of system (7), under the following constraints on the parameters of the problem

$$a_1 = -\frac{m_2 s_1}{m_1},\; b_1 = -\frac{m_2 s_2}{m_1},\; c_1 = -\frac{m_2 s_3}{m_1}$$

and the relations $b_{21} = b_{12}, b_{31} = b_{13}, b_{32} = b_{23}$ (we assume that the cosine matrix β is symmetric), a Gröbner basis was constructed with respect to an eliminating ordering of the variables with the help of the built-in function

$$\begin{aligned}
&\mathit{GroebnerBasis}[\mathit{polys}_1, \{b_{11}, b_{33}\}, \{\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, a_{13}, a_{23}, a_{33}\},\\
&\quad \text{CoefficientDomain} \to \text{RationalFunctions},\\
&\quad \text{MonomialOrder} \to \text{EliminationOrder}]
\end{aligned}$$

Here $polys_1$ is the list of the polynomials of system (7).

After the transformation of the basis into a lexicographical one by means of

$$\begin{aligned}
&\mathit{GroebnerBasis}[\mathit{polys}_2, \{\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, a_{13}, a_{23}, a_{33}, b_{11}, b_{33}\},\\
&\quad \text{CoefficientDomain} \to \text{RationalFunctions}],
\end{aligned}$$

where $polys_2$ is the list of polynomials obtained at the previous step, we have a system dividing into two subsystems. One of them is presented below.

$$\begin{aligned}
&c_2(a_2 b_{12} + b_2(b_{22} - b_{33})) + c_2^2 b_{23} - b_2(a_2 b_{13} + b_2 b_{23}) = 0,\\
&a_2^2 b_{12} - b_2(b_{12} b_2 + b_{13} c_2) + a_2(b_2(b_{22} - b_{11}) + c_2 b_{23}) = 0,\\
&\lambda_1 b_2 a_{33} - \lambda_0 g m_2 c_2(a_2 b_{12} + b_2 b_{22} + c_2 b_{23}) = 0,\\
&\lambda_0 g m_2\,(a_2 b_{12} + b_2 b_{22} + b_{23} c_2) - a_{23}\lambda_1 = 0,\\
&\lambda_0 g m_2 a_2(a_2 b_{12} + b_2 b_{22} + c_2 b_{23}) - \lambda_1 b_2 a_{13} = 0,\\
&b_2 c_2 g^2 m_2^2 \lambda_0^2 - \lambda_1 \lambda_7 = 0,\; a_2 c_2 g^2 m_2^2 \lambda_0^2 - \lambda_1 \lambda_6 = 0,
\end{aligned} \tag{8}$$

$$a_2b_2g^2\lambda_0^2m_2^2-\lambda_1\lambda_5=0,\ c_2^2\,g^2m_2^2\lambda_0^2-\lambda_1\lambda_4=0,$$
$$b_2^2\,g^2\lambda_0^2m_2^2-\lambda_1\lambda_3=0,\ a_2^2\,g^2m_2^2\lambda_0^2-\lambda_1\lambda_2=0.$$

All computations have been performed on a computer with an Intel Core i7 CPU (3.6 GHz) and 32 GB of RAM. The total computation time is 46 s.

Then, expressions (5) are added to the first five equations of system (8) and a lexicographical basis for the polynomials of the resulting system with respect to $a_{11} > a_{12} > a_{21} > a_{22} > a_{31} > a_{13} > a_{23} > a_{33} > b_{11} > b_{33} > b_{13} > b_{22} > b_{23} > \lambda_1$ is constructed. Again we obtain a system splitting into two subsystems. Below, both subsystems are represented.

$$\begin{aligned}
&(a_2^2+b_2^2+c_2^2)\,g^2m_2^2\lambda_0^2-\lambda_1^2=0,\\
&a_2^2b_{12}^2+b_2^2(b_{12}^2+b_{23}^2)+c_2^2b_{23}^2+2(a_2c_2b_{12}b_{23}\mp a_2b_2b_{12}\mp b_2c_2b_{23})=0,\\
&-a_2b_{12}-b_2(b_{22}\mp 1)-c_2b_{23}=0,\\
&a_2^2(b_2b_{13}+c_2b_{12})+b_2^2(a_2b_{23}+c_2b_{12})+c_2^2(a_2b_{23}+b_2b_{13})\mp 2a_2b_2c_2=0,\\
&-a_2^2(a_2b_{12}-b_2(b_{33}\pm 1)+c_2b_{23})+b_2^2(c_2b_{23}-a_2b_{12})+b_2c_2^2(b_{33}\mp 1)=0,\\
&-a_2^2b_2(b_{11}\mp 1)+b_2^2(c_2b_{23}-a_2b_{12})+c_2^2(a_2b_{12}-b_2(b_{11}\pm 1)+c_2b_{23})=0,\\
&(a_2^2+b_2^2+c_2^2)\,gm_2\lambda_0a_{33}\mp c_2\lambda_1=0,\\
&\pm b_2\lambda_1-(a_2^2+b_2^2+c_2^2)\,gm_2\lambda_0a_{23}=0,\\
&\pm a_2\lambda_1-(a_2^2+b_2^2+c_2^2)\,gm_2\lambda_0a_{13}=0,\\
&(a_2^2+b_2^2)(a_{31}^2+a_{32}^2-1)+c_2^2(a_{31}^2+a_{32}^2)=0,\\
&a_2^2(a_{22}^2+a_{32}^2-1)+(b_2a_{22}+c_2a_{32})^2=0,\\
&(a_2^2+b_2^2)(a_{21}+a_{32}(a_{22}a_{31}-a_{21}a_{32}))+b_2c_2a_{31}+c_2^2a_{32}(a_{22}a_{31}-a_{21}a_{32})=0,\\
&a_2a_{12}+b_2a_{22}+c_2a_{32}=0,\\
&a_2(a_2^2a_{11}+a_2c_2a_{31}+b_2^2a_{11})(a_{32}^2-1)+[b_2(a_2^2+b_2^2)a_{22}a_{31}+c_2^2(a_2a_{11}a_{32}\\
&+b_2a_{22}a_{31})+c_2(b_2^2a_{31}+c_2^2a_{31})a_{32}]a_{32}=0.
\end{aligned}\tag{9}$$

Next, we find $\lambda_1=\pm\sqrt{a_2^2+b_2^2+c_2^2}\,m_2g\lambda_0$ from the first equation of (9). Under the above values of λ_1, the latter 13 equations of each of the subsystems together with equations $p_i=q_i=r_i=0$ $(i=1,2)$ and the relations $b_{21}=b_{12}$, $b_{31}=b_{13}, b_{32}=b_{23}$ determine the four IMs of codimension 22 of the differential equations (2)–(4). It is verified by direct computation according to the definition of IM. Let us consider one of these IMs, e.g., the one defined by the equations

$$\begin{aligned}
&p_1=0,\ p_2=0,\ q_1=0,\ q_2=0,\ r_1=0,\ r_2=0,\\
&b_{21}-b_{12}=0,\ b_{31}-b_{13}=0, b_{32}-b_{23}=0,\\
&a_2^2b_{12}^2+b_2^2(b_{12}^2+b_{23}^2)+c_2^2b_{23}^2+2(a_2c_2b_{12}b_{23}+a_2b_2b_{12}+b_2c_2b_{23})=0,\\
&-a_2b_{12}-b_2(b_{22}+1)-c_2b_{23}=0,\\
&a_2^2(b_2b_{13}+c_2b_{12})+b_2^2(a_2b_{23}+c_2b_{12})+c_2^2(a_2b_{23}+b_2b_{13})+2a_2b_2c_2=0,\\
&-a_2^2(a_2b_{12}-b_2(b_{33}-1)+c_2b_{23})+b_2^2(c_2b_{23}-a_2b_{12})+b_2c_2^2(b_{33}+1)=0,\\
&-a_2^2b_2(b_{11}+1)+b_2^2(c_2b_{23}-a_2b_{12})+c_2^2(a_2b_{12}-b_2(b_{11}-1)+c_2b_{23})=0,\\
&\sqrt{a_2^2+b_2^2+c_2^2}\,a_{33}+c_2=0,\\
&\sqrt{a_2^2+b_2^2+c_2^2}\,a_{23}+b_2=0,\\
&\sqrt{a_2^2+b_2^2+c_2^2}\,a_{13}+a_2=0,\\
&(a_2^2+b_2^2)(a_{31}^2+a_{32}^2-1)+c_2^2(a_{31}^2+a_{32}^2)=0,
\end{aligned}\tag{10}$$

$a_2^2(a_{22}^2 + a_{32}^2 - 1) + (b_2 a_{22} + c_2 a_{32})^2 = 0,$
$(a_2^2 + b_2^2)(a_{21} + a_{32}(a_{22}a_{31} - a_{21}a_{32})) + b_2 c_2 a_{31} + c_2^2 a_{32}(a_{22}a_{31} - a_{21}a_{32}) = 0,$
$a_2 a_{12} + b_2 a_{22} + c_2 a_{32} = 0,$
$a_2(a_2^2 a_{11} + a_2 c_2 a_{31} + b_2^2 a_{11})(a_{32}^2 - 1) + [b_2(a_2^2 + b_2^2)a_{22}a_{31} + c_2^2(a_2 a_{11} a_{32}$
$+b_2 a_{22} a_{31}) + c_2(b_2^2 a_{31} + c_2^2 a_{31})a_{32}]a_{32} = 0.$

The differential equations $\dot{a}_{32} = 0,\ \dot{b}_{12} = 0$ on this IM have the following family of solutions:

$$a_{32} = a_{32}^0 = \text{const},\ b_{12} = b_{12}^0 = \text{const}. \tag{11}$$

From a geometric point of view, Eqs. (10) in the space R^{24} define a two-dimensional surface whose points correspond to the fixed points of the phase space of the system under study.

