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Kaïs Ammari
Editor

Research in PDEs and Related Fields

The 2019 Spring School, Sidi Bel Abbès, Algeria

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Preface

This volume constitutes the proceedings of the spring school: “Trends in PDE’s and Related Fields”.

This spring school was held at the University of Sidi Bel Abbès, Algeria, in the period 08–10 April 2019 (see http://conf-sidi-bel-abbes.ur-acedp.org/ for details).

The spring school consisted of two mini-courses, seven invited talks on the theme, and some short talks. This volume gathered the mini-course and the five contributions.

The invited speakers agreed to write review papers related to their contributions to the spring school, while others have written more traditional research papers, which constitute the last part of this volume. They represent recent and new works on the topic of mathematical control theory and related fields.

We believe that this volume therefore provides an accessible summary of a wide range of active research topics, along with some exciting new results, and we hope that it will prove a useful resource for both graduate students new to the area and more established researchers.

The spring school brought together internationally leading researchers and young researchers who came from all around the world. The organizers’ intention was to provide a wide angle snapshot of this exciting and fast-moving area and facilitate the exchange of ideas on recent advances in its various aspects. The numerous formal, informal, and sometimes lively discussions that resulted from this interaction were for us a sign that we achieved something in the direction of fulfilling this aim.

Our second aim was to ensure that the diffusion of these recent results was not limited to established researchers in the area and those present at the spring school but also available to newcomers and more junior members of the research community. This was reflected by the presence of many unfamiliar and/or young faces in the audience. The present proceedings should hopefully complete the fulfillment of our second aim.

This spring school would not have materialized without the help and support of the following institutions.

We are very grateful to Laboratoire d’Analyse et de Contrôle des EDP at the University of Sidi Bel Abbès and to the Research Lab ACPDE, Analysis and Control

of Partial Differential Equations, at the University of Monastir for their financial supports without whom this spring school would not be accessible without fees.

We would also like to thank all the participants of the spring school who have made this event a success, the contributors to these proceedings.

Monastir, Tunisia
May 2022

Kaïs Ammari

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Chérif Amrouche

2010 Mathematics Subject Classification 35L05, 34K35

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1.1 Sobolev Spaces

Let us introduce the following Sobolev spaces: for any $1 < p < \infty$

$$W^{m,p}(\Omega) = \left\{u \in \mathscr{D}'(\Omega);\ \forall\, |\alpha| \leq m,\ D^\alpha u \in L^p(\Omega)\right\}$$

and

$$W^{s,p}(\Omega) = \left\{u \in W^{m,p}(\Omega);\ \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+\sigma p}} < \infty,\ \forall\, |\alpha| = m\right\},$$

where $m \in \mathbb{N}$, $s = m + \sigma$, $0 < \sigma < 1$ and Ω is an open set of $\mathbb{R}^N$. Equipped with the graph norm, they are Banach spaces.

When $\Omega = \mathbb{R}^N$, using the Fourier transform, we define for any real number s the space

$$H^s(\mathbb{R}^N) = \left\{u \in \mathscr{S}'(\mathbb{R}^N);\ \int_{\mathbb{R}^N} (1 + |\xi|^2)^s\, |\hat{u}(\xi)|^2\, d\xi < \infty\right\},$$

which is an Hilbert space for the norm:

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$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}|^2 dx \right)^{1/2}.$$

By Plancherel's theorem we prove that $W^{s,2}(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ for all $s \geq 0$ and this identity is algebraical and topological. So, in the case $p = 2$, we denote more simply the space $W^{s,2}(\Omega)$ by $H^s(\Omega)$.

Definition 1.1 For $s > 0$ and $1 \leq p < \infty$, we denote

$$W_0^{s,p}(\Omega) = \overline{\mathscr{D}(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}},$$

and its topological dual space

$$W^{-s,p'}(\Omega) = \left[W_0^{s,p}(\Omega)\right]',$$

where p' is the conjugate of p: $1/p + 1/p' = 1$. For $p = 2$, we will write $H_0^s(\Omega)$ and $H^{-s}(\Omega)$, respectively.

Proposition 1.2 *Suppose $T \in \mathscr{D}'(\Omega)$. Then $T \in W^{-m,p'}(\Omega)$, with $m \in \mathbb{N}^*$, if and only if*

$$T = \sum_{|\alpha| \leq m} D^\alpha f_\alpha, \quad \text{with} \quad f_\alpha \in L^{p'}(\Omega).$$

1.2 First Properties

It will be assumed from now on that Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary.

Let us consider the following space

$$\mathscr{D}(\overline{\Omega}) = \left\{ v_{|\Omega};\ v \in \mathscr{D}(\mathbb{R}^N) \right\}.$$

Theorem 1.3

(i) The space $\mathscr{D}(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ for any $s > 0$ (even if Ω is unbounded).
(ii) The space $\mathscr{D}(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$ for any $s \in \mathbb{R}$.

As consequence, we have the following property: for any $s > 0$

$$W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N) \quad \text{and} \quad W^{-s,p'}(\mathbb{R}^N) = \left[W^{s,p}(\mathbb{R}^N)\right]'.$$

But in general, for any $s > 0$, we have $W_0^{s,p}(\Omega) \subsetneq W^{s,p}(\Omega)$.

Definition 1.4 For $s > 0$, we set

$$\widetilde{W}^{s,p}(\Omega) = \left\{u \in W^{s,p}(\Omega);\ \widetilde{u} \in W^{s,p}(\mathbb{R}^N)\right\},$$

where $\widetilde{u}$ is the extension by 0 of u outside of Ω.

The space $\widetilde{W}^{s,p}(\Omega)$ is a Banach space for the norm

$$\|u\|_{\widetilde{W}^{s,p}(\Omega)} = \|\widetilde{u}\|_{W^{s,p}(\mathbb{R}^N)}\,.$$

It is easy to verify that for any nonnegative integer m

$$W_0^{m,p}(\Omega) \hookrightarrow \widetilde{W}^{m,p}(\Omega) \tag{1}$$

and for any $u \in W_0^{m,p}(\Omega)$ we have

$$\|u\|_{\widetilde{W}^{m,p}(\Omega)} = \|u\|_{W^{m,p}(\Omega)}\,. \tag{2}$$

When $s = m + \sigma$ with $0 < \sigma < 1$, we can show that

$$\|u\|_{\widetilde{W}^{s,p}(\Omega)} \simeq \|u\|_{W^{s,p}(\Omega)} + \sum_{|\alpha|=m} \left\| \frac{D^\alpha u}{\varrho^\sigma} \right\|_{L^p(\Omega)}, \tag{3}$$

where $\varrho(x) = d(x, \Gamma)$ and $\Gamma = \partial\Omega$.

Theorem 1.5 *The space $\mathscr{D}(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$ (even if Ω is unbounded).*

From (1), (2) and the definition of $W_0^{m,p}(\Omega)$, we deduce the following: for any $m \in \mathbb{N}^*$,

$$\widetilde{W}^{m,p}(\Omega) = W_0^{m,p}(\Omega). \tag{4}$$

Theorem 1.6 *For any $0 < s \leq 1/p$, the space $\mathscr{D}(\Omega)$ is dense in $W^{s,p}(\Omega)$, which means that*

$$W_0^{s,p}(\Omega) = W^{s,p}(\Omega). \tag{5}$$

Theorem 1.7 *Let $0 < s \leq 1$ and $u \in W_0^{s,p}(\Omega)$. Then*

$$\frac{u}{\varrho^s} \in L^p(\Omega) \Longleftrightarrow s \neq 1/p$$

and in this case

$$\left\| \frac{u}{\varrho^s} \right\|_{L^p(\Omega)} \leq C\, |u|_{W^{s,p}(\Omega)},$$

where the notation $|\cdot|$ *denotes the semi-norm of* $W^{s,p}(\Omega)$.

The case $s=1$ is known as Hardy's inequality: for all $u \in W_0^{1,p}(\Omega)$,

$$\left\| \frac{u}{\varrho} \right\|_{L^p(\Omega)} \leq C\, \|\nabla u\|_{L^p(\Omega)}\,.$$

Using again a Hardy's inequality, we prove the following result:

Theorem 1.8 *Let* $s > 0$ *and* $u \in W_0^{s,p}(\Omega)$. *Then for any* $|\alpha| \leq s$, *we have*

$$\frac{D^\alpha u}{\varrho^{s-|\alpha|}} \in L^p(\Omega) \Longleftrightarrow s - 1/p \notin \mathbb{N}. \tag{6}$$

From (3) and (6), we deduce the following identity:

$$\widetilde{W}^{s,p}(\Omega) = W_0^{s,p}(\Omega) \tag{7}$$

which holds for any $s > 0$ satisfying $s - 1/p \notin \mathbb{N}$.

Proposition 1.9

(i) For any $1 \leq j \leq N$ *and for any* $s \in \mathbb{R}$, *the operator*

$$\frac{\partial}{\partial x_j} : W^{s,p}(\mathbb{R}^N) \longrightarrow W^{s-1,p}(\mathbb{R}^N) \tag{8}$$

is continuous.

(ii) However, if we replace $\mathbb{R}^N$ *by* Ω, *Property (8) takes place unless* $s = 1/p$.

Sketch of the Proof of Point (ii)

1. Case $s = m+\sigma$, with $m \in \mathbb{N}^*$ and $0 \leq \sigma < 1$. Let $u \in W^{s,p}(\Omega)$. By definition, we know that

$$u \in W^{m,p}(\Omega) \quad \text{and} \quad \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+\sigma p}} < \infty, \quad \forall\, |\alpha| = m.$$

So for any $1 \leq j \leq N$

$$\frac{\partial u}{\partial x_j} \in W^{m-1,p}(\Omega) \quad \text{and} \quad \int_\Omega \int_\Omega \frac{\left|D^\alpha \frac{\partial u}{\partial x_j}(x) - D^\alpha \frac{\partial u}{\partial x_j}(y)\right|^p}{|x-y|^{N+\sigma p}} < \infty,$$

for all $|\alpha| = m-1$. Consequently $\frac{\partial u}{\partial x_j} \in W^{s-1,p}(\Omega)$.

2. Case $s \leq 0$. Let $u \in W^{s,p}(\Omega)$. Since $-s+1 \geq 1$, for any $\varphi \in \mathscr{D}(\Omega)$, we get:

$$\begin{aligned}\left|\langle \frac{\partial u}{\partial x_j}, \varphi\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)}\right| &= \left|-\langle u, \frac{\partial \varphi}{\partial x_j}\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)}\right| \\ &\leq \|u\|_{W^{s,p}(\Omega)} \left\|\frac{\partial \varphi}{\partial x_j}\right\|_{W_0^{-s,p'}(\Omega)} \\ &\leq \|u\|_{W^{s,p}(\Omega)} \|\varphi\|_{W_0^{-s+1,p'}(\Omega)}.\end{aligned}$$

We conclude by using the density of $\mathscr{D}(\Omega)$ in $W_0^{-s+1,p'}(\Omega)$.

3. Case $0 < s < 1$. Let $u \in W^{s,p}(\Omega)$. Recall that Ω being Lipschitz open set, there exists an extension operator

$$\forall t \geq 0, \quad P : W^{t,p}(\Omega) \longrightarrow W^{t,p}(\mathbb{R}^N)$$

which is linear, continuous, and satisfying

$$Pv_{|\Omega} = v, \quad \text{for any } v \in W^{t,p}(\Omega).$$

As $Pu \in W^{s,p}(\mathbb{R}^N)$, we get $\frac{\partial Pu}{\partial x_j} \in W^{s-1,p}(\mathbb{R}^N)$. But

$$\left(\frac{\partial Pu}{\partial x_j}\right)_{|\Omega} = \frac{\partial u}{\partial x_j},$$

where $\frac{\partial u}{\partial x_j}$ is the restriction to Ω of the distribution $T = \frac{\partial Pu}{\partial x_j} \in W^{s-1,p}(\mathbb{R}^N)$. More precisely, we have:

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \langle \frac{\partial u}{\partial x_j}, \varphi\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)} = \langle T, \widetilde{\varphi}\rangle_{\mathscr{D}'(\mathbb{R}^N)\times\mathscr{D}(\mathbb{R}^N)}.$$

That implies

$$\left|\langle \frac{\partial u}{\partial x_j}, \varphi\rangle\right| \leq \|T\|_{W^{s-1,p}(\mathbb{R}^N)} \|\widetilde{\varphi}\|_{W^{1-s,p'}(\mathbb{R}^N)} = \|T\|_{W^{s-1,p}(\mathbb{R}^N)} \|\varphi\|_{\widetilde{W}^{1-s,p'}(\Omega)}.$$

We have shown that $\frac{\partial u}{\partial x_j} \in \left[\widetilde{W}^{1-s,p'}(\Omega)\right]'$. But

$$\left[\widetilde{W}^{1-s,p'}(\Omega)\right]' = \left[W_0^{1-s,p'}(\Omega)\right]' \Longleftrightarrow 1-s \neq 1/p',$$

i.e., $s \neq 1/p$. □

Remark 1 The above proof shows that

$$u \in W^{1/p,p}(\Omega) \Longrightarrow \frac{\partial u}{\partial x_j} \in \left[\widetilde{W}^{1/p',p'}\right]'.$$

In particular,

$$u \in H^{1/2}(\Omega) \Longrightarrow \frac{\partial u}{\partial x_j} \in \left[\widetilde{H}^{1/2}(\Omega)\right]',$$

where we remark also that

$$\widetilde{H}^{1/2}(\Omega) \hookrightarrow H^{1/2}(\Omega) = H_0^{1/2}(\Omega).$$

This embedding being dense, we get by duality

$$H^{-1/2}(\Omega) = \left[H_0^{1/2}(\Omega)\right]' \hookrightarrow \left[\widetilde{H}^{1/2}(\Omega)\right]'.$$

Corollary 1.10 *Let $s > 0$. The following characterization holds:*

$$u \in \widetilde{W}^{s,p}(\Omega) \Longleftrightarrow u \in W_0^{s,p}(\Omega) \quad \textit{and} \quad \textit{for any } |\alpha| = m, \quad \frac{D^\alpha u}{\varrho^\sigma} \in L^p(\Omega),$$

where $s = m + \sigma$, $m \in \mathbb{N}$ and $0 \le \sigma < 1$.

1.3 Traces

Firstly, recall the following inclusions:

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow \mathscr{C}^0(\mathbb{R}^N) \quad \text{if} \quad s > \frac{N}{p}.$$

So that if $u \in W^{s,p}(\mathbb{R}^N)$ with $s > \frac{N}{p}$, the restriction of u to the hyperplane $x_N = 0$ is well defined. But the continuity with respect to all variables is not necessary. It is enough to have the continuity with respect to the variable x_N. This is possible as soon as $s > 1/p$.

Actually, we have the following result:

Theorem 1.11

(i) *Suppose that $s - 1/p = k + \sigma$, with $k \in \mathbb{N}$ and $0 < \sigma < 1$ (which implies, in particular, that $s - 1/p \notin \mathbb{N}$). Then the mapping*

$$u \overset{\gamma}{\longmapsto} (\gamma_0 u, \gamma_1 u, \dots, \gamma_k u),$$

where

$$\gamma_0 u(x) = u(x', 0), x' = (x_1, \dots, x_{N-1}), \quad \textit{and} \quad \gamma_j u(x') = \frac{\partial^j u}{\partial x_N^j}(x', 0),$$

defined for $u \in \mathscr{D}(\mathbb{R}^N)$, has a unique extension

$$W^{s,p}(\mathbb{R}^n) \longrightarrow \prod_{j=0}^{k} W^{s-j-1/p,p}(\mathbb{R}^{N-1})$$

which is continuous and where k is the integer part of $s > 0$.

(ii) Moreover this operator has a right continuous inverse R:

$$\begin{cases} \forall\, \boldsymbol{g} = (g_0, \dots, g_k) \in \displaystyle\prod_{j=0}^{k} W^{s-j-1/p,p}(\mathbb{R}^{N-1}), \quad \gamma R\boldsymbol{g} = \boldsymbol{g} \\ \|R\boldsymbol{g}\|_{W^{s,p}(\mathbb{R}^N)} \leq C_N \displaystyle\sum_{j=0}^{k} \|g_j\|_{W^{s-j-1/p,p}(\mathbb{R}^{N-1})} . \end{cases}$$

Remark 2 For $p = 2$, the above result can be proved using the Fourier transform.

This result can be extended to the case where Ω is a bounded open subset of $\mathbb{R}^N$, with a $\mathscr{C}^{k,1}$ boundary (see the definition below).

Definition 1.12 Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω is Lipschitz (respectively of class $\mathscr{C}^{k,1}$, $k \in \mathbb{N}^\star$) if for every $x \in \Gamma$, there exists a neighborhood V of x in $\mathbb{R}^N$ and orthonormal coordinates $\{y_1, \dots, y_N\}$ satisfying:

(i) V is an hypercube

$$V = \left\{(y_1, \dots, y_N) \in \mathbb{R}^N;\ |y_j| < a_j,\ 1 \leq j \leq N\right\},$$

(ii) there exists a function φ defined in

$$V' = \left\{y' \in \mathbb{R}^{N-1};\ |y_j| < a_j, 1 \leq j \leq N-1\right\},$$

such that φ and φ^{-1} are Lipschitz (respectively, $\mathscr{C}^{k,1}$) and satisfying (Fig. 1)

$$\forall\, y' \in V', \quad \left|\varphi(y')\right| \leq \frac{1}{2} a_N$$

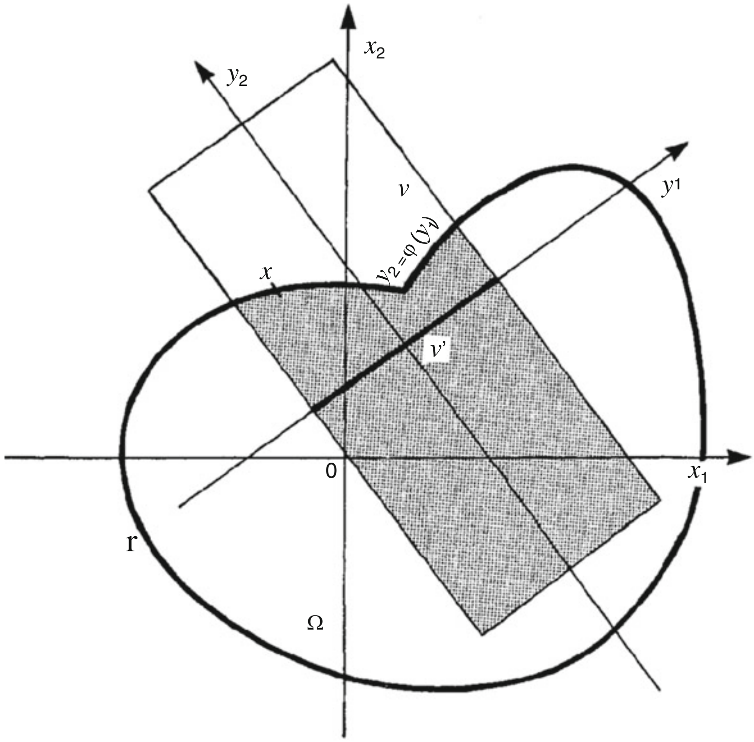


Fig. 1

$$\Omega \cap V = \left\{(y', y_N) \in V;\ y_N < \varphi(y')\right\}$$

$$\Gamma \cap V = \left\{(y', y_n) \in V;\ y_N = \varphi(y')\right\}.$$

Let

$$\begin{aligned} \Phi : V' &\longrightarrow \Gamma \cap V \\ y' &\longmapsto (y', \varphi(y')). \end{aligned}$$

Definition 1.13 Suppose that Ω is an open subset of $\mathbb{R}^N$ of class $\mathscr{C}^{k,1}$, with $k \in \mathbb{N}$ and let $0 < s \leq k+1$. We introduce the following space

$$W^{s,p}(\Gamma) = \left\{u \in L^p(\Gamma);\ u \circ \Phi \in W^{s,p}(V' \cap \Phi^{-1}(\Gamma \cap V))\right\}$$

for any (V, φ) verifying the previous definition.

Let (V_j, φ_j), $1 \leq j \leq J$, be any atlas of Γ for which each pair (V_j, φ_j) satisfies the above definition. One possible Banach norm for $W^{s,p}(\Gamma)$ is given by:

$$\|u\|_{W^{s,p}(\Gamma)} = \sum_{j=1}^{J} \left\|u \circ \Phi_j\right\|_{W^{s,p}(V'_j \cap \Phi_j^{-1}(\Gamma \cap V_j))}$$

which is equivalent when $0 < s < 1$ to the norm

$$\left(\|u\|^p_{L^p(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{N-1+sp}} d\sigma_x d\sigma_y\right)^{1/p}.$$

We are now in position to extend Theorem 1.11 to the case where $\mathbb{R}^{N-1}$ is replaced by an $N-1$-dimensional manifold of $\mathbb{R}^N$, but which is sufficiently regular. This simply uses changes of variables.

If locally Γ is represented by the pair (V, φ) with φ and φ^{-1} Lipschitz, then a unit outward normal vector can be defined as follows:

$$\text{for } y' \in V', \quad \boldsymbol{\nu}(y', \varphi(y')) = \frac{(-\nabla'\varphi(y'), 1)}{\sqrt{1 + |\nabla'\varphi(y')|^2}}.$$

One can then extend this vector in all V by setting

$$\boldsymbol{\nu}(y', y_N) = \boldsymbol{\nu}(y', \varphi(y')),\ y \in V.$$

As $\Gamma \subset \cup_{j=1}^J V_j$, we know that there exist functions $\mu_0, \mu_1, \ldots, \mu_J \in \mathscr{C}^\infty(\mathbb{R}^N)$ such that

(i) for all $j = 0, \ldots, J, \quad 0 \le \mu_j \le 1 \quad \text{and} \quad \sum_{j=1}^{J} \mu_j = 1$

(ii) supp μ_j is compact and supp $\mu_j \subset V_j$ for any $j \ge 1$ and supp $\mu_0 \subset \Omega$.

This partition of unity then allows to extend $\boldsymbol{\nu}$ in a neighborhood of $\overline{\Omega}$ as follows: $\boldsymbol{\nu} = \sum_{j=0}^{J} (\mu_j \boldsymbol{\nu})$. It is then easy to verify that $\boldsymbol{\nu} \in L^\infty(\overline{\Omega})$ if Γ is Lipschitz and $\boldsymbol{\nu} \in \mathscr{C}^{k-1,1}(\overline{\Omega})$ if Γ is $\mathscr{C}^{k,1}$.

We are now ready to establish the following result:

Theorem 1.14 (Traces) *Let Ω be an open subset of $\mathbb{R}^N$ of class $\mathscr{C}^{k,1}$, with $k \in \mathbb{N}$. Let $s > 0$ satisfying $s \le k + 1$ and $s - 1/p = \ell + \sigma$ with $0 < \sigma < 1$ and $\ell \in \mathbb{N}$. Then the mapping*

$$u \stackrel{\gamma}{\longmapsto} (\gamma_0 u, \gamma_1 u, \ldots, \gamma_\ell u)$$

defined for $\mathscr{C}^{k,1}$ has a unique continuous extension as an operator from $W^{s,p}(\Omega)$ into $\prod_{j=0}^{\ell} W^{s-j-1/p,p}(\Gamma)$ where

$$\gamma_1 u = \frac{\partial u}{\partial \boldsymbol{\nu}} = \nabla u \cdot \boldsymbol{\nu}, \quad \gamma_j u = \frac{\partial^j u}{\partial \boldsymbol{\nu}^j}.$$

Moreover this operator has a right continuous inverse R (not depending of p).

Case Ω Lipschitz. Suppose $1/p < s \leq 1$. We have the following properties:

(i) If $u \in W^{s,p}(\Omega)$, then $u_{|\Gamma} \in W^{s-1/p,p}(\Gamma)$.
(ii) If $g \in W^{s-1/p,p}(\Gamma)$, then there exists $u \in W^{s,p}(\Omega)$ such that $u = g$ on Γ and satisfying the estimate

$$\|u\|_{W^{s,p}(\Omega)} \leq C \ \|g\|_{W^{s-1/p,p}(\Gamma)}.$$

Case Ω of class $\mathscr{C}^{1,1}$.

(i) Let $u \in W^{s,p}(\Omega)$. If $1/p < s \leq 2$, then $u_{|\Gamma} \in W^{1-1/p}(\Gamma)$. Moreover, for any $g \in W^{s-1/p,p}(\Gamma)$, there exists $u \in W^{s,p}(\Omega)$ such that $u = g$ on Γ, with

$$\|u\|_{W^{s,p}(\Omega)} \leq C \ \|g\|_{W^{s-1/p,p}(\Gamma)}.$$

(ii) Let $u \in W^{s,p}(\Omega)$. If $1 + 1/p < s \leq 2$, then $\frac{\partial u}{\partial \boldsymbol{\nu}} \in W^{s-1-1/p,p}(\Gamma)$. Moreover, for any $g_0 \in W^{s-1/p,p}(\Gamma)$ and $g_1 \in W^{s-1-1/p,p}(\Gamma)$, there exists $u \in W^{s,p}(\Omega)$ such that

$$u = g_0 \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = g_1 \quad \text{on } \Gamma$$

with

$$\|u\|_{W^{s,p}(\Omega)} \leq C \ \left(\|g_0\|_{W^{s-1/p,p}(\Gamma)} + \|g_1\|_{W^{s-1-1/p,p}(\Gamma)}\right).$$

Theorem 1.15 *Suppose that Ω is an open subset of $\mathbb{R}^N$ of class $\mathscr{C}^{k,1}$, with $k \in \mathbb{N}$. Let $s > 0$ such that $s - 1/p \notin \mathbb{N}$ and $s - 1/p = \ell + \sigma$, where $0 < \sigma < 1$ and $\ell \geq 0$ is an integer. Then we have the following characterization for $s \leq k + 1$:*

$$W_0^{s,p}(\Omega) = \left\{u \in W^{s,p}(\Omega);\, \gamma_0 u = \gamma_1 u = \ldots = \gamma_\ell u = 0\right\}.$$

1.4 Interpolation

We will consider here only the case of spaces $H^s(\Omega)$, with Ω bounded open Lipschitz of $\mathbb{R}^N$.

Recall that for every $s > 0$ there exists a continuous linear operator:

$$P : H^s(\Omega) \longrightarrow H^s(\mathbb{R}^N)$$

satisfying

$$\forall u \in H^s(\Omega), \quad Pu_{|\Omega} = u.$$

Theorem 1.16 *[Interpolation Inequality] Let s_1, s_2, s_3 with $0 \leq s_1 < s_2 < s_3$. Then*

$$\forall \varepsilon > 0, \quad \|u\|_{W^{s_2,p}(\Omega)} \leq \varepsilon \|u\|_{W^{s_3,p}(\Omega)} + K\varepsilon^{-\frac{s_2-s_1}{s_3-s_2}} \|u\|_{W^{s_1,p}(\Omega)},$$

where $K = K(\Omega, s_1, s_2, s_3, p)$.

The above inequality is a consequence of the compactness of the embedding of $W^{s_3,p}(\Omega)$ into $W^{s_2,p}(\Omega)$.

Recall now that we have different ways to define the Sobolev space $H^m(\Omega)$, for $m \in \mathbb{N}$:

$$\begin{aligned} &u \in H^m(\Omega) \iff \forall |\alpha| \leq m,\ D^\alpha u \in L^2(\Omega),\\ &u \in H^m(\Omega) \iff u = U_{|\Omega} \text{ with } U \in H^m(\mathbb{R}^N),\\ &u \in H^m(\mathbb{R}^N) \iff u \in \mathscr{S}'(\mathbb{R}^N) \quad \text{and} \quad (1+|\xi|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^N). \end{aligned} \tag{9}$$

In the case of fractional Sobolev spaces $H^s(\Omega)$, with $s = m+\sigma$, $m \in \mathbb{N}$, $0 < \sigma < 1$, we have:

$$\begin{aligned} &u \in H^s(\Omega) \iff u \in H^m(\Omega) \quad \text{and} \quad \forall |\alpha| = m,\ \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{N+2\sigma}} < \infty\\ &u \in H^s(\Omega) \iff u = U_{|\Omega} \quad \text{with} \quad U \in H^s(\mathbb{R}^N),\\ &u \in H^s(\mathbb{R}^N) \iff u \in \mathscr{S}'(\mathbb{R}^N) \quad \text{and} \quad (1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^N). \end{aligned} \tag{10}$$

We can also get this space by interpolation:

$$H^s(\Omega) = \Big[H^m(\Omega), L^2(\Omega)\Big]_\mu,\ 0 < \mu < 1 \quad (1-\mu)m = s$$

and more generally we have for any $0 < \mu < 1$

$$\big[H^{s_1}(\Omega), H^{s_2}(\Omega)\big]_\mu = H^{(1-\mu)s_1+\mu s_2}(\Omega).$$

Concerning the interpolation of spaces $H_0^m(\Omega)$, we have:

$$\big[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)\big]_\mu = H_0^{(1-\mu)s_1+\mu s_2}(\Omega) \quad \text{if} \quad (1-\mu)s_1 + \mu s_2 \notin \frac{1}{2} + \mathbb{N}$$

and

$$\big[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)\big]_\mu = \widetilde{H}^{(1-\mu)s_1+\mu s_2}(\Omega) \quad \text{otherwise},$$

with equivalent norms.

1.5 Transposition

Let V and H be two Hilbert spaces on $\mathbb{R}$ and $A \in \mathscr{L}(V, H)$. For every fixed $g \in H'$, we consider the following mapping

$$\begin{aligned} V &\longrightarrow \mathbb{R} \\ x &\longmapsto \langle g, Ax \rangle_{H' \times H} \end{aligned}$$

which defines a linear and continuous form on V that we denote by ${}^tAg$:

$$\langle {}^tAg, x \rangle_{V' \times V} = \langle g, Ax \rangle_{H' \times H}.$$

Remark 3 If $A : V \longrightarrow H$ is an isomorphism, then we can define the transpose of A^{-1} and we easily verify that

$${}^tA^{-1} = \left({}^tA\right)^{-1} \quad \text{and} \quad {}^tA : H' \longrightarrow V' \text{ is an isomorphism.}$$

1.6 Inequalities

They are fundamental tools in the study of partial differential equations:

(i) **Poincaré's Inequality.** Let Ω be an open space bounded in at least one direction. Then there exists a constant $C \geq 0$, depending on the diameter of Ω such that

$$\forall u \in W_0^{1,p}(\Omega), \quad \|u\|_{L^p(\Omega)} \leq C \ \|\nabla u\|_{L^p(\Omega)}.$$

(ii) **Poincaré-Wirtinger's Inequality.** Let Ω be a Lipschitz bounded domain of $\mathbb{R}^N$. Then there exists a constant $C(\Omega) \geq 0$ such that

$$\forall u \in W^{1,p}(\Omega), \quad \inf_{K \in \mathbb{R}} \|u + K\|_{L^p(\Omega)} \leq C(\Omega) \ \|\nabla u\|_{L^p(\Omega)}.$$

(iii) **Hardy's Inequality.** Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^N$. Then there exists a constant $C(\Omega) \geq 0$ such that

$$\forall u \in W_0^{1,p}(\Omega), \quad \left\| \frac{u}{\varrho} \right\|_{L^p(\Omega)} \leq C(\Omega) \ \|\nabla u\|_{L^p(\Omega)}.$$

(iv) **Calderòn–Zygmund's Inequality.**

$$\forall u \in \mathscr{D}(\Omega), \quad \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(\Omega)} \leq C(\Omega) \ \|\Delta u\|_{L^p(\Omega)}.$$

1.7 Weak Solutions

Consider the following problems:

$$(P_D) \quad -\Delta u = f \quad \text{in } \Omega \qquad \text{and} \quad u = g \quad \text{on } \Gamma$$

and

$$(P_N) \quad -\Delta u = f \quad \text{in } \Omega \qquad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{\nu}} = h \quad \text{on } \Gamma,$$

where Ω is a Lipschitz bounded domain of $\mathbb{R}^N$, f, g, and h are given.

Theorem 1.17 *Given any $f \in H^{-1}(\Omega)$ and any $g \in H^{1/2}(\Gamma)$, there exists a unique solution $u \in H^1(\Omega)$ to Problem (P_D). Moreover*

$$\|u\|_{H^1(\Omega)} \leq C(\Omega) \left(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\right).$$

Proof Using Theorem 1.14, there exists $u_g \in H^1(\Omega)$ such that

$$u_g = g \quad \text{on } \Gamma \quad \text{with} \quad \left\|u_g\right\|_{H^1(\Omega)} \leq C(\Omega) \, \|g\|_{H^{1/2}(\Gamma)}.$$

Setting

$$f_g = -\Delta u_g = -\text{div}\, \nabla u_g \in H^{-1}(\Omega),$$

the problem becomes: Find $v \in H_0^1(\Omega)$ solution of

$$(P_D^0) \quad -\Delta v = f - f_g \quad \text{in } \Omega \qquad \text{and} \quad v = 0 \quad \text{on } \Gamma.$$

This last problem is equivalent to the following variational formulation:

$$(FV)_D \begin{cases} \text{Find } v \in H_0^1(\Omega) \text{ such that} \\ \forall \varphi \in H_0^1(\Omega), \quad \displaystyle\int_\Omega \nabla v \cdot \nabla \varphi dx = \langle f - f_g, \varphi \rangle_{H^{-1}(\Omega \times H_0^1(\Omega)}. \end{cases}$$

Applying Lax–Milgram Lemma or Riesz Theorem, we prove the existence of a unique solution $v \in H_0^1(\Omega)$ satisfying $(FV)_D$.

Note that the bilinear form

$$a(v, \varphi) = \int_\Omega \nabla v \cdot \nabla \varphi dx$$

is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ and coercive on $H_0^1(\Omega)$ thanks to Poincaré's inequality. In addition, this form allows to define a scalar product on Hilbert's space $H_0^1(\Omega)$. □

Remark 4

(i) If Ω is of class $\mathscr{C}^1$, $f \in W^{-1,p}(\Omega)$ and $g \in W^{1-1/p,p}(\Gamma)$ with $1 < p < \infty$, then there exists a unique solution $u \in W^{1,p}(\Omega)$ to (P_D).
(ii) When Ω is only Lipschitz, this regularity result holds for $p \in]2-\varepsilon', 2+\varepsilon[$ where ε and $\varepsilon' > 0$ are depending on Ω and $2-\varepsilon'$ and $2+\varepsilon$ are conjugate.

Concerning the Neumann problem, the approach is a bit more complicated. Indeed, if we are looking for a solution $u \in H^1(\Omega)$ only, the boundary condition on the normal derivative does not make sense, since the functions of $L^2(\Omega)$ do not have any trace at the boundary. Here, in fact, if one set $\boldsymbol{v} = \nabla u$ we have

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \boldsymbol{v} \cdot \boldsymbol{\nu} \text{ on } \Gamma.$$

Definition 1.18

$$H(\text{div};\ \Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega);\ \text{div}\, \boldsymbol{v} \in L^2(\Omega) \right\}.$$

It is a Hilbert space for the scalar product

$$((\boldsymbol{v}, \boldsymbol{w}))_{H(\text{div};\ \Omega)} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} dx + \int_\Omega (\text{div}\, \boldsymbol{v})(\text{div}\, \boldsymbol{w}) dx.$$

Proposition 1.19

(i) The space $\mathscr{D}(\overline{\Omega})$ *is dense in* $H(\text{div};\ \Omega)$.
(ii) The linear mapping

$$\boldsymbol{v} \longmapsto \boldsymbol{v} \cdot \boldsymbol{\nu},$$

defined on $\mathscr{D}(\overline{\Omega})^N$, *can be uniquely extended into a linear mapping of* $H(div; \Omega)$ *in* $H^{-1/2}(\Gamma) := \left[H^{1/2}(\Gamma)\right]'$.
(iii) In addition, we have the following Green's formula (or Stokes' formula):

$$\forall \varphi \in H^1(\Omega),\ \forall \boldsymbol{v} \in H(\text{div};\ \Omega), \quad \int_\Omega \boldsymbol{v} \cdot \nabla\varphi\, dx + \int_\Omega \varphi\, \text{div}\, \boldsymbol{v}\, dx = \langle \boldsymbol{v} \cdot \boldsymbol{\nu}, \varphi \rangle_\Gamma$$

where $\langle \cdot, \cdot \rangle_\Gamma$ *denotes the duality brackets* $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

Corollary 1.20 *Let* $u \in H^1(\Omega)$ *be such that* $\Delta u \in L^2(\Omega)$. *Then* $\frac{\partial u}{\partial \boldsymbol{\nu}} \in H^{-1/2}(\Gamma)$. *Moreover for any* $\varphi \in H^1(\Omega)$, *we have the following Green formula:*

$$\int_\Omega \varphi \Delta u \, dx + \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \langle \frac{\partial u}{\partial \boldsymbol{\nu}}, \varphi \rangle_\Gamma .$$

Proof It suffices to apply Proposition 1.19 by setting $\boldsymbol{v} = \nabla u$. □

As a Consequence we can show that for any $f \in L^2(\Omega)$ and for any $g \in H^{-1/2}(\Gamma)$, the problems

$$(P_N) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ -\Delta u = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = g \quad \text{on } \Gamma \end{cases}$$

and

$$(Q_N) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx + \langle g, \varphi \rangle_\Gamma \end{cases}$$

are equivalent, so that any solution of one is a solution of the other.

Remark 5

(i) The open Ω being bounded, the constant functions belong to $H^1(\Omega)$. So that if u is a solution of (Q_N), taking $\varphi = 1$, the data f and g must satisfy the (necessary) compatibility condition:

$$\int_\Omega f \, dx + \langle g, 1 \rangle_\Gamma = 0.$$

(ii) The implication $(P_N) \implies (Q_N)$ results from Corollary 1.20. The reverse implication also uses Green's formula and the surjectivity of the trace operator of $H^1(\Omega)$ into $H^{1/2}(\Gamma)$.

Theorem 1.21 *Let Ω be a bounded, connected, and Lipschitzian open of $\mathbb{R}^N$, with $N \geq 2$. Let $f \in L^2(\Omega)$, $g \in H^{-1/2}(\Gamma)$ satisfying the compatibility condition*

$$\int_\Omega f \, dx + \langle g, 1 \rangle_\Gamma = 0.$$

Then Problem (P_N) has a solution $H^1(\Omega)$, unique to an additive constant, verifying the estimate:

$$\|\nabla u\|_{L^2(\Omega)} \leq C(\Omega) \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma)} \right).$$

Proof According to Poincaré-Wirtinger's inequality, we have

$$\inf_{K \in \mathbb{R}} \|u + K\|_{H^1(\Omega)} \leq C(\Omega) \, \|\nabla u\|_{L^2(\Omega)} .$$

So that the bilinear form

$$a(u,\varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx$$

is coercive on the quotient space $V = H^1(\Omega)/_{\mathbb{R}}$. It is then sufficient to apply Lax–Milgram on the Hilbert space V. □

Remark 6

(i) We could have chosen as space V the space $H^1(\Omega) \cap L^2_0(\Omega)$ where

$$L^2_0(\Omega) = \left\{ v \in L^2(\Omega); \int_\Omega v \, dx = 0 \right\},$$

which is a Hilbert space and then use the inequality:

$$\forall\, v \in H^1(\Omega) \cap L^2_0(\Omega), \quad \|v\|_{H^1(\Omega)} \le C \, \|\nabla v\|_{L^2(\Omega)} .$$

(ii) We could have taken f in a space larger than $L^2(\Omega)$. More precisely if $f \in L^{(2^*)'}(\Omega)$, where $(2^*)'$ is the conjugate of 2^* defined by

$$\frac{1}{2^*} = \begin{cases} \frac{1}{2} - \frac{1}{N} & \text{if } N \ge 3 \\ \varepsilon > 0 & \text{arbitrary if } N = 2, \end{cases}$$

i.e., $(2^*)' = \frac{2N}{N+2}$ if $N \ge 3$ and $(2^*)' > 1$ if $N = 2$.

(iii) In L^p-theory, we have existence results in $W^{1,p}(\Omega)$ when Ω is $\mathscr{C}^1$ and $1 < p < \infty$ or when Ω is $\mathscr{C}^{0,1}$ and $2 - \varepsilon' < p < 2 + \varepsilon$.

In the same spirit, we can consider the case of Fourier-Robin boundary condition:

$$(P_{FR}) \begin{cases} \text{Find } u \in H^1(\Omega) \\ -\Delta u = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} + \alpha u = g \quad \text{on } \Gamma, \end{cases}$$

where α is a positive function defined on Γ, which can be formulated in an equivalent way by:

$$(Q_{FR}) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ \forall\, \varphi \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Gamma \alpha u \varphi \, dx = \int_\Omega f \varphi \, dx + \langle g, \varphi \rangle_\Gamma . \end{cases}$$

1.8 Strong Solutions

Theorem 1.22 *Let Ω be a bounded open of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^N$. Let $f \in L^2(\Omega)$ and $g \in H^{/2}(\Gamma)$. Then the solution u given by Theorem 1.17 belongs to $H^2(\Omega)$ and verifies the estimate:*

$$\|u\|_{H^2(\Omega)} \leq C(\Omega) \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Gamma)}\right).$$

Proof Firstly, we note that

$$L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \quad \text{and} \quad H^{3/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$$

so that the problem (P_D) has a unique solution $u \in H^1(\Omega)$.

We shift the data $g \in H^{3/2}(\Gamma)$ by $u_g \in H^2(\Omega)$ and we set again $u = v + u_g$, so that $v \in H^1(\Omega)$ vérifies:

$$\begin{cases} -\Delta v = f + \Delta u_g \in L^2(\Omega), \\ v = 0 \quad \text{on } \Gamma. \end{cases}$$

So, we need to show that $v \in H^2(\Omega)$. One of the methods to establish this regularity consists in using the technique of the differential quotients.

The complete proof being long and tedious, we will admit it. □

Remark 7 We can also establish the existence of solutions in $W^{2,p}(\Omega)$ when the data f and g verify:

$$f \in L^p(\Omega) \quad \text{and} \quad g \in W^{2-1/p,p}(\Gamma)$$

and the domain Ω is of class $\mathscr{C}^{1,1}$.

1.9 Very Weak Solutions

We assume here that Ω is a bounded open of class $\mathscr{C}^{1,1}$ and we are interested in the homogeneous problem

$$(P_D^H) \begin{cases} \text{Find } u \in L^2(\Omega) \\ -\Delta u = 0 \quad \text{in } \Omega, \\ u = g \quad \text{on } \Gamma, \end{cases}$$

where $g \in H^{-1/2}(\Gamma)$.

Remark 8 As the function u belongs “only” to $L^2(\Omega)$, the boundary condition $u = g$ on Γ has *a priori* no sense. But we will see that in fact, we can make sense of the trace of a harmonic function in $L^2(\Omega)$ and (we can in fact weaken this last hypothesis).

Lemma 1.23

(i) The space $\mathscr{D}(\overline{\Omega})$ is dense in the space

$$E(\Omega;\,\Delta) = \left\{v \in L^2(\Omega);\;\; \Delta v \in L^2(\Omega)\right\}.$$

(ii) The mapping $v \longmapsto v_{|\Gamma}$ defined on $\mathscr{D}(\overline{\Omega})$ can be uniquely extended into a continuous linear mapping of $E(\Omega;\,\Delta)$ into $H^{-1/2}(\Gamma)$.

(iii) In addition, we have the following Green's formula:

$$\begin{cases} \forall\, v \in E(\Omega;\,\Delta), \quad \forall\, \varphi \in H^2(\Omega) \cap H^1_0(\Omega) \\ \displaystyle\int_\Omega v \Delta\varphi \, dx - \int_\Omega \varphi \Delta v \, dx = \langle v,\, \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}. \end{cases}$$

Proof

(i) The idea is to use the Hahn–Banach theorem. So let $\ell \in [E(\Omega;\,\Delta)]'$ vanishing on $\mathscr{D}(\overline{\Omega})$ and show that it cancels on $E(\Omega;\,\Delta)$.

We know that there exist $(f, g) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$\forall\, v \in E(\Omega;\,\Delta), \quad \langle \ell, v\rangle = \int_\Omega f v \, dx + \int_\Omega g \Delta v \, dx.$$

Let $\widetilde{f}$ and $\widetilde{g}$ the extensions by 0 outside of Ω of f and g, respectively. Then, for any $v \in \mathscr{D}(\mathbb{R}^N)$

$$\langle \ell,\, v_{|\Omega}\rangle = \int_\Omega f v \, dx + \int_\Omega g \Delta v \, dx = \int_{\mathbb{R}^N} \widetilde{f} v \, dx + \int_{\mathbb{R}^N} \widetilde{g} \Delta v \, dx,$$

i.e.,

$$\Delta \widetilde{g} = -\widetilde{f} \text{ in } \mathbb{R}^N.$$

As $\widetilde{g} \in L^2(\mathbb{R}^N)$ and $\Delta\widetilde{g} \in L^2(\mathbb{R}^N)$, then $\widetilde{g} \in H^2(\mathbb{R}^N)$. Therefore, $g \in H^2(\Omega)$. The extension $\widetilde{g}$, by 0 outside of Ω, belongs to $H^2(\mathbb{R}^N)$. We know then that $g \in H^2_0(\Omega)$. By definition, there exists a sequence $(g_k)_k$ of functions of $\mathscr{D}(\Omega)$ such that $g_k \longrightarrow g$ in $H^2(\Omega)$.

Finally, let $v \in E(\Omega;\,\Delta)$. So,

$$\langle \ell, v\rangle = \lim_{k \Longrightarrow \infty} \left[\int_\Omega -v \Delta v_k \, dx + \int_\Omega g_k \Delta v \, dx\right] = \lim_{k \Longrightarrow \infty} 0 = 0.$$

(ii) Let $v \in \mathscr{D}(\overline{\Omega})$ fixed and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Then

$$\int_\Omega v \Delta \varphi \, dx - \int_\Omega \varphi \Delta v \, dx = \int_\Gamma v \frac{\partial \varphi}{\partial \boldsymbol{\nu}}.$$

Now let $\mu \in H^{1/2}(\Gamma)$. According to the trace theorem and since Ω is of class $\mathscr{C}^{1,1}$, there exists $\varphi \in H^2(\Omega)$ verifying

$$\begin{cases} \varphi = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial \boldsymbol{\nu}} = \mu \quad \text{on } \Gamma, \\ \|\varphi\|_{H^2(\Omega)} \leq C \ \|\mu\|_{H^{1/2}(\Gamma)} . \end{cases}$$

Thus, using the Cauchy–Schwarz inequality

$$\left| \langle v, \mu \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \right| = \left| \int_\Gamma v \mu \right| = \left| \int_\Gamma v \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \right|$$

$$\leq C(\Omega) \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)} \right)^{1/2} \ \|\varphi\|_{H^2(\Omega)}$$

$$\leq C(\Omega) \, \|v\|_{E(\Omega; \Delta)} \, \|\mu\|_{H^{1/2}(\Gamma)} .$$

This shows that the linear mapping

$$\begin{aligned} \mathscr{D}(\overline{\Omega}) &\longrightarrow H^{-1/2}(\Gamma) \\ v &\longmapsto v_{|\Gamma} \end{aligned}$$

is continuous when $\mathscr{D}(\overline{\Omega})$ is equipped with the norm of $E(\Omega; \Delta)$. We finish the proof by using the density of $\mathscr{D}(\overline{\Omega})$ in $E(\Omega; \Delta)$.

