Mathematization: A Crosscutting Theme to Enhance the Curricular Coherence

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Quantitative reasoning played a crucial role in the development and revolution of scientific knowledge in the history of science (Crombie, 1961; Jin et al., 2019a; Kline, 1982). It has been emphasized as an important learning goal for K-12 students for many years (NGSS Lead States, 2013; National Research Council [NRC], 1996, 2000). In science education literature, the term mathematization of science, or mathematization in short, is often used to refer to the specialized ways that scientists use to quantify phenomena and construct knowledge; it emphasizes the relationship between quantitative reasoning and science disciplinary knowledge (e.g., Kline, 1982; Lehrer & Schauble, 1998). Therefore, in this chapter, we use this term to refer to quantitative reasoning in science.

Researchers describe scientists' specialized ways of using quantitative reasoning with different terms such as mathematical deduction (Kind & Osborne, 2017; Osborne et al., 2018), mathematization (Kline, 1982), postulation exemplified by the Greek mathematical sciences (Crombie, 1961; Hacking, 1994), and quantification (Crombie, 1961). Nevertheless, they all emphasize a process of quantification: Scientists generate mathematical descriptions of phenomena in the material world.

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G. Karagöz Akar et al. (eds.), *Quantitative Reasoning in Mathematics and Science Education*, Mathematics Education in the Digital Era 21,
https://doi.org/10.1007/978-3-031-14553-7_10

In those descriptions, entities (e.g., matter, energy, and force) are represented by algebraic symbols and numeric values; and the relationships among those entities are represented by mathematical equations, tables, and graphs. Scientists generate concepts, principles, and theories to conceptualize those mathematical relationships. The value of mathematical descriptions resides in their accuracy, universality, and deductive logic (Pereira de Ataide & Greca, 2013). Due to this value, mathematical descriptions allow precise predictions and generation of new concepts; they also provide an objective base for scientific argumentation and discussion (Holton & Brush, 2006; Kline, 1990; Osborne et al., 2018). Although existing literature of scientists' mathematization provides concrete ideas about the quantification process, additional effort is needed to identify key components that differentiate that quantification process from our everyday intuitive thinking. Such information will help teachers and researchers design more targeted instruction on quantitative reasoning.

Researchers have investigated how students use mathematization to solve problems and explain phenomena. These studies have documented the expert-novice differences across physics (Bing & Redish, 2009; Chi et al., 1981; Kuo et al., 2013; Niss, 2017; Schuchardt & Schunn, 2016; Sherin, 2001; Tuminaro & Redish, 2004, 2007), chemistry (Dori & Hameiri, 2003; Kozma & Russell, 1997; Schuchardt & Schunn, 2016; Taasoobshirazi & Glynn, 2009), and biology (Schuchardt & Schunn, 2016). While experts incorporate conceptual understanding of scientific knowledge with mathematical representations, novices tend to select mathematical equations based on surface features of the scenario and manipulate the mathematical symbols/equations without understanding their scientific meaning. These expert-novice differences are largely due to the different epistemological perspectives—while experts view mathematics and science as integrated, students tend to see mathematics as a mere instrument for calculation (Bing & Redish, 2009). Additionally, using graphs presents significant challenge for many students. Most existing studies on students' use of graphs were conducted in the context of kinematics. These studies show that students often misinterpret graphs as pictures (e.g., viewing a velocity–time graph as a picture of the object's trajectory) and do not use scientific ideas to interpret the relationships presented in the graphs such as slope, trends, and patterns (Beichner, 1994; Kozhevnikov et al., 2007; Planinic et al., 2012). Although empirical studies have generated significant findings about students' mathematization, more research is needed to investigate how students develop from their novice thinking to expert thinking is limited.

We addressed these two needs in a Mathematical Thinking in Science project. We used a learning progression (LP) approach to investigate student development of mathematization in physical and life sciences. LPs are "descriptions of successively more sophisticated ways of thinking about how learners develop key disciplinary concepts and practices within a grade level and across multiple grades" (Fortus & Krajcik, 2012, p. 784). It is well-recognized that coherence in science curriculum leads to high-quality instruction and student achievement (Fortus & Krajcik, 2012; Schmidt et al., 2005). Existing literature emphasizes two aspects of curricular coherence—logical coherence and cognitive coherence (Fortus & Krajcik, 2012; Schmidt et al., 2005; Shwartz, et al., 2008; Sikorski & Hammer, 2017). That is, curriculum, instruction, and assessment are aligned based not only on the logical structure and organization of the discipline but also on the cognitive theories about student learning of the disciplinary knowledge and practices. Science LPs are rooted in foundational theories about disciplinary knowledge and cognition. Therefore, they are powerful in enhancing curricular coherence (Jin et al., 2019b).