Next, let us find λ_j $(j = 2, \ldots, 7)$ from the latter six equations of (8) when $\lambda_1 = \sqrt{a_2^2 + b_2^2 + c_2^2}\, m_2 g \lambda_0$ and substitute them into the integral Ω. Having added a combination of the integrals V_1, V_2 to the resulting expression, we have:

$$\begin{aligned}\Omega_1 = \lambda_0\Big[H - \frac{1}{2}\sqrt{a_2^2 + b_2^2 + c_2^2}\, g m_2 V_3 - \frac{g m_2(a_2^2 V_7 + b_2^2 V_8 + c_2^2 V_9)}{2\sqrt{a_2^2 + b_2^2 + c_2^2}} \\ - \frac{g m_2(a_2 b_2 V_{10} + a_2 c_2 V_{11} + b_2 c_2 V_{12})}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\Big] - \lambda_8 V_1^2 - \lambda_9 V_2^2.\end{aligned} \tag{12}$$

Using the maps of an atlas on IM (10), e.g.,

$$\begin{aligned}
&p_1 = 0,\ p_2 = 0,\ q_1 = 0,\ q_2 = 0,\ r_1 = 0,\ r_2 = 0,\\
&a_{11} = \frac{a_2 c_2 z_1 \pm b_2 z a_{32}}{(a_2^2 + b_2^2)\, z^{1/2}},\ a_{12} = -\frac{a_2 c_2 a_{32} \mp b_2 z_1}{a_2^2 + b_2^2}, a_{13} = -a_2 z^{-1/2},\\
&a_{21} = \frac{b_2 c_2 z_1 \mp a_2 z a_{32}}{(a_2^2 + b_2^2)\, z^{1/2}},\ a_{22} = -\frac{b_2 c_2 a_{32} \pm a_2 z_1}{a_2^2 + b_2^2},\ a_{23} = -b_2 z^{-1/2},\\
&a_{31} = -z_1 z^{-1/2},\ a_{33} = -c_2 z^{-1/2},\\
&b_{11} = -\frac{a_2(a_2 + b_2 b_{12}) + c_2 z_2}{a_2^2 + c_2^2},\ b_{13} = \frac{a_2 z_2 - (a_2 + b_2 b_{12})\, c_2}{a_2^2 + c_2^2},\\
&b_{21} = b_{12},\ b_{22} = \frac{c_2 z_2 - b_2(a_2 b_{12} + b_2)}{b_2^2 + c_2^2},\ b_{23} = -\frac{a_2 c_2 b_{12} + b_2(c_2 + z_2)}{b_2^2 + c_2^2},\\
&b_{31} = \frac{a_2 z_2 - (a_2 + b_2 b_{12})\, c_2}{a_2^2 + c_2^2},\ b_{32} = -\frac{a_2 c_2 b_{12} + b_2(c_2 + z_2)}{b_2^2 + c_2^2},\\
&b_{33} = \frac{1}{(a_2^2 + c_2^2)(b_2^2 + c_2^2)}[a_2^2(b_2(b_2 + a_2 b_{12}) - c_2 z_2)\\
&\quad +b_2(a_2(b_2^2 + 2c_2^2)b_{12} + b_2 c_2 z_2) - c_2^4],
\end{aligned}$$

it is not difficult to show that the integral Ω_1 takes a stationary value on this IM. Here $z_1 = [(1 - a_{32}^2)(a_2^2 + b_2^2) - c_2^2 a_{32}^2]^{1/2}$, $z_2 = [c_2^2 - b_{12}(a_2^2 b_{12} + 2a_2 b_2 + (b_2^2 + c_2^2)b_{12})]^{1/2}$, $z = a_2^2 + b_2^2 + c_2^2$.

Equations (10) together with (11) allow one to obtain up to the eight families of solutions of the equations of motion (2)–(4). We represent one of them, e.g.,

$$
\begin{aligned}
&p_1 = 0,\ p_2 = 0,\ q_1 = 0,\ q_2 = 0,\ r_1 = 0,\ r_2 = 0,\\
&a_{11} = \frac{a_2 c_2 D_1 + a_{32}^0 b_2 D}{(a_2^2 + b_2^2)\, D^{1/2}},\ a_{12} = -\frac{a_2 a_{32}^0 c_2 - b_2 D_1}{a_2^2 + b_2^2}, a_{13} = -a_2 D^{-1/2},\\
&a_{21} = \frac{b_2 c_2 D_1 - a_2 a_{32}^0 D}{(a_2^2 + b_2^2)\, D^{1/2}},\ a_{22} = -\frac{a_{32}^0 b_2 c_2 + a_2 D_1}{a_2^2 + b_2^2},\ a_{23} = -b_2 D^{-1/2},\\
&a_{31} = -D_1 D^{-1/2},\ a_{32} = a_{32}^0,\ a_{33} = -c_2 D^{-1/2},\\
&b_{11} = -\frac{a_2(a_2 + b_2 b_{12}^0) + c_2 D_2}{a_2^2 + c_2^2},\ b_{12} = b_{12}^0,\ b_{13} = \frac{a_2 D_2 - (a_2 + b_2 b_{12}^0)\, c_2}{a_2^2 + c_2^2},\\
&b_{21} = b_{12}^0,\ b_{22} = \frac{c_2 D_2 - b_2(a_2 b_{12}^0 + b_2)}{b_2^2 + c_2^2},\ b_{23} = -\frac{a_2 c_2 b_{12}^0 + b_2(c_2 + D_2)}{b_2^2 + c_2^2},\\
&b_{31} = \frac{a_2 D_2 - (a_2 + b_{12}^0 b_2)\, c_2}{a_2^2 + c_2^2},\ b_{32} = -\frac{a_2 c_2 b_{12}^0 + b_2(c_2 + D_2)}{b_2^2 + c_2^2},\\
&b_{33} = \frac{1}{(a_2^2 + c_2^2)(b_2^2 + c_2^2)}[a_2^2(b_2(b_2 + a_2 b_{12}^0) - c_2 D_2)\\
&\quad + b_2(a_2(b_2^2 + 2c_2^2) b_{12}^0 + b_2 c_2 D_2) - c_2^4].
\end{aligned}
\tag{13}
$$

The rest of the solutions differs from the above by the signs of the expressions. Here a_{32}^0 and b_{12}^0 are the parameters of the families, $D_1 = [(1 - a_{32}^{0^2})(a_2^2 + b_2^2) - a_{32}^{0^2} c_2^2]^{1/2}$, $D_2 = [c_2^2 - b_{12}^0(a_2^2 b_{12}^0 + 2a_2 b_2 + b_{12}^0(b_2^2 + c_2^2))]^{1/2}$, $D = a_2^2 + b_2^2 + c_2^2$. Solutions (13) are real, in particular, when the following conditions hold:

$$
a_2 \neq 0,\ b_2 \neq 0,\ c_2 \neq 0 \text{ and } -\sigma_1 \le a_{32}^0 \le \sigma_1 \text{ and } -\sigma_2 \le b_{12}^0 \le \sigma_2,
$$

where $\sigma_1 = \sqrt{(a_2^2 + b_2^2)\, D^{-1}}, \sigma_2 = \sqrt{(a_2^2 + c_2^2)(b_2^2 + c_2^2)}\, D^{-1}, \sigma = a_2 b_2\, D^{-1}$.

It is easy to verify by direct computation that the integral Ω_1 also takes a stationary value on the elements of the family of solutions (13). From a mechanical point of view, the elements of this family correspond to equilibria of the mechanical system under consideration.

By means of relations (1), the family of solutions (13) can be represented in the initial variables (Euler's angles). Since these solutions are rather cumbersome we give here the expressions obtained under some constraints on the parameters of the problem. For instance, under the following conditions

$$
b_{12}^0 = -\frac{2a_2 b_2}{2a_2^2 + b_2^2},\ c_2 = a_2,
$$

the one-parametric families of solutions correspond to the family of solutions (13) in the initial variables:

$$
\varphi_1 = \pm \arccos\left(\pm \frac{b_2}{\sqrt{a_2^2 + b_2^2}}\right),\ \psi_1 = \pm \arccos\left(\pm \frac{a_{32}^0 \sqrt{2a_2^2 + b_2^2}}{\sqrt{a_2^2 + b_2^2}}\right),
$$

$$\theta_1 = \mp \arccos\left(-\frac{a_2}{\sqrt{2a_2^2+b_2^2}}\right),\ \varphi_2 = \mp \arccos\left(\pm\frac{2a_2}{\sqrt{4a_2^2+b_2^2}}\right),$$
$$\psi_2 = \mp \arccos\left(\mp\frac{2a_2}{\sqrt{4a_2^2+b_2^2}}\right),\ \theta_2 = \mp \arccos\left(-\frac{2a_2^2}{2a_2^2+b2^2}\right), \tag{14}$$

where a_{32}^0 is the parameter of the families.

When $b_{12}^0 = \frac{2a_2b_2}{a_2^2+2b_2^2},\ c_2 = b_2$, we have the families of solutions:

$$\varphi_1 = \pm \arccos\left(\pm\frac{b_2}{\sqrt{a_2^2+b_2^2}}\right),\ \psi_1 = \pm \arccos\left(\pm\frac{a_{32}^0\sqrt{a_2^2+2b_2^2}}{\sqrt{a_2^2+b_2^2}}\right),$$
$$\theta_1 = \mp \arccos\left(-\frac{a_2}{\sqrt{a_2^2+2b_2^2}}\right),\ \varphi_2 = \mp \arccos\left(\pm\frac{a_2^2+b_2^2}{\sqrt{(a_2^2+b_2^2)^2+a_2^2b_2^2}}\right),$$
$$\psi_2 = \mp \arccos\left(\mp\frac{a_2^2+b_2^2}{\sqrt{(a_2^2+b_2^2)^2+a_2^2b_2^2}}\right),$$
$$\theta_2 = \mp \arccos\left(a_2^2\left(\frac{1}{a_2^2+2b_2^2}-\frac{2}{a_2^2+b_2^2}\right)\right). \tag{15}$$

Substituting expressions (14) and (15) into the equations of motion (2)–(4) written in Euler's variables shows that these equations are identically satisfied.

