Sobolev Spaces and Elliptic Boundary Value Problems

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1 Sobolev Spaces, Inequalities, Dirichlet, and Neumann Problems for the Laplacian

1.1 Sobolev Spaces

Let us introduce the following Sobolev spaces: for any $1 < p < \infty$

$$W^{m,p}(\Omega) = \left\{u \in \mathscr{D}'(\Omega);\ \forall\, |\alpha| \leq m,\ D^\alpha u \in L^p(\Omega)\right\}$$

and

$$W^{s,p}(\Omega) = \left\{u \in W^{m,p}(\Omega);\ \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+\sigma p}} < \infty,\ \forall\, |\alpha| = m\right\},$$

where $m \in \mathbb{N}$, $s = m + \sigma$, $0 < \sigma < 1$ and Ω is an open set of $\mathbb{R}^N$. Equipped with the graph norm, they are Banach spaces.

When $\Omega = \mathbb{R}^N$, using the Fourier transform, we define for any real number s the space

$$H^s(\mathbb{R}^N) = \left\{u \in \mathscr{S}'(\mathbb{R}^N);\ \int_{\mathbb{R}^N} (1 + |\xi|^2)^s\, |\hat{u}(\xi)|^2\, d\xi < \infty\right\},$$

which is an Hilbert space for the norm:

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$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (1+|\xi|^2)^s |\hat{u}|^2 dx\right)^{1/2}.$$

By Plancherel's theorem we prove that $W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ for all $s \geq 0$ and this identity is algebraical and topological. So, in the case $p = 2$, we denote more simply the space $W^{s,2}(\Omega)$ by $H^s(\Omega)$.

Definition 1.1 For $s > 0$ and $1 \leq p < \infty$, we denote

$$W_0^{s,p}(\Omega) = \overline{\mathscr{D}(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}},$$

and its topological dual space

$$W^{-s,p'}(\Omega) = \left[W_0^{s,p}(\Omega)\right]',$$

where p' is the conjugate of p: $1/p + 1/p' = 1$. For $p = 2$, we will write $H_0^s(\Omega)$ and $H^{-s}(\Omega)$, respectively.

Proposition 1.2 *Suppose $T \in \mathscr{D}'(\Omega)$. Then $T \in W^{-m,p'}(\Omega)$, with $m \in \mathbb{N}^*$, if and only if*

$$T = \sum_{|\alpha| \leq m} D^\alpha f_\alpha, \quad \text{with} \quad f_\alpha \in L^{p'}(\Omega).$$

1.2 First Properties

It will be assumed from now on that Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary.

Let us consider the following space

$$\mathscr{D}(\overline{\Omega}) = \left\{v_{|\Omega};\ v \in \mathscr{D}(\mathbb{R}^N)\right\}.$$

Theorem 1.3

(i) The space $\mathscr{D}(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ for any $s > 0$ (even if Ω is unbounded).
(ii) The space $\mathscr{D}(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$ for any $s \in \mathbb{R}$.

As consequence, we have the following property: for any $s > 0$

$$W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N) \quad \text{and} \quad W^{-s,p'}(\mathbb{R}^N) = \left[W^{s,p}(\mathbb{R}^N)\right]'.$$

But in general, for any $s > 0$, we have $W_0^{s,p}(\Omega) \subsetneq W^{s,p}(\Omega)$.

Definition 1.4 For $s > 0$, we set

$$\widetilde{W}^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\Omega);\ \widetilde{u} \in W^{s,p}(\mathbb{R}^N) \right\},$$

where $\widetilde{u}$ is the extension by 0 of u outside of Ω.

The space $\widetilde{W}^{s,p}(\Omega)$ is a Banach space for the norm

$$\|u\|_{\widetilde{W}^{s,p}(\Omega)} = \|\widetilde{u}\|_{W^{s,p}(\mathbb{R}^N)}\,.$$

It is easy to verify that for any nonnegative integer m

$$W_0^{m,p}(\Omega) \hookrightarrow \widetilde{W}^{m,p}(\Omega) \tag{1}$$

and for any $u \in W_0^{m,p}(\Omega)$ we have

$$\|u\|_{\widetilde{W}^{m,p}(\Omega)} = \|u\|_{W^{m,p}(\Omega)}\,. \tag{2}$$

When $s = m + \sigma$ with $0 < \sigma < 1$, we can show that

$$\|u\|_{\widetilde{W}^{s,p}(\Omega)} \simeq \|u\|_{W^{s,p}(\Omega)} + \sum_{|\alpha|=m} \left\| \frac{D^\alpha u}{\varrho^\sigma} \right\|_{L^p(\Omega)}, \tag{3}$$

where $\varrho(x) = d(x, \Gamma)$ and $\Gamma = \partial\Omega$.

Theorem 1.5 *The space $\mathscr{D}(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$ (even if Ω is unbounded).*

From (1), (2) and the definition of $W_0^{m,p}(\Omega)$, we deduce the following: for any $m \in \mathbb{N}^*$,

$$\widetilde{W}^{m,p}(\Omega) = W_0^{m,p}(\Omega). \tag{4}$$

Theorem 1.6 *For any $0 < s \le 1/p$, the space $\mathscr{D}(\Omega)$ is dense in $W^{s,p}(\Omega)$, which means that*

$$W_0^{s,p}(\Omega) = W^{s,p}(\Omega). \tag{5}$$

Theorem 1.7 *Let $0 < s \le 1$ and $u \in W_0^{s,p}(\Omega)$. Then*

$$\frac{u}{\varrho^s} \in L^p(\Omega) \Longleftrightarrow s \neq 1/p$$

and in this case

$$\left\| \frac{u}{\varrho^s} \right\|_{L^p(\Omega)} \leq C\, |u|_{W^{s,p}(\Omega)},$$

where the notation $|\cdot|$ *denotes the semi-norm of* $W^{s,p}(\Omega)$.

The case $s = 1$ is known as Hardy's inequality: for all $u \in W_0^{1,p}(\Omega)$,

$$\left\| \frac{u}{\varrho} \right\|_{L^p(\Omega)} \leq C\, \|\nabla u\|_{L^p(\Omega)}\,.$$

Using again a Hardy's inequality, we prove the following result:

Theorem 1.8 *Let* $s > 0$ *and* $u \in W_0^{s,p}(\Omega)$. *Then for any* $|\alpha| \leq s$, *we have*

$$\frac{D^\alpha u}{\varrho^{s-|\alpha|}} \in L^p(\Omega) \Longleftrightarrow s - 1/p \notin \mathbb{N}. \tag{6}$$

From (3) and (6), we deduce the following identity:

$$\widetilde{W}^{s,p}(\Omega) = W_0^{s,p}(\Omega) \tag{7}$$

which holds for any $s > 0$ satisfying $s - 1/p \notin \mathbb{N}$.

Proposition 1.9

(i) For any $1 \leq j \leq N$ *and for any* $s \in \mathbb{R}$, *the operator*

$$\frac{\partial}{\partial x_j} : W^{s,p}(\mathbb{R}^N) \longrightarrow W^{s-1,p}(\mathbb{R}^N) \tag{8}$$

is continuous.

(ii) However, if we replace $\mathbb{R}^N$ *by* Ω, *Property (8) takes place unless* $s = 1/p$.

Sketch of the Proof of Point (ii)

1. Case $s = m+\sigma$, with $m \in \mathbb{N}^*$ and $0 \leq \sigma < 1$. Let $u \in W^{s,p}(\Omega)$. By definition, we know that

$$u \in W^{m,p}(\Omega) \quad \text{and} \quad \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+\sigma p}} < \infty, \quad \forall\, |\alpha| = m.$$

So for any $1 \leq j \leq N$

$$\frac{\partial u}{\partial x_j} \in W^{m-1,p}(\Omega) \quad \text{and} \quad \int_\Omega \int_\Omega \frac{\left| D^\alpha \frac{\partial u}{\partial x_j}(x) - D^\alpha \frac{\partial u}{\partial x_j}(y) \right|^p}{|x-y|^{N+\sigma p}} < \infty,$$

for all $|\alpha| = m - 1$. Consequently $\frac{\partial u}{\partial x_j} \in W^{s-1,p}(\Omega)$.

2. Case $s \le 0$. Let $u \in W^{s,p}(\Omega)$. Since $-s+1 \ge 1$, for any $\varphi \in \mathscr{D}(\Omega)$, we get:

$$\begin{aligned}\left|\langle \frac{\partial u}{\partial x_j}, \varphi\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)}\right| &= \left|-\langle u, \frac{\partial \varphi}{\partial x_j}\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)}\right| \\ &\le \|u\|_{W^{s,p}(\Omega)} \left\|\frac{\partial \varphi}{\partial x_j}\right\|_{W_0^{-s,p'}(\Omega)} \\ &\le \|u\|_{W^{s,p}(\Omega)} \|\varphi\|_{W_0^{-s+1,p'}(\Omega)}.\end{aligned}$$

We conclude by using the density of $\mathscr{D}(\Omega)$ in $W_0^{-s+1,p'}(\Omega)$.

