

# Constructing Covariational Relationships and Distinguishing Nonlinear and Linear Relationships



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[T]o ground the development of algebraic thinking on the notion of functions and functional relationships without, in turn, grounding these on understandings of quantities and quantitative reasoning in dynamic situations, is like building a house starting with the second floor. The house will not stand. (Thompson & Thompson, 1995, p. 98)

## 1 Introduction

Mathematics generally (Crisp et al., 2009; Sass, 2015), and algebra specifically (Loveless, 2013), serve as gatekeepers that have restricted student access to STEM fields. Thus, it is more important than ever that K-12 education supports students in developing foundational knowledge and ways of thinking that support their algebra learning. However, current algebra curricula and teaching often present an abstract, static, symbolic, and largely procedural mathematics (e.g., Hiebert et al., 2005; Litke, 2020). To increase STEM opportunity, pre-algebra and algebra instruction must help students develop ways of thinking that are meaningful, accessible, and applicable broadly across STEM fields. One of these ways of thinking is *covariational reasoning*, the ability to construct and reason about relationships between quantities changing together. Students are not currently being provided sufficient opportunities to reason about covarying quantities (Frank & Thompson, 2021; Smith & Thompson, 2008; Thompson & Harel, 2021). As reflected in the opening quote, the lack of opportunities to reason about covarying quantities may explain much of students'

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difficulties with algebra; students are not being provided with a foundation on which to build their more formal algebra knowledge.

In this chapter, we propose that middle school students would be better served first having opportunities to reason about dynamically changing quantities to construct basic types of covariational relationships. These covariational relationships can serve as the foundation for students' activity as they begin to use multiple representations (e.g., graphs, tables, equations) to represent such relationships. In the following sections, we first outline our theoretical framework, which details the requisite meanings students need to construct, reason about, and represent covariational relationships between continuously changing quantities. We then outline a task sequence we iteratively designed and tested to support students in constructing, coordinating, and graphically representing covarying quantities. Throughout, we use two students' activity to exemplify the productivity of this task sequence. We conclude by discussing implications for student reasoning and highlighting the potential for such activity to serve as a foundation for students developing meanings for various functional relationships.

## 2 Theoretical Background

Prior to presenting our task sequence, we describe constructs relevant to how students construct, coordinate, and represent covarying quantities. We then describe how students can leverage their covariational reasoning to characterize basic types of covariational relationships and differentiate between nonlinear and linear relationships. We conclude by characterizing the requisite meanings students must maintain to represent such relationships graphically.

### 2.1 *Foundations of Covariational Reasoning*

In this section, we first describe the theoretical framework we use when characterizing students' quantitative and variational reasoning. We then present our coordination of the frameworks from Carlson et al. (2002) and Thompson and Carlson (2017) that we leverage.

#### 2.1.1 Quantitative and Variational Reasoning

Several researchers (see Thompson & Carlson, 2017 for a review) have begun to explore ways in which students' quantitative reasoning (Thompson, 2011) can support their development of productive meanings for various mathematical ideas. Adopting this theory, we contend that quantities are conceptual entities a student constructs to make sense of some phenomenon. A student's quantitative

reasoning can involve numerical and non-numerical reasoning (Johnson, 2012; Moore et al., 2019), but the essence of quantitative reasoning is non-numerical (Smith & Thompson, 2008). There are numerous ways students can reason about magnitudes (see Thompson et al., 2014), but we are particularly interested in students conceiving of an increasing or decreasing amount of a measurable attribute of an object or phenomenon. Furthermore, although a student can reason quantitatively about static quantities (e.g., comparing two static lengths to determine the measure of one length in terms of the other), we attend to students' variational reasoning about conceived dynamic quantities.

In this report, we leverage and refine Thompson and Carlson's (2017) variational reasoning framework (Table 1) based on our attempts to analyze student activity using this framework. Specifically, we add smooth variational reasoning, which entails a student reasoning about the variation of a quantity's magnitude or value as changing smoothly across an interval. Such reasoning is not as sophisticated as smooth continuous variation. Smooth continuous variation entails smooth variation with an additional anticipation that any smaller sub-interval would also entail smooth and continuous variation; we only characterize a student as engaging in smooth continuous variation if the student explicitly describes such smaller sub-intervals. In our research, we often observed students engaging in smooth variation without explicitly considering or describing sub-intervals, thereby creating a need for a new level within the framework. We note our characterization of smooth variational reasoning is more sophisticated than gross variation and chunky continuous variation, as the student anticipates the quantity takes on magnitudes or values while changing between intervals of a fixed size (Thompson, personal communication).

We note that while Thompson and Carlson (2017) use the term levels in both their variational and covariational reasoning framework, these levels are not necessarily hierarchical. Students do not need to move sequentially up from the lowest to the highest level. Thompson and Carlson (2017) cautioned researchers not see these levels as

a learning progression in the sense that one level should be targeted instructionally before the next higher level. As Castillo-Garsow et al. (2013) point out, teachers should emphasize smooth variation in their talk and actions whenever they can. Students will reason at the level they will, and if at some point in time they reason variationally at the highest level, they get all other levels for free. (p. 440)

Consistent with this recommendation, we design tasks we intend to promote smooth variation in students' reasoning.

To exemplify several of the levels in Table 1, and the distinctions we make moving forward, we will use the *Triangle/Rectangle Task*. In the *Triangle/Rectangle Task*, students are presented with a GeoGebra applet which presents an (apparently)

**Table 1** An amended version of Thompson and Carlson's (2017) major levels of variational reasoning (p. 440), with changes in bold

Level	Description
Smooth continuous variation	The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously
<b>Smooth variation (added)</b>	<b>The person thinks of variation of a variable's value as changing smoothly across an interval without considering sub-intervals within the interval. The person anticipates that the variable changes from <math>a</math> to <math>b</math> smoothly by taking all values between <math>a</math> and <math>b</math> without attending to sub-intervals within the interval from <math>a</math> to <math>b</math></b>
Chunky continuous variation	The person thinks of variation of a variable's value as changing by intervals of a fixed size...The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2... and so on, "come along" by virtue of each being part of a chunk...but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values
Gross variation	The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing
Discrete variation	The person envisions a variable as taking specific values. The person sees the variable's value changing from $a$ to $a_n$ by taking values $a_1, a_2, \dots, a_n$ but does not envision the variable taking any value between $a_i$ and $a_{i+1}$ ( $i = 1, 2, \dots, n$ )
No variation	The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario
Variable as symbol	The person understands a variable as being just a symbol that has nothing to do with variation

smoothly<sup>1</sup> growing triangle and rectangle (Fig. 1; <https://www.geogebra.org/m/cxevsyc>). The two shapes have equal base lengths (highlighted in pink) defined by the slider value ( $a$ ), which ranges from 0 to 5. The shorter slider allows students to change the increment with which the  $a$ -values change, from apparently smoothly (in increments of 0.01) to larger chunks (in increments of 1.0). We ask students to consider variations in each shape's base length and area as the animations play to support students in conceiving of each shape's base length and area as quantities in the situation.

<sup>1</sup> We acknowledge that, due to the digital nature of the task, all quantities vary according to the discrete parameters set in the applet, which is why we refer to the quantities as varying (apparently) smoothly. Hereafter, we will use smoothly to convey the (apparently) smooth nature of the variations in the applets.

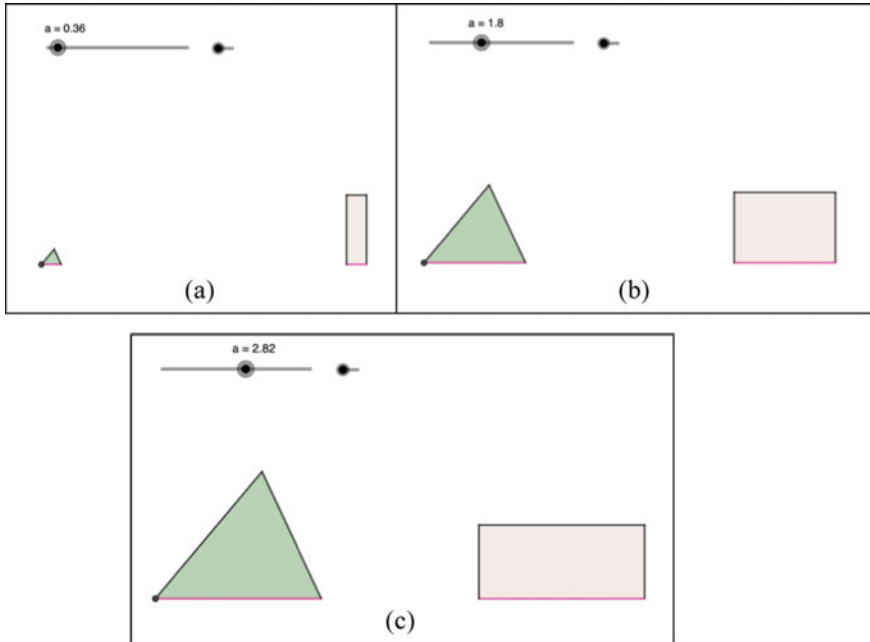


Fig. 1 Screenshots of the triangle/rectangle task

Particular to students’ variational reasoning, *gross variational reasoning* is common when students begin to conceive quantities in the situation. For example, a student may initially conceive that each shape has an increasing area as the base length is increasing. If a student conceives area as varying smoothly as the slider increases from 0 to 5 without providing any description of how the area is changing within subintervals, we only classify the student as engaging in *smooth variational reasoning*. Evidence for smooth variational reasoning may entail students using smooth hand motions or active voice to describe how quantities are changing (e.g., motioning to represent an interval of changing  $a$ -values, describing “the  $a$ -value starts at 0 and increases until it reaches 5”). We classify a student as engaging in *smooth continuous variational reasoning* if the student anticipates and explicitly describes variations of area within smaller subintervals (e.g., describing what could be happening to area as the base length varies from 1.77 to 1.78).