3.2 The Usage of the Equations of Motion

When $p_i = q_i = r_i = 0$ $(i = 1, 2)$, the equations of motion (2) take the form:

$$\begin{array}{l}
a_{33}(b_1m_1 + m_2(a_2b_{12} + b_2b_{22} + c_2b_{32} + s_2)) \\
-a_{23}(c_1m_1 + m_2(a_2b_{13} + b_2b_{23} + c_2b_{33} + s_3)) = 0, \\
a_{13}(c_1m_1 + m_2(a_2b_{13} + b_2b_{23} + c_2b_{33} + s_3)) \\
-a_{33}m_2(a_2b_{11} + b_2b_{21} + c_2b_{31} + s_1) - m_1a_1a_{33} = 0, \\
a_{23}(a_1m_1 + a_{23}m_2(a_2b_{11} + b_2b_{21} + c_2b_{31} + s_1)) \\
-a_{13}(b_1m_1 + m_2(a_2b_{12} + b_2b_{22} + c_2b_{32} + s_2)) = 0, \\
b_2(a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33}) - c_2(a_{13}b_{21} + a_{23}b_{22} + a_{33}b_{23}) = 0, \\
c_2(a_{13}b_{11} + a_{23}b_{12} + a_{33}b_{13}) - a_2(a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33}) = 0, \\
a_2(a_{13}b_{21} + a_{23}b_{22} + a_{33}b_{23}) - b_2(a_{13}b_{11} + a_{23}b_{12} + a_{33}b_{13}) = 0.
\end{array} \tag{16}$$

System (16) is multiparametric (11 parameters).

We assume that the elements of the cosine matrix β are related as follows:

$$\begin{array}{l}
b_1m_1 + m_2(a_2b_{12} + b_2b_{22} + c_2b_{32} + s_2) = 0, \\
a_1m_1 + m_2(a_2b_{11} + b_2b_{21} + c_2b_{31} + s_1) = 0, \\
b_2b_{31} - c_2b_{21} = 0,\ c_2b_{12} - a_2b_{32} = 0,\ a_2b_{22} - b_2b_{12} = 0.
\end{array} \tag{17}$$

Add relations (17) and (5) to Eqs. (16) and construct a lexicographical basis with respect to $a_{11} > a_{12} > a_{21} > a_{22} > a_{23} > a_{31} > a_{13} > a_{33} > b_{11} > b_{12} >$

$b_{13} > b_{21} > b_{22} > b_{23} > b_{31} > b_{32} > b_{33} > s_1 > s_2 > s_3$ for the polynomials of the resulting system. As a result, we have the system of equations dividing into two subsystems. Both subsystems are given below:

$$\begin{aligned}
&-c_1 m_1 - m_2 s_3 = 0,\ -a_1 m_1 - m_2 s_1 = 0,\\
&(b_1 m_1 + m_2 s_2)^2 - (a_2^2 + b_2^2 + c_2^2)\, m_2^2 = 0,\\
&b_2^2 - (b_2^2 + c_2^2)\, b_{33}^2 = 0,\\
&-b_1 c_2 m_1 - m_2((a_2^2 + b_2^2 + c_2^2)\, b_{32} + c_2 s_2) = 0,\\
&a_2^2 c_2^2 - (b_2^2 + c_2^2)(a_2^2 + b_2^2 + c_2^2)\, b_{31}^2 = 0,\\
&b_2 b_{23} + c_2 b_{33} = 0,\\
&b_1 b_2 m_1 + (a_2^2 + b_2^2 + c_2^2)\, m_2 b_{22} + b_2 m_2 s_2 = 0,\\
&b_2 b_{31} - c_2 b_{21} = 0,\ b_{13} = 0,\\
&-a_2^2 b_{12} m_2 - b_{12}(b_2^2 + c_2^2)\, m_2 - a_2(b_1 m_1 + m_2 s_2) = 0,\\
&-a_2 c_2 b_{11} - (b_2^2 + c_2^2)\, b_{31} = 0,\\
&a_{33} = 0,\ a_{13} = 0,\ 1 - a_{31}^2 - a_{32}^2 = 0,\ a_{23} \pm 1 = 0,\\
&a_{22} = 0,\ a_{21} = 0,\ a_{12}^2 + a_{32}^2 - 1 = 0,\\
&a_{11} + (a_{12} a_{31} - a_{11} a_{32})\, a_{32} = 0.
\end{aligned} \tag{18}$$

From the first three equations of system (18), we find the constraints on the parameters of the problem

$$s_1 = -\frac{a_1 m_1}{m_2},\ s_2 = -\frac{b_1 m_1 \pm \sqrt{a_2^2 + b_2^2 + c_2^2}\, m_2}{m_2},\ s_3 = -\frac{c_1 m_1}{m_2}$$

under which the latter 17 equations of the system together with the relations $p_i = q_i = r_i = 0$ $(i = 1, 2)$ determine the four one-dimensional IMs of the equations of motion (2)–(4). It is easy to verify by direct computation according to the definition of IM. Below, the equations of one of these IMs are represented.

$$\begin{aligned}
&p_1 = 0,\ p_2 = 0,\ q_1 = 0,\ q_2 = 0,\ r_1 = 0,\ r_2 = 0,\\
&b_2^2 - (b_2^2 + c_2^2)\, b_{33}^2 = 0,\\
&-b_1 c_2 m_1 - m_2 (a_2^2 + b_2^2 + c_2^2)\, b_{32} + c_2 (b_1 m_1 + \sqrt{a_2^2 + b_2^2 + c_2^2}\, m_2) = 0,\\
&a_2^2 c_2^2 - (b_2^2 + c_2^2)(a_2^2 + b_2^2 + c_2^2)\, b_{31}^2 = 0,\\
&b_2 b_{23} + c_2 b_{33} = 0,\\
&b_1 b_2 m_1 + (a_2^2 + b_2^2 + c_2^2)\, m_2 b_{22} - b_2 (b_1 m_1 + \sqrt{a_2^2 + b_2^2 + c_2^2}\, m_2) = 0,\\
&b_2 b_{31} - c_2 b_{21} = 0,\ b_{13} = 0,\\
&(a_2 \sqrt{a_2^2 + b_2^2 + c_2^2} - a_2^2 b_{12} - b_{12}(b_2^2 + c_2^2))\, m_2 = 0,\\
&-a_2 c_2 b_{11} - (b_2^2 + c_2^2)\, b_{31} = 0,\\
&a_{33} = 0,\ a_{13} = 0,\ 1 - a_{31}^2 - a_{32}^2 = 0,\ a_{23} - 1 = 0,\\
&a_{22} = 0,\ a_{21} = 0,\ a_{12}^2 + a_{32}^2 - 1 = 0,\\
&a_{11} + (a_{12} a_{31} - a_{11} a_{32})\, a_{32} = 0.
\end{aligned} \tag{19}$$

The differential equation $\dot{a}_{32} = 0$ on IM (19) has the family of solutions:

$$a_{32} = a_{32}^0 = \text{const.} \tag{20}$$

Thus, from a geometrical point of view, Eqs. (19) in the space R^{24} define a curve whose points correspond to the fixed points of the phase space of the system under study.

Applying the technique [5], we find the combination of the integrals

$$2\Omega_2 = 2\lambda_0 H - \lambda_2 V_1^2 - \lambda_3 V_2^2 - \lambda_4 V_3^2 - \lambda_5 V_7^2$$
$$-gm_2\lambda_0\sqrt{a_2^2+b_2^2+c_2^2}\,V_8 - \lambda_7 V_9^2$$

which takes a stationary value on IM (19). It can be verified by direct computation, using the maps of an atlas on this IM, e.g.,

$$\begin{array}{l}
p_1=0,\ p_2=0,\ q_1=0,\ q_2=0,\ r_1=0,\ r_2=0,\\
a_{11}=\mp a_{32},\ a_{12}=\mp\sqrt{1-a_{32}^2},\ a_{13}=0,\ a_{21}=0,\\
a_{22}=0,\ a_{23}=1,\ a_{31}=-\sqrt{1-a_{32}^2},\ a_{33}=0,\\
b_{11}=\bar{D}\bar{D}_1,\ b_{12}=a_2\bar{D},\ b_{13}=0,\\
b_{21}=-a_2b_2\bar{D}\bar{D}_1^{-1},\ b_{22}=b_2\bar{D},\ b_{23}=\mp c_2\bar{D}_1^{-1},\\
b_{31}=-a_2c_2\bar{D}\bar{D}_1^{-1}, b_{32}=c_2\bar{D},\ b_{33}=\pm b_2\bar{D}_1^{-1}.
\end{array}$$

Here $\bar{D}=(a_2^2+b_2^2+c_2^2)^{-1/2}$, $\bar{D}_1=(b_2^2+c_2^2)^{1/2}$.