3. Case $0 < s < 1$. Let $u \in W^{s,p}(\Omega)$. Recall that Ω being Lipschitz open set, there exists an extension operator

$$\forall t \ge 0, \quad P : W^{t,p}(\Omega) \longrightarrow W^{t,p}(\mathbb{R}^N)$$

which is linear, continuous, and satisfying

$$Pv_{|\Omega} = v, \quad \text{for any } v \in W^{t,p}(\Omega).$$

As $Pu \in W^{s,p}(\mathbb{R}^N)$, we get $\frac{\partial Pu}{\partial x_j} \in W^{s-1,p}(\mathbb{R}^N)$. But

$$\left(\frac{\partial Pu}{\partial x_j}\right)_{|\Omega} = \frac{\partial u}{\partial x_j},$$

where $\frac{\partial u}{\partial x_j}$ is the restriction to Ω of the distribution $T = \frac{\partial Pu}{\partial x_j} \in W^{s-1,p}(\mathbb{R}^N)$. More precisely, we have:

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \langle \frac{\partial u}{\partial x_j}, \varphi\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)} = \langle T, \widetilde{\varphi}\rangle_{\mathscr{D}'(\mathbb{R}^N)\times\mathscr{D}(\mathbb{R}^N)}.$$

That implies

$$\left|\langle \frac{\partial u}{\partial x_j}, \varphi\rangle\right| \le \|T\|_{W^{s-1,p}(\mathbb{R}^N)} \|\widetilde{\varphi}\|_{W^{1-s,p'}(\mathbb{R}^N)} = \|T\|_{W^{s-1,p}(\mathbb{R}^N)} \|\varphi\|_{\widetilde{W}^{1-s,p'}(\Omega)}.$$

We have shown that $\frac{\partial u}{\partial x_j} \in \left[\widetilde{W}^{1-s,p'}(\Omega)\right]'$. But

$$\left[\widetilde{W}^{1-s,p'}(\Omega)\right]' = \left[W_0^{1-s,p'}(\Omega)\right]' \Longleftrightarrow 1-s \ne 1/p',$$

i.e., $s \ne 1/p$. □

Remark 1 The above proof shows that

$$u \in W^{1/p,p}(\Omega) \Longrightarrow \frac{\partial u}{\partial x_j} \in \left[\widetilde{W}^{1/p',p'}\right]'.$$

In particular,

$$u \in H^{1/2}(\Omega) \Longrightarrow \frac{\partial u}{\partial x_j} \in \left[\widetilde{H}^{1/2}(\Omega)\right]',$$

where we remark also that

$$\widetilde{H}^{1/2}(\Omega) \hookrightarrow H^{1/2}(\Omega) = H_0^{1/2}(\Omega).$$

This embedding being dense, we get by duality

$$H^{-1/2}(\Omega) = \left[H_0^{1/2}(\Omega)\right]' \hookrightarrow \left[\widetilde{H}^{1/2}(\Omega)\right]'.$$

Corollary 1.10 *Let $s > 0$. The following characterization holds:*

$$u \in \widetilde{W}^{s,p}(\Omega) \Longleftrightarrow u \in W_0^{s,p}(\Omega) \quad \textit{and} \quad \textit{for any } |\alpha| = m, \quad \frac{D^\alpha u}{\varrho^\sigma} \in L^p(\Omega),$$

where $s = m + \sigma$, $m \in \mathbb{N}$ and $0 \le \sigma < 1$.

1.3 Traces

Firstly, recall the following inclusions:

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow \mathscr{C}^0(\mathbb{R}^N) \quad \text{if} \quad s > \frac{N}{p}.$$

So that if $u \in W^{s,p}(\mathbb{R}^N)$ with $s > \frac{N}{p}$, the restriction of u to the hyperplane $x_N = 0$ is well defined. But the continuity with respect to all variables is not necessary. It is enough to have the continuity with respect to the variable x_N. This is possible as soon as $s > 1/p$.

Actually, we have the following result:

Theorem 1.11

(i) Suppose that $s - 1/p = k + \sigma$, with $k \in \mathbb{N}$ and $0 < \sigma < 1$ (which implies, in particular, that $s - 1/p \notin \mathbb{N}$). Then the mapping

$$u \overset{\gamma}{\longmapsto} (\gamma_0 u, \gamma_1 u, \dots, \gamma_k u),$$

where

$$\gamma_0 u(x) = u(x', 0), x' = (x_1, \dots, x_{N-1}), \quad \text{and} \quad \gamma_j u(x') = \frac{\partial^j u}{\partial x_N^j}(x', 0),$$

defined for $u \in \mathscr{D}(\mathbb{R}^N)$*, has a unique extension*

$$W^{s,p}(\mathbb{R}^n) \longrightarrow \prod_{j=0}^{k} W^{s-j-1/p,p}(\mathbb{R}^{N-1})$$

which is continuous and where k *is the integer part of* $s > 0$.

(ii) Moreover this operator has a right continuous inverse R*:*

$$\begin{cases} \forall \boldsymbol{g} = (g_0, \dots, g_k) \in \prod_{j=0}^{k} W^{s-j-1/p,p}(\mathbb{R}^{N-1}), \quad \gamma R\boldsymbol{g} = \boldsymbol{g} \\ \|R\boldsymbol{g}\|_{W^{s,p}(\mathbb{R}^N)} \leq C_N \sum_{j=0}^{k} \|g_j\|_{W^{s-j-1/p,p}(\mathbb{R}^{N-1})} . \end{cases}$$

Remark 2 For $p = 2$, the above result can be proved using the Fourier transform.

This result can be extended to the case where Ω is a bounded open subset of $\mathbb{R}^N$, with a $\mathscr{C}^{k,1}$ boundary (see the definition below).

Definition 1.12 Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω is Lipschitz (respectively of class $\mathscr{C}^{k,1}$, $k \in \mathbb{N}^\star$) if for every $x \in \Gamma$, there exists a neighborhood V of x in $\mathbb{R}^N$ and orthonormal coordinates $\{y_1, \dots, y_N\}$ satisfying:

(i) V is an hypercube

$$V = \left\{(y_1, \dots, y_N) \in \mathbb{R}^N; \ |y_j| < a_j, \ 1 \leq j \leq N\right\},$$

(ii) there exists a function φ defined in

$$V' = \left\{y' \in \mathbb{R}^{N-1}; \ |y_j| < a_j, 1 \leq j \leq N-1\right\},$$

such that φ and φ^{-1} are Lipschitz (respectively, $\mathscr{C}^{k,1}$) and satisfying (Fig. 1)

$$\forall\, y' \in V', \quad \left|\varphi(y')\right| \leq \frac{1}{2} a_N$$

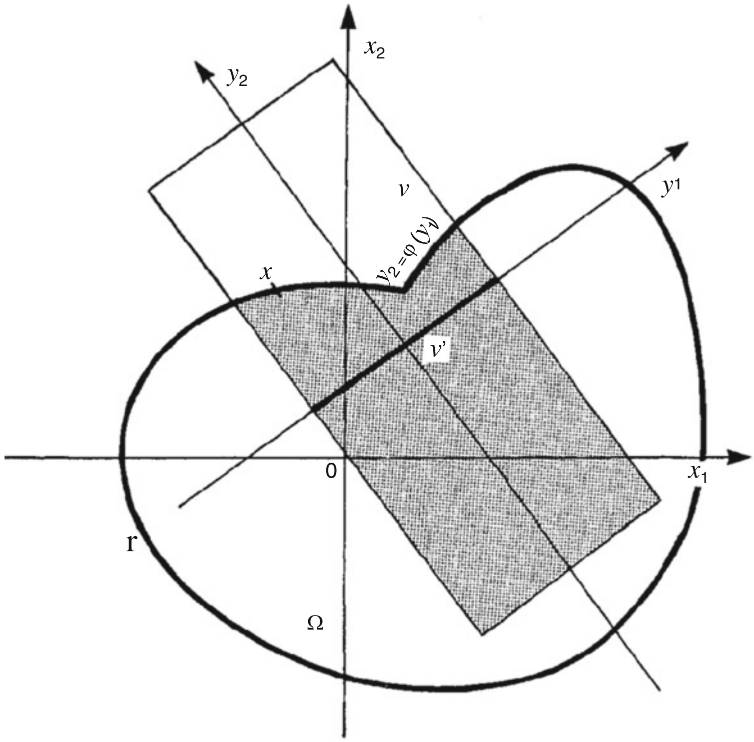


Fig. 1

$$\Omega \cap V = \left\{(y', y_N) \in V;\ \ y_N < \varphi(y')\right\}$$

$$\Gamma \cap V = \left\{(y', y_n) \in V;\ \ y_N = \varphi(y')\right\}.$$

Let

$$\begin{aligned} \Phi : V' &\longrightarrow \Gamma \cap V \\ y' &\longmapsto (y', \varphi(y')). \end{aligned}$$

Definition 1.13 Suppose that Ω is an open subset of $\mathbb{R}^N$ of class $\mathscr{C}^{k,1}$, with $k \in \mathbb{N}$ and let $0 < s \leq k + 1$. We introduce the following space

$$W^{s,p}(\Gamma) = \left\{u \in L^p(\Gamma);\ u \circ \Phi \in W^{s,p}(V' \cap \Phi^{-1}(\Gamma \cap V))\right\}$$

for any (V, φ) verifying the previous definition.

Let (V_j, φ_j), $1 \leq j \leq J$, be any atlas of Γ for which each pair (V_j, φ_j) satisfies the above definition. One possible Banach norm for $W^{s,p}(\Gamma)$ is given by:

$$\|u\|_{W^{s,p}(\Gamma)} = \sum_{j=1}^{J} \left\|u \circ \Phi_j\right\|_{W^{s,p}(V'_j \cap \Phi_j^{-1}(\Gamma \cap V_j))}$$

which is equivalent when $0 < s < 1$ to the norm

$$\left(\|u\|^p_{L^p(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{N-1+sp}} d\sigma_x d\sigma_y\right)^{1/p}.$$

We are now in position to extend Theorem 1.11 to the case where $\mathbb{R}^{N-1}$ is replaced by an $N-1$-dimensional manifold of $\mathbb{R}^N$, but which is sufficiently regular. This simply uses changes of variables.

If locally Γ is represented by the pair (V, φ) with φ and φ^{-1} Lipschitz, then a unit outward normal vector can be defined as follows:

$$\text{for } y' \in V', \quad \boldsymbol{\nu}(y', \varphi(y')) = \frac{(-\nabla'\varphi(y'), 1)}{\sqrt{1 + |\nabla'\varphi(y')|^2}}.$$

One can then extend this vector in all V by setting

$$\boldsymbol{\nu}(y', y_N) = \boldsymbol{\nu}(y', \varphi(y')), y \in V.$$

As $\Gamma \subset \cup_{j=1}^J V_j$, we know that there exist functions $\mu_0, \mu_1, \ldots, \mu_J \in \mathscr{C}^\infty(\mathbb{R}^N)$ such that

(i) for all $j = 0, \ldots, J, \quad 0 \le \mu_j \le 1 \quad \text{and} \quad \sum_{j=1}^{J} \mu_j = 1$

(ii) supp μ_j is compact and supp $\mu_j \subset V_j$ for any $j \ge 1$ and supp $\mu_0 \subset \Omega$.