If a student is constrained to reasoning about incremental changes of a fixed size of the base length (e.g., 0.01, 1.0, or some other value), then we would categorize the student’s reasoning as entailing *chunky continuous variation*. It is common for students to exhibit chunky continuous variational reasoning when they describe how the area changes if the short slider is set to integer values (e.g., describing “the area of the triangle jumps by bigger amounts”). Although such reasoning is chunky, chunky reasoning is critical to differentiate between different patterns of growth (Vishnubhotla, 2020). Hence, several tasks are designed to support students in engaging in

both smooth and chunky thinking. However, we always start by presenting continuously changing phenomena as we concur with Castillo-Garsow et al. (2013), who contended:

smooth thinking would entail a capacity to think in chunks (or, at least at its foundation, one chunk). In contrast, in our experience with students, chunky thinking does not seem to entail a capacity to think smoothly, nor does chunky thinking seem to provide a cognitive root for smooth thinking. (p. 36)

With the idea that smooth thinking can entail chunky thinking, we design tasks that allow students to first experience and possibly conceptualize a smoothly changing phenomenon. Only after such opportunities do we introduce features of the applet that allow the quantities to change in chunks.

### 2.1.2 Covariational Reasoning: Coordinating Frameworks

As our goal is for students to coordinate and represent (at least) *two* varying quantities, we also attend to students' covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). In this section, we first offer an overview of covariational reasoning and multiplicative objects. We then provide the theoretical underpinnings for how we design tasks to support students' covariational reasoning by relating our interpretations of Carlson et al.'s (2002) framework with Thompson and Carlson's (2017) covariational reasoning framework (Table 2).

Researchers (Carlson et al., 2002; Saldanha & Thompson, 1998) have contended covariational reasoning is developmental. Saldanha and Thompson (1998) described that initially a student is likely to coordinate two quantities by thinking, "of one, then the other, then the first, then the second, and so on" (p. 299). Through this process, the student can develop an operative image of covariation that entails a relationship between quantities that results from imaging both quantities being tracked for some duration. Elaborating on their description of covariational reasoning Saldanha and Thompson (1998) stated:

[Covariational reasoning] entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (p. 299)

Elaborating on their use of *multiplicative object*, which stems from Piaget's notion of 'and' as a multiplicative operator, Thompson et al. (2017) noted, "A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one and the other" (p. 98). Hence, covariational reasoning entails understanding the simultaneity of two quantities' magnitudes or values in relation to each other.

As we are particularly interested in supporting students in conceiving smoothly changing phenomena, we use dynamic applets that could support students in anticipating that quantities covary smoothly, as recommended by others (Castillo-Garsow et al., 2013; Johnson, 2020; Stevens et al., 2017). We agree with Castillo-Garsow

**Table 2** An amended version of Thompson and Carlson’s (2017) major levels of covariational reasoning (p. 441), with changes in bold

Level	Description
Smooth continuous covariation	Envisioning changes in one quantity’s or variable’s value as happening simultaneously with changes in another variable’s value, and the person envisions both variables varying <b>via smooth continuous variation</b>
<b>Smooth covariation (added)</b>	<b>Envisioning changes in one quantity’s or variable’s value as happening simultaneously with changes in another variable’s value, and the person envisions both variables varying via smooth variation</b>
Chunky continuous covariation	Envisioning changes in one variable’s value as happening simultaneously with changes in another variable’s value, with both variables varying with chunky continuous variation
Coordination of values	Coordinating the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y)
Gross coordination of values	Forming a gross image of quantities’ values varying together. Envisioning a loose link between the overall changes in two quantities’ values. The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, non-multiplicative link between the overall changes in two quantities’ values
Pre-coordination of values	Envisioning two variables’ values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects
No coordination	Having no image of variables varying together

et al.’s (2013) assertion that smooth thinking can serve as a cognitive root for chunky thinking, and therefore design tasks to provide students opportunities to first anticipate quantities changing smoothly. Specifically, we design tasks to support students in moving from gross coordination of values to smooth or smooth continuous covariational reasoning, initially bypassing chunky continuous covariational reasoning (Table 2; Thompson & Carlson, 2017). *Gross coordination of values* (Table 2), which Carlson et al. (2002) referred to as coordinating *direction of change*, is common when students are first conceiving a relationship between covarying quantities (e.g., “the triangle’s area and base length are both increasing”).

After students conceive of the directional changes in two quantities, they can further conceive the relationship via smooth or smooth continuous covariational reasoning. Due to our addition of smooth variational reasoning, we also amend Thompson and Carlson’s (2017) covariational framework to differentiate between smooth covariation and smooth continuous covariation (Table 2). The primary distinction we make is to differentiate conceiving each quantity at the smooth variational or smooth continuous variational level. We would characterize a student who consistently attends to simultaneous variations in two quantities across an interval,

without describing subintervals, as engaging in smooth covariational reasoning. For example, a student may describe that, until a ball thrown in the air reaches its maximum height, the ball's height increases and its velocity decreases without explicitly describing the ball's height or velocity within any sub-interval. We characterize such reasoning as smooth covariational reasoning unless a student explicitly describes the quantities values within sub-intervals.

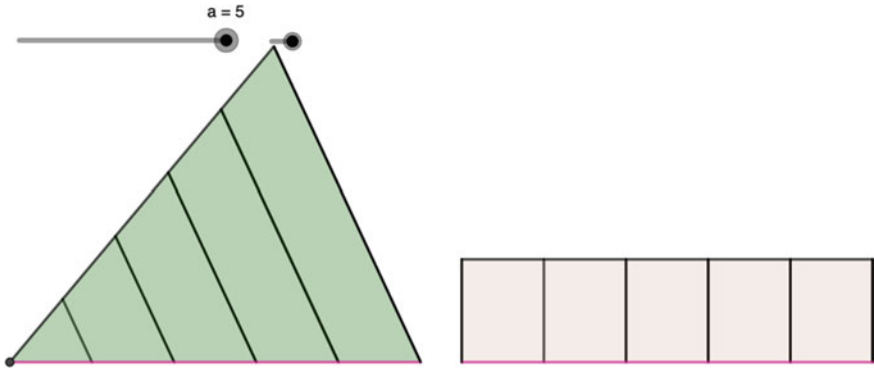
After students have had opportunities to engage in smooth (or smooth continuous) covariational reasoning, we provide opportunities for students to engage in *chunky continuous covariational reasoning* (Thompson & Carlson, 2017). Particularly, we are interested in supporting students in reasoning about what Carlson et al. (2002) referred to as *amounts of change* (e.g., the *change* in a triangle's area *increases* as the base length *increases in equal successive amounts*). Although there are ways students may engage in chunky continuous covariation without attending to the amounts of change of one quantity for equal changes in the second quantity, reasoning about amounts of change is a particular form of chunky continuous covariational reasoning (Thompson, personal communication). In this paper, when we refer to a student engaging in chunky continuous covariation, we refer specifically to a student reasoning about amounts of change as described by Carlson et al. (2002). As we describe in the next section, such reasoning can productively interplay with students' smooth reasoning as they conceive of different types of covariational relationships (e.g., Paoletti & Moore, 2017).

## ***2.2 Using Direction and Amounts of Change to Conceive the Basic Types of Covariational Relationships and Distinguish Between Nonlinear and Linear Relationships***

Smooth covariational reasoning can support students in conceiving of the directional change of two quantities. A student can conceive that as the first quantity increases or decreases, the second quantity increases, decreases, or remains constant. If both quantities are changing (i.e., the second quantity is not constant), engaging in chunky continuous covariational reasoning is necessary to further characterize a covariational relationship. Specifically, students can examine equal successive changes in the first quantity to explore whether the amounts of change of the second quantity are increasing, decreasing, or remaining constant. For example, in the *Triangle/Rectangle Task*, a student may conceive that, as the base length increases, the triangle's area increases by increasing amounts (as represented by the consecutive trapezoids on the triangle in Fig. 2). For the rectangle, the student may identify that, as the base length increases, the rectangle's area increases by equal amounts (as represented by the five consecutive rectangles in Fig. 2).