Equations (19) together with (20) allow one to obtain up to the eight families of solutions for the equations of motion (2)–(4). One of them is given by:

$$\begin{array}{l}
p_1=0,\ p_2=0,\ q_1=0,\ q_2=0,\ r_1=0,\ r_2=0,\\
a_{11}=-a_{32}^0,\ a_{12}=-\sqrt{1-a_{32}^{0^2}},\ a_{13}=0,\ a_{21}=0,\ a_{22}=0,\\
a_{23}=1,\ a_{31}=-\sqrt{1-a_{32}^{0^2}},\ a_{32}=a_{32}^0,\ a_{33}=0,\\
b_{11}=\dfrac{\sqrt{b_2^2+c_2^2}}{\sqrt{a_2^2+b_2^2+c_2^2}},\ b_{12}=\dfrac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ b_{13}=0,\\
b_{21}=-\dfrac{a_2b_2}{\sqrt{(b_2^2+c_2^2)(a_2^2+b_2^2+c_2^2)}},\ b_{22}=\dfrac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\\
b_{23}=-\dfrac{c_2}{\sqrt{b_2^2+c_2^2}},\ b_{31}=-\dfrac{a_2c_2}{\sqrt{(b_2^2+c_2^2)(a_2^2+b_2^2+c_2^2)}},\\
b_{32}=\dfrac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ b_{33}=\dfrac{b_2}{\sqrt{b_2^2+c_2^2}}.
\end{array} \tag{21}$$

Here a_{32}^0 is the parameter of the family.

The integral Ω_2 takes a stationary value on the elements of the family of solutions (21). From a mechanical point of view, the elements of this family correspond to equilibria of the mechanical system under consideration.

The following families of solutions correspond to the family of solutions (21) in the initial variables:

$$\begin{array}{l}
\varphi_1=0,\ \psi_1=-\arccos(-a_{32}^0),\ \theta_1=\dfrac{\pi}{2},\ \varphi_2=0,\\
\psi_2=\arccos\left(\dfrac{\sqrt{b_2^2+c_2^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}\right),\ \theta_2=-\arccos\left(\dfrac{b_2}{\sqrt{b_2^2+c_2^2}}\right);\\
\varphi_1=\pm\pi,\ \psi_1=\arccos(\,a_{32}^0),\ \theta_1=-\dfrac{\pi}{2},\ \varphi_2=\pm\pi,
\end{array}$$

$$\psi_2 = -\arccos\left(-\frac{\sqrt{b_2^2+c_2^2}}{\sqrt{a_2^2+b_2^2+c_2^2}}\right),\ \theta_2 = \arccos\left(\frac{b_2}{\sqrt{b_2^2+c_2^2}}\right).$$

4 On the Stability of Solutions

The integrals Ω_1 and Ω_2 taking stationary values both on IMs (10) and (19) and the elements of the families of solutions (13) and (21) can be used to obtain the sufficient conditions of their stability by the Routh–Lyapunov method [6]. In such a way, the sufficient conditions of stability for the permanent rotation of the system of two bodies were derived in [5]. In this work, such approach did not allow us to solve the question of stability for the solutions. The stability analysis of solutions (13) and (21) has been performed by the linear approximation method [7].

First, let us investigate the family of solutions (21). This problem is solved on the IM given by the equations [5]:

$$\begin{aligned}
&V_1-1=0,\ V_2-1=0,\ V_4-1=0,\ V_5-1=0,\ V_6-1=0,\\
&b_{11}b_{12}+b_{21}b_{22}+b_{31}b_{32}=0,\ b_{11}b_{13}+b_{21}b_{23}+b_{31}b_{33}=0,\\
&b_{12}b_{13}+b_{22}b_{23}+b_{32}b_{33}=0,\ b_{13}-b_{21}b_{32}+b_{22}b_{31}=0.
\end{aligned} \tag{22}$$

We write the equations of motion (2)–(4) on IM (22). To do this, the variables $a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, b_{11},\ b_{13}, b_{23}, b_{33}$ are eliminated from them with the help of (22). The differential equations in the Poisson form take the form:

$$\begin{aligned}
&\dot a_{32} = \frac{a_{32}a_{33}(a_{23}p_1-a_{13}q_1)+\sqrt{(a_{13}^2+a_{23}^2)(1-a_{32}^2)-a_{32}^2a_{33}^2}\,(a_{13}p_1+a_{23}q_1)}{a_{13}^2+a_{23}^2},\\
&\dot a_{13} = a_{23}r_1-a_{33}q_1,\ \dot a_{23} = a_{33}p_1-a_{13}r_1,\ \dot a_{33} = a_{13}q_1-a_{23}p_1,\\
&\dot b_{21} = b_{31}p_2+\frac{(b_{21}b_{22}+b_{31}b_{32})\,r_2}{b_{12}},\ \dot b_{31} = -\frac{b_{12}b_{21}p_2+b_{21}b_{22}q_2+b_{31}b_{32}q_2}{b_{12}},\\
&\dot b_{12} = b_{22}r_2-b_{32}q_2,\ \dot b_{22} = b_{32}p_2-b_{12}r_2,\ \dot b_{32} = b_{12}q_2-b_{22}p_2.
\end{aligned}$$

The differential equations in the Lagrange form are rather cumbersome and are not given here.

The following family of solutions

$$\begin{aligned}
&p_1=0,\ p_2=0,\ q_1=0,\ q_2=0,\ r_1=0,\ r_2=0,\\
&a_{32}=a_{32}^0,\ a_{13}=0,\ a_{23}=1,\ a_{33}=0,\\
&b_{12}=\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ b_{21}=-\frac{a_2b_2}{\sqrt{(b_2^2+c_2^2)(a_2^2+b_2^2+c_2^2)}},\\
&b_{22}=\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ b_{31}=-\frac{a_2c_2}{\sqrt{(b_2^2+c_2^2)(a_2^2+b_2^2+c_2^2)}},\\
&b_{32}=\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}
\end{aligned} \tag{23}$$

corresponds to solutions (21) on IM (22).

Let us consider the special case $a_2 = b_2 = c_2$. Taking into account the above restrictions, we write the differential equations linearized in the neighbourhood of the elements of the family of solutions (23):

$$\begin{aligned}
&\Big(A_1+\frac{1}{6}(4A_2+B_2+C_2)+\frac{(b_1^2+c_1^2)\,m_1^2}{m_2}-3c_2^2m_2\Big)\,\dot y_{10}+\sqrt{2}\,\Big(\frac{A_2}{\sqrt{3}}-\sqrt{2}z_1\Big)\,\dot y_{11}\\
&+\Big(\frac{2A_2-B_2-C_2}{3\sqrt{2}}-\frac{a_1b_1m_1^2}{m_2}\Big)\dot y_{12}+\Big(z_1-\frac{B_2}{\sqrt{6}}\Big)\,\dot y_{13}+\Big(\frac{B_2-C_2}{2\sqrt{3}}-\frac{a_1c_1m_1^2}{m_2}\Big)\,\dot y_{14}\\
&+\Big(z_1-\frac{C_2}{\sqrt{6}}\Big)\,\dot y_{15}+\sqrt{\frac{3}{2}}\,c_2m_2g(y_9-y_7)=0,\\
&\Big(\frac{2A_2-B_2-C_2}{3\sqrt{2}}-\frac{a_1b_1m_1^2}{m_2}\Big)\,\dot y_{10}+\Big(\frac{A_2}{\sqrt{3}}+\sqrt{2}\,a_1c_2m_1\Big)\,\dot y_{11}\\
&+\Big(\frac{1}{3}(A_2+3B_1+B_2+C_2)+\frac{(a_1^2+c_1^2)\,m_1^2}{m_2}\Big)\,\dot y_{12}+\Big(\frac{B_2}{\sqrt{3}}-z_2\Big)\,\dot y_{13}\\
&+\Big(\frac{C_2-B_2}{\sqrt{6}}-\frac{b_1c_1m_1^2}{m_2}\Big)\,\dot y_{14}+\Big(\frac{C_2}{\sqrt{3}}+z_3\Big)\,\dot y_{15}=0,\\
&\Big(\frac{B_2-C_2}{2\sqrt{3}}-\frac{a_1c_1m_1^2}{m_2}\Big)\,\dot y_{10}+\Big(\frac{C_2-B_2}{\sqrt{6}}-\frac{b_1c_1m_1^2}{m_2}\Big)\,\dot y_{12}+\frac{\sqrt{6}z_1-B_2}{\sqrt{2}}\,\dot y_{13}\\
&+\Big(\frac{1}{2}(B_2+2C_1+C_2)+\frac{(a_1^2+b_1^2)\,m_1^2}{m_2}-3c_2^2m_2\Big)\,\dot y_{14}+\frac{C_2-\sqrt{6}z_1}{\sqrt{2}}\,\dot y_{15}\\
&+\frac{c_2m_2g}{\sqrt{2}}\,(y_7+y_9-2y_5)=0,\\
&\sqrt{2}\,\Big(\frac{A_2}{\sqrt{3}}-\sqrt{2}z_1\Big)\,\dot y_{10}+A_2\dot y_{11}+\Big(\frac{A_2}{\sqrt{3}}+\sqrt{2}\,a_1c_2m_1\Big)\,\dot y_{12}\\
&+c_2m_2g\,(\sqrt{2}y_4-y_7+y_9)=0,\\
&\Big(z_1-\frac{B_2}{\sqrt{6}}\Big)\,\dot y_{10}+\Big(\frac{B_2}{\sqrt{3}}-z_2\Big)\,\dot y_{12}+B_2\dot y_{13}+\frac{\sqrt{6}z_1-B_2}{\sqrt{2}}\,\dot y_{14}\\
&+c_2m_2g\,\Big(\frac{\sqrt{3}\,y_2-y_4}{\sqrt{2}}+y_5-y_9\Big)=0,\\
&\Big(z_1-\frac{C_2}{\sqrt{6}}\Big)\,\dot y_{10}+\Big(\frac{C_2}{\sqrt{3}}+z_3\Big)\,\dot y_{12}+\frac{C_2-\sqrt{6}\,z_1}{\sqrt{2}}\,\dot y_{14}+C_2\dot y_{15}\\
&+c_2m_2g\,\Big(y_7-\frac{\sqrt{3}\,y_2-y_4}{\sqrt{2}}-y_5\Big)=0,\\
&\dot y_1-\sqrt{1-a_{32}^{0\,2}}\,y_{12}=0,\;\;\dot y_2-y_{14}=0,\;\;\dot y_3=0,\;\;\dot y_4+y_{10}=0,\\
&\dot y_6+\frac{y_{11}+2y_{15}}{\sqrt{6}}=0,\;\;\dot y_8-\frac{y_{11}+2y_{13}}{\sqrt{6}}=0,\;\;\dot y_5+\frac{y_{13}-y_{15}}{\sqrt{3}}=0,\\
&\dot y_7+\frac{y_{15}-y_{11}}{\sqrt{3}}=0,\;\;\dot y_9+\frac{y_{11}-y_{13}}{\sqrt{3}}=0.
\end{aligned}\tag{24}$$

Here $y_i\,(i=1,\ldots,15)$ are the deviations from the unperturbed motion, $z_1 = c_2(b_1m_1+\sqrt{3}c_2m_2)/\sqrt{2}$, $z_2 = c_2m_1(a_1+\sqrt{3}c_1)/\sqrt{2}$, $z_3 = c_2m_1(\sqrt{3}c_1-a_1)/\sqrt{2}$.