This partition of unity then allows to extend $\boldsymbol{\nu}$ in a neighborhood of $\overline{\Omega}$ as follows: $\boldsymbol{\nu} = \sum_{j=0}^{J}(\mu_j \boldsymbol{\nu})$. It is then easy to verify that $\boldsymbol{\nu} \in L^\infty(\overline{\Omega})$ if Γ is Lipschitz and $\boldsymbol{\nu} \in \mathscr{C}^{k-1,1}(\overline{\Omega})$ if Γ is $\mathscr{C}^{k,1}$.

We are now ready to establish the following result:

Theorem 1.14 (Traces) *Let Ω be an open subset of $\mathbb{R}^N$ of class $\mathscr{C}^{k,1}$, with $k \in \mathbb{N}$. Let $s > 0$ satisfying $s \le k + 1$ and $s - 1/p = \ell + \sigma$ with $0 < \sigma < 1$ and $\ell \in \mathbb{N}$. Then the mapping*

$$u \stackrel{\gamma}{\longmapsto} (\gamma_0 u, \gamma_1 u, \ldots, \gamma_\ell u)$$

defined for $\mathscr{C}^{k,1}$ has a unique continuous extension as an operator from $W^{s,p}(\Omega)$ into $\prod_{j=0}^{\ell} W^{s-j-1/p,p}(\Gamma)$ where

$$\gamma_1 u = \frac{\partial u}{\partial \boldsymbol{\nu}} = \nabla u \cdot \boldsymbol{\nu}, \quad \gamma_j u = \frac{\partial^j u}{\partial \boldsymbol{\nu}^j}.$$

Moreover this operator has a right continuous inverse R (not depending of p).

Case Ω Lipschitz. Suppose $1/p < s \leq 1$. We have the following properties:

(i) If $u \in W^{s,p}(\Omega)$, then $u_{|\Gamma} \in W^{s-1/p,p}(\Gamma)$.
(ii) If $g \in W^{s-1/p,p}(\Gamma)$, then there exists $u \in W^{s,p}(\Omega)$ such that $u = g$ on Γ and satisfying the estimate

$$\|u\|_{W^{s,p}(\Omega)} \leq C \ \|g\|_{W^{s-1/p,p}(\Gamma)}.$$

Case Ω of class $\mathscr{C}^{1,1}$.

(i) Let $u \in W^{s,p}(\Omega)$. If $1/p < s \leq 2$, then $u_{|\Gamma} \in W^{1-1/p}(\Gamma)$. Moreover, for any $g \in W^{s-1/p,p}(\Gamma)$, there exists $u \in W^{s,p}(\Omega)$ such that $u = g$ on Γ, with

$$\|u\|_{W^{s,p}(\Omega)} \leq C \ \|g\|_{W^{s-1/p,p}(\Gamma)}.$$

(ii) Let $u \in W^{s,p}(\Omega)$. If $1 + 1/p < s \leq 2$, then $\frac{\partial u}{\partial \boldsymbol{\nu}} \in W^{s-1-1/p,p}(\Gamma)$. Moreover, for any $g_0 \in W^{s-1/p,p}(\Gamma)$ and $g_1 \in W^{s-1-1/p,p}(\Gamma)$, there exists $u \in W^{s,p}(\Omega)$ such that

$$u = g_0 \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = g_1 \quad \text{on } \Gamma$$

with

$$\|u\|_{W^{s,p}(\Omega)} \leq C \ \left(\|g_0\|_{W^{s-1/p,p}(\Gamma)} + \|g_1\|_{W^{s-1-1/p,p}(\Gamma)}\right).$$

Theorem 1.15 *Suppose that Ω is an open subset of $\mathbb{R}^N$ of class $\mathscr{C}^{k,1}$, with $k \in \mathbb{N}$. Let $s > 0$ such that $s - 1/p \notin \mathbb{N}$ and $s - 1/p = \ell + \sigma$, where $0 < \sigma < 1$ and $\ell \geq 0$ is an integer. Then we have the following characterization for $s \leq k + 1$:*

$$W_0^{s,p}(\Omega) = \left\{u \in W^{s,p}(\Omega);\, \gamma_0 u = \gamma_1 u = \ldots = \gamma_\ell u = 0\right\}.$$

1.4 Interpolation

We will consider here only the case of spaces $H^s(\Omega)$, with Ω bounded open Lipschitz of $\mathbb{R}^N$.

Recall that for every $s > 0$ there exists a continuous linear operator:

$$P : H^s(\Omega) \longrightarrow H^s(\mathbb{R}^N)$$

satisfying

$$\forall u \in H^s(\Omega), \quad Pu_{|\Omega} = u.$$

Theorem 1.16 *[Interpolation Inequality] Let s_1, s_2, s_3 with $0 \le s_1 < s_2 < s_3$. Then*

$$\forall \varepsilon > 0, \quad \|u\|_{W^{s_2,p}(\Omega)} \le \varepsilon \|u\|_{W^{s_3,p}(\Omega)} + K\varepsilon^{-\frac{s_2-s_1}{s_3-s_2}} \|u\|_{W^{s_1,p}(\Omega)},$$

where $K = K(\Omega, s_1, s_2, s_3, p)$.

The above inequality is a consequence of the compactness of the embedding of $W^{s_3,p}(\Omega)$ into $W^{s_2,p}(\Omega)$.

Recall now that we have different ways to define the Sobolev space $H^m(\Omega)$, for $m \in \mathbb{N}$:

$$\begin{aligned} &u \in H^m(\Omega) \iff \forall |\alpha| \le m,\ D^\alpha u \in L^2(\Omega),\\ &u \in H^m(\Omega) \iff u = U_{|\Omega} \text{ with } U \in H^m(\mathbb{R}^N),\\ &u \in H^m(\mathbb{R}^N) \iff u \in \mathscr{S}'(\mathbb{R}^N) \quad \text{and} \quad (1+|\xi|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^N). \end{aligned} \tag{9}$$

In the case of fractional Sobolev spaces $H^s(\Omega)$, with $s = m+\sigma$, $m \in \mathbb{N}$, $0 < \sigma < 1$, we have:

$$\begin{aligned} &u \in H^s(\Omega) \iff u \in H^m(\Omega) \quad \text{and} \quad \forall |\alpha| = m,\ \int_\Omega\int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{N+2\sigma}} < \infty\\ &u \in H^s(\Omega) \iff u = U_{|\Omega} \quad \text{with} \quad U \in H^s(\mathbb{R}^N),\\ &u \in H^s(\mathbb{R}^N) \iff u \in \mathscr{S}'(\mathbb{R}^N) \quad \text{and} \quad (1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^N). \end{aligned} \tag{10}$$

We can also get this space by interpolation:

$$H^s(\Omega) = \Big[H^m(\Omega), L^2(\Omega)\Big]_\mu,\ 0 < \mu < 1 \quad (1-\mu)m = s$$

and more generally we have for any $0 < \mu < 1$

$$\big[H^{s_1}(\Omega), H^{s_2}(\Omega)\big]_\mu = H^{(1-\mu)s_1+\mu s_2}(\Omega).$$

Concerning the interpolation of spaces $H_0^m(\Omega)$, we have:

$$\big[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)\big]_\mu = H_0^{(1-\mu)s_1+\mu s_2}(\Omega) \quad \text{if} \quad (1-\mu)s_1 + \mu s_2 \notin \frac{1}{2} + \mathbb{N}$$

and

$$\big[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)\big]_\mu = \widetilde{H}^{(1-\mu)s_1+\mu s_2}(\Omega) \quad \text{otherwise},$$

with equivalent norms.

1.5 Transposition

Let V and H be two Hilbert spaces on $\mathbb{R}$ and $A \in \mathscr{L}(V, H)$. For every fixed $g \in H'$, we consider the following mapping

$$\begin{aligned} V &\longrightarrow \mathbb{R} \\ x &\longmapsto \langle g, Ax \rangle_{H' \times H} \end{aligned}$$

which defines a linear and continuous form on V that we denote by ${}^t Ag$:

$$\langle {}^t Ag, x \rangle_{V' \times V} = \langle g, Ax \rangle_{H' \times H}.$$

Remark 3 If $A : V \longrightarrow H$ is an isomorphism, then we can define the transpose of A^{-1} and we easily verify that

$${}^t A^{-1} = \left({}^t A\right)^{-1} \quad \text{and} \quad {}^t A : H' \longrightarrow V' \text{ is an isomorphism.}$$

1.6 Inequalities

They are fundamental tools in the study of partial differential equations:

(i) **Poincaré's Inequality.** Let Ω be an open space bounded in at least one direction. Then there exists a constant $C \geq 0$, depending on the diameter of Ω such that

$$\forall u \in W_0^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \leq C \ \|\nabla u\|_{L^p(\Omega)}.$$

(ii) **Poincaré-Wirtinger's Inequality.** Let Ω be a Lipschitz bounded domain of $\mathbb{R}^N$. Then there exists a constant $C(\Omega) \geq 0$ such that

$$\forall u \in W^{1,p}(\Omega), \quad \inf_{K \in \mathbb{R}} \|u + K\|_{L^p(\Omega)} \leq C(\Omega) \ \|\nabla u\|_{L^p(\Omega)}.$$

(iii) **Hardy's Inequality.** Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^N$. Then there exists a constant $C(\Omega) \geq 0$ such that

$$\forall u \in W_0^{1,p}(\Omega), \quad \left\| \frac{u}{\varrho} \right\|_{L^p(\Omega)} \leq C(\Omega) \ \|\nabla u\|_{L^p(\Omega)}.$$

(iv) **Calderòn–Zygmund's Inequality.**

$$\forall u \in \mathscr{D}(\Omega), \quad \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(\Omega)} \leq C(\Omega) \ \|\Delta u\|_{L^p(\Omega)}.$$

1.7 Weak Solutions

Consider the following problems:

$$(P_D) \quad -\Delta u = f \quad \text{in } \Omega \qquad \text{and} \quad u = g \quad \text{on } \Gamma$$

and

$$(P_N) \quad -\Delta u = f \quad \text{in } \Omega \qquad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = h \quad \text{on } \Gamma,$$

where Ω is a Lipschitz bounded domain of $\mathbb{R}^N$, f, g, and h are given.