Table 3 presents the basic types of covariational relationships that a student can conceive by focusing explicitly on directional and amounts of change. We note that,





**Fig. 2** A screenshot of the triangle/rectangle task showing amounts of change of area for equal integer changes of base length

**Table 3** The basic types of covariational relationships

First quantity	Directional change in second quantity	Amounts of change in second quantity
Increasing or decreasing	Constant	N/A
	Increasing	By equal amounts
		By increasing amounts
		By decreasing amounts
	Decreasing	By equal amounts
		By increasing amounts
By decreasing amounts		

due to the prevalence of students’ presuming all relationships are linear after learning about linear functions (e.g., De Bock et al., 2007; Esteley et al., 2010), we intentionally provide students repeated opportunities to construct various nonlinear relationships prior to considering a linear relationship. As described in the introduction, we conjecture providing students opportunities to construct these basic types of covariational relationships can serve as the foundation for students’ meanings for various nonlinear and linear relationships. Based on prior experiences, and as exemplified in the Faucet Task (Sect. 4.1), we found it important to provide students with repeated opportunities to construct and reason about different directional relationships first. After such experiences, students can experience an intellectual need (Harel, 2008) to further characterize such relationships via the amounts of change of the second quantity with respect to the first.

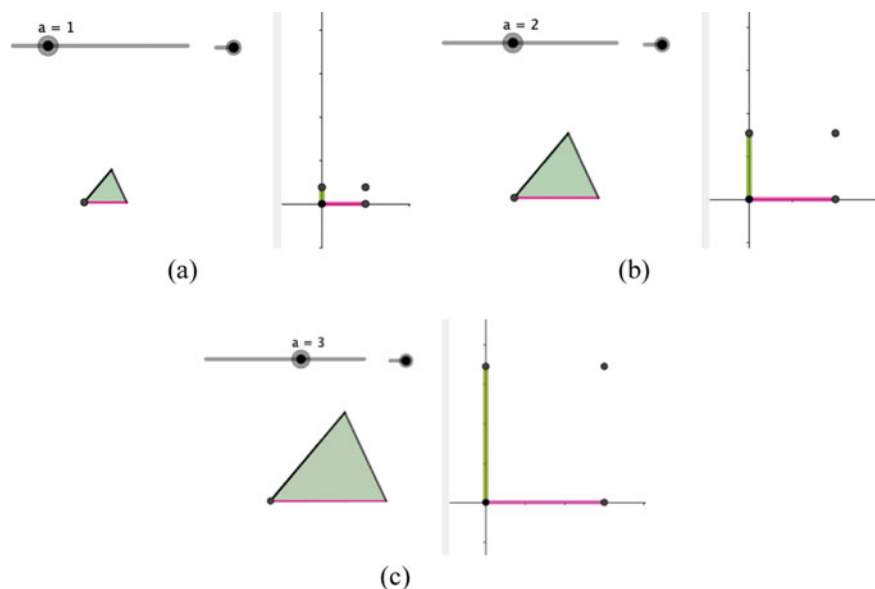
### 2.3 Representing Covariational Relationships Graphically

In the above descriptions, we characterize students' reasoning about quantities in situations. To represent a covariational relationship graphically, it is important to attend to students' meanings for the underlying coordinate system. Lee and colleagues (Lee, 2016; Lee et al., 2020; Paoletti et al., 2022) distinguished two types of coordination that result in two uses of coordinate systems in students' thinking: spatial coordinate systems and quantitative coordinate systems. Spatial coordination refers to an individual using a coordinate system to represent a physical space or phenomenon. The resulting spatial coordinate system organizes the space (or an analogous space) in which the phenomenon is conceived (e.g., a map).

Students must construct a quantitative coordinate system to represent two quantities that are not established spatially in a physical space (e.g., temperature, pressure). To construct a quantitative coordinate system, a student must first establish quantitative frames of reference (Joshua et al., 2015) within the situation. They can then disembed the quantities from the situation while maintaining an awareness of the situational quantities (Steffe & Olive, 2010) and project the quantities onto the quantitative coordinate system. Produced graphs in a quantitative coordinate system are not projections of physical phenomena onto the same space containing the original objects or phenomena.

To construct a quantitative coordinate system in the context of area and base length of the triangle in the *Triangle/Rectangle Task* (Fig. 3a–c), a student must first conceive of the triangle's area and base length as quantities. Then, intending to represent the quantities on a coordinate system, the student must consider representing each magnitude (or value) with a corresponding line segment. The student may disembed area and base length from the situation and represent them with a green segment on the vertical axis and a pink segment on the horizontal axis, respectively (e.g., the segments on the axes in Fig. 3a–c). The student can then anticipate that variations in the quantities' magnitudes (or values) correspond to variations in the segments' magnitudes (or values). For example, the student may leverage their situational understanding to argue that if they move the slider to the right, the green segment will go up and the pink segment will go to the right as the area and base length are both increasing. As representing nonlinear quantities using linear segments is non-trivial (Johnson et al., 2017; Paoletti et al., accepted), we provide students repeated opportunities to consider how line segments could be used to represent such quantities, often using tasks similar to the 'Which One' task described by Moore and colleagues (Liang & Moore, 2020; Stevens et al., 2017).

With a quantitative coordinate system in mind, a student can then conceive of a point as a multiplicative object (Lee, 2016; Lee et al., 2020; Thompson, 2011) that simultaneously represents the two segments' magnitudes on the axes. This point reflects the multiplicative object the student conceived of when reasoning covariationally by representing the two quantities' simultaneous values at every moment. For example, a student may argue that moving the slider to the right will result in the



**Fig. 3** Screenshots of the triangle/rectangle task applet with a coordinate system shown

point representing (triangle's base length, triangle's area) in Fig. 3 moving diagonally up and to the right because the triangle's area and base length both increase.

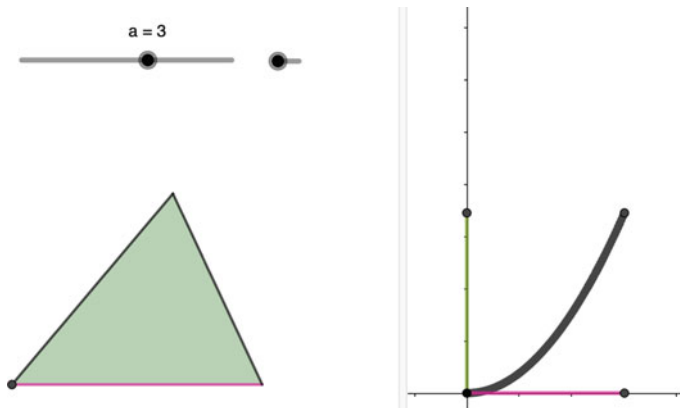
### 2.3.1 Emergent Graphical Shape Thinking

Leveraging the aforementioned descriptions of covariational reasoning, Moore and colleagues have differentiated between students' static and emergent graphical shape thinking (Moore, 2021; Moore & Thompson, 2015). Moore and Thompson (2015) described *emergent graphical shape thinking* (hereafter emergent thinking) as:

understanding a graph simultaneously as what is made (a trace) and how it is made (covariation)... [E]mergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities. (p. 785)

Students' conceptions of quantities, coordinate systems, and points as multiplicative objects are all critical to their emergent thinking. Prior to conceiving a graph as an emergent trace of covarying quantities, students must construct quantities and consider how such quantities could be represented on a coordinate system.<sup>2</sup> They must then conceive of a point as a multiplicative object simultaneously representing two quantities. With such a conception in mind, a student can conceive of a graph in

<sup>2</sup> Paoletti et al. (2018) characterize ways students could engage in emergent thinking in both spatial and quantitative coordinate systems.



**Fig. 4** Trace of a point simultaneously representing the triangle's changing area and side length

terms of an emergent, progressive trace constituted by the point's movement dictated by the covarying quantities' magnitudes represented on the axes.

To support students' emergent thinking, we use GeoGebra's 'trace' feature to trace the point's motion. For example, we have students trace the point in Fig. 3a–c to produce the graph in Fig. 4 representing a record of the relationship between the triangle's base length and area.

In addition to having students observe points producing emergent traces in multiple contexts, we leverage two other techniques to support their emergent thinking. First, as we have contended elsewhere (e.g., Paoletti, 2019; Paoletti & Moore, 2017), students' reasoning about the same final graph as being producible by different traces is a strong indication of a student engaging in emergent thinking; hence, we often provide students with opportunities to engage in such reasoning. For example, in the context of the *Triangle/Rectangle Task*, this opportunity can entail having the animation play in reverse (with  $a$  going from 5 to 0).

Second, we often leverage animations, applets, or videos with deliberate pauses. Such pauses provide students opportunities to explicitly attend to the two quantities under consideration (e.g., if the animation pauses and the quantities stop varying, then the point does not move). For example, we use a video of the growing triangle that deliberately pauses for several seconds at two  $a$ -values to provide students with opportunities to consider how such pauses impact their graphs. Such opportunities help address a common difficulty students experience with graphs, namely, reasoning univariationally about one quantity with respect to time (e.g., Carlson et al., 2003; Leinhardt et al., 1990; Paoletti, 2015). Students using such reasoning might expect that, if the animation pauses, the graph should contain a straight horizontal line to represent the pause.

### 2.3.2 Differentiating Between Nonlinear and Linear Relationships Graphically

Particular to differentiating between nonlinear and linear relationships, once a student has conceived of each type of relationship situationally (as described in Sect. 2.2), they can consider how such changes will constrain the movement of the segments and the point in the coordinate system. For example, in the *Triangle/Rectangle Task*, a student may conceive that the increasing changes in the triangle's area will correspond to increasing jumps of the segment representing area along the vertical axis (shown in Fig. 5a). These changes will therefore create points with increasing vertical changes for equal horizontal changes, represented by the large green dots in the coordinate system in Fig. 5a. The student may then leverage their smooth covariational reasoning to anticipate the smooth nature of the increasing quantities to draw a smooth (concave up) curve representing the relationship. The student can engage in similar reasoning for the growing rectangle where the vertical changes are equal, thereby creating a straight graph (Fig. 5b). In both cases, the student understands the shape of the graph is dictated by the relationship between the covarying quantities, which is indicative of emergent thinking.