In the project, we defined mathematization based on a historical analysis and Thompson's theory of quantitative reasoning in mathematics (Thompson, 1993, 2011, Thompson, et al., 2014). We then used a learning progression (LP) approach to study student development in mathematization across several topics in physical sciences and life sciences (heat and temperature, kinetic and gravitational potential energy, and elastic energy in physical sciences; the carbon cycle and interdependent relationships in life sciences). Our study suggests that, by using an LP approach, mathematization can be used as a crosscutting theme to align curriculum, instruction, and assessment. In this chapter, we summarize the major findings of this work, including the definition of mathematization, the LP for mathematization, and preliminary evidence of mathematization as a crosscutting theme. Based on these results, we discuss the possibility and benefits of using mathematization as a crosscutting theme for building curricular coherence.

1 Defining Mathematization

We intended to develop a functional definition of mathematization that reflects how scientists used quantitative reasoning to construct scientific knowledge. To do so, we conducted a historical analysis. We identified and examined five events across physics, biology, astronomy, and chemistry. These five events include the development of the ideal gas law, Mendel's discovery of the laws of hybridization, Newton's derivation of universal gravitation from Kepler's law of planetary motion, the chemical revolution initiated by Lavoisier, and the paradigm shift from Aristotelian to Newtonian theories about forces and motion. They played a critical role in the knowledge development and revolution in the history of science. Our analysis focused on how measurement and quantification enabled the generation of fundamental ideas in science. Details of the analysis can be found in our previous publication (Jin et al., 2019a). In this chapter, we summarize one event that has led to the overthrown of the phlogiston theory and the establishment of modern chemistry—Antoine Lavoisier's chemical revolution.

Both phlogiston theorists and Lavoisier investigated phenomena of burning, calcination, and breathing. However, the ways of reasoning used in their investigations are vastly different. Take burning as an example. The phlogiston theorists observed that some materials were combustible, while other materials were not. To explain this observation, they conjectured those combustible materials must contain some type of essence. They named this essence phlogiston. The ashes after combustion often weigh less than the combustible material. To explain this phenomenon, phlogiston theorists supposed that phlogiston must escape into the air. These qualitative conjectures constitute the phlogiston theory: Materials that are rich in phlogiston can burn; when a material burns, its phlogiston is liberated into the air and only ashes are left.

Unlike the phlogiston theorists, Lavoisier used quantitative reasoning to analyze burning. He conducted experiments in closed systems and with accurate measurement. He studied burning of different materials in a closed vessel system (Holton & Brush, 2001). In the burning iron experiment, burning 100 grains [a unit of mass] of iron produced 135 or 136 grains oxide of iron. At the same time, the diminution of air was found to be exactly 70 cubical inches, which weighed 35 grains. Lavoisier analyzed the relationships among several quantities: The mass of iron, the volume of air, the mass of air, and the mass of oxide of iron. After many similar experiments, he found a mathematical pattern: the total mass of materials is conserved in burning. To explain this pattern, Lavoisier proposed a new theory of combustion, the oxygen theory: The total mass is conserved before and after the combustion because oxygen is involved in combustion and the mass of oxygen should be included in the calculation.

Thompson's theory of quantitative reasoning in mathematics (Thompson, 1993, 2011; Thompson & Carlson, 2017) offers unique insights for us to identify key components that differentiates Lavoisier's and other scientists' mathematization from the intuitive reasoning patterns that once appeared and then became obsolete in the history of science. Thompson (1993) defines quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 165). In explaining this definition, Thompson emphasizes two ideas. First, a key characteristic of quantity is its measurability (Thompson, 1993, p. 165):

> Quantity is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it (Thompson, 1989, in press). Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them.

This concept of **measurability**, or measurable quantities/variables, is one component that differentiates mathematization from intuitive reasoning. While phlogiston theorists focused the analysis on qualitative attributes (e.g., combustible materials turn into ashes; some materials are combustible, while others are not), Lavoisier analyzed measured variables (i.e., the mass of iron, the volume of air, the mass of air, and the mass of oxide of iron). As another example, consider two responses to the following question: "Does a person have more energy after a night's sleep?"

> Response A: After a night's sleep, a person will have *less* energy than the night before since a certain amount of energy stored in the person's body has been used to support body functions such as heart beating and breathing.
>
> Student B: The person has *more* energy because people normally feel more energetic after a good night's sleep.

In Response A, energy is treated as a measurable quantity because the response is about how the total amount of energy changes and where the reduced amount of energy goes. Response B does not treat energy as a measurable quantity because it uses a qualitative reason (i.e., feeling more energetic) to explain why the person has more energy after a night's sleep.