Theorem 1.17 *Given any $f \in H^{-1}(\Omega)$ and any $g \in H^{1/2}(\Gamma)$, there exists a unique solution $u \in H^1(\Omega)$ to Problem (P_D). Moreover*

$$\|u\|_{H^1(\Omega)} \leq C(\Omega) \left(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\right).$$

Proof Using Theorem 1.14, there exists $u_g \in H^1(\Omega)$ such that

$$u_g = g \quad \text{on } \Gamma \quad \text{with} \quad \left\|u_g\right\|_{H^1(\Omega)} \leq C(\Omega)\, \|g\|_{H^{1/2}(\Gamma)}.$$

Setting

$$f_g = -\Delta u_g = -\text{div}\, \nabla u_g \in H^{-1}(\Omega),$$

the problem becomes: Find $v \in H_0^1(\Omega)$ solution of

$$(P_D^0) \quad -\Delta v = f - f_g \quad \text{in } \Omega \qquad \text{and} \quad v = 0 \quad \text{on } \Gamma.$$

This last problem is equivalent to the following variational formulation:

$$(FV)_D \begin{cases} \text{Find } v \in H_0^1(\Omega) \text{ such that} \\ \forall \varphi \in H_0^1(\Omega), \quad \displaystyle\int_\Omega \nabla v \cdot \nabla \varphi dx = \langle f - f_g, \varphi \rangle_{H^{-1}(\Omega \times H_0^1(\Omega)}. \end{cases}$$

Applying Lax–Milgram Lemma or Riesz Theorem, we prove the existence of a unique solution $v \in H_0^1(\Omega)$ satisfying $(FV)_D$.

Note that the bilinear form

$$a(v, \varphi) = \int_\Omega \nabla v \cdot \nabla \varphi dx$$

is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ and coercive on $H_0^1(\Omega)$ thanks to Poincaré's inequality. In addition, this form allows to define a scalar product on Hilbert's space $H_0^1(\Omega)$. □

Remark 4

(i) If Ω is of class $\mathscr{C}^1$, $f \in W^{-1,p}(\Omega)$ and $g \in W^{1-1/p,p}(\Gamma)$ with $1 < p < \infty$, then there exists a unique solution $u \in W^{1,p}(\Omega)$ to (P_D).
(ii) When Ω is only Lipschitz, this regularity result holds for $p \in]2-\varepsilon', 2+\varepsilon[$ where ε and $\varepsilon' > 0$ are depending on Ω and $2-\varepsilon'$ and $2+\varepsilon$ are conjugate.

Concerning the Neumann problem, the approach is a bit more complicated. Indeed, if we are looking for a solution $u \in H^1(\Omega)$ only, the boundary condition on the normal derivative does not make sense, since the functions of $L^2(\Omega)$ do not have any trace at the boundary. Here, in fact, if one set $\boldsymbol{v} = \nabla u$ we have

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \boldsymbol{v} \cdot \boldsymbol{\nu} \text{ on } \Gamma.$$

Definition 1.18

$$H(\text{div};\ \Omega) = \Big\{ \boldsymbol{v} \in L^2(\Omega);\ \text{div}\, \boldsymbol{v} \in L^2(\Omega) \Big\}.$$

It is a Hilbert space for the scalar product

$$((\boldsymbol{v}, \boldsymbol{w}))_{H(\text{div};\ \Omega)} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} dx + \int_\Omega (\text{div}\, \boldsymbol{v})(\text{div}\, \boldsymbol{w}) dx.$$

Proposition 1.19

(i) The space $\mathscr{D}(\overline{\Omega})$ is dense in $H(\text{div};\ \Omega)$.
(ii) The linear mapping

$$\boldsymbol{v} \longmapsto \boldsymbol{v} \cdot \boldsymbol{\nu},$$

defined on $\mathscr{D}(\overline{\Omega})^N$, can be uniquely extended into a linear mapping of $H(div; \Omega)$ in $H^{-1/2}(\Gamma) := \big[H^{1/2}(\Gamma)\big]'$.
(iii) In addition, we have the following Green's formula (or Stokes' formula):

$$\forall \varphi \in H^1(\Omega),\ \forall \boldsymbol{v} \in H(\text{div};\ \Omega), \quad \int_\Omega \boldsymbol{v} \cdot \nabla \varphi\, dx + \int_\Omega \varphi\, \text{div}\, \boldsymbol{v}\, dx = \langle \boldsymbol{v} \cdot \boldsymbol{\nu}, \varphi \rangle_\Gamma$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality brackets $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Corollary 1.20 *Let $u \in H^1(\Omega)$ be such that $\Delta u \in L^2(\Omega)$. Then $\frac{\partial u}{\partial \boldsymbol{\nu}} \in H^{-1/2}(\Gamma)$. Moreover for any $\varphi \in H^1(\Omega)$, we have the following Green formula:*

$$\int_{\Omega} \varphi \Delta u \, dx + \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \langle \frac{\partial u}{\partial \boldsymbol{\nu}}, \varphi \rangle_{\Gamma}.$$

Proof It suffices to apply Proposition 1.19 by setting $\boldsymbol{v} = \nabla u$. □

As a Consequence we can show that for any $f \in L^2(\Omega)$ and for any $g \in H^{-1/2}(\Gamma)$, the problems

$$(P_N) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ -\Delta u = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = g \quad \text{on } \Gamma \end{cases}$$

and

$$(Q_N) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \forall \varphi \in H^1(\Omega), \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx + \langle g, \varphi \rangle_{\Gamma} \end{cases}$$

are equivalent, so that any solution of one is a solution of the other.

Remark 5

(i) The open Ω being bounded, the constant functions belong to $H^1(\Omega)$. So that if u is a solution of (Q_N), taking $\varphi = 1$, the data f and g must satisfy the (necessary) compatibility condition:

$$\int_{\Omega} f \, dx + \langle g, 1 \rangle_{\Gamma} = 0.$$

(ii) The implication $(P_N) \Longrightarrow (Q_N)$ results from Corollary 1.20. The reverse implication also uses Green's formula and the surjectivity of the trace operator of $H^1(\Omega)$ into $H^{1/2}(\Gamma)$.

Theorem 1.21 *Let Ω be a bounded, connected, and Lipschitzian open of $\mathbb{R}^N$, with $N \geq 2$. Let $f \in L^2(\Omega)$, $g \in H^{-1/2}(\Gamma)$ satisfying the compatibility condition*

$$\int_{\Omega} f \, dx + \langle g, 1 \rangle_{\Gamma} = 0.$$

Then Problem (P_N) has a solution $H^1(\Omega)$, unique to an additive constant, verifying the estimate:

$$\|\nabla u\|_{L^2(\Omega)} \leq C(\Omega) \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma)} \right).$$

Proof According to Poincaré-Wirtinger's inequality, we have

$$\inf_{K \in \mathbb{R}} \|u + K\|_{H^1(\Omega)} \leq C(\Omega) \, \|\nabla u\|_{L^2(\Omega)}.$$

So that the bilinear form

$$a(u, \varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx$$

is coercive on the quotient space $V = H^1(\Omega)/_{\mathbb{R}}$. It is then sufficient to apply Lax–Milgram on the Hilbert space V. □

Remark 6

(i) We could have chosen as space V the space $H^1(\Omega) \cap L^2_0(\Omega)$ where

$$L^2_0(\Omega) = \left\{ v \in L^2(\Omega);\ \int_\Omega v \, dx = 0 \right\},$$

which is a Hilbert space and then use the inequality:

$$\forall\, v \in H^1(\Omega) \cap L^2_0(\Omega), \quad \|v\|_{H^1(\Omega)} \leq C \; \|\nabla v\|_{L^2(\Omega)} .$$

(ii) We could have taken f in a space larger than $L^2(\Omega)$. More precisely if $f \in L^{(2^*)'}(\Omega)$, where $(2^*)'$ is the conjugate of 2^* defined by

$$\frac{1}{2^*} = \begin{cases} \frac{1}{2} - \frac{1}{N} & \text{if } N \geq 3 \\ \varepsilon > 0 & \text{arbitrary if } N = 2, \end{cases}$$

i.e., $(2^*)' = \frac{2N}{N+2}$ if $N \geq 3$ and $(2^*)' > 1$ if $N = 2$.

(iii) In L^p-theory, we have existence results in $W^{1,p}(\Omega)$ when Ω is $\mathscr{C}^1$ and $1 < p < \infty$ or when Ω is $\mathscr{C}^{0,1}$ and $2 - \varepsilon' < p < 2 + \varepsilon$.

In the same spirit, we can consider the case of Fourier-Robin boundary condition:

$$(P_{FR}) \begin{cases} \text{Find } u \in H^1(\Omega) \\ -\Delta u = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} + \alpha u = g \quad \text{on } \Gamma, \end{cases}$$

where α is a positive function defined on Γ, which can be formulated in an equivalent way by:

$$(Q_{FR}) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \forall\, \varphi \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Gamma \alpha u \varphi \, dx = \int_\Omega f \varphi \, dx + \langle g, \varphi \rangle_\Gamma . \end{cases}$$

1.8 Strong Solutions

Theorem 1.22 *Let Ω be a bounded open of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^N$. Let $f \in L^2(\Omega)$ and $g \in H^{/2}(\Gamma)$. Then the solution u given by Theorem 1.17 belongs to $H^2(\Omega)$ and verifies the estimate:*

$$\|u\|_{H^2(\Omega)} \leq C(\Omega) \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Gamma)}\right).$$

Proof Firstly, we note that

$$L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \quad \text{and} \quad H^{3/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$$

so that the problem (P_D) has a unique solution $u \in H^1(\Omega)$.