## 3 Methods, Participants, and Analysis

The results reported here are a part of a larger design-based research study (Cobb et al., 2003) involving six small group teaching experiments (Steffe & Thompson, 2000). The goal of the study was to examine ways to support middle school students developing various mathematical ideas via their variational and covariational reasoning. We iteratively designed, tested, and redesigned tasks and a task

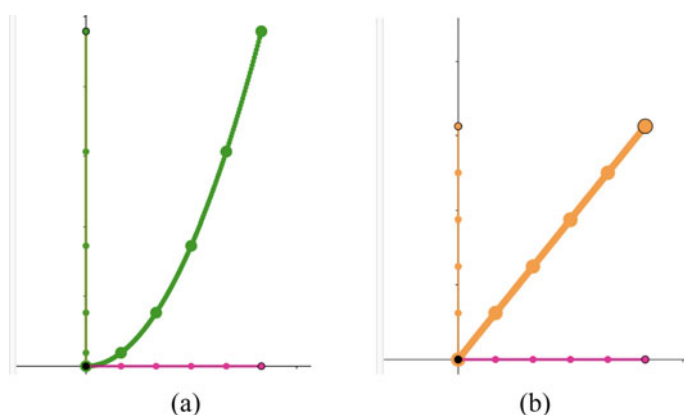


Fig. 5 Graphs representing nonlinear and linear relationships for the *Triangle/Rectangle Task*

sequence that was productive for this goal. In this report, we describe the final task sequence we designed and use student data to exemplify ways students engage with the task sequence. We briefly describe the subjects and data analysis below.

### ***3.1 Subjects and Setting***

We conducted the teaching experiments in a school in the Northeastern U.S. that hosts a diverse student population with over 75% students of color and over 67% students who qualify for free or reduced-price lunch. We chose to work with middle school students because they had not taken or completed Algebra I. We asked teachers to recommend students who would be willing to participate and could articulate their thinking.

In this report, we focus on two male students, Vicente (Hispanic) and Lajos (Asian) (pseudonyms), as they engaged with the task sequence. The students participated in 10 teaching episodes that mostly occurred one week apart (February through May), though, due to scheduling constraints (e.g., spring break), some sessions occurred two weeks apart. Each session lasted approximately 40 min. We only report on their activity in the first 8 sessions, as it is critical to their differentiating between linear and nonlinear relationships. Table 4 provides an overview of the 8 sessions including the time span and the task the students were engaged in during each session.

### ***3.2 Data Analysis***

We employed on-going and retrospective analyses to characterize models of each student's reasoning. During each phase of analysis, we conducted conceptual analysis—"building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 105). To accomplish this, we analyzed the records using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Specifically, we watched all videos and identified instances that provided insights into each student's reasoning about, coordination of, or representations of varying and covarying quantities. We developed tentative models of each student's mathematics with special attention to the students' covariational reasoning and emergent thinking, including the possibility that they were distinguishing between the different types of relationships (e.g., linear and nonlinear). To test these models, we returned to the previously identified instances, searching for supporting or contradicting instances. When evidence contradicted our models, we revised the models based on interpretation of latter instances. This iterative process resulted in viable models of each student's mathematics. We reiterate that our goal in this paper is not to present complete models of the students' mathematics but to use evidence from these models to exemplify the efficacy of the task sequence.

**Table 4** A summary of the teaching experiment sessions

Session	Month	Task	Reasoning supported
1	February	Faucet task	Construct numerous directional relationships
2	March	Faucet task	Construct a quantitative coordinate system and graph directional relationships
3	March	Faucet task	Reason emergently to represent directional relationships
4	March	Growing triangle task	Construct an increasing by increasing amounts relationship
5	March	Pausing triangle task Shrinking triangle task	Explicitly reason emergently Construct a decreasing by decreasing amounts relationship
6	April	Growing trapezoid task	Construct an increasing by decreasing amounts relationship
7	April	Triangle/rectangle task	Construct an increasing by constant amounts relationship Differentiate between nonlinear and linear relationships
8	April	Triangle/rectangle task	Graph nonlinear and linear relationships

#### 4 Building to Nonlinear and Linear Growth: A Task Sequence with Student Work

In this section, we describe most of the final task sequence that supported students reasoning covariationally to construct and graphically represent many of the basic types of covariational relationships (Table 3) and differentiate between nonlinear and linear relationships. For each task, we first describe the task and how it relates to our goals for student learning. We then present results highlighting how Vicente and Lajos engaged in the task and connect their activity back to our goals and theoretical framework. Notably, all tasks were situated in an experientially real situation (Gravemeijer & Doorman, 1999) that entailed quantities we intended to be conceived as varying smoothly (i.e., entail smooth variation). As we are interested in middle school students' initial variational and covariational reasoning, we did not design tasks with eliciting smooth continuous variational reasoning in mind.

## 4.1 *The Faucet Task: Gross Covariational Reasoning and Emergent Thinking*

We designed the *Faucet Task* (<https://ggbm.at/rdxkrwek>; see Fig. 6 for screenshots of initial applet) for two primary purposes. First, the *Faucet Task* provides students repeated opportunities to engage in smooth covariational reasoning in which one quantity (temperature) increased, decreased, or remained constant as the other quantity (amount of water) increased or decreased, reflecting the directional relationships in Table 3. Second, we designed this task to have students consider how to represent two changing quantities as an emergent trace in a quantitative coordinate system.

### 4.1.1 Students' Quantification and Directional Covariation in the *Faucet Task*

To support students' quantification, at the outset of the *Faucet Task*, we present students with a GeoGebra applet intending to represent a faucet with water coming out (Fig. 6). Students can use red and blue sliders to smoothly turn the hot and cold knobs on and off. As they do so, the rectangle below the faucet smoothly increases or decreases in width to represent the changing amount of water leaving the faucet. Further, the color of the water changes to represent the water's changing temperature (darker red for hotter, darker blue for colder). Initially, our goal is to provide students with the opportunity to construct quantities within the situation.

After Vicente and Lajos explored the applet, the teacher-researcher (TR) asked them what quantities they can measure, with the intention of discussing water temperature and amount of water leaving the faucet (e.g., flow rate, water pressure). Vicente quickly identified "temperature or direction of the knobs" as quantities we could consider, with Lajos adding "the degree of the [knob]". Shortly thereafter, Vicente described the "speed" of the water as another quantity, which we referred to as the "amount of water" or "volume" throughout the rest of the task. Both students had constructed amount of water and water temperature as quantities in the situation.



**Fig. 6** Several screenshots for Scenario A of the *Faucet Task* with the cold-water knob being turned all the way on



After students describe water temperature and amount of water, the TR further describes the faucet system in relation to an “engineering problem.” The applet reflects that, situationally, if only cold water is turned on then the temperature of the water leaving the faucet is the constant temperature of groundwater. Similarly, if only hot water is turned on, then the temperature of the water is the constant temperature determined by the hot water heater’s settings. This conversation includes describing why water feels as if it is warming up when the hot knob is first turned due to the loss of heat of stagnant water in the hot water pipe. By describing the situation, we intend to provide opportunities for students to conceive a situation in which one quantity (temperature) remains constant while the second quantity (amount of water) varies.

After this conversation, we begin to pose questions intended to support students’ covariational reasoning. For each prompt, both the hot and cold knobs start halfway on. We ask students to predict what will happen to water temperature and amount of water leaving the faucet for the following prompts, with the directional relationship between (amount of water, temperature) noted in brackets:

- (A) they turn the cold knob all the way on [increasing, decreasing] (i.e., Fig. 6),
- (B) they turn the cold knob off [decreasing, increasing],
- (C) they turn the hot knob all the way on [increasing, increasing], and
- (D) they turn the hot knob off [decreasing, decreasing].

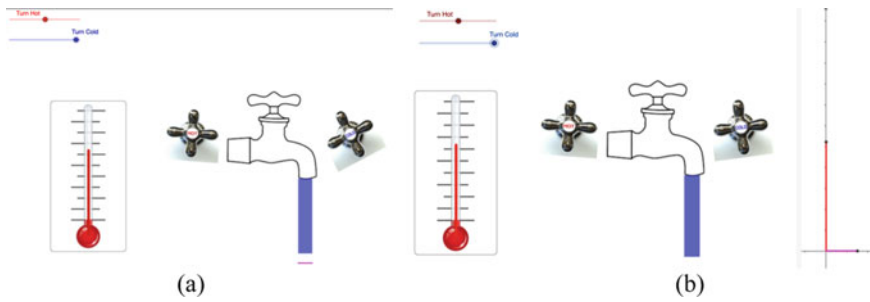
Additionally, we ask students to explore how the same two quantities will vary when:

- (E) the hot knob stays off and the cold knob is turned on and/or off [increasing/decreasing, constant] and
- (F) the cold knob stays off and the hot knob is turned on and/or off [increasing/decreasing, constant].