(iii) Immediate.

□

Theorem 1.24 *Let Ω be a bounded open of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^N$ and let $g \in H^{-1/2}(\Gamma)$. Then, the problem (P_D^0) has a unique solution $u \in L^2(\Omega)$ verifying the estimate*

$$\|u\|_{L^2(\Omega)} \leq C(\Omega) \ \|g\|_{H^{-1/2}(\Gamma)} .$$

Proof From Green's formula above, it is easy to see that $u \in L^2(\Omega)$ is a solution of the problem (P_D^0) if and only if

$$\forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega), \qquad \int_\Omega u \Delta \varphi \, dx = \langle g, \frac{\partial \varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma. \tag{11}$$

Indeed, let $u \in L^2(\Omega)$ be a solution of (P_D^0). Green's formula implies that (11) takes place.

Conversely, let $u \in L^2(\Omega)$ be a solution of (11). Then, for all $\varphi \in \mathscr{D}(\Omega)$, we have

$$0 = \int_\Omega u \Delta\varphi \, dx = \langle \Delta u, \varphi \rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)},$$

i.e.,

$$\Delta u = 0 \quad \text{in } \Omega. \tag{12}$$

Let now $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$. From (12) and Green's formula above, we deduce successively that:

$$0 = \int_\Omega \varphi \Delta u \, dx = \int_\Omega u \Delta\varphi \, dx - \langle u, \frac{\partial\varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma$$

then

$$\langle u, \frac{\partial\varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma = \langle g, \frac{\partial\varphi}{\partial \boldsymbol{\nu}} \rangle_\Gamma.$$

From the surjectivity of the trace mapping $v \mapsto (v_{|\Gamma}, \frac{\partial v}{\partial \boldsymbol{\nu}})$ from $H^2(\Omega)$ into $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ we know that

$$\forall \mu \in H^{1/2}(\Gamma), \quad \langle u, \mu \rangle_\Gamma = \langle g, \mu \rangle_\Gamma,$$

i.e., $u = g$ in $H^{-1/2}(\Gamma)$. □

Remark 9 A similar result can be established for the Neumann problem (P^0_N) with boundary data h in $H^{-3/2}(\Gamma)$ and satisfying the compatibility condition $\langle h, 1 \rangle_\Gamma = 0$.

1.10 Solutions in $H^s(\Omega)$, with $0 < s < 2$

We have established in the previous paragraphs the existence of solutions in $H^1(\Omega)$, $H^2(\Omega)$, and $L^2(\Omega)$ under generally optimal assumptions (except for the Neumann problem).

We will now consider the case of solutions in $H^s(\Omega)$ with $0 < s < 2$ and $s \neq 1$. The main ingredient is to use interpolation (complex here).

Theorem 1.25 *Let Ω be a bounded open of class $\mathscr{C}^{1,1}$.*

(i) Suppose that $\frac{1}{2} < s < 2$. Then the operators

$$\begin{aligned}
&\Delta : H^s(\Omega) \cap H^1_0(\Omega) \longrightarrow H^{s-2}(\Omega) = \left[H^{2-s}_0(\Omega)\right]' \quad \text{if } 1 < s < 2 \text{ and } s \neq \tfrac{3}{2},\\
&\Delta : H^{3/2}_0(\Omega) \longrightarrow \left[H^{1/2}_{00}(\Omega)\right]',\\
&\Delta : H^{2-s}_0(\Omega) \longrightarrow H^{-s}(\Omega) = \left[H^s_0(\Omega)\right]' \quad \text{if } 1 < s < \tfrac{3}{2},
\end{aligned} \tag{13}$$

are isomorphisms.

(ii) For any $g \in H^s(\Gamma)$*, with* $-\frac{1}{2} < s < \frac{3}{2}$*, Problem* (P^H_D) *has a unique solution* $u \in H^{s+\frac{1}{2}}(\Omega)$.

Remark 10 What happens if Ω is only Lipschitz? For what values of s can we have $u \in H^s(\Omega)$?

2 The Stokes Problem with Various Boundary Conditions

We are interested here in the study of the Stokes problem:

$$(S) \begin{cases} \text{Find } (\boldsymbol{u}, \pi) \quad \text{satisfying} \\ -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \end{cases}$$

with one of the following boundary conditions on Γ:

(i) $\boldsymbol{u} = \boldsymbol{0}$ (Dirichlet boundary condition)
(ii) $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ (Navier type boundary condition)
(iii) $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $(\mathbb{D}\boldsymbol{u})\boldsymbol{v} + \alpha \boldsymbol{u}_\tau = \boldsymbol{0}$ (Navier boundary condition)
(iv) $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ and $\pi = \pi_0$ (pressure boundary condition).

Here $\boldsymbol{u}$ denotes the velocity field, π the pressure field, Ω a connected bounded open set we assume at least Lipschitz.

Recall that

$$\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u}, \quad \mathbf{curl}\, \boldsymbol{u} = \nabla \times \boldsymbol{u} \quad \text{and} \quad \mathbb{D}\boldsymbol{u} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right).$$

The notation $\boldsymbol{u}_\tau$ denotes the tangential component of $\boldsymbol{u}$: $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}$. Finally $\boldsymbol{f}$ and α are given on Ω and Γ, respectively.

Remark 11

(i) We limit ourselves here, with the exception of pressure, to the case of homogeneous boundary conditions.
(ii) If the boundary of Ω is flat (like a cube, for example, or half space), the above boundary conditions are more easily written. When $\Omega = \mathbb{R}^3_+$, the Navier type boundary condition is equivalent to:

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0$$

and that of Navier at:

$$u_3 = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial x_3} - \alpha u_1 = \frac{\partial u_2}{\partial x_3} - \alpha u_2 = 0.$$

2.1 The Problem (*S*) *with Dirichlet Boundary Condition*

As for the Laplace equation with the Dirichlet boundary condition, we will assume

$$\boldsymbol{f} \in H^{-1}(\Omega)^3$$

and so look for $\boldsymbol{u} \in H_0^1(\Omega)^3$ verifying (S). Here we have in addition the constraint

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega$$

and the Lagrange multiplier π. First of all, as π must verify

$$\nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{u} \in H^{-1}(\Omega)^3$$

it is, therefore, reasonable to look for π in $L^2(\Omega)$. Moreover, it is easy to verify that such π satisfies:

$$\forall\, \boldsymbol{v} \in H_0^1(\Omega)^3, \quad \langle \nabla \pi, \boldsymbol{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = -\int_\Omega \pi \operatorname{div} \boldsymbol{v}\, dx.$$

The space

$$V = \left\{ \boldsymbol{v} \in H_0^1(\Omega)^3;\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \right\}$$

being a subspace of $H_0^1(\Omega)^3$ is, therefore, a Hilbert space. Moreover

$$\forall\, \boldsymbol{v} \in V, \quad \langle \nabla \pi, \boldsymbol{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = 0.$$

We are now able to propose a variational formulation of Problem (S):

$$(P_D^0) \begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that} \\ \forall\, \boldsymbol{v} \in V, \quad \displaystyle\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}\, dx = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \end{cases}$$

where we note that the pressure π has “disappeared.”

Lemma 2.1 *The problem*

$$(S_D^0)\begin{cases}\text{Find } (\boldsymbol{u},\pi)\in H_0^1(\Omega)^3\times L^2(\Omega)\\ -\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}\quad\text{in }\Omega,\\ \operatorname{div}\boldsymbol{u}=0\quad\text{in }\Omega\end{cases}$$

is equivalent to the problem (P_D^0).

Proof The implication $(S_D^0)\Longrightarrow(P_D^0)$ is immediate. Conversely, let $\boldsymbol{u}$ be a solution of (P_D^0). Then, in particular,

$$\forall\,\boldsymbol{v}\in\mathscr{D}(\Omega)^3\quad\text{such that}\quad\operatorname{div}\boldsymbol{v}=0\quad\text{in }\Omega,$$

we have

$$\langle-\Delta\boldsymbol{u}-\boldsymbol{f},\boldsymbol{v}\rangle_{\mathscr{D}'(\Omega)^3\times\mathscr{D}(\Omega)^3}=0. \tag{14}$$

As $-\Delta\boldsymbol{u}-\boldsymbol{f}\in H^{-1}(\Omega)^3$ and the space

$$\mathcal{V}(\Omega)=\Big\{\boldsymbol{v}\in\mathscr{D}(\Omega)^3;\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega\Big\}$$

is dense in the space V, then the relation (14) takes place for all $\boldsymbol{v}$. Then we know that there exists $\pi\in L^2(\Omega)$, unique up to an additive constant, because Ω is connected, such that

$$-\Delta\boldsymbol{u}-\boldsymbol{f}=\nabla(-\pi)\quad\text{in }\Omega$$

(this result is called "De Rham's version of the theorem" in $H^{-1}(\Omega)^N$). And finally, as $\boldsymbol{u}\in V$, then

$$\operatorname{div}\boldsymbol{u}=0\text{ in }\Omega\quad\text{and}\quad\boldsymbol{u}=\boldsymbol{0}\quad\text{on }\Gamma.$$

This ends the proof of the lemma. □

Theorem 2.2 *For any* $\boldsymbol{f}\in H^{-1}(\Omega)^3$, *the Stokes problem* (P_D^0) *has a unique solution* $\boldsymbol{u}\in V$ *vérifying further*

$$\|\boldsymbol{u}\|_{H^1(\Omega)^3}\le C(\Omega)\,\|\boldsymbol{f}\|_{H^{-1}(\Omega)^3}\,.$$

Proof Simply apply Lax–Milgram theorem. □

Remark 12 The theory is well known for everything that concerns the regularity of solutions when the data are:

- solutions in $W^{1,p}(\Omega)^3 \times L^p(\Omega)$
- solutions in $W^{2,p}(\Omega)^3 \times L^p(\Omega)$

with $1 < p < \infty$.

In particular, if $\boldsymbol{f} \in L^2(\Omega)^3$ and Ω is of class $\mathscr{C}^{1,1}$, then $\boldsymbol{u} \in H^2(\Omega)^3$ and $\pi \in H^1(\Omega)$.

2.2 The Stokes Problem with Navier Type Boundary Condition

Here we are still interested in Stokes' problem, but with the following boundary condition:

$$\boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma.$$

In order to take into account this condition at the boundary, it is important to write the Laplacian operator in the form:

$$-\Delta = \mathbf{curl\, curl} \; - \nabla\, \text{div}.$$

On the other hand, if we study the existence of weak solutions $\boldsymbol{u}$ in $H^1(\Omega)^3$, it will be necessary to give a meaning to the condition at the boundary

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

Recall the following Green formulas:

(i) If $\boldsymbol{v} \in L^2(\Omega)^3$ and $\mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)^3$, then $\boldsymbol{v} \times \boldsymbol{\nu} \in H^{-1/2}(\Gamma)^3$ and

$$\forall\, \boldsymbol{\varphi} \in H^1(\Omega)^3, \quad \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx - \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx = \langle \boldsymbol{v} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality brackets $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

(ii) If $\boldsymbol{v} \in L^2(\Omega)^3$ and $\text{div}\, \boldsymbol{v} \in L^2(\Omega)$, then $\boldsymbol{v} \cdot \boldsymbol{\nu} \in H^{-1/2}(\Gamma)$ and

$$\forall\, \varphi \in H^1(\Omega), \quad \int_\Omega \boldsymbol{v} \cdot \nabla \varphi\, dx + \int_\Omega \varphi\, \text{div}\, \boldsymbol{v}\, dx = \langle \boldsymbol{v} \cdot \boldsymbol{\nu}, \varphi \rangle_\Gamma.$$

Remark 13 If $\boldsymbol{v} \in L^2(\Omega)^3$ and $\mathbf{curl}\, \boldsymbol{v} \in L^{6/5}(\Omega)^3$ (respectively, $\text{div}\; \boldsymbol{v} \in L^{6/5}(\Omega)$), then

$$\boldsymbol{v} \times \boldsymbol{\nu} \in H^{-1/2}(\Gamma)^3 \text{ (resp. } \boldsymbol{v} \cdot \boldsymbol{\nu} \in H^{-1/2}(\Gamma))$$

and Green's formulas above remain valid.

Proposition 2.3 *Let* $\boldsymbol{v} \in L^2(\Omega)^3$ *such that* $\mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega)^3$ *and* $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \in L^{6/5}(\Omega)^3$. *Then* $\mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{\nu} \in H^{-1/2}(\Gamma)^3$ *and we have the following Green formula:*

$$\forall\,\boldsymbol{\varphi} \in H^1(\Omega)^3,\ \int_\Omega \mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\boldsymbol{\varphi} - \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} = \langle \mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle_\Gamma.$$

Proof It suffices to put $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{v}$ and use the previous reminders. □

We are now able to propose a variational formulation for the Stokes problem (S) with the Navier type homogeneous condition. To do this, we set

$$V = \Big\{ \boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega),\ \text{div}\ \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \Big\}$$

equipped with the graph norm:

$$\|\boldsymbol{v}\|_V = \Big(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)^3} \Big)^{1/2}$$

which makes it a Hilbert space.

We suppose $\boldsymbol{f} \in L^{6/5}(\Omega)^3$ and we consider the following variational formulation:

$$(P^0_{TN}) \begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that for any } \boldsymbol{v} \in V, \\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx. \end{cases}$$

Questions

(i) Is the problem (P^0_{TN}) equivalent to the problem (S^0_{TN})?
(ii) If so, is the bilinear form

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) &\longmapsto \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, dx \end{aligned}$$

coercive?

Remark 14 As with the Neumann problem for the Laplacian, the boundary condition

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma$$

is "hidden" in the variational formulation.

Answers to the Above Questions

In order to study Problem (P^0_{TN}), we have to describe with more precision the geometry of the domain. We first need the following definition.

Definition 2.4 A bounded domain in $\mathbb{R}^3$ is called pseudo-$\mathscr{C}^{0,1}$ (respectively, pseudo-$\mathscr{C}^{1,1}$) if for any point $\boldsymbol{x}$ on the boundary there exists an integer $r(\boldsymbol{x})$ equal to 1 or 2 and a strictly positive real number λ_0 such that for all real numbers λ with $0 < \lambda < \lambda_0$, the intersection of Ω with the ball with center $\boldsymbol{x}$ and radius λ, has $r(\boldsymbol{x})$ connected components, each one being $\mathscr{C}^{0,1}$ (resp. $\mathscr{C}^{1,1}$).

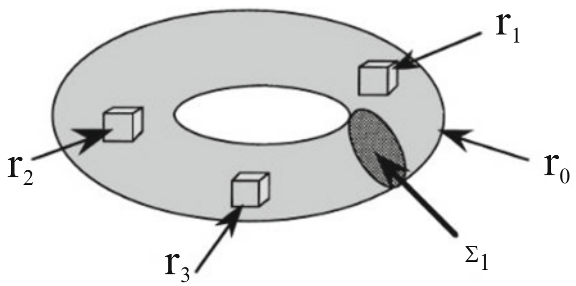
Hypothesis There exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called "*cuts*," contained in Ω, such that:

(i) each surface Σ_j is an open part of a smooth manifold $\mathcal{M}_j$,
(ii) the boundary of Σ_j is contained in $\partial\Omega$ for $1 \leq j \leq J$,
(iii) the intersection $\bar{\Sigma}_i \cap \bar{\Sigma}_j$ is empty for $i \neq j$,
(iv) the open set

$$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$$

is pseudo-$\mathscr{C}^{0,1}$ (respectively, pseudo-$\mathscr{C}^{1,1}$) simply connected.

Example for $J = 1$ and $I = 3$



Theorem 2.5 *Let Ω be a bounded open $\mathscr{C}^{1,1}$ set.*

(i) Let $\boldsymbol{v} \in L^2(\Omega)^3$ such that $div\, \boldsymbol{v} \in L^2(\Omega)$, $\mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)$ and satisfying in addition

$$\boldsymbol{v} \cdot \boldsymbol{\nu} \in H^{1/2}(\Gamma) \quad (\textit{respectively, } \boldsymbol{v} \times \boldsymbol{\nu} \in H^{1/2}(\Gamma)^3).$$

Then $\boldsymbol{v} \in H^1(\Omega)^3$ and we have the following estimates:

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega)(\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{v} \cdot \boldsymbol{\nu}\|_{H^{1/2}(\Gamma)}) \tag{15}$$

and

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega) \left[\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{v} \times \boldsymbol{\nu}\|_{H^{1/2}(\Gamma)}) \right]. \tag{16}$$

(ii) Under the above assumptions, if in addition $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$ on Γ, then we have the following estimate:

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega) \left(\|div\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \sum_{j=1}^{J} \left| \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{\nu} \right| \right) \tag{17}$$

and if $\boldsymbol{v} \times \boldsymbol{\nu} = \mathbf{0}$ on Γ, then we have the following estimate:

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \leq C(\Omega) \left(\|div\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} + \sum_{i=1}^{J} \left| \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{\nu} \right| \right). \tag{18}$$

Remark 15

(i) Suppose that

$$\boldsymbol{v} \in L^2(\Omega)^3, \; \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega) \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)^3$$

with

$$\boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

Let us then extend $\boldsymbol{v}$ by $\mathbf{0}$ outside of Ω. It is easy to show that this extension verifies:

$$\widetilde{\boldsymbol{v}} \in L^2(\mathbb{R}^3)^3, \; \mathrm{div}\; \widetilde{\boldsymbol{v}} \subset L^2(\mathbb{R}^3) \quad \text{and} \quad \mathbf{curl}\, \widetilde{\boldsymbol{v}} \in L^2(\mathbb{R}^3)^3.$$

As $-\Delta = \mathbf{curl}\,\mathbf{curl} - \nabla\, \mathrm{div}$, then $\Delta\widetilde{\boldsymbol{v}} \in H^{-1}(\mathbb{R}^3)^3$ and

$$\widetilde{\boldsymbol{v}} - \Delta\widetilde{\boldsymbol{v}} \in H^{-1}(\mathbb{R}^3)^3,$$

which means that $\widetilde{\boldsymbol{v}} \in H^1(\mathbb{R}^3)^3$ and, therefore, $\boldsymbol{v} \in H^1_0(\Omega)^3$.

(ii) Now note that if $\boldsymbol{u} \in \mathscr{D}(\mathbb{R}^3)^3$, then

$$\begin{aligned} \int_\Omega |\nabla \boldsymbol{u}|^2 \; dx &= -\int_{\mathbb{R}^3} \boldsymbol{u} \cdot \Delta \boldsymbol{u}\, dx = \int_{\mathbb{R}^3} [\boldsymbol{u} \cdot (\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u}) - \boldsymbol{u} \cdot \Delta \mathrm{div}\, \boldsymbol{u}]\, dx \\ &= \int_{\mathbb{R}^3} \left(|\mathbf{curl}\, \boldsymbol{u}|^2 + |\mathrm{div}\, \boldsymbol{u}|^2 \right) dx. \end{aligned}$$

Since $\mathscr{D}(\mathbb{R}^3)^3$ is dense in $H^1(\mathbb{R}^3)^3$, we deduce that:

$$\forall \boldsymbol{u} \in H^1(\mathbb{R}^3)^3, \int_{\mathbb{R}^3} |\nabla \boldsymbol{u}|^2 \, dx = \int_{\mathbb{R}^3} \left(|\mathbf{curl}\, \boldsymbol{u}|^2 + |\mathrm{div}\, \boldsymbol{u}|^2 \right) dx.$$

(iii) Back to point (i) of the remark: since $\boldsymbol{v} \in H^1_0(\Omega)^3$, we have:

$$\|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} = \|\nabla \widetilde{\boldsymbol{v}}\|_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left(|\mathbf{curl}\, \widetilde{\boldsymbol{v}}|^2 + |\mathrm{div}\, \widetilde{\boldsymbol{v}}|^2 \right) dx,$$

which gives the relation

$$\int_\Omega |\nabla \boldsymbol{v}|^2 \, dx = \int_\Omega \left(|\mathbf{curl}\, \boldsymbol{v}|^2 + |\mathrm{div}\, \boldsymbol{v}|^2 \right) dx.$$

Note that this last relation can also be directly established if $\boldsymbol{v} \in \mathscr{D}(\Omega)^3$ and then, by density of $\mathscr{D}(\Omega)$ in $H^1_0(\Omega)^3$, for any $\boldsymbol{v} \in H^1_0(\Omega)^3$.

Remark 16

(i) If Ω is simply connected, then for any $\boldsymbol{v} \in H^1(\Omega)^3$ such that $\boldsymbol{v} \cdot \boldsymbol{\nu} = 0$ on Γ, the inequality (17) is written

$$\|\boldsymbol{v}\|_{H^1(\Omega)^3} \leq C(\Omega) \left(\|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} \right).$$

(ii) If Γ is connected ($I = 1$), then for any $\boldsymbol{v} \in H^1(\Omega)^3$ such that $\boldsymbol{v} \times \boldsymbol{\nu} = 0$ on Γ, the inequality (18) is written

$$\|\boldsymbol{v}\|_{H^1(\Omega)^3} \leq C(\Omega) \left(\|\mathrm{div}\, \boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} \right).$$

Proposition 2.6 *Let Ω be a bounded open subset of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^3$. Then the bilinear form*

$$(\boldsymbol{u}, \boldsymbol{v}) \longmapsto \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, dx$$

is coercive on the following spaces V and on W, respectively:

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^3;\ \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \text{ and } \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{\nu} = 0, 1 \leq j \leq J \right\}$$

$$W = \left\{ \boldsymbol{v} \in H^1(\Omega)^3; \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{\nu} = \mathbf{0} \text{ on } \Gamma \text{ and } \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{\nu} = 0, 1 \leq j \leq I \right\}.$$

We are now able to study the problem (P^0_{TN}). We start with the simplest case where Ω is simply connected.

Theorem 2.7 *Let Ω be a bounded open domain of class $\mathscr{C}^{1,1}$ of $\mathbb{R}^3$. Suppose that Ω is simply connected.*

(i) Then for any $\boldsymbol{f} \in L^{6/5}(\Omega)^3$, Problem (P^0_{TN}) admits a unique solution verifying the estimate

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C(\Omega)\ \|\boldsymbol{f}\|_{L^{6/5}(\Omega)}\,.$$

(ii) The problem (P^0_{TN}) is equivalent to the problem (S^0_{TN}).

(iii) If moreover Ω is of class $\mathscr{C}^{1,1}$ then the solution $(\boldsymbol{u},\pi) \in W^{2,6/5}(\Omega)^3 \times W^{1,6/5}(\Omega)$.

Proof

(i) The open Ω being simply connected, then

$$V = \Big\{\boldsymbol{v} \in H^1(\Omega)^3;\ \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega,\ \ \boldsymbol{v}\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Gamma\Big\}$$

and V is an Hilbert space. Then let us put

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx.$$

Proposition 2.6 shows that the form a is coercive on V. Finally, the form $\ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx$ is clearly continuous because the continuous embedding $H^1(\Omega)^3 \hookrightarrow L^6(\Omega)^3$. The Lax–Milgram theorem implies the existence of a unique solution of Problem (P^0_{TN}).

(ii) Let us first show that

$$(S^0_{TN}) \Longrightarrow (P^0_{TN}).$$

Set

$$H = \Big\{\boldsymbol{v} \in L^6(\Omega)^3;\ \operatorname{div}\boldsymbol{v} \in L^2(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma\Big\}.$$

We know that $\mathscr{D}(\Omega)^3$ is dense in H. So we can show that the dual of H can be characterized as follows:

$$H' = \Big\{\boldsymbol{g} + \nabla\chi;\ \ \boldsymbol{g} \in L^{6/5}(\Omega)^3 \text{ and } \chi \in L^2(\Omega)\Big\}$$

(similar proof to the characterization of the dual $H^{-1}(\Omega)$ of $H^1_0(\Omega)$).

Let now $(\boldsymbol{u}, \pi) \in V \times L^2(\Omega)$ solution of (S^0_{TN}). Then for any $\boldsymbol{v} \in V$

$$\langle \nabla \pi, \boldsymbol{v} \rangle_{H' \times H} = - \int_\Omega \pi \operatorname{div} \boldsymbol{v} \, dx = 0.$$

Therefore,

$$-\Delta \boldsymbol{u} = \nabla \pi - \boldsymbol{f} \in H'.$$

We need the following lemma:

Lemma 2.8

(i) The space $\mathscr{D}(\overline{\Omega})^3$ *is dense in the following space*

$$E = \left\{ \boldsymbol{v} \in H^1(\Omega)^3; \quad \Delta \boldsymbol{v} \in H' \right\}.$$

(ii) The mapping

$$\boldsymbol{v} \longmapsto \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{\nu}$$

defined on $\mathscr{D}(\overline{\Omega})^3$ *can be uniquely extended into a continuous linear mapping from E into* $H^{-1/2}(\Gamma)^3$.

(iii) Moreover, for any $\boldsymbol{\varphi} \in H^1(\Omega)^3$ *such that*

$$div\, \boldsymbol{\varphi} = 0 \ in\ \Omega \quad and \quad \boldsymbol{\varphi} \cdot \boldsymbol{\nu} = 0 \ on\ \Gamma$$

and for any $\boldsymbol{v} \in E$*, we have the following Green formula*

$$-\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{H' \times H} = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, dx + \langle \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{\nu}, \boldsymbol{\varphi} \rangle_\Gamma,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ *denotes the duality brackets* $H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)^3$.

We return to the proof of the theorem. Since $\boldsymbol{u} \in H^1(\Omega)^3$ and $\Delta \boldsymbol{u} \in H'$, i.e., $\boldsymbol{u} \in E$, we can use this lemma to deduce on the one hand that the condition $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ has a meaning in $H^{-1/2}(\Gamma)^3$ and, on the other hand, that

$$\forall\, \boldsymbol{v} \in V, \quad \langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{H' \times H} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx,$$

i.e., $\boldsymbol{u}$ is solution of (P^0_{TN}).

Conversely, let $\boldsymbol{u} \in V$ solution of Problem (P^0_{TN}). Then

$$\operatorname{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma$$

and

$$\forall\, \boldsymbol{v} \in \mathscr{D}(\Omega)^3 \text{ with } \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega$$

we have

$$\langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3}.$$

That gives

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\mathscr{D}'(\Omega)^3 \times \mathscr{D}(\Omega)^3}.$$

So there exists, by De Rham's theorem, a function π in $L^2(\Omega)$, unique up to an additive constant, such that

$$-\Delta \boldsymbol{u} - \boldsymbol{f} = \nabla(-\pi) \quad \text{in } \Omega \tag{19}$$

(note that $L^{6/5}(\Omega) \hookrightarrow H^{-1}(\Omega)$).

It remains to show that $\boldsymbol{u}$ vérifies:

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

For that, from (19) and use the formula of Green of the first lemma, one deduces that

$$\forall\, \boldsymbol{v} \in V, \quad \langle -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{v} \rangle_{H' \times H} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx + \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{v} \rangle_\Gamma$$

that is to say that

$$\forall\, \boldsymbol{v} \in V, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx + \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{v} \rangle_\Gamma = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx.$$

But $\boldsymbol{u}$ being solution of (P^0_{TN}), then

$$\forall\, \boldsymbol{v} \in V, \quad \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{v} \rangle_\Gamma = 0.$$

Now let it be $\boldsymbol{\mu} \in H^{1/2}(\Gamma)$. We know that there exists

$$\boldsymbol{w} \in H^1(\Omega)^3, \ \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega, \ \boldsymbol{w} = \boldsymbol{\mu}_\tau \text{ on } \Gamma,$$

where $\boldsymbol{\mu}_\tau = \boldsymbol{\mu} - (\boldsymbol{\mu} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ the tangential component of $\boldsymbol{\mu}$ on Γ. As $\boldsymbol{w} \in V$, we have:

$$\langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{\mu} \rangle_\Gamma = \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{\mu}_\tau \rangle_\Gamma = \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu}, \boldsymbol{w} \rangle_\Gamma = 0,$$

which means that

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

(iii) The regularity $W^{1,6/5}(\Omega)$ of π is due to the fact that π satisfies:

$$\operatorname{div}(\nabla\pi - \boldsymbol{f}) = 0 \quad \text{in } \Omega \quad \text{and} \quad (\nabla\pi - \boldsymbol{f}) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma.$$

Setting $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$, the regularity $W^{2,6/5}(\Omega)^3$ of $\boldsymbol{u}$ is a consequence of the following properties:

$$\boldsymbol{z} \in L^{6/5}(\Omega)^3, \ \operatorname{div} \boldsymbol{z} = 0, \ \mathbf{curl}\, \boldsymbol{z} \in L^{6/5}(\Omega)^3 \quad \text{and} \quad \boldsymbol{z} \times \boldsymbol{\nu} = \mathbf{0} \ \text{ on } \Gamma.$$

□

Case Ω non Simply Connected
We then show that the kernel:

$$K_T(\Omega) = \Big\{ \boldsymbol{v} \in L^2(\Omega)^3;\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \Big\}$$

is of finite dimension and that the dimension corresponds to the number of cuts Σ_j necessary to obtain an open set $\overset{\circ}{\Omega} = \Omega \setminus \cup_{j=1}^{J} \Sigma_j$ simply connected.

As a consequence, if

$$V = \Big\{ \boldsymbol{v} \in H^1(\Omega)^3; \quad \operatorname{div} \boldsymbol{v} = 0 \text{ in } \ \Omega \text{ and } \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \Big\},$$

then, to prove that Problem (P^0_{TN}) admits a solution, it is necessary that $\boldsymbol{f}$ satisfies the following compatibility condition:

$$\forall\, \boldsymbol{v} \in K_T(\Omega), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx = 0.$$

Moreover, if such a solution $\boldsymbol{u}$ exists, it is unique up to an additive element of $K_T(\Omega)$.

2.3 The Stokes Problem with Navier Boundary Condition

We recall the Navier condition:

$$[2(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu}]_\tau + \alpha \boldsymbol{u}_\tau = \mathbf{0} \quad \text{on } \Gamma,$$

where

$$\mathbb{D}\boldsymbol{u} = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)_{1\le i,j\le 3}$$

is the deformation tensor, α defined on Γ is the friction coefficient and $\boldsymbol{u}_\tau$ is the tangential component of $\boldsymbol{u}$. To simplify, we will consider here only the case $\alpha = 0$.

Note that when $\operatorname{div} \boldsymbol{u} = 0$ in Ω, then $2\operatorname{div} \mathbb{D}\boldsymbol{u} = \Delta\boldsymbol{u}$.

Lemma 2.9 *If* $(\boldsymbol{u}, \pi) \in H^1(\Omega)^3 \times L^2(\Omega)$ *is such that*

$$-\Delta\boldsymbol{u} + \nabla\pi \in L^{6/5}(\Omega)^3$$

then

$$[(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu}]_\tau \in H^{-1/2}(\Gamma)^3$$

and

$$for\ any\ \boldsymbol{\varphi} \in H^1(\Omega)^3 \quad such\ that \quad \operatorname{div} \boldsymbol{\varphi} = 0 \quad in\ \Omega \quad and \quad \boldsymbol{\varphi}\cdot\boldsymbol{\nu} = 0 \quad on\ \Gamma$$

we have the Green's formula:

$$\int_\Omega (-\Delta\boldsymbol{u} + \nabla\pi)\cdot\boldsymbol{\varphi}\,dx = 2\int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi}\,dx - 2\langle[(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu}]_\tau\,, \boldsymbol{\varphi}\rangle_\Gamma,$$

where $\langle\cdot,\cdot\rangle_\Gamma$ *denotes the duality brackets* $H^{-1/2}(\Gamma)^3 \times H^{-1/2}(\Gamma)^3$.

With this Green's formula, the Stokes problem can be formulated as:

$$(P_N^0)\ \begin{cases} \text{Find } \boldsymbol{u} \in V, \text{ such that for any } \boldsymbol{\varphi} \in V, \\ 2\displaystyle\int_\Gamma \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx. \end{cases}$$

Set

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{\varphi}\,dx.$$

When Ω is not axisymmetric, then this form is coercive on V due to Korn's inequality:

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \backsimeq \|\mathbb{D}\boldsymbol{u}\|_{L^2(\Omega)}\,.$$

While if Ω is axisymmetric, this is not the case anymore. We must then quotient by some finite dimensional kernel.

Remark 17 In fact, on Γ we have the relation:

$$[2(\mathbb{D}\boldsymbol{u})\boldsymbol{v}]_\tau = \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{v} - \Lambda\boldsymbol{u},$$

where Λ is an operator of order 0:

$$\Lambda\boldsymbol{u} = \sum_{k=1}^{2}\left(\boldsymbol{u}_\tau \cdot \frac{\partial \boldsymbol{v}}{\partial s_k}\right)\boldsymbol{\tau}_k,$$

where $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is a base of the tangent plane to Γ at point x and (s_1, s_2) are local coordinates in this tangent plane.

This means that on the questions of regularity, they can be reduced to those concerning the Navier type condition.

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Survey on the Decay of the Local Energy for the Solutions of the Nonlinear Wave Equation

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1 Introduction and Preliminaries

We survey the results of the local energy of the solutions of the semilinear wave equation (subcritical and critical) and the critical Klein–Gordon wave equation in [3–7].

Indeed, we consider the following problems:

$$\begin{cases} \Box u + \chi(x)\, u|u|^{p-1} = 0 \text{ on } \mathbb{R}\times\Omega, \\ u = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ u(0,x) = u^0(x) \in H_D(\Omega) \text{ and } \partial_t u(0,x) = u^1(x) \in L^2(\Omega), \end{cases} \tag{1}$$

where $\Omega = \mathbb{R}^3 \backslash O$ and O is a strictly convex compact with smooth boundary $\partial\Omega$, $O \subset B_R$ for some $R > 0$ and $2 < p \leq 5$. The function χ is a positive and of class $\mathcal{C}^1$, with compact support such that $\mathrm{supp}\chi \subset B_R$. Here the function χ is allowed to be equal to 1 near $\partial\Omega$. We denote by $H = H_D(\Omega) \times L^2(\Omega)$ the completion of $(C_0^\infty(\Omega))^2$ with respect to the norm

$$\|(\varphi_1, \varphi_2)\|_H^2 = \int_\Omega (|\nabla\varphi_1|^2 + |\varphi_2|^2)dx.$$

More precisely we have the following theorem:

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Theorem 1.1 *Given R and R_0 two positive real numbers, there exist $C > 0$ and $\alpha > 0$ such that inequality*

$$E_R(u(t)) \le Ce^{-\alpha t} E(u(0)) \tag{2}$$

holds for every u solution to (6) in the "Shatah–Struwe" class with initial data $\varphi = (\varphi_1, \varphi_2)$ supported in B_R and satisfying

$$E(\varphi) = \frac{1}{2}\int_\Omega \left(|\varphi_2|^2 + |\nabla_x \varphi_1|^2\right) dx + \int_\Omega \chi(x)\frac{|\varphi_1|^{p+1}}{p+1} dx \le R_0. \tag{3}$$

Next we consider the case $\chi(x) = \lambda \in \mathbb{R}$, $p > 1 + \sqrt{2}$ and for small data we obtain the polynomial decay of the local energy.

We are interested also in the following system:

$$\begin{cases} \Box u + \chi_1 u + \chi_2 u^5 = 0, & \text{on } \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u^0(x) \in H^1(\mathbb{R}^3) & \text{and} \quad \partial_t u(0,x) = u^1(x) \in L^2(\mathbb{R}^3), \end{cases} \tag{4}$$

where $\Box = \partial_t^2 - \triangle$, χ_1 and χ_2 are positives functions, of class C^1, with compact support such that $supp\chi_1 \cup supp\chi_2 \subset B_R$ for some $R > 0$ and satisfying

$$x \cdot \nabla \chi_1(x) \le 0 \quad \text{and} \quad x \cdot \nabla \chi_2(x) \le 4, \; \forall x \in \mathbb{R}^3. \tag{5}$$

We obtain the exponential decay of the local energy for the solutions of the solution of the critical Klein–Gordon equation by combining the time global Strichartz norms with the exponential decay of the local energy of the solutions of the Klein–Gordon equation.

The paper is organized as follows: In Sect. 2 we give the results on the Scattering. In Sect. 3 we prove the result on Exponential decay of the local energy of the solution for the wave equation. Section 4 is devoted the polynomial decay of the local energy of the semilinear wave equation with arbitrary exponent. In Sect. 5, we give the proof of the exponential decay of the local energy of solution for the Klein–Gordon equation.

2 Scattering for the Subcritical and Critical Wave Equation

We consider the following nonlinear wave equation,

$$\begin{cases} \Box u + \chi(x) u|u|^{p-1} = 0 \text{ on } \mathbb{R} \times \Omega, \\ u = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ u(0,x) = u^0(x) \in H_D(\Omega) \text{ and } \partial_t u(0,x) = u^1(x) \in L^2(\Omega), \end{cases} \tag{6}$$

where $\Omega = \mathbb{R}^3 \backslash O$ and O is a strictly convex compact with smooth boundary $\partial\Omega$, $O \subset B_R$ for some $R > 0$ and $p > 2$. The function χ is a positive and of class $\mathcal{C}^1$, with compact support such that $\text{supp}\chi \subset B_R$. Here the function χ is allowed to be equal to 1 near $\partial\Omega$. We denote by $H = H_D(\Omega) \times L^2(\Omega)$ the completion of $(C_0^\infty(\Omega))^2$ with respect to the norm

$$\|(\varphi_1, \varphi_2)\|_H^2 = \int_\Omega (|\nabla\varphi_1|^2 + |\varphi_2|^2) dx.$$

Global existence and uniqueness of the solutions to the Cauchy problem (6) has been studied in [4, 14, 26].

Consequently, for every initial data (u^0, u^1) in the energy space H and in the case $2 < p < 5$ (respectively, in the critical case $p = 5$), system (6) admits a unique solution $u \in \mathcal{C}(\mathbb{R}, H_D(\Omega))$ (respectively, in the "Shatah–Struwe" class, that is

$$u \in C(\mathbb{R}, H_D(\Omega)) \cap L^5_{loc}(\mathbb{R}, L^{10}(\Omega)),\ \partial_t u \in C(\mathbb{R}, L^2(\Omega)).$$

The global energy of u at time t is defined by

$$E(u(t)) = \frac{1}{2} \int_\Omega \left(|\partial_t u(t)|^2 + |\nabla_x u(t)|^2\right) dx + \frac{1}{p+1} \int_\Omega \chi(x)\, |u(t)|^{p+1}\, dx, \tag{7}$$

which is time independent.

We define the local energy by

$$E_\rho(u(t)) = \frac{1}{2} \int_{\Omega\cap B_\rho} \left(|\partial_t u(t)|^2 + |\nabla_x u(t)|^2\right) dx + \frac{1}{p+1} \int_{\Omega\cap B_\rho} \chi(x)\, |u(t)|^{p+1}\, dx, \tag{8}$$

where B_ρ is a ball of radius $\rho > 0$ containing the obstacle O.

For every $t \in \mathbb{R}$, we define the wave operator $U(t)$ by

$$\begin{array}{rccl} U(t): & H & \longrightarrow & H \\ & (\varphi_1, \varphi_2) & \longmapsto & U(t)(\varphi_1, \varphi_2) = (u(t), \partial_t u(t)), \end{array}$$

where u is the solution of (6) in the "Shatah–Struwe" class with initial data $\varphi = (\varphi_1, \varphi_2)$.

The family $(U(t))_{t\in\mathbb{R}}$ forms a one parameter continuous group on H, to which we will refer as the nonlinear wave group.

Let us consider the wave equation in exterior domain

$$\begin{cases} \Box u = 0 \text{ on } \mathbb{R} \times \Omega, \\ u = 0 \text{ in } \mathbb{R} \times \partial\Omega, \\ u(0, x) = \varphi_1(x) \in H_D(\Omega) \text{ and } \partial_t u(0, x) = \varphi_2(x) \in L^2(\Omega). \end{cases} \tag{9}$$

We denote $U_L(t)$ the linear wave group.

We first recall the following result due to the author [4] (see [7] for the subcritical case) who prove that u is equivalent for the energy norm, as $t \to +\infty$, to a solution of the linear equation:

Theorem 2.1 *The nonlinear wave group outside a compact convex obstacle is asymptotically complete with respect to the linear wave group in the same domain. More precisely, with the notations defined above we have*

(a) *If u is the solution of (6) (with $2 < p \leq 5$), then there exists a unique finite energy solution u_+ of*

$$\begin{cases} \Box u_+ = 0 \ on \ \mathbb{R} \times \Omega, \\ u_+ = 0 \ in \ \mathbb{R} \times \partial\Omega \end{cases}$$

such that $E_c((u_+ - u)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0$, where

$$E_c((u_+ - u)(t)) = \frac{1}{2} \int_{\Omega} \left(|\partial_t (u_+ - u)(t)|^2 + |\nabla_x (u_+ - u)(t)|^2 \right) dx.$$

(b) *The wave operator defined by*

$$\Omega_+ : \begin{array}{ccc} H & \longrightarrow & H \\ (u_{/t=0}, \partial_t u_{/t=0}) & \longmapsto & (u_{+/t=0}, \partial_t u_{+/t=0}) \end{array}$$

is a bijection.

(c) *Similar results to a) and b) hold if t goes to $-\infty$.*

We note that a large number of works have been devoted to the theory of scattering for the nonlinear wave equation. In addition to the works of Bahouri and Gérard [2], Ginibre and Vélo [14], and Pecher [24] which concerned the semilinear wave equation in free space, we have the results obtained by Nakanishi [22] for the Klein–Gordon equation.

To establish Theorem 2.1 we prove first that the Strichartz norms of the solutions to (6) are global in time. The scattering theorem then follows as in [2].

Remark 2.2 The result of the Theorem 2.1 remains true if $\chi = 1$ on Ω.

In order to prove that the Strichartz norms for the solutions of (6) are global in time, we recall the following theorem due to H.Smith and C.D.Sogge [27].

Theorem 2.3 *Let u be a solution of the following system*

$$(S) \begin{cases} \Box u = F(t, x) \ \in L^1(\mathbb{R}, L^2(\Omega)) \ on \ \mathbb{R} \times \Omega \\ u = 0 \ \ on \ \mathbb{R} \times \partial\Omega \\ u(0, x) = u^0(x) \in H_D(\Omega) \ and \ \partial_t u(0, x) = u^1(x) \in L^2(\Omega) \end{cases}.$$

Given $r \geq 2$, q given by $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, there exists $C_r > 0$ such that,

$$\|u\|_{L^q(\mathbb{R},L^{3r}(\Omega))} \leq C_r(E(u)(0)^{1/2} + \|F\ (t,x)\|_{L^1(\mathbb{R},L^2(\Omega))}).$$

From this theorem we deduce

Proposition 2.4 *Given $r \geq 2$, q given by $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, there exists $C_r > 0$, such that for every $T \geq 0$, for every u solution of (S) we have*

$$\|u\|_{L^q([0,T],L^{3r}(\Omega))} \leq C_r(E(u(0))^{1/2} + \|F\ (t,x)\|_{L^1([0,T],L^2(\Omega))}). \tag{10}$$

Proof For $T > 0$, we define a cutoff function by

$$\chi_T = \begin{cases} 1 \text{ if } 0 \leq t \leq T \\ 0 \text{ if not} \end{cases},$$

and we consider, v_T the solution of the system

$$\begin{cases} \Box v_T + \chi_T F(t,x) = 0 \text{ on } \mathbb{R} \times \Omega \\ v_T = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ (v_T(0,x), \partial_t v_T(0,x)) = (u^0(x), u^1(x)) \end{cases},$$

where u is the solutions of (S) with initial data $(u^0(x), u^1(x))$ in H. By virtue of local time Strichartz estimate of [27], we have $\chi_T F(t,x) \in L^1(\mathbb{R}, L^2(\Omega))$ and thanks to the previous theorem, we deduce

$$\|v_T\|_{L^q(\mathbb{R},L^{3r}(\Omega))} \leq C_r(E(v_T(0))^{1/2} + \|\chi_T F(t,x)\|_{L^1(\mathbb{R},L^2(\Omega))})$$

and, therefore,

$$\|v_T\|_{L^q([0,T],L^{3r}(\Omega))} \leq C_r(E(u)(0)^{1/2} + \|F(t,x)\|_{L^1([0,T],L^2(\Omega))}).$$

Since $u = v_T$ on $[0,T] \times \Omega$, we obtain

$$\|u\|_{L^q([0,T],L^{3r}(\Omega))} \leq C_r(E(u)(0)^{1/2} + \|F(t,x)\|_{L^1([0,T],L^2(\Omega))}).$$

□

2.1 The Subcritical Case

We are interesting to the semilinear system:

$$\begin{cases} \Box u + f(x,u) = 0 \text{ on } \mathbb{R} \times \Omega \\ u = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ u(0,x) = u^0(x) \in H_D(\Omega) \text{ and } \partial_t u(0,x) = u^1(x) \in L^2(\Omega) \end{cases}. \tag{11}$$

Here f is defined by $f(x,u) = \chi(x)\, g(u)$, where χ is a function of class C^1 with compact support in B_R satisfying $\chi \geq 0$ and $\frac{\partial \chi}{\partial r} = \chi_r \leq 0$.

Moreover g is a function from $\mathbb{R}$ to $\mathbb{R}$ of class C^1 satisfying

$$\begin{cases} \text{for every } s \in \mathbb{R},\ \ sg(s) \geq 0 \\ \text{there exists } \delta > 0 \text{ and } p_0 \geq 1,\ \text{ such that, } sg(s) - 2G(s) \geq \delta\,|s|^{p_0+1} \\ \text{where } G(s) = \int_0^s g(t)\,dt \end{cases} \tag{12}$$

and

$$\begin{cases} \text{there exist } p_1, p; 2 < p_1 \leq p < 5, \text{ such that,} \\ \text{for every } s \in \mathbb{R},\ \ \left|g^{(j)}(s)\right| \leq C(|s|^{p_1-j} + |s|^{p-j}),\ j = 0, 1. \end{cases} \tag{13}$$

The typical example is $f(x,u) = \chi(x)\,|u|^{p-1}\,u$.

The global energy of u at time t is defined by:

$$E(u(t)) = \frac{1}{2}\int_\Omega |\partial_t u(t)|^2 + |\nabla_x u(t)|^2 + \int_\Omega \chi(x) G(u(t))dx \tag{14}$$

and is time independent.