Second, understanding **relational complexity** is crucial for analyzing a network of quantities and quantitative relationships (Thompson, 1993; Thompson & Carlson, 2017). This understanding involves coordination of two aspects of quantitative difference: (1) difference as the amount left over after a comparison and (2) quantitative difference as an item in a relational structure. In this sense, understanding relational complexity is not just about obtaining the result of subtracting. It includes understanding the relationships among multiple differences in a structure. Thompson discusses relational complexity in contexts involving subtraction and addition. We modified and applied this component to fit scientific contexts. We define relational complexity as the complexity involved in the kinds of relationships that play an important role in scientific conceptualization. These relationships include quantitative conservation, *extensive* versus *intensive* variables, change versus the rate of change, proportionality, exponential growth, quadratic relationships, and so on. The historical analysis is about quantitative conservation. Lavoisier's notion of conservation is quantitative because it is based on calculation of numerical values measured in experiments. Phlogiston theorists hold a notion of 'qualitative conservation'. They recognize that the ashes cannot weigh more than the combustible material. Something must come out from the material and that something must go somewhere. They label that something as phlogiston. This type of conservation is qualitative because it is based on humans' perception of less or more. Unlike Lavoisier's quantitative conservation, this "qualitative conservation" does not involve numerical values measured in experiments or real-world situations.

A third component involved in Lavoisier's mathematization is **conceptualization**. Lavoisier identified a quantitative relationship in his experiments—quantitative conservation, that is, the mass of materials before and after combustion is conserved. He then conceptualized this relationship into the oxygen theory of combustion: Oxygen is involved in combustion. If we calculate all substances involved in combustion, we will find that mass is conserved before and after the combustion. In the history of science, many concepts, principles, and theories were conceptualized from quantitative relationships. They are usually counter-intuitive, and therefore present significant challenges to students. For example, a student who understands relational complexity will understand the scientific implication of the equation of kinetic energy ($E = \frac{1}{2} mv^2$) and explain that doubling the vehicle speed can lead to collision damage that is much larger than doubling (due to the quadratic relationship between energy and speed). However, a student who does not recognize the relational complexity involved in the same equation may know that a higher vehicle speed is associated with more damage, but the student would not recognize the scientific implication of the quadratic relationship between the speed and the energy.

The above discussion suggests three components of mathematization of science—**measurable variables, relational complexity, and scientific conceptualization**.

Subsequently, we define mathematization of science as *abstracting measurable variables from 'messy' phenomena, identifying mathematical relationships among the variables, and using scientific ideas to conceptualize the mathematical relationships.*

2 The Learning Progression for Mathematization of Science

In the Mathematical Thinking in Science project, we developed an LP for mathematization in across topics in physical and life sciences: heat and temperature, kinetic and gravitational potential energy, and elastic energy in physical sciences; the carbon cycle and interdependent relationships in life sciences. We first carried out an interview, where 44 students from suburban and urban high schools each completed a set of mathematization tasks. Based on the interview data, we developed an initial mathematization LP.

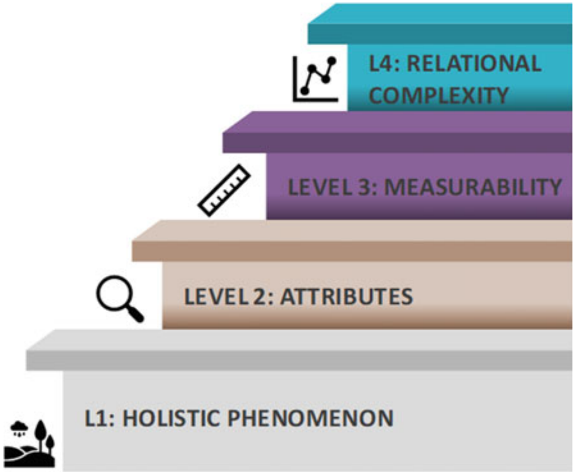
Next, we conducted a large-scale field study. In the study, 57 assessment items, including 36 physical science items and 21 life science items were assembled into multiple computer-delivered tests forms, based on the courses taught by the participating teachers. In addition, most students took 24 mathematics assessment items developed by Wylie et al. (2015). The mathematics items assess student understanding of linear functions and proportional reasoning. These two concepts are essential in middle school mathematics curriculum. They also constitute the foundational knowledge for students to learn and conceptualize a variety of mathematical relationships in high school science. Therefore, they are used as a proxy for students' mathematics baseline understanding. The test forms were administered to 5353 students from 22 high schools in 14 US states. Among these students, 34% were in 11th grade, 27% in 10th grade, and 24% in 12th grade. Urban, suburban, and rural schools participated in the pilot study. Approximately 65% of the students were White, 10% Asian or Asian American, 8% African American, 8% Hispanic or Latino. We used students' assessment responses to revise the LP. The assessment results also provide two pieces of evidence that the mathematization LP is applicable to topics in both physical sciences and life sciences.

In this section, we use students' assessment responses to illustrate the LP levels. Then, we provide the evidence for using mathematization as a crosscutting theme across science topics and disciplines.