Our goal is to support students in engaging in directional covariational reasoning with either quantity increasing, decreasing, or remaining constant reflecting each directional relationship in Table 3. Additionally, we intend to foreshadow for students the notion that graphs can be producible in different directions (i.e., support their emergent thinking).

Vicente and Lajos had little difficulty describing how each quantity changes as the knobs are turned. For instance, addressing Prompt B, Vicente quickly described, “I think that it’ll only be hot water [left running]. So temperature will increase, but volume will decrease because it’s less water.” Addressing Prompt D, he described, “it’ll be more cold, like the temperature will go down. And I think less water will be pouring out of the faucet.” Further, when asked to address Prompts E and F, Vicente identified in each case that temperature would remain the same while the amount of water varied. For instance, addressing Prompt F, Vicente described, “Temperature is going to stay the same, and less water will be coming out as you turn the knob.”

There are several notable features from Vicente’s activity. First, based on the active nature of his utterances describing changing quantities (e.g., “temperature will increase, but volume will decrease,” “less water will be coming out as you turn the knob”), we infer Vicente engaged in (at least) smooth variational reasoning as he



**Fig. 7** **a** The Faucet Task applet with an additional thermometer and pink segment below the water and **b** the next applet showing red and pink segments on the vertical and horizontal axis

developed smooth images of change, including varying temperature, amount of water, and the turning of the knobs. We note that since Vicente never explicitly referred to sub-intervals of either quantity, we do not classify his reasoning as smooth continuous variational reasoning; student reasoning compatible with Vicente's motivated our modification of Thompson and Carlson's (2017) variational reasoning framework.

Second, Vicente's quantitative understanding of the situation supported him in reaching numerous (accurate) conclusions regarding the directional relationships between temperature and amount of water. This included situations in which one quantity changed as the other quantity remained constant. Third, Vicente consistently described how both the amount of water and temperature changed as a knob was turned without ever referring to sub-intervals, which is indicative of his engaging in (at least) smooth *covariational* reasoning.

#### 4.1.2 Students Developing Graphing Meanings Via the *Faucet Task*

Once students have described each relationship covariationally, we ask a series of prompts designed to support students in constructing a quantitative coordinate system. First, we present a revised applet that includes a thermometer to gauge the water temperature and a horizontal pink line segment below the rectangle corresponding to the rectangle's width to represent the amount of water leaving the faucet (Fig. 7a).<sup>3</sup> We present these segments for two reasons. First, the red segment provides students a way to describe temperature changing without referring to the color of the water. Second, by using vertical and horizontal linear segments to represent the quantities' magnitudes, we intend to foreshadow a similar representation in the next applet. In that applet, we present students what we intend to be a quantitative coordinate system, with temperature represented by a red segment on the vertical axis and amount of water represented by a pink segment on the horizontal axis (i.e., Fig. 7b).

<sup>3</sup> Depending on time constraints, we sometimes have the thermometer and pink segment as two different tasks and sometimes present them simultaneously, as we did in this case.

Lajos and Vicente interpreted the segment lengths as representing variations in the disembodied quantities. Describing a situation in which the hot knob is turned on, Lajos described, “the temperature will increase [*motioning his finger in an upward direction*], and the pink segment [*putting two fingers together then moving them apart horizontally*] will get wider.” Lajos characterized each segments’ variation based on his conception of the quantities in the situation. Also, like Vicente, Lajos’s words (e.g., “temperature will increase”) and actions (e.g., smooth motions with his fingers) are indicative of at least smooth variational reasoning.

After students describe what each segment represents situationally, we move to the next applet showing a red and a pink segment on the vertical and horizontal axis, respectively (Fig. 7b). Hoping to support students in conceiving of a quantitative coordinate system, the TR asks them to describe what will happen to each segment for Prompts A–F described above.

After observing the applet, Lajos and Vicente described how the segments vary based on their understanding of how the quantities change situationally. For example, when tasked with describing how the segments vary for Prompt D, the following conversation ensued:

- Lajos The temperature will decrease and [*pause*] the water will decrease.  
TR So the amount of water will decrease, and you said, why will the temperature decrease?  
Lajos Because since the cold is still on, and temp. The hot water will, ah, you’re turning it off, and the cold is still on so it will decrease.

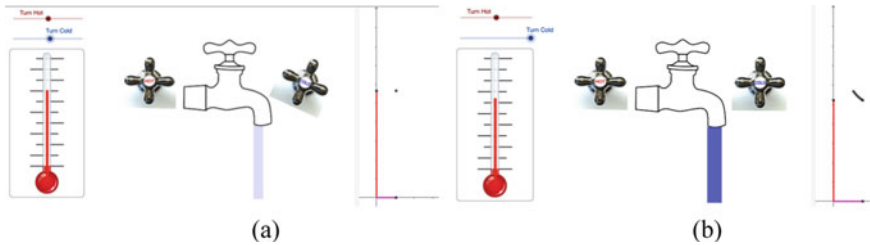
[*The TR asked Lajos to describe what that means for the segments.*]

- Lajos Down.  
TR Yeah, this one [*pointing to the red segment on the vertical axis*] will definitely move down, but when you say this one [*pointing to the pink segment on the horizontal axis*] will move down, what does that mean?  
Lajos Like smaller.  
TR Smaller, so it’ll go down?  
Lajos Like to the left.

We infer Lajos disembodied amount of water and water temperature from the situation as he interpreted the varying segment on each axis, which is critical to his constructing a quantitative coordinate system.

After describing how each segment varies for Prompts A–F, we intend to support students in constructing a point as a multiplicative object simultaneously representing water temperature and amount of water. Hence, we present another applet that now includes a point in the coordinate system with a horizontal magnitude corresponding to the endpoint of the pink segment and a vertical magnitude corresponding to the endpoint of the red segment (Fig. 8a). We first have students turn the knobs and observe the movement of the point.

Constructing a point as a multiplicative object is non-trivial. For example, after Vicente and Lajos turned each knob and observed the point, the following conversation ensued:



**Fig. 8** **a** The applet with the point shown and **b** the applet with one emergent trace resulting from turning the cold on

TR So, Vicente, what do you think you've got about [the point]?

Vic. I think as, as the water gets warmer [*turning hot knob on*], [the point] moves farther away from [the vertical axis]. It's still like in the same spot but like it goes farther away.

TR Ohhh, why do you think it might be moving to the right?

Vic. Maybe, because of this [*motioning the cursor over the pink segment on the horizontal axis*].

Whereas previously, Vicente always attended to variations in both quantities, when initially making sense of the point's motion, he only attended to the horizontal motion dictated by the pink segment. We often observe such reasoning when students are first considering how to represent two quantities on a quantitative coordinate system.

Immediately after the above interaction, Vicente again attended to only one quantity as he described the point moving left when the amount of water decreased. The TR attempted to draw his attention to the vertical motion of the point by providing a prompt analogous to Prompt C (hot on):

TR There is some cold water right now. Does the temperature go up or down as I turn hot all the way on?

Vic. It's going to go up.

TR It's going to go up. So this red segment is going to go up. So, what do you think is going to happen to that point in terms of moving up, down?

Lajos Go like [*Vicente interjects, Lajos continues*] away from the [*motioning away from the horizontal axis*].

TR It's going to go away because there's more water but will it go away like to the right and down or to the right and up?

Lajos & Vic. [*simultaneously*] To the right and up [*each moving his finger in the air to the right then up*].

After conceiving of the point's movement as being constrained by the varying magnitudes on each axis (i.e., as a multiplicative object), each student described the point's motion in the quantitative coordinate system so that the point represented the covarying magnitudes in this and other cases.

Once students have conceived of the point as a multiplicative object, we support them in engaging in emergent shape thinking. To do this, we use the ‘Trace’ feature of GeoGebra to trace the point as the students again address (at least a subset of) Prompts A–F (Fig. 8b shows the resulting trace for Prompt A). Our goal is to support students in imagining the graph as being produced by the trace of the point as it moves based on the two quantities. Further, the variety of prompts ensures students have opportunities to engage with graphs tracing in multiple directions.

After having numerous opportunities to observe how graphs are created via the changing quantities in the situation, we present students with several completed graphs (e.g., Fig. 9) and ask them to predict how the knobs began and what action occurred to produce the graph. Our goals are to explore how students interpret a graph representing covarying quantities and to examine if students consider reasoning emergently to describe different scenarios that create the same final graph. If the students do not consider more than one scenario, the TR can raise a second scenario as a hypothetical classmate’s solution and ask students to comment on this solution.

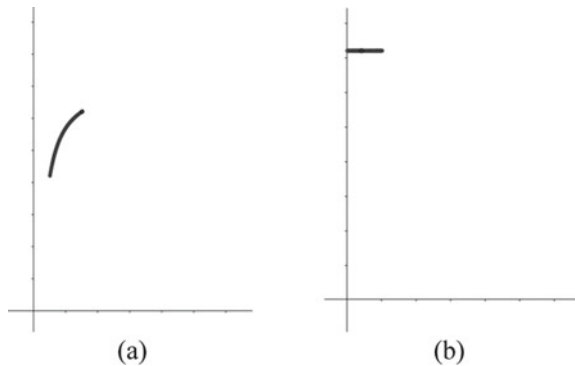
Addressing the first of these tasks (Fig. 9a), Vicente accurately argued, “I think the hot water is going to be turned on... because it looks like the temperature is going up... there’s more water coming out, it’s going to the right.” After this, the TR posited that a pair of their classmates had argued the graph was made from turning hot water off and asked Vicente and Lajos if these students could have been correct. Responding to this, and indicative of reasoning about a single graph being traceable in multiple directions, Vicente immediately responded:

[M]aybe backwards. Maybe they could be thinking about it in reverse because, so you turn hot off right? [TR agrees] So that means if you turn it off there will be less water, so you go to the left [motioning from the top right point leftwards], and the temperature is going to go down [tracing along the curve from the top right point down to the left]... [T]hey’re imagining it backwards.