We define the local energy for every $\rho > 0$ by:

$$E_\rho(u(t)) = \frac{1}{2}\int_{\Omega\cap B_\rho} |\partial_t u(t)|^2 + |\nabla_x u(t)|^2 + \int_{\Omega\cap B_\rho} \chi(x) G(u(t))dx. \tag{15}$$

2.1.1 Prisized Morawetz Estimate

We adapt the proof of Morawetz in [20] to obtain the following result:

Proposition 2.5 *The solution u of (11) satisfies*

$$\chi u \in L^{p_0+1}(\mathbb{R}_+ \times \Omega). \tag{16}$$

Proof We denote

$$\begin{cases} Q(u) = \chi G(u), \\ Q_u(u) = \chi g(u), \\ Q_r(u) = \chi_r G(u). \end{cases}$$

Moreover arguing by density we can suppose the initial data φ in $(\mathcal{C}_0^\infty(\Omega))^2$. Thanks to Morawetz multiplier, we have

$$\begin{aligned}2r^{-1}(x.\nabla u+u)(\Box u+Q_u(u)) &= 2(r^{-1}\partial_t((x.\nabla u+u)\partial_t u))-2r^{-3}(x.\nabla u)^2\\ &+2r^{-1}\,|\nabla u|^2+2r^{-1}(uQ_u-2Q)-2Q_r(u)\\ &+div\left\{r^{-1}(-(\partial_t u)^2x-2(x.\nabla u)\nabla u\right.\\ &\left.+\,|\nabla u|^2\,x-2u\nabla u-r^{-2}u^2x+2Q.x)\right\}.\end{aligned}$$

Integrating over Ω, one has

$$\begin{aligned}&-\frac{d}{dt}\int_\Omega\left[2r^{-1}(x.\nabla u+u)\partial_t u)\right]dx\\ &=\int_{\partial\Omega}(-2r^{-1}(x.\nabla u)\nabla u+r^{-1}\,|\nabla u|^2x).\nu d\sigma+2\int_\Omega r^{-1}(|\nabla u|^2-(r^{-1}x.\nabla u)^2)dx\\ &+2\int_\Omega r^{-1}(uQ_u-2Q)dx-2\int_\Omega Q_r(u)dx.\end{aligned}$$

On the other hand

$$(-2r^{-1}(x.\nabla u)\nabla u+r^{-1}\,|\nabla u|^2x)=-\left\langle r^{-1}x,\nu\right\rangle\left|\frac{\partial u}{\partial n}\right|^2,$$

where ν is the outward normal; so

$$\begin{aligned}&-\frac{d}{dt}\int_\Omega\left[2r^{-1}(x.\nabla u+u)\partial_t u)\right]dx-2\int_\Omega r^{-1}(|\nabla u|^2-u_r^2)dx\\ &=2\int_\Omega r^{-1}(uQ_u-2Q)dx-2\int_\Omega Q_r(u)dx-\int_{\partial\Omega}\left\langle r^{-1}x,\nu\right\rangle\left|\frac{\partial u}{\partial n}\right|^2d\sigma.\end{aligned}$$

Since O is convex, we have $\left\langle r^{-1}x,\nu\right\rangle\le 0$; in addition, by hypothesis on χ and f, $(uQ_u-2Q)r^{-1}\ge 0$ and $Q_r(u)\le 0$.

This yields

$$\int_\Omega r^{-1}(uQ_u-2Q)dx\le-\frac{d}{dt}\int_\Omega\left[2r^{-1}(x.\nabla u+u)\partial_t u)\right]dx.$$

For every t_0, we have

$$\int_{\Omega(t=t_0)}\left[r^{-1}(x.\nabla u+u)\partial_t u)\right]dx\le 2E(0).$$

Hence for $T \geq 0$, $\int_0^T \int_\Omega r^{-1}(uQ_u - 2Q)dxdt \leq 4E(0)$,
which yields for every $\Omega_\rho = \Omega \cap B_\rho$

$$\int_0^T \int_{\Omega_\rho} (uQ_u - 2Q)dxdt \leq 4\rho E(0).$$

Now, by (12) there exists $p_0 \geq 1$ such that,

$$uQ_u - 2Q = \chi ug - 2\chi G \geq C\delta\chi\, |u|^{p_0+1}$$

which gives

$$\begin{aligned}\int_0^T \int_\Omega \chi\, |u|^{p_0+1}\, dxdt &\leq C_{R,\delta} \int_0^T \int_\Omega (uQ_u - 2Q)dxdt \\ &\leq C_{R,\delta}\; E(0).\end{aligned}$$

□

2.1.2 Global Time Strichartz Norms

Let us recall the following bootstrap lemma (see [2]).

Lemma 2.6 *Let $M(t)$ be a nonnegative continuous function in $[0, T]$ such that, for every $t \in [0, T]$,*

$$M(t) \leq a + bM(t)^\theta,$$

where $a, b > 0$ and $\theta > 1$ are constants such that,

$$a < (1 - \frac{1}{\theta})\frac{1}{(\theta b)^{1/\theta - 1}}, \quad M(0) \leq \frac{1}{(\theta b)^{1/\theta - 1}}.$$

Then for every $t \in [0, T]$, we have

$$M(t) \leq \frac{\theta}{\theta - 1}a.$$

Proposition 2.7 *Let u be a solution of (11), then for every $r > 2$ and q given by $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, we have:*

$$f(x, u) \in L^1(\mathbb{R}, L^2(\Omega)) \text{ et } u \in L^q(\mathbb{R}, L^{3r}(\Omega)). \tag{17}$$

Proof We recall that u is solution of

$$\begin{cases} \Box u + f(x,u) = 0 \text{ on } \mathbb{R} \times \Omega \\ u = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ u(0,x) = u^0(x) \in H_D(\Omega) \text{ and } \partial_t u(0,x) = u^1(x) \in L^2(\Omega), \end{cases}$$

with $f(x,u) = \chi(x)\, g(u)$ where g verifies, in particular,

$$\begin{cases} \text{there exist } p_1 \text{ and } p;\ 2 < p_1 \le p < 5, \text{ such that,} \\ \forall s \in \mathbb{R},\ \ |g(s)| \le C(|s|^{p_1} + |s|^p) \\ \text{and} p_0 \ge 1,\, \delta > 0 \text{ such that } sg(s) - 2G(s) \ge \delta\, |s|^{p_0+1}. \end{cases}$$

Thanks to Hölder's inequality, we obtain for every $\lambda \ge 1$:

$$\begin{aligned} \|f(x,u)\|_{L^1([T,S],L^2(\Omega))} &\le \|\chi u\|^p_{L^p([T,S],L^{2p\lambda}(\Omega))} \\ &\le C(\|\chi u\|^{\eta_1}_{L^{p_0+1}([T,S]\times\Omega)} \|u\|^{\theta_1}_{L^{q_1}([T,S],L^{3r_1}(\Omega))} \\ &\quad + \|\chi u\|^{\eta_2}_{L^{p_0+1}([T,S]\times\Omega)} \|u\|^{\theta_2}_{L^{q_2}([T,S],L^{3r_2}(\Omega))}), \end{aligned} \tag{18}$$

with $q_i = \frac{8p_i\lambda - 2p_0\lambda - 3p_0 - 2\lambda - 3}{p_i\lambda - p_0\lambda + 5\lambda - 3}$, $r_i = \frac{2(8p_i\lambda - 2p_0\lambda - 3p_0 - 2\lambda - 3)}{3(2\lambda p_i + 1 - p_0 - 4\lambda)}$, $\theta_i = \frac{8p_i\lambda - 2p_0\lambda - 3p_0 - 2\lambda - 3}{\lambda(7-p_0)}$ and $\eta_i = \frac{p_0\lambda - p_i\lambda + 3p_0 + 2\lambda + 3}{\lambda(7-p_0)}$, such that $\frac{1}{q_i} + \frac{1}{r_i} = \frac{1}{2}$, for $i = 1, 2$ and $p_2 = p$.

By proposition 2.4, and thanks to the conservation of global energy, we obtain

$$\begin{aligned} &\|u\|_{L^{q_1}([T,S],L^{3r_1}(\Omega))} + \|u\|_{L^{q_2}([T,S],L^{3r_2}(\Omega))} \\ &\le C_r(E(u(T))^{1/2} + \|f(x,u)\|_{L^1([T,S],L^2(\Omega))}) \\ &\le C_r(E(u(0))^{1/2} + \|\chi u\|^{\eta_1}_{L^{p_0+1}([T,S]\times\Omega)} \|u\|^{\theta_1}_{L^{q_1}([T,S],L^{3r_1}(\Omega))} \\ &\|\chi u\|^{\eta_2}_{L^{p_0+1}([T,S]\times\Omega)} \|u\|^{\theta_2}_{L^{q_2}([T,S],L^{3r_2}(\Omega))}). \end{aligned}$$

Thanks to the inequality:: $1 + x^a + y^b \le (1 + x + y)^a$ for every x and $y \ge 0$, and $a > b \ge 0$, we obtain

$$\begin{aligned} &1 + \|u\|_{L^{q_1}([T,S],L^{3r_1}(\Omega))} + \|u\|_{L^{q_2}([T,S],L^{3r_2}(\Omega))} \\ &\le C_r((E(u(0)))^{1/2} + \|\chi u\|^{\eta'}_{L^{p_0+1}([T,S]\times\Omega)} \\ &\times(1 + \|u\|_{L^{q_1}([T,S],L^{3r_1}(\Omega))} + \|u\|_{L^{q_2}([T,S],L^{3r_2}(\Omega))})^{\theta'}), \end{aligned} \tag{19}$$

where $\eta' = \min(\eta_1, \eta_2)$ and $\theta' = \max(\theta_1, \theta_2)$.

We choose T large enough so that the conditions of lemma 2.6 are satisfied, then

$$\|u\|_{L^{q_1}([T,S],L^{3r_1}(\Omega))} + \|u\|_{L^{q_2}([T,S],L^{3r_2}(\Omega))} \le C_{r,E_0}.$$

Finally back to (19) and (10), we deduce that:

$f(x,u) \in L^1(\mathbb{R}_+, L^2(\Omega))$ and $u \in L^q(\mathbb{R}_+, L^{3r}(\Omega))$; $\forall\, r > 2$ and q such that $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$.

□

2.1.3 The Proof of Theorem 2.1

First we give the following properties:

Proposition 2.8 *If $F \in L^1(\mathbb{R}, L^2(\Omega))$, then there exists a unique finite energy solution w of*

$$\begin{cases} \Box w = F \text{ on } \mathbb{R} \times \Omega \\ w = 0 \text{ on } \mathbb{R} \times \partial\Omega. \end{cases} \tag{20}$$

Satisfying $E_0(w(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0$ *where* $E_0(w(t)) = \frac{1}{2} \int_\Omega |\partial_t w(t)|^2 + |\nabla_x w(t)|^2 dx.$

This result has already been used in free space by [2].

Proof Let w_T be a solution of $\begin{cases} \Box w_T = F \text{ on } \mathbb{R} \times \Omega \\ w_T = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ E_0(w_T(T)) = 0. \end{cases}$

$(w_T(0), \partial_t w_T(0))$ is a Cauchy sequence in H.

In fact; let $T > S$:

$$\begin{cases} \Box (w_T - w_S) = 0 \text{ on } \mathbb{R} \times \Omega \\ w_T - w_S = 0 \text{ on } \mathbb{R} \times \partial\Omega \end{cases}.$$

This system is conservative then,

$$\begin{aligned} E_0((w_T - w_S)(0)) &= E_0((w_T - w_S)(T)) \\ &\leq E_0(w_S(T)) \\ &\leq E_0(w_S(S)) + \|F\|_{L^1([S,T[,L^2(\Omega))} = \|F\|_{L^1([S,T[,L^2(\Omega))}. \end{aligned}$$

The second member goes to 0, when S, T goes to infinity, since $F \in L^1(\mathbb{R}, L^2(\Omega))$.

Finally w_T converges to w solution of

$$\begin{cases} \Box w = F \text{ on } \mathbb{R} \times \Omega \\ w = 0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases}$$

and

$$E_0^{\frac{1}{2}}(w(t)) \leq E_0^{\frac{1}{2}}((w - w_T)(t)) + \|F\|_{L^1([T.t[,L^2(\Omega))}$$

$$\leq \left(E_0^{\frac{1}{2}}((w - w_T)(0)) + \|F\|_{L^1([T.t[,L^2(\Omega))} \right) \underset{t \longrightarrow +\infty}{\longrightarrow} 0.$$

□

Remark 2.9 In addition we have, for all $T \geq 0$, $r > 2$, and q given by $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$,

$$\|w\|_{L^q([T,+\infty[,L^{3r}(\Omega))} \leq C_r \|F\|_{L^1([T,+\infty[,L^2(\Omega))} \,. \tag{21}$$

We come back to the proof of Theorem 2.1.

Proof $f(x, u) \in L^1(\mathbb{R}, L^2(\Omega)$ for every u solution of (11), therefore, there exists a unique w_+ such that, $\Box w_+ - f\,(x, u) = 0$ and $E_0(w_+(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0.$

$$u_+ = u + w_+ \text{ satisfies then } \begin{cases} \Box u_+ = 0 \text{ on } \mathbb{R} \times \Omega, \\ u_+ = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ E_0((u - u_+)\,(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

To prove the uniqueness of u_+; we consider u_+^1 and u_+^2 such that,

$$\begin{cases} \Box u_+^i = 0 \text{ on } \mathbb{R} \times \Omega, \\ u_+^i = 0 \text{ on } \mathbb{R} \times \partial\Omega, \quad \text{for } i = 1, 2, \\ E_0(\left(u - u_+^i\right)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Then $v = u_+^1 - u_+^2$ satisfies

$$\begin{cases} \Box v = 0 \text{ on } \mathbb{R} \times \Omega, \\ v = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ E_0((v)\,(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Using the conservation of the energy, we conclude that $v = 0$. So the wave operator

$$\begin{array}{rccc} \Omega_+ : & H & \longrightarrow & H \\ & (u_{/t=0}, \partial u_{/t=0}) & \longmapsto & (u_{+/t=0}, \partial_t u_{+/t=0}) \end{array}$$

is well defined.

Conversely, let v a finite energy solution of

$$\begin{cases} \Box v = 0 \text{ sur } \mathbb{R} \times \Omega \\ v = 0 \text{ sur } \mathbb{R} \times \partial\Omega. \end{cases}$$

We will prove that there exists a unique v_+, satisfying

$$\begin{cases} \Box v_+ + f(x, v_+) = 0 \text{ on } \mathbb{R} \times \Omega \\ u_+ = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ E_0((v_+ - v)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0 \end{cases}.$$

Let $\delta > 0$ to be fixed later, and $T \geq 0$ such that

$$\|v\|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \|v\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))} \leq \delta,$$

for r et r_1 such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ and $\frac{1}{p_1} + \frac{1}{r_1} = \frac{1}{2}$.

Setting

$$B_\delta = \left\{ v \in H_D \times L^2,\ \|v\|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \|v\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))} \leq \delta \right\},$$

and

$$\begin{aligned} S : B_\delta &\longrightarrow B_\delta \\ w &\longmapsto \widetilde{w} \end{aligned} \tag{22}$$

where $\widetilde{w}$ is defined by

$$\begin{cases} \Box \widetilde{w} + f(x, v + w) = 0 \text{ on } \mathbb{R} \times \Omega \\ \widetilde{w} = 0 \text{ on } \mathbb{R} \times \partial\Omega \\ E_0(\widetilde{w}(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0 \end{cases}.$$

Using the fact that $\frac{2p}{3r} < 1$ and $\frac{2p_1}{3r_1} < 1$ we deduce

$$\begin{aligned} \|f(x, v + w)\|_{L^1([T,+\infty[,L^2(\Omega))} &\leq C(\|v + w\|^p_{L^p([T,+\infty[,L^{3r}(\Omega))} \\ &\quad + \|v + w\|^{p_1}_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))}) \\ &\leq C^1_{R_0,p,p_1}(\delta^{p_1-1} + \delta^{p-1})\delta, \end{aligned}$$

which, by hyperbolic inequality, yields

$$E_0^{1/2}(\tilde{w}(T)) \leq 2C(\delta^p + \delta^{p_1}).$$

This allows one to apply the fixed point argument to find v_+ on the interval $[T, \infty[$.

Indeed by Remark 2.9 for w_1 and w_2 in B_δ, we have

$$\begin{aligned} &\|\widetilde{w_1} - \widetilde{w_2}\|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \|\widetilde{w_1} - \widetilde{w_2}\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))} \\ &\leq C_{p,p_1} \|f(x, v + w_1) - f(x, v + w_2)\|_{L^1([T,+\infty[,L^2(\Omega))}. \end{aligned} \tag{23}$$

In addition, the relation (13) yields

$$\begin{aligned}
&|f(x, v+w_1) - f(x, v+w_2)| \\
&= \left| \int_0^1 \frac{d}{d\lambda} f(x, \lambda (v+w_1) + (1-\lambda)(v+w_2)) \right| \\
&= \left| \chi \int_0^1 (w_1 - w_2) . \nabla g(\lambda (v+w_1) + (1-\lambda)(v+w_2)) \right| \\
&\leq C\chi \, |w_1 - w_2| \left((|v+w_1| + |v+w_2|)^{p-1} + (|v+w_1| + |v+w_2|)^{p_1-1} \right).
\end{aligned}$$

Let β and β' such that $\frac{1}{\frac{3r}{2}} + \frac{1}{\beta} = \frac{1}{\frac{3r1}{2}} + \frac{1}{\beta'} = 1$, then

$$\begin{aligned}
&\| f(x, v+w_1) - f(x, v+w_2) \|_{L^1([T,+\infty[,L^2(\Omega))} \\
&\leq C(\|w_1 - w_2\|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \|w_1 - w_2\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))}) \times \\
&(\| (|v+w_1| + |v+w_2|)\, \chi \|^{p-1}_{L^p([T,+\infty[,L^{2\beta(p-1)}(\Omega))} \\
&+ \| (|v+w_1| + |v+w_2|)\, \chi \|^{p_1-1}_{L^{p_1}([T,+\infty[,L^{2\beta'(p_1-1)}(\Omega))}).
\end{aligned}$$

Since $2\beta \,(p-1) \leq 3r$ and $2\beta' \,(p_1 - 1) \leq 3r_1$ then

$$\begin{aligned}
&\| f(x, v+w_1) - f(x, v+w_2) \|_{L^1([T,+\infty[,L^2(\Omega))} \\
&\leq C_{R,p}(\|w_1 - w_2\|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \|w_1 - w_2\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))}) \\
&(\| |v+w_1| + |v+w_2| \|^{p-1}_{L^p([T,+\infty[,L^{3r}(\Omega))} + \| |v+w_1| + |v+w_2| \|^{p_1-1}_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))}.
\end{aligned} \tag{24}$$

Combining (23) and (24) we obtain

$$\begin{aligned}
&\| \widetilde{w_1} - \widetilde{w_2} \|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \| \widetilde{w_1} - \widetilde{w_2} \|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))} \\
&\leq \ C^2_{R_0,p,p_1} (\delta^{p_1-1} + \delta^{p-1})(\|w_1 - w_2\|_{L^p([T,+\infty[,L^{3r}(\Omega))} \\
&+ \|w_1 - w_2\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))}).
\end{aligned} \tag{25}$$

So choosing δ such that $\delta^{p_1-1} + \delta^{p-1} < \min(\frac{1}{C^1_{R_0,p,p_1}}, \frac{1}{C^2_{R,p,p_1}})$, we see that S is well defined, and with Lipschitz constant smaller than 1. As a consequence there exists w_+ with finite energy satisfying

$$\begin{cases}
\Box w_+ + f(x, v+w_+) = 0 \text{ on } \mathbb{R} \times \Omega \\
w = 0 \ \text{ on } \mathbb{R} \times \partial\Omega \\
E_0(w_+(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0.
\end{cases}$$

Setting $v_+ = v + w_+$ and extending v_+ to $\mathbb{R} \times \Omega$, using existence and uniqueness for the Cauchy problem, we conclude that there exists v_+ satisfying

$$\begin{cases} \Box v_+ + f(x, v_+) = 0 \text{ on } \mathbb{R} \times \Omega \\ v_+ = 0 \ \text{ on } \ \mathbb{R} \times \partial\Omega \\ E_0((v_+ - v)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Moreover $E_0((v_+)(0))$ is controlled by $E_0((v_+)(T))$ which is finite.

For the uniqueness of v_+, we consider v_+^1 and v_+^2 such that

$$\begin{cases} \Box v_+^i + f(x, v_+^i) = 0 \text{ on } \mathbb{R} \times \Omega, \\ v_+^i = 0 \ \text{ on } \ \mathbb{R} \times \partial\Omega, \quad \text{for } i = 1, 2, \\ E_0(\big(v_+^i - v\big)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

$u = v_+^1 - v_+^2$ satisfies then

$$\begin{cases} \Box u + f(x, v_+^1) - f(x, v_+^2) = 0 \text{ on } \mathbb{R} \times \Omega, \\ u = 0 \ \text{ on } \ \mathbb{R} \times \partial\Omega, \\ E_0((u)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

As in (25), we can prove that

$$\begin{aligned} &\|u\|_{L^p([T,+\infty[,L^{3r}(\Omega))} + \|u\|_{L^{p_1}([T,+\infty[,L^{3r_1}(\Omega))} \\ &\leq C_{R,p}\Big(\|u\|_{L^p(L^{3r})} + \|u\|_{L^{p_1}(L^{3r_1})}\Big)\Big(\|v_+^1\|_{L^p(L^{3r})}^{p-1} + \|v_+^2\|_{L^{p_1}(L^{3r_1})}^{p_1-1}\Big). \end{aligned}$$

Then choosing $T > 0$ such that

$$C_{R,p}\Big(\|v_+^1\|_{L^p(L^{3r})}^{p-1} + \|v_+^2\|_{L^{p_1}(L^{3r_1})}^{p_1-1}\Big) < 1,$$

we obtain $v_+^1(t) = v_+^2(t)$ $\forall t \geq T$ and conclude by uniqueness for the Cauchy problem (11).

Finally we conclude that the wave operator Ω_+ is a bijection, and using the same method we get a similar result when t goes to $-\infty$. □

2.2 The Critical Case

Here we suppose that $p = 5$.

2.2.1 Global Time Strichartz Norms

Let

$$e(t) = \int_{\substack{|x|\le t\\ x\in\Omega}} \left[\frac{1}{2}\left|\nabla_x u(t,x)\right|^2 + \frac{1}{2}\left|\partial_t u(t,x)\right|^2 + \frac{1}{6}\chi(x)\left|u(t,x)\right|^6\right] dx.$$

We then obtain the following lemma which is similar to the one in [2] in free space.

Lemma 2.10 *There exists $D > 0$ such that, for all $b > a > R$, for every solution u to $\Box u + \chi(x)u^5 = 0$, with $u \in C([a,b], H_D(\Omega)) \cap L^5([a,b], L^{10}(\Omega))$, $\partial_t u \in C([a,b], L^2(\Omega))$ we have*

$$\int_{\substack{|x|\le b\\ x\in\Omega}} \chi(x)\left|u(b,x)\right|^6 dx \le D\left[\frac{a}{b}(e(a)+e(a)^{1/3}) + e(b) - e(a) + (e(b)-e(a))^{1/3}\right].$$

Proof We use the notations of [26]. Let

$$K_a^b = \{(t,x), a \le t \le b, |x| \le t\} \cap \Omega$$

the truncated light cone,

$$M_a^b = \{(t,x), a \le t \le b, |x| = t\} \cap \Omega$$

the "mantle" associated with K_a^b, and

$$D(t) = \{(t,x), |x| \le t\} \cap \Omega$$

its spacelike sections. We note that

$$\partial K_a^b = D(a) \cup D(b) \cup M_a^b \cup \left\{(t,x) \in K_a^b \cap \partial\Omega\right\}.$$

We start with an initial data in $\left(C_0^\infty(\Omega)\right)^2$, hence the associated solution is of class C^∞(see [26]).

Multiplying equation (6) by $\partial_t u$ and using an argument of scaling we obtain for all $x_0 \in supp(\chi)$ which can be chosen and fixed

$$div_{t,x}(\frac{1}{2}\left|\nabla_x u(t,x)\right|^2 + \frac{1}{2}\left|\partial_t u(t,x)\right|^2 + \frac{1}{6}\chi(x-x_0)\left|u(t,x)\right|^6, -\partial_t u\nabla_x u) = 0, \tag{26}$$

then we integrate (26) over the truncated cone K_a^b to obtain the classical energy identity

$$e(b)-e(a)=\int_{M_a^b}(\frac{1}{2}\left|\frac{x}{t}\partial_t u+\nabla_x u\right|^2+\frac{1}{6}\chi(x-x_0)\,|u|^6)\frac{d\sigma}{\sqrt{2}}. \tag{27}$$

Moreover, multiplying (6) by $Lu=(t\partial_t+x\cdot\nabla+1)u$, we obtain

$$div_{t,x}(tQ+\partial_t uu,-tP)+(\frac{1}{3}-\frac{1}{6}x\cdot\nabla\chi(x-x_0))u^6=0, \tag{28}$$

where

$$Q=\frac{1}{2}(|\nabla_x u|^2+|\partial_t u|^2)+\frac{1}{6}\chi(x-x_0)u^6+\partial_t u\frac{x}{t}\cdot\nabla_x u$$

and

$$P=\frac{x}{t}(\frac{1}{2}(|\partial_t u|^2-|\nabla_x u|^2)-\frac{1}{6}\chi(x-x_0)u^6)+\nabla_x u(\partial_t u+\frac{x}{t}\cdot\nabla_x u+\frac{u}{t}).$$

Integrating (28) over K_a^b we obtain

$$\begin{aligned}0&=\int_{D(b)}(bQ+(\partial_t u)u)dx-\int_{D(a)}(aQ+(\partial_t u)u)dx\\&\quad-\int_{M_a^b}(tQ+\partial_t uu+x\cdot P)\frac{d\sigma}{\sqrt{2}}+\int_{(t,x)\in K_a^b\cap\partial\Omega}\nu\cdot(-tP)d\sigma\\&\quad+\int_{K_a^b}(\frac{1}{3}-\frac{1}{6}x\cdot\nabla\chi(x-x_0))u^6dxdt\\&=I+II+III+IV+V,\end{aligned} \tag{29}$$

where ν is the outward unit normal to O.

We start with the term III. Since $t=|x|$ on M_a^b, we can write

$$III=-\int_{M_a^b}(|x|\,|\partial_t u|^2+2(\partial_t u)x\cdot\nabla_x u+\frac{|x\cdot\nabla_x u|^2}{|x|}+u\frac{x\cdot\nabla_x u}{|x|}+(\partial_t u)u)\frac{d\sigma}{\sqrt{2}}.$$

We parameterize M_a^b by

$$\Omega\ni y\longrightarrow(|y|\,,y),\quad a\le|y|\le b,$$

and let $v(y)=u(|y|\,,y)$. Then

$$d\sigma=\sqrt{2}dy\quad\text{and}\quad y\cdot\frac{\nabla v}{|y|}=\frac{x\cdot\nabla_x u}{|x|}+\partial_t u.$$

This yields

$$III = -\int_{\substack{y\in\Omega \\ a\le|y|\le b}} \left\{ \frac{|y\cdot\nabla v|^2}{|y|} + v\frac{y\cdot\nabla v}{|y|} \right\} dy$$
$$= -\int_{\substack{y\in\Omega \\ a\le|y|\le b}} \frac{|v+y\cdot\nabla v|^2}{|y|} dy + \int_{\substack{y\in\Omega \\ a\le|y|\le b}} \left(\frac{v^2}{|y|} + v\frac{y\cdot\nabla v}{|y|} \right) dy.$$

By integrating by parts, one sees that

$$\int_{\substack{y\in\Omega \\ a\le|y|\le b}} v\frac{y\cdot\nabla v}{|y|} dy = \frac{1}{2}\int_{\substack{y\in\Omega \\ |y|=b}} v^2 d\sigma - \frac{1}{2}\int_{\substack{y\in\Omega \\ |y|=a}} v^2 d\sigma - \int_{\substack{y\in\Omega \\ a\le|y|\le b}} \frac{v^2}{|y|} dy.$$

So if we switch back to the original coordinates, we have

$$III = -\int_{M_a^b} \left|\frac{1}{t}Lu\right|^2 t\frac{d\sigma}{\sqrt{2}} + \frac{1}{2}\int_{\partial D_b} u^2 d\sigma - \frac{1}{2}\int_{\partial D_a} u^2 d\sigma. \tag{30}$$

Now, we rewrite the first and second term of (29) as

$$I + II = H(b) - H(a) - \frac{1}{b}\int_{D_b} (x\cdot\nabla_x uu + \frac{3}{2}u^2)dx + \frac{1}{a}\int_{D_a} (x\cdot\nabla_x uu + \frac{3}{2}u^2)dx, \tag{31}$$

where

$$H(t) = \int_{D(t)} (t\left[\frac{1}{2}\left|\frac{1}{t}Lu\right|^2 + \frac{1}{2}\left(|\nabla_x u|^2 - \left|\frac{x\cdot\nabla_x u}{t}\right|^2\right) + \chi(x-x_0)\frac{|u|^6}{6}\right] + \frac{u^2}{t})dx.$$

As above, a simple integration by parts gives

$$\int_{D(t)} (x\cdot\nabla_x uu + \frac{3}{2}u^2)dx = \frac{t}{2}\int_{\partial D(t)} u^2 d\sigma. \tag{32}$$

Therefore, we obtain from (29)–(32),

$$H(b) - H(a) + \int_{K_a^b} (\frac{1}{3} - \frac{1}{6}x\cdot\nabla\chi(x-x_0))u^6 dxdt = \int_{M_a^b} \left|\frac{1}{t}Lu\right|^2 t\frac{d\sigma}{\sqrt{2}} + \int_{(t,x)\in K_a^b\cap\partial\Omega} \nu\cdot tPd\sigma. \tag{33}$$

We note that on $[a,b]\times\Omega$,

$$\nabla_x u = (\partial_\nu u)\nu,\ \ u = \partial_t u = 0 \text{ and } (\nu\cdot x) \le 0 \text{ for every } x \in \partial\Omega,$$

which yields

$$\nu\cdot tP = \frac{1}{2}(\nu\cdot x)(\partial_\nu u)^2 \le 0. \tag{34}$$

The Hölder's inequality gives

$$\int_{D(t)} \chi(x - x_0)\frac{u^6}{6}dx \le \frac{1}{t}H(t) \le C_1(e(t) + e(t)^{1/3}) \tag{35}$$

and then

$$\begin{aligned}\int_{M_a^b} \left|\frac{1}{t}Lu\right|^2 t\frac{d\sigma}{\sqrt{2}} &\le \int_{M_a^b} \left(2b\left|\frac{x}{t}\partial_t u + \nabla_x u\right|^2 + 2\frac{u^2}{t}\right)\frac{d\sigma}{\sqrt{2}} \\ &\le bC_2([e(b) - e(a)] + [e(b) - e(a)]^{1/3}).\end{aligned} \tag{36}$$

Combining (33)–(36) we obtain the result for smooth initial data.

Now, by density argument and the continuity of the nonlinear map:

$$\begin{aligned} F: \quad H \quad &\longrightarrow C([0, T], H) \\ (\varphi, \psi) &\longmapsto \quad (u, \partial_t u), \end{aligned}$$

where u is the solution to (6) such that $(u, \partial_t u)_{/t=0} = (\varphi, \psi)$(see [10]), the result of this lemma holds for every data in H. □

Remark 2.11 We note that the main estimate in Lemma 2.10 is true for any star shaped obstacle. However, we will use in this article Lemma 2.10 for convex obstacles which is sufficient to prove our result.

We will now apply Theorem 2.3 and Lemma 2.10 to prove the following proposition.

Proposition 2.12 *Let u be a solution to (6), then*

$$\int_\Omega \chi(x)\,|u(t,x)|^6\,dx \underset{t\to\pm\infty}{\longrightarrow} 0, \tag{37}$$

and for all $q > 2$ and r such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ we have

$$u \in L^q(\mathbb{R}_+, L^{3r}(\Omega)). \tag{38}$$

Proof The classical energy identity (27) shows that $e(t)$ is a nondecreasing bounded function of t, hence it has a limit L as $t \to +\infty$. Applying Lemma 2.10 with $a = \varepsilon T$, $B = T$ and passing to the limit as $T \to +\infty$, we obtain

$$\limsup_{T\longrightarrow+\infty} \int_{\substack{|x|\le T \\ x\in\Omega}} \chi(x)\,|u(T,x)|^6\,dx \le D\varepsilon(L + L^{1/3}) \tag{39}$$

for every $\varepsilon > 0$, hence the left-hand side of (39) is 0. Using invariance of (6) by time translations, we get for every $R > 0$,

$$\lim_{T \longrightarrow +\infty} \int_{\substack{|x| \leq R+T \\ x \in \Omega}} \chi(x) \, |u(T,x)|^6 \, dx = 0. \tag{40}$$

On the other hand, the energy identity outside a forward wave cone leads to

$$\int_{\substack{|x| \geq T+R \\ x \in \Omega}} \chi(x) \, |u(T,x)|^6 \, dx \leq \int_{\substack{|x| \geq R \\ x \in \Omega}} \left[3 \, |\partial_t u(0,x)|^2 + 3 \, |\nabla_x u(0,x)|^2 + |u(0,x)|^6 \right] dx \tag{41}$$

which goes to 0 as $R \longrightarrow +\infty$.

Finally (37) follows by combining (40) and (41).

Now applying Proposition 2.1 with $q = r = 4$ we have

$$\begin{aligned} \|u\|_{L^4([T,S],L^{12}(\Omega))} &\leq C(E(u(T))^{1/2} + \left\| \chi(x) u^5 \right\|_{L^1([T,S],L^2(\Omega))}) \\ &\leq C(E(u(0))^{1/2} + \|\chi(x) u\|^5_{L^5([T,S],L^{10}(\Omega))} \end{aligned}$$

which yields by Hölder's inequality

$$\|u\|_{L^4([T,S],L^{12}(\Omega))} \leq C(E(u(T))^{1/2} + \|\chi u\|_{L^\infty([T,S],L^6(\Omega))} \, \|u\|^4_{L^4([T,S],L^{12}(\Omega))}).$$

Then, by choosing then T large enough and using Lemma 2.6 and (37), we deduce $u \in L^4(\mathbb{R}_+, L^{12}(\Omega))$ and by Hölder's inequality that $u \in L^5(\mathbb{R}_+, L^{10}(\Omega))$. Finally (38) follows by virtue of Theorem 2.3. □

2.2.2 The Proof of Theorem 2.1 in the Case $p = 5$

The arguments in the proof below are contained in [7]. We include them for the convenience of the reader and to make the paper self-contained.

First, we consider w_T the solution in the "Shatah–Struwe" class of the following system

$$\begin{cases} \Box w_T = -\chi(x) u^5 \text{ on } \mathbb{R} \times \Omega, \\ w_T = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ E_c(w_T(T)) = 0. \end{cases}$$

Here, the solution w_T converges to w_+ solution of $\begin{cases} \Box w_+ = -\chi(x) u^5 \text{ on } \mathbb{R} \times \Omega, \\ w_+ = 0 \text{ in } \mathbb{R} \times \partial\Omega \end{cases}$ "as t goes to infinity."

Moreover, we have

$$E_c^{\frac{1}{2}}(w_+(t)) \leq E_c^{\frac{1}{2}}((w_+ - w_T)(t)) + \left\| \chi(x) u^5 \right\|_{L^1([T.t[,L^2(\Omega))}$$
$$\leq \left(E_c^{\frac{1}{2}}((w_+ - w_T)(0)) + \|u\|_{L^5([T.t[,L^{10}(\Omega))} \right) \underset{t \longrightarrow +\infty}{\longrightarrow} 0.$$

Let $u_+ = u + w_+$ which satisfies

$$\begin{cases} \Box u_+ = 0 \text{ on } \mathbb{R} \times \Omega, \\ u_+ = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ E_c((u - u_+)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

To prove the uniqueness of u_+; we define u_+^1 and u_+^2 as the solutions of

$$\begin{cases} \Box u_+^i = 0 \text{ on } \mathbb{R} \times \Omega, \\ u_+^i = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ E_c(\left(u - u_+^i\right)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0 \end{cases} \quad \text{for } i = 1, 2$$

Then $v = u_+^1 - u_+^2$ satisfies

$$\begin{cases} \Box v = 0 \text{ on } \mathbb{R} \times \Omega, \\ v = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ E_c((v)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Using the conservation of the energy, we conclude that $v = 0$. So the wave operator

$$\begin{aligned} \Omega_+ : \quad & H \longrightarrow H \\ & (u_{/t=0}, \partial u_{/t=0}) \longmapsto (u_{+/t=0}, \partial_t u_{+/t=0}) \end{aligned}$$

is well defined. Conversely, let v be a finite energy solution of

$$\begin{cases} \Box v = 0 \text{ on } \mathbb{R} \times \Omega, \\ v = 0 \text{ in } \mathbb{R} \times \partial\Omega. \end{cases}$$

We will prove that there exists a unique v_+ satisfying

$$\begin{cases} \Box v_+ + \chi(x) v_+^5 = 0 \text{ on } \mathbb{R} \times \Omega, \\ v_+ = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ E_c((v_+ - v)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Let $\delta > 0$ to be fixed later, and $T \geq 0$ such that $\|v\|_{L^5([T,+\infty[,L^{10}(\Omega))} \leq \delta$.

Next, we define the set B_δ by:

$$B_\delta = \left\{ w \in L^5(L^{10}) \text{ such that } \|w\|_{L^5([T,+\infty[,L^{10}(\Omega))} \leq \delta \right\}, \text{ and}$$

$$S : w \longrightarrow \widetilde{w},$$

where $w \in B_\delta$ and $\widetilde{w}$ is defined by

$$\begin{cases} \Box \widetilde{w} + \chi(x)(v+w)^5 = 0 \text{ on } \mathbb{R} \times \Omega, \\ \widetilde{w} = 0 \ \text{ in } \ \mathbb{R} \times \partial\Omega, \\ E_c(\widetilde{w}(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

By virtue of Proposition 2.4 we have

$$\begin{aligned} \|\widetilde{w}\|_{L^5([T,+\infty[,L^{10}(\Omega))} &\leq C \left\| \chi(x)(v+w)^5 \right\|_{L^1([T,+\infty[,L^2(\Omega))} \\ &\leq C(\|v\|^5_{L^5([T,+\infty[,L^{10}(\Omega))} + \|w\|^5_{L^5([T,+\infty[,L^{10}(\Omega))}), \end{aligned}$$

for which we apply the hyperbolic inequality and we obtain that

$$E_c^{1/2}(\widetilde{w}(T)) \leq 2C\delta^5.$$

This allows one to apply the fixed point argument and then v_+ on $[T, +\infty[$. Indeed, using Proposition 2.4, we have for every w_1 and w_2 in B_δ, we have

$$\begin{aligned} &\|\widetilde{w_1} - \widetilde{w_2}\|_{L^5([T,+\infty[,L^{10}(\Omega))} \qquad (42) \\ &\leq C_1 \left\| \chi(x)((v+w_1)^5 - (v+w_2)^5) \right\|_{L^1([T,+\infty[,L^2(\Omega))} \\ &\leq C_2 \left\| \chi(x) \left|w_1 - w_2\right| (|v+w_1| + |v+w_2|)^4 \right\|_{L^1([T,+\infty[,L^2(\Omega))} \\ &\leq C_2 \|w_1 - w_2\|_{L^5([T,+\infty[,L^{10}(\Omega))} \| |v+w_1| + |v+w_2| \|^4_{L^5([T,+\infty[,L^{10}(\Omega))} \\ &\leq C_3 \delta^4 \|w_1 - w_2\|_{L^5([T,+\infty[,L^{10}(\Omega))} . \end{aligned}$$

So choosing δ such that $C_3\delta^4 < 1$, we see that

$$S : B_\delta \longrightarrow B_\delta$$

is well defined, with Lipschitz constant smaller than 1. As a consequence there exists w_+ with finite energy satisfying

$$\begin{cases} \Box w_+ + \chi(x)(v+w_+)^5 = 0 \text{ on } [T, +\infty[\times \Omega, \\ w = 0 \ \text{ in } \ [T, +\infty[\times \partial\Omega \\ E_c(w_+(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Setting $v_+ = v + w_+$, extending v_+ to $\mathbb{R} \times \Omega$and using the existence and uniqueness of the Cauchy problem in the "Shatah–Struwe" class, we conclude that there exists v_+ satisfying

$$\begin{cases} \Box v_+ + \chi(x) v_+^5 = 0 \text{ on } \mathbb{R} \times \Omega, \\ v_+ = 0 \ \text{ in } \ \mathbb{R} \times \partial\Omega \\ E_c((v_+ - v)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

Moreover, $E_c((v_+)(0))$ is controlled by $E_c((v_+)(T))$ which is finite.

For the uniqueness of v_+, we consider v_+^1 and v_+^2 such that

$$\begin{cases} \Box v_+^i + \chi(x)(v_+^i)^5 = 0 \text{ on } \mathbb{R} \times \Omega, \\ v_+^i = 0 \ \text{ in } \ \mathbb{R} \times \partial\Omega \\ E_c(\big(v_+^i - v\big)(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0, \end{cases} \quad \text{for } i = 1, 2$$

and then set $u = v_+^1 - v_+^2$ which satisfies

$$\begin{cases} \Box u + \chi(x)(v_+^1)^5 - \chi(x)(v_+^2)^5 = 0 \text{ on } \mathbb{R} \times \Omega, \\ u = 0 \ \text{ in } \ \mathbb{R} \times \partial\Omega \\ E_c(u(t)) \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{cases}$$

As in (42), we can prove that

$$\|u\|_{L^5([T,+\infty[,L^{10}(\Omega))} \le C \, \|u\|_{L^5(L^{10})} \big(\left\| v_+^1 \right\|_{L^5(L^{10})}^4 + \left\| v_+^2 \right\|_{L^5(L^{10})}^4\big),$$

then choosing $T > 0$ such that

$$C\big(\left\| v_+^1 \right\|_{L^5(L^{10})}^4 + \left\| v_+^2 \right\|_{L^5(L^{10})}^4\big) < 1,$$

we obtain $v_+^1(t) = v_+^2(t)$ for every $t \ge T$, and we conclude by the uniqueness of the Cauchy problem (6) in the "Shatah–Struwe" class.

Finally, we obtain that the wave operator Ω_+ is a bijection, and using the same method we get a similar result when t goes to $-\infty$.

3 Exponential Decay for the Local Energy of the Subcritical and Critical Wave Equation with Localized Semilinearity

The main result of this work is to prove that the decay of the local energy of the solutions of (6) is of exponential type.

More precisely we have the following theorem:

Theorem 3.1 ([3, 7]) *Given R and R_0 two positive real numbers, there exist $C > 0$ and $\alpha > 0$ such that inequality*

$$E_R(u(t)) \leq Ce^{-\alpha t} E(u(0)) \tag{43}$$

holds for every u solution to (6) in the "Shatah–Struwe" class with initial data $\varphi = (\varphi_1, \varphi_2)$ supported in B_R and satisfying

$$E(\varphi) = \frac{1}{2} \int_\Omega \left(|\varphi_2|^2 + |\nabla_x \varphi_1|^2 \right) dx + \int_\Omega \chi(x) \frac{|\varphi_1|^{p+1}}{p+1} dx \leq R_0. \tag{44}$$

Remark 3.2 Theorems 2.1 and 3.1 remain true if $O = \emptyset$; that is the free space.

For the literature we quote essentially the results of Jeng-Eng-Lin [18], C. Morawetz [21], and W. Strauss [28] and which obtained various rates of decay (from polynomial to exponential) in free space. We note that the results of the decay of the local energy for the solutions of the semilinear wave equation are less provided.

We discuss now the methods used to establish Theorem 3.1.

Let $\varphi \in H$ with support in B_R, clearly $\varphi \in K$. For all $h \in H$, we have $P^+ h = h$ on B_R.

Consequently $U(t)\varphi = Z(t)\varphi$ on B_R, so

$$E_R(U(t)\varphi) = E_R(Z(t)\varphi) \leq E(Z(t)\varphi).$$

We note that to prove the exponential decay of the local energy, it is then enough to prove that $E(Z(t)\varphi)$ decays exponentially. Consequently, by the semi-group property it suffices to prove that: for every $E_0 > 0$ and $\varphi \in K$ verifying $E(\varphi) \leq E_0$,

$$E(Z(T)\varphi) \leq CE(\varphi),$$

for some $T > 0$ and $0 < C < 1$. In the subcritical case, we argue by contradiction and we use the result proved by the author and M. Daoulatli in [7] that the nonlinear Lax–Phillips semi-group $Z(t)$ is compact for some $T > 0$ when $2 < p < 5$: The proof was based on the properties of the microlocal defect measures of P. Gérard [12] and used in crucial way, the subcritical nature of the equation.

Obviously, this is not possible when $p = 5$; we will overcome this difficulty with the help of the energy balance Theorem proved by B. Dehman and P. Gérard [8] which is adapted in our case. We prove in this case that for some sequences of initial data $Z(T)$ is compact "at infinity" which is sufficient to obtain an absurdity.

3.1 Nonlinear Lax–Phillips Theory

Let us consider the free wave equation

$$(E_0)\begin{cases}\partial_t^2 u - \Delta u = 0 \ \text{ on } \mathbb{R}\times\mathbb{R}^3\\ u(0)=\varphi_1\ ,\ \partial_t u(0)=\varphi_2 \ \text{ on } \mathbb{R}^3\end{cases} \tag{45}$$

with $\varphi=(\varphi_1,\varphi_2)\in H_0$; the completion of $(C_0^\infty(\mathbb{R}^3))^2$ with respect to the norm

$$\|\varphi\|^2=\int_{\mathbb{R}^d}(|\nabla\varphi_1|^2+|\varphi_2|^2)dx.$$

We denote $U_0(t)$ the free wave group.

We recall now a classical result for the wave equation in the free space with odd dimension.

Theorem 3.3 (Huygens' Principle [16]) *If the initial data $\varphi\in H_0$ is supported in the ball B_R, then the corresponding solution of wave equation in free space vanishes in the cone*

$$\{(t,x)\in\mathbb{R}\times\mathbb{R}^3\ ;\ |t|>R\ et\ |x|\le|t|-R\}. \tag{46}$$

Following Lax and Phillips, we denote:

$$D_+^0=\{\varphi=(\varphi_1,\varphi_2)\in H_0 \text{ such that } U_0(t)\varphi=0 \text{ on } |x|\le t\ ,\ t\ge 0\} \tag{47}$$

the space of outgoing data, and

$$D_-^0=\{\varphi=(\varphi_1,\varphi_2)\in H_0 \text{ such that } U_0(t)\varphi=0 \text{ on } |x|\le -t\ ,\ t\le 0\} \tag{48}$$

the space of incoming data associated with the solutions of (E_0).

Let us now consider the wave equation in outside domain

$$(E_L)\begin{cases}\Box u=0 \ \text{ on } \ \mathbb{R}\times\Omega\\ u=0 \ \text{ on } \mathbb{R}\times\partial\Omega\\ u(0,x)=u^0(x)\in H_D(\Omega)\ and\ \partial_t u(0,x)=u^1(x)\in L^2(\Omega).\end{cases} \tag{49}$$

We recall that the system is conservative; more precisely

$$E_0(u(t))=\frac{1}{2}\int_\Omega|\partial_t u(t)|^2+|\nabla_x u(t)|^2=E_0(u(0)). \tag{50}$$

We denote $U_L(t)$ the linear wave group.