The characteristic equation of system (24) is

$$|A\lambda + B| = \lambda^7(f_0\lambda^8 + f_2\lambda^6 + f_4\lambda^4 + f_6\lambda^2 + f_8) = 0,$$

where $A,\ B$ are the matrices of the 15th order: A is the matrix composed of the coefficients of the derivatives of system (24), B is the matrix of the coefficients of y_i; f_0, f_2, f_4, f_6, f_8 are the expressions of $a_1, b_1, c_1, c_2, g, A_l, B_l, C_l, m_l (l = 1, 2)$. These are rather cumbersome and are not represented here.

Let us find the number of linearly independent eigenvectors corresponding to the multiple root $\lambda = 0$. To do this, we compute the rank of the matrix $A\lambda + B$ when $\lambda = 0$, using the built-in function *MatrixRank*. The rank r is 10 (the "Maple" function *Rank* gives the same result). The number of linearly independent eigenvectors is $15 - r = 5$. The multiple of the root $\lambda = 0$ is 7. Thus, the Jordan form of the matrix of linear system (24) is non-diagonal. The instability of the elements of the family under study in the linear approximation thus follows.

In an analogous way, the instability in the linear approximation for the elements of the family of solutions (13) in the special case $c_2 = b_2 = a_2$, $b^0_{12} = -2/3$, $s_1 = -s_2$ was proved.

5 Conclusion

In the problem of the rotation of two connected rigid bodies in a uniform gravity field, the Lagrange function and the equations of motion for the mechanical system have been derived in a symbolic form with the help of the software package written in the CAS "Mathematica" language. Using the Gröbner basis method, the stationary solutions and IMs of the equations have been found. From a mechanical point of view, these solutions correspond to equilibria of the mechanical system. Their instability in the linear approximation has been proved. Many questions concerning the stability of the obtained solutions remain yet unresolved. The analysis of the dynamics of the mechanical system in other force fields is also of interest. They will be addressed in our future work.

Appendix

$$\begin{aligned}
\Phi_1 = {} & q_1r_2[(A_2-B_2)(b_{12}b_{21}+b_{11}b_{22}) + b_{33}C_2 - m_2((b_2(b_{12}-b_{23}b_{31}+b_{21}b_{33})\\
& -a_2(b_{22}+b_{13}b_{31}-b_{11}b_{33}))s_1 - (b_2(b_{11}-b_{23}b_{32}+b_{22}b_{33})\\
& -a_2(b_{21}+b_{13}b_{32}-b_{12}b_{33}))\,s_2)]\\
& +p_2q_1[A_2b_{13} + (b_{22}b_{31}+b_{21}b_{32})(B_2-C_2) + m_2((b_2(b_{32}-b_{13}b_{21}\\
& +b_{11}b_{23}) - c_2(b_{22}+b_{13}b_{31}-b_{11}b_{33}))s_1 - (b_2(b_{31}-b_{13}b_{22}+b_{12}b_{23})\\
& -c_2(b_{21}+b_{13}b_{32}-b_{12}b_{33}))s_2)]\\
& -p_1r_1[A_2b_{11}b_{12} + B_2b_{21}b_{22} + C_2b_{31}b_{32} - m_2((b_2(b_{11}b_{33}-b_{13}b_{31})\\
& +a_2(b_{23}b_{31}-b_{21}b_{33}) + c_2(b_{13}b_{21}-b_{11}b_{23}))s_1 + (b_2(b_{13}b_{32}-b_{12}b_{33})\\
& +a_2(b_{22}b_{33}-b_{23}b_{32}) + c_2(b_{12}b_{23}-b_{13}b_{22}) + s_1)s_2)]
\end{aligned}$$

$$+q_1q_2[B_2b_{23}-(b_{12}b_{31}+b_{11}b_{32})(A_2-C_2)+m_2((a_2(b_{13}b_{21}-b_{11}b_{23}-b_{32})$$
$$+c_2(b_{12}-b_{23}b_{31}+b_{21}b_{33}))s_1+(a_2(b_{31}-b_{13}b_{22}+b_{12}b_{23})$$
$$-c_2(b_{11}-b_{23}b_{32}+b_{22}b_{33}))s_2)]$$
$$+p_2q_2[(B_2-A_2)b_{31}+m_2((b_{13}b_2+a_2b_{23})s_2-(b_{12}b_2+a_2b_{22})s_3)]$$
$$+p_2r_2[(A_2-C_2)b_{21}+m_2((a_2b_{33}+b_{13}c_2)s_2-(a_2b_{32}+b_{12}c_2)s_3)]$$
$$+q_2r_2[(C_2-B_2)b_{11}+m_2((b_2b_{33}+b_{23}c_2)s_2-(b_2b_{32}+b_{22}c_2)s_3)]$$
$$+p_2r_1[(b_{23}b_{31}+b_{21}b_{33})(B_2-C_2)-A_2b_{12}+m_2((b_2(b_{12}b_{21}-b_{11}b_{22}+b_{33})$$
$$-c_2(b_{23}-b_{12}b_{31}+b_{11}b_{32}))s_1+(c_2(b_{21}+b_{13}b_{32}-b_{12}b_{33})$$
$$-b_2(b_{31}-b_{13}b_{22}+b_{12}b_{23}))s_3)]$$
$$-m_2[c_2(p_2^2+q_2^2)(b_{33}s_2-b_{32}s_3)+b_2(p_2^2+r_2^2)(b_{23}s_2-b_{22}s_3)$$
$$+a_2(q_2^2+r_2^2)(b_{13}s_2-b_{12}s_3)]$$
$$-2p_1r_2[-b_{11}(A_2-B_2)b_{21}+m_2((a_2(b_{13}b_{31}-b_{11}b_{33})+b_2(b_{23}b_{31}-b_{21}b_{33}))s_2$$
$$+(a_2(b_{11}b_{32}-b_{12}b_{31})+b_2(b_{21}b_{32}-b_{22}b_{31}))s_3)]$$
$$+r_1r_2[(A_2-B_2)(b_{13}b_{21}+b_{11}b_{23})-C_2b_{32}+m_2((a_2(b_{23}-b_{12}b_{31}+b_{11}b_{32})$$
$$-b_2(b_{13}+b_{22}b_{31}-b_{21}b_{32}))s_1+(a_2(b_{21}+b_{13}b_{32}-b_{12}b_{33})+b_2(b_{11}-b_{23}b_{32}$$
$$+b_{22}b_{33}))s_3)]$$
$$+2p_1p_2[(B_2-C_2)b_{21}b_{31}+m_2((b_2(b_{13}b_{21}-b_{11}b_{23})+(b_{13}b_{31}-b_{11}b_{33})c_2)s_2$$
$$+(b_2(b_{11}b_{22}-b_{12}b_{21})+c_2(b_{11}b_{32}-b_{12}b_{31}))s_3)]$$
$$-2p_1q_2[A_2b_{11}b_{31}-b_{11}b_{31}C_2+m_2((a_2(b_{13}b_{21}-b_{11}b_{23})+c_2(b_{21}b_{33}-b_{23}b_{31}))s_2$$
$$+(a_2(b_{11}b_{22}-b_{12}b_{21})+c_2(b_{22}b_{31}-b_{21}b_{32}))s_3)]$$
$$-q_2r_1[B_2b_{22}+(b_{13}b_{31}+b_{11}b_{33})(A_2-C_2)+m_2((a_2(b_{12}b_{21}-b_{11}b_{22}+b_{33})$$
$$-c_2(b_{13}+b_{22}b_{31}-b_{21}b_{32}))s_1+(a_2(b_{13}b_{22}-b_{12}b_{23}-b_{31})+c_2(b_{11}-b_{23}b_{32}$$
$$+b_{22}b_{33}))s_3)]$$
$$+p_1q_1[A_2b_{11}b_{13}+B_2b_{21}b_{23}+b_{31}b_{33}C_2-m_2((b_2(b_{12}b_{31}-b_{11}b_{32})$$
$$+a_2(b_{21}b_{32}-b_{22}b_{31})+c_2(b_{11}b_{22}-b_{12}b_{21}))s_1+(b_2(b_{13}b_{32}-b_{12}b_{33})$$
$$+a_2(b_{22}b_{33}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_3)]$$
$$+q_1^2[A_2b_{12}b_{13}+B_2b_{22}b_{23}+b_{32}b_{33}C_2+m_2((b_2(b_{11}b_{32}-b_{12}b_{31})$$
$$+a_2(b_{22}b_{31}-b_{21}b_{32})+c_2(b_{12}b_{21}-b_{11}b_{22}))s_2-(b_2(b_{11}b_{33}-b_{13}b_{31})$$
$$+a_2(b_{23}b_{31}-b_{21}b_{33})+c_2(b_{13}b_{21}-b_{11}b_{23})+s_2)s_3)]$$
$$-r_1^2[A_2b_{12}b_{13}+B_2b_{22}b_{23}+C_2b_{32}b_{33}-m_2((b_2(b_{12}b_{31}-b_{11}b_{32})$$
$$+a_2(b_{21}b_{32}-b_{22}b_{31})+c_2(b_{11}b_{22}-b_{12}b_{21}))s_2+(b_2(b_{11}b_{33}-b_{13}b_{31})$$
$$+a_2(b_{23}b_{31}-b_{21}b_{33})+c_2(b_{13}b_{21}-b_{11}b_{23})+s_2)s_3)]$$
$$+q_1r_1[C_1-B_1-A_2(b_{12}^2-b_{13}^2)-B_2(b_{22}^2-b_{23}^2)-C_2(b_{32}^2-b_{33}^2)$$
$$+m_2(s_2(2(b_2(b_{11}b_{33}-b_{13}b_{31})$$
$$+a_2(b_{23}b_{31}-b_{21}b_{33})+(b_{13}b_{21}-b_{11}b_{23})c_2)+s_2)+2(b_2(b_{11}b_{32}-b_{12}b_{31})$$
$$+a_2(b_{22}b_{31}-b_{21}b_{32})+c_2(b_{12}b_{21}-b_{11}b_{22}))s_3-s_3^2)]$$
$$+g[m_1(b1a_{33}-c1a_{23})+m_2(a_{33}(a_2b_{12}+b_2b_{22}+c_2b_{32}+s_2)$$
$$-a_{23}(a_2b_{13}+b_2b_{23}+c_2b_{33}+s_3))],$$