Hence, Vicente, and later Lajos, reasoned about covarying quantities to describe two possible emergent traces producing the same final graph.

Additionally, each student correctly described numerous ways a straight horizontal line could be produced from the situation, with Lajos arguing that “everything

**Fig. 9** Two examples of graph interpretation tasks



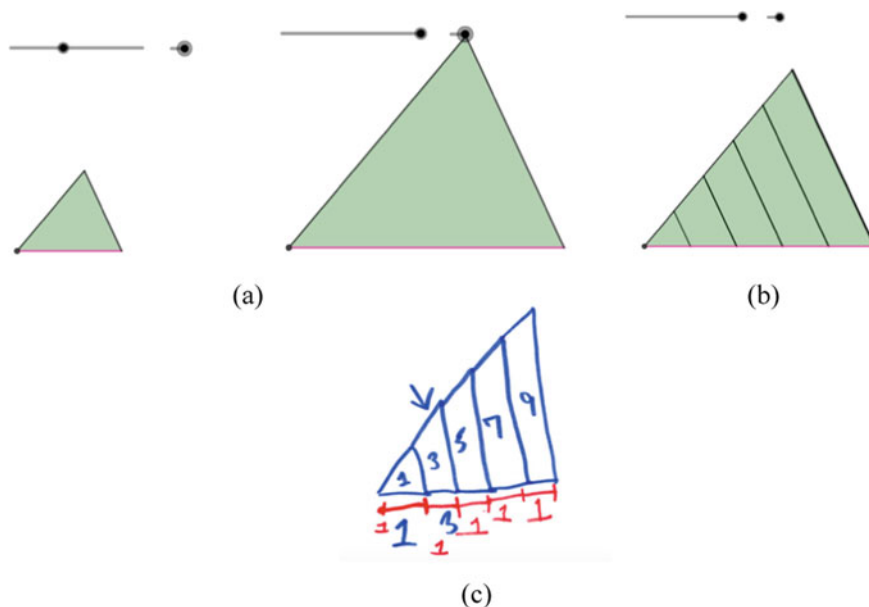
turned off then turn hot all the way on,” would produce the graph in Fig. 9b. Hence, the students repeatedly engaged in smooth covariational reasoning in which temperature increased, decreased, or remained constant as the amount of water increased or decreased, reflecting the directional relationships in Table 3.

## 4.2 *The Growing Triangle Task*

After the *Faucet Task*, we have students address the *Growing Triangle Task*. This task provides students additional opportunities to reason emergently about a smoothly changing phenomenon in a quantitative coordinate system. Additionally, we designed the task to extend their covariational reasoning by supporting them in reasoning about amounts of change to construct and accurately represent such a relationship. Specifically, we intended to support students in conceiving that the triangle’s area grows by increasing amounts for equal changes in its base length; this relationship is the first type of nonlinear covariational relationship we have students construct.

To support students in imagining and anticipating smooth variation, we first have them interact with a dynamic GeoGebra applet (<https://www.geogebra.org/m/you25d2my>) showing a smoothly growing scalene triangle (Fig. 10a). We ask them what quantities they could measure in this situation. To support them in attending to and coordinating area and base length (i.e., to reason covariationally), we highlight the triangle’s base length in pink and area in green. After describing the directional change of area and side length, we specifically ask students to identify if, for equal changes in the base length, the area increases by (a) more, (b) less, or (c) the same amount. As described in Sect. 2.1, we included a second smaller slider which allows students to increase the increment by which the pink length increases (e.g., to integer chunks versus smoothly). We have the ‘trace’ option available so that students can visually identify the increasing amounts of change of area in the applet (i.e., the increasing size of the consecutive trapezoids shown in Fig. 10b). This feature of the task supports students in conceiving of increasing changes in area.

Consistent with constructing quantities in the situation, when asked what quantities they could measure, Vicente quickly described, “all sides are increasing,” and, as the base length increases, “the area gets bigger.” Later in the session, with the pink length increasing in one unit chunks, Vicente described, “[the area jump] starts with small and... keeps getting bigger and bigger.” After conceiving of the area increasing by more, Vicente and Lajos approximated values for each of the amount of change amounts to numerically represent the increasing changes in the triangle’s area, as shown in Fig. 10c (i.e., 1, 3, 5, 7, and 9 represent the amounts of change). Hence, while initially leveraging smooth variation (e.g., “area gets bigger”), the students began to leverage chunky variational reasoning as they conceived of and numerically represented the amounts of change of area increasing by more.



**Fig. 10** **a** Two screenshots of the *Growing Triangle Task*, **b** the triangle shown in the applet with chunky changes, and **c** the students’ work approximating area values

### 4.2.1 Graphing the Relationship in the *Growing Triangle Task*

Once students describe the amounts of change in area as increasing, we present them with what we intend to be a quantitative coordinate system with the side length represented by a pink line segment on the horizontal axis. We ask them to describe how the increasing change in the triangle’s area will correspond to the movement of a segment representing area on the vertical axis. Our intent is to offer students repeated opportunities to reason about covariational relationships and consider how to represent a quantity’s magnitude (or value) with a corresponding line segment.

Indicative of disembedding area from the situation and representing it with a segment’s magnitude on the vertical axis, each student described that the increasing changes in the triangle’s area will correspond to increasing jumps of the segment on the vertical axis. For instance, referring to the approximated area values, Lajos motioned his finger by three units in an upward direction along the vertical axis from a point representing the area when the side length was one unit. Lajos described that “the area is four...the area would go to nine [motioning his pointing finger by five units in an upward direction along the vertical axis from the point placed by the TR at (0, 4)].” While Lajos described how the segment representing area increased by more, Vicente simultaneously motioned his finger by one unit to the right on the horizontal axis, indicative of reasoning about the horizontal segment varying by equal amounts. We infer Lajos engaged in numeric chunky continuous covariational reasoning to describe how the segments varied, and hence conceived of a quantitative

coordinate system. Further, the students extended this activity by creating points in the coordinate system that simultaneously represented side length and area.

The next several prompts are designed to support students' emergent thinking. After students represent specific base length and area magnitudes via points as multiplicative objects, we change the small slider to present the triangle growing smoothly. We then show the dynamic point representing the two quantities' magnitudes in the coordinate system and use the 'trace' feature to allow the students to observe how the point moves with the intention of supporting the students in conceiving the graph as being produced by an emergent trace.

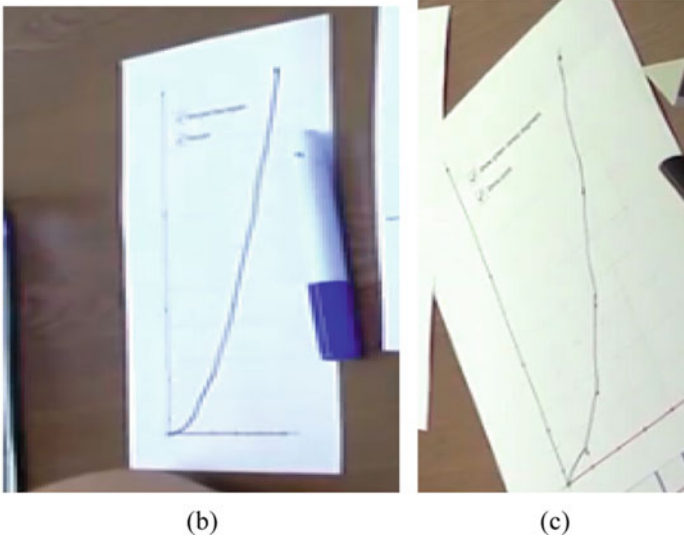
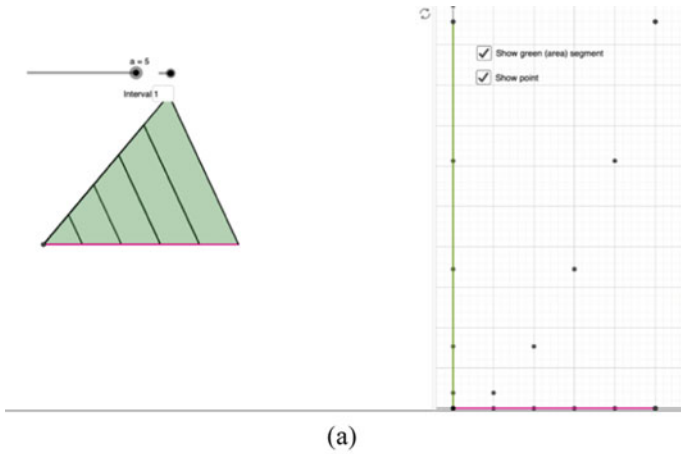
When asked to explain why the graph contains more than the five chunky points shown in Fig. 11a, Vicente claimed "it's [the point] just not skipping, it'll go like this [*motioning his pointer finger in a curve that passes through the five points as though sketching a smooth concave up curve*] and Lajos added, "it's [the point is] tracing, tracing slowly up [*motioning his fingers in the air as though sketching a smooth concave up curve*]." After this, each student sketched a smooth curve (Fig. 11b, c) joining the five points on a given handout. Further, Vicente claimed, "[the area] is not going to be just here and here [*pointing to consecutive points shown on the graph*]" and explained that "like [the area] could be at 50, but at some point it has to be smaller than that like 49, 48." Realizing that Vicente spontaneously began to describe smaller sub-intervals of the changing area, the TR asked him if the area's value must ever be 48.5. Possibly indicative of Vicente engaging in smooth continuous variational reasoning, he quickly agreed that the area must take on such a value. However, the TR did not provide additional follow-up questions to allow us to claim definitively if Vicente was engaging in smooth continuous variational reasoning.