In order to study the influence of the obstacle, Lax and Phillips introduced the spaces of outgoing and incoming data associated with the solutions of problem (E_L) by

$$D_+^R = \{\varphi = (\varphi_1, \varphi_2) \in H_0\, ; U_L(t)\varphi = 0 \text{ on } |x| \leq t + R,\ t \geq 0\} \tag{51}$$

$$D_-^R = \{\varphi = (\varphi_1, \varphi_2) \in H_0\, ; U_L(t)\varphi = 0 \text{ on } |x| \leq -t + R,\ t \leq 0\}\,. \tag{52}$$

We identify H to a subspace of H_0 with the help of the following continuation operator

$$E : H \longrightarrow H_0 : \ E\varphi = \begin{cases} \varphi \text{ on } \ \Omega \\ 0 \text{ on } {}^C\Omega. \end{cases}$$

Then we remark that the subspace of outgoing and incoming data associated with (E_L) coincide, respectively, with $U_0(R)D_+^0$ et $U_0(-R)D_-^0$ and $U_0(-R)D_-^0$.

Moreover, they satisfy the following properties.

(a) D_+^R and D_-^R are closed in H.
(b) D_+^R and D_-^R are orthogonal and

$$D_+^R \oplus D_-^R \oplus \left(\left(D_+^R\right)^\perp \cap \left(D_-^R\right)^\perp\right) = H. \tag{53}$$

Remarks 3.4

(1) The solutions of (6) and (9) verify the finite speed propagation property.
(2) The nonlinearity being localized in a ball B_R, it is easy to see that $U(t) = U_L(t)$ on D_+^R and $U(-t) = U_L(-t)$ on D_-^R for every $t \geq 0$. In particular, this yields

$$U(t) \text{ operates on } D_+^R \text{ and } U(-t) \text{ operates on } D_-^R \text{ for every } t \geq 0. \tag{54}$$

(3) We remind that $P^+\left[P^-\right]$ is the orthogonal projection of H onto the orthogonal complement of $D_+^R\left[D_-^R\right]$ and thanks to (53), it is clear that

$$P^+\varphi \in \left(D_+^R\right)^\perp \cap \left(D_-^R\right)^\perp \text{ if } \varphi \in \left(D_-^R\right)^\perp. \tag{55}$$

(4) $U(t)$ operates on D_+^R for $t \geq 0$, so $supp(U(t)\varphi) \cap supp\,(\chi) = \varnothing$ for every $t \geq 0$ and $\varphi \in D_+^R$. Using then the uniqueness for the Cauchy problem (in the "Shatah–Struwe" class in the critical case), we obtain: for every φ in H and for every $t \in \mathbb{R}_+$,

$$\begin{aligned} U(t)\,\varphi &= U(t)\,P^+\varphi + U(t)\left(I - P^+\right)\varphi \\ &= U(t)\,P^+\varphi + U_0(t)\left(I - P^+\right), \varphi \end{aligned} \tag{56}$$

where $\left(I - P^{+}\right)$ denotes the orthogonal projection on D_{+}^{R} and $U_0\,(t)$ is the free wave group.

(5) $U(t)$ operates on $\left(D_{-}^{R}\right)^{\perp}$ $\left[\left(D_{+}^{R}\right)^{\perp}\right]$ for every $t \geq 0$ $[t \leq 0]$.

By analogy with the linear case, we define the nonlinear Lax–Phillips semi-group by

$$Z(t) = P^{+}U(t)P^{-}, \quad \text{for } t \geq 0. \tag{57}$$

In order to prove that $Z(t)$ is a semi-group, we need the following lemma (we take $f(x, u) = \chi(x)|u|^4 u$ in the critical case):

Lemma 3.5 *Given $(\varphi, \psi) \in H^2$, and $t \in \mathbb{R}$ we have:*

$$\begin{aligned}&\langle U(t)\varphi, U(t)\psi\rangle_H - \langle \varphi, \psi\rangle_H \\ &= -\int_0^t \langle f(x, u(s)), \partial_t v(s)\rangle_{L^2} + \langle \partial_t u(s), f(x, v(s))\rangle_{L^2}\, ds,\end{aligned} \tag{58}$$

where we denoted $U(t)\varphi = (u(t), \partial_t u(t))$ and $U(t)\psi = (v(t), \partial_t v(t))$.

Proof By density argument, it suffices to prove the result for (φ, ψ) in $(C_0^\infty(\Omega))^2$.

Thanks to Green formula

$$\begin{aligned}\frac{d}{dt}\langle U(t)\varphi, U(t)\psi\rangle_H &= \frac{d}{dt}\left(\langle \nabla u, \nabla v\rangle_{L^2} + \langle \partial_t u, \partial_t v\rangle_{L^2}\right)\\ &= \left\langle \partial_t^2 u, \partial_t v\right\rangle + \left\langle \partial_t u, \partial_t^2 v\right\rangle - \langle \partial_t u, \triangle v\rangle - \langle \triangle u, \partial_t v\rangle.\end{aligned}$$

Since u and v verify the system (E),

$$\frac{d}{dt}\langle U(t)\varphi, U(t)\psi\rangle_H = -\langle f\,(x, u)\,, \partial_t v\rangle_{L^2} - \langle \partial_t u, f\,(x, v)\rangle_{L^2}$$

and the result follows. □

Proposition 3.6 (1) *$Z(t)D_{+}^{R} = Z(t)D_{-}^{R} = \{0\}$, for every $t \geq 0$.*

(2) *$Z(t)$ operates on $K = \left(D_{+}^{R}\right)^{\perp} \cap \left(D_{-}^{R}\right)^{\perp}$.*

(3) *$(Z(t))_{t \geq 0}$ is a continuous semi-group on K, satisfying $E(Z(t)\varphi) \leq E(\varphi)$ for every $t \geq 0$, and $\varphi \in K$.*

Proof

(1) First it is clear that $Z(t)\varphi = 0$ if $\varphi \in D_{-}^{R}$.

On the other hand, let $\varphi \in D_{+}^{R}$, $\varphi \in (D_{-}^{R})^{\perp}$ since D_{+}^{R} and D_{-}^{R} are orthogonal, so $Z(t)\varphi = P^{+}U(t)\varphi = 0$, for $t \geq 0$, due to (54).

(2) By virtue of (55) to prove that $Z(t)$ operates on K, it suffices to verify that $U(t)$ operates on $\left(D_-^R\right)^{\perp}$; for every $t \geq 0$.

Let $\varphi \in \left(D_-^R\right)^{\perp}$ and $\psi \in D_-^R$; according to lemma (3.5) we have:

$$\begin{aligned}&\langle U(t)\varphi, \psi\rangle_H - \langle \varphi, U(-t)\psi\rangle_H \\ &= \int_0^t \langle f(x, u(s)), \partial_t v(s-t)\rangle_{L^2}\, ds - \int_0^t \langle \partial_t u(s), f(x, v(s-t))\rangle_{L^2}\, ds.\end{aligned}$$

Thanks to (54) $U(s-t)\psi \in D_-^R$ for every $s \leq t$
and $supp(f) \cap supp(U(s-t)\psi) = \emptyset$,
then

$$\langle U(t)\varphi, \psi\rangle_H = \langle \varphi, U(-t)\psi\rangle_H = 0.$$

3. $Z(t)$ is obviously continuous. And we just prove that $Z(t_1+t_2) = Z(t_1)Z(t_2)$ for $t_1, t_2 \geq 0$.

Let $\varphi \in K$, by (56) we have:

$$\begin{aligned}Z(t_1+t_2)\varphi &= P^+\, U(t_1)U(t_2)\varphi \\ &= P^+\, U(t_1)P^+\, U(t_2)\varphi + P^+\, U_0(t_1)(I-P^+)U(t_2)\varphi,\end{aligned}$$

since

$$U_0(t_1)(I-P^+)U(t_2)\varphi \in D_+^R,$$

we obtain

$$Z(t_1+t_2)\varphi = P^+\, U(t_1)Z(t_2)\varphi = Z(t_1)Z(t_2)\varphi.$$

□

Remark 3.7 In the proof of proposition (3.6), we obtained that $U(t)$ operates on $\left(D_-^R\right)^{\perp}$, for every $t \geq 0$. Similarly, it is possible to prove that $U(t)$ operates on $\left(D_+^R\right)^{\perp}$ for every $t \leq 0$.

The proposition below shows that $Z(t)\varphi$ goes to 0 as $t \longrightarrow +\infty$ for all $\varphi \in K$. This result is useful to deduce the exponential decay for the local energy of the solutions to (6).

Proposition 3.8

(1) *For all* $\rho \geq R$ *and* $\varphi \in H$

$$\lim_{t\longrightarrow+\infty} \|U(t)\varphi\|_{H(B_\rho\cap\Omega)} = 0.$$

(2) *For all* $\varphi \in K$, $\lim_{t \longrightarrow +\infty} \|Z(t)\varphi\|_H = 0$.

Proof

(1) Taking φ in H, and applying Theorem 2.1, we can find ψ in H such that,

$$\|U(t)\varphi - U_L(t)\psi\|_H \underset{t \longrightarrow +\infty}{\longrightarrow} 0$$

then

$$\|U(t)\varphi\|_{H(B_\rho \cap \Omega)} \leq \|U(t)\varphi - U_L(t)\psi\|_{H(B_\rho \cap \Omega)} + \|U_L(t)\psi\|_{H(B_\rho \cap \Omega)} \underset{t \longrightarrow +\infty}{\longrightarrow} 0, \tag{59}$$

since the last term of the right-hand side of (59) converges to 0, by the classical Lax–Phillips theory [16].

(2) For all $\varphi \in K = (D_+^R)^\perp \cap (D_-^R)^\perp$ and $t \geq 2R$ we have

$$Z(t)\varphi = P^+ MU(t-2R)\varphi + P^+ U_0(2R)U(t-2R)\varphi,$$

where $M = U(2R) - U_0(2R)$.

By remarks 3.4, $U(t-2R)\varphi \in (D_-^R)^\perp$. Moreover

$$U_0(2R)(D_-^R)^\perp \subset D_+^R \text{ (see [1] lemma 4.2)},$$

hence

$$Z(t)\varphi = P^+ MU(t-2R)\varphi.$$

Using the finite speed propagation property and the fact that the nonlinearity is supported in B_R, we get

$$\begin{aligned} \|Z(t)\varphi\|_H &= \|Z(t)\varphi\|_{H_0} \\ &= \left\|P^+ MU(t-2R)\varphi\right\|_{H_0} \\ &\leq \|MU(t-2R)\varphi\|_{H_0} \\ &= \|MU(t-2R)\varphi\|_{H_0(B_{3R})} \\ &= \|U(t)\varphi - U_0(2R)U(t-2R)\varphi\|_{H_0(B_{3R})} \\ &\leq \|U(t)\varphi\|_{H(B_{3R})} + \|U(t-2R)\varphi\|_{H(B_{5R})} \underset{t \longrightarrow +\infty}{\longrightarrow} 0. \end{aligned} \tag{60}$$

Here H_0 denotes the completion of $\left(C_0^\infty\left(\mathbb{R}^3\right)\right)^2$ with respect to the norm

$$\|\varphi\|^2 = \|(\varphi_2, \varphi_2)\|^2 = \int_{\mathbb{R}^3} (|\nabla \varphi_1(x)|^2 + |\varphi_2|^2) dx.$$

□

3.2 Exponential Decay for the Local Energy of the Subcritical Wave Equation

3.2.1 The Compactness of $Z(T)$

Definition 3.9 We denote by T_R the minimal time needed by all the "generalized" geodesics starting from B_R at $(t = 0)$ to leave the ball B_R: T_R is called the escape time.

By analogy with proposition 5.1 of [1], we obtain

Proposition 3.10 *Let $(\varphi_n)_n$ a bounded sequence in K, then there exists a subsequence of φ_n (still denoted φ_n) and φ in K such that φ_n converges weakly to φ in H and $(U(t)\varphi_n)_1$ converges to $(U(t)\varphi)_1$ strongly in $H^1_{loc}(\tilde{K}(T))$ for $T \geq T_R + 3R$, where $\tilde{K}(T) = \{|x| \leq t - T + R,\ t \geq T\}$, and $(U(t)\varphi_n)_1$ is the first component of the vector $U(t)\varphi$.*

The proof of this proposition is based on the notion of microlocal defect measures. These measures were introduced by P. Gérard in [12, 13]. And G. Lebeau proved there propagation near the boundary for the Dirichlet problem [17]).

Proof of Proposition 3.10 The sequence $(\varphi_n)_n$ is bounded in the closed subspace K, then there exists a subsequence still denoted $(\varphi_n)_n$, $\varphi \in K$ such that

$$\varphi_n \underset{n \longrightarrow +\infty}{\rightharpoonup} \varphi.$$

Let $T_0 \geq T$ and $(u_n)_{n\in\mathbb{N}}$ (respectively,$(v_n)_{n\in\mathbb{N}}$) the sequence of solutions of the system (E) (respectively,(E_L)) associated with the sequence $\varphi_n)_{n\in\mathbb{N}}$. By virtue of (14) (respectively, (50)), $(u_n)_{n\in\mathbb{N}}$ (respectively, $(u_n)_{n\in\mathbb{N}}$) is bounded in $H^1_{loc}([0, T_0] \times \Omega)$; then it admits a subsequence, still denoted $(u_n)_{n\in\mathbb{N}}$ (respectively, $(u_n)_{n\in\mathbb{N}}$) that converges weakly to u (respectively, v).

Let μ (respectivement μ_L) the microlocal defect measure associated with $(u_n - u)_{n\in\mathbb{N}}$ (respectivement $(v_n - v)_{n\in\mathbb{N}}$) in $H^1_{loc}([0, T_0] \times \Omega)$.

We will prove that $\mu = 0$ in $\tilde{K}(T) = \{|x| \leq t - T + R\ , t \geq T\}$.

Let $q \in T^*(\tilde{K}(T))$ and λ a generalized bicharacteristic starting at q.

The obstacle is strictly convex and then nontrapping. So if λ is traced backwards in time, it does not meet $\partial\Omega$ or meets $\partial\Omega$ at $t_0 > 2R$, consequently $\lambda_0 = \lambda_{/t=0} \notin B(0, R)$. Since $\text{Supp}(f) \subset B(0, R)$, we have $u_n = v_n$ near λ_0, and we conclude that $\mu = \mu_L$ near λ_0.

On the other hand $\varphi_n \in K$, and $Z_L(t) = P^+U_L(t)P^-$ is compact for $T \geq T_R + 3R$, (see [20]), so $\mu_L = 0$ near q. By propagation of the support of μ_L, (see [17]), this gives $\mu = \mu_L = 0$ near λ_0. To finish the proof we give the following lemma, where the subcritical power of the nonlinearity plays a crucial role. □

Lemma 3.11 *Under conditions of the proposition 3.10 we have:*

$$f(x,u_n)-f(x,u) \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \; in \;\; L^1([0,T_0]\,,L^2(\Omega))). \tag{61}$$

We postpone the proof of this lemma and use it to prove that $\mu = 0$ near q.

By (61), we have $(u_n - u)_{n\in N}$ is "linearizable" [13], then μ propagates along the bicharacteristic. And we conclude that $\mu = 0$ near q since $\mu = 0$ near λ_0.

Proof of lemma 3.11 We write $f(x,u_n) - f(x,u) = \chi\,(u_n - u)\,h\,(u_n,u)$ with $h\,(u_n,u)$ verifying

$$|h\,(u_n,u)| \le C((|u|+|u_n|)^{p-1} + (|u|+|u_n|)^{p_1-1}) \tag{62}$$

(hypotheses (13)). Applying Hölder inequality for $\varepsilon > 0$ and β such that, $\frac{1}{6-\varepsilon} + \frac{1}{2\beta} = \frac{1}{2}$, $1/\alpha + 1/\gamma = 1$.

$$\|f(x,u_n)-f(x,u)\|_{\;L^1([0,T_0],L^2(\Omega)))} \le C\,\|\chi\,(u_n-u)\|_{L^\infty(0,T_0;L^{6-\varepsilon})}\,\|\chi h\|_{L^\alpha(0,T_0;L^{2\beta})}$$

The compactness of $H^1_{loc} \to L^{6-\varepsilon}_{loc}$ gives that

$$\|\chi\,(u_n-u)\|_{L^\infty(0,T_0;L^{6-\varepsilon})} \underset{n\to+\infty}{\to} 0$$

Thus, it remains to prove that $\|\chi h\|_{L^\alpha(0,T_0;L^{2\beta})}$ is bounded. We obtain, by (62)

$$\begin{aligned}\|\chi h\|_{L^\alpha(0,T_0;L^{2\beta})} \le\; & \|\chi(|u|+|u_n|)\|^{p-1}_{L^{\alpha(p-1)}(0,T_0;L^{2\beta(p-1)})}\,T_0^{1/\gamma}\\ &+\|\chi(|u|+|u_n|)\|^{p_1-1}_{L^{\alpha(p_1-1)}(0,T_0;L^{2\beta(p_1-1)})}\,T_0^{1/\gamma}.\end{aligned}$$

Now we estimate one of the members of the right-hand side of (63).

(a) If $\beta(p-1) \le 3$, then

$$\begin{aligned}\|\chi(|u|+|u_n|)\|_{L^{\alpha(p-1)}(0,T_0;L^{2\beta(p-1)})} &\le C_{T_0,\beta}(\|u\|_{L^\infty(L^6)}+\|u_n\|_{L^\infty(L^6)})\\ &\le 2C_{T_0,\beta}R_0,\end{aligned}$$

where R_0 is the radius of the ball of the energy space in which we choose the initial data.

(b) In the other case, we chose α such that, $\frac{2}{p-1} < \alpha < \frac{2\beta}{\beta(p-1)-3}$, and we use Hölder inequality, to obtain a Strichartz norm. Indeed we chose r such that, $\frac{1}{r} + \frac{1}{\alpha(p-1)} = \frac{1}{2}$, with $3r > 2\beta(p-1)$,

$$\begin{aligned}\|\chi(|u|+|u_n|)\|_{L^{\alpha(p-1)}(0,T_0;L^{2\beta(p-1)})} &\le C(\|u\|_{L^{\alpha(p-1)}(0,T_0;L^{3r})}\\ &\quad+\|u_n\|_{L^{\alpha(p-1)}(0,T_0;L^{3r})})\\ &\le C_{T_0,\beta,R_0}.\end{aligned}$$

□

Lemma 3.12 *Let $(\varphi_n)_n$ a bounded sequence in H, then there exists a subsequence still denoted $(\varphi_n)_n$ and $\varphi \in H$, such that,*

$$M_L\varphi_n \underset{n\longrightarrow+\infty}{\longrightarrow} M_L\varphi,$$

where $M_L = U(2R) - U_L(2R)$.

Proof Setting $U_L(t)\varphi_n = \begin{pmatrix} v_n \\ \partial_t v_n \end{pmatrix}$, $U(t)\varphi_n = \begin{pmatrix} u_n \\ \partial_t u_n \end{pmatrix}$.

$(\varphi_n)_n$ a bounded sequence in H, then there exists a subsequence still denoted $(\varphi_n)_n$ and $\varphi \in H$ such that $\varphi_n \underset{n\longrightarrow+\infty}{\rightharpoonup} \varphi$ in H.

By virtue of (14) and (50) $(u_n)_{n\in\mathbb{N}}$ is bounded (respectively, $(v_n)_{n\in\mathbb{N}}$) in $H^1_{loc}([0,T]\times\Omega)$ for all $T \geq 0$, then there exists a subsequence, still denoted $(u_n)_n$ (respectively, $(v_n)_n$) weakly converging to u (respectively, v). Then $u - u_n + v - v_n$ satisfies

$$\begin{cases} \Box(u - u_n + v - v_n) + \chi(x)(g(u) - g(u_n)) = o \text{ on } D'([0,T]\times\Omega) \\ u - u_n + v - v_n = 0 \text{ on } \mathbb{R}\times\partial\Omega \\ (u - u_n + v - v_n)_{/t=0} = 0 \text{ and } \partial_t(u - u_n + v - v_n)_{/t=0} = 0, \end{cases}$$

where $f(x,u) - f(x,u_n)) \in L^1([0,T], L^2(\Omega))$, for all $T \geq 0$.

The hyperbolic inequality and lemma 3.11 then yield the desired result

$$\begin{aligned} \|M_L\varphi_n - M_L\varphi\|_H &\leq \sup_{[0,2R]} \|U_L(t)(\varphi_n - \varphi) - U(t)\varphi_n + U(t)\varphi)\|_H \\ &\leq C_R \|f(x,u) - f(x,u_n))\|_{L^1([0,2R],L^2(\Omega))} \underset{n\longrightarrow+\infty}{\longrightarrow} 0. \end{aligned}$$

□

Proposition 3.13 *$Z(T)$ is compact operator on K for $T \geq T_R + 9R$.*

Proof Let $(\varphi_n)_n$ be a bounded sequence in K; by proposition 3.10 there exist a subsequence, still denoted (φ_n) and $\varphi \in K$ such that, for all $T \geq T_R + 9R$,

$$U(t - 2R)\varphi_n \longrightarrow U(t - 2R)\varphi \text{ dans } H_{B_{5R}}. \tag{63}$$

On the other hand

$$Z(t)\varphi_n = P^+M_LU(t - 2R)\varphi_n + P^+(U_L(2R) - U_0(2R))U(t - 2R)\varphi_n.$$

Now by the lemma 3.12, we have

$$P^+M_LU(t - 2R)\varphi_n \underset{n\longrightarrow+\infty}{\longrightarrow} P^+M_LU(t - 2R)\varphi.$$

Furthermore the finite speed propagation property guarantees that the support of $(U_L(2R) - U_0(2R))\, U\,(t-2R)\,\varphi_n$ is in B_{3R}. Using the continuity of P^+ and (63), we get that for all $t \geq\ T_R\ +9R$

$$P^+(U_L(2R)-U_0(2R))U(t-2R)\varphi_n \underset{n\longrightarrow+\infty}{\longrightarrow} P^+(U_L(2R)-U_0(2R))U(t-2R)\varphi.$$

And we conclude that,

$$Z(t)\varphi_n \underset{n\longrightarrow+\infty}{\longrightarrow} Z(t)\varphi \ \text{ dans }\ H, \quad \text{if } t \geq\ T_R + 9R.$$

□

3.2.2 Proof of Theorem 3.1

In the following subsubsection we identify $U(t)\varphi$ and $Z(t)\varphi$ with their first components.

Let $\varphi \in H$ with support in B_R, clearly $\varphi \in K$. For all $h \in H$, we have $P^+h = h$ on B_R. Consequently $U(t)\varphi = Z(t)\varphi$ on B_R, so

$$E_R(U(t)\varphi) = E_R(Z(t)\varphi) \leq E(Z(t)\varphi).$$

To prove the exponential decay of the local energy, it is then enough to prove that $E(Z(t)\varphi)$ decays exponentially. Consequently, by the semi-group property it suffices to prove that: for every $E_0 > 0$ and $\varphi \in K$ verifying $E(\varphi) \leq E_0$

$$E(Z(T)\varphi) \leq CE(\varphi), \quad \text{for some } T > 0 \quad \text{and } 0 < C < 1.$$

For that we argue by contradiction: We fix E_0 and we suppose that for every T, and for every $0 < C < 1$, there exist φ such that,

$$E(Z(T)\varphi) \geq CE(\varphi) \text{ and } E(\varphi) \leq E_0. \tag{64}$$

Then we obtain two sequences $C_n \underset{n\longrightarrow+\infty}{\longrightarrow} 1$ and φ_n verifying

$$E\,(Z\,(n)\,\varphi_n) \geq C_n E\,(\varphi_n)\,.$$

On the other hand if $t\ \leq n$

$$E\,(Z\,(t)\,\varphi_n) \geq C_n E\,(Z\,(n)\,\varphi_n) \geq C_n E\,(\varphi_n)\,,$$

then

$$E_0 \geq E\,(\varphi_n) \geq E\,(Z\,(t)\,\varphi_n) \geq C_n E\,(\varphi_n)\,. \tag{65}$$

(φ_n) is a bounded sequence in H, so by proposition 3.10 there exists a subsequence, still denoted (φ_n) and $\varphi \in K$ such that,

$$\varphi_n \rightharpoonup \varphi \text{ in } K \text{ and } Z(t)\varphi_n \underset{n\longrightarrow+\infty}{\longrightarrow} Z(t)\varphi \text{ in } H, \text{ for all } T \geq T_R + 9R. \tag{66}$$

Combining (65) and (66), we obtain

$$E\left(\varphi_n\right) \underset{n\longrightarrow+\infty}{\longrightarrow} E\left(Z\left(t\right)\varphi\right) \text{ for all } t \geq TR + 9R.$$

On the other hand

$$E(Z(t)\varphi_n) \geq C_n E(U(t)\varphi_n) \tag{67}$$

$$= C_n[E(Z(t)\varphi_n) + \frac{1}{2}\|(P^+ - I)U(t)\varphi_n\|_H^2], \tag{68}$$

then

$$\frac{1}{2}C_n\|(P^+ - I)U(t)\varphi_n\|_H^2 \leq (1 - C_n)E(Z(t)\varphi_n).$$

Passing to the limit in the last inequality, we obtain

$$\|(P^+ - I)U(t)\varphi_n\| \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

In other words $U(t)\varphi \in \left(D_+^R\right)^\perp$, for every $t \geq T_R + 9R$.

By Remark 3.7, we conclude that

$$E(Z(t)\varphi) = E(U(t)\varphi) = E\left(\varphi\right), \quad \text{for every } \ t \geq 0. \tag{69}$$

Since φ_n^1 is bounded in H_D, then

$$\int_{\Omega\cap B_R} \chi(x)G(\varphi_n^1)dx \underset{n\longrightarrow+\infty}{\longrightarrow} \int_{\Omega\cap B_R} \chi(x)G(\varphi^1)dx, \tag{70}$$

where we denoted $\varphi_n = (\varphi_n^1, \varphi_n^2)$ and $\varphi = (\varphi^1, \varphi^2)$. So by combining (69) and (70), we obtain

$$\varphi_n \underset{n\longrightarrow+\infty}{\longrightarrow} \varphi \in H.$$

Finally there exists $\varphi \in H$, and u solution of

$$\begin{cases} \Box u + f(x,u) = 0, \\ u = 0 \quad \text{on} \quad \mathbb{R} \times \partial\Omega, \\ (u(0), \partial_t u(0)) = \varphi, \\ E(Z(t)\varphi) = E(\varphi), \ \forall t \geq 0. \end{cases}$$

1st case: $E(\varphi) = \lim_{n \longrightarrow +\infty} E(\varphi_n) = \alpha^2 > 0$.

This case is impossible, since by Proposition 3.8, we have $\lim_{t \longrightarrow +\infty} E(Z(t)\varphi) = 0$, which contradicts $E(\varphi) > 0$.

2nd case: $E(\varphi) = \lim_{n \longrightarrow +\infty} E(\varphi_n) = 0$. Let $\alpha_n^2 = E(\varphi_n)$.

$v_n = \frac{u_n}{\alpha_n}$ satisfies

$$(S) \begin{cases} \Box v_n + \frac{1}{\alpha_n} f(x, \alpha_n v_n) = 0, \\ v_{n/\partial\Omega} = 0, \\ (v_n(0), \partial_t v_n(0)) = \frac{\varphi_n}{\alpha_n} = \psi_n \in K, \\ \widetilde{E}_n(\psi_n) = \frac{1}{2} \int_\Omega \left(\left| \partial_t \psi_{2,n} \right|^2 + \left| \nabla_x \psi_{1,n} \right|^2 \right) dx + \frac{1}{\alpha_n^2} \int_\Omega \chi(x) G(\alpha_n \psi_{1,n}) dx = 1. \end{cases}$$

Denote $V_n(t) = (v_n, \partial_t v_n) = V_n^0(t) + W_n(t)$, where $V_n^0(t) = (v_n^0, \partial_t v_n^0)$ and $W_n(t) = (w_n, \partial_t w_n)$, with

$$\begin{cases} \Box v_n^0 = 0 \\ v_{n/\partial\Omega}^0 = 0 \\ (v_n^0(0), \partial_t v_n^0(0)) = \frac{\varphi_n}{\alpha_n} \end{cases} \quad \text{and} \quad \begin{cases} \Box w_n + \frac{1}{\alpha_n} f(x, \alpha_n v_n) = 0 \\ w_{n/\partial\Omega} = 0 \\ (w_n(0), \partial_t w_n(0)) = 0. \end{cases}$$

It is clear that

$$\left\| \frac{1}{\alpha_n} f(x, \alpha_n v_n) \right\|_{L^1([0,T], L^2(\Omega))} \underset{n \longrightarrow +\infty}{\longrightarrow} 0 \quad \text{for every } T \geq 0,$$

then $W_n(t) \underset{n \longrightarrow +\infty}{\longrightarrow} 0$ in $L^\infty([0,T], H)$, that is

$$\sup_{0 \leq t \leq T} \left| E\left(\left(v_n - v_n^0 \right)(t) \right) \right| \underset{n \to +\infty}{\to} 0 \tag{71}$$

due to the hyperbolic inequality.

Let $t \geq 0$, by virtue of (14) (respectively, (50), $(V_n(t))_n$ (respectively, $(V_n^0(t))_n$) is bounded in H and admits a subsequence that converges weakly to $V(t)$ (respectively, $V^0(t)$).

Therefore, (71) yields

$$V(t) = V(t)^0 \quad \text{for every } t \geq 0.$$

Coming back to the contradiction argument developed above, we can find $\psi \in K$, $\psi_n \rightharpoonup \psi$ and $P^+ Vn(T) \rightharpoonup P^+ V(T) = P^+ V^0(T)$ in H, where

$$\begin{cases} \Box v = 0 \\ v_{/\partial\Omega} = 0 \\ (v(0), \partial_t v(0)) = \psi. \end{cases}$$

By the compactness of $Z_L(t)$ for $T \geq T_R + 3R$, (see [20])

$$Z_L(T)\psi_n \underset{n \longrightarrow +\infty}{\longrightarrow} Z_L(T)\psi = P^+ V(T), \quad \text{pour tout } T \geq T_R + 3R.$$

On the other hand $\|\psi_n\|_H \leq 1$ and $\|\psi_n\|_H \underset{n \longrightarrow +\infty}{\longrightarrow} 1$, which gives using (65)

$$\|Z_L(T)\psi\|_H = \|\psi\|_H = 1,$$

and contradicts the result of Melrose (see [20]).

3.3 Exponential Decay for the Local Energy of the Critical Wave Equation

In the following section we identify $U(t)\varphi$ and $Z(t)\varphi$ with their first components.

Let $\varphi \in H$ with support in B_R; clearly $\varphi \in K$. Moreover for all $h \in H$, we have $P^+ h = h$ on B_R. Consequently $U(t)\varphi = Z(t)\varphi$ on B_R, so

$$E_R(U(t)\varphi) = E_R(Z(t)\varphi) \leq E(Z(t)\varphi).$$

Thus it is enough to prove the exponential decay of $E(Z(t)\varphi)$. Furthermore, by the semi-group property it suffices to prove: for every $E_0 > 0$ there exist $T > 0$ and $0 < C < 1$ such that,

$$E(Z(T)\varphi) \leq C E(\varphi) \text{ for every } \varphi \in K \text{ satisfying } E(\varphi) \leq E_0.$$

For that we argue by contradiction: We fix $E_0 > 0$ and we suppose that for every T and for every $0 < C < 1$, there exists φ such that,

$$E(Z(T)\varphi) \geq C E(\varphi) \text{ and } E(\varphi) \leq E_0. \tag{72}$$

Then we obtain two sequences $C_n \underset{n \longrightarrow +\infty}{\longrightarrow} 1$, and $(\varphi_n)_n$ with

$$E\left(Z\left(n\right)\varphi_n\right) \geq C_n E\left(\varphi_n\right).$$

Therefore, for every $t \leq n$

$$\begin{aligned} E(Z(t)\varphi_n) &\geq C_n E\left(\varphi_n\right) \\ &= C_n E(U(t)\varphi_n) \\ &= C_n \left(\frac{1}{2} \left\| U(t)\varphi_n \right\|_H^2 + \frac{1}{6} \int_\Omega \chi(x) \left| (U\,(t)\,\varphi_n)_1 \right|^6 dx \right) \\ &= C_n \left(E(Z(t)\varphi_n) + \frac{1}{2} \left\| (P^+ - I)U(t)\varphi_n \right\|_H^2 \right), \end{aligned}$$

then

$$\frac{1}{2} C_n \left\| (P^+ - I)U(t)\varphi_n \right\|_H^2 \leq (1 - C_n)E(Z(t)\varphi_n) \underset{n \longrightarrow +\infty}{\longrightarrow} 0. \tag{73}$$

(φ_n) is a bounded sequence in H, so there exists a subsequence, still denoted (φ_n) and $\varphi \in K$ such that $\varphi_n \underset{n \longrightarrow +\infty}{\rightharpoonup} \varphi$ in K. And thanks to Corollary A.1

$$(P^+ - I)U(t)\varphi_n \underset{n \longrightarrow +\infty}{\rightharpoonup} (P^+ - I)U(t)\varphi, \text{ for every } t \geq 0.$$

Combining with (73), we obtain $E(Z(t)\varphi) = E(U(t)\varphi) = E\left(\varphi\right)$, for every $t \geq 0$.

Using then Proposition 3.8, we easily obtain that the weak limit φ of the sequence φ_n is 0.

To finish the proof of Theorem 3.1 we need the following proposition.

Proposition 3.14 *Let $(\varphi_n)_n$ a bounded sequence in K such that $\varphi_n \rightharpoonup 0$ then there exists a positive and nondecreasing sequence $\left(\alpha_j\right)$ satisfying*

$$\lim_{j \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} \left\| U(\alpha_j)\varphi_n \right\|_{H(B_{5R})} = 0. \tag{74}$$

We postpone the proof of this proposition.

End of Proof of Theorem 3.1 We write as in (60)

$$\begin{aligned} \left\| Z(\alpha_j + 2R)\varphi_n \right\|_H &= \left\| Z(\alpha_j + 2R)\varphi_n \right\|_{H_0} \\ &= \left\| P^+ M U(\alpha_j)\varphi_n \right\|_{H_0} \\ &\leq \left\| M U(\alpha_j)\varphi_n \right\|_{H_0} \\ &= \left\| M U(\alpha_j)\varphi_n \right\|_{H_{0(B_{3R})}} \\ &\leq \left\| U(\alpha_j + 2R)\varphi_n \right\|_{H(B_{3R})} + \left\| U(\alpha_j)\varphi_n \right\|_{H(B_{5R})}, \end{aligned}$$

where $M = U(2R) - U_0(2R)$. As the first term of the last inequality is controlled by the second (the finite speed propagation property) we deduce that

$$\lim_{n\longrightarrow+\infty} E(Z(\alpha_j+2R)\varphi_n)$$
$$= \lim_{n\longrightarrow+\infty}(\frac{1}{2}\left\|Z(\alpha_j+2R)\varphi_n\right\|_H^2 + \frac{1}{6}\int_\Omega \chi(x)\left|(U(\alpha_j+2R)\varphi_n)_1\right|^6 dx) \underset{j\longrightarrow+\infty}{\longrightarrow} 0. \tag{75}$$

Now we rewrite the right-hand term of (75) as

$$E(Z(\alpha_j+2R)\varphi_n) = E(\varphi_n) - \frac{1}{2}\left\|(P^+ - I)U(\alpha_j+2R)\varphi_n\right\|_H^2.$$

Passing then to the limit, first as $n \longrightarrow +\infty$, then as $j \longrightarrow +\infty$ we obtain $\lim_{n\longrightarrow+\infty} E(\varphi_n) = 0$.

Let $\beta_n^2 = E(\varphi_n)$. $v_n = \frac{u_n}{\beta_n}$ satisfies

$$(S)\begin{cases} \Box v_n + \beta_n^4\chi(x)v_n^5 = 0 \text{ on } \mathbb{R}\times\Omega \\ v_n = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ (v_n(0), \partial_t v_n(0)) = \frac{\varphi_n}{\beta_n} = \psi_n \in K \\ \widetilde{E}_n(v_n) = \frac{1}{2}\int_\Omega \left(|\partial_t v_n|^2 + |\nabla_x v_n|^2\right)dx + \frac{1}{6}\int_\Omega \beta_n^4\chi(x)v_n^6 dx = 1. \end{cases}$$

Denote $V_n(t) = (v_n(t), \partial_t v_n(t)) = V_n^0(t) + W_n(t)$, where $V_n^0(t) = (v_n^0(t), \partial_t v_n^0(t))$, and $W_n(t) = (w_n(t), \partial_t w_n(t))$ with

$$\begin{cases} \Box v_n^0 = 0 \text{ on } \mathbb{R}\times\Omega \\ v_n^0 = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ (v_n^0(0), \partial_t v_n^0(0)) = \frac{\varphi_n}{\beta_n} \end{cases} \quad\text{and}\quad \begin{cases} \Box w_n + \beta_n^4\chi(x)v_n^5 = 0 \text{ on } \mathbb{R}\times\Omega \\ w_n = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ (w_n(0), \partial_t w_n(0)) = 0. \end{cases}$$

Strichartz inequality (see Corollary 2.2 in [26] or Proposition 2.1 in [4]) applied to system (S) gives

$$\|v_n\|_{L^5([0,T],L^{10}(\Omega))} \le C(E(\frac{\varphi_n}{\beta_n}) + \beta_n^4\left\|\chi(x)v_n^5\right\|_{L^1([0,T],L^2(\Omega))})$$
$$\le C(1 + \beta_n^4\|v_n\|^5_{L^5([0,T],L^{10}(\Omega))}).$$

Since $\beta_n \underset{n\longrightarrow+\infty}{\longrightarrow} 0$, a classical bootstrap argument shows that $\chi(x)v_n^5$ is bounded in $L^1([0,T], L^2(\Omega))$ for every $T \ge 0$, which yields due to the hyperbolic inequality

$$\sup_{0\le t\le T}\left|E\left(\left(v_n - v_n^0\right)(t)\right)\right| \underset{n\to+\infty}{\to} 0. \tag{76}$$

Now, for $t \geq 0$, $(V_n(t))_n$ $\big($respectively, $V_n^0(t)\big)$ is bounded in H and admits then a subsequence weakly converging to $V(t)$ $\big($respectively, $V^0(t)\big)$. Moreover (76) gives

$$V_n^0(t) \underset{n \longrightarrow +\infty}{\rightharpoonup} V^0(t) = V(t) \quad \text{for every } t \geq 0,$$

and by the compactness of $Z_L(t)$ [20], we have

$$P^+ V_n^0(t) \underset{n \longrightarrow +\infty}{\longrightarrow} P^+ V^0(t), \quad \text{for every } t \geq T_R + 9R.$$

Then, according to (76)

$$P^+ V_n(t) \underset{n \longrightarrow +\infty}{\longrightarrow} P^+ V^0(t), \quad \forall t \geq T_R + 9R.$$

Coming back to the contradiction argument developed above, we have

$$C_n \leq \widetilde{E}_n(P^+ V_n(t)) \leq 1,$$

and passing to the limit we get

$$\frac{1}{2} \left\| P^+ V^0(t) \right\|_H^2 = 1. \tag{77}$$

Using again the fact $\widetilde{E}_n\,(v_n) = 1$, we obtain

$$\| V_n(t) \|_H \leq \sqrt{2}$$

then

$$\left\langle V_n(t), V^0(t) \right\rangle \leq \| V_n(t) \|_H \left\| V^0(t) \right\|_H \leq \sqrt{2} \left\| V^0(t) \right\|_H,$$

and using $V_n(t) \underset{n \longrightarrow +\infty}{\rightharpoonup} V^0(t)$ which implies, in particular,

$$\left\langle V_n(t), V^0(t) \right\rangle \underset{n \longrightarrow +\infty}{\longrightarrow} \left\| V^0(t) \right\|_H^2,$$

then we obtain $\left\| V^0(t) \right\|_H \leq \sqrt{2}$. Combining this with (77) we deduce that we can find $\psi = V(0) \in K$ such that

$$\| Z_L(t) \psi \|_H = \| \psi \|_H = \sqrt{2} \;\text{ for every } t \geq 0,$$

which contradicts the result of Melrose (see [20]). □

In order to prove Proposition 3.14 we will need the following Proposition due to B. Dehman and P. Gérard. They proved this result for $\Omega = \mathbb{R}^3$, but one can see that, with slight modifications, the proof remains valid when Ω is the exterior of convex obstacle.

Proposition 3.15 (Adapted from [8]) *Let* (r_n) *be a sequence of solutions of*

$$\Box r_n + \chi(x) r_n^5 = f_n,$$

in the "Shatah–Struwe" class and we assume that $(r_n(0), \partial_t r_n(0)) \rightharpoonup (r_0, r_1)$ *in* $H_D(\Omega) \times L^2(\Omega)$ *and* $f_n \longrightarrow 0$ *strongly in* $L^1_{loc}(\mathbb{R}_+, L^2(\Omega))$.

Let r *be the "Shatah–Struwe" solution of*

$$\Box r + \chi(x) r^5 = 0,\ r(0) = r_0,\ \partial_t r(0) = r_1$$

and $\tilde{r}_n$ *the "Shatah–Struwe" solution of*

$$\Box \tilde{r}_n + \chi(x)\tilde{r}_n^5 = 0,\ \tilde{r}_n(0) = r_n(0) - r_0,\ \partial_t \tilde{r}_n(0) = \partial_t r_n(0) - r_1.$$

Then for every $T > 0$,

$$\sup_{0\le t\le T} \left\| \nabla_{x,t} r_n - \nabla_{x,t} r - \nabla_{x,t}\tilde{r}_n \right\|_{L^2(\Omega)} + \| r_n - r - \tilde{r}_n\|_{L^5(\mathbb{R},L^{10}(\Omega))} \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

We come back now to the proof of proposition 3.14.

Proof of Proposition 3.14 Let $(u_n)_{n\in\mathbb{N}}$ (respectively,$(v_n)_{n\in\mathbb{N}}$) the sequence of solutions to (6) (with $p = 5$) (respectively,(E_L)) associated with the sequence of initial data $(\varphi_n)_{n\in\mathbb{N}}$, in the sense that

$$\begin{cases} \Box u_n + \chi(x) u_n^5 = 0 \text{ on } \mathbb{R}\times\Omega \\ u_n = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ (u_n(0), \partial_t u_n(0)) = \varphi_n \end{cases} \quad\text{and}\quad \begin{cases} \Box v_n = 0 \text{ on } \mathbb{R}\times\Omega \\ v_n = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ (v_n(0), \partial_t v_n(0)) = \varphi_n. \end{cases}$$

$r_n = u_n - v_n$ satisfies then

$$\begin{cases} \Box r_n + \chi(x) r_n^5 = \chi(x)((u_n - v_n)^5 - u_n^5) = f_n \text{ on } \mathbb{R}\times\Omega \\ r_n = 0 \text{ in } \mathbb{R}\times\partial\Omega \\ (r_n, \partial_t r_n)_{/t=0} = 0. \end{cases}$$

Due to Proposition 3.10 (which is easily adapted in our context) $v_n \longrightarrow 0$ on $H^1_{loc}\left(\tilde{K}(T)\right)$ for $T \ge T_0 = T_R + 3R$, where $\tilde{K}(T) = \{(t,x) \in \mathbb{R}\times\Omega \,/\, |x| \le t - T + R, t \ge T\}$, then

$$f_n = \chi(x) \sum_{p=0}^{4} C_5^p u_n^p v_n^{5-p} \underset{n \longrightarrow +\infty}{\longrightarrow} 0 \text{ in } L^1_{loc}([T_0, +\infty[\,, L^2(\Omega)).$$

Indeed by Hölder's inequality, Strichartz estimates, and corollary A.2 one can see that
$\left\| \chi(x) u_n^p v_n^{5-p} \right\|_{L^1_{loc}([T_0,+\infty[,L^2(\Omega))}$ converges to 0.

Applying then Proposition 3.15, we obtain

$$\|r_n - \tilde{r}_n\|_{H^1_{loc}([T_0,T]\times\Omega)} \underset{n \longrightarrow +\infty}{\longrightarrow} 0 \text{ for every } T \geq T_0, \tag{78}$$

where $\tilde{r}_n$ satisfies

$$\begin{cases} \Box \tilde{r}_n + \chi(x) \tilde{r}_n^5 = 0 \text{ on } \mathbb{R} \times \Omega \\ \tilde{r}_n = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ (\tilde{r}_n, \partial_t \tilde{r}_n)_{/t=T_0} = (r_n, \partial_t r_n)_{/t=T_0}. \end{cases}$$

Combining then (78) with the fact that $supp(r_n(t)) \subset B_{R+t}$ for every $t \geq 0$, we see that

$$\tilde{r}_n \underset{n \longrightarrow +\infty}{\longrightarrow} 0 \text{ in } H^1_{loc}(|x| > R + t, \; t \geq T_0). \tag{79}$$

Moreover, we recall that the energy density of $(\tilde{r}_n)$ is given by

$$e_n(t,x) = \frac{1}{2}\left[|\partial_t \tilde{r}_n(t,x)|^2 + |\nabla_x \tilde{r}_n(t,x)|^2 \right] + \frac{1}{6} \chi(x) \left| \tilde{r}_n(t,x) \right|^6,$$

and $e(t,x)$ the weak limit of $e_n(t,x)$.

We note that the conclusion of Theorem 7 in [8] remains valid in our situation, that is

$$e(t,x) = \sum_{j=1}^{+\infty} e^{(j)}(t,x) + e_f(t,x), \tag{80}$$

where $e^{(j)}$ is the limit energy density of the nonlinear concentrating wave $q_n^{(j)}$ solution to

$$\begin{cases} \Box q_n^{(j)} + \chi(x) \left(q_n^{(j)} \right)^5 = 0 \text{ on } \mathbb{R}\times\Omega, \; q_n^{(j)} = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ (q_n^{(j)}(0), \partial_t q_n^{(j)}(0)) = (p_n^{(j)}(0), \partial_t p_n^{(j)}(0)), \end{cases}$$

$(p_n^{(j)})$ is the solution of (128) associated with $\left(\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)}\right)$ and e_f is the limit energy density of a sequence of solutions of the linear wave equation $\widetilde{w}_n$, namely

$$e_f(t,x) = \int_{\xi \in S^2} \mu(t,x,d\xi)$$

with $\mu(t,x,d\xi) = \mu_+(t,x,d\xi) + \mu_-(t,x,d\xi)$ and $\mu_\pm$ are positive measures on $\Omega \times S^2$.

Consequently, using (80) we obtain

$$\left\| q_n^{(j)} \right\|_{H^1_{loc}(|x|>R+t,\ t\geq T_0)} \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \text{ and } \|\widetilde{w}_n\|_{H^1_{loc}(|x|>R+t,\ t\geq T_0)} \underset{n\longrightarrow+\infty}{\longrightarrow} 0. \tag{81}$$

On the other hand, taking $\chi = \chi(x^{(j)})$, where $x^{(j)} = \lim\limits_{n\longrightarrow+\infty} x_n^{(j)}$ and using Theorem 1 in [4] (or also Theorem 2 in [10] with slight modifications), we obtain

$$\int_\Omega (\left|\partial_t(q_n^{(j)} - v_n^{(j)})(t,x)\right|^2 + \left|\nabla_x(q_n^{(j)} - v_n^{(j)})(t,x)\right|^2)dx \underset{n\longrightarrow+\infty}{\longrightarrow} 0,\ t \in \left]t_\infty^{(j)}, T\right], \tag{82}$$

for every $T > t_\infty^{(j)}$, where $v_n^{(j)}$ is a sequence of finite energy solutions of the linear wave equation and $t_\infty^{(j)} = \lim\limits_{n\longrightarrow+\infty} t_n^{(j)}$ witch verifies $t_\infty^{(j)} \geq T_0$, in fact the decomposition of the energy density is only made in the region $t \geq T_0$.