3 The LP for Mathematization of Science

The learning progression contains four levels, with each level describing a characteristic way of reasoning that students use to solve scientific problems and to explain real-world phenomena (Fig. 1). These four levels are named holistic phenomenon,

Fig. 1 The learning progression for mathematization for problem-solving



attributes, measurability, and relational complexity. Together, they present a developmental trend, where students progress from intuitive qualitative reasoning to scientific quantitative reasoning.

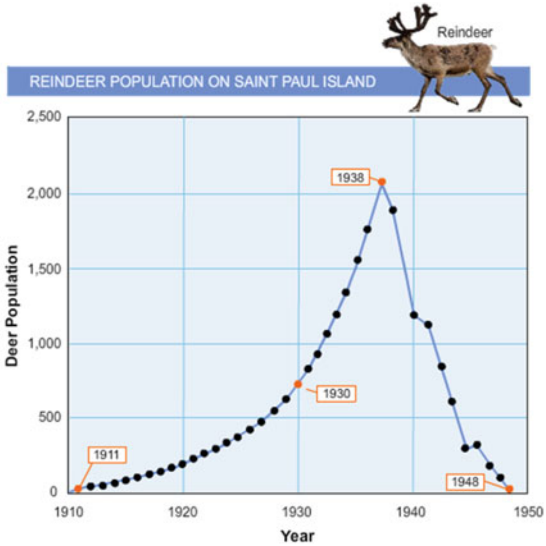
- *Level 1. Holistic Phenomenon*: At Level 1, students do not 'analyze', meaning that they do not identify any relevant attributes of the phenomena. Instead, they describe the phenomenon, or tell a story related to the phenomenon, or express personal feeling about the phenomenon.
- *Level 2. Attributes*: Students identify relevant attributes of a phenomenon considering their everyday concepts. However, they do not 'quantify', meaning that they treat those attributes as qualitative characteristics rather than measurable quantities/variables. The phlogiston theorists' analysis of burning is an example of reasoning about attributes.
- *Level 3. Measurability*: Students analyze phenomena in terms of measurable quantities/variables. They can abstract some relevant variables from the messy phenomena and identify some mathematical relationships. However, conceptualizing the mathematical relationships in terms of scientific ideas presents significant challenge for them.
- *Level 4. Relational Complexity*: Students distinguish among different types of quantities/variables and understand the complex relationships among those quantities/variables. The complex relationships include relationships between change and rate of change (e.g., velocity and acceleration), distinctions between extensive and intensive variables (e.g., thermal energy and temperature; mass and density), proportional relationship (e.g., gravitational potential energy is proportional to height), quadratic relationship (e.g., the relationship between kinetic energy and speed of a car), exponential relationship (e.g., the population size and the time), and so on.

We use students' responses in two assessment items, one in physical sciences and the other in life sciences, to illustrate this LP. In these responses, pseudonyms are used to protect the identity of the students. As presented below, the life science item (Fig. 2) asks students to mathematize the growth of reindeer population. In the item, the relevant variables are birth rate, the number of births, death rate, the number of deaths, the population size, and the population growth rate. The relationships among these variables are presented in the graph in Fig. 2. More specifically, the part about the exponential growth of the reindeer population shows an important mathematical pattern—the slope of the graph increases over time, meaning the population growth becomes more rapid over time, or in other words, the population growth rate increases. The conceptualization of this mathematical pattern is: Given adequate resources and an absence of predators, the reindeer population would increase exponentially for a long time. In such situation, while the birth rate and death rate (the number of births/deaths per reindeer per year) do not change, the total number of organisms increases, causing the rate of population growth (i.e., the absolute growth rate) to increase.

Table 1 presents students' responses that were scored at each level of the LP. The responses at Level 1 indicate that Diego did not identify any qualitative factors or attributes that explain the observed pattern—the population grew faster in timespan 2 than in timespan 1. Instead, he claimed that the observed pattern is due to reindeer's intention to increase their population. Diego treated the phenomenon holistically and did not analyze and abstract any variables or attributes. The responses at Level 2 suggests that Cindy identified two factors affecting the reindeer population: predation and starvation. She further explained how these qualitative factors affect the reindeer population. Cindy did not reason about any quantitative relationships or measurable variables. She only reasoned at a qualitative level. The responses at Level 3 suggest that Mike reasoned at a quantitative level. He explained that during the timespan 2, the reindeer has adapted to their surroundings; consequently, the reindeer were able to increase the breeding rate significantly, which caused the reindeer population to increase more rapidly. This explanation focuses on the relationship between two measurable variables—the breeding rate and the population growth rate; the increase of breeding rate caused the increase of population growth rate. Although Mike began to reason about the quantitative relationships between measurable variables, he was not successful in identifying and conceptualizing the relational complexity involved in the problem. The responses at Level 4 show that Amber was able to identify relevant measurable variables and conceptualize the complex relationships among the variables. Amber explained that, although individual reindeer produced offspring at the same rate, the total number of reindeers increased. As a result, the population size increased exponentially. Her explanation targets the complex relationship among three measurable variables—the reproduction rate per individual reindeer, the population size, and the population growth rate.