$$\Phi_2=p_1r_2[(A_2-B_2)(b_{12}b_{21}+b_{11}b_{22})-C_2b_{33}+m_2((a_2(b_{13}b_{31}-b_{11}b_{33}-b_{22})$$
$$+b_2(b_{12}+b_{23}b_{31}-b_{21}b_{33}))s_1+(a_2(b_{21}-b_{13}b_{32}+b_{12}b_{33})-b_2(b_{11}+b_{23}b_{32}$$
$$-b_{22}b_{33}))s_2)]$$
$$+p_1p_2[(b_{22}b_{31}+b_{21}b_{32})(B_2-C_2)-A_2b_{13}+m_2((c_2(b_{22}-b_{13}b_{31}+b_{11}b_{33})$$
$$-b_2(b_{13}b_{21}-b_{11}b_{23}+b_{32}))s_1+(b_2(b_{13}b_{22}-b_{12}b_{23}+b_{31})-c_2(b_{21}-b_{13}b_{32}$$
$$+b_{12}b_{33}))s_2)]$$
$$-p_1q_2[B_2b_{23}+(A_2-C_2)(b_{12}b_{31}+b_{11}b_{32})-m_2((a_2(b_{13}b_{21}-b_{11}b_{23}+b_{32})$$
$$-c_2(b_{12}+b_{23}b_{31}-b_{21}b_{33}))s_1-(a_2(b_{13}b_{22}-b_{12}b_{23}+b_{31})-c_2(b_{11}+b_{23}b_{32}$$
$$-b_{22}b_{33}))s_2)]$$

$$
\begin{aligned}
&+q_1r_1[A_2b_{11}b_{12}+B_2b_{21}b_{22}+C_2b_{31}b_{32}+m_2((a_2(b_{21}b_{33}-b_{23}b_{31})\\
&+b_2(b_{13}b_{31}-b_{11}b_{33})-c_2(b_{13}b_{21}-b_{11}b_{23}))s_1-(a_2(b_{22}b_{33}-b_{23}b_{32})\\
&+b_2(b_{13}b_{32}-b_{12}b_{33})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_2)]\\
&+r_2^2m_2[(a_2b_{13}+b_2b_{23})s_1-(a_2b_{11}-b_2b_{21})s_3]\\
&-p_2q_2[(A_2-B_2)b_{32}+m_2((a_2b_{23}+b_2b_{13})s_1-(a_2b_{21}+b_2b_{11})s_3)]\\
&+q_2^2m_2[(a_2b_{13}+c_2b_{33})s_1-(a_2b_{11}+c_2b_{31})s_3]\\
&+p_2^2m_2[(b_2b_{23}+c_2b_{33})s_1-(b_2b_{21}+c_2b_{31})s_3]\\
&+2q_1r_2[(A_2-B_2)b_{12}b_{22}-m_2((a_2(b_{12}b_{33}-b_{13}b_{32})+b_2(b_{22}b_{33}-b_{23}b_{32}))s_1\\
&+(a_2(b_{11}b_{32}-b_{12}b_{31})+b_2(b_{21}b_{32}-b_{22}b_{31}))s_3)]\\
&+r_1r_2((A_2-B_2)(b_{13}b_{22}+b_{12}b_{23})+C_2b_{31}+m_2((a_2(b_{23}-b_{12}b_{31}+b_{11}b_{32})\\
&-b_2(b_{13}+b_{22}b_{31}-b_{21}b_{32}))s_2-(a_2(b_{22}-b_{13}b_{31}+b_{11}b_{33})-b_2(b_{12}+b_{23}b_{31}\\
&-b_{21}b_{33}))s_3))\\
&+p_2r_2[(A_2-C_2)b_{22}-m_2((a_2b_{33}+b_{13}c_2)s_1-(a_2b_{31}+c_2b_{11})s_3)]\\
&-q_2r_2((B_2-C_2)b_{12}+m_2((b_2b_{33}+c_2b_{23})s_1-(b_2b_{31}+c_2b_{21})s_3))\\
&-2q_1q_2[(A_2-C_2)b_{12}b_{32}+m_2((a_2(b_{12}b_{23}-b_{13}b_{22})+c_2(b_{23}b_{32}-b_{22}b_{33}))s_1\\
&+(a_2(b_{11}b_{22}-b_{12}b_{21})+c_2(b_{22}b_{31}-b_{21}b_{32}))s_3)]\\
&+p_2r_1[A_2b_{11}+(B_2-C_2)(b_{23}b_{32}+b_{22}b_{33})+m_2((b_2(b_{12}b_{21}-b_{11}b_{22}+b_{33})\\
&-c_2(b_{23}-b_{12}b_{31}+b_{11}b_{32}))s_2-(b_2(b_{13}b_{21}-b_{11}b_{23}+b_{32})+c_2(b_{22}-b_{13}b_{31}\\
&+b_{11}b_{33}))s_3)]\\
&+q_2r_1[B_2b_{21}-(b_{13}b_{32}+b_{12}b_{33})(A_2-C_2)-m_2((a_2(b_{12}b_{21}-b_{11}b_{22}+b_{33})\\
&-c_2(b_{13}+b_{22}b_{31}-b_{21}b_{32}))s_2-(a_2(b_{13}b_{21}-b_{11}b_{23}+b_{32})-c_2(b_{12}+b_{23}b_{31}\\
&-b_{21}b_{33}))s_3)]\\
&+2p_2q_1[b_{22}b_{32}(B_2-C_2)+m_2((b_2(b_{12}b_{23}-b_{13}b_{22})+c_2(b_{12}b_{33}-b_{13}b_{32}))s_1\\
&+(b_2(b_{11}b_{22}-b_{12}b_{21})+c_2(b_{11}b_{32}-b_{12}b_{31}))s_3)]\\
&+r_1^2[A_2b_{11}b_{13}+B_2b_{21}b_{23}+C_2b_{31}b_{33}+m_2((a_2(b_{22}b_{31}-b_{21}b_{32})\\
&+b_2(b_{11}b_{32}-b_{12}b_{31})+c_2(b_{12}b_{21}-b_{11}b_{22}))s_1-(a_2(b_{22}b_{33}-b_{23}b_{32})\\
&+b_2(b_{13}b_{32}-b_{12}b_{33})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_3)]\\
&-p_1^2(A_2b_{11}b_{13}+B_2b_{21}b_{23}+C_2b_{31}b_{33}-m_2((a_2(b_{21}b_{32}-b_{22}b_{31})\\
&+b_2(b_{12}b_{31}-b_{11}b_{32})+c_2(b_{11}b_{22}-b_{12}b_{21}))s_1+(a_2(b_{22}b_{33}-b_{23}b_{32})\\
&+b_2(b_{13}b_{32}-b_{12}b_{33})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_3))\\
&+p_1q_1(-A_2b_{12}b_{13}-B_2b_{22}b_{23}-C_2b_{32}b_{33}+m_2((a_2(b_{21}b_{32}-b_{22}b_{31})\\
&+b_2(b_{12}b_{31}-b_{11}b_{32})+c_2(b_{11}b_{22}-b_{12}b_{21}))s_2+(a_2(b_{23}b_{31}-b_{21}b_{33})\\
&+b_2(b_{11}b_{33}-b_{13}b_{31})+c_2(b_{13}b_{21}-b_{11}b_{23})+s_2)s_3))\\
&+p_1r_1[A_1-C_1+A_2(b_{11}^2-b_{13}^2)+B_2(b_{21}^2-b_{23}^2)+C_2(b_{31}^2-b_{33}^2)\\
&-m_2\,((2(a_2(b_{22}b_{33}-b_{23}b_{32})+b_2(b_{13}b_{32}-b_{12}b_{33})+c_2(b_{12}b_{23}-b_{13}b_{22}))\\
&+s_1)s_1-(2(a_2(b_{21}b_{32}-b_{22}b_{31})+b_2(b_{12}b_{31}-b_{11}b_{32})+c_2(b_{11}b_{22}-b_{12}b_{21}))\\
&+s_1)s_1+s_3)s_3)]-g[\,m_1(a_1a_{33}-c_1a_{13})-m_2(a_{13}(a_2b_{13}+b_2b_{23}+c_2b_{33}+s_3)\\
&-a_{33}(a_2b_{11}+b_2b_{21}+c_2b_{31}+s_1))],
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3=\;&r_2^2\,m_2[(a_2b_{11}+b_2b_{21})s_2-(a_2b_{12}+b_2b_{22})s_1]\\
&+q_2^2\,m_2[(a_2b_{11}+c_2b_{31})s_2-(a_2b_{12}+c_2b_{32})s_1]\\
&+p_2^2\,m_2[(b_2b_{21}+c_2b_{31})s_2-(b_2b_{22}+c_2b_{32})s_1]\\
&+p_2q_2\,[(B_2-A_2)b_{33}+m_2((b_2b_{12}+a_2b_{22})s_1-(b_2b_{11}+a_2b_{21})s_2)]\\
&+2r_1r_2\,[b_{13}(A_2-B_2)b_{23}+m_2((a_2(b_{13}b_{32}-b_{12}b_{33})+b_2(b_{23}b_{32}-b_{22}b_{33}))s_1\\
&+(a_2(b_{11}b_{33}-b_{13}b_{31})+b_2(b_{21}b_{33}-b_{23}b_{31}))s_2)]\\
&+p_2r_2\,[b_{23}(A_2-C_2)+m_2((a_2b_{32}+c_2b_{12})s_1-(a_2b_{31}+c_2b_{11})s_2)]\\
&+q_2r_2\,[(C_2-B_2)b_{13}+m_2((b_2b_{32}+b_{22}c_2)s_1-(b_2b_{31}+b_{21}c_2)s_2)]\\
&-2p_2r_1\,[(C_2-B_2)b_{23}b_{33}+m_2((b_2(b_{13}b_{22}-b_{12}b_{23})+c_2(b_{13}b_{32}-b_{12}b_{33}))s_1
\end{aligned}
$$