Denote $\mu_j,\ j \geq 1$ (respectively,μ) the microlocal defect measures associated with $\left(q_n^{(j)}\right)_n$ (respectively,$\widetilde{w}_n$). The result (82) implies that μ_j is also attached to the sequence $v_n^{(j)}$ on the time interval $\left]t_\infty^{(j)}, T\right]$. Let $q \in T^*(\tilde{K}(T_{t_\infty^{(j)}+R} + t_\infty^{(j)}))$ (recall that $T_{t_\infty^{(j)}+R}$ is given by Definition 3.9) and λ a generalized bicharacteristic starting at q. The obstacle is strictly convex and then nontrapping; so if λ is traced backwards in time, it does not meet $\partial\Omega$ or meets $\partial\Omega$.

But in the two cases $\lambda_0 = \lambda_{/t=t_\infty^{(j)}} \in \left\{|x| > R+t,\ t \geq t_\infty^{(j)}\right\}$ and in view of (81), we get

$$\mu_j = \mu = 0 \text{ on } \left\{|x| > R+t,\ t \geq t_\infty^{(j)}\right\}.$$

Applying then the linear result of G.Lebeau (see [10]) for propagation of the support of μ_j (respectively,μ) we deduce that $\mu_j = \mu = 0$ on $\tilde{K}(T_{t_\infty^{(j)}+R} + t_\infty^{(j)})$. Hence

$$e^{(j)} = e_f = 0 \text{ on } \tilde{K}(T_{t_\infty^{(j)}+R} + t_\infty^{(j)}),$$

consequently

$$\sum_{p=1}^{j} e^{(p)} = 0 \text{ on } \tilde{K}(\max_{1\leq p\leq j}(T_{t_\infty^{(p)}+R} + t_\infty^{(p)})). \tag{83}$$

On the other hand, by (80) we have

$$\forall \varepsilon > 0, \exists j_0 \in \mathbb{N} \text{ such that for every } \ j \geq j_0 \ \sum_{p \geq j+1} e^{(p)} \leq \frac{\varepsilon}{vol(B_{5R})}, \tag{84}$$

then (83) and (84) gives

$$\lim_{j \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} \left\| \tilde{r}_n(\alpha_j) \right\|_{H(B_{5R})} = 0,$$

where $\alpha_j = \max_{1 \leq p \leq j} T_{t_\infty^{(p)}+R} + t_\infty^{(p)}$.

Consequently, we get by (78) the same limit for r_n. Finally, recalling that $u_n = r_n + v_n$ and $v_n \longrightarrow 0$ on $H^1_{loc}\left(\tilde{K}(T)\right)$ for $T \geq T_0$ we obtain the desired result. □

4 Polynomial Decay for the Local Energy of the Semilinear Wave Equation with Small Data

Now we consider the following nonlinear wave equation,

$$\begin{cases} \Box u + \lambda u \, |u|^{p-1} = 0, \ \text{ in } \mathbb{R} \times \mathbb{R}^3, \\ u(0, x) = f(x) \in C^1(\mathbb{R}^3) \ \text{ and } \ \partial_t u(0, x) = g(x) \in C^0(\mathbb{R}^3), \end{cases} \tag{85}$$

where $\lambda \in \mathbb{R}$.

We assume that $|f(x)| \leq \dfrac{\varepsilon}{(1+|x|)^{p-1}}$ and $|g(x)| + |\nabla f(x)| \leq \dfrac{\varepsilon}{(1+|x|)^p}$, for some $\varepsilon > 0$.

We keep the global energy and the local energy of u at time t as in ***Sect. 2 and we take here $\chi(x) = \lambda$ and $\Omega = \mathbb{R}^3$.

Now, we define the following functional space which is inspired from the space introduced in [24]

$$X^p_{\delta,R} = \left\{ \begin{array}{l} u \text{ s.t. } \nabla^l_x u(t,x) \in C^0(\mathbb{R} \times \mathbb{R}^3), \ 0 \leq l \leq 1, \\ \|u\|_{V_p} \leq \delta \ \text{ and } \ \|\partial_t u\|_{V_p} + \|\nabla_x u\|_{V_p} \leq R \end{array} \right\},$$

where we denoted

$$\|u\|_{V_p} := \sup_{\substack{t \in \mathbb{R} \\ x \in \mathbb{R}^3}} [(1+|x|+|t|)(1+||x|-|t||)^{p-2}|u(t,x)|].$$

We establish the global well posedness and the local energy decay for (85).

Theorem 4.1 *Assume that* $p > 1+\sqrt{2}$. *Then there exist* $\varepsilon_0 > 0$, δ *and* $R > 0$ *such that, for every* $\varepsilon \in \,]0, \varepsilon_0[$ *the system* (1.1) *admits a unique solution in the space* $X^p_{\delta,R}$. *Moreover, there exists a constant* $C = C(\rho, \varepsilon_0) > 0$ *such that following* *inequality*

$$E_\rho(u)(t) = \frac{1}{2}\int_{B\rho}(|\partial_t u(t,x)|^2+|\nabla_x u(t,x)|^2)dx+\frac{\lambda}{p+1}\int_{B\rho}|u(t,x)|^{p+1} \leq \frac{C}{(1+t)^{2p-2}}$$

holds for every u solution of (85).

Remark 4.2

(1) The results in Theorem 4.1 complete the work of Pecher [24] who proves the global well posedness and scattering for $p \in \left]1+\sqrt{2}, 3\right[$.
(2) Let also indicate that optimality of the decay rate is still an open problem.
(3) The proof of theorem 4.1 is based on a fixed point process and uses in crucial way the properties of the fundamental solution of the wave operator on $\mathbb{R}^3$.

4.1 Fundamental Lemmas

In this subsection, we give some preliminary lemmas.

Lemma 4.3 ([24]) *If h is a continuous function and $r = |x|$, then*

$$\int_{|y-x|=t} h(|y|)dS_y = \frac{2\pi t}{r}\int_{|r-t|}^{r+t}\sigma h(\sigma)d\sigma.$$

Lemma 4.4 *Assume $p > 1+\sqrt{2}$ and define*

$$g(\sigma,s) = \frac{\sigma}{(1+\sigma+s)^p(1+|s-\sigma|)^{p(p-2)}}.$$

Then for some $C = C(p)$ the following inequality holds

$$\int_0^t (\int_{|r-t+s|}^{r+t-s} g(\sigma,s)d\sigma)ds \leq \frac{c_0 r}{(1+r+t)(1+|r-t|)^{(p-2)}} = N(r,t) \quad \textit{for } r \geq 0,\ t \in \mathbb{R}_+.$$

Proof The region of integration is divided into three parts as follows:

$$0 \leq r \leq t-1,\ t-1 \leq r \leq t+1 \textit{ and } r \geq t+1.$$

We just treat the first case and we note that the other cases can be treated in the same way.

We substitute $\gamma = s+\sigma$, $\beta = s-\sigma$

$$\int_0^t (\int_{|r-t+s|}^{r+t-s} g(\sigma,s)d\sigma)ds \leq \int_0^t (\int_{|r-t+s|}^{r+t-s} \frac{d\sigma}{(1+\sigma+s)^{p-1}(1+|s-\sigma|)^{p(p-2)}})ds$$

$$\leq \int_{t-r}^{t+r} \frac{d\gamma}{(1+\gamma)^{p-1}} \int_{-\infty}^{t-r} \frac{d\beta}{(1+|\beta|)^{p(p-2)}}$$
$$\leq C \int_{t-r}^{t+r} \frac{d\gamma}{(1+\gamma)^{p-1}}.$$

If $1+t-r \geq \frac{1+t+r}{2}$, i.e., $1+t \geq 3r$ one can estimate

$$\int_{t-r}^{t+r} \frac{d\gamma}{(1+\gamma)^{p-1}} \leq \frac{2r}{(1+t-r)^{p-1}} \leq \frac{4r}{(1+t+r)(1+t-r)^{p-2}}.$$

Whereas in the case $1+t-r \geq \frac{1+t+r}{2}$, i.e., $1+t \leq 3r$ one estimates by

$$\int_{t-r}^{t+r} \frac{d\gamma}{(1+\gamma)^{p-1}} = \frac{1}{p-2}\left[\frac{1}{(1+t-r)^{p-2}} - \frac{1}{(1+t+r)^{p-2}}\right]$$
$$\leq \frac{c}{(1+t-r)^{p-2}} \leq \frac{cr}{(1+t+r)(1+t-r)^{p-2}}.$$

This completes the proof of Lemma 4.4. □

Remark 4.5 As a direct consequence of Lemma 4.4 we define

$$V_p = \left\{u \in C^0\left(\mathbb{R}\times\mathbb{R}^3\right) \setminus \ \|u\|_{V_p} < \infty\right\}.$$

Note that $\|\cdot\|_{V_p}$ is an algebra norm.

Lemma 4.6 *Let u_0 be the solution of the following linear wave equation*

$$\begin{cases} \partial_t^2 u_0 - \Delta u_0 = 0, \\ u_0(x,0) = f(x) \in C^1(\mathbb{R}^3),\ \partial_t u_0(x,0) = g(x) \in C^0(\mathbb{R}^3), \end{cases}$$

and take $\varepsilon > 0$ and $k > 2$ such that

$$|f(x)| \leq \frac{\varepsilon}{(1+|x|)^{k-1}} \quad \textit{and} \quad |g(x)| + |\nabla f(x)| \leq \frac{\varepsilon}{(1+|x|)^k},\ \textit{for all } x \in \mathbb{R}^3.$$

Then

$$|u_0(x,t)| \leq \frac{C\varepsilon}{(1+|x|+t)(1+||x|-t|)^{k-2}}\ ,\ \textit{for } x \in \mathbb{R}^3 \textit{ and } t \in \mathbb{R}_+,\ \ \textit{i.e.,}\ \ u_0 \in V_k.$$

Proof According to the classical representation formula, we have

$$u_0(x,t) = \frac{t}{4\pi}\int_{|y|=1} g(x+ty)dS_y + \frac{\partial}{\partial t}(\frac{t}{4\pi}\int_{|y|=1} f(x+ty)dS_y)$$

$$= \frac{t}{4\pi}\int_{|y|=1} g(x+ty)dS_y + \frac{1}{4\pi}\int_{|y|=1} f(x+ty)dS_y + \frac{t}{4\pi}\int_{|y|=1} (\nabla_x f(x+ty),\xi)dS_y$$
$$= \frac{1}{4\pi t}\int_{|x-y|=t} g(y)dS_y + \frac{1}{4\pi t^2}\int_{|x-y|=1} f(y)dS_y + \frac{t}{4\pi}\int_{|y|=1} (\nabla_x f(x+ty),y)dS_y$$
$$= I_1 + I_2 + I_3.$$

We treat the first term as follows

$$|I_1| = \frac{C}{t}\int_{|y-x|=t} |g(y)|\,dS_y \le \frac{C\varepsilon}{t}\int_{|y-x|=t} \frac{dS_y}{(1+|y|)^k} = \frac{2\pi C\varepsilon}{r}\int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1+\sigma)^k}.$$

If $r \ge 1$ and $r \ge \frac{t}{2}$ we estimate

$$\frac{1}{r}\int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1+\sigma)^k} \le \frac{1}{r}\int_{|r-t|}^{+\infty} \frac{d\sigma}{(1+\sigma)^{k-1}} \le \frac{c}{r(1+|r-t|)^{k-2}} \le \frac{c}{(1+r+t)(1+|r-t|)^{k-2}}.$$

If $r \le \frac{t}{2}$ or $\frac{t}{2} \le r \le 1$ we have

$$\frac{1}{r}\int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1+\sigma)^k} \le \frac{1}{r}\int_{|r-t|}^{r+t} \frac{d\sigma}{(1+|t-r|)^{k-1}} \le \frac{2}{(1+|r-t|)^{k-1}}$$
$$\le \frac{c}{(1+r+t)(1+|r-t|)^{k-2}}.$$

Finally the second and third terms can be handled in the same way.

The proof of Lemma 4.6 is achieved. □

4.2 *Proof of Theorem 4.1: Existence and Decay of the Local Energy*

We denote by

$$\mathcal{E}(u)(t,x) = \frac{\lambda}{4\pi}\int_0^t \frac{1}{t-\tau}\Big(\int_{|y-x|=t-\tau} u^p(\tau,y)dS_y\Big)d\tau,$$

where u satisfies (85).

In order to run a fixed point theorem we estimate for $u \in V_p$

$$|\mathcal{E}(u)(t,x)| \le C\frac{1}{4\pi}\int_0^t \frac{1}{t-\tau}\Big(\int_{|y-x|=t-\tau} \left|u^p(\tau,y)\right| dS_y\Big)d\tau$$
$$\le C\frac{1}{4\pi}\int_0^t \frac{1}{t-\tau}\Big(\int_{|y-x|=t-\tau} \frac{dS_y}{(1+|y|+\tau)^p(1+||y|-\tau|)^{p(p-2)}}\Big)d\tau\ \|u\|_{V_p}^p$$

$$\leq \frac{C_1}{r}\int_0^t (\int_{|r-t+\tau|}^{r+t-\tau} \frac{\sigma d\sigma}{(1+\sigma+\tau)^p(1+|\sigma-\tau|)^{p(p-2)}})d\tau \, \|u\|_{V_p}^p$$

$$\leq \frac{C_2}{(1+r+t)(1+|r-t|)^{(p-2)}} \, \|u\|_{V_p}^p$$

which gives

$$\|\mathcal{E}(u)\|_{V_p} \leq C \, \|u\|_{V_p}^p . \tag{86}$$

One can easily verify that $\partial_{x_k}\mathcal{E}(u) = \mathcal{E}(\partial_{x_k}(u^p))$.

Consequently one proves

$$\left\| \partial_{x_k}\mathcal{E}(u) \right\|_{V_p} \leq C \, \|u\|_{V_p}^{p-1} \left\| \partial_{x_k} u \right\|_{V_p} . \tag{87}$$

On the other hand

$$|(\mathcal{E}(u) - \mathcal{E}(v))(t,x)| \leq C \frac{1}{4\pi}\int_0^t \frac{1}{t-\tau}(\int_{|y-x|=t-\tau} \left|u^p - v^p\right|(\tau,y)dS_y)d\tau$$

$$\leq C \frac{1}{4\pi}\int_0^t \frac{1}{t-\tau}(\int_{|y-x|=t-\tau} \left|(u-v)(u^{p-1}+v^{p-1})\right|(\tau,y)dS_y)d\tau.$$

Thus

$$\|\mathcal{E}(u) - \mathcal{E}(v)\|_{V_p} \leq C(\|u\|_{V_p}^{p-1} + \|v\|_{V_p}^{p-1}) \, \|u-v\|_{V_p} , \tag{88}$$

and one easily verifies

$$\begin{aligned} \left\| \partial_{x_K}\mathcal{E}(u) - \partial_{x_K}\mathcal{E}(v) \right\|_{V_p} &\leq C[\|u-v\|_{V_p} (\|u\|_{V_p}^{p-2} + \|v\|_{V_P}^{p-2}) \left\| \partial_{x_K} u \right\|_{V_p} \\ &+ \left\| \partial_{x_K} u - \partial_{x_K} v \right\|_{V_p} (\|u\|_{V_p}^{p-1} + \|v\|_{V_p}^{p-1})]. \end{aligned} \tag{89}$$

Then we write

$$\mathcal{E}(u)(t,x) = \frac{\lambda}{4\pi}\int_0^t (t-\tau)(\int_{|y|=1} u^p(\tau, x+(t-\tau)y)dS_y)d\tau.$$

It is easy to check that

$$\partial_t\mathcal{E}(u)(t,x) = p\mathcal{E}(\partial_t u \, u^{p-1})(t,x) + \frac{1}{4\pi t}\int_{|x-y|=t} u^p(0,y)dS_y.$$

As $|u^p(0,y)| \leq \frac{C}{(1+|y|)^{p(p-1)}}$, we deduce that

$$|\partial_t \mathcal{E}(u)(t,x)| \leq p \left|\mathcal{E}(\partial_t u\, u^{p-1})(t,x)\right| + \frac{C}{t}\int_{|y-x|=t} \frac{dS_y}{(1+|y|)^{p(p-1)}} \|u\|_{V_p}^p$$
$$\leq p \left|\mathcal{E}(\partial_t u\, u^{p-1})(t,x)\right| + \frac{C}{r}\int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1+\sigma)^{p+1}} \|u\|_{V_p}^p$$

since $p > 1+\sqrt{2}$.

Similarly to the proof of Lemma 4.6 and in order to estimate the second term of the last inequality, we distinguish the two following cases:

If $r \geq 1$ and $r \geq t/2$ we obtain

$$\frac{1}{r}\int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1+\sigma)^{p+1}} \leq \frac{1}{r}\int_{|r-t|}^{+\infty} \frac{d\sigma}{(1+\sigma)^{p}} \leq \frac{c}{r(1+|r-t|)^{p-1}} \leq \frac{c}{(1+r+t)(1+|r-t|)^{p-1}}.$$

If $r \leq \frac{t}{2}$ or $\frac{t}{2} \leq r \leq 1$ it follows that

$$\frac{1}{r}\int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1+\sigma)^{p+1}} \leq \frac{1}{r}\int_{|r-t|}^{r+t} \frac{d\sigma}{(1+|t-r|)^{p}} \leq \frac{2}{(1+|r-t|)^{p}} \leq \frac{c}{(1+r+t)(1+|r-t|)^{p-1}},$$

and we obtain

$$\|\partial_t \mathcal{E}(u)\|_{V_p} \leq C\, \|u\|_{V_p}^{p-1} (\|\partial_t u\|_{V_p} + \|u\|_{V_P}). \tag{90}$$

Finally we write

$$(\partial_t \mathcal{E}(u) - \partial_t \mathcal{E}(v))(t,x) = p\mathcal{E}(\partial_t u(u^{p-1} - v^{p-1}))(t,x) + p\mathcal{E}(v^{p-1}(\partial_t u - \partial_t v))(t,x)$$
$$+ \frac{1}{4\pi t}\int_{|x-y|=t} (u^p(0,y) - v^p(0,y))dS_y.$$

Consequently

$$\|\partial_t \mathcal{E}(u) - \partial_t \mathcal{E}(v)\|_{V_p} \leq C\, \|\partial_t u\|_{V_p} \|u-v\|_{V_p} (\|u\|_{V_p}^{p-2} + \|v\|_{V_p}^{p-2}) +$$
$$\|\partial_t u - \partial_t v\|_{V_p} \|v\|_{V_p}^{p-1} + \|u-v\|_{V_p} (\|u\|_{V_p}^{p-1} + \|v\|_{V_p}^{p-1}). \tag{91}$$

The rest of the proof is standard.

The estimates (86), (87), and (90) show that for an arbitrary given R one has

$$\|\mathcal{E}(u)\|_{V_p} \leq C\delta^p, \quad \left\|\partial_{x_k}\mathcal{E}(u)\right\|_{V_p} \leq C\delta^{p-1}R \quad \text{and} \quad \|\partial_t(\mathcal{E}(u))\|_{V_p} \leq C(\delta^p + \delta^{p-1}R).$$

So $u_0 + \mathcal{E}(u) \in X$ if $u \in X^p_{\delta,R}$. Now we take $\delta > 0$ small enough, say,

$$C\delta^{p-1} \leq \frac{1}{4} \quad \text{and} \quad C\delta^{p-2}R \leq \frac{1}{4},$$

and we consider the sequence $u_{n+1} = u_0 + \mathcal{E}(u_n)$, $n \geq 0$.

By (88), (89), and (91), we have $\|u_{n+1} - u_n\|_{V_p} \leq \frac{1}{2} \|u_n - u_{n-1}\|_{V_p}$.

Consequently, we have

$$\|u_{n+1} - u_n\|_{V_p} \leq \frac{c}{2^n}, \ \left\|\partial_{x_k}(u_{n+1} - u_n)\right\|_{V_p} \leq \frac{c}{2^n} + \frac{1}{2}\left\|\partial_{x_k}(u_n - u_{n-1})\right\|_{V_p}.$$

Thus, we deduce that

$$\left\|\partial_{x_k}(u_{n+1} - u_n)\right\|_{V_p} \leq \frac{cn}{2^n} \ \text{ and } \ \|\partial_t(u_{n+1} - u_n)\|_{V_p} \leq \frac{cn}{2^n}.$$

We then conclude that (u_n) converges in $X^p_{\delta,R}$ to u which is the unique solution of the system (85).

Finally as u in $X^p_{\delta,R}$ then for $t \geq 0$ and $x \in B(0,\rho)$ we have

$$|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 \leq \frac{C}{(1+t)^{2p-2}},$$

and then

$$|u(t,x)|^{p+1} \leq \frac{C}{(1+|x|+|t|)^{p+1}(1+||x|-|t||)^{(p+1)(p-2)}}.$$

This gives the energy decay.

5 Decay of the Local Energy for the Solutions of the Critical Klein–Gordon Equation

In this section, we are interested in the following system:

$$\begin{cases} \Box u + \chi_1 u + \chi_2 u^5 = 0, & \text{on } \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u^0(x) \in H^1(\mathbb{R}^3) \quad \text{and} & \partial_t u(0,x) = u^1(x) \in L^2(\mathbb{R}^3), \end{cases} \tag{92}$$

where $\Box = \partial_t^2 - \triangle$, χ_1 and χ_2 are positives functions, of class C^1, with compact support such that $supp\chi_1 \cup supp\chi_2 \subset B_R$ for some $R > 0$ and satisfying

$$x \cdot \nabla\chi_1(x) \leq 0 \quad \text{and} \quad x \cdot \nabla\chi_2(x) \leq 4, \ \forall x \in \mathbb{R}^3. \tag{93}$$

We denote by $H = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ endowed with the norm

$$\|(\varphi_1, \varphi_2)\|_H^2 = \int_{\mathbb{R}^3} \left(|\nabla\varphi_1|^2 + |\varphi_2|^2\right) dx.$$

It is by now well-known that for every initial data $(u^0, u^1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$; system (92) admits a unique solution u in the "Shatah–Struwe" class, that is

$$u \in C(\mathbb{R}, H^1(\mathbb{R}^3)) \cap L^5_{loc}(\mathbb{R}, L^{10}(\mathbb{R}^3)),\ \partial_t u \in C(\mathbb{R}, L^2(\mathbb{R}^3)).$$

The global energy of u at time t is defined by

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\partial_t u\,(t)|^2 + |\nabla_x u\,(t)|^2 + \chi_1(x)|u(t)|^2\right) dx + \frac{1}{6}\int_{\mathbb{R}^3} \chi_2(x)\,|u\,(t)|^6\, dx, \tag{94}$$

which is independent of time.

We define the local energy by

$$E_R(u(t)) = \frac{1}{2}\int_{B_R} \left(|\partial_t u\,(t)|^2 + |\nabla_x u\,(t)|^2 + \chi_1(x)|u(t)|^2\right) dx + \frac{1}{6}\int_{B_R} \chi_2(x)\,|u\,(t)|^6\, dx, \tag{95}$$

where B_R is a ball of radius R.

For every $t \in \mathbb{R}$, we define the nonlinear Klein–Gordon operator $U(t)$ by

$$\begin{array}{rccl} U(t): & H & \longrightarrow & H \\ & (\varphi_1, \varphi_2) & \longmapsto & U\,(t)\,(\varphi_1, \varphi_2) = (u\,(t)\,, \partial_t u\,(t))\,, \end{array}$$

where u is the solution of (92) in the "Shatah–Struwe" class with initial data $\varphi = (\varphi_1, \varphi_2)$.

$(U\,(t))_{t\in\mathbb{R}}$ forms a one parameter continuous group on H, which will be referred as the nonlinear group.

Our major concern is to prove exponential decay of the local energy. More precisely, we have:

Theorem 5.1 *For all $R > 0$, there exist $\alpha > 0$ and $c > 0$ such that*

$$E_R(u(t)) \leq Ce^{-\alpha t} E(0) \tag{96}$$

holds for every u solution to (92) with initial data $(u^0, u^1) \in H$ supported in B_R.

The literature is less provided for Klein–Gordon equation. We quote essentially the work of C. Morawetz [21] and B. Dehman and P. Gérard [12]. Furthermore, a recent result by M. Malloug [19] which establishes the exponential decay of the local energy for the damped Klein–Gordon equation in exterior domain and R-S-O. Nunes and W-D. Bastos [23] obtain the polynomial decay of the local energy for the linear Klein–Gordon equation.

In order to prove the main result in this section, first we use an argument inspired from the works of [2, 4] to establish that the Strichartz norms of the solutions to

(92) are global in time. Then, we introduce the Lax–Phillips semi-group $Z_{KG}(t)$ and a similar argument to that giving in [23] and we get the exponential decay of the localized linear Klein–Gordon equation.

More precisely, we consider the system,

$$\begin{cases} \Box u + \chi_1 u = 0, & \text{on } \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u^0(x) \in H^1(\mathbb{R}^3) \quad \text{and} & \partial_t u(0,x) = u^1(x) \in L^2(\mathbb{R}^3), \end{cases} \tag{97}$$

where χ_1 is a function of class C^1 with compact support such that $supp\chi_1 \subset B_R$, for some $R > 0$. We denote by E_L the global energy of u solution of (9) at time t defined by

$$E_L(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\partial_t u\,(t)|^2 + |\nabla_x u\,(t)|^2 + \chi_1(x)|u(t)|^2 \right) dx,$$

and we define the local energy by

$$E_{L,R}(u(t)) = \frac{1}{2} \int_{B_R} \left(|\partial_t u\,(t)|^2 + |\nabla_x u\,(t)|^2 + \chi_1(x)|u(t)|^2 \right) dx,$$

where B_R is a ball of radius R.

We prove the following theorem:

Theorem 5.2 *Let $R > 0$, there exist $\alpha > 0$ and $c > 0$ such that*

$$E_{L,R}(u(t)) \leq Ce^{-\alpha t} E_L(0) \tag{98}$$

holds for every u solution to (97) with initial data $(u^0, u^1) \in H$ supported in B_R.

5.1 *Strichartz Norms Global in Time*

The main concern of this section is to prove that Strichartz norms for the solutions of (92) are global in time, we recall the following theorem due to J. Zhang and J. Zheng.

Theorem 5.3 ([29, Zhang–Zheng])

Let (X, g) be a nontrapping scattering manifold of dimension $n \geq 3$. Suppose that u is the solution to the Cauchy problem:

$$(S) \begin{cases} \partial_t^2 u - \triangle_g u + u = F\,(t, z)\,, & (t, z) \in I \times X, \\ u(0) = u_0(z), & \partial_t u(0) = u_1(z), \end{cases}$$

where $\triangle_g$ denotes the Laplacian on the manifold X with scattering metric g. For some initial data $u_0 \in H^s$, $u_1 \in H^{s-1}$, the time interval $I \subseteq \mathbb{R}$ and

$F \in L_t^{\widetilde{q}'}(I; L_z^{\widetilde{r}'}(X))$ then

$$\|u(t,z)\|_{L_t^q(I;L_z^r(X))} + \|u(t,z)\|_{C(I;H^s(X))} \lesssim \|u_0\|_{H^s(X)} + \|u_1\|_{H^{s-1}(X)} + \|F\|_{L_t^{\widetilde{q}'}(I;L_z^{\widetilde{r}'}(X))},$$

where the pairs (q,r), $(\widetilde{q},\widetilde{r}) \in [2,+\infty]^2$ satisfy the KG-admissible condition with $0 \le \theta \le 1$,

$$\frac{2}{q} + \frac{n-1+\theta}{r} \le \frac{n-1+\theta}{2}, \quad (q,r,n,\theta) \ne (2,\infty,3,0), \tag{99}$$

and the gap condition

$$\frac{1}{q} + \frac{n+\theta}{r} = \frac{n+\theta}{2} - s = \frac{1}{\widetilde{q}'} + \frac{n+\theta}{\widetilde{r}'} - 2. \tag{100}$$

Remark 5.4 We note that in the case of flat Euclidean space, where $X = \mathbb{R}^3$ and $g_{jk} = \delta_{jk}$ the previous result for problem (S) remains valid with slight modification in the context of our problem (92), i.e., by taking $F = \chi_2 u^5$ and replacing the linear term u by $\chi_1 u$.

Now, notice the following corollary, which is adapted from Theorem 5.3.

Corollary 5.5 *Given a time interval $I \subseteq \mathbb{R}$. The inequality*

$$\|u(t,x)\|_{L_t^q(I;L_x^r(\mathbb{R}^3))} + \|u(t,x)\|_{C(I;H^1(\mathbb{R}^3))} \lesssim E(u(0))^{1/2} + \|\chi_2 u^5\|_{L_t^1(I;L_x^2(\mathbb{R}^3))}$$

holds for every u solution to (92). Here the pair $(q,r) \in]2,+\infty] \times [2,+\infty[$ and satisfies

$$\frac{1}{q} + \frac{3}{r} = \frac{1}{2}. \tag{101}$$

For the proof of our main result, we need the following lemma:

Lemma 5.6 *Let*

$$e(t) = \frac{1}{2} \int\limits_{\substack{|x|\le t \\ x\in\mathbb{R}^3}} \left(|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 + \chi_1(x)|u(t,x)|^2\right) dx + \frac{1}{6} \int\limits_{\substack{|x|\le t \\ x\in\mathbb{R}^3}} \chi_2(x)\,|u(t,x)|^6\,dx.$$

There exists $D > 0$ such that, for all $b > a > R$, and every solution u to $\Box u + \chi_1 u + \chi_2(x) u^5 = 0$, with $u \in C([a,b], H^1(\mathbb{R}^3)) \cap L^5([a,b], L^{10}(\mathbb{R}^3))$, $\partial_t u \in C([a,b], L^2(\mathbb{R}^3))$ we have

$$\int_{\substack{|x|\leq a \\ x\in\mathbb{R}^3}} \chi_2(x)\,|u(a,x)|^6\,dx \leq D\left[\frac{b}{a}(e(b)+e(b)^{1/3})\right].$$

Remark 5.7 We note that Lemma 5.6 is similar to that 2.10. Here, we choose a suitable multiplier for the equation of Klein–Gordon.

Proof Let

$$K_a^b = \left\{(t,x)\in\mathbb{R}\times\mathbb{R}^3, a\leq t\leq b, |x|\leq t\right\},$$

the truncated light cone,

$$M_a^b = \left\{(t,x)\in\mathbb{R}\times\mathbb{R}^3, a\leq t\leq b, |x| = t\right\},$$

the “mantle” associated with K_a^b, and

$$D(t) = \left\{(t,x)\in\mathbb{R}\times\mathbb{R}^3, |x|\leq t\right\},$$

its spacelike sections. We note that

$$\partial K_a^b = D(a)\cup D(b)\cup M_a^b.$$

We start with initial data in $\left(C_0^\infty\left(\mathbb{R}^3\right)\right)^2$, hence the associated solution is of class C^∞.

Multiplying equation (92) by $\partial_t u$, we obtain

$$div_{t,x}\left(\frac{1}{2}\,|\nabla_x u(t,x)|^2+\frac{1}{2}\,|\partial_t u(t,x)|^2+\frac{1}{2}\chi_1\,|u(t,x)|^2+\frac{1}{6}\chi_2(x)\,|u(t,x)|^6\,, -\partial_t u\nabla_x u\right) = 0, \tag{102}$$

then we integrate (102) over the truncated cone K_a^b to obtain the classical energy identity

$$e(b)-e(a) = \int_{M_a^b}\left(\frac{1}{2}\left|\frac{x}{t+1}\partial_t u+\nabla_x u\right|^2+\frac{1}{2}\chi_1\,|u(t,x)|^2+\frac{1}{6}\chi_2(x)\,|u|^6\right)\frac{d\sigma}{\sqrt{2}}. \tag{103}$$

Moreover, multiplying (92) by $Lu = (-t\partial_t + x\cdot\nabla + 1)u$, we get

$$div_{t,x}\,(tQ+\partial_t uu, -tP)+\left(\frac{2}{3}\chi_2-\frac{1}{6}x\cdot\nabla\chi_2\right)u^6+|\partial_t u|^2+|\nabla_x u|^2-(x.\nabla\chi_1)\,u^2 = 0, \tag{104}$$

where

$$Q = -\left[\frac{1}{2}(|\nabla_x u|^2 + |\partial_t u|^2 + \chi_1 |u|^2) + \frac{1}{6}\chi_2(x)u^6\right] + \partial_t u \frac{x}{t} \cdot \nabla_x u$$

and

$$P = \frac{x}{t}\left(\frac{1}{2}\left(|\partial_t u|^2 - |\nabla_x u|^2 - \chi_1 |u|^2\right) - \frac{1}{6}\chi_2(x)u^6\right) + \nabla_x u \left(-\partial_t u + \frac{x}{t} \cdot \nabla_x u + \frac{u}{t}\right).$$

Integrating (104) over K_a^b we obtain

$$\begin{aligned} 0 &= \int_{D(b)} (bQ + (\partial_t u)u)\, dx - \int_{D(a)} (aQ + (\partial_t u)u)\, dx \\ &\quad - \int_{M_a^b} (tQ + \partial_t u u + x \cdot P)\, \frac{d\sigma}{\sqrt{2}} + \int_{K_a^b} \left(|\partial_t u|^2 + |\nabla_x u|^2\right) dxdt \\ &\quad + \int_{K_a^b} \left[\left(\frac{2}{3}\chi_2 - \frac{1}{6}x \cdot \nabla \chi_2\right) u^6 - (x.\nabla \chi_1)u^2\right] dxdt \\ &= I + II + III + IV + V. \end{aligned} \tag{105}$$

We start with the term III. Since $t = |x|$ on M_a^b, we can write

$$III = \int_{M_a^b} |x| \left(\chi_1 |u|^2 + \frac{1}{3}\chi_2 u^6\right) \frac{d\sigma}{\sqrt{2}} - \int_{M_a^b} u \frac{x \cdot \nabla_x u}{|x|} + (\partial_t u)u \frac{d\sigma}{\sqrt{2}}.$$

We parameterize M_a^b by

$$y \longmapsto (|y|\,, y), \quad a \leq |y| \leq b,$$

and let $v(y) = u(|y|\,, y)$. Then

$$d\sigma = \sqrt{2}dy \quad \text{and} \quad y \cdot \frac{\nabla v}{|y|} = \frac{x \cdot \nabla_x u}{|x|} + \partial_t u.$$

Integrating by parts, one sees that

$$\int_{\substack{y \in \mathbb{R}^3 \\ a \leq |y| \leq b}} v \frac{y \cdot \nabla v}{|y|} dy = \frac{1}{2} \int_{\substack{y \in \mathbb{R}^3 \\ |y| = b}} v^2 d\sigma - \frac{1}{2} \int_{\substack{y \in \mathbb{R}^3 \\ |y| = a}} v^2 d\sigma - \int_{\substack{y \in \mathbb{R}^3 \\ a \leq |y| \leq b}} \frac{v^2}{|y|} dy.$$

So if we go back to the original coordinates, we have

$$III = \int_{M_a^b} |x| \left(\chi_1 u^2 + \frac{u^2}{|x|^2} + \frac{1}{3}\chi_2 u^6\right) \frac{d\sigma}{\sqrt{2}} - \frac{1}{2} \int_{\partial D_b} u^2 d\sigma + \frac{1}{2} \int_{\partial D_a} u^2 d\sigma. \tag{106}$$

Now, we rewrite the first and second term of (105) as

$$I+II=-H(b)+H(a)+\frac{1}{b}\int_{D_b}\left(x\cdot\nabla_x uu+\frac{3}{2}u^2\right)dx-\frac{1}{a}\int_{D_a}\left(x\cdot\nabla_x uu+\frac{3}{2}u^2\right)dx, \tag{107}$$

where

$$H(t)=\int_{D(t)} t\left[\frac{1}{2}\left|\frac{1}{t}Lu\right|^2+\frac{1}{2}\left(|\nabla_x u|^2-\left|\frac{x\cdot\nabla_x u}{t}\right|^2+\chi_1|u|^2\right)+\chi_2\frac{|u|^6}{6}\right]+\frac{u^2}{t}dx. \tag{108}$$

An integration by parts gives

$$\int_{D(t)}\left(x\cdot\nabla_x uu+\frac{3}{2}u^2\right)dx=\frac{t}{2}\int_{\partial D(t)}u^2 d\sigma. \tag{109}$$

Make use of (105), (106), (107), and (109), we get

$$H(a)-H(b)+\int_{M_a^b} t\left(\chi_1(x)u^2+\frac{u^2}{t^2}+\frac{1}{3}\chi_2(x)u^6\right)\frac{d\sigma}{\sqrt{2}}+\int_{K_a^b}\left(|\partial_t u|^2+|\nabla_x u|^2\right)dxdt$$
$$+\int_{K_a^b}\left[\left(\frac{2}{3}-\frac{1}{6}x\cdot\nabla\chi_2(x)\right)u^6-(x.\nabla\chi_1(x))\,u^2\right]dxdt=0.$$

Using hypothesis (93), $\left[\left(\frac{2}{3}-\frac{1}{6}x\cdot\nabla\chi_2(x)\right)u^6-(x.\nabla\chi_1(x))\,u^2\right]\geqslant 0$ and we conclude that

$$H(a)\le H(b). \tag{110}$$

On the other hand, in view of (108) we have the following double inequality

$$\int_{D(t)}\chi_2(x)\frac{u^6}{6}dx\le\frac{1}{t}H(t)\le C_1\left(e(t)+e(t)^{1/3}\right). \tag{111}$$

The last inequality being a simple consequence of Hölder's inequality applied to $\int_{D(t)}u^2dx$.

Consequently, dividing by a, we find from (110) and (111)

$$\int_{D(a)}\chi_2(x)|u(a,x)|^6dx\le D\left[\frac{b}{a}\left(e(b)+e(b)^{1/3}\right)\right]. \tag{112}$$

Now, by a density argument and the continuity of the nonlinear map

$$\begin{aligned} F:\quad H &\longrightarrow C\left([0,T],H\right)\\ (\varphi,\psi) &\longmapsto (u,\partial_t u), \end{aligned}$$

where u is the solution of (92) such that $(u, \partial_t u)_{/t=0} = (\varphi, \psi)$, the result of this lemma holds for every data in H. □

To prove that the Strichartz norms are global in time we follow the same program of that in Proposition 2.12 and we obtain the following proposition:

Proposition 5.8 *Let u be a solution to (92), then*

$$\int_{\mathbb{R}^3} \chi_2(x)\,|u(t,x)|^6\,dx \underset{t\to\pm\infty}{\longrightarrow} 0, \tag{113}$$

and for all $(q, r) \in]2, \infty] \times [2, \infty[$ *satisfying (101), we have*

$$u \in L^q(\mathbb{R}_+, L^r(\mathbb{R}^3)). \tag{114}$$

5.2 Exponential Decay of the Local Energy of Localized Linear Klein–Gordon Equation

The goal of this subsection is to prove the exponential decay of the local energy for the localized linear Klein–Gordon equation,

$$\begin{cases} \Box u + \chi_1 u = 0, & \text{on } \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u^0(x) \in H^1(\mathbb{R}^3) \quad \text{and} & \partial_t u(0,x) = u^1(x) \in L^2(\mathbb{R}^3), \end{cases} \tag{115}$$

where χ_1 is a function of class C^1 with compact support such that $supp \chi_1 \subset B_R$, for some $R > 0$. We equipped the Hilbert space $H = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with the scalar product

$$((u_1, u_2), (v_1, v_2)) = \int_{\mathbb{R}^3} \nabla u_1(x)\overline{\nabla v_1(x)} + \chi_1(x) u_1(x)\overline{v_1(x)} + u_2(x)\overline{v_2(x)}dx.$$

Obviously, this scalar product is equivalent to the classical one. We denote by E_L the global energy of u solution of (115) at time t defined by

$$E_L(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\partial_t u\,(t)|^2 + |\nabla_x u\,(t)|^2 + \chi_1(x)|u(t)|^2\right) dx,$$

and we define the local energy by

$$E_{L,R}(u(t)) = \frac{1}{2}\int_{B_R} \left(|\partial_t u\,(t)|^2 + |\nabla_x u\,(t)|^2 + \chi_1(x)|u(t)|^2\right) dx,$$

where B_R is a ball of radius R.

For that, we prove the following theorem:

Theorem 5.9 *Let $R > 0$, there exist $\alpha > 0$ and $c > 0$ such that*

$$E_{L,R}(u(t)) \leq Ce^{-\alpha t} E_L(0) \tag{116}$$

holds for every u solution to (115) with initial data $(u^0, u^1) \in H$ supported in B_R.

5.2.1 Semi-Group of Lax–Phillips Adapted to Localized Linear Klein–Gordon Equation

In this part we will show that the solution of (9) is generated by a semi-group of contractions that we note $U_{KG}(t) : t \geq 0$. Then we introduce the Lax–Phillips semi-group $Z_{KG}(t)$ adapted to our case.

Proposition 5.10 *The operator*

$$A_{KG} = \begin{pmatrix} 0 & I \\ \triangle - \chi_1 I & 0 \end{pmatrix}$$

of domain $D(A_{KG}) = \{\varphi = (\varphi_1, \varphi_2) \in H$ such that $A_{KG}\varphi \in H\}$, is maximal dissipative.

Proof Let $\varphi = (\varphi_1, \varphi_2) \in D(A_{KG})$. We have

$$(A_{KG}\varphi, \varphi) = \int_{\mathbb{R}^3} \nabla\varphi_2(x)\overline{\nabla\varphi_1(x)} + \Delta\varphi_1(x)\overline{\varphi_2(x)}dx.$$

By integrating by parts, one sees that $Re\,(A_{KG}\varphi, \varphi) = 0$ and, therefore, A_{KG} is a dissipative operator.

Now, in order to prove that $Im\,(I - A_{KG}) = H$, we consider $g = (g_1, g_2) \in H$ and $\varphi = (\varphi_1, \varphi_2) \in D(A_{KG})$ satisfying

$$\varphi - A_{KG}\varphi = g. \tag{117}$$

This is equivalent to

$$\begin{cases} \varphi_2 = \varphi_1 - g_1, \\ -\Delta\varphi_1 + (1 + \chi_1)\varphi_1 = g_1 + g_2. \end{cases} \tag{118}$$

Let introduce now, the bilinear form $b(h, \psi)$ on $H^1(\mathbb{R}^3)$ as follows:

$$b(h, \psi) = \int_{\mathbb{R}^3} \nabla h\overline{\nabla\psi}dx + \int_{\mathbb{R}^3} (1 + \chi_1)\, h\overline{\psi}dx. \tag{119}$$

We have

$$|b(h,\psi)| \leq c\|h\|_{H^1(\mathbb{R}^3)}\|\psi\|_{H^1(\mathbb{R}^3)},$$

so b is continuous, and $b(h,h) \geq c\|h\|_{H^1(\mathbb{R}^3)}$, i.e., b is coercive.

Moreover, $g_1 + g_2 =: \tilde{g} \in L^2(\mathbb{R}^3)$, then the linear form

$$\begin{aligned} l : H^1 &\to \mathbb{C} \\ \psi &\longmapsto (\tilde{g},\psi) \end{aligned}$$

is continuous. By the Lax–Milgram theorem, there exists a unique solution $\varphi_1 \in H^1(\mathbb{R}^3)$ of the variational problem

$$b(\varphi_1,\psi) = l(\psi),\ \forall\psi \in H^1(\mathbb{R}^3).$$

Using again the system (118), one can easily find that $\varphi = (\varphi_1,\varphi_2) \in D(A_{KG})$.

This permit to conclude the proof. □

Remark 5.11 According to the Hille–Yosida theorem, A_{KG} generates a C_0-contraction semi-group $\left(U_{KG}(t)\right)_{t\geq 0}$.

At present, we set for $t \geq 0$ $Z_{KG}(t) = P^+U_{KG}(t)P^-$, where P^+ and P^- are, respectively, orthogonal projection on $(D_+)^\perp$ and $(D_-)^\perp$.

The following proposition gives some properties of the operator $Z_{KG}(t)$.

Proposition 5.12

(1) $Z_{KG}(t)D_+ = Z_{KG}(t)D_- = \{0\}$, *for every* $t \geq 0$.
(2) $Z_{KG}(t)$ *operates on* $K = (D_+)^\perp \cap (D_-)^\perp$.
(3) $(Z_{KG}(t))_{t\geq 0}$ *is a continuous semi-group on* K.

The arguments (with slight modifications) in the proof below are contained in [1]. We include them for the reader's convenience to make the paper self-contained.

Proof

(1) Let $\varphi \in D_-$ then by definition of P^- we have: $Z_{KG}(t)\varphi = 0$. Let $\varphi \in D_+$, since D_+ and D_- are orthogonal then $P^-\varphi = \varphi$ and so to deduce that $Z_{KG}(t)\varphi = 0$, it is enough to verify $U_{KG}(t)D_+ \subset D_+$. Let $\varphi \in D_+$ and $U_{KG}(t)\varphi = \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix}$ the corresponding KG group where $u(t)$ is the solution of (9). Since $\varphi \in D_+$ then $u(t,x) = 0$ for $|x| \leq t + R$ and $t \geq 0$. As $supp(\chi_1 u) \subset B_R$ then u verifies:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0) = \varphi_1 \ \ \partial_t u(0) = \varphi_2 \text{ on } \mathbb{R}^3. \end{cases}$$

According to the uniqueness of the solution of Eq. (9), we conclude that $U_{KG}(t)\varphi = U(t)\varphi$. Which gives $U_{KG}(t)\varphi \in D_+$ because $U(t)\varphi \in D_+$.

(2) Let $\varphi \in K = (D_+)^\perp \cap (D_-)^\perp$, show that $Z_{KG}(t)\varphi \in K$. It is easy to see that $Z_{KG}(t)\varphi \in (D_+)^\perp$. In fact, let $g \in D_+$, we have

$$(Z_{KG}(t)\varphi, g) = (P^+U_{KG}(t)\varphi, g) = (U_{KG}(t)\varphi, P^+g) = 0,$$

which shows that $Z_{KG}(t)\varphi \in (D_+)^\perp$. It remains to verify that $Z_{KG}(t)\varphi \in (D_-)^\perp$.