A physical science item is provided in Fig. 3. This item assesses how well students identify and differentiate between heat/energy and temperature. High school students are expected to understand the following distinctions among heat, energy, and temperature (Kesidou & Duit, 1993, p. 90):

In 1911, scientists released 25 reindeer on Saint Paul Island, a small Alaskan island. There were no predators of reindeer on the island. Scientists collected data on the reindeer population over many years. The graph below shows the scientists' data.



1. Please compare the population growth in these two timespans.
   Time span 1: 1911 to 1932
   Time span 2: 1932 to 1938
   Which of the three patterns below best describes the changes in reindeer population?

   A. The population grew faster in timespan 1 than in timespan 2.
   B. The population grew faster in timespan 2 than in timespan 1.
   C. The population grew at the same rate in these two timespans.

2. Why do you think the reindeers on Saint Paul Island exhibited this pattern?

Fig. 2 The life sciences item

Heat is the form of energy that is transported from one system to another due to temperature differences. From the physicist's point of view, heat is a process variable. Therefore, it is wrong to state that a body contains a certain amount of heat. But it makes sense to view heat as an extensive quantity. If a specific amount of heat (Q1) is transported and if this is followed by another amount of heat (Q2) the total amount of heat transported is Q1 + Q2. Temperature, on the other hand, is an intensive quantity. If two bodies at temperature T are brought into contact, then the temperature of the two bodies is still T.

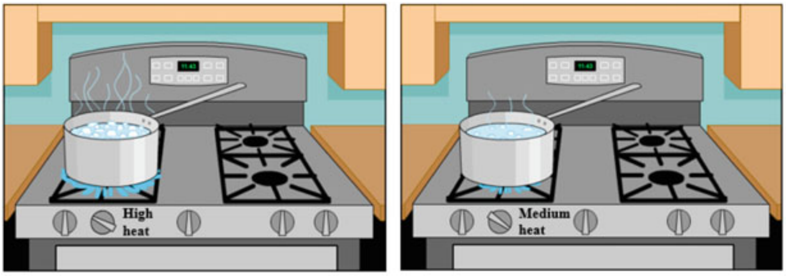
Table 1 Middle and high school students' responses in the life science task

Learning progression levels	Responses
Level 4. Relational complexity	Responses from Amber Choice: B. The population grew faster in timespan 2 than in timespan 1 Explanation: the population increased exponentially because more individuals means that there is a greater number of animals capable of producing offspring. When they produce this amount of offspring, the population will increase, and then those offspring will go on to have offspring of their own, showing population growth
Level 3. Measurability	Responses from mike Choice: B. The population grew faster in timespan 2 than in timespan 1 Explanation: I believe the reindeer on Saint Paul grew faster of the course of time span 2 because the reindeer need to be adjusted to their environmet [environment]. The island only hosted 25 reindeer to start, but as the reindeer adapted themselves to their surroundings, they were able to utilize whatever that helped them breed at a significantly faster rate
Level 2. Attributes	Responses from Cindy Choice: A. The population grew faster in timespan 1 than in timespan 2 Explanation: they exhibit this pattern because there were no predators which says that they won't die, but there are other problems when there aren't predators, because they will they [then] die from starvation and etc.
Level 1. Holistic phenomena	Responses from Diego Choice: B. The population grew faster in timespan 2 than in timespan 1 Explanation: they exhibited this because they wanted to increase their population

Note that the distinction between heat as a process variable and energy as a status variable is not assessed in the item illustrated in Fig. 3. The item focuses on the distinction between heat/energy and temperature: the former are extensive variables, while the latter is an intensive variable. Successful mathematization involves identifying and distinguishing variables from three observations. The first observation is the oven setting (or the fire), which indicates the amount of heat transferred into the water in the pot. Since the oven is set at high heat in Situation 1 and at medium heat in Situation 2, less heat is transferred into the water in Situation 2. The second observation is that, in both situations, the water is at the boiling stage, indicating the water temperature as 100 °C. The third observation is the degree of vigorousness in boiling, which indicates how much evaporation is going on. In Situation 1, more energy/heat is used to evaporate the water, so the water boils more vigorously.

Table 2 provides students' responses at each LP level. The responses at Level 1 indicate that Zane did not identify any attributes or variables to support his claim

Paulo put a pot with water on the stove at high heat. After a few minutes, the water started to boil vigorously. Paulo turned down the heat setting to medium, and the water kept boiling but less vigorously.



Situation 1: The water boiled vigorously. Situation 2: The water kept boiling but less vigorously.