#### 4.2.2 The Pausing and Shrinking Triangle Tasks: Examining Students' Emergent Reasoning

To investigate the extent to which the students are attending to the two intended quantities and to examine their potential emergent thinking, we include two follow-up tasks. In the *Pausing Triangle Task*, we present a video showing the same smoothly growing triangle. However, twice in this video, the triangle's growth pauses. In the *Shrinking Triangle Task*, we present a video showing the same triangle, but with its side length and area decreasing from their maximum values until they are both zero. In each case, we intend to support the students in attending to the two changing quantities in the situation and considering how the new features of the situation (pauses, going in reverse) do or do not influence either their graph or the trace producing their graph. An added affordance of the *Shrinking Triangle Task* is that students have the opportunity to describe a decreasing by decreasing amounts relationship; this relationship is the second type of nonlinear covariational relationship we have students construct.

Addressing these tasks, each student exhibited emergent thinking as they related their original graph to these new situations. For the *Pausing Triangle Task*, Lajos argued this new situation would have a different trace but produce the same graph. He explained, "it [the moving point] would stop for a few seconds here [*marking a*





**Fig. 11** a One screenshot from the *Growing Triangle* applet that includes a graphical representation and points produced by the 5 equal changes of side length, b, c Lajos’s and Vicente’s graphs, respectively

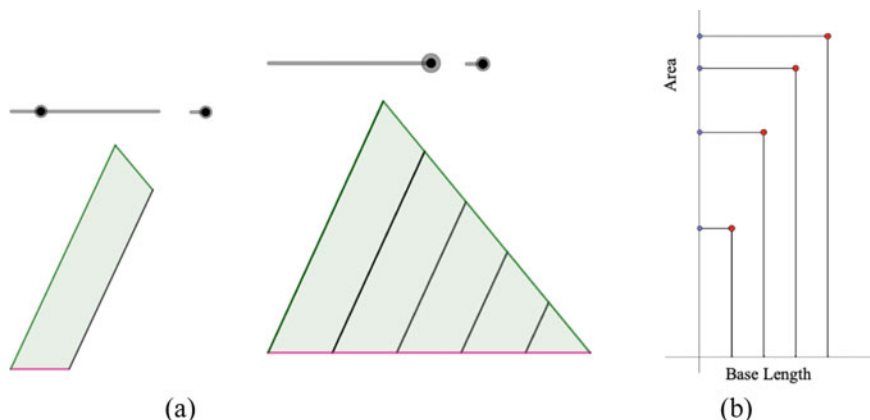
*point on the curve in Fig. 11b*] and then keeps going [*tracing the pen on the curve*] then stops [*stops pen along the curve*] and then keeps going [*moves the pen along the curve*].” Addressing the *Shrinking Triangle Task*, Lajos re-traced the original graph from the top right to the bottom left while claiming, “[the point] would start right up there and it would go reverse and go back down.” Likewise, Vicente claimed that the graph “would go backwards.” We infer each student was reasoning emergently as he created and interpreted graphs representing the triangle’s varying base length and area. Further, when addressing the *Shrinking Triangle Task*, the students explicitly

described that the triangle's area decreased by less for equal decreases in the base length. Hence, the students constructed a second type of nonlinear covariational relationship.

### 4.3 The Growing Trapezoid Task: An Increasing by Less Relationship

After engaging in the prior tasks, we hope students will begin to spontaneously examine the direction and amounts of change of a relationship and will leverage emergent thinking when prompted to graphically represent a relationship. We use the *Growing Trapezoid Task* (<https://ggbm.at/jbk6kw8f>) to examine this possibility. In this task, we present students with a smoothly growing figure that starts as a line, becomes a trapezoid, and increases until it results in a triangle (Fig. 12a). Although the resultant triangle is the same triangle as in the *Growing Triangle Task*, in this situation, area increases by less for equal changes in the pink length, which is a third type of covariational relationship in the task sequence.

Relevant to the students' quantitative and covariational reasoning, each student quickly responded that the "area gets bigger." Vicente described the area increases by "smaller amounts," and Lajos elaborated the consecutive amounts of change in the area "are smaller." Particular to their graphing activity, the students plotted points on the vertical axis in a way that was indicative of leveraging a quantitative coordinate system. Specifically, with prompting from the TR, Vicente and Lajos worked together to connect the increasing amounts of change represented on the vertical axis in their graph in the *Growing Triangle Task* to corresponding decreasing amounts of change in this task (e.g., approximating the size of the final jump for their graph in the



**Fig. 12** **a** The *Growing Trapezoid Task* and **b** a recreation of Vicente's work showing area increasing by smaller amounts and points produced as multiplicative objects

*Growing Triangle Task* and using this magnitude for the first jump in this task). Leveraging this reasoning, Vicente plotted several points on the vertical axis that he conceived “jump by less.” Using these points, Vicente plotted points in the coordinate system representing simultaneously the growing trapezoid’s base length and area (Fig. 12b), which we infer represented the trapezoid’s area increasing by decreasing amounts for equal changes in base length.

#### 4.4 *The Triangle/Rectangle Task*

In addition to providing another opportunity to reason emergently, the *Triangle/Rectangle Task* provides students an occasion to construct a linear covariational relationship, a fourth type of covariational relationship in the sequence. Further, the task provides opportunities to compare linear and nonlinear growth and consider how each type of relationship can be represented via graphs as emergent traces. In the *Triangle/Rectangle Task*, we present students with a smoothly growing rectangle next to the original triangle from the *Growing Triangle Task*; both figures have equal pink base lengths. We ask students to describe how each area is changing and to graphically represent the relationships between area and side length for each growing shape.

Once Lajos and Vicente described that each area is increasing, the TR began to pose questions to investigate their covariational reasoning. For instance, after Vicente and Lajos described the area of the rectangle as increasing, the following conversation ensued:

TR How is the area of the rectangle increasing?

Vic. I think for the rectangle, I think that it’s increasing by, keeps increasing by the same amount.

TR Increasing by the same amount?

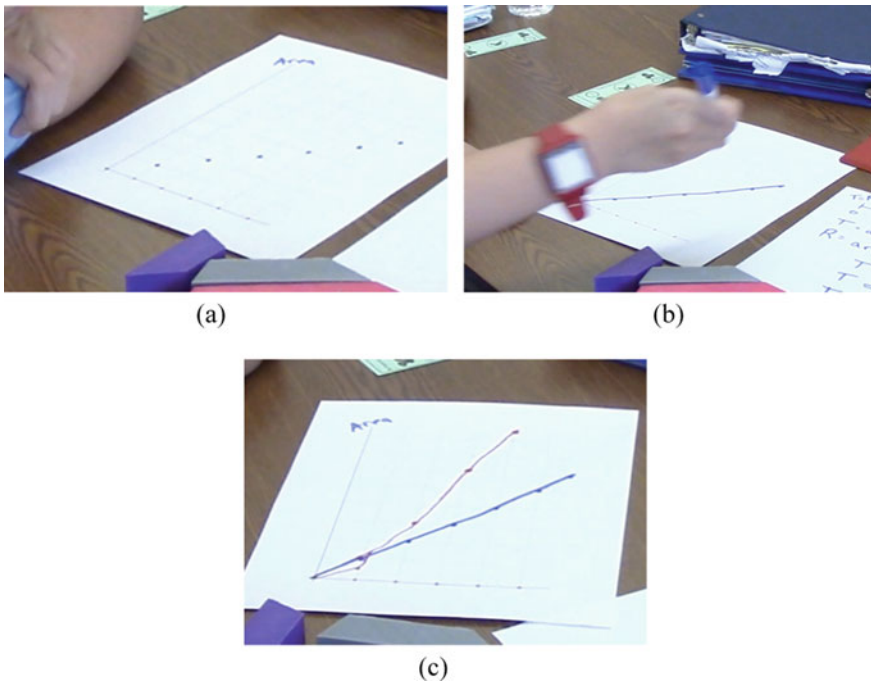
Vic. Yeah, ‘cause it keeps adding that one block [*pointing to the smallest amount of change rectangle*] over and over again [*motions hand over successive amounts of change rectangles, shown in Fig. 2*].

We infer Vicente (and later Lajos) engaged in chunky continuous covariational reasoning to describe the area of the rectangle increasing by equal amounts for equal changes in base length.