Let $g \in D_-$, we have

$$\begin{aligned}(Z_{KG}(t)\varphi, g) &= (P^+U_{KG}(t)\varphi, g) = (U_{KG}(t)\varphi, P^+g) = (U_{KG}(t)\varphi, g)\\ &= (\varphi, U^*_{KG}(t)g).\end{aligned}$$

To complete the proof of (2), we give the following lemma: □

Lemma 5.13 *Let $U^*_{KG}(t)$ the adjoint operator of $U_{KG}(t)$. Then $U^*_{KG}(t)\varphi = U(-t)\varphi$, $\forall \varphi \in D_-$.*

Proof Since $U_{KG}(t)$ is a semi-group generated by A_{KG}, then $U_{KG^*}(t)$ is a semi-group generated by A^*_{KG}. Let $g = (g_1, g_2) \in D_-$, we put

$$U^*_{KG}(t)g = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

such that

$$\begin{cases} \partial_t v_1 = -v_2, \\ \partial_t v_2 = -\Delta v_1 - \chi_1 v_1, \end{cases}$$

which implies

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 - \chi_1 v_1 = 0, \\ \partial_t v_1 = -v_2. \end{cases}$$

So, we have $U^*_{KG}(t)g = \begin{pmatrix} v_1(t) \\ -\partial_t v_1(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \partial_t w(t) \end{pmatrix}$, $v(t)$ is the solution of:

$$\begin{cases} \partial_t^2 v - \Delta v - \chi_1 v = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^3, \\ (v(0), \partial_t v(0)) = (g_1, -g_2). \end{cases}$$

Similarly $U(-t)g = \begin{pmatrix} w(t) \\ -\partial_t w(t) \end{pmatrix}$, and $w(t)$ the solution of

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^3, \\ (w(0), \partial_t w(0)) = (g_1, -g_2). \end{cases}$$

Setting $\widetilde{v}(t) = v(-t)$ and $\widetilde{w}(t) = w(-t)$ with $t \leq 0$, then $\widetilde{v}(t)$ and $\widetilde{w}(t)$ verify the following equations:

$$\begin{cases} \partial_t^2 \widetilde{v} - \Delta \widetilde{v} - \chi_1 \widetilde{v} = 0, & \text{on } \mathbb{R}_- \times \mathbb{R}^3, \\ (\widetilde{v}(0), \partial_t \widetilde{v}(0)) = (g_1, -g_2), \end{cases}$$

$$\begin{cases} \partial_t^2 \widetilde{w} - \Delta \widetilde{w} = 0, & \text{on } \mathbb{R}_- \times \mathbb{R}^3, \\ (\widetilde{w}(0), \partial_t \widetilde{w}(0)) = (g_1, g_2). \end{cases}$$

Since $g \in D_-$ then $U(-t)g = 0$ on $|x| \leq t + R$ and $t \geq 0$, so $\widetilde{w}(t) = 0$ on $|x| \leq -t + R$ and $t \leq 0$. But we have $supp\chi_1 \subset B_R$ then by uniqueness of the solution we deduce that $\widetilde{w}(t) = \widetilde{v}(t)$ for $t \leq 0$, hence $w(t) = v(t)$ for $t \geq 0$, and $U^*_{KG}(t)g = U(-t)g$. □

Now let us go back to the second point proof of the proposition, according to Lax and Phillips (see Theorem 2.1 in [16, chapter V]) we have, $U(-t)D_- \subset D_-$ for all $t \geq 0$. We deduce then that $U^*_{KG}(t)g \in D_-$ and since $\varphi \in (D_-)^\perp$ then $(Z_{KG}(t)\varphi, g) = (\varphi, U^*_{KG}(t)g) = 0$ which shows that $Z_{Kg}(t)\varphi \in (D_-)^\perp$.

3) Let $s \geq 0$, $t \geq 0$ and $\varphi \in K$. We have

$$\begin{aligned} Z_{KG}(t)Z_{KG}(s)\varphi &= P^+U_{KG}(t)P^-Z_{KG}(s)\varphi \\ &= P^+U_{KG}(t)P^+U_{KG}(s)P^-\varphi \\ &= P^+U_{KG}(t)P^+U_{KG}(s)\varphi. \end{aligned}$$

Since $P^+U_{KG}(t)P^+ = P^+U_{KG}(t)$ (because $(P^+ - I)$ is the orthogonal projection on D_+) then

$$Z_{KG}(t)Z_{KG}(s)\varphi = P^+U_{KG}(t)U_{KG}(s)\varphi = Z_{KG}(t+s).$$

The following lemma will be used to establish the exponential decay of the semi-group $Z_{KG}(t)$:

Lemma 5.14 *We have*

(a) $U_{KG}(t)(D_-)^\perp \subset (D_-)^\perp$ *and* $U(t)(D_-)^\perp \subset (D_-)^\perp$ *for all* $t \geq 0$.
(b) $U(t)(D_-)^\perp \subset D_+$ *for all* $t \geq 2R$.
(c) *If we put* $M = U_{KG}(2R) - U(2R)$ *then we have,*

$$M\varphi = 0 \quad \text{for } |x| \geq 3R \quad \text{and } \|M\varphi\| \leq 2\|\varphi\|_{5R}.$$

(d) $Z_{KG}(t) = P^+MU_{KG}(t-4R)MP^-, \forall t \geq 4R$.

Proof

(a) Let $\varphi \in (D_-)^\perp$ and $g \in D_-$, we have $(U_{KG}(t)\varphi, g) = (\varphi, U^*_{KG}(t)g)$. As $U^*_{KG}(t)g \in D_-$, then $(U_{KG}(t)\varphi, g) = 0$ and consequently $U_{KG}(t)\varphi \in (D_-)^\perp$.
In the same way we show the other inclusions.

(b) It suffices to show that $U(2R)(D_-)^\perp \subset D_+$, according to the theory of representation ([28]), the spaces D_- and D_+ corresponding, respectively, to sub-spaces $L^2(]-\infty, -R] \times \mathcal{S}^2)$ and $L^2([R, +\infty] \times \mathcal{S}^2)$. Since the group $U(t)$ operates like translation on the right on L^2 then $U(2R)(D_-)^\perp$ is represented by $L^2([R, +\infty] \times \mathcal{S}^2)$ which proves the second point.

(c) Let $\varphi \in H$, by a domain of dependence argument (see [16]), we see that

$$U(t)\varphi = U_{KG}(t)\varphi \quad \text{on } |x| > t + R,\ t \geq 0.$$

In particular, for $t = 2R,\ U(2R)\varphi = U_{KG}(2R)\varphi$ on $|x| > 3R$.
Another application of the principle of domain of dependence shows that

$$\|U(2R)\varphi\|_{3R} \leq \|\varphi\|_{5R},$$

and

$$\|U_{KG}(2R)\varphi\|_{3R} \leq \|\varphi\|_{5R},$$

then

$$\|M\varphi\| = \|M\varphi\|_{3R} \leq 2\|\varphi\|_{5R}.$$

(d) We have

$$\begin{aligned} &P^+ MU(t-4R)MP^-\varphi \\ &= Z_{KG}(t)\varphi + P^+U(2R)U_{KG}(t-4R)U(2R)P^-\varphi \\ &- P^+U_{KG}(t-2R)U(2R)P^-\varphi - P^+U(2R)U_{KG}(t-2R)P^-\varphi. \end{aligned}$$

Using (b), $U(2R)P^-\varphi \in D_+$, therefore, the second and the third terms are equal to 0. Similarly, using a) we deduce $U_{KG}(t-2R)P^-\varphi \in (D_-)^\perp$.
Using again the argument in (b) we get $U(2R)U_{KG}(t-2R)P^-\varphi \in D_+$ which shows that the last term is equal to 0. □

5.2.2 Proof of Theorem 5.9

In order to prove Theorem 5.9 we will need the following theorem due to Nunes and Bastos which establish the polynomial decay of local energy. They proved this result for the linear Klein–Gordon equation in $\mathbb{R}^n$, $n \geq 1$, but we can see that, with slight modifications, the theorem and its proof remain valid in the context of our problem. Let us state the theorem which in our case.

Theorem 5.15 ([23, Nunes–Bastos])
Let $\Omega \subset \mathbb{R}^3$, be a bounded domain. There exists positive constants $T_0 > d(\Omega)$ and $K > 0$ depending on Ω, c and T_0 such that for every $u^0, u^1 \in C_0^\infty(\mathbb{R}^3)$ with supp $u^0 \cup$ supp $u^1 \subseteq \Omega$, the solution u to the Cauchy problem (115) satisfies

$$E(u(t)) \leq \frac{K}{t^3}\left(\|u^0\|^2_{H^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)}\right) \tag{120}$$

for every $t > T_0$.

Remarks 5.16

(1) The truncation function χ_1 does not have impact on the proof of theorem; following [25] and using the well-known representation for the solutions of the wave equation see [9], and also an integral representation of Bessel's functions (see [11, p. 437]) we obtain the desired result.
(2) Using spectral approach in our case, more precisely, the method used by Malloug [19] we can find the polynomial decay of the local energy for Klein–Gordon equation.
(3) This result combined with the properties of semi-group $Z_{KG}(t)$ will allow us to show that the decay of energy is in fact, exponential.

We come back now to the proof of Theorem 5.9.

For $\rho > 0$, we put $H_\rho = \{\varphi \in H; \varphi$ with support in $B_\rho\}$. Let $\varphi \in H_R$ and $g \in D_+ \cup D_-$, then $g = 0$ on B_R, so $(\varphi, g) = 0$ and consequently $\varphi \in K$. On the other hand, for a given h of H, $P^+h = h$ on B_R, thus we obtain

$$U_{KG}(t)\varphi = Z_{KG}(t)\varphi \quad \text{on } B_R.$$

We have $\|U_{KG}(t)\varphi\|_R = \|Z_{KG}(t)\varphi\|_R \leq \|Z_{KG}(t)\varphi\|$. Hence to get the exponential decay of the local energy, it is enough to prove the exponential decay of $\|Z_{KG}(t)\|$.

By applying the estimate (120) of Theorem 5.15 with $\Omega = B_\rho$, and choosing $\rho = 5R$ and T sufficiently large such that

$$\|U_{KG}(t)g\|_{5R} \leq \frac{1}{8}\|g\|, \qquad g \in H_{5R}. \tag{121}$$

Let $\varphi \in H$, by the previous lemma we have

$$\begin{aligned}
\|Z_{KG}(T+4R)\varphi\| &= \|P^+ M U_{KG}(T) M P^- \varphi\| \\
&\leq \|M U_{KG}(T) M P^- \varphi\| \\
&= \|M U_{KG}(T) M P^- \varphi\|_{3R} \\
&\leq 2\|U_{KG}(T) M P^- \varphi\|_{5R}
\end{aligned}$$

$$\leq \frac{1}{4}\|MP^{-}\varphi\| \leq \frac{1}{2}\|P^{-}\varphi\| \leq \frac{1}{2}\|\varphi\|.$$

Put $T' = T + 4R$; let $t > 0$, there exists $k \in \mathbb{N}$ such that $kT' \leq t \leq (k+1)T'$ and one deduces

$$\|Z_{KG}(t)\varphi\| \leq \|Z_{KG}(kT')\varphi\| \leq \|Z_{KG}(T')\varphi\|^k \leq \frac{1}{2^k}\|\varphi\| \leq Ce^{-\alpha t}\|\varphi\|.$$

Thus, Theorem 5.9 holds.

We are now ready to prove our main result.

5.3 *Proof of Theorem 5.1*

We come back to the proof of the main result of this section: By combining the results of the global time Strichartz norms and the exponential decay of the local energy of the linear Klein–Gordon equation, we deal with the nonlinear term $\chi_2 u^5$ as a source term and using the Gronwall lemma in a crucial way, we obtain the exponential decay of the local energy for the solutions of (92). Thanks to the Duhamel's Formula, the nonlinear equation (92) for $u = U(t)\psi$ can be written as

$$u(t) = U_L(t)\psi + \int_0^t U_L(t-s)I\chi_2 u^5(s)ds,$$

where $U_L(t)$ is the linear evolution group, $\chi_2 = \chi_2(x)$ is the localizer, and I is the mapping defined by $Iu = (0, u)$. Fix a ball B_R, an energy bound $E(\psi) \leq R_0$, and a smooth cut-off function K that satisfies $K(x) = 1$ on $supp(\psi)$. From now we denote by C any constant which may depend on B_R, R_0, and K.

By the support property of K, we have

$$Ku(t) = KU_L(t)\psi + \int_0^t KU_L(t-s)KI\chi_2 u^5(s)ds,$$

and using the fact that the local energy of $U_L(t)$ decay exponentially

$$\|KU_L(t)K\phi\|_E \leq Ce^{-\beta t}\|\phi\|_E,$$

for some constant $\beta > 0$, here E denotes the energy space. Applying this estimate to the above integral identity, we obtain

$$\|Ku\|_{L^\infty_{(t)}(E)} \leq \|KU_L(t)\psi\|_{L^\infty_{(t)}(E)} + C\int_0^t e^{-\beta(t-s)}\|\chi_2 u^5\|_{L^1_{(s)}(L^2)}ds. \tag{122}$$

For any $t \geq 0$, where $L^p_{(s)}(X) := L^p(s, s+1; X)$, and the spatial domain $\mathbb{R}^3$ is omitted. As a consequence of the global time Strichartz in time, we deduce for any $\varepsilon > 0$ that there exists $T > 0$ such that

$$\|\chi_2^{1/6} u\|_{\mathrm{L}^\infty(\mathrm{T},\infty,\mathrm{L}^6)} < \varepsilon.$$

Similarly, if ε is sufficiently small, we can bound any other space-time norms of the Strichartz type.

For example, we have $\|u\|_{L^{5/2}(T,\infty,L^{30})} < C$. By the Hölder's inequality, we have for any interval $I \subseteq (T, \infty)$

$$\|\chi u^5\|_{L^1(I,L^2)} \leq C\|\chi^{1/6} u\|^{5/2}_{L^\infty(I,L^6)} \|u\|^{5/2}_{L^{5/2}(I,L^{30})} \leq C\|\chi^{1/6} u\|^{5/2}_{L^\infty(I,L^6)}. \tag{123}$$

Since $|\chi^{1/6} u| \leq C|Ku|$ by the support property, the Sobolev inequality implies that

$$\|\chi u^5\|_{L^1(I,L^2)} \leq C\varepsilon^{3/2} \|Ku\|_{L^\infty(I,E)}. \tag{124}$$

We apply these bounds to (122), translating t by T. Denoting by

$$f(t) := \|Ku\|_{L^\infty_{(t+T)}(E)}, \qquad g(t) := \|KU_L(t-T)\|_{L^\infty_{(t+T)}(E)}. \tag{125}$$

We obtain the following integral inequality

$$f(t) \leq g(t) + C\varepsilon^{3/2} \int_0^t e^{-\beta(t-s)} f(s) ds, \tag{126}$$

so

$$f(t) \leq Ce^{-\beta t} + C\varepsilon^{3/2} \int_0^t e^{-\beta(t-s)} f(s) ds,$$

which is equivalent to

$$e^{\beta t} f(t) \leq C + C\varepsilon^{3/2} \int_0^t e^{\beta s} f(s) ds.$$

By virtue of Gronwall Lemma, we obtain

$$f(t) \leq Ce^{(c\varepsilon^{3/2}-\beta)t}, \quad \text{for } t \geq 0.$$

We choose ε so $c\varepsilon^{3/2} - \beta < -\frac{\beta}{2}$, to get the exponential decay for $f(t)$, which implies that of $E(Ku(t))$.

Appendix

In this appendix, we give some important theorems of the literature that we have used in several proofs in this article. We give them in our context, i.e., for the solutions of the critical wave equation with localized semilinearity near a convex obstacle. All these results are borrowed from [2, 8, 10], we note that (with slight modifications of their proofs) these results remain valid in the context of our problem.

Let us first introduce some vocabulary.

A scale h is a sequence $(h_n)_n$ of positive numbers going to 0 if n goes to infinity; a core is a convergent sequence $z = (t_n, x_n)$ of $\mathbb{R}_t \times \mathbb{R}^3_x$.

(h, z) and (h', z') are orthogonal if

$$\text{either } \left|\log(\frac{h_n}{h'_n})\right| \underset{n\to+\infty}{\to} +\infty \text{ or } h = h' \text{ and } \frac{\left|t_n - t'_n\right| + \left|x_n - x'_n\right|}{h_n} \underset{n\to+\infty}{\to} +\infty. \tag{127}$$

The linear concentrating wave associated with $(\varphi, \psi, h_n, x_n, t_n)$ is the solution of the following linear wave equation

$$\begin{cases} \Box p_n = 0 \text{ on } \mathbb{R}\times\Omega, \;\; p_n = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ (p_n(t_n), \partial_t p_n(t_n)) = \left(\frac{1}{h_n^{1/2}} P_\Omega\left(\varphi\left(\frac{\cdot - x_n}{h_n}\right)\right); \frac{1}{h_n^{3/2}} 1_\Omega(\cdot)\psi\left(\frac{\cdot - x_n}{h_n}\right)\right), \end{cases} \tag{128}$$

where Ω is the exterior of a compact, strictly convex, smooth domain of $\mathbb{R}^3$ and P_Ω is the orthogonal projection from $\dot{H}^1(\mathbb{R}^3)$ to $H_D(\Omega)$.

The nonlinear concentrating wave associated with p_n is the solution of the following equation

$$\begin{cases} \Box q_n + \chi(x) q_n^5 = 0 \text{ on } \mathbb{R}\times\Omega, \;\; q_n = 0 \text{ in } \mathbb{R} \times \partial\Omega \\ (q_n(0), \partial_t q_n(0)) = (p_n(0), \partial_t p_n(0)). \end{cases} \tag{129}$$

We recall that the energy of any function u solution to (128) or (129) is defined by:

$$E_0(u)(t) = \frac{1}{2}\int_\Omega \left(|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2\right) dx.$$

Finally, we assume that the initial data (φ_n, ψ_n) is compact at infinity, in the sense that

$$\overline{\lim_{n\longrightarrow+\infty}} \int_{|x|\geq R} \left(|\nabla\varphi_n(x)|^2 + |\psi_n(x)|^2\right) dx \underset{R\longrightarrow+\infty}{\longrightarrow} 0. \tag{130}$$

We recall the following theorem witch is adapted from Theorems 1 and 3 in [10].

Theorem 5.17 *Let v_n be the solution of*

$$\begin{cases} \Box v_n = 0 \; on \; \mathbb{R}\times\Omega, \; v_n = 0 \; in \; \mathbb{R}\times\partial\Omega \\ \quad (v_n(0), \partial_t v_n(0)) = (\varphi_n, \psi_n) \end{cases}$$

satisfying $\sup_n E_0(v_n) < +\infty$ *and (130). Then there exists a finite energy solution to the linear wave equation* v*, orthogonal concentrating data* $\left(\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)}\right)$, *for* $j \in \mathbb{N}^*$*, such that* v_n *can be decomposed as follows, up to the extraction of a subsequence: for any* $l \in \mathbb{N}^*$,

$$v_n = v + \sum_{j=1}^{l} p_n^{(j)} + w_n^{(l)},$$

where $p_n^{(j)}$ *is the linear concentrating wave associated with* $\left(\varphi^{(j)}, \psi^{(j)}, h_n^{(j)}, x_n^{(j)}, t_n^{(j)}\right)$ *and the remainder* $w_n^{(l)}$ *satisfies, for every* $T > 0$,

$$\overline{\lim_{n\longrightarrow+\infty}} \left\| w_n^{(l)} \right\|_{L^\infty([-T,T],L^6(\Omega))} \underset{l\to+\infty}{\to} 0. \tag{131}$$

Moreover, denote u_n *a solution in the "Shatah–Struwe" class of*

$$\begin{cases} \Box u_n + \chi(x)u_n^5 = 0 \; on \; \mathbb{R}\times\Omega, \; u_n = 0 \; in \; \mathbb{R}\times\partial\Omega \\ \quad (u_n(0), \partial_t u_n(0)) = (\varphi_n, \psi_n) \end{cases} \tag{132}$$

satisfying $\sup_n E_0(u_n) < +\infty$ *and (130). Then up to the extraction of a subsequence, we can write, for any* $l \in \mathbb{N}^*$,

$$u_n = u + \sum_{j=1}^{l} q_n^{(j)} + w_n^{(l)} + r_n^{(l)}, \tag{133}$$

where u *is a solution of a nonlinear wave equation,* $q_n^{(j)}$ *is the nonlinear concentrating wave equation associated with* $p_n^{(j)}$ *and for every* $T > 0$,

$$\overline{\lim_{n\longrightarrow+\infty}} \Big(\sup_{-T\le t\le T} E_0(r_n^{(l)}, t)^{1/2} + \left\| r_n^{(l)} \right\|_{L^5([-T,T],L^{10}(\Omega))} \Big) \underset{l\to+\infty}{\to} 0. \tag{134}$$

Let us note that this result, which describes the high frequency approximation of the solutions of the critical wave equation, is easily applicable in our context, i.e., in the presence of the function χ. Indeed, looking carefully to the proof of Theorem 3, one observes that the behavior of a profile concentrating at $x_n^{(j)} \longrightarrow x_\infty^{(j)}$ depends locally only on $\chi(x_\infty^{(j)})$ while the behavior is nonlinear and does not have any effect while the profile is close to linear.

Now, notice the following corollaries.

Corollary 1 (Adapted from Corollary 1 in [2]) *Let (u_n) be a sequence of solution in the "Shatah–Struwe" class to* (1.1). *We assume that $(\varphi_n, \psi_n) \rightharpoonup (\varphi, \psi)$ in $H_D(\Omega) \times L^2(\Omega)$. Then $u_n \rightharpoonup u$, where u is the solution in the "Shatah–Struwe" class of*

$$\begin{cases} \Box u + \chi(x) u^5 = 0 \text{ on } \mathbb{R}\times\Omega \\ (u(0), \partial_t u(0)) = (\varphi, \psi). \end{cases} \tag{135}$$

Corollary 2 (Adapted from Corollary 2 in [2]) *There exists a nondecreasing function $A : [0, +\infty[\longrightarrow [0, +\infty[$ such that, for every Shatah–Struwe solution u to (1.1),*

$$\|u\|_{L^5(\mathbb{R}, L^{10}(\Omega))} \leq A\left(E(u)\right).$$

Let now (u_n) be a sequence of solutions to (135). We recall that the energy density of u_n is given by

$$e_n(t,x) = \frac{1}{2}\left[|\partial_t u_n(t,x)|^2 + |\nabla_x u_n(t,x)|^2\right] + \frac{1}{6}\chi(x)\,|u_n(t,x)|^6,$$

and we say that $e(t,x)$ is the limit energy density of the sequence (u_n) if $e_n(t,x)$ converges weakly to $e(t,x)$.

We finally come to the "energy balance theorem" witch is adapted from Theorem 7 in [8].

Theorem 5.18 *Let (u_n) be a bounded sequence in the "Shatah–Struwe" class, solution of (135) and satisfying $\sup_n E_0(u_n) < +\infty$, $u_n(0)$, $\partial_t u_n(0)$ are supported in a fixed compact of Ω and $u_n \rightharpoonup 0$. Then we can write the limit energy density of (u_n) as*

$$e(t,x) = \sum_{j=1}^{+\infty} e^{(j)}(t,x) + e_f(t,x), \tag{136}$$

where $e^{(j)}$ is the limit energy density of the nonlinear concentrating wave $q_n^{(j)}$ and e_f is the limit energy density of a sequence of solutions of linear wave equation $\widetilde{w}_n$, namely

$$e_f(t,x) = \int_{\xi\in S^2} \mu(t,x,d\xi)$$

with $\mu(t,x,d\xi) = \mu_+(t,x,d\xi) + \mu_-(t,x,d\xi)$ and $\mu_\pm$ are positive measures on $\Omega \times S^2$.

This theorem remains valid in our case. Indeed its proof is based on lemma A.3 of [6] which we easily adapt to our work by extending the solutions by 0 outside Ω. More precisely, using the notations of [8], let $\varphi(t,x) \in C_0^\infty(\mathbb{R}\times\Omega)$, $\psi(t,x) \in C_0^\infty(\mathbb{R}\times\Omega)$ such that $\mathrm{supp}(\varphi) \subset \{(t,x) \setminus \psi \equiv 1\}$ and $\tilde{v}_{\pm,n}^{(j)}$ (respectively, $\tilde{w}_n^{(l)}$) the extensions by 0 of $v_{\pm,n}^{(j)}$ (respectively, $w_n^{(l)}$), outside Ω. We have

$$\Box\left(\psi\tilde{w}_n^{(l)}\right) = [\Box,\psi]\,w_n^{(l)} \underset{n\to+\infty}{\to} 0 \text{ in } L^2(\mathbb{R}\times\Omega),$$

which yields (as in [8]) the desired result, i.e.,

$$\varphi b\left(v_{\pm,n}^{(j)}, w_n^{(l)}\right) = \varphi b\left(\psi\tilde{v}_{\pm,n}^{(j)}, \psi\tilde{w}_n^{(l)}\right) \underset{n\to+\infty}{\to} 0 \text{ in } L^1(\mathbb{R}\times\Omega).$$

Finally, let us indicate that the analogue of the lemma A2 in [8] is in [10] (lemma 3.7 p. 35).

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A Spectral Numerical Method to Approximate the Boundary Controllability of the Wave Equation with Variable Coefficients

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1 Introduction

Consider the one-dimensional wave equation with a given potential $a(x) \in L^\infty(0,1)$ and a boundary control at the extreme $x = 1$,

$$\begin{cases} u''(t,x) - u_{xx}(t,x) + a(x)u(t,x) = 0 & t \in (0,T), \quad x \in (0,1), \\ u(t,0) = 0, \qquad u(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1}$$

where $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ are the initial data. Here, $H^{-1}(0,1)$ represents the dual space of $H_0^1(0,1)$. The following controllability result is known to hold [10]: given $T > 2$, for any initial data $(u^0, u^1) \in L^2 \times H^{-1}(0,1)$, there exists a control $f \in L^2(0,T)$ such that the solution u of system (1) satisfies

$$u(x,T) = u_t(x,T) = 0, \quad x \in (0,1).$$

This is usually known as the null controllability problem. It is well known that, due to the time reversibility of the wave equation, this is equivalent to control any initial data to any other final one, not necessarily zero (see, for example, [8] or [11]).

We are interested in the numerical approximation of the control f. A natural numerical approach consists in substituting the continuous wave equation by a consistent discrete approximation, depending on a discretization parameter $h \to 0$,

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and stating the corresponding results at the discrete level. When this discrete system is controllable, it can be used to obtain numerical approximations of the controls and the controlled solutions.

It turns out that this is not the case when considering usual discretization of the wave equation based on finite differences or classical finite elements. This has been observed by a number of authors in different situations (see [11] or [6] and the references therein for a complete description and number of examples).

This has a dramatic consequence in the applications. For example, in the context of the boundary controllability, if we compute the boundary controls of the discrete approximation, their norms grow exponentially as the discretization parameter $h \to 0$, for some initial data, and do not produce any approximation of the control for the continuous system.

These phenomena were first described in a series of papers by R. Glowinski and J.-L Lions (see the review paper [7] and the references therein). They also proposed a number of cures to deal with this lack of uniformity, as bigrid approximations or Tychonoff regularization methods. Later on, other methods have been proposed: filtering, projection of the controlled solution, etc. (see the review [11]).

In [2] and [4], a numerical method based on a mixed finite element formulation for (1) was found to give a corresponding semi-discrete approximation of the controlled wave equation with the property that the discrete controls converge to the continuous one as $h \to 0$. This was proved for the constant coefficients wave equation in one dimension and two dimension in a square. As far as we know, this is the only discretization of the wave equation for which such property holds. The proof in the one-dimensional case is based on a Fourier series argument and requires a detailed spectral analysis, while the two-dimensional case relies on a discrete version of the classical multipliers method. This mixed finite elements method has been recently extended to the variable coefficient equation (1) (see [3]).

In this chapter, we propose a Fourier approach to this controllability problem in the one-dimensional case. It consists basically in a projection method over the finite dimensional space generated by the first eigenfunctions of the Laplacian. In this way, the controllability problem is reduced to a controllability one for a finite dimensional system for which we apply the well-known results. In particular, we write the explicit formula for the controls. This method has been previously considered in the literature by F. Bourquin et al. in [12] and [13] to approximate the boundary control of the wave equation but without potential. Here we focus on the numerical implementation of the method and give some experiments that show its efficiency. This provides numerical evidences of the convergence. Convergence estimates of the discrete controls to the continuous one can be found in [1] or [12] for slightly different problems.

Note that the projection method described in this work is not restricted to the one-dimensional case and it can be extended to several dimensions, as long as we can compute efficiently the eigenfunctions of the Laplacian. In some special domains as balls or intervals, these are explicit and the method is easily adapted.

The rest of the chapter is divided as follows. In Sect. 2, we introduce the Fourier projection method and deduce the discretization of (1). In Sect. 3, we recall the controllability results for the finite dimensional systems and apply them to the discrete problem. In Sect. 4, we show some numerical experiments that illustrate the efficiency of the method. Finally, in Sect. 5, we give an appendix with the MATLAB code.

2 Numerical Approximation of the Control Problem

In this section, we introduce a numerical approximation of the control problem (1) that provides a new discrete control problem that we solve in the next section. The solutions of this discrete control problem are taken as numerical approximations of the original control problem.

We assume that f is compactly supported in $t \in (0, T)$. This condition is required for the controls that we construct in the next section. We first homogenize the boundary condition by introducing the following system:

$$\begin{cases} h''(x) - a(x)h(x) = 0, & x \in (0,1) \\ h(0) = 0, & h(1) = 1. \end{cases} \tag{2}$$

Then, the change of variables

$$v'' = u - h(x)f(t)$$

transforms system (1) into

$$\begin{cases} v''(t,x) - v_{xx}(t,x) + a(x)v(t,x) = -h(x)f(t), & t \in (0,T), \ x \in (0,1) \\ v(t,0) = v(t,1) = 0, & t \in (0,T) \\ v(0,x) = v^0(x), \ v'(0,x) = v^1(x) & x \in (0,1), \end{cases} \tag{3}$$

where the initial data $v^i(x)$ $(i = 1, 2)$ are obtained from $u^i(x)$ as the solutions of the elliptic problems

$$\begin{cases} (v^i)''(x) - a(x)v^i(x) = u^i(x), & x \in (0,1) \\ v^i(0) = v^i(1) = 0, & i = 1, 2. \end{cases} \tag{4}$$

Remark 1 We can also consider the apparently more natural change of variables $v = u - h(x)f(t)$, but this produces a system where the control is $f''(t)$, instead of $f(t)$, and therefore a control problem with less smooth data.

Now we use the Fourier representation of solutions. For any time $t \in [0, T]$, we can write $v(t, x)$ as a linear combination of the elements in the orthogonal basis $\{\sin(k\pi x)\}_{k=1}^{\infty}$ in $L^2(0, 1)$. Therefore,

$$v(t,x) = \sum_{k=1}^{\infty} v_k(t) \sin(k\pi x).$$

Substituting this expression in the first equation in (1), we formally obtain

$$\sum_{k=1}^{\infty}\left(v_k''(t)+k^2\pi^2 v_k(t)+a(x)v_k(t)\right)\sin(k\pi x)+h(x)f(t)=0,$$

which we project in the finite dimensional space X_N spanned by $\{\sin(k\pi x)\}_{k=1}^{N}$. In this way, we obtain

$$v_n''(t)+n^2\pi^2 v_n(t)+2\sum_{k=1}^{\infty}v_k(t)\int_0^1 a(x)\sin(k\pi x)\sin(n\pi x)\,dx$$
$$+2f(t)\int_0^1 h(x)\sin(n\pi x)\,dx=0,\quad n=1,\ldots,N. \tag{5}$$

We now truncate the sum, leaving the first N terms,

$$v_n''(t)+n^2\pi^2 v_n(t)+2\sum_{k=1}^{N}v_k(t)\int_0^1 a(x)\sin(k\pi x)\sin(n\pi x)\,dx$$
$$+2f(t)\int_0^1 h(x)\sin(n\pi x)\,dx=0.\quad n=1,\ldots,N. \tag{6}$$

It is convenient to write this system in matrix form

$$V''=-(D+P_a)V-f(t)F_h, \tag{7}$$

where

$$V=\begin{pmatrix} v_1\\ v_2\\ \ldots\\ v_N\end{pmatrix},\quad D=\pi^2\begin{pmatrix}1^2 & 0 & \ldots & 0\\ 0 & 2^2 & \ldots & 0\\ \ldots & & & \\ 0 & 0 & \ldots & N^2\end{pmatrix}, \tag{8}$$

P_a is the $N\times N$ matrix with components

$$(P_a)_{ij}=2\int_0^1 a(x)\sin(i\pi x)\sin(j\pi x)\,dx, \tag{9}$$

and W_h is the column vector with components

$$(F_h)_n=2\int_0^1 h(x)\sin(n\pi x)\,dx,\quad n=1,\ldots,N. \tag{10}$$

To compute the initial condition for (7), we consider the Fourier representation of $v_0(x)$ and $v^1(x)$,

$$v^0(x) = \sum_{k=1}^{\infty} u_k^0 \sin(k\pi x), \quad v^1(x) = \sum_{k=1}^{\infty} u_k^1 \sin(k\pi x),$$

where $\{v_k^0\}_{k=1}^N$ and $\{v_k^1\}_{k=1}^N$ are the corresponding Fourier coefficients.

Therefore,

$$V(0) = V^0 = \begin{pmatrix} v_1^0 \\ v_2^0 \\ \cdots \\ v_N^0 \end{pmatrix}, \qquad V'(0) = V^1 = \begin{pmatrix} v_1^1 \\ v_2^1 \\ \cdots \\ v_N^1 \end{pmatrix}.$$

Finally, we obtain the following second-order system for V, which is a finite dimensional approximation of (3):

$$\begin{cases} V'' = -(D + P_a)V - F_h f(t), \\ V(0) = V^0, \quad V'(0) = V^1. \end{cases} \tag{11}$$

Here, the vector F_h defined in (10) is not completely explicit since $h(x)$ is the solution of the boundary value problem (2). The same can be said about the initial data (V^0, V^1). In order to find a numerical approximation for these vectors, we adopt again the projection method. We start with F_h that requires to solve (2).

We first homogenize the boundary condition. Consider

$$g(x) = h(x) - x,$$

and then g is solution of

$$\begin{cases} g''(x) - a(x)g(x) = xa(x), \quad x \in (0, 1) \\ h(0) = g(1) = 0. \end{cases} \tag{12}$$

Now we write

$$g(x) = \sum_{k=1}^{\infty} g_k \sin(k\pi x), \quad g_k = 2\int_0^1 g(x)\sin(k\pi x)\,dx,$$

which we replace in (12)

$$\sum_{k=1}^{\infty} g_k(-k^2\pi^2 - a(x))\sin(k\pi x) = xa(x).$$

Truncating the series up to the first N terms and projection on the subspace generated by the first N functions $\{\sin(n\pi x)\}_{n=1}^{N}$, we easily obtain the system

$$-n^2\pi^2 g_n - 2\sum_{k=1}^{N} a(x)\sin(k\pi x)\sin(n\pi x) = 2\int_0^1 xa(x)\sin(n\pi x)\,dx.$$

This can be written in matrix form

$$-(D + P_a)G = F_a, \tag{13}$$

where G and F_a are the column vectors with components $(G)_j = g_j$ and

$$(F_a)_j = 2\int_0^1 xa(x)\sin(j\pi x). \tag{14}$$

Finally,

$$\begin{aligned}(F_h)_j &= 2\int_0^1 h(x)\sin(j\pi x)\,dx = 2\int_0^1 g(x)\sin(j\pi x)\,dx + 2\int_0^1 x\sin(j\pi x)\,dx\\ &= g_j + \frac{(-1)^{j+1}}{j\pi}.\end{aligned} \tag{15}$$

Thus, in order to find F_h in (11), we have to solve first the linear system (13) to obtain G and then use formula (15).

We now compute (V^0, V^1) from (4). Following the previous approach, we easily see that V^i $(i = 1, 2)$ is solution of

$$-(D + P_a)V^i = U^i, \tag{16}$$

where U^i is the column vector with the first N components of $u^i(x)$, i.e.,

$$(U^i)_k = 2\int_0^1 u^i(x)\sin(k\pi x)\,dx, \quad i = 1, 2, \qquad k = 1, \ldots, N. \tag{17}$$

Thus, in order to find a finite dimensional approximation of (1), we have to follow the following steps:

Algorithm 1 Numerical approximation of the controlled solution

Step 1 Choose N, the dimension of the projecting space X_N.
Step 2 Compute (V^0, V^1) the initial data of system (11)

(1) Compute the vectors U^0 and U^1 given by (17), i.e., the Fourier coefficients of the initial data $u^0(x)$ and $u^1(x)$.

(2) Compute the diagonal matrix D, given in (8), and P_a the matrix with components in (9).
(3) Solve the linear systems (16) to obtain (V^0, V^1).

Step 3 Compute F_h, the secondhand term of system (11).

(1) Compute the vector F_a whose components are given by (14), i.e., the Fourier coefficients of the function $xa(x)$.
(2) Solve the linear system (13) to obtain G.
(3) Use formula (15) to obtain F_h.

Step 4 Solve (11) to obtain $V(t)$ (we assume $f(t)$ known). In the experiments below, we used the Newmark method.

Step 5 Compute the approximation $v^N(x)$ from the Fourier coefficients

$$v^N(x,t) = \sum_{k=1}^{N} v_k \sin(k\pi x).$$

Step 6 Compute the approximation of $g(x)$, $g^N(x)$ from the Fourier coefficients. These are the components of G

$$g^N(x) = \sum_{k=1}^{N} G_k \sin(k\pi x).$$

Step 7 The approximate solution of (1), u^N, is given by

$$u^N(x,t) = (v^N)''(x,t) + (g(x)+x)f(t) = \sum_{k=1}^{N} \left[(v_k)''(t) + G_k f(t)\right] \sin(k\pi x) + xf(t).$$

Concerning the control problem, we see that, as we assume that f is compactly supported in $t \in (0,T)$, any control f for (1) is also a control for (3) that drives (V^0, V^1) to $(0,0)$ and reciprocally. Therefore, the numerical approximation of the control f associated with (3) provides also a numerical approximation for the control in (1).

The discrete control problem associated with (3) is that given $T > 2$, for any initial data $(V^0, V^1) \in \mathbb{R}^N \times \mathbb{R}^N$, find $f^N \in L^2(0,T)$ such that the solution V of system (11) satisfies

$$V(T) = V'(T) = 0.$$

As we have said before, this f^N provides an approximation of the control f associated with (1).

3 Minimal L^2-Weighted Controls

In this section, we introduce a class of controls for both (1) and (11), which are compactly supported in $t \in [0, T]$.

For $T > 2$, the minimal time to have controllability of system (1), we consider a positive smooth weight function $\eta \geq 0$ with compact support in $(0, T)$, i.e., $\eta \in C_0^\infty(0, T)$, and such that $\eta(t) > \eta_0 > 0$ in a subinterval $[\delta, T - \delta] \subset (0, T)$ with δ such that $T - 2\delta > 2$.

Definition 2 Let η be the weight function introduced above. For any $(y_0, y_1) \in L^2 \times H^{-1}$, we define the minimal L^2-weighted control $f(t)$ associated with (1) as the control that minimizes the following L^2-weighted norm:

$$\|f\|_h^2 = \int_0^T \frac{|f(t)|^2}{\eta(t)} dt. \tag{18}$$

Analogously, for any $(V^0, V^1) \in \mathbb{R}^N \times \mathbb{R}^N$, the minimal L^2-weighted control $f^N(t)$ associated with (11) is defined as the control that minimizes the above L^2-weighted norm.

Minimal L^2-weighted controls were introduced in [5] to recover smooth controls from smooth data. Here, we adopt the same idea but in order to have compactly supported controls. A similar technique was used in [2] to prove convergence of controls in a different context.

The existence and uniqueness of the minimal L^2-weighted control for system (1) is easily obtained when $T > 2$ from the results in [10]. In fact, the main ingredient is the following observability inequality:

$$\|(\varphi^0, \varphi^1)\|^2_{H_0^1 \times L^2} \leq C \int_0^T |\varphi_x(1, t)|^2 dt,$$

which holds for all the solutions of the adjoint uncontrolled problem

$$\begin{cases} \varphi''(t, x) - \varphi_{xx}(t, x) + a(x)\varphi(t, x) = 0 & t \in (0, T), \ x \in (0, 1) \\ \varphi(t, 0) = 0, \qquad \varphi(t, 1) = 0, & t \in (0, T) \\ \varphi(0, x) = \varphi^0(x), \ \varphi'(0, x) = \varphi^1(x) & x \in (0, 1) \end{cases} \tag{19}$$

with a constant $C > 0$, which only depends on $T > 2$ and the L^∞-norm of $a(x)$.

Concerning the approximate system (11), the existence and uniqueness of weighted controls is more involved (see [1]). In order to see that, we write system (11) in the usual form, i.e., as a first-order system, to apply the control techniques for ordinary differential systems. Thus, we write

$$\begin{cases} y' = Ay + Bf, & t \in [0, T], \\ y(0) = y_0 \in \mathbb{R}^{2N}, \end{cases} \tag{20}$$

where

$$y = \begin{pmatrix} V \\ V' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ D + P_a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ F_h \end{pmatrix}. \tag{21}$$

Then, the existence of controls is equivalent to the following observability inequality:

$$\|\varphi_0\|^2_{\mathbb{R}^n} \leq c_T \int_0^T |B^*(t)\varphi(t)|^2 dt, \quad \forall \varphi_0 \in \mathbb{R}^{2N}, \tag{22}$$

where φ is the solution of the adjoint system with final data φ_0,

$$\begin{cases} -\varphi' = A^*(t)\varphi, & t \in [0, T], \\ \varphi(T) = \varphi_0. \end{cases}$$

Here, A^* is the transpose matrix and $\varphi_0 \in \mathbb{R}^{2N}$.

It is well known that such observability inequality can be reduced to the so-called Kalman rank condition, which reads

$$rank[B \ AB \ A^2B \ \ldots \ A^{2N-1}B] = 2N.$$

Either the observability inequality (22) or the rank condition is difficult to establish in this type of problems. We refer to [1] for the existence result of such control problem. In [1], it is proved that the constant in (22) can be chosen uniform with respect to N as long as $T > T_0$ with T_0 sufficiently large. This result is used to prove the convergence of the minimal L^2-weighted controls for (11) to those of the corresponding limit problem (1), as $N \to \infty$.

Here, we assume that the discrete approximate control problem has solution, and therefore, there exists a unique minimal L^2-weighted control for (11). Then, we focus on implementation issues and numerical simulation.

It turns out that for ordinary differential controllability systems, as (20) the minimal L^2-weighted control can be computed explicitly. Let $F(t)$ be the fundamental solution of the uncontrolled equation, i.e., $S(t) = e^{At}$, which solves

$$\begin{cases} S' = AS, \ t \in [0, T], \\ S(0) = I, \end{cases}$$

where I is the identity matrix. Then, the solution of the controlled system (20) can be written with the Duhamel formula

$$y(t) = F(t)y_0 + \int_0^t S(t-s)Bf(s)ds, \quad t \in [0, T].$$

On the other hand, if we define the controllability Gramiam Q_T by

$$Q_T = \int_0^T \eta(t) S(t) B B^* S^*(t) dt, \tag{23}$$

then, when the system is controllable, this matrix Q_T is nonsingular and the control with minimal L^2-weighted norm is given by

$$f(t) = -\eta(t) B^* S^*(T-t) Q_T^{-1} S(T) y_0. \tag{24}$$

Thus, in order to find a finite dimensional approximation of the control, we follow the steps:

Algorithm 2 Numerical approximation of the control

Step 1 Choose N, the dimension of the projecting space X_N.
Step 2 Compute (V^0, V^1) the initial data of system (11)

(1) Compute the vectors U^0 and U^1 given by (17), i.e., the Fourier coefficients of the initial data $u^0(x)$ and $u^1(x)$.
(2) Compute the diagonal matrix D, given in (8), and P_a the matrix with components in (9).
(3) Solve the linear systems (16) to obtain (V^0, V^1).

Step 3 Compute F_h, the secondhand term of system (11).

(1) Compute the vector F_a whose components are given by (14), i.e., the Fourier coefficients of the function $xa(x)$.
(2) Solve the linear system (13) to obtain G.
(3) Use formula (15) to obtain F_h.

Step 4 Choose the final time T and the discretization parameter dt
Step 5 Define the cutoff function $\eta(t)$ at the discrete time mesh.
Step 6 Compute the matrixes A and B and the vector y^0 as in (20)–(21).
Step 7 Approximate the Gramiam in (23) with the trapezoidal rule.
Step 8 Compute the control with formula (24).

We refer to the appendix for a MATLAB code which computes the control and solution following the above steps.

4 Numerical Experiments

In this section, we show some experiments that illustrate the method. The aim of the first two experiments is to check how the regularity of the initial data affects the approximation of the control, as we consider more Fourier coefficients. The third experiment shows the behavior of the controls as we consider larger potentials. Finally, the last experiment aims to check the well-known homogenization result when consider an oscillating potential.

Table 1 Experiment 1. $\|u(x,T)\|_{L^2}$ for the controlled solution

$\|u(x,T)\|_{L^2}$	$N_f = 10$	$N_f = 50$	$N_f = 100$
$dt = 10^{-2}$	3.6×10^{-3}	1.34×10^{-2}	1.35×10^{-2}
$dt = 10^{-3}$	4.38×10^{-4}	7.62×10^{-4}	1.20×10^{-3}

Table 2 Experiment 1. Condition number for the controllability matrix Q_T

Cond(Q_T)	$N_f = 10$	$N_f = 50$	$N_f = 100$
$dt = 10^{-2}$	7.42×10^4	4.72×10^7	2.91×10^{15}
$dt = 10^{-3}$	2.34×10^4	1.36×10^7	2.19×10^8

Experiment 1 Here, we consider as initial data the function

$$u^0(x) = \max\{0, 1 - 4|x - 1/2|\}, \qquad u^1(x) = 0,$$

which is Lipschitz continuous but not in $C^1[0,1]$. The final time is chosen to $T = 2.5$, and the time step is set to $dt = 10^{-2}, 10^{-3}$ and the space discretization to compute the Fourier coefficients $dx = 10^{-3}$. The potential $a(x)$ is given by

$$a(x) = 50\chi_{(1/2,1/4)}(x),$$

where $\chi_{(1/2,1/4)}(x)$ is the characteristic function of the interval $(1/2, 1/4)$.

We compute the control for a different number of Fourier coefficients. In Table 1, we show the norm of the controlled solution at time T for different time steps and Fourier coefficients. We see that decreasing the time step the error decreases but not when considering a larger projection space with more Fourier coefficients.

In Table 2, the condition number of the controllability matrix Q_T is given. We observe that this condition number grows very fast, with the number of Fourier coefficients, and this is probably the reason why the norm of the controlled solution at $t = T$ does not decrease. For larger values in the number of Fourier coefficients N_f, the condition number is too large and the control becomes wildly oscillating. One way to decrease this condition number is to consider a smaller time step dt, as illustrated in Table 2.

In Fig. 1, the solution and controls for $N_f = 10, 100$ Fourier coefficients are drawn.

Experiment 2 Here, we consider the same parameters as in the previous experiment but with a more singular discontinuous initial data,

$$u^0(x) = \begin{cases} x, & \text{if } x < 1/2 \\ 0, & \text{if } x \geq 1/2 \end{cases}, \qquad u^1(x) = 0.$$

We compute the control for a different number of Fourier coefficients. In Table 2, we show the norm of the controlled solution at time T. The condition number of the controllability matrix Q_T is the same as in the previous experiment. The results are very similar to those of the previous experiment (Table 3).