We shift the data $g \in H^{3/2}(\Gamma)$ by $u_g \in H^2(\Omega)$ and we set again $u = v + u_g$, so that $v \in H^1(\Omega)$ vérifies:

$$\begin{cases} -\Delta v = f + \Delta u_g \in L^2(\Omega), \\ v = 0 \quad \text{on } \Gamma. \end{cases}$$

So, we need to show that $v \in H^2(\Omega)$. One of the methods to establish this regularity consists in using the technique of the differential quotients.

The complete proof being long and tedious, we will admit it. □

Remark 7 We can also establish the existence of solutions in $W^{2,p}(\Omega)$ when the data f and g verify:

$$f \in L^p(\Omega) \quad \text{and} \quad g \in W^{2-1/p,p}(\Gamma)$$

and the domain Ω is of class $\mathscr{C}^{1,1}$.

1.9 Very Weak Solutions

We assume here that Ω is a bounded open of class $\mathscr{C}^{1,1}$ and we are interested in the homogeneous problem

$$(P_D^H) \begin{cases} \text{Find } u \in L^2(\Omega) \\ -\Delta u = 0 \quad \text{in } \Omega, \\ u = g \quad \text{on } \Gamma, \end{cases}$$

where $g \in H^{-1/2}(\Gamma)$.

Remark 8 As the function u belongs "only" to $L^2(\Omega)$, the boundary condition $u = g$ on Γ has *a priori* no sense. But we will see that in fact, we can make sense of the trace of a harmonic function in $L^2(\Omega)$ and (we can in fact weaken this last hypothesis).

Lemma 1.23

(i) The space $\mathscr{D}(\overline{\Omega})$ is dense in the space

$$E(\Omega;\,\Delta) = \left\{ v \in L^2(\Omega);\ \ \Delta v \in L^2(\Omega) \right\}.$$

(ii) The mapping $v \longmapsto v_{|\Gamma}$ defined on $\mathscr{D}(\overline{\Omega})$ can be uniquely extended into a continuous linear mapping of $E(\Omega;\,\Delta)$ into $H^{-1/2}(\Gamma)$.

(iii) In addition, we have the following Green's formula:

$$\begin{cases} \forall\, v \in E(\Omega;\,\Delta), \quad \forall\, \varphi \in H^2(\Omega) \cap H_0^1(\Omega) \\ \displaystyle\int_\Omega v \Delta \varphi \, dx - \int_\Omega \varphi \Delta v \, dx = \langle v,\, \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}. \end{cases}$$

Proof

(i) The idea is to use the Hahn–Banach theorem. So let $\ell \in [E(\Omega;\,\Delta)]'$ vanishing on $\mathscr{D}(\overline{\Omega})$ and show that it cancels on $E(\Omega;\,\Delta)$.

We know that there exist $(f, g) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$\forall\, v \in E(\Omega;\,\Delta), \quad \langle \ell, v \rangle = \int_\Omega f v \, dx + \int_\Omega g \Delta v \, dx.$$

Let $\widetilde{f}$ and $\widetilde{g}$ the extensions by 0 outside of Ω of f and g, respectively. Then, for any $v \in \mathscr{D}(\mathbb{R}^N)$

$$\langle \ell,\ v_{|\Omega} \rangle = \int_\Omega f v \, dx + \int_\Omega g \Delta v \, dx = \int_{\mathbb{R}^N} \widetilde{f} v \, dx + \int_{\mathbb{R}^N} \widetilde{g} \Delta v \, dx,$$

i.e.,

$$\Delta \widetilde{g} = -\widetilde{f} \text{ in } \mathbb{R}^N.$$

As $\widetilde{g} \in L^2(\mathbb{R}^N)$ and $\Delta \widetilde{g} \in L^2(\mathbb{R}^N)$, then $\widetilde{g} \in H^2(\mathbb{R}^N)$. Therefore, $g \in H^2(\Omega)$. The extension $\widetilde{g}$, by 0 outside of Ω, belongs to $H^2(\mathbb{R}^N)$. We know then that $g \in H_0^2(\Omega)$. By definition, there exists a sequence $(g_k)_k$ of functions of $\mathscr{D}(\Omega)$ such that $g_k \longrightarrow g$ in $H^2(\Omega)$.

Finally, let $v \in E(\Omega;\,\Delta)$. So,

$$\langle \ell, v \rangle = \lim_{k \Longrightarrow \infty} \left[\int_\Omega -v \Delta v_k \, dx + \int_\Omega g_k \Delta v \, dx \right] = \lim_{k \Longrightarrow \infty} 0 = 0.$$

(ii) Let $v \in \mathscr{D}(\overline{\Omega})$ fixed and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Then

$$\int_\Omega v \Delta\varphi \, dx - \int_\Omega \varphi \Delta v \, dx = \int_\Gamma v \frac{\partial \varphi}{\partial \boldsymbol{\nu}}.$$

Now let $\mu \in H^{1/2}(\Gamma)$. According to the trace theorem and since Ω is of class $\mathscr{C}^{1,1}$, there exists $\varphi \in H^2(\Omega)$ verifying

$$\begin{cases} \varphi = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{\nu}} = \mu \quad \text{on } \Gamma, \\ \|\varphi\|_{H^2(\Omega)} \leq C \ \|\mu\|_{H^{1/2}(\Gamma)} . \end{cases}$$

Thus, using the Cauchy–Schwarz inequality

$$\left|\langle v, \mu\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\right| = \left|\int_\Gamma v\mu\right| = \left|\int_\Gamma v \frac{\partial \varphi}{\partial \boldsymbol{\nu}}\right|$$

$$\leq C(\Omega) \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)}\right)^{1/2} \ \|\varphi\|_{H^2(\Omega)}$$

$$\leq C(\Omega) \, \|v\|_{E(\Omega;\Delta)} \, \|\mu\|_{H^{1/2}(\Gamma)} .$$

This shows that the linear mapping

$$\begin{aligned} \mathscr{D}(\overline{\Omega}) &\longrightarrow H^{-1/2}(\Gamma) \\ v &\longmapsto v_{|\Gamma} \end{aligned}$$

is continuous when $\mathscr{D}(\overline{\Omega})$ is equipped with the norm of $E(\Omega; \Delta)$. We finish the proof by using the density of $\mathscr{D}(\overline{\Omega})$ in $E(\Omega; \Delta)$.

(iii) Immediate.

□

Theorem 1.24 *Let Ω be a bounded open of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^N$ and let $g \in H^{-1/2}(\Gamma)$. Then, the problem (P_D^0) has a unique solution $u \in L^2(\Omega)$ verifying the estimate*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) \ \|g\|_{H^{-1/2}(\Gamma)} .$$

Proof From Green's formula above, it is easy to see that $u \in L^2(\Omega)$ is a solution of the problem (P_D^0) if and only if

$$\forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega), \qquad \int_\Omega u \Delta\varphi \, dx = \langle g, \frac{\partial \varphi}{\partial \boldsymbol{\nu}}\rangle_\Gamma. \tag{11}$$

Indeed, let $u \in L^2(\Omega)$ be a solution of (P_D^0). Green's formula implies that (11) takes place.

Conversely, let $u \in L^2(\Omega)$ be a solution of (11). Then, for all $\varphi \in \mathscr{D}(\Omega)$, we have

$$0 = \int_\Omega u \Delta\varphi \, dx = \langle \Delta u, \varphi \rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)},$$

i.e.,

$$\Delta u = 0 \quad \text{in } \Omega. \tag{12}$$

Let now $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. From (12) and Green's formula above, we deduce successively that:

$$0 = \int_\Omega \varphi \Delta u \, dx = \int_\Omega u \Delta\varphi \, dx - \langle u, \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma$$

then

$$\langle u, \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma = \langle g, \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma.$$

From the surjectivity of the trace mapping $v \mapsto (v_{|\Gamma}, \frac{\partial v}{\partial \boldsymbol{\nu}})$ from $H^2(\Omega)$ into $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ we know that

$$\forall \mu \in H^{1/2}(\Gamma), \quad \langle u, \mu \rangle_\Gamma = \langle g, \mu \rangle_\Gamma,$$

i.e., $u = g$ in $H^{-1/2}(\Gamma)$. □

Remark 9 A similar result can be established for the Neumann problem (P_N^0) with boundary data h in $H^{-3/2}(\Gamma)$ and satisfying the compatibility condition $\langle h, 1 \rangle_\Gamma = 0$.

1.10 Solutions in $H^s(\Omega)$, with $0 < s < 2$

We have established in the previous paragraphs the existence of solutions in $H^1(\Omega)$, $H^2(\Omega)$, and $L^2(\Omega)$ under generally optimal assumptions (except for the Neumann problem).

We will now consider the case of solutions in $H^s(\Omega)$ with $0 < s < 2$ and $s \neq 1$. The main ingredient is to use interpolation (complex here).

Theorem 1.25 *Let Ω be a bounded open of class $\mathscr{C}^{1,1}$.*

(i) Suppose that $\frac{1}{2} < s < 2$. Then the operators

$$\begin{aligned}
&\Delta : H^s(\Omega) \cap H_0^1(\Omega) \longrightarrow H^{s-2}(\Omega) = \left[H_0^{2-s}(\Omega)\right]' \quad \text{if } 1 < s < 2 \text{ and } s \neq \tfrac{3}{2},\\
&\Delta : H_0^{3/2}(\Omega) \longrightarrow \left[H_{00}^{1/2}(\Omega)\right]',\\
&\Delta : H_0^{2-s}(\Omega) \longrightarrow H^{-s}(\Omega) = \left[H_0^s(\Omega)\right]' \quad \text{if } 1 < s < \tfrac{3}{2},
\end{aligned} \tag{13}$$

are isomorphisms.