$$
\begin{aligned}
&+(b_2(b_{11}b_{23}-b_{13}b_{21})+c_2(b_{11}b_{33}-b_{13}b_{31}))s_2)]\\
&+2q_2r_1\,[(C_2-A_2)b_{13}b_{33}+m_2((a_2(b_{13}b_{22}-b_{12}b_{23})+c_2(b_{22}b_{33}-b_{23}b_{32}))s_1\\
&+(a_2(b_{11}b_{23}-b_{13}b_{21})+c_2(b_{23}b_{31}-b_{21}b_{33}))s_2)]\\
&+p_1^2\,[A_2b_{11}b_{12}+B_2b_{21}b_{22}+C_2b_{31}b_{32}+m_2((b_2(b_{13}b_{31}-b_{11}b_{33})\\
&+a_2(b_{21}b_{33}-b_{23}b_{31})+c_2(b_{11}b_{23}-b_{13}b_{21}))s_1-(b_2(b_{13}b_{32}-b_{12}b_{33})\\
&+a_2(b_{22}b_{33}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_2)]\\
&-q_1^2\,[A_2b_{11}b_{12}+B_2b_{21}b_{22}+C_2b_{31}b_{32}-m_2((b_2(b_{11}b_{33}-b_{13}b_{31})\\
&+a_2(b_{23}b_{31}-b_{21}b_{33})+c_2(b_{13}b_{21}-b_{11}b_{23}))s_1+(b_2(b_{13}b_{32}-b_{12}b_{33})\\
&+a_2(b_{22}b_{33}-b_{23}b_{32})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_2)]\\
&+q_1r_2\,[(A_2-B_2)(b_{13}b_{22}+b_{12}b_{23})-C_2b_{31}+m_2((b_2(b_{13}-b_{22}b_{31}+b_{21}b_{32})\\
&-a_2(b_{23}+b_{12}b_{31}-b_{11}b_{32}))s_2+(a_2(b_{22}+b_{13}b_{31}-b_{11}b_{33})\\
&-b_2(b_{12}-b_{23}b_{31}+b_{21}b_{33}))s_3)]\\
&+p_1r_2\,[(A_2-B_2)(b_{13}b_{21}+b_{11}b_{23})+C_2b_{32}+m_2((b_2(b_{13}-b_{22}b_{31}+b_{21}b_{32})\\
&-a_2(b_{23}+b_{12}b_{31}-b_{11}b_{32}))s_1+(a_2(b_{21}\\
&-b_{13}b_{32}+b_{12}b_{33})-b_2(b_{11}+b_{23}b_{32}-b_{22}b_{33}))s_3)]\\
&+p_2q_1\,[(b_{23}b_{32}+b_{22}b_{33})(B_2-C_2)-A_2b_{11}+m_2((b_2(b_{12}b_{21}-b_{11}b_{22}-b_{33})\\
&+c_2(b_{23}+b_{12}b_{31}-b_{11}b_{32}))s_2+(b_2(b_{32}-b_{13}b_{21}+b_{11}b_{23})\\
&-c_2(b_{22}+b_{13}b_{31}-b_{11}b_{33}))s_3)]\\
&+p_1p_2\,[A_2b_{12}+(B_2-C_2)(b_{23}b_{31}+b_{21}b_{33})+m_2((c_2(b_{23}+b_{12}b_{31}-b_{11}b_{32})\\
&-b_2(b_{33}-b_{12}b_{21}+b_{11}b_{22}))s_1+(b_2(b_{13}b_{22}-b_{12}b_{23}+b_{31})\\
&-c_2(b_{21}-b_{13}b_{32}+b_{12}b_{33}))s_3)]\\
&-q_1q_2\,[B_2b_{21}+(A_2-C_2)(b_{13}b_{32}+b_{12}b_{33})-m_2((a_2(b_{33}-b_{12}b_{21}+b_{11}b_{22})\\
&-c_2(b_{13}-b_{22}b_{31}+b_{21}b_{32}))s_2+(a_2(b_{13}b_{21}-b_{11}b_{23}-b_{32})\\
&+c_2(b_{12}-b_{23}b_{31}+b_{21}b_{33}))s_3)]\\
&+p_1q_2(B_2b_{22}-(A_2-C_2)(b_{13}b_{31}+b_{11}b_{33})+m_2((a_2(b_{33}-b_{12}b_{21}+b_{11}b_{22})\\
&-c_2(b_{13}-b_{22}b_{31}+b_{21}b_{32}))s_1-(a_2(b_{31}+b_{13}b_{22}-b_{12}b_{23})\\
&-c_2(b_{11}+b_{23}b_{32}-b_{22}b_{33}))s_3))\\
&-q_1r_1\,[A_2b_{11}b_{13}+B_2b_{21}b_{23}+C_2b_{31}b_{33}-m_2((a_2(b_{21}b_{32}-b_{22}b_{31})\\
&+b_2(b_{12}b_{31}-b_{11}b_{32})+c_2(b_{11}b_{22}-b_{12}b_{21}))s_1+(a_2(b_{22}b_{33}-b_{23}b_{32})\\
&+b_2(b_{13}b_{32}-b_{12}b_{33})+c_2(b_{12}b_{23}-b_{13}b_{22})+s_1)s_3)]\\
&+p_1r_1\,[A_2b_{12}b_{13}+B_2b_{22}b_{23}+C_2b_{32}b_{33}+m_2((a_2(b_{22}b_{31}-b_{21}b_{32})\\
&+b_2(b_{11}b_{32}-b_{12}b_{31})+c_2(b_{12}b_{21}-b_{11}b_{22}))s_2-(a_2(b_{23}b_{31}-b_{21}b_{33})\\
&+b_2(b_{11}b_{33}-b_{13}b_{31})+c_2(b_{13}b_{21}-b_{11}b_{23})+s_2)s_3)]\\
&-p_1q_1\,[A_1-B_1-A_2(b_{11}^2-b_{12}^2)+B_2(b_{21}^2-b_{22}^2)+C_2(b_{31}^2-b_{32}^2)\\
&-m_2((2(a_2(b_{22}b_{33}-b_{23}b_{32})+b_2(b_{13}b_{32}-b_{12}b_{33})+c_2(b_{12}b_{23}-b_{13}b_{22}))+s_1)s_1\\
&+(2(b_2(a_2(b_{21}b_{33}-b_{23}b_{31})+b_{13}b_{31}-b_{11}b_{33})+c_2(b_{11}b_{23}-b_{13}b_{21}))-s_2)s_2)]\\
&+g(m_1(a_1a_{23}-b_1a_{13})+m_2(a_{23}(a_2b_{11}+b_2b_{21}+c_2b_{31}+s_1)\\
&-a_{13}(a_2b_{12}+b_2b_{22}+c_2b_{32}+s_2))),
\end{aligned}
$$