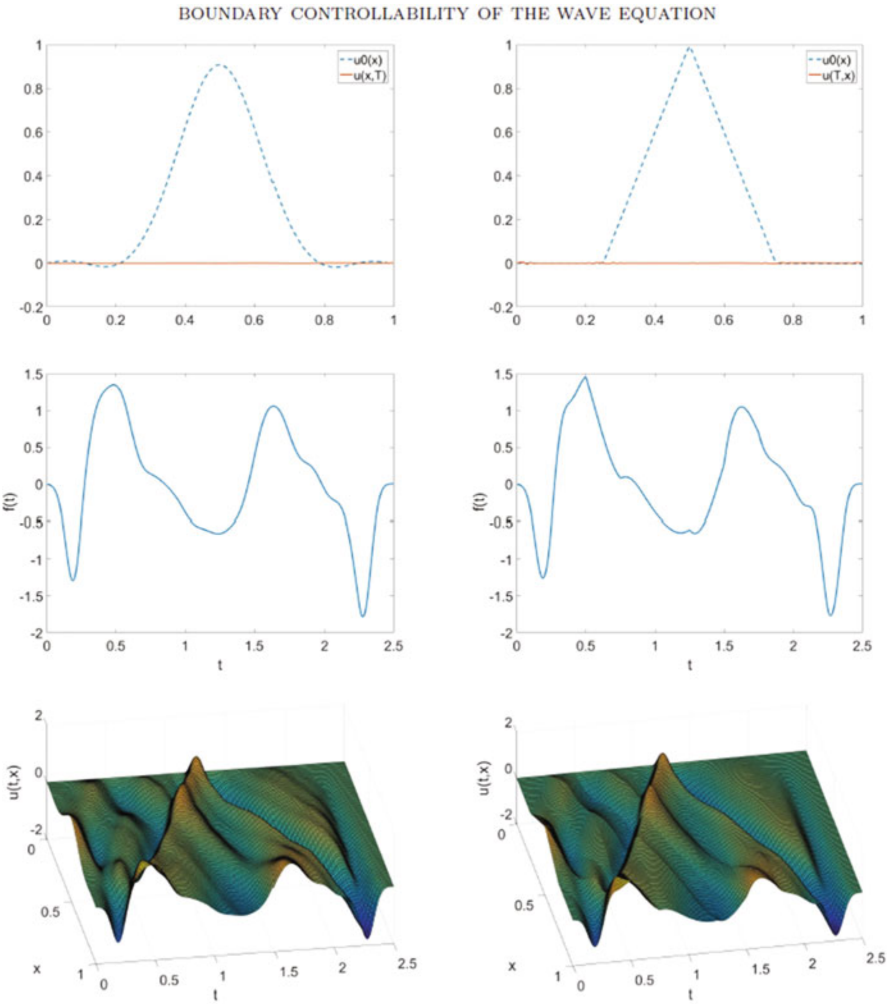

Fig. 1 Experiment 1. Initial position $u^0(x)$ and controlled solution at $t = T$ (upper), control (medium), and space-time controlled solution (lower). The left column corresponds to $N_f = 10$ Fourier coefficients and the right one to $N_f = 100$. The time step is set to $dt = 10^{-3}$

Table 3 Experiment 2. $\|u(x,T)\|_{L^2}$ for the controlled solution

$\|u(x,T)\|_{L^2}$	$N_f = 10$	$N_f = 50$	$N_f = 100$
$dt = 10^{-2}$	2.70×10^{-2}	3.70×10^{-2}	4.05×10^{-2}
$dt = 10^{-3}$	1.55×10^{-4}	9.45×10^{-4}	1.14×10^{-2}

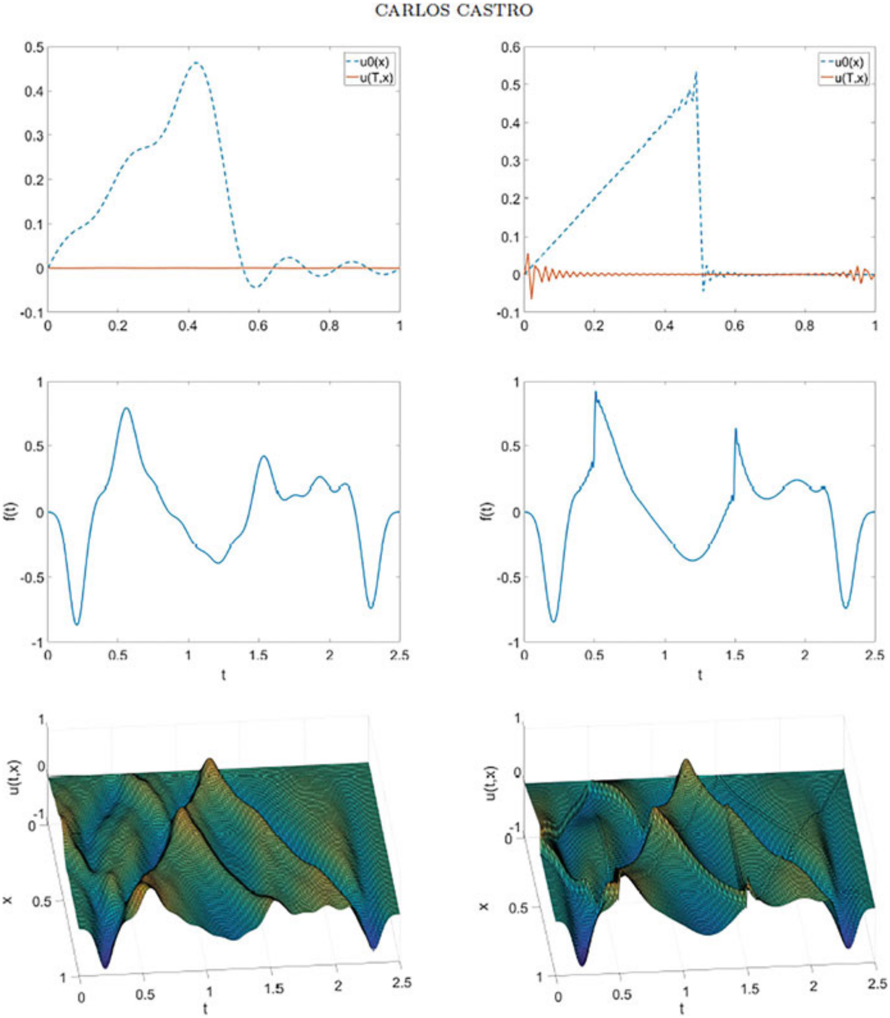

Fig. 2 Experiment 2. Initial position $u^0(x)$ and controlled solution at $t = T$ (upper), control (medium), and space-time controlled solution (lower). The left column corresponds to $N_f = 10$ Fourier coefficients and the right one to $N_f = 100$. The time step is set to $dt = 10^{-3}$

In Fig. 1, the solution and controls for $N_f = 10, 100$ Fourier coefficients are drawn. We see how oscillations in the control and the solution are larger in this more singular case and increase as the number of Fourier coefficients increases (Fig. 2).

We can solve this system of ordinary differential equations with the Newark method (see, for instance, [9]) with parameters $\beta = 1/2$ and $\gamma = 1/4$ that provides a second-order accurate method.

Table 4 Experiment 3. $\|u(x,T)\|_{L^2}$ for the controlled solution

$\|u(x,T)\|_{L^2}$	$N_f = 10$	$N_f = 50$	$N_f = 100$
$a_0 = 1$	1.68×10^{-4}	1.40×10^{-3}	1.15×10^{-2}
$a_0 = 10$	2.61×10^{-4}	1.30×10^{-3}	1.14×10^{-2}
$a_0 = 10^2$	2.09×10^{-4}	1.20×10^{-3}	1.14×10^{-2}

Experiment 3 Here, we consider see how the control is affected by a large potential. We consider the same data as in Experiment 2 but with a potential

$$a(x) = a_0 \chi_{[1/2,1/4]}(x), \tag{25}$$

for different values of a_0.

We compute the control for a different number of Fourier coefficients. In Table 4, we show the norm of the controlled solution at time T. We observe that the efficiency of the method is not affected by large positive potentials. For larger values of a_0, we require a finer mesh to compute the Fourier coefficients (in this experiment, we have considered $N = 10^3$).

It is interesting to observe that the method works fine also for "small" non-positive potentials. However, in this case, the method becomes unstable when we take very large negative potentials. In our experiments, this starts with the potential in (25) with $a_0 = -10$. This is due to the condition number of the controllability Gramiam, which becomes very large in this case.

In Fig. 1, the controls for N_f 10, 100 Fourier coefficients are drawn (Fig. 3).

Experiment 4 Here, we see how the control is affected by a very oscillating potential. We consider the same data as in experiment 2 but with a potential

$$a(x) = 1 + 10\sin(m\pi x),$$

for different values of m. It is well known that the control of (1) converges, as $m \to \infty$, to the control of the limit equation where the potential is replaced by the homogenized one, in this case the constant potential with value the average $a_{av} = \int_0^1 a(x)dx = 1$.

We compute the control when projecting in a space with a different number of Fourier coefficients. In Table 5, we show the norm of the difference between the control f_m and the one corresponding to the limit homogenized equation f_∞. This illustrates the convergence to the homogenized limit as $m \to \infty$.

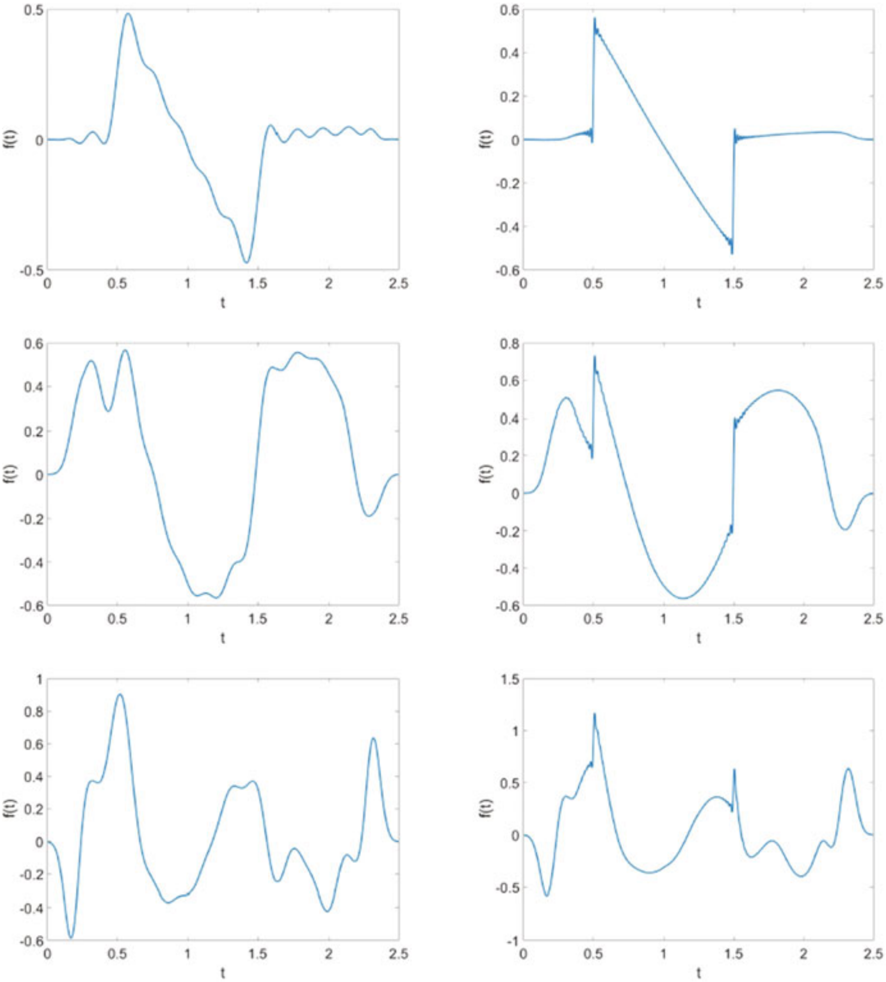

Fig. 3 Experiment 3. Controls for $a_0 = 1$ (upper), $a_0 = 10$ (medium), and $a_0 = 100$ (lower). The left column corresponds to $N_f = 10$ Fourier coefficients and the right one to $N_f = 100$. The time step is set to $dt = 10^{-3}$

Table 5 Experiment 4. $\|f_m - f_\infty\|_{L^2(0,T)}$ for the controlled solution

$\|f_m - f_\infty\|_{L^2(0,T)}$	$N_f = 10$	$N_f = 50$	$N_f = 100$
$m = 1$	1.43	1.43	1.43
$m = 10$	3.40×10^{-2}	3.7×10^{-2}	3.70×10^{-2}
$m = 10^2$	1.77×10^{-6}	3.42×10^{-5}	8.02×10^{-5}

5 Appendix

The program below is a combination of the two algorithms presented in Sects. 2 and 3 above. It computes the approximation of the boundary control of the wave equation with a potential $a(x)$ (following Algorithm 2) and then simulates the controlled solution (following Algorithm 1). The experiments in this paper are obtained with this code.

```
% Spectral approximation and simulation of the wave
% equation with a boundary control f
% Data problem
clear all
L=1;
T=2.5;
Nf=50; % number of Fourier coefficients

%% Discretization data
N=100;
h=L/N;
x=0:h:L;
dt=0.01;
t=0:dt:T;
Nt=length(t);

%% initial data
u0=max(0,1-4*abs(x-1/2));
u1=0*x;
%y0=sin(5*pi*x);
% Fourier coefficients
%y=0;
U0=zeros(Nf,1);
U1=zeros(Nf,1);
for k=1:Nf
    fk=sin(k*pi*x);
    U0(k)=2*trapz(x,fk.*u0);
    U1(k)=2*trapz(x,fk.*u1);
end

%% Potential
a=abs(x-1/2)<0.25;

%% Matrixes
D=diag((1:Nf).^2*pi^2); % Laplacian
Pa=zeros(Nf,Nf);        % Potential
for ii=1:Nf
```

```
        for jj=1:Nf
            fi=sin(ii*pi*x);
            fj=sin(jj*pi*x);
            Pa(ii,jj)=2*trapz(x,a.*fi.*fj);
        end
end

%% Initial data for V
V0=-(D+Pa)\U0;
V1=-(D+Pa)\U1;

%% Calculus of Fh:
% Fourier coefficients of xa(x)
Fa=zeros(Nf,1);
for k=1:Nf
    fi=sin(k*pi*x);
    Fa(k)=2*trapz(x,fi.*x.*a);
end
% Compute G
G=-(D+Pa)\Fa;
% Compute Fh
Fh=zeros(Nf,1);
for ii=1:Nf
    Fh(ii)=G(ii)+(-1)^(ii+1)/(ii*pi);
end

%% eta function
eta0=1;
eta=eta0*exp(-(t-1/4).^2/0.01);
indi=(t>1/4);
eta(indi)=eta0;
eta_i=fliplr(eta);
indi2=(t>=T/2);
eta(indi2)=eta_i(indi2);
%plot(t,eta)

%% Control Matrixes
% initial data
y0=[V0;V1];
% Matrix A
A=[zeros(Nf,Nf),eye(Nf);-(D+Pa),zeros(Nf,Nf)];
B=[zeros(Nf,1);-Fh];

%% Gramiam QT(t)
```

```
QT=zeros(2*Nf,2*Nf);
for jj=1:length(t)
        St=expm(A*t(jj));
       QT=QT+dt*eta(jj)*St*(B*B')*St';
end

%% Control
f1=expm(A*T)*y0;
f2=QT\f1;
f=t*0;
for jj=1:length(t)
     f(jj)=-B'*(expm(A*(T-t(jj))))'*f2*eta(jj);
end
figure(1)
plot(t,f)

%% Solve the 2nd linear system
% V''=-(D+Pa)V-Fh f(t)
% V(0)=V0; V'(0)=V1
% Newmark method
% Newmark parameters
beta=0.25;
gamma=.5;
% de = displacement at n
% dep= displacement at n+1
de=V0;
ve=V1;
K=-(D+Pa);
ac=K*de-Fh*f(1);

% Time iterations
sol(:,1)=ac;
for iter =1:Nt-1
    det = de + dt*ve + (.5*dt^2)*(1-2*beta)*ac;
    vet = ve + (1-gamma)*dt*ac;

    % New ac
    ac = (eye(Nf)-(beta*dt^2)*K)\(-Fh*f(iter+1)+K*det);

    % New de and ve
    de = det+beta*(dt^2)*ac;
    ve = vet+gamma*dt*ac;
    sol(:,iter+1)=ac;
end
```

```
%% Aproximation in x-variable
solx=zeros(Nt,N+1);
funh=x*0;
for ii=1:Nf
    fi=sin(ii*pi*x);
    coef=sol(ii,:);
    solx=solx+(coef'+G(ii)*f')*fi;
end
solx=solx+f'*x;
[xx,tt]=meshgrid(x,t);
figure(2)
surf(xx,tt,solx)
```

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Aggregation Equation and Collapse to Singular Measure

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1 Introduction

This work is concerned with the study of the two-dimensional aggregation equation with the Newtonian potential:

$$\begin{cases} \partial_t \rho + \mathrm{div}(v\,\rho) = 0,\ t \geq 0, x \in \mathbb{R}^2, \\ v(t,x) = -\frac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\rho(t,y)dy, \\ \rho(0,x) = \rho_0(x). \end{cases} \tag{1}$$

This model with more general potential interactions is used to explain some behavior in physics and population dynamics. As a matter of fact, it appears in vortex densities in superconductors [1, 21, 27], material sciences [26, 33], cooperative controls and biological swarming [2, 11, 12, 24, 31, 32, 35], etc. During the last few decades, a lot of intensive research activity has been devoted to explore several mathematical and numerical aspects of this equation. It is known according to [8, 33] that classical solutions can be constructed for short time. They develop finite time singularity if and only if the initial data is strictly positive at some points and the blowup time is explicitly given by $T_\star = \frac{1}{\max \rho_0}$. This follows from the equivalent form

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$$\partial_t \rho + v \cdot \nabla \rho = \rho^2,$$

which, written with Lagrangian coordinates, gives exactly a Riccati equation. Note that similarly to Yudovich result for Euler equations [37], weak unique solutions in $L^1 \cap L^\infty$ can be constructed following the same strategy; for more details see [3, 5–8, 20, 22, 23, 28, 29]. Since L^1 norm is conserved at least at the formal level, then lot of efforts were done in order to extend the classical solutions beyond the first blowup time. In [34], Poupaud established the existence of global generalized solutions with defect measure when the initial data is a nonnegative bounded Radon measure. He also showed that when the second moment of the initial data is bounded, then for such solutions atomic part appears in finite time. This result is at some extent in contrast with what is established for Euler equations. Indeed, according to Delort's result [19], global weak solutions without defect measure can be established when the initial vorticity is a nonnegative bounded Radon measure and the associated velocity has finite local energy. During the time, those solutions do not develop atomic part contrary to the aggregation equation. This illustrates somehow the gap between both equations not only at the level of classical solutions but also for the weak solutions. The literature dealing with measure valued solutions for the aggregation equation with different potentials is very abundant, and we refer the reader to the papers [10, 13–15, 30] and the references therein.

Now, we shall discuss another subject concerning the aggregation patches. Assume that the initial data takes the patch form

$$\rho_0 = \mathbf{1}_{D_0}$$

with D_0 a bounded domain, then solutions can be uniquely constructed up to the time $T^\star = 1$, and one can check that

$$\rho(t) = \frac{1}{1-t}\mathbf{1}_{D_t} \quad \text{with} \quad (\partial_t + v \cdot \nabla)\mathbf{1}_{D_t} = 0.$$

Note that v is computed from ρ through the Biot–Savart law. To filter the time factor in the velocity field and find analogous equation to Euler equations, it is more convenient to rescale the time as it was done in [8]. Indeed, set

$$\tau = -\ln(1-t), \quad u(\tau, x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\mathbf{1}_{\tilde{D}_\tau}(y)dy, \quad \tilde{D}_\tau = D_t,$$

and then we get

$$(\partial_\tau + u \cdot \nabla)\mathbf{1}_{\tilde{D}_\tau} = 0, \tilde{D}_0 = D_0.$$

We observe that with this formulation, the blowup occurs at infinite time and so the solutions do exist globally in time. To alleviate the notations, we shall write this latter equation with the initial variables. Hence, the vortex patch problem is reduced to understanding the evolution equation

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \ t \geq 0, \\ v(t,x) = -\frac{1}{2\pi} \int_{D_t} \frac{x-y}{|x-y|^2} dy, \\ \rho(0) = \mathbf{1}_{D_0}. \end{cases} \tag{2}$$

Let us point out that the area of the domain D_t shrinks to zero exponentially, that is,

$$\forall t \geq 0, \quad \|\rho(t)\|_{L^1} = e^{-t}|D_0|. \tag{3}$$

The solution to this problem is global in time and takes the form $\rho(t) = \mathbf{1}_{D_t}$, $D_t = \psi(t, D_0)$, where ψ denotes the flow associated with the velocity v. Similarly to Euler equations [4, 16], Bertozzi, Garnett, Laurent, and Verdera proved in [9] the global-in-time persistence of the boundary regularity in Hölder spaces C^{1+s}, $s \in (0,1)$. However, the asymptotic behavior of the patches for large time is still not well-understood despite some interesting numerical simulations giving some indications on the concentration dynamics. Notice first that the area of the patch shrinks to zero which entails that the associated domains will converge in Hausdorff distance to negligible sets. The geometric structure of such sets is not well explored, and hereafter we will give two pedagogic and interesting simple examples illustrating the concentration, and one can find more details in [8]. The first example is the disk which shrinks to its center leading after normalization procedure to the convergence to Dirac mass. The second one is the ellipse patch which collapses to a segment along the big axis, and the normalized patch converges weakly to Wigner's semicircle law of density

$$x_1 \mapsto \frac{2\sqrt{\mathbf{x_0}^2 - x_1^2}}{\pi \mathbf{x_0}^2} \mathbf{1}_{[-x_0, x_0]}, \ \mathbf{x_0} = a - b.$$

It seems that the mechanisms governing the concentration are very complex and related in part for some special class to the initial distribution of the local mass. Indeed, the numerical experiments implemented in [8] for some regular shapes indicate that generically the concentration is organized along a skeleton structure. The aim of this chapter is to investigate this phenomenon and try to give a complete answer for special class of initial data where the concentration occurs along disjoint segments lying in the same line. More precisely, we will deal with a onefold symmetric patch, and by rotation invariant we can suppose that its axis of symmetry coincides with the real axis. We assume in addition that the boundary of the upper part is the graph of a slightly smooth function with small amplitude. Then, we will show that we can track the dynamics of the graph globally in time and prove that the normalized solution converges weakly toward a probability measure supported in the union of disjoint segments lying in the real axis. The results will be formulated rigorously in Sect. 2. The chapter is organized as follows. In the next section, we formulate the graph equation and state our main results. In Sects. 3 and 4, we shall discuss basic tools that we use frequently throughout the chapter. In Sect. 5, we prove the local well-posedness for the graph equation. The global existence with small initial data is proved in Sect. 6. The last section deals with the asymptotic behavior of the normalized density and its convergence toward a singular measure.

2 Graph Reformulation and Main Results

The main purpose of this section is to describe the boundary motion of the patch associated with Eq. (2) under suitable symmetry structure. One of the basic properties of the aggregation equation that we shall use in a crucial way concerns its group of symmetry which is much more rich than Euler equations. Actually, in addition to rotation and translation invariance, the aggregation equation is in fact invariant by reflection. To check this property and without loss of generality, we can look for the invariance with respect to the real axis. Set

$$X = (x, y) \in \mathbb{R}^2 \quad \text{and} \quad \overline{X} = (x, -y),$$

and introduce

$$\widehat{\rho}(t, X) = \rho(t, \overline{X}), \quad \widehat{v}(t, X) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{X - Y}{|X - Y|^2}\widehat{\rho}(t, Y)dY.$$

Using straightforward change of variables, it is quite easy to get

$$v(t, X) = \overline{\widehat{v}(t, \overline{X})}, \quad \operatorname{div}\big(v\,\rho\big)(t, X) = \operatorname{div}\big(\widehat{v}\widehat{\rho}\big)(t, \overline{X}).$$

Therefore, we find that $\widehat{\rho}$ satisfies also the aggregation equation

$$\partial_t\widehat{\rho} + \operatorname{div}\big(\widehat{v}\,\widehat{\rho}\big) = 0.$$

Combining this property with the uniqueness of Yudovich solutions, it follows that if the initial data belongs to $L^1 \cap L^\infty$ and admits an axis of symmetry, then the solution remains invariant with respect to the same axis. In the framework of the vortex patches, this result means that if the initial data are given by $\rho_0 = \mathbf{1}_{D_0}$ and the domain D_0 is symmetric with respect to the real axis, the domain D_t defining the solution $\rho(t) = \mathbf{1}_{D_t}$ remains symmetric with respect to the same axis for any positive time. Recall that in the form (2) Yudovich type solutions are global in time. To be precise about the terminology, here, contrary to the standard definition of domain in topology which means a connected open set, we mean by domain any measurable set of strictly positive measure. In addition, a patch whose domain is symmetric with respect to the real axis (or any axis) is called onefold symmetric.

In the current study, we shall focus on the domains D_0 such that the boundary part lying in the upper half-plane is described by the graph of a C^1 positive function $f_0 : \mathbb{R} \to \mathbb{R}_+$ with compact support. This is equivalent to say

$$D_0 = \Big\{(x, y) \in \mathbb{R}^2;\ x \in \text{supp } f_0,\ -f_0(x) \le y \le f_0(x)\Big\}.$$

We point out that concretely we shall consider the evolution not of D_0 but of its extended set defined by

$$\widehat{D}_0 = \Big\{(x, y) \in \mathbb{R}^2;\ x \in \mathbb{R},\ -f_0(x) \le y \le f_0(x)\Big\}.$$

This does not matter since the domain D_t remains symmetric with respect to the real axis, and then we can simply track its evolution by knowing the dynamics of its extended domain: we just remove the extra lines located on the real axis.

One of the main purpose of this chapter is to follow the dynamics of the graph and investigate local and global well-posedness issues in different function spaces. In the next lines, we shall derive the evolution equation governing the motion of the initial graph f_0. Assume that in a short time interval $[0, T]$, the part of the boundary in the upper half-plane is described by the graph of a C^1-function $f_t : \mathbb{R} \to \mathbb{R}_+$. This forces the points of the boundary of ∂D_t located on the real axis to be cusp singularities. As a material point located at the boundary remains on the boundary, then, any parametrization $s \mapsto \gamma_t(s)$ of the boundary should satisfy

$$\big(\partial_t\gamma_t(s) - v(t, \gamma_t(s))\big) \cdot \vec{n}(\gamma_t(s)) = 0,$$

with $\vec{n}(\gamma_t)$ being a normal unit vector to the boundary at the point $\gamma_t(s)$. Now, take the parametrization in the graph form $\gamma_t : x \mapsto \big(x, f(t, x)\big)$, and then the preceding equation reduces to the nonlinear transport equation

$$\begin{cases} \partial_t f(t, x) + u_1(t, x)\partial_x f(t, x) = u_2(t, x),\ t \ge 0, x \in \mathbb{R} \\ f(0, x) = f_0(x), \end{cases} \tag{4}$$

where $(u_1, u_2)(t, x)$ is the velocity $(v_1, v_2)(t, X)$ computed at the point $X = (x, f(t, x))$. Sometimes, along this chapter, we use the following notations:

$$f_t(x) = f(t, x) \quad \text{and} \quad f'(t, x) = \partial_x f(t, x).$$

To reformulate Eq. (4) in a closed, form we shall recover the velocity components with respect to the graph parametrization. We start with the computation of $v_1(X)$. Here, for the sake of simplicity, we drop the time parameter from the graph and the domain of the patch. One writes, according to Fubini's theorem and canonical change of variables,

$$v_1(X) = \frac{1}{2\pi}\int_{\mathbb{R}} \Big\{\arctan\Big(\frac{f(y) - f(x)}{y - x}\Big) + \arctan\Big(\frac{f(y) + f(x)}{y - x}\Big)\Big\}dy$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \arctan\left(\frac{f(x+y) - f(x)}{y} \right) + \arctan\left(\frac{f(x+y) + f(x)}{y} \right) \right\} dy.$$

Similarly, we obtain

$$v_2(x, f(x)) = \frac{1}{4\pi} \int_{\mathbb{R}} \log\left(\frac{y^2 + \big(f(x+y) - f(x)\big)^2}{y^2 + \big(f(x+y) + f(x)\big)^2} \right) dy.$$

With the notation adopted before for (u_1, u_2), we finally get the formulas

$$u_1(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \arctan\left(\frac{f_t(x+y) - f_t(x)}{y} \right) + \arctan\left(\frac{f_t(x+y) + f_t(x)}{y} \right) \right\} dy$$

$$u_2(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}} \log\left(\frac{y^2 + \big(f_t(x+y) - f_t(x)\big)^2}{y^2 + \big(f_t(x+y) + f_t(x)\big)^2} \right) dy. \tag{5}$$

We emphasize that for the coherence of the model, the graph equation (4) is supplemented with the initial condition $f_0(x) \geq 0$. According to Proposition 5.2, the positivity is preserved for enough smooth solutions. Furthermore, once again according to this proposition, we have a maximum principle estimate:

$$\forall t \geq 0,\ \forall x \in \mathbb{R}, \quad 0 \leq f(t, x) \leq \|f_0\|_{L^\infty}.$$

Notice that the model remains meaningful even when the function f_t changes the sign. In this case, the geometric domain of the patch is simply obtained by looking to the region delimited by the curve of f_t and its symmetric with respect to the real axis. This is also equivalent to deal with positive function f_t, but its graph will be less regular and belongs only to the Lipschitz class. Another essential element that will be analyzed later in Proposition 5.2 concerns the support of the solutions which remains confined through the time. More precisely, if $\operatorname{supp} f_0 \subset [a, b]$ with $a < b$, then provided that the graph exists for $t \in [0, T]$ one has

$$\operatorname{supp} f(t) \subset [a, b].$$

This follows from the fact that the flow associated with the horizontal velocity u_1 is contractive on the boundary. It is not clear whether global weak solutions satisfying the maximum principle can be constructed. However, to deal with classical solutions, one should control higher regularity of the graph, and it seems from the transport structure of the equation that the optimal scaling for local well-posedness theory is Lipschitz class. Thus, in what follows, we say that a function space is critical if it scales as Lipschitz class and subcritical if it scales above like Hölder spaces $C^{1+s}, s > 0$. Denote by $g(t,x) = \partial_x f(t,x)$ the slope of the graph, and then it is quite obvious from (4) that

$$\partial_t g + u_1 \partial_x g = -\partial_x u_1 g + \partial_x u_2. \tag{6}$$

For the computation of the source term, we proceed in a classical way using the differentiation under the integral sign, and we get successively

$$\begin{aligned} 2\pi \partial_x u_1(x) &= \text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) - f'(x)}{y^2 + (f(x+y) - f(x))^2} y dy \\ &+ \text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) + f'(x)}{y^2 + (f(x+y) + f(x))^2} y dy \end{aligned} \tag{7}$$

and

$$\begin{aligned} 2\pi \partial_x u_2(x) &= \text{p.v.} \int_{\mathbb{R}} \frac{\big(f(x+y) - f(x)\big)\big(f'(x+y) - f'(x)\big)}{y^2 + (f(x+y) - f(x))^2} dy \\ &- \text{p.v.} \int_{\mathbb{R}} \frac{\big(f(x+y) + f(x)\big)\big(f'(x+y) + f'(x)\big)}{y^2 + (f(x+y) + f(x))^2} dy, \end{aligned} \tag{8}$$

where the notation p.v. is the Cauchy principal value. It is worth pointing out that the first two integrals appearing in the right-hand side of the expressions of $\partial_x u_1$ and $\partial_x u_2$ are in fact connected to Cauchy operator associated with the curve f defined in (24). This operator is well studied in the literature, and some details will be given later in Sect. 4. Next, we shall check that the integrals appearing in the right-hand side of the preceding formulas can actually be restricted over a compact set related to the support of f. Let $[-M, M]$ be a symmetric segment containing the set $K_0 - K_0$, with K_0 being the convex hull of the support of f_0 denoted by supp f_0. It is clear that the support of $\partial_x u_1 f'$ is contained in K_0, and thus, for $x \in K_0$, one has

$$\text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y) - f'(x)}{y^2 + (f(x+y) - f(x))^2} y dy = \text{p.v.} \int_{-M}^{M} \frac{f'(x+y) - f'(x)}{y^2 + (f(x+y) - f(x))^2} y dy.$$

Consequently, we obtain for $x \in \mathbb{R}$

$$2\pi f'(x)\partial_x u_1(x) = f'(x)\text{p.v.}\int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)-f(x))^2}ydy$$
$$+ f'(x)\text{p.v.}\int_{-M}^{M} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2}ydy.$$

Coming back to the integral representation defining $\partial_x u_2$, one can see, using a cancellation between both integrals, that the support of $\partial_x u_2$ is contained in K_0. Furthermore, for $x \in K_0$, one may write

$$2\pi\,\partial_x u_2(x) = \text{p.v.}\int_{-M}^{M} \frac{\big(f(x+y)-f(x)\big)\big(f'(x+y)-f'(x)\big)}{y^2+(f(x+y)-f(x))^2}dy$$
$$- \text{p.v.}\int_{-M}^{M} \frac{\big(f(x+y)+f(x)\big)\big(f'(x+y)+f'(x)\big)}{y^2+(f(x+y)+f(x))^2}dy.$$

Gathering the preceding identities, we deduce that

$$2\pi\big(-\partial_x u_1 f'(x) + \partial_x u_2\big) = F(x) - G(x) \tag{9}$$

with

$$F(x) \triangleq \text{p.v.}\int_{-M}^{M} \frac{\big[f(x+y)-f(x)-yf'(x)\big]\big(f'(x+y)-f'(x)\big)}{y^2+(f(x+y)-f(x))^2}dy$$

and

$$G(x) \triangleq \text{p.v.}\int_{-M}^{M} \frac{\big[f(x+y)+f(x)+yf'(x)\big]\big(f'(x+y)+f'(x)\big)}{y^2+(f(x+y)+f(x))^2}dy.$$

One should keep in mind, and this will be useful for some points, that the foregoing integrals can also be extended to the full real axis. Sometimes, in order to reduce the size of the integral representation, we use the notations

$$\Delta_y^{\pm} f(x) = f(x+y) \pm f(x). \tag{10}$$

Thus, F and G take the form

$$F(x) = \text{p.v.} \int_{-M}^{M} \frac{\big[\Delta_y^- f(x) - yf'(x)\big]\Delta_y^- f'(x)}{y^2 + (\Delta_y^- f(x))^2} dy \tag{11}$$

and

$$G(x) = \text{p.v.} \int_{-M}^{M} \frac{\big[\Delta_y^+ f(x) + yf'(x)\big]\Delta_y^+ f'(x)}{y^2 + (\Delta_y^+ f(x))^2} dy. \tag{12}$$

The first main result of this chapter is devoted to the local well-posedness issue. We shall discuss two results related to subcritical and critical regularities. Denote by X one of the following spaces: Hölder spaces $C^s(\mathbb{R})$ with $s \in (0,1)$ or Dini space $C^\star(\mathbb{R})$. For more details about classical properties of these spaces, we refer the reader to Sect. 3.

Theorem 2.1 *Let f_0 be a positive compactly supported function such that $f_0' \in X$. Then, the following results hold true:*

(1) *Equation (4) admits a unique local solution such that $f' \in L^\infty([0,T], X)$, where the time existence T is related to the norm $\|f_0'\|_X$ and the size of the support of f_0. In addition, the solution satisfies the maximum principle*

$$\forall\, t \in [0,T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

(2) *There exists a constant $\varepsilon > 0$ depending only on s and the size of the support of f_0 such that if*

$$\|f_0'\|_{C^s} < \varepsilon, \tag{13}$$

then Eq. (4) admits a unique global solution $f' \in L^\infty(\mathbb{R}_+; C^s(\mathbb{R}))$. Moreover,

$$\forall\, t \ge 0, \quad \|\partial_x f(t)\|_{L^\infty} \le C_0 e^{-t}$$

with C_0 a constant depending only on $\|f_0'\|_{C^s}$.

Before outlining the strategy of the proofs, some comments are in order. Now, we shall give some details about the proofs. First we establish local-in-time a priori estimates based on the transport structure of the equation combined with some refined studies on modified curved Cauchy operators implemented in Sect. 4 and essentially based on standard arguments from singular integrals. The construction of the solutions is slightly intricate than the usual schemes used for transport equations.

This is due to the fact that the establishment of the a priori estimates is not only energetic. First, at some levels, we use some nonlinear rigidity of the equation like in Theorem 2.1-(3) where the factor f' behind the operator should be the derivative of the function f that appears inside the operator. Second, we use at some point the fact that the support is confined in time. Last, we use at different steps the positivity of the solution. Hence, it seems quite difficult to find a linear scheme taking into account of those constraints. The idea is to implement a nonlinear scheme with two regularizing parameters ε and n. The first one is used to smooth out the singularity of the kernel and the second to smooth the solution through a nonlinear scheme. We first establish that one has uniform a priori estimates on n but on some small interval depending on ε. We are also able to pass to the limit on n and get a solution for a modified nonlinear problem. Second, we check that the a priori estimates still be valid uniformly on ε. This ensures that the time existence can be in fact pushed up to the time given by the a priori estimates obtained for the initial equation (4). As a consequence, we get a uniform time existence with respect to ε, and finally we establish the convergence toward a solution of the initial value problem using standard compactness arguments.

The global existence for small initial data requires much more careful analysis because there is no apparent dissipation or damping mechanisms in the equation. Notice that the estimates of the source term G contain some linear parts as it is stated in Proposition 5.1. The basic ingredient to get rid of those linear parts is to implement a kind of linearization allowing to capture a weak damping effect in G that can just absorb the growth of the linear part. We do not know if the damping proved for lower regularity still happen in the resolution space. As to the nonlinear terms, they are always associated with some subcritical norms, and thus using an interpolation argument with the exponential decay of the L^1 norm, we get a global-in-time control that leads to the global existence.

The second result that we shall discuss deals with the asymptotic behavior of the solutions to (2) and (4). We shall study the collapse of the support to a collection of disjoint segments located at the axis of symmetry. Another interesting issue that will be covered by this discussion concerns the characterization of the limit behavior of the probability measure

$$dP_t \triangleq e^t \frac{\mathbf{1}_{D_t}}{|D_0|} dA, \tag{14}$$

with dA being Lebesgue measure, and $|D_0|$ denotes the Lebesgue measure of D_0. Our result reads as follows.

Theorem 2.2 *Let f_0 be a positive compactly supported function such that $f_0' \in C^s(\mathbb{R})$, with $s \in (0, 1)$. Assume that* $\operatorname{supp} f_0$ *is the union of n-disjoint segments, satisfying the smallness condition* (13). *Then, there exists a compact set $D_\infty \subset \mathbb{R}$ composed of exactly of n-disjoint segments and a constant $C > 0$ such that*

$$\forall t \geq 0, \quad d_H(D_t, D_\infty) \leq Ce^{-t}, \quad |D_\infty| \geq \frac{1}{2}|D_0|,$$

with d_H being the Hausdorff distance, and $|D_\infty|$ is the one-dimensional Lebesgue measure of D_∞. In addition, the probability measures $\{dP_t\}_{t\geq 0}$ defined in (14) *converge weakly as t goes to $+\infty$ to the probability measure*

$$dP_\infty := \Phi\, \delta_{D_\infty \otimes \{0\}},$$

with Φ being a compactly supported function in D_∞ belonging to $C^\alpha(\mathbb{R})$, for any $\alpha \in (0, 1)$, which can be expressed in the form

$$\Phi(x) = \frac{f_0(\psi_\infty^{-1}(x))}{\|f_0\|_{L^1}} e^{g(x)}, \tag{15}$$

with g a function that can be implicitly recovered from the full dynamics of solution $\{f_t, t \geq 0\}$ and

$$\psi_\infty = \lim_{t\to+\infty} \psi(t).$$

Note that $\psi(t)$ is the one-dimensional flow associated with u_1 defined in (29) *and*

$$D_t = \Big\{(x, y),\ x \in \operatorname{supp} f_t;\ -f_t(x) \leq y \leq f_t(x)\Big\}.$$

The proof of the collapse of the support to a disjoint union of segments can be easily derived from the formula (15), which ensures that the support of the limit measure is exactly the image of the support of f_0 by the limit flow ψ_∞, which is a homeomorphism of the real axis. To get the convergence with the Hausdorff distance, we just use the exponential damping of the amplitude of the curve. As to the characterization of the limit measure, it is based on the exponential decay of the amplitude of graph combined with the scattering as t goes to infinity of the normalized solution $e^t f(t)$. In fact, we prove that the density is nothing but the formal quantity

$$\Phi(x) = 2 \lim_{t\to+\infty} e^t f(t, x)$$

whose existence is obtained using the transport structure of the equation through the method of characteristics combined with the damping effects of the nonlinear source terms.

3 Dini and Hölder Spaces

In this section, we set up some function spaces that we shall use and review some of their important properties. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and we define its modulus of continuity $\omega_f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\omega_f(r) = \sup_{|x-y|\le r} |f(x) - f(y)|.$$

This is a nondecreasing function satisfying $\omega_f(0) = 0$ and sub-additive, that is, for $r_1, r_2 \ge 0$, we have

$$\omega_f(r_1 + r_2) \le \omega_f(r_1) + \omega_f(r_2). \tag{16}$$

Now we intend to recall Dini and Hölder spaces. Dini space denoted by $C^\star(\mathbb{R})$ is the set of continuous bounded functions f such that

$$\|f\|_{L^\infty} + \|f\|_D < \infty \quad \text{with} \quad \|f\|_D = \int_0^1 \frac{\omega_f(r)}{r} dr.$$

Another space that we frequently use throughout this chapter is the Hölder space. Let $s \in (0, 1)$, and we denote by $C^s(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{L^\infty} + \|f\|_s < \infty \quad \text{with} \quad \|f\|_s = \sup_{0<r<1} \frac{\omega_f(r)}{r^s}.$$

Let K be a compact set of $\mathbb{R}$, and we define $C_K^\star$ as the subspace of $C^\star(\mathbb{R})$ whose elements are supported in K. Note that $C_K^\star \hookrightarrow L^\infty(\mathbb{R})$, which means that a constant C depending only on the diameter of the compact K exists such that

$$\forall f \in C_K^\star, \quad \|f\|_{L^\infty} \le C\|f\|_D. \tag{17}$$

This follows easily from the observation

$$\forall r \in (0, 1/2], \quad \omega(r) \ln 2 \le \|f\|_D.$$

From (17), we deduce that for any $A \ge 1$

$$\begin{aligned}\int_0^A \frac{\omega_f(r)}{r} dr &\le \|f\|_D + 2\|f\|_{L^\infty} \ln A \\ &\le C\|f\|_D\big(1 + \ln A\big). \end{aligned} \tag{18}$$

Coming back to the definition of Dini semi-norm, one deduces the product laws: for $f, g \in C^\star_K$,

$$\|fg\|_D \le \|f\|_{L^\infty}\|g\|_D + \|g\|_{L^\infty}\|f\|_D \quad \text{and} \quad \|fg\|_D \le C\|f\|_D\|g\|_D. \tag{19}$$

Another useful space is C^s_K, which is the subspace of $C^s(\mathbb{R})$ whose functions are supported on the compact K. It is quite obvious that

$$C^s_K \hookrightarrow C^\star_K \hookrightarrow L^\infty. \tag{20}$$

We point out that all these spaces are complete. Another property that will be very useful is the following composition law. If $f \in C^s(\mathbb{R})$ with $0 < s < 1$ and $\psi : \mathbb{R} \to \mathbb{R}$ a Lipschitz function, then $f \circ \psi \in C^s(\mathbb{R})$ and

$$\|f \circ \psi\|_s \le \big[\|f\|_s + 2\|f\|_{L^\infty}\big]\|\nabla\psi\|^s_{L^\infty}. \tag{21}$$

It is worth pointing out that in the case of Dini space $C^\star(\mathbb{R})$, we get more precise estimate of logarithmic type,

$$\|f \circ \psi\|_D \le C\big(\|f\|_D + \|f\|_{L^\infty}\big)\Big(1 + \ln_+\big(\|\nabla\psi\|_{L^\infty}\big)\Big), \tag{22}$$

with the notation

$$\ln_+ x \triangleq \begin{cases} \ln x, & \text{if} \quad x \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$

Another estimate of great interest is the following product law:

$$\|fg\|_s \le \|f\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|f\|_s. \tag{23}$$

In the next task, we will be concerned with a pointwise estimate connecting a positive smooth function to its derivative and explore how this property is affected by the regularity. This kind of property will be required in Sect. 4 in studying Cauchy operators with special forms. The following result will be very useful later.

Lemma 3.1 *Let K be a compact set of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous positive function supported in K such that $f' \in C^\star(\mathbb{R})$. Then, we have*

$$\forall\, x \in \mathbb{R}, \quad |f'(x)| \le C\frac{\|f'\|_D + \|f'\|_{L^\infty}}{1 + \ln_+\big(\frac{\|f'\|_D}{f(x)}\big)}.$$

A weak version of this inequality is

$$\forall x \in \mathbb{R}, \quad |f'(x)| \le C\frac{\big(\|f'\|_D + \|f'\|_{L^\infty}\big)\big(1 + \ln_+(1/\|f'\|_D)}{1 + \ln_+\big(\frac{1}{f(x)}\big)},$$

with C an absolute constant. If in addition $f' \in C^s(\mathbb{R})$ *with* $s \in (0,1)$, *then*

$$\forall x \in \mathbb{R}, \quad |f'(x)| \le C\|f'\|_s^{\frac{1}{1+s}}[f(x)]^{\frac{s}{1+s}}$$

and the constant C depends only on s.

4 Modified Curved Cauchy Operators

This section is devoted to the study of some variants of Cauchy operators which are closely connected to the operators arising in (7) and (8). Let us first recall the classical Cauchy operator associated with the graph of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{C}_f g(x) = \int_{\mathbb{R}} \frac{g(x+y) - g(x)}{y + i(f(x+y) - f(x))} dy, \tag{24}$$

which is well defined at least for smooth function g. According to a famous theorem by Coifman, McIntosh, and Meyer [17], this operator can be extended as a bounded operator from L^p to L^p for $1 < p < \infty$. By adapting the proof of the paper by Wittmann [36], this operator can also be extended continuously from C^s_K to $C^s(\mathbb{R})$ for $0 < s < 1$, provided that f belongs to $C^{1+s}(\mathbb{R})$. However, this operator fails to be extended continuously from Dini space $C^\star_K$ to itself as it can be checked from Hilbert transform. The structure of the operators that we have to deal with, as one may observe from the expression of F following (9), is slightly different from the Cauchy operators. It can be associated with the truncated bilinear Cauchy operator defined as follows: for given $M > 0$, $\theta \in [0,1]$,

$$\mathcal{C}_f^\theta(g,h)(x) = \int_{-M}^{M} \frac{\big(g(x+\theta y) - g(x)\big)\big(h(x+y) - h(x)\big)}{y + i(f(x+y) - f(x))} dy.$$

The real and imaginary parts of this operator are given, respectively, by

$$\mathcal{C}_f^{\theta,\Re}(g,h)(x) = \int_{-M}^{M} \frac{y\big(g(x+\theta y) - g(x)\big)\big(h(x+y) - h(x)\big)}{y^2 + [f(x+y) - f(x)]^2} dy \tag{25}$$

and

$$\mathcal{C}_f^{\theta,\Im}(g,h)(x) = -\int_{-M}^{M} \frac{\big(f(x+y) - f(x)\big)\big(g(x+\theta y) - g(x)\big)\big(h(x+y) - h(x)\big)}{y^2 + [f(x+y) - f(x)]^2} dy.$$

In what follows, we denote by X one of the spaces C_K^s, with $0 < s < 1$ or $C_K^\star$. The result that we shall discuss deals with the continuity of the preceding bilinear operators on the spaces X. The proof of the next result is detailed in [25].