1. Do you think the temperature of the water is the same in these two situations?
   A. The water temperature in Situation 1 and the water temperature in Situation 2 are the same.
   B. The water temperature in Situation 1 and the water temperature in Situation 2 are different.
2. [Different sets of questions are shown when the student chooses A or B.]

   Choosing A: If the water temperatures in these two situations are the same, why does the water in Situation 2 boil less vigorously than the water in Situation 1?

   Choosing B: What evidence can be used to support the claim that the water temperature in Situation 1 and the water temperature in Situation 2 are different? Please explain why this evidence can be used to support the claim.

Fig. 3 The physical sciences item

that the water temperatures in both situations are the same. Instead, he described macroscopic observations in an everyday activity—boiling water to cook pasta. The responses at Level 2 shows that Mia associated ‘boiling more vigorously’ with ‘being hotter’ and with higher temperature. As such, Mia treated temperature as hotness, which is a qualitative attribute and therefore is not measured and has no numerical values. The responses at Level 3 show that Lucy reasoned about the values of temperature, indicating that she recognized measurability as a key characteristic of variables. However, she does not differentiate between energy/heat and temperature in terms of extensive and intensive variables. Instead, she assumed that more heat input causes the water to boil more vigorously; and that water boiling more vigorously has a higher temperature. However, she did learn that boiling water has a temperature of 100 °C. To reconcile the discrepancy, she conceptualized a new theory—water begins to boil at 100 °C, and the temperature of the water will keep increasing when the water is boiling more vigorously. This way, the input energy/heat, degree of boiling, and temperature are equivalent. The exemplar responses at Level 4 suggest that Kai was able to identify and differentiate heat/energy and temperature. Although more energy goes to the water in Situation 1 than Situation 2, the water temperature stayed the same (100 °C) in the two situations. The reason is that the input energy is used to make water evaporate. Because more evaporation happens in Situation 1 than Situation 2, we observe that the water in Situation 1 boils more vigorously.

Table 2 Students' responses in a physical science task

Learning progression levels	Responses
Level 4. Relational complexity	Responses from Kai Choice: A. The water temperature in situation 1 and the water temperature in situation 2 are the same Explanation (if the water temperatures in these two situations are the same, why does the water in situation 2 boil less vigorously than the water in situation 1?): because with more heat the water is turning to steam more quickly, but at water's boiling point no matter how much heat is added it does not increase in temperature in this state
Level 3. Measurability	Responses from Lucy Choice: B. The water temperature in situation 1 and the water temperature in situation 2 are different Explanation (what evidence can be used to support the claim that the water temperature in situation 1 and the water temperature in situation 2 are different? please explain why this evidence can be used to support the claim.): the evidence that can be used to support this claim is that they tell you in situation 1 the water is boiling but in situation 2 it is not boiling as much. this evidence can be used because boiling starts to occur at 100 °C but that is not just where it stops
Level 2. Attributes	Responses from Mia Choice: B. The water temperature in situation 1 and the water temperature in situation 2 are different Explanation (what evidence can be used to support the claim that the water temperature in situation 1 and the water temperature in situation 2 are different? Please explain why this evidence can be used to support the claim.): the water in situation 1 was boiling much more vigoriously [vigorously] than the water in situation 2. This means that situation 1 had significantly more energy to use, meaning that it was hotter
Level 1. Holistic phenomena	Responses from Zane Choice: A. The water temperature in situation 1 and the water temperature in situation 2 are the same Explanation (if the water temperatures in these two situations are the same, why does the water in situation 2 boil less vigorously than the water in situation 1?): your pot of water is on the stove, you've turned on the maximum heat, and the wait for boiling begins. You are staring impatiently at the pot when the water looks like it's starting to swirl. You're anxious to see the bubbles that signify that you can put your pasta into that water

4 Evidence for Mathematization to Be Used as a Crosscutting Theme

For mathematization to be used as a crosscutting theme, a framework of mathematization must be developed to guide the development of curriculum, instruction, and assessment across topics and disciplines. In the project, we conducted quantitative analyses of the student assessment data. Our analyses provide two pieces of evidence that the mathematization LP is applicable to topics in both physical sciences and life sciences. Therefore, the mathematization LP is a potential framework to guide the development of curriculum, instruction, and assessment across science topics and disciplines. In this chapter, we describe these two pieces of evidence.

First, we scored the item responses in terms of the four levels of the LP (score 1 for Level 1 responses, etc.) and used the item response theory (IRT) models to analyze those scores. The results suggested that the mathematization LP *can* be used to evaluate student proficiency in both physical science topics and life science topics. More specifically, the Rasch model was used to fit dichotomous items; the Partial Credit model (Masters, 1982) was used for polytomous items. Results of the IRT analysis are presented in Wright maps (Fig. 4). The Wright maps provide quantified locations of item difficulties and students' performances on the same scale, called the logit scale. The left side of the Wright Map displays the distribution of students' performance estimates while the right side represents the distribution of the Thurstonian thresholds for each item. Each item has two to four threshold values. These values are 1, 2, 3, or 4, representing the transition between a zero score (responses such as "I don't know" or random letters) and Level 1, between Level 1 and Level 2, between Level 2 and Level 3, and between Level 3 and Level 4, respectively. For example, the location of the second threshold (labeled as 2) for a life science item, "LS18", is close to zero logit. This suggests that students located at zero logit value of performance have about 50% chance of transition from Level 2 to Level 3 for LS18.