(ii) For any $g \in H^s(\Gamma)$, with $-\frac{1}{2} < s < \frac{3}{2}$, Problem (P_D^H) has a unique solution $u \in H^{s+\frac{1}{2}}(\Omega)$.

Remark 10 What happens if Ω is only Lipschitz? For what values of s can we have $u \in H^s(\Omega)$?

2 The Stokes Problem with Various Boundary Conditions

We are interested here in the study of the Stokes problem:

$$(S)\begin{cases} \text{Find } (\boldsymbol{u}, \pi) \quad \text{satisfying}\\ -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{in } \Omega,\\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \end{cases}$$

with one of the following boundary conditions on Γ:

(i) $\boldsymbol{u} = \boldsymbol{0}$ (Dirichlet boundary condition)
(ii) $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ and $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \boldsymbol{0}$ (Navier type boundary condition)
(iii) $\boldsymbol{u} \cdot \boldsymbol{\nu} = 0$ and $(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu} + \alpha \boldsymbol{u}_\tau = \boldsymbol{0}$ (Navier boundary condition)
(iv) $\boldsymbol{u} \times \boldsymbol{\nu} = \boldsymbol{0}$ and $\pi = \pi_0$ (pressure boundary condition).

Here $\boldsymbol{u}$ denotes the velocity field, π the pressure field, Ω a connected bounded open set we assume at least Lipschitz.

Recall that

$$\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u}, \quad \mathbf{curl}\, \boldsymbol{u} = \nabla \times \boldsymbol{u} \quad \text{and} \quad \mathbb{D}\boldsymbol{u} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right).$$

The notation $\boldsymbol{u}_\tau$ denotes the tangential component of $\boldsymbol{u}$: $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$. Finally $\boldsymbol{f}$ and α are given on Ω and Γ, respectively.

Remark 11

(i) We limit ourselves here, with the exception of pressure, to the case of homogeneous boundary conditions.
(ii) If the boundary of Ω is flat (like a cube, for example, or half space), the above boundary conditions are more easily written. When $\Omega = \mathbb{R}^3_+$, the Navier type boundary condition is equivalent to:

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0$$

and that of Navier at:

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial x_3} - \alpha u_1 = \frac{\partial u_2}{\partial x_3} - \alpha u_2 = 0.$$

2.1 The Problem (*S*) *with Dirichlet Boundary Condition*

As for the Laplace equation with the Dirichlet boundary condition, we will assume

$$\boldsymbol{f} \in H^{-1}(\Omega)^3$$

and so look for $\boldsymbol{u} \in H_0^1(\Omega)^3$ verifying (S). Here we have in addition the constraint

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega$$

and the Lagrange multiplier π. First of all, as π must verify

$$\nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{u} \in H^{-1}(\Omega)^3$$

it is, therefore, reasonable to look for π in $L^2(\Omega)$. Moreover, it is easy to verify that such π satisfies:

$$\forall\, \boldsymbol{v} \in H_0^1(\Omega)^3, \quad \langle \nabla \pi, \boldsymbol{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = -\int_\Omega \pi \operatorname{div} \boldsymbol{v}\, dx.$$

The space

$$V = \left\{ \boldsymbol{v} \in H_0^1(\Omega)^3;\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \right\}$$

being a subspace of $H_0^1(\Omega)^3$ is, therefore, a Hilbert space. Moreover

$$\forall\, \boldsymbol{v} \in V, \quad \langle \nabla \pi, \boldsymbol{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0.$$

We are now able to propose a variational formulation of Problem (S):

$$(P_D^0) \begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that} \\ \forall\, \boldsymbol{v} \in V, \quad \displaystyle\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \end{cases}$$

where we note that the pressure π has “disappeared.”

Lemma 2.1 *The problem*

$$(S_D^0)\begin{cases}\textit{Find } (\boldsymbol{u},\pi)\in H_0^1(\Omega)^3\times L^2(\Omega)\\ -\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}\quad \textit{in } \Omega,\\ \operatorname{div}\boldsymbol{u}=0\quad \textit{in } \Omega\end{cases}$$

is equivalent to the problem (P_D^0).

Proof The implication $(S_D^0)\Longrightarrow(P_D^0)$ is immediate. Conversely, let $\boldsymbol{u}$ be a solution of (P_D^0). Then, in particular,

$$\forall\,\boldsymbol{v}\in\mathscr{D}(\Omega)^3\quad\text{such that}\quad\operatorname{div}\boldsymbol{v}=0\quad\text{in }\Omega,$$

we have

$$\langle-\Delta\boldsymbol{u}-\boldsymbol{f},\boldsymbol{v}\rangle_{\mathscr{D}'(\Omega)^3\times\mathscr{D}(\Omega)^3}=0. \tag{14}$$

As $-\Delta\boldsymbol{u}-\boldsymbol{f}\in H^{-1}(\Omega)^3$ and the space

$$\mathcal{V}(\Omega)=\left\{\boldsymbol{v}\in\mathscr{D}(\Omega)^3;\,\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega\right\}$$

is dense in the space V, then the relation (14) takes place for all $\boldsymbol{v}$. Then we know that there exists $\pi\in L^2(\Omega)$, unique up to an additive constant, because Ω is connected, such that

$$-\Delta\boldsymbol{u}-\boldsymbol{f}=\nabla(-\pi)\quad\text{in }\Omega$$

(this result is called "De Rham's version of the theorem" in $H^{-1}(\Omega)^N$). And finally, as $\boldsymbol{u}\in V$, then

$$\operatorname{div}\boldsymbol{u}=0\text{ in }\Omega\quad\text{and}\quad\boldsymbol{u}=\boldsymbol{0}\quad\text{on }\Gamma.$$

This ends the proof of the lemma. □

Theorem 2.2 *For any* $\boldsymbol{f}\in H^{-1}(\Omega)^3$, *the Stokes problem* (P_D^0) *has a unique solution* $\boldsymbol{u}\in V$ *vérifying further*

$$\|\boldsymbol{u}\|_{H^1(\Omega)^3}\le C(\Omega)\,\|\boldsymbol{f}\|_{H^{-1}(\Omega)^3}\,.$$

Proof Simply apply Lax–Milgram theorem. □

Remark 12 The theory is well known for everything that concerns the regularity of solutions when the data are:

- solutions in $W^{1,p}(\Omega)^3 \times L^p(\Omega)$
- solutions in $W^{2,p}(\Omega)^3 \times L^p(\Omega)$

with $1 < p < \infty$.

In particular, if $\boldsymbol{f} \in L^2(\Omega)^3$ and Ω is of class $\mathscr{C}^{1,1}$, then $\boldsymbol{u} \in H^2(\Omega)^3$ and $\pi \in H^1(\Omega)$.

2.2 The Stokes Problem with Navier Type Boundary Condition

Here we are still interested in Stokes' problem, but with the following boundary condition:

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma.$$

In order to take into account this condition at the boundary, it is important to write the Laplacian operator in the form:

$$-\Delta = \mathbf{curl}\,\mathbf{curl} - \nabla\, \mathrm{div}.$$

On the other hand, if we study the existence of weak solutions $\boldsymbol{u}$ in $H^1(\Omega)^3$, it will be necessary to give a meaning to the condition at the boundary

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

Recall the following Green formulas:

(i) If $\boldsymbol{v} \in L^2(\Omega)^3$ and $\mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)^3$, then $\boldsymbol{v} \times \boldsymbol{\nu} \in H^{-1/2}(\Gamma)^3$ and

$$\forall\, \boldsymbol{\varphi} \in H^1(\Omega)^3, \quad \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx - \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx = \langle \boldsymbol{v} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality brackets $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

(ii) If $\boldsymbol{v} \in L^2(\Omega)^3$ and $\mathrm{div}\, \boldsymbol{v} \in L^2(\Omega)$, then $\boldsymbol{v} \cdot \boldsymbol{\nu} \in H^{-1/2}(\Gamma)$ and

$$\forall\, \varphi \in H^1(\Omega), \quad \int_\Omega \boldsymbol{v} \cdot \nabla \varphi\, dx + \int_\Omega \varphi\, \mathrm{div}\, \boldsymbol{v}\, dx = \langle \boldsymbol{v} \cdot \boldsymbol{\nu}, \varphi \rangle_\Gamma.$$

Remark 13 If $\boldsymbol{v} \in L^2(\Omega)^3$ and $\mathbf{curl}\, \boldsymbol{v} \in L^{6/5}(\Omega)^3$ (respectively, $\mathrm{div}\, \boldsymbol{v} \in L^{6/5}(\Omega)$), then

$$\boldsymbol{v} \times \boldsymbol{\nu} \in H^{-1/2}(\Gamma)^3 \text{ (resp. } \boldsymbol{v} \cdot \boldsymbol{\nu} \in H^{-1/2}(\Gamma))$$

and Green's formulas above remain valid.

Proposition 2.3 *Let* $\boldsymbol{v} \in L^2(\Omega)^3$ *such that* $\mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega)^3$ *and* $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \in L^{6/5}(\Omega)^3$. *Then* $\mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{\nu} \in H^{-1/2}(\Gamma)^3$ *and we have the following Green formula:*

$$\forall\, \boldsymbol{\varphi} \in H^1(\Omega)^3, \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} - \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} = \langle \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle_\Gamma.$$

Proof It suffices to put $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{v}$ and use the previous reminders. □

We are now able to propose a variational formulation for the Stokes problem (S) with the Navier type homogeneous condition. To do this, we set

$$V = \left\{ \boldsymbol{v} \in L^2(\Omega)^3;\, \mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega),\, \text{div } \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \right\}$$

equipped with the graph norm:

$$\|\boldsymbol{v}\|_V = \left(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)^3} \right)^{1/2}$$

which makes it a Hilbert space.