To investigate the students’ emergent shape thinking, we then asked the students to graph the relationship between area and side length for each growing shape on a handout with the area and side length represented on the vertical and horizontal axes, respectively. Watching the applet with side length increasing in increments of 1, Lajos leveraged chunky continuous variational reasoning as he used his fingers to indicate the segment representing the rectangle’s area would jump by equal amounts along the vertical axis. Justifying these equal changes along the vertical axis, he described, “Because all of the [smaller] rectangles (shown in Fig. 2) are equal sized, so it [the increase in area] has to be the same amount.” After this, Vicente also engaged

in chunky continuous covariational reasoning as he plotted points that represented the area changes described by Lajos vertically and corresponding equal side length changes horizontally (Fig. 13a). As Vicente plotted points, Lajos described that the points correspond to “both of them” referring to the rectangle’s base length and area. After Vicente plotted the last point, the TR changed the smaller slider to change the applet from playing in chunks to smoothly and asked “and what if I have it playing, sort of, smoothly?” Immediately, and indicative of engaging in smooth covariational reasoning and emergent shape thinking, Lajos picked up the marker, said “it would be like this,” and sketched a straight line through the points Vicente plotted (Fig. 13b).

Shortly after this, the TR prompted the students to graphically represent the triangle’s area and side length on the same coordinate system. The students recalled their work in the prior sessions to sketch a smooth concave up curve to represent this relationship (Fig. 13c). Hence, the students were able to leverage a combination of their chunky continuous and smooth covariational reasoning to conceive of both nonlinear and linear relationships. Further, they graphically represented each relationship via an emergent trace on a quantitative coordinate system.



**Fig. 13** a Vicente’s plotted points, b Lajos finishing sketching a straight line, and c the pair’s graph with the curve representing the triangle’s base length and area

## 5 Discussion

We first discuss contributions this chapter provides to the literature on students’ covariational reasoning. We then relate our task design to the theoretical framework and provide implications for developing other mathematical ideas. We conclude with areas for future research.

### 5.1 Middle School Students’ Covariational Reasoning

In this chapter, we explored the possibility of middle school students constructing and reasoning about basic types of covariational relationships (Table 3), which supports them in differentiating between nonlinear and linear relationships. Through the task sequence, students had repeated opportunities to construct numerous directional relationships. Such activity was foundational for the students’ later activity as they characterized covariational relationships with differing amounts of change; Table 5 presents all of the directional and amounts of change relationships Lajos and Vicente constructed. Further, we described how such covariational reasoning supported students’ emergent reasoning as they accurately constructed and interpreted graphs tracing in numerous directions with varying concavities.

**Table 5** The directional and amounts of change covariational relationships Vicente and Lajos constructed with the first quantity in the *Faucet Task* being amount of water

First quantity	Directional change in second quantity (task where student experienced)	Amounts of change in second quantity (task where student experienced)
Increasing	Constant ( <i>Faucet Task</i> , Scenarios E and/or F)	–
	Increasing, ( <i>Faucet Task</i> , Scenario C; <i>Growing Triangle Task</i> ; <i>Growing Trapezoid Task</i> )	Increasing by the same amount (rectangle’s area in <i>Triangle/Rectangle</i> task)
		Increasing by more (triangle’s area in <i>Growing Triangle</i> task)
		Increasing by less (trapezoid’s area in <i>Growing Trapezoid</i> task)
Decreasing ( <i>Faucet Task</i> , Scenario A)	–	
Decreasing	Constant ( <i>Faucet Task</i> , Scenarios E and/or F)	–
	Increasing ( <i>Faucet Task</i> , Scenario B)	–
	Decreasing ( <i>Faucet Task</i> , Scenario D; <i>Shrinking Triangle Task</i> )	Decreasing by less (triangle’s shrinking area in <i>Shrinking Triangle</i> task)

Throughout our presentation of the tasks, we explicitly connected our task design to our theoretical framing. Compatible with the Learning Through Activity framework described by Simon and colleagues (e.g., Simon et al., 2018), our goal was to design a task sequence that supported students in gradually developing ways of thinking that would eventually lead to their constructing sophisticated mathematical understandings. We intend for such descriptions to serve as a resource for other researchers' and teachers' efforts at adapting these tasks or designing new tasks that could afford similar shifts in students' reasoning.

In addition to providing empirical examples of middle school students constructing numerous covariational relationships, we extend Thompson and Carlson's (2017) variational and covariational framework to include smooth variational and smooth covariational reasoning. We provide empirical examples of middle school students first engaging in gross and smooth covariational reasoning prior to engaging in chunky continuous covariational reasoning. Consistent with the conjecture of Castillo-Garsow et al. (2013), smooth reasoning seemed to entail a capacity to engage in chunky reasoning, with the latter reasoning supporting students in further characterizing their conceived relationships. These forms of reasoning interplayed productively with the students' meanings for quantitative coordinate systems and points as multiplicative objects as students constructed and interpreted graphs as emergent traces, "with the trace being a record of the relationship between covarying quantities" (Moore & Thompson, 2015, p. 785).

## 5.2 *Task Design in Relation to Our Theoretical Framework*

We highlight that each part of the *Faucet Task* involved (almost all) of Prompts (A)–(F). We conjecture these repeated opportunities were critical for the students' developing graphing meanings as they considered how to represent a relationship via a point as a multiplicative object constrained by the motions of segments on axes. Further, their directional covariational reasoning in the *Faucet Task* laid the foundation for their later activity discerning the amounts of change of one quantity with respect to a second quantity in the tasks that followed. In these latter tasks, the students leveraged a combination of chunky continuous and smooth covariational reasoning to construct and graphically represent numerous nonlinear and linear relationships.

We contend the ability for students to change the intervals by which an applet's parameter varied from smooth to chunky was critical. By first engaging with smoothly changing phenomena, Vicente and Lajos developed smooth images of change. However, and as we contended elsewhere (Paoletti & Moore, 2017), smooth thinking alone is not sufficient for discerning the amounts of change of one quantity with respect to another. Hence, changing the parameter via the slider also supported students in developing chunky images of the situation, which was critical to them constructing the different covariational relationships in Table 5.

Relatedly, across all students we interviewed in the larger design experiment, using smoothly changing phenomena supported students in developing smooth images of

change. In contrast, we conjecture tasks which present a table of values, regardless of the teacher's or researcher's intention, are more likely to elicit (at best) chunky covariational reasoning from students. We contend that creating mental imagery of smoothly changing phenomena from a table of values is possible, but non-trivial; providing students with dynamic representations of smoothly changing phenomena is invaluable to their development of smooth variational and covariational reasoning (Castillo-Garsow et al., 2013; Johnson, 2020; Stevens et al., 2017).

### ***5.3 Implications for Developing Other Mathematical Ideas***

An immediate consequence of constructing various nonlinear and linear relationships is that students can experience an intellectual need (Harel, 2008) to further differentiate between types of covariational relationship (or function) classes that exhibit similar change patterns. For example, both quadratic and exponential relationships can exhibit growth such that the second quantity increases by an increasing amount for equal changes of the first quantity. As described by Vishnubhotla (2020), once students identify such a similarity, they may further explore numeric relationships to identify patterns. Hence, once students have repeated experiences constructing and representing covarying quantities, other representations, such as tables of values, can be useful as they further differentiate between various forms of change (beyond linear versus nonlinear).

To exemplify this, we turn to Vicente and Lajos's activity described in Sect. 4.2. Specifically, after identifying numeric values for specific amounts of change (+ 1, + 3, + 5, etc.) the pair identified that these amounts of changes were changing by a constant amount. As our goal in this study did not entail students developing meanings for quadratic relationships, we did not design tasks or prompts to explore this reasoning further. However, such activity supported this pair, and other students (Mohamed et al., 2020), in identifying the defining characteristic of quadratic growth (Ellis, 2011; Lobato et al., 2012). Hence, the presented task sequence has the potential to lay a foundation for students developing meanings for specific nonlinear relationships.

### ***5.4 Concluding Remarks and Areas for Future Research***

We contend that providing middle school students opportunities to reason about dynamically changing quantities to construct basic types of covariational relationships can serve as a foundation for their developing meanings for various functional relationships (Thompson & Thompson, 1995). There is a further need to develop or adapt tasks that extend middle school students' covariational and emergent thinking to support them in developing meanings for other relationships as described by other

researchers, including quadratic relationships (Ellis, 2011; Lobato et al., 2012), exponential relationships (Confrey & Smith, 1994, 1995; Ellis et al., 2015; Thompson, 2008), and possibly even trigonometric relationships (Moore, 2014).

In addition to designing or adapting tasks to foster students' thinking, there is also a need to investigate ways to scale a task sequence like this one to be effective in larger settings (e.g., whole class). Based on a pilot whole class teaching experiment with 6<sup>th</sup>-grade students, we conjecture there is a need to provide students with sufficient opportunities to reflect on their activity for them to develop stable meanings that entail covariational reasoning and emergent thinking. Such reflective activities can further support students in developing stable meanings for graphs, relationships, and various relationship classes. Hence, we call for further research on how to make task sequences like the one presented in this chapter both productive in whole class settings and usable by middle school teachers everywhere. Such research has the potential to impact the teaching and learning of middle school mathematics across the world, as such an approach can support all students in developing foundational knowledge and ways of thinking that are critical to their algebra learning.

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