Wright Maps allow a visual determination of whether the LP levels for mathematization of science were differentiated from each other. Undifferentiated levels in the Wright map would indicate that the scoring rubric or the LP is not empirically supported and should not be used to evaluate students' performance. The two Wright maps (Fig. 4) provide the following evidence. For all items, the data supports the hypothesized order of the four learning progression levels. For most items, the data shows that the learning progression levels are differentiated from each other. It is also important to note that the data of some items do not support the distinction between adjacent levels. For example, for a physical science item, "PS21", the threshold 1 and the threshold 2 are located remarkably close to each other. This evidence indicates that the transition between the zero score and Level 1 and the transition between Level 1 and Level 2 are not clearly distinguishable. One possible cause is that the small number of responses at those levels caused unreliable estimates. The distinction between levels is clear in the Wright map for the life science items, but not in the Wright map for the physical science items. This is probably because the physical

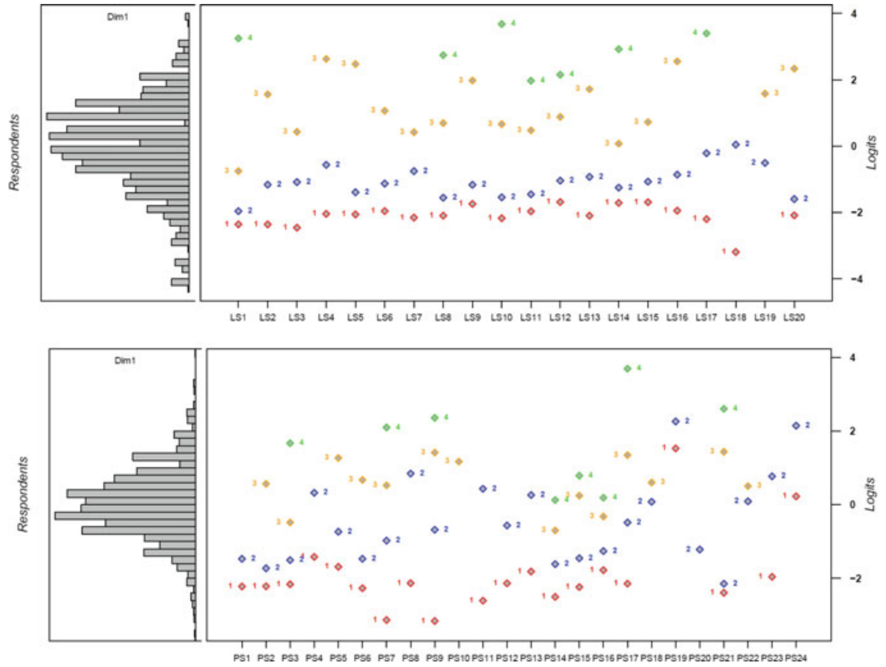


Fig. 4 Wright maps of mathematization of science in life science (upper panel) and in physical science (lower panel)

science assessments contain more multiple-choice items and multiple-choice items are not as effective as constructed response items. This may also be due to that more topics are involved in physical science assessments. In summary, the Wright Maps in both the life science and physical science domains support the internal structure of the assessment (Wilson, 2004) that the learning progression levels provide useful measures of mathematization of science.

Second, more advanced IRT modeling revealed the potential that the mathematization LP is applicable in both life sciences and physical sciences. More specifically, the same data was analyzed through a special type of IRT model (Shin et al., 2017) to investigate the relationships between students' mathematics ability and their mathematization proficiency in physical/life sciences. For this analysis, the same set of math item parameters were used to put mathematization in physical science and mathematization in life science on the same scale in relation to the mathematics ability measure. Next, thresholds between two adjacent learning progression levels were computed. Table 3 provides the estimated thresholds in the life science and the physical science referenced to the mathematics items.

As shown in Table 3, thresholds were estimated as the median values across items on the logit scale (Shin et al., 2012). Thus, the differences between thresholds are allowed to be varied (e.g., Level 1 could have a smaller range than Level 2). For Thresholds 1 and 2, the values in two different disciplines were estimated to be quite

Table 3 Estimated thresholds of the learning progression for mathematization

Thresholds	Life sciences	Physical sciences
Threshold 1	− 2.08	− 2.16
Threshold 2	− 1.09	− 0.96
Threshold 3	0.97	0.55
Threshold 4	2.92	2.09

similar, although not identical. This pattern is not observed for Thresholds 3 and 4. This may be due to that many physical science items were multiple-choice items and that the physical science items were divided into multiple topics to fit the courses taught by the participating teachers.