We suppose $\boldsymbol{f} \in L^{6/5}(\Omega)^3$ and we consider the following variational formulation:

$$(P^0_{TN}) \begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that for any } \boldsymbol{v} \in V, \\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx. \end{cases}$$

Questions

(i) Is the problem (P^0_{TN}) equivalent to the problem (S^0_{TN})?
(ii) If so, is the bilinear form

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) &\longmapsto \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx \end{aligned}$$

coercive?

Remark 14 As with the Neumann problem for the Laplacian, the boundary condition

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma$$

is "hidden" in the variational formulation.

Answers to the Above Questions

In order to study Problem (P^0_{TN}), we have to describe with more precision the geometry of the domain. We first need the following definition.

Definition 2.4 A bounded domain in $\mathbb{R}^3$ is called pseudo-$\mathscr{C}^{0,1}$ (respectively, pseudo-$\mathscr{C}^{1,1}$) if for any point $\boldsymbol{x}$ on the boundary there exists an integer $r(\boldsymbol{x})$ equal to 1 or 2 and a strictly positive real number λ_0 such that for all real numbers λ with $0 < \lambda < \lambda_0$, the intersection of Ω with the ball with center $\boldsymbol{x}$ and radius λ, has $r(\boldsymbol{x})$ connected components, each one being $\mathscr{C}^{0,1}$ (resp. $\mathscr{C}^{1,1}$).

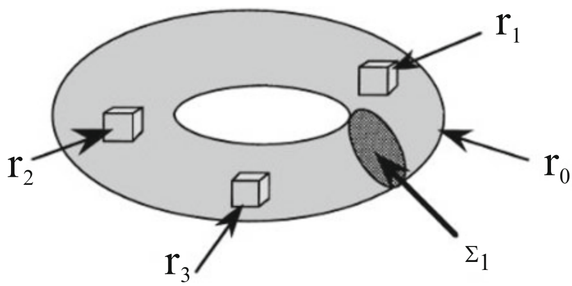
Hypothesis There exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called "*cuts*," contained in Ω, such that:

(i) each surface Σ_j is an open part of a smooth manifold $\mathcal{M}_j$,
(ii) the boundary of Σ_j is contained in $\partial\Omega$ for $1 \leq j \leq J$,
(iii) the intersection $\bar{\Sigma}_i \cap \bar{\Sigma}_j$ is empty for $i \neq j$,
(iv) the open set

$$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$$

is pseudo-$\mathscr{C}^{0,1}$ (respectively, pseudo-$\mathscr{C}^{1,1}$) simply connected.

Example for $J = 1$ and $I = 3$



Theorem 2.5 *Let Ω be a bounded open $\mathscr{C}^{1,1}$ set.*

(i) Let $\boldsymbol{v} \in L^2(\Omega)^3$ such that $div\, \boldsymbol{v} \in L^2(\Omega)$, $\mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)$ and satisfying in addition

$$\boldsymbol{v} \cdot \boldsymbol{\nu} \in H^{1/2}(\Gamma) \quad (\textit{respectively, } \boldsymbol{v} \times \boldsymbol{\nu} \in H^{1/2}(\Gamma)^3).$$

Then $\boldsymbol{v} \in H^1(\Omega)^3$ and we have the following estimates:

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega)(\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{v} \cdot \boldsymbol{\nu}\|_{H^{1/2}(\Gamma)}) \tag{15}$$

and

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega) \Big[\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{v} \times \boldsymbol{\nu}\|_{H^{1/2}(\Gamma)})\Big]. \tag{16}$$

(ii) Under the above assumptions, if in addition $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$ *on* Γ*, then we have the following estimate:*

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega) \left(\|div\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \sum_{j=1}^{J} \left| \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{\nu} \right| \right) \tag{17}$$

and if $\boldsymbol{v} \times \boldsymbol{\nu} = \mathbf{0}$ *on* Γ*, then we have the following estimate:*

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega) \left(\|div\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \sum_{i=1}^{J} \left| \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{\nu} \right| \right). \tag{18}$$

Remark 15

(i) Suppose that

$$\boldsymbol{v} \in L^2(\Omega)^3,\ \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega) \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)^3$$

with

$$\boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

Let us then extend $\boldsymbol{v}$ by $\mathbf{0}$ outside of Ω. It is easy to show that this extension verifies:

$$\widetilde{\boldsymbol{v}} \in L^2(\mathbb{R}^3)^3,\ \mathrm{div}\ \widetilde{\boldsymbol{v}} \subset L^2(\mathbb{R}^3) \quad \text{and} \quad \mathbf{curl}\, \widetilde{\boldsymbol{v}} \in L^2(\mathbb{R}^3)^3.$$

As $-\Delta = \mathbf{curl}\,\mathbf{curl} - \nabla\, \mathrm{div}$, then $\Delta \widetilde{\boldsymbol{v}} \in H^{-1}(\mathbb{R}^3)^3$ and

$$\widetilde{\boldsymbol{v}} - \Delta \widetilde{\boldsymbol{v}} \in H^{-1}(\mathbb{R}^3)^3,$$

which means that $\widetilde{\boldsymbol{v}} \in H^1(\mathbb{R}^3)^3$ and, therefore, $\boldsymbol{v} \in H^1_0(\Omega)^3$.

(ii) Now note that if $\boldsymbol{u} \in \mathscr{D}(\mathbb{R}^3)^3$, then

$$\begin{aligned}\int_\Omega |\nabla \boldsymbol{u}|^2\, dx &= -\int_{\mathbb{R}^3} \boldsymbol{u} \cdot \Delta \boldsymbol{u}\, dx = \int_{\mathbb{R}^3} [\boldsymbol{u} \cdot (\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u}) - \boldsymbol{u} \cdot \Delta \mathrm{div}\, \boldsymbol{u}]\, dx \\ &= \int_{\mathbb{R}^3} \left(|\mathbf{curl}\, \boldsymbol{u}|^2 + |\mathrm{div}\, \boldsymbol{u}|^2 \right) dx.\end{aligned}$$

Since $\mathscr{D}(\mathbb{R}^3)^3$ is dense in $H^1(\mathbb{R}^3)^3$, we deduce that:

$$\forall \boldsymbol{u} \in H^1(\mathbb{R}^3)^3, \int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 \; dx = \int_{\mathbb{R}^3} \Big(|\mathbf{curl}\, \boldsymbol{u}|^2 + |\mathrm{div}\, \boldsymbol{u}|^2 \Big) \; dx.$$

(iii) Back to point (i) of the remark: since $\boldsymbol{v} \in H^1_0(\Omega)^3$, we have:

$$\|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} = \|\nabla \widetilde{\boldsymbol{v}}\|_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \Big(|\mathbf{curl}\, \widetilde{\boldsymbol{v}}|^2 + |\mathrm{div}\, \widetilde{\boldsymbol{v}}|^2 \Big) \; dx,$$

which gives the relation

$$\int_{\Omega} |\nabla \boldsymbol{v}|^2 \; dx = \int_{\Omega} \Big(|\mathbf{curl}\, \boldsymbol{v}|^2 + |\mathrm{div}\, \boldsymbol{v}|^2 \Big) \; dx.$$

Note that this last relation can also be directly established if $\boldsymbol{v} \in \mathscr{D}(\Omega)^3$ and then, by density of $\mathscr{D}(\Omega)$ in $H^1_0(\Omega)^3$, for any $\boldsymbol{v} \in H^1_0(\Omega)^3$.

Remark 16

(i) If Ω is simply connected, then for any $\boldsymbol{v} \in H^1(\Omega)^3$ such that $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$ on Γ, the inequality (17) is written

$$\|\boldsymbol{v}\|_{H^1(\Omega)^3} \leq C(\Omega) \; \Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} \Big) \, .$$

(ii) If Γ is connected ($I = 1$), then for any $\boldsymbol{v} \in H^1(\Omega)^3$ such that $\boldsymbol{v} \times \boldsymbol{\nu} = 0$ on Γ, the inequality (18) is written

$$\|\boldsymbol{v}\|_{H^1(\Omega)^3} \leq C(\Omega) \; \Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} \Big) \, .$$

Proposition 2.6 *Let Ω be a bounded open subset of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^3$. Then the bilinear form*

$$(\boldsymbol{u}, \boldsymbol{v}) \longmapsto \int_{\Omega} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, dx$$

is coercive on the following spaces V and on W, respectively:

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^3; \; \mathrm{div}\, \boldsymbol{v} = 0 \textit{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \textit{ on } \Gamma \textit{ and } \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{\nu} = 0, 1 \leq j \leq J \right\}$$

$$W = \left\{ \boldsymbol{v} \in H^1(\Omega)^3; \mathrm{div}\, \boldsymbol{v} = 0 \textit{ in } \Omega, \boldsymbol{v} \times \boldsymbol{\nu} = \mathbf{0} \textit{ on } \Gamma \textit{ and } \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{\nu} = 0, 1 \leq j \leq I \right\}.$$

We are now able to study the problem (P^0_{TN}). We start with the simplest case where Ω is simply connected.

Theorem 2.7 *Let Ω be a bounded open domain of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^3$. Suppose that Ω is simply connected.*

(i) Then for any $\boldsymbol{f} \in L^{6/5}(\Omega)^3$, Problem (P^0_{TN}) admits a unique solution verifying the estimate

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C(\Omega)\ \|\boldsymbol{f}\|_{L^{6/5}(\Omega)}\,.$$

(ii) The problem (P^0_{TN}) is equivalent to the problem (S^0_{TN}).

(iii) If moreover Ω is of class $\mathscr{C}^{1,1}$ then the solution $(\boldsymbol{u}, \pi) \in W^{2,6/5}(\Omega)^3 \times W^{1,6/5}(\Omega)$.