$$
\begin{aligned}
\Phi_4 = {} & (A_2-B_2+C_2)(b_{21}p_1+b_{22}q_1+b_{23}r_1)\,r_2\\
&-(A_2+B_2-C_2)(b_{31}p_1+b_{32}q_1+b_{33}r_1)\,q_2+(C_2-B_2)\,q_2r_2\\
&+r_1^2\,[(C_2-B_2)b_{23}b_{33}+m_2((b_2(b_{13}b_{22}-b_{12}b_{23})+c_2(b_{13}b_{32}-b_{12}b_{33}))s_1\\
&+(b_2(b_{11}b_{23}-b_{13}b_{21})+c_2(b_{11}b_{33}-b_{13}b_{31}))s_2)]\\
&-p_1q_1[(B_2-C_2)(b_{22}b_{31}+b_{21}b_{32})-m_2((b_2(b_{13}b_{21}-b_{11}b_{23})\\
&+c_2(b_{13}b_{31}-b_{11}b_{33}))s_1+(b_2(b_{12}b_{23}-b_{13}b_{22})+c_2(b_{12}b_{33}-b_{13}b_{32}))s_2)]\\
&+p_1^2\,[(C_2-B_2)b_{21}b_{31}+m_2((b_2(b_{11}b_{23}-b_{13}b_{21})+c_2(b_{11}b_{33}-b_{13}b_{31}))s_2\\
&+(b_2(b_{12}b_{21}-b_{11}b_{22})+c_2(b_{12}b_{31}-b_{11}b_{32}))s_3)]\\
&-q_1^2\,[(B_2-C_2)b_{22}b_{32}-m_2((b_2(b_{13}b_{22}-b_{12}b_{23})+c_2(b_{13}b_{32}-b_{12}b_{33}))s_1
\end{aligned}
$$

$$
\begin{aligned}
&+(b_2(b_{12}b_{21}-b_{11}b_{22})+c_2(b_{12}b_{31}-b_{11}b_{32}))s_3)]\\
&-q_1r_1\,[(B_2-C_2)(b_{23}b_{32}+b_{22}b_{33})-m_2((b_2(b_{11}b_{22}-b_{12}b_{21})\\
&+c_2(b_{11}b_{32}-b_{12}b_{31}))s_2+(b_2(b_{13}b_{21}-b_{11}b_{23})+(b_{13}b_{31}-b_{11}b_{33})c_2)s_3)]\\
&-p_1r_1\,[(b_{23}b_{31}+b_{21}b_{33})(B_2-C_2)-m_2((b_2(b_{11}b_{22}-b_{12}b_{21})\\
&+c_2(b_{11}b_{32}-b_{12}b_{31}))s_1+(b_2(b_{12}b_{23}-b_{13}b_{22})+c_2(b_{12}b_{33}-b_{13}b_{32}))s_3)]\\
&+gm_2(b_2(a_{13}b_{31}+a_{23}b_{32}+a_{33}b_{33})-c_2(a_{13}b_{21}+a_{23}b_{22}+a_{33}b_{23})),
\end{aligned}
$$

$$
\begin{aligned}
\Phi_5 =\ &(A_2+B_2-C_2)(b_{31}p_1+b_{32}q_1+b_{33}r_1)\,p_2\\
&+(A_2-B_2-C_2)(b_{11}p_1+b_{12}q_1+b_{13}r_1)\,r_2+(A_2-C_2)\,p_2r_2\\
&+r_1^2\,[(A_2-C_2)\,b_{13}b_{33}+m_2((a_2(b_{12}b_{23}-b_{13}b_{22})+c_2(b_{23}b_{32}-b_{22}b_{33}))s_1\\
&+(a_2(b_{13}b_{21}-b_{11}b_{23})+c_2(b_{21}b_{33}-b_{23}b_{31}))s_2)]\\
&+p_1q_1\,[(A_2-C_2)(b_{12}b_{31}+b_{11}b_{32})+m_2((a_2(b_{11}b_{23}-b_{13}b_{21})\\
&+c_2(b_{23}b_{31}-b_{21}b_{33}))s_1+(a_2(b_{13}b_{22}-b_{12}b_{23})+c_2(b_{22}b_{33}-b_{23}b_{32}))s_2)]\\
&+q_1^2\,[b_{12}b_{32}(A_2-C_2)+m_2((a_2(b_{12}b_{23}-b_{13}b_{22})+c_2(b_{23}b_{32}-b_{22}b_{33}))s_1\\
&+(a_2(b_{11}b_{22}-b_{12}b_{21})+c_2(b_{22}b_{31}-b_{21}b_{32}))s_3)]\\
&+p_1^2\,[(A_2-C_2)\,b_{11}b_{31}+m_2((a_2(b_{13}b_{21}-b_{11}b_{23})+c_2(b_{21}b_{33}-b_{23}b_{31}))s_2\\
&+(a_2(b_{11}b_{22}-b_{12}b_{21})+c_2(b_{22}b_{31}-b_{21}b_{32}))s_3)]\\
&+q_1r_1\,[(A_2-C_2)(b_{13}b_{32}+b_{12}b_{33})+m_2((a_2(b_{12}b_{21}-b_{11}b_{22})\\
&+c_2(b_{21}b_{32}-b_{22}b_{31}))s_2+(a_2(b_{11}b_{23}-b_{13}b_{21})+c_2(b_{23}b_{31}-b_{21}b_{33}))s_3)]\\
&+p_1r_1\,[(A_2-C_2)(b_{13}b_{31}+b_{11}b_{33})+m_2((a_2(b_{12}b_{21}-b_{11}b_{22})\\
&+c_2(b_{21}b_{32}-b_{22}b_{31}))s_1+(a_2(b_{13}b_{22}-b_{12}b_{23})+c_2(b_{22}b_{33}-b_{23}b_{32}))s_3)]\\
&-gm_2\,[a_2(a_{13}b_{31}+a_{23}b_{32}+a_{33}b_{33})-c_2(a_{13}b_{11}+a_{23}b_{12}+a_{33}b_{13})],
\end{aligned}
$$

$$
\begin{aligned}
\Phi_6 =\ &-(A_2-B_2-C_2)(b_{11}p_1+b_{12}q_1+b_{13}r_1)\,q_2\\
&-(A_2-B_2+C_2)(b_{21}p_1+b_{22}q_1+b_{23}r_1)\,p_2-(A_2-B_2)p_2q_2\\
&-r_1^2\,[(A_2-B_2)b_{13}b_{23}-m_2((a_2(b_{12}b_{33}-b_{13}b_{32})+b_2(b_{22}b_{33}-b_{23}b_{32}))s_1\\
&+(a_2(b_{13}b_{31}-b_{11}b_{33})+b_2(b_{23}b_{31}-b_{21}b_{33}))s_2)]\\
&-p_1q_1\,[(A_2-B_2)(b_{12}b_{21}+b_{11}b_{22})-m_2((a_2(b_{11}b_{33}-b_{13}b_{31})\\
&+b_2(b_{21}b_{33}-b_{23}b_{31}))s_1+(a_2(b_{13}b_{32}-b_{12}b_{33})+b_2(b_{23}b_{32}-b_{22}b_{33}))s_2)]\\
&-q_1^2\,[(A_2-B_2)b_{12}b_{22}-m_2((a_2(b_{12}b_{33}-b_{13}b_{32})+b_2(b_{22}b_{33}-b_{23}b_{32}))s_1\\
&+(a_2(b_{11}b_{32}-b_{12}b_{31})+b_2(b_{21}b_{32}-b_{22}b_{31}))s_3)]\\
&-p_1^2\,[(A_2-B_2)b_{11}b_{21}-m_2((a_2(b_{13}b_{31}-b_{11}b_{33})+b_2(b_{23}b_{31}-b_{21}b_{33}))s_2\\
&+(a_2(b_{11}b_{32}-b_{12}b_{31})+b_2(b_{21}b_{32}-b_{22}b_{31}))s_3)]\\
&-q_1r_1\,[(A_2-B_2)(b_{13}b_{22}+b_{12}b_{23})-m_2((a_2(b_{12}b_{31}-b_{11}b_{32})\\
&+b_2(b_{22}b_{31}-b_{21}b_{32}))s_2+(a_2(b_{11}b_{33}-b_{13}b_{31})+b_2(b_{21}b_{33}-b_{23}b_{31}))s_3)]\\
&-p_1r_1\,[(A_2-B_2)(b_{13}b_{21}+b_{11}b_{23})-m_2((a_2(b_{12}b_{31}-b_{11}b_{32})\\
&+b_2(b_{22}b_{31}-b_{21}b_{32}))s_1+(a_2(b_{13}b_{32}-b_{12}b_{33})+b_2(b_{23}b_{32}-b_{22}b_{33}))s_3)]\\
&+gm_2(a_2(a_{13}b_{21}+a_{23}b_{22}+a_{33}b_{23})-b_2(a_{13}b_{11}+a_{23}b_{12}+a_{33}b_{13})).
\end{aligned}
$$

References

1. Banshchikov, A.V., Burlakova, L.A., Irtegov, V.D., Titorenko, T.N.: Software Package LinModel for the Analysis of the Dynamics of Large Dimensional Mechanical Systems. Certificate of State Registration of Software Programs. FGU-FIPS. 2008610622 (2008)

2. Burov, A.A.: Linear invariant relations in the problem of the motion of a bundle of two bodies. Dokl. Phys. **65**(4), 147–148 (2020). https://doi.org/10.1134/S1028335820040035
3. Gutnik, S.A., Sarychev, V.A.: Application of computer algebra methods to investigation of stationary motions of a system of two connected bodies moving in a circular orbit. Comput. Math. Math. Phys. **60**(1), 74–81 (2020). https://doi.org/10.1134/S0965542520010091
4. Irtegov, V.D., Titorenko, T.N.: About stationary movements of the generalized Kovalevskaya top and their stability. Mech. Solids **54**(1), 81–91 (2019). https://doi.org/10.3103/S0025654419010072
5. Irtegov, V., Titorenko, T.: On the study of the motion of a system of two connected rigid bodies by computer algebra methods. In: Boulier, F., England, M., Sadykov, T.M., Vorozhtsov, E.V. (eds.) CASC 2020. LNCS, vol. 12291, pp. 266–281. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-60026-6_15
6. Lyapunov, A.M.: On permanent helical motions of a rigid body in fluid. Collected Works, USSR Acad. Sci., Moscow-Leningrad **1**, 276–319 (1954). (in Russian)
7. Lyapunov, A.M.: The General Problem of the Stability of Motion. Taylor & Francis, London (1992)