5 Conclusions

Quantitative reasoning played a crucial role in the development and revolution of scientific knowledge (Crombie, 1961; Kline, 1982). It is also an essential learning goal for K-12 students (NGSS Lead States, 2013; NRC, 1996). In this chapter, we use mathematization of science to refer to quantitative reasoning in science, because it was used to refer to the specialized ways of reasoning that scientists used to quantify phenomena and construct knowledge (Kline, 1982; Lehrer & Schauble, 1998). We conducted an analysis of five events that played critical role in the development and revolution of scientific knowledge. This analysis is inspired by Thompson's theory of quantitative reasoning in mathematics (Thompson, 1993, 2011). We found Thompson's ideas about measurability and relational complexity very useful for us to understand how mathematics and quantitative reasoning were used in the history of science. Our historical analysis suggests three components that differentiate mathematization from people's everyday intuitive reasoning. These three components are measurability, relational complexity, and scientific conceptualization. Together, they illustrate a quantification process, by which scientists abstract measurable variables from messy phenomena and observations; use mathematical operations to identify relationships among those variables; and conceptualize concepts, principles, and theories to explain the identified relationships. By using this quantification process, scientists have made significant breakthroughs in the history of science (Jin et al., 2019a). Mathematization is also one of the six styles of scientific reasoning embedded in all science disciplines (Crombie, 1994; Kind & Osborne, 2017; Osborne et al., 2018). Like other styles of reasoning, the value of mathematization includes "explaining the diversity to be found within the sciences, elegantly capturing the forms of reasoning, and helping to identify the intellectual achievement that the sciences represent" (Osborne & Rafanelli, 2019, p. 530). Therefore, mathematization, as well as other styles of reasoning, are good candidates for crosscutting themes to build curricular coherence.

Schmidt et al. (2005) compared the mathematics and science standards of the United States with those of top-achieving countries in the Third International Mathematics and Science Study (TIMSS). They found that, "coherence is one of the most critical, if not the single most important, defining elements of high-quality standards" (p. 554). They further point out that the U.S. has a 'mile-wide inch-deep' science curriculum that covers a wide range of science topics, but the topics are not organized in ways reflecting the logical nature of the disciplinary content. In other words, the U.S. curriculum is not coherent. The current science standards, NGSS, present a significant improvement in curricular coherence. NGSS were developed under the guidance of the NRC Framework. In the Framework, the three dimensions of science learning (two to four core ideas in each discipline, eight scientific and engineering practices, and seven crosscutting concepts) are integrated to achieve the logical coherence in science disciplines; learning progressions for the components in each dimension are used to ensure the cognitive coherence in science learning. Moreover, the seven crosscutting concepts (patterns; cause and effect; energy and matter, etc.) "provide students with connections and intellectual tools that are related across the differing areas of disciplinary content and can enrich their application of practices and their understanding of core ideas" (NRC, 2012, p. 233).

Osborne and colleagues (Kind & Osborne, 2017; Osborne & Rafanelli, 2019; Osborne et al., 2018) propose using the six styles of reasoning to replace the seven crosscutting concepts as crosscutting themes. While we agree with Saleh and colleagues (Saleh et al., 2019) that the crosscutting concepts have been proved effective when being used as a crosscutting theme, we also believe it is valuable to explore styles of reasoning as alternative crosscutting themes to build curricular coherence. After all, diversity drives innovation and advancements. A variety of approaches are needed to promote teaching and learning of science. For example, if mathematization is taught and assessed consistently across science topics and disciplines, students will learn to use mathematization more effectively. They will also develop deep understanding of the content knowledge in different topics and disciplines, because mathematization requires using disciplinary knowledge to explain mathematical relationships.

For mathematization to be used as a crosscutting theme, evidence in both logical coherence and cognitive coherence should be provided. In terms of logical coherence, researchers have conducted extensive and thorough analysis of scientific knowledge and found that mathematization is embedded in the knowledge across science disciplines (Crombie, 1961; Kline, 1982; Jin et al., 2019a). In terms of cognitive coherence, research of student understanding must be conducted to show that mathematization can be taught and assessed across topics and disciplines. Our research provides preliminary evidence for the cognitive coherence. We developed an LP that describes and evaluates student performance in terms of four levels of achievement—holistic phenomenon, attributes, measurability, and relational complexity. Moving up these levels, students demonstrate increasingly sophisticated mastery of mathematization. Our analyses of students' assessment data suggest that the mathematization LP can be used to assess mathematization across several topics in physical and life sciences.

Further research is needed to use the LP to guide assessment and instruction in more topics.

Acknowledgements This work was supported by Institute of Education Sciences under Grant R305A160219.

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