Proof

(i) The open Ω being simply connected, then

$$V = \Big\{\boldsymbol{v} \in H^1(\Omega)^3;\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega,\ \ \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma\Big\}$$

and V is an Hilbert space. Then let us put

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx.$$

Proposition 2.6 shows that the form a is coercive on V. Finally, the form $\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx$ is clearly continuous because the continuous embedding $H^1(\Omega)^3 \hookrightarrow L^6(\Omega)^3$. The Lax–Milgram theorem implies the existence of a unique solution of Problem (P^0_{TN}).

(ii) Let us first show that

$$(S^0_{TN}) \Longrightarrow (P^0_{TN}).$$

Set

$$H = \Big\{\boldsymbol{v} \in L^6(\Omega)^3;\ \operatorname{div} \boldsymbol{v} \in L^2(\Omega),\ \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\Big\}.$$

We know that $\mathscr{D}(\Omega)^3$ is dense in H. So we can show that the dual of H can be characterized as follows:

$$H' = \Big\{\boldsymbol{g} + \nabla\chi;\ \boldsymbol{g} \in L^{6/5}(\Omega)^3 \text{ and } \chi \in L^2(\Omega)\Big\}$$

(similar proof to the characterization of the dual $H^{-1}(\Omega)$ of $H^1_0(\Omega)$).

Let now $(\boldsymbol{u}, \pi) \in V \times L^2(\Omega)$ solution of (S^0_{TN}). Then for any $\boldsymbol{v} \in V$

$$\langle \nabla \pi, \boldsymbol{v} \rangle_{H' \times H} = -\int_\Omega \pi \operatorname{div} \boldsymbol{v} \, dx = 0.$$

Therefore,

$$-\Delta \boldsymbol{u} = \nabla \pi - \boldsymbol{f} \in H'.$$

We need the following lemma:

Lemma 2.8

(i) The space $\mathscr{D}(\overline{\Omega})^3$ *is dense in the following space*

$$E = \left\{ \boldsymbol{v} \in H^1(\Omega)^3; \quad \Delta \boldsymbol{v} \in H' \right\}.$$

(ii) The mapping

$$\boldsymbol{v} \longmapsto \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{\nu}$$

defined on $\mathscr{D}(\overline{\Omega})^3$ *can be uniquely extended into a continuous linear mapping from* E *into* $H^{-1/2}(\Gamma)^3$.

(iii) Moreover, for any $\boldsymbol{\varphi} \in H^1(\Omega)^3$ *such that*

$$div\, \boldsymbol{\varphi} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{\varphi} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma$$

and for any $\boldsymbol{v} \in E$*, we have the following Green formula*

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{H' \times H} = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, dx + \langle \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ *denotes the duality brackets* $H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)^3$.

We return to the proof of the theorem. Since $\boldsymbol{u} \in H^1(\Omega)^3$ and $\Delta \boldsymbol{u} \in H'$, i.e., $\boldsymbol{u} \in E$, we can use this lemma to deduce on the one hand that the condition $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ has a meaning in $H^{-1/2}(\Gamma)^3$ and, on the other hand, that

$$\forall\, \boldsymbol{v} \in V, \quad \langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{H' \times H} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx,$$

i.e., $\boldsymbol{u}$ is solution of (P^0_{TN}).

Conversely, let $\boldsymbol{u} \in V$ solution of Problem (P^0_{TN}). Then

$$\operatorname{div}\, \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma$$

and

$$\forall\, \boldsymbol{v} \in \mathscr{D}(\Omega)^3 \text{ with } \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega$$

we have

$$\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3}.$$

That gives

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3}.$$

So there exists, by De Rham's theorem, a function π in $L^2(\Omega)$, unique up to an additive constant, such that

$$-\Delta \boldsymbol{u} - \boldsymbol{f} = \nabla(-\pi) \quad \text{in } \Omega \tag{19}$$

(note that $L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega)$).

It remains to show that $\boldsymbol{u}$ vérifies:

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

For that, from (19) and use the formula of Green of the first lemma, one deduces that

$$\forall\, \boldsymbol{v} \in V, \quad \langle -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{v} \rangle_{H' \times H} = \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx + \langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{v} \rangle_\Gamma$$

that is to say that

$$\forall\, \boldsymbol{v} \in V, \quad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx + \langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{v} \rangle_\Gamma = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

But $\boldsymbol{u}$ being solution of (P^0_{TN}), then

$$\forall\, \boldsymbol{v} \in V, \quad \langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{v} \rangle_\Gamma = 0.$$

Now let it be $\boldsymbol{\mu} \in H^{1/2}(\Gamma)$. We know that there exists

$$\boldsymbol{w} \in H^1(\Omega)^3, \ \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega, \ \boldsymbol{w} = \boldsymbol{\mu}_\tau \text{ on } \Gamma,$$

where $\boldsymbol{\mu}_\tau = \boldsymbol{\mu} - (\boldsymbol{\mu} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ the tangential component of $\boldsymbol{\mu}$ on Γ. As $\boldsymbol{w} \in V$, we have:

$$\langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{\mu} \rangle_\Gamma = \langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{\mu}_\tau \rangle_\Gamma = \langle \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{w} \rangle_\Gamma = 0,$$

which means that

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

(iii) The regularity $W^{1,6/5}(\Omega)$ of π is due to the fact that π satisfies:

$$\operatorname{div}(\nabla\pi - \boldsymbol{f}) = 0 \quad \text{in } \Omega \quad \text{and} \quad (\nabla\pi - \boldsymbol{f}) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma.$$

Setting $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$, the regularity $W^{2,6/5}(\Omega)^3$ of $\boldsymbol{u}$ is a consequence of the following properties:

$$\boldsymbol{z} \in L^{6/5}(\Omega)^3,\ \operatorname{div}\boldsymbol{z} = 0,\ \mathbf{curl}\,\boldsymbol{z} \in L^{6/5}(\Omega)^3 \quad \text{and} \quad \boldsymbol{z} \times \boldsymbol{\nu} = \mathbf{0} \ \text{ on } \Gamma.$$

□

Case Ω non Simply Connected
We then show that the kernel:

$$K_T(\Omega) = \left\{\boldsymbol{v} \in L^2(\Omega)^3;\ \operatorname{div}\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\right\}$$

is of finite dimension and that the dimension corresponds to the number of cuts Σ_j necessary to obtain an open set $\overset{\circ}{\Omega} = \Omega \setminus \cup_{j=1}^{J}\Sigma_j$ simply connected.

As a consequence, if

$$V = \left\{\boldsymbol{v} \in H^1(\Omega)^3; \quad \operatorname{div}\boldsymbol{v} = 0 \text{ in } \ \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\right\},$$

then, to prove that Problem (P^0_{TN}) admits a solution, it is necessary that $\boldsymbol{f}$ satisfies the following compatibility condition:

$$\forall\, \boldsymbol{v} \in K_T(\Omega), \qquad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx = 0.$$

Moreover, if such a solution $\boldsymbol{u}$ exists, it is unique up to an additive element of $K_T(\Omega)$.

2.3 *The Stokes Problem with Navier Boundary Condition*

We recall the Navier condition:

$$[2(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu}]_\tau + \alpha \boldsymbol{u}_\tau = \mathbf{0} \quad \text{on } \Gamma,$$

where

$$\mathbb{D}\boldsymbol{u} = \left(\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right)_{1 \le i,j \le 3}$$

is the deformation tensor, α defined on Γ is the friction coefficient and $\boldsymbol{u}_\tau$ is the tangential component of $\boldsymbol{u}$. To simplify, we will consider here only the case $\alpha = 0$.

Note that when div $\boldsymbol{u} = 0$ in Ω, then 2div $\mathbb{D}\boldsymbol{u} = \Delta \boldsymbol{u}$.

Lemma 2.9 *If* $(\boldsymbol{u}, \pi) \in H^1(\Omega)^3 \times L^2(\Omega)$ *is such that*

$$-\Delta \boldsymbol{u} + \nabla \pi \in L^{6/5}(\Omega)^3$$

then

$$[(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu}]_\tau \in H^{-1/2}(\Gamma)^3$$

and

$$\textit{for any } \boldsymbol{\varphi} \in H^1(\Omega)^3 \quad \textit{such that} \quad \text{div } \boldsymbol{\varphi} = 0 \quad \textit{in } \Omega \quad \textit{and} \quad \boldsymbol{\varphi} \cdot \boldsymbol{\nu} = 0 \quad \textit{on } \Gamma$$

we have the Green's formula:

$$\int_\Omega (-\Delta \boldsymbol{u} + \nabla \pi) \cdot \boldsymbol{\varphi} \, dx = 2 \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} \, dx - 2 \langle [(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu}]_\tau \,, \boldsymbol{\varphi} \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ *denotes the duality brackets* $H^{-1/2}(\Gamma)^3 \times H^{-1/2}(\Gamma)^3$.

With this Green's formula, the Stokes problem can be formulated as:

$$(P_N^0) \begin{cases} \text{Find } \boldsymbol{u} \in V, \text{ such that for any } \boldsymbol{\varphi} \in V, \\ 2 \displaystyle\int_\Gamma \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \, dx. \end{cases}$$

Set

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi} \, dx.$$

When Ω is not axisymmetric, then this form is coercive on V due to Korn's inequality:

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \backsimeq \|\mathbb{D}\boldsymbol{u}\|_{L^2(\Omega)} \,.$$

While if Ω is axisymmetric, this is not the case anymore. We must then quotient by some finite dimensional kernel.

Remark 17 In fact, on Γ we have the relation:

$$[2(\mathbb{D}\boldsymbol{u})\boldsymbol{v}]_\tau = \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{v} - \Lambda \boldsymbol{u},$$

where Λ is an operator of order 0:

$$\Lambda \boldsymbol{u} = \sum_{k=1}^{2} \left(\boldsymbol{u}_\tau \cdot \frac{\partial \boldsymbol{v}}{\partial s_k} \right) \boldsymbol{\tau}_k,$$

where $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is a base of the tangent plane to Γ at point x and (s_1, s_2) are local coordinates in this tangent plane.

This means that on the questions of regularity, they can be reduced to those concerning the Navier type condition.

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