Proposition 4.1 *Let K be a compact set of $\mathbb{R}$ and f be a compactly supported function such that $f' \in X$. Then the following assertions hold true.*

(1) *The bilinear operator $\mathcal{C}_f^\theta : X \times X \to X$ is well defined and continuous. More precisely, there exists a constant C independent of θ such that for any $g, h \in X$*

$$\|\mathcal{C}_f^{\theta,\Re}(g,h)\|_X \le C\big(1 + \|f'\|_{L^\infty}\|f'\|_X\big)\Big(\|g\|_D\|h\|_X + \|h\|_D\|g\|_X\Big)$$

and

$$\|\mathcal{C}_f^{\theta,\Im}(g,h)\|_X \le C\|f'\|_X\big(1 + \|f'\|_{L^\infty}^2\big)\Big(\|g\|_D\|h\|_X + \|g\|_X\|h\|_D\Big).$$

The second kind of Cauchy integrals that we have to deal with and related to the integral terms in (7) and (8) is given by the following linear operators:

$$T_f^{\alpha,\beta} g(x) = \text{p.v.} \int_{\mathbb{R}} \frac{y\, g(\alpha x + \beta y)}{y^2 + [f(x) + f(x+y)]^2} dy$$

with α and β being two given parameters. The continuity of these operators in classical Banach spaces is not in general easy to establish and could fail for some special cases. We point out that it is not our purpose in this exposition to implement a complete study of those operators. A more complete theory may be achieved, but this topic exceeds the scope of this chapter, and we shall restrict ourselves to some special configurations that fit with the application to the aggregation equation. Our result in this direction reads as follows, and for the proof, we refer the reader to [25].

Theorem 4.2 *Let $\alpha, \beta \in [0,1]$, K be a compact set of $\mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}_+$ be a compactly supported continuous positive function such that $f' \in C_K^\star$. Then, the following assertions hold true:*

(1) *The operator $T_f^{\alpha,\beta} : C_K^\star \to L^\infty(\mathbb{R})$ is well defined and continuous*

$$\|T_f^{\alpha,\beta} g\|_{L^\infty} \le C\Big(1 + \|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty}\|f'\|_D\Big)\|g\|_D$$

with C a constant depending only on K and not on α and β.

(2) *The modified operator $f' T_f^{\alpha,\beta} : C_K^\star \to C_K^\star$ is continuous. More precisely,*

$$\|f' T_f^{\alpha,\beta} g\|_D \le C\|f'\|_D\Big(C_\beta \ln_+(1/\|f'\|_D) + \|f'\|_D^{14}\Big)\|g\|_D$$

with C a constant depending only on K and

$$C_\beta \triangleq \begin{cases} (1-\ln\beta), & \beta\in(0,1] \\ 1, & \beta=0. \end{cases}$$

(3) *Let* $s\in(0,1)$ *and assume that* $f'\in C_K^s$, *and then* $f'T_f^{\alpha,\beta}: C_K^s\to C_K^s(\mathbb{R})$ *is well defined and continuous. More precisely, there exists a constant* C *depending only on the compact* K *and* s *such that*

$$\|f'T_f^{\alpha,\beta}g\|_s \le C\Big(C_\beta\|f'\|_{L^\infty}^{\frac{1}{1+s}}+\|f'\|_s^{14}\Big)\|g\|_s. \tag{26}$$

In addition, one has the refined estimate

$$\|f'T_f^{\alpha,\beta}g\|_s \le C\|f'\|_{L^\infty}^{\frac{1}{2+s}}\Big[\|f'\|_s^{\frac{1}{2+s}}C_\beta+\|f'\|_s^{14}\Big]\|g\|_s \\ + C\|g\|_{L^\infty}^{\frac{1}{2+s}}\|g\|_s^{\frac{1+s}{2+s}}\|f'\|_s, \tag{27}$$

with

$$C_\beta \triangleq \begin{cases} \beta^{-\frac{1}{2}}, & \beta\in(0,1] \\ 1, & \beta=0. \end{cases}$$

5 Local Well-Posedness

The main goal of this section is to discuss the local well-posedness result stated in the first part of Theorem 2.1. The approach that we follow is classical and will be done in several steps. We start with a priori estimates of smooth solutions in suitable Banach spaces. As to the rigorous proof about the existence, it can be implemented in a classical way, and for more details about that, we refer the reader to the paper [25]. To go ahead this program, we need first the following result where we provide a priori estimates of the source terms F and G described in (11) and (12). The proof is described in [25].

Proposition 5.1 *Let* K *be a compact set of* $\mathbb{R}$ *and* $s\in(0,1)$. *We denote by* X *one of the spaces* $C_K^\star$ *and* C_K^s. *There exists a constant* $C>0$ *depending only on* K *such that the following estimates hold true:*

(1) *For any* $f\in X$, *we have*

$$\|F\|_{L^\infty}\le C\|f'\|_{L^\infty}\|f'\|_D, \quad \|F\|_X\le C\|f'\|_D\big(\|f'\|_X+\|f'\|_X^3\big).$$

(2) *For any* $f\in X$, *we have*

$$\|G\|_{L^\infty} \le C\|f'\|_{L^\infty}\Big(1+\|f'\|_D^3\Big), \quad \|G\|_X \le C\big(1+\|f'\|_D^{\frac{1}{3}}\big)\Big(\|f'\|_X+\|f'\|_X^{16}\Big).$$

Now, we shall give the a priori estimates needed for the local well-posedness.

Proposition 5.2 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth solution for the graph equation (4). Assume that the initial data is positive and with compact support K_0. Then, the following assertions hold true:*

(1) *For any $t \in [0, T]$, the function f_t is positive and*

$$\forall t \in [0, T], \quad \|f(t)\|_{L^\infty} \le \|f_0\|_{L^\infty}.$$

(2) *For any $t \in [0, T]$, we have*

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} e^{-t}.$$

(3) *The support a supp f_t is contained in the convex hull of K_0, that is,*

$$\forall t \in [0, T], \quad \text{supp } f(t) \subset \text{Conv} K_0.$$

(4) *Set $X = C_K^\star$ or $X = C_K^s$, with $s \in (0, 1)$. If $f_0' \in X$, then there exists T depending only on $\|f_0'\|_X$ such that $f' \in L^\infty([0, T]; X)$.*

Proof (**1**) To get the first part about the persistence of the positivity, we shall prove that

$$\forall x \in \mathbb{R}, \quad u_2(t, x) = f(t, x)U(t, x) \tag{28}$$

with

$$\|U(t)\|_{L^\infty} \le C\big(1 + \|f'(t)\|_D^6\big)$$

and C being a constant depending only on the size of the support of f_t. Note from the point (**2**) of the current proposition that the support of f_t is contained in a fixed compact, and therefore the constant C can be taken independent of the time variable. Assume for a while (28), and let us see how to propagate the positivity. Denote by ψ the flow associated with the velocity u_1, that is, the solution of the ODE

$$\partial_t \psi(t, x) = u_1(t, \psi(t, x)), \quad \psi(0, x) = x. \tag{29}$$

Recall that

$$u_1(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Big\{ \arctan\Big(\frac{f(t, x+y) - f(t, x)}{y}\Big) - \arctan\Big(\frac{f(t, x+y) + f(t, x)}{y}\Big)\Big\} dy.$$

Set

$$\eta(t,x)=f(t,\psi(t,x)),$$

and then

$$\begin{aligned}\partial_t\eta(t,x)&=u_2(t,\psi(t,x))\\&=\eta(t,x)U(t,\psi(t,x)).\end{aligned}\tag{30}$$

Consequently,

$$\eta(t,x)=f_0(x)e^{\int_0^t U(\tau,\psi(\tau,x))d\tau}.$$

Since the flow $\psi(t):\mathbb{R}\to\mathbb{R}$ is a diffeomorphism, we get the representation

$$f(t,x)=f_0\big(\psi^{-1}(t,x)\big)e^{\int_0^t U[\tau,\psi(\tau,\psi^{-1}(t,x))]d\tau}.\tag{31}$$

As an immediate consequence, we get the persistence through the time of the positivity of the solution. Let us now come back to the proof of the identity (28). To alleviate the notation, we remove the variable t from the functions. Applying Taylor formula to the function,

$$\tau\in[0,f(x)]\mapsto g(\tau)\triangleq\log\left[\frac{y^2+\big(\tau-f(x+y)\big)^2}{y^2+\big(\tau+f(x+y)\big)^2}\right]$$

yields to

$$\begin{aligned}-2\pi u_2(x)&=f(x)\int_0^1\int_{-M}^M\frac{f(x+y)-\tau f(x)}{y^2+\big[f(x+y)-\tau f(x)\big]^2}d\tau dy\\&+f(x)\int_0^1\int_{-M}^M\frac{f(x+y)+\tau f(x)}{y^2+\big[f(x+y)+\tau f(x)\big]^2}d\tau dy\\&\triangleq f(x)V_1(x)+f(x)V_2(x).\end{aligned}$$

Using once again the Taylor formula, we get the following expressions:

$$V_1(x)=\int_0^1\int_{-M}^M\frac{(1-\tau)f(x)}{y^2+\big[(1-\tau)f(x)+y\int_0^1 f'(x+\theta y)d\theta\big]^2}d\tau dy$$

$$+ \text{p.v.} \int_0^1 \int_{-M}^{M} \frac{y \int_0^1 f'(x+\theta y)d\theta}{y^2 + \big[f(x+y) - \tau f(x)\big]^2} d\tau dy$$
$$\triangleq V_{1,1}(x) + V_{1,2}(x)$$

and

$$V_2(x) = \int_0^1 \int_{-M}^{M} \frac{(1+\tau)f(x)}{y^2 + \big[(1+\tau)f(x) + y\int_0^1 f'(x+\theta y)d\theta\big]^2} d\tau dy$$
$$+ \text{p.v.} \int_0^1 \int_{-M}^{M} \frac{y \int_0^1 f'(x+\theta y)d\theta}{y^2 + \big[f(x+y) + \tau f(x)\big]^2} d\tau dy$$
$$\triangleq V_{2,1}(x) + V_{2,2}(x).$$

To estimate $V_{1,1}$ and $V_{2,1}$, we can assume that $f(x) > 0$. Then, making the change of variables $z \mapsto y = (1-\tau)f(x)z$ leads to

$$V_{1,1}(x) = \int_0^1 \int_{-\frac{M}{(1-\tau)f(x)}}^{\frac{M}{(1-\tau)f(x)}} \frac{d\tau dz}{z^2 + \big[1 + z\int_0^1 f'\big(x + \theta(1-\tau)f(x)z\big)d\theta\big]^2}. \tag{32}$$

We deduce that

$$\|V_{1,1}\|_{L^\infty} \leq C\big(1 + \|f'\|_{L^\infty}^2\big). \tag{33}$$

Similarly, we get

$$\|V_{2,1}\|_{L^\infty} \leq C\big(1 + \|f'\|_{L^\infty}^2\big). \tag{34}$$

Let us now bound $V_{j,2}$, $j = 1, 2$. First, by symmetry, we write

$$V_{1,2}(x) = \int_0^1 \int_0^M \frac{y\int_0^1 f'(x+\theta y)d\theta\big[f(x-y) - f(x+y)\big]\psi_\tau(x,y)}{\Big(y^2 + [f(x+y) - \tau f(x)]^2\Big)\Big(y^2 + [f(x-y) - \tau f(x)]^2\Big)} dy d\tau$$
$$+ \int_0^1 \int_0^M \frac{y\int_0^1 \big[f'(x+\theta y) - f'(x-\theta y)\big]d\theta}{y^2 + [f(x-y) - \tau f(x)]^2} dy d\tau,$$

where

$$\psi_\tau(x,y) = f(x+y) + f(x-y) - 2\tau f(x)$$
$$= 2(1-\tau)f(x) + y\int_0^1 \big[f'(x+\theta y) - f'(x-\theta y)\big]d\theta.$$

Thus

$$\|V_{1,2}\|_{L^\infty} \le C\int_0^1\int_0^M \frac{\|f'\|_{L^\infty}^2 y^2\big[(1-\tau)f(x)+y\omega_{f'}(y)\big]}{\Big(y^2+[f(x+y)-\tau f(x)]^2\Big)\Big(y^2+[f(x-y)-\tau f(x)]^2\Big)}dyd\tau$$
$$+ C\int_0^1\int_0^M \frac{\omega_{f'}(y)}{y}dyd\tau.$$

Similarly to $V_{1,1}$, one gets

$$\int_0^1\int_0^M \frac{y^2(1-\tau)f(x)\,dyd\tau}{\Big(y^2+[f(x+y)-\tau f(x)]^2\Big)\Big(y^2+[f(x-y)-\tau f(x)]^2\Big)} \le C\big(1+\|f'\|_{L^\infty}^4\big).$$

It follows that

$$\|V_{1,2}\|_{L^\infty} \le C\|f'\|_{L^\infty}^2\Big(1+\|f'\|_{L^\infty}^4+\int_0^M \frac{\omega_{f'}(y)}{y}dy\Big)+C\|f'\|_D$$
$$\le C\|f'\|_{L^\infty}^2\Big(1+\|f'\|_{L^\infty}^4+\|f'\|_D\Big)+C\|f'\|_D. \tag{35}$$

The estimate of $V_{2,2}$ can be done in a similar way, and one obtains

$$\|V_{2,2}\|_{L^\infty} = C\|f'\|_{L^\infty}^2\Big(1+\|f'\|_{L^\infty}^4+\|f'\|_D\Big)+C\|f'\|_D. \tag{36}$$

Combining both last estimates with (33) and (34), we finally get according to the embedding (17)

$$\|U\|_{L^\infty} \le C\big(1+\|f'\|_D^6\big),$$

where the constant C depends only on the size of the support of f.

Now, let us establish the maximum principle. From (5) combined with the positivity of f_t, one gets

$$\forall t\in[0,T],\ \forall x\in\mathbb{R}\quad u_2(t,x)\le 0.$$

Coming back to (30), we deduce that

$$\partial_t\eta(t,x)\le 0,$$

which implies in turn that

$$\forall t\in[0,T],\ \forall x\in\mathbb{R}\quad f(t,x)\le f_0\big(\psi^{-1}(t,x)\big).$$

Combined with the positivity of $f(t)$, we deduce immediately the maximum principle

$$\forall t \in [0, T], \quad \|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}.$$

Now, we intend to provide more refined identity that we shall use later in studying the asymptotic behavior of the solution. Actually, we have

$$u_2(t,x) = -f(t,x)\big(1 + R(t,x)\big), \tag{37}$$

with

$$\|R(t)\|_{L^\infty} \leq C\|f'(t)\|_D\Big(1 + \|f'(t)\|_{L^\infty}^5\Big).$$

First, note that $R = \sum_{i,j=1}^{2} V_{i,j}$. The estimates of $V_{1,2}$ and $V_{2,2}$ are done in (35) and (36). However, to deal with $V_{1,1}$ and similarly $V_{2,1}$, we return to the expression (32). Set

$$\tau \mapsto K(\tau) = \frac{1}{z^2 + \big[1 + z\tau\big]^2}.$$

Easy computations show the existence of a positive constant C such that

$$\begin{aligned}\forall \tau, z \in \mathbb{R}, \quad |K'(\tau)| &= \frac{2|z||1 + z\tau|}{\big(z^2 + [1 + zt]^2\big)^2}\\ &\leq \frac{1}{z^2 + [1 + z\tau]^2}\\ &\leq C\frac{1 + \tau^2}{1 + z^2}.\end{aligned}$$

Applying the mean value theorem yields

$$|K(\tau) - \frac{1}{1 + z^2}| \leq C|\tau|\frac{1 + \tau^2}{1 + z^2}.$$

Therefore, we get

$$\left|V_{1,1}(x) - \int_0^1 \int_{-\frac{M}{(1-\tau)f(x)}}^{\frac{M}{(1-\tau)f(x)}} \frac{dzd\tau}{1 + z^2}\right| \leq C\|f'\|_{L^\infty}\Big(1 + \|f'\|_{L^\infty}^2\Big),$$

which implies that

$$\left|V_{1,1}(x)-\pi\right| \leq C\|f'\|_{L^\infty}\Big(1+\|f'\|_{L^\infty}^2\Big)+C\|f\|_{L^\infty}. \tag{38}$$

Similarly, we obtain

$$\left|V_{2,1}(x)-\pi\right| \leq C\|f'\|_{L^\infty}\Big(1+\|f'\|_{L^\infty}^2\Big)+C\|f\|_{L^\infty}. \tag{39}$$

Putting together (35), (36), (38), and (39), we get (37).

(**2**) Integrating the Eq. (2) in the space variable, we get after integration by parts

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\rho(t,x)dx &= \int_{\mathbb{R}}\operatorname{div} v(t,x)\rho(t,x)dx\\ &= -\int_{\mathbb{R}}\rho^2(t,x)dx\\ &= -\int_{\mathbb{R}}\rho(t,x)dx,\end{aligned}$$

where in the last line we have used that for the characteristic function one has $\rho^2 = \rho$. The time decay follows then easily.

(**3**) According to the representation of the solution given by (31), we have easily that the support of $f(t)$ is the image by the flow $\psi(t)$ of the initial support, that is,

$$K_t = \psi(t, K_0). \tag{40}$$

We have to check that if $K_0 \subset [a,b]$, with $a < b$, then $K_t \subset [a,b]$. To do so, it is enough to prove that

$$\psi(t,[a,b]) \subset [a,b].$$

This means that the flow is contractive on the boundary of the support. As $\psi(t)$ is a homeomorphism then necessary $\psi(t,[a,b]) = [\psi(t,a), \psi(t,b)]$. Hence, to get the desired inclusion, it suffices to establish that

$$a_t \triangleq \psi(t,a) \geq a \quad \text{and} \quad b_t \triangleq \psi(t,b) \leq b.$$

This reduces to study the derivative in time of a_t and b_t. First, one has

$$\dot{a}_t = u_1(t,a_t) \quad \text{and} \quad \dot{b}_t = u_1(t,b_t).$$

Since $f(t, y) = 0, \forall y \notin (a_t, b_t)$ and f_t is positive everywhere, then

$$u_1(t, a_t) = \frac{1}{\pi} \int_0^{b_t - a_t} \arctan\Big(\frac{f_t(a_t + y)}{y}\Big) dy \geq 0.$$

Hence, $\dot{a}_t \geq 0$, and therefore $a_t \geq a$, for any $t \in [0; T]$.

Similarly, we get

$$u_1(t, b_t) = -\frac{1}{\pi} \int_0^{b_t - a_t} \arctan\Big(\frac{f_t(b_t - y)}{y}\Big) dy \leq 0,$$

which implies that $b_t \leq b$, for any $t \in [0; T]$. This ends the proof of the point (**2**).

(**4**) Recall from (6) and (9) that $g \triangleq f'$ satisfies the equation

$$\partial_t g + u_1 \partial_1 g = \frac{1}{2\pi}\big(F - G\big).$$

Set $h(t, x) = g(t, \psi(t, x))$, where ψ is the flow defined in (29). Then,

$$\partial_t h(t, x) = \frac{1}{2\pi}\Big(F\big(t, \psi(t, x)\big) - G\big(t, \psi(t, x)\big)\Big).$$

Thus,

$$g(t, x) = g_0(\psi^{-1}(t, x) + \frac{1}{2\pi} \int_0^t (F - G)\big(\tau, \psi\big(\tau, \psi^{-1}(t, x)\big) d\tau.$$

Recall the classical estimate

$$\big\| \partial_x \big[\psi\big(\tau, \psi^{-1}(t, \cdot)\big)\big] \big\|_{L^\infty} \leq e^{\int_\tau^t \|\partial_x u_1(t', \cdot)\|_{L^\infty} dt'}, \tag{41}$$

which we may combine with the composition laws (21) and (22) to get

$$\|g(t)\|_X \leq Ce^{V(t)}\Big[\|g_0\|_X + \int_0^t \|(F-G)(\tau)\|_X d\tau\Big], \quad V(t) \triangleq \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty} d\tau. \tag{42}$$

To estimate $\|\partial_x u_1(t)\|_{L^\infty}$, we come back to (7). The first integral term can be restricted to a compact set $[-M, M]$ and thus

$$\begin{aligned} \left| \text{p.v.} \int_{-M}^M \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) - f(x))^2} y dy \right| &\leq 2 \int_0^M \frac{\omega_{f'}(y)}{y} dy \\ &\leq C\|f'\|_D. \end{aligned}$$

As to the second term, the integral can be restricted to $[-M, M]$ and we simply write

$$\text{p.v.} \int_{\mathbb{R}} \frac{f'(x+y)+f'(x)}{y^2+(f(x+y)+f(x))^2} y dy = \text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)+f(x))^2} y dy$$
$$+ \text{p.v.} \int_{\mathbb{R}} \frac{2f'(x)}{y^2+(f(x+y)+f(x))^2} y dy.$$

The first term of the right-hand side is controlled as before

$$\left|\text{p.v.} \int_{-M}^{M} \frac{f'(x+y)-f'(x)}{y^2+(f(x+y)+f(x))^2} y dy\right| \le C\|f'\|_D.$$

However, for the last term, it can be estimated as in the proof of Theorem 4.2-(1). One gets

$$\left|\text{p.v.} \int_{\mathbb{R}} \frac{y}{y^2+(f(x+y)+f(x))^2} dy\right| \le C\Big(\|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty}\|f'\|_D + \|f'\|_{L^\infty}\Big).$$

Hence, using the embedding $X \hookrightarrow C_K^\star \hookrightarrow L^\infty$, we find

$$\begin{aligned}\|\partial_x u_1(t)\|_{L^\infty} &\le C\Big(\|f'\|_D + \|f'\|_{L^\infty}\|f'\|_D\Big)\\ &\le C\Big(\|f'(t)\|_X + \|f'(t)\|_X^2\Big), \end{aligned} \tag{43}$$

which implies that

$$V(t) \le Ct\Big(\|f'\|_{L_t^\infty X} + \|f'\|_{L_t^\infty X}^2\Big). \tag{44}$$

Using Proposition 5.1, we obtain

$$\|(F-G)(t)\|_X \le C\Big(\|f'(t)\|_X + \|f'(t)\|_X^{17}\Big). \tag{45}$$

Plugging (44) and (45) into (42), we obtain

$$\|f'\|_{L_T^\infty X} \le e^{CT\left(\|f'\|_{L_T^\infty X}+\|f'\|_{L_T^\infty X}^2\right)}\Big[\|f_0'\|_X + T\Big(\|f'\|_{L_T^\infty X} + \|f'\|_{L_T^\infty X}^{17}\Big)\Big].$$

This shows the existence of small T depending only on $\|f_0'\|_X$ and such that

$$\|f'\|_{L_T^\infty X} \le 2\|f_0'\|_X,$$

which ends the proof of the proposition. □

6 Global Well-Posedness

We are concerned here with the global existence of strong solutions already constructed in Theorem 2.1. This will be established under a smallness condition on the initial data, and it is probable that for arbitrary large initial data the graph structure might be destroyed in finite time. The basic ingredient that allows to balance the energy amplification during the time evolution is a damping effect generated by the source terms. Note that this damping effect is plausible from the graph equation (4) according to the identity (37). However, as we shall see in the next section, it is quite complicate to extend this behavior for higher regularity at the level of the resolution space due to the existence of linear part in the source term governing the motion of the slope (6). This part could in general bring an amplification in time of the energy. To circumvent this difficulty, we establish a weakly damping property of the linearized operator associated with the source term that we combine with the time decay of the solution for weak regularity using an interpolation argument.

6.1 *Weak and Strong Damping Behavior of the Source Term*

Note from Proposition 5.1 that F does not contribute at the linear level, which is not the case of the functional G. We shall prove that actually there is no linear contribution for G. This will be done by establishing a damping property that occurs at least at the linear level. This is described by the following proposition whose proof is done in [25].

Proposition 6.1 *Let K be a compact set of $\mathbb{R}$ and $s \in (0,1)$. Then, for any $f \in C^s_K$, we have the decomposition*

$$G(x) = 2\pi f'(x) + L(x) + N(x)$$

with

$$\|L\|_s \le 2\pi\big(\|f'\|_s + 2\|f'\|_{L^\infty}\big) + C\|f'\|^s_{L^\infty}\|f'\|_s \quad \textit{and} \quad \|N\|_s \le C\|f'\|_D^{\frac{1}{3}}\Big(\|f'\|_s + \|f'\|_s^{16}\Big),$$

where $C > 0$ is a constant depending only on K and s. Moreover,

$$\|L\|_{L^\infty} \le C\min\big(\|f\|^s_{L^\infty}\|f'\|_s, \|f'\|_{L^\infty}\big) \quad \textit{and} \quad \|N\|_{L^\infty} \le C\|f'\|_{L^\infty}\Big(\|f'\|_D + \|f'\|_D^3\Big).$$

6.2 Global a Priori Estimates

The main goal of this section is to show how we may use the weakly damping effect of the source terms stated in Proposition 6.1 in order to get global a priori estimates when the initial data is small enough. The basic result reads as follows and the proof can be found in [25].

Proposition 6.2 *Let K be a compact set of $\mathbb{R}$ and $s \in (0, 1)$. There exists a constant $\varepsilon > 0$ such that if $\|f_0'\|_s \leq \varepsilon$, then Eq. (4) admits a unique global solution*

$$f' \in L^\infty(\mathbb{R}_+; C_K^s).$$

Moreover, there exists a constant C_0 depending on the initial data such that

$$\forall\, t \geq 0, \quad \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}.$$

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Geometric Control of Eigenfunctions of Schrödinger Operators

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1 Introduction

The purpose of this note is to explore the connections between three problems arising in (linear) PDE that involve a compact Riemannian manifold (M, g), an open subset $\omega \subseteq M$, and the **Laplace–Beltrami** operator (or simply, the Laplacian) Δ_x on M associated with the metric g (complemented with suitable boundary conditions when $\partial M \neq \emptyset$):

- Uniform decay for solutions to the **damped wave equation** (with damping coefficient supported on ω)
- Exact controllability of the **Schrödinger equation** (with controls supported on ω)
- Observability of **eigenfunctions of the Laplacian** (from the open set ω)

The first of these problems involves the **damped wave equation**:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) + a(x)\partial_t u(t,x) = 0, & (t,x) \in \mathbb{R} \times M, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (u^0, u^1) \in H_0^1(M) \times L^2(M). \end{cases}$$

Here, the **damping coefficient** $a \in L^\infty(M)$ is supposed to be nonnegative; for instance,

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$$a := c\mathbb{1}_\omega, \qquad \text{for some open set } \omega \subset M \text{ and some } c > 0,$$

and when $\partial M = \emptyset$, one defines $H_0^1(M) := H^1(M)/\mathbb{R}$.

It is easy to check that the **energy**

$$\mathcal{E}(t,u) := \int_M (|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2)dx$$

of any non-constant solution u decays

$$\frac{d}{dt}\mathcal{E}(t,u) = -2\int_M a(x)|\partial_t u(t,x)|^2 dx < 0.$$

It is natural to ask whether or not this decay is uniform with respect to the initial data:

$$\mathcal{E}(t,u) \le f(t)\mathcal{E}(0,u), \qquad \text{for some } f(t) \to 0^+ \text{ as } t \to +\infty,$$

for every initial datum $(u^0, u^1) \in H_0^1(M) \times L^2(M)$. If such an $f(t)$ exists, it is known (see for instance [31]) that it must be of the form:

$$f(t) = Me^{-\alpha t}, \qquad \text{for some } \alpha, M > 0,$$

hence the terminology **exponential decay** or stabilization.

It is possible to show that uniform decay for solutions to the damped wave equation holds if and only if the following **observability estimate**

$$\mathcal{E}(0,v) \le C_{T,\omega}\int_0^T \int_\omega |\partial_t v(t,x)|^2 dxdt \qquad (O_{\mathrm{W}}(\omega))$$

is verified for some $T, C_{T,\omega} > 0$ uniformly for every solution of the **free wave equation**:

$$\begin{cases} \partial_t^2 v(t,x) - \Delta_x v(t,x) = 0, \quad (t,x) \in \mathbb{R} \times M, \\ (v(0,\cdot), \partial_t v(0,\cdot)) = (v^0, v^1) \in H_0^1(M) \times L^2(M). \end{cases} \qquad (1)$$

Again, see [31] for a proof.

In order to describe the second problem we are interested in, we must introduce the forced **Schrödinger equation**:

$$\begin{cases} i\partial_t \Psi(t,x) + \Delta_x \Psi(t,x) = F(t,x), \\ \Psi(0,\cdot) = \Psi^0, \end{cases}$$

where the forcing term satisfies

$$F \in L^2_{\text{loc}}(\mathbb{R} \times M), \quad F(t,x) = 0 \quad \text{for a.e. } x \in M \setminus \omega.$$

This equation is said to be **exactly (null) controllable** from ω at some time $T > 0$ provided that given any $\Psi^0 \in L^2(M)$, it is always possible to find a forcing term F as above such that the corresponding solution Ψ satisfies

$$\Psi(T,\cdot) = 0.$$

An application of the closed graph theorem shows that controllability from ω at time T is equivalent to the existence of a constant $C = C_{T,\omega} > 0$ such that every $\psi^0 \in L^2(\Omega)$ satisfies the **observability estimate**

$$||\psi^0||^2_{L^2(M)} \leq C \int_0^T \int_\omega |\psi(t,x)|^2 dx\, dt, \tag{$O_{\text{S}}(\omega)$}$$

where ψ is the solution of the **homogeneous Schrödinger equation**

$$\begin{cases} i\partial_t \psi\,(t,x) + \Delta_x \psi\,(t,x) = 0, \\ \psi(0,\cdot) = \psi^0. \end{cases} \tag{2}$$

The proof of this fact is standard, but the reader may consult for instance the introduction of [28].

This estimate is essentially the same as ($O_{\text{W}}(\omega)$), when the wave propagator associated with $-\Delta_x$ is replaced by the Schrödinger group.

The last problem of the list we presented at the beginning of this introduction involves eigenfunctions of Δ_x. Since M is compact, the Laplacian can be diagonalized and its spectrum is discrete. There exists an orthonormal basis (φ_k) of $L^2(M)$ consisting of **eigenfunctions** of $-\Delta_x$:

$$-\Delta_x \varphi_k(x) = \lambda_k \varphi_k(x), \quad x \in M,$$

with

$$0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty.$$

When $\partial M \neq \emptyset$, we will assume in what follows that $-\Delta_x$ is the self-adjoint extension of the Laplacian over $\mathcal{C}_c^\infty(M)$ obtained by imposing Dirichlet boundary conditions. An important question in Quantum Mechanics is that of understanding the localization properties of high-frequency eigenfunctions. The following definition is aimed at quantifying this property, following the same spirit as ($O_{\text{W}}(\omega)$), ($O_{\text{S}}(\omega)$).

Definition 1 (Observability of Eigenfunctions) We say that the eigenfunctions of the Laplacian are observable from ω if a constant $C = C_\omega > 0$ exits such that

$$||\varphi||_{L^2(M)} \leq C_\omega \|\varphi\|_{L^2(\omega)} \qquad (O_\mathrm{E}(\omega))$$

holds for every $L^2(M)$-eigenfunction of the Laplacian:

$$-\Delta_x \varphi(x) = \lambda\varphi(x), \quad x \in M.$$

Note that it is crucial in this definition that the constant C_ω is required to be uniform with respect to the eigenvalue λ.

If φ is an eigenfunction of the Laplacian with $\|\varphi\|_{L^2(M)} = 1$ and eigenvalue λ, then

- $v(t, \cdot) := e^{-it\sqrt{\lambda}}\varphi$ is a solution to the wave equation (1) and
$$\mathcal{E}(0, v) = \lambda.$$
- $\psi(t, \cdot) = e^{-it\lambda}\varphi$ is a solution to the Schrödinger equation (2).

Note that

$$|\partial_t v(t, \cdot)|^2 = \lambda|\varphi|^2, \quad |\psi(t, \cdot)|^2 = |\varphi|^2;$$

therefore,

$$\boxed{(O_\mathrm{W}(\omega)) \text{ or } (O_\mathrm{S}(\omega)) \implies (O_\mathrm{E}(\omega)).}$$

2 The Geometric Control Condition

Let us briefly recall at this point some geometric notions that we will use in the sequel. From now on, SM denotes the sphere bundle of M:

$$SM = \bigsqcup_{x\in M} S_x M, \quad S_x M := \{v \in TM \ : \ \|v\|_x = 1\},$$

where $\|\cdot\|_x$ denotes the norm on $T_x M$ induced by the Riemannian metric g. Given any $z_0 := (x_0, v_0) \in SM$, there exists a unique geodesic $\gamma_{z_0}(t)$ satisfying

$$\gamma_{z_0}(0) = x_0, \quad \dot{\gamma}_{z_0}(0) = v_0.$$

The map ϕ_t that associates with every $z_0 \in SM$ the point

$$\phi_t(z_0) := (\gamma_{z_0}(t), \dot{\gamma}_{z_0}(t)) \in SM$$

is called the **geodesic flow** on SM.

The open subset $\omega \subseteq M$ satisfies the **Geometric Control Condition (GCC)** provided that every geodesic of (M, g) intersects ω. Since M is compact, the GCC can be also written in the form it is sometimes found in the literature:

$$\exists T_0 > 0, \quad K_\omega^{T_0} := \{z_0 \in SM \ : \ \phi_t(z_0) \notin S\omega, \ \forall t \in [0, T_0]\} = \emptyset, \tag{GCC}$$

since $K_\omega^{T_0}$ consists of those points $z_0 \in SM$ with the property that the geodesic of length T_0 issued from z_0 does not enter the region ω.

The GCC is **sufficient and (almost) necessary** for the uniform decay of damped waves.

Theorem 2 *The following statements hold:*

- *Suppose that ω satisfies* (GCC)*. Then,* ($O_W(\omega)$) *holds.*
- *Suppose that $M \setminus \overline{\omega}$ contains a geodesic. Then,* ($O_W(\omega)$) *does not hold.*

This result was proved in [31] for manifolds without boundary. The case of manifolds with boundary was established in [5]. Note that the necessary and sufficient conditions are slightly different, and in fact the necessary condition can be refined, see, for instance, [7, 35].

The GCC is **sufficient** for the controllability of the Schrödinger equation, as was first proved by Lebeau [20].

Theorem 3 *Suppose that ω satisfies* (GCC)*. Then,* ($O_S(\omega)$) *holds.*

As a consequence,

$$\boxed{(O_W(\omega)) \implies (O_S(\omega)) \implies (O_E(\omega)).}$$

However, GCC **is not necessary in general** in order to have ($O_S(\omega)$). The simplest setting where this kind of behavior takes place is the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Theorem 4 *Suppose that $M = \mathbb{T}^d$ equipped with the flat metric. Then,* ($O_S(\omega)$) *holds for every open set $\omega \subseteq \mathbb{T}^d$ and therefore so does* ($O_E(\omega)$).

Therefore, ($O_S(\omega)$) and ($O_E(\omega)$) **hold unconditionally**, and this is **not the case** for ($O_W(\omega)$). This was first proved by Jaffard [18] for the free Laplacian when $d = 2$ and generalized to the multidimensional case by Komornik [19]. The proof is based on the theory of nonharmonic Fourier series. This type of result holds for general Quantum Completely Integrable systems (in particular for manifolds with completely integrable geodesic flow). This was done in [2] using tools from Microlocal Analysis.

One can also find examples of manifolds for which ($O_S(\omega)$) holds under a weaker but nontrivial condition than (GCC). The simplest of such examples is the Euclidean disk $\mathbb{D} := \{z \in \mathbb{R}^2 \ : \ |z| < 1\}$, see [3]; one also has this type of behavior in products of spheres [16]. Estimates ($O_S(\omega)$) and ($O_E(\omega)$) do not hold for every open set ω, yet (GCC) is not necessary. The precise result on the disk is the following.

Theorem 5 ([3]) *Suppose that $M = \overline{\mathbb{D}}$ equipped with the flat metric and that $\omega \subseteq \overline{\mathbb{D}}$ is open. Estimates* ($O_{\mathrm{S}}(\omega)$) *and* ($O_{\mathrm{E}}(\omega)$) *hold if and only if $\omega \cap \partial\overline{\mathbb{D}} \neq \emptyset$.*

The fact that $\omega \cap \partial\overline{\mathbb{D}} \neq \emptyset$ is necessary for ($O_{\mathrm{E}}(\omega)$) is known for a long time and is related to the existence of sequences of eigenfunctions, called *whispering gallery modes*, that concentrate on the boundary of the disk. In polar coordinates, these modes are

$$\varphi_k(re^{i\theta}) = c_k J_k(\alpha_k r)e^{ik\theta},$$

where $k \in \mathbb{N}$, J_k is the k-th Bessel function, α_k is its first positive zero, and c_k is chosen such that $\|\varphi_k\|_{L^2(\mathbb{D})} = 1$. One can check that

$$-\Delta_z \varphi_k(z) = \alpha_k^2 \varphi_k(z), \quad z \in \mathbb{D}, \quad \varphi_k|_{\partial\overline{\mathbb{D}}} = 0,$$

and that, for any open set ω such that $\overline{\omega} \subset \mathbb{D}$,

$$\lim_{k\to\infty} \int_\omega |\varphi_k(z)|^2 dz = 0.$$

Therefore, the main contribution of Theorem 5 is proving the reverse implication. This is done following the strategy proposed in [1, 2, 24], which uses tools from Microlocal Analysis rather than the arithmetic properties of eigenvalues and eigenfunctions. It is worth noting that, as a consequence of this fact, the results of [1–3] still hold when $-\Delta_z$ is replaced by a Schrödinger operator $-\Delta_z + V$, $V \in \mathcal{C}^\infty(M; \mathbb{R})$.

The examples we have examined so far are such that ($O_{\mathrm{S}}(\omega)$) and ($O_{\mathrm{E}}(\omega)$) hold under the same condition on ω, but ($O_{\mathrm{W}}(\omega)$) holds under a strictly stronger condition on ω than ($O_{\mathrm{S}}(\omega)$).

Let us finish this section with a third example, suppose now $M = \mathbb{S}^d$ is the round sphere. In this case, (GCC) is necessary for ($O_{\mathrm{E}}(\omega)$) (and therefore for ($O_{\mathrm{S}}(\omega)$):

$$\boxed{\text{if } (M, g) = (\mathbb{S}^d, \text{can}), \text{ then} \quad (O_{\mathrm{W}}(\omega)) \iff (O_{\mathrm{S}}(\omega)) \iff (O_{\mathrm{E}}(\omega)).}$$

This is the content of the next result, which is well known. See [4, 23] for similar results in a more general context.

Theorem 6 *Suppose that $\mathbb{S}^d \setminus \overline{\omega}$ contains a geodesic. Then,* ($O_{\mathrm{E}}(\omega)$) *does not hold.*

Proof Write the sphere as

$$\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : |x| = 1\}.$$

Let

$$\varphi_k(x) = c_k(x_1 + ix_2)^k, \quad \text{with } c_k := \sqrt{\frac{\Gamma(k + (d+1)/2)}{2\pi^{\frac{d+1}{2}} k!}} \sim k^{\frac{d-1}{4}}.$$

This function is a spherical harmonic and therefore an eigenfunction of the Laplacian:

$$-\Delta_x \varphi_k(x) = k(k+d-1)\varphi_k(x), \quad x \in \mathbb{S}^d, \quad ||\varphi_k||_{L^2(\mathbb{S}^d)} = 1.$$

Clearly,

$$|\varphi_k(x)|^2 = (c_k)^2(|x_1|^2 + |x_2|^2)^k = (c_k)^2(1 - |x'|^2)^k,$$

where $x = (x_1, x_2, x')$. This shows that $|\varphi_k|^2$ concentrates on the equator $\{x' = 0\}$.

If $\overline{\omega} \cap \{x' = 0\} = \emptyset$, then no constant $C > 0$ can exist such that

$$||\varphi_k||_{L^2(\mathbb{S}^d)} \leq C\|\varphi_k\|_{L^2(\omega)}$$

holds uniformly in $k \in \mathbb{N}$, since

$$\lim_{k\to\infty} \int_\omega |\varphi_k(x)|^2 dx = 0, \quad \text{and} \quad ||\varphi_k||_{L^2(\mathbb{S}^d)} = 1.$$

Since any other geodesic of $\mathbb{S}^d$ can be obtained by applying a rotation to $\{x' = 0\}$ and the composition of a spherical harmonic with a Euclidean rotation is again a spherical harmonic, the claim follows. □

3 Are There Examples for Which ($O_\mathrm{E}(\omega)$) Holds and ($O_\mathrm{S}(\omega)$) Does Not?

There are indeed geometries (M, g) and open sets $\omega \subset M$ for which ($O_\mathrm{E}(\omega)$) holds and ($O_\mathrm{S}(\omega)$) fails. This was first shown in [27]. The geometric setting is particularly simple.

Theorem 7 ([27]) *There exist infinitely many surfaces of revolution M with the following properties:*

- ($O_\mathrm{W}(\omega)$) *and* ($O_\mathrm{S}(\omega)$) *hold if and only if ω satisfies* (GCC).
- *There are infinitely ω such that* ($O_\mathrm{E}(\omega)$) *holds, but* ($O_\mathrm{S}(\omega)$) *does not.*

All these examples are **Zoll surfaces**, i.e., surfaces all of whose geodesics are closed (just like the sphere). The particular class of Zoll surfaces that are useful to our purposes are surfaces of revolution, or equivalently, Riemannian metrics on $\mathbb{S}^2$ that are invariant by rotations along the z-axis (a surface of revolution is isometric to $\mathbb{S}^2$ equipped with such a metric). Zoll surfaces of revolution $(\mathbb{S}^2, g)$ are characterized by the existence of a parametrization (r, θ) of the sphere (θ is the longitude and r is a function of the geodesic distance to north pole) such that the Riemannian metric g is given by

$$g = (1 + h(\cos r))^2 dr \otimes dr + \sin^2 r d\theta \otimes d\theta,$$

where h is an odd function from $[-1, 1]$ to $(-1, 1)$ with $h(1) = h(-1) = 0$ (see [6, Proposition 4.10, Corollary 4.16]).

The reader may consult [6] for a comprehensive account on the subject of Zoll manifolds. Let us just mention here that the existence of orientable Zoll surfaces that are not surfaces of revolution is known (see [15]), although they all are topologically spheres. Additional visual examples of this class of surfaces can be found in [17].

In order to get an insight on the main ideas of the proof of Theorem 7, it is useful to place ourselves in an a priori simpler setting: the sphere $M = \mathbb{S}^d$ with its usual metric, but, instead of considering eigenfunctions of the Laplacian $-\Delta_x$, one looks at eigenfunctions of a Schrödinger operator $-\Delta_x + V$ where $V \in \mathcal{C}^\infty(\mathbb{S}^d; \mathbb{R})$. It is not hard to prove that the addition of this potential does not affect the conditions on ω under which ($O_S(\omega)$) holds.

Theorem 8 ([25, 27]) *Suppose that* (M, g) *is a Zoll manifold,* $V \in \mathcal{C}^\infty(M; \mathbb{R})$*, and that* $\omega \subseteq M$ *is an open set such that*

$$\|\psi^0\|^2_{L^2(M)} \leq C \int_0^T \int_\omega |e^{it(\Delta_x - V)}\psi^0(x)|^2 dx\, dt$$

holds uniformly on $\psi^0 \in L^2(M)$ *for some choice of* $T, C > 0$*. Then,* $\gamma \cap \overline{\omega} \neq \emptyset$ *for every geodesic* γ *of* M.

We are going to show that this is no longer the case for ($O_E(\omega)$). In fact, it is possible to construct explicit examples of potentials V such that the open sets ω for which ($O_E(\omega)$) holds satisfy much weaker assumptions than (GCC). A particularly extreme situation is the one described by the following result.

Theorem 9 *There exists a family of potentials* $\mathcal{T} \subset \mathcal{C}^\infty(\mathbb{S}^2; \mathbb{R})$ *such that, for every* $V \in \mathcal{T}$*, there exist three distinct geodesics of* $\mathbb{S}^2$*,* γ_1, γ_2*, and* γ_3 *with* $\gamma_1 \cap \gamma_2 \cap \gamma_3 = \emptyset$ *such that, for every open set* $\omega \subseteq \mathbb{S}^2$ *satisfying*

$$\omega \cap \gamma_i \neq \emptyset, \quad i = 1, 2, 3,$$

the estimate

$$\|\varphi\|_{L^2(\mathbb{S}^2)} \leq C \|\varphi\|_{L^2(\omega)}$$

holds for some $C > 0$ *uniformly for every solution to*

$$(-\Delta_x + V(x))\varphi(x) = \lambda \varphi(x), \quad x \in \mathbb{S}^2.$$

A sketch of the proof of this result will be presented in the next section (full details can be found in [29]. Let us just mention here that the set ω can be very small: if $P \in \gamma_1 \cap \gamma_2$ and $Q \in \gamma_2 \cap \gamma_3$, then ω can be any arbitrarily small neighborhood of $\{P, Q\}$.

The key ingredient in the proof of Theorem 9 is establishing that observability for eigenfunctions of the Schrödinger operator $-\Delta_x + V$ holds under a geometric assumption on ω that involves the **Radon transform** or **X-Ray transform** of the perturbation:

$$\mathcal{I}(V)(x,v) = \frac{1}{2\pi}\int_0^{2\pi} V(\gamma_{(x,v)}(s))ds, \quad (x,v) \in S\mathbb{S}^2,$$

which is nothing but the average of V along geodesics of the sphere.

The function $\mathcal{I}(V)$ defines a Hamiltonian vector field $X_{\mathcal{I}(V)}$ on $S\mathbb{S}^2$ (with respect to the symplectic form obtained by identification of T^*M and TM using the Riemannian metric g). Its flow φ_s^V **commutes** with the geodesic flow. As a consequence, it transforms orbits of the geodesic flow into orbits of the geodesic flow. In other words, given geodesic γ_0, write

$$(\gamma_s, \dot{\gamma}_s) := \varphi_s^V(\gamma_0, \dot{\gamma}_0),$$

then γ_s is a geodesic for every $s \in \mathbb{R}$.

This flow on the space of geodesics induces a new geometric condition on ω, which we name the **V-Geometric Control Condition**, that holds provided that

$$K_\omega^V := \{\gamma_0 \text{ geodesic } : \ \phi_s^V(\gamma_0, \dot{\gamma}_0) \cap S\omega \neq \emptyset, \ \forall s \in \mathbb{R}\} = \emptyset. \qquad (V\text{-GCC})$$

In other words, ω satisfies (V-GCC) provided that, given any geodesic γ_0, one can find $s \in \mathbb{R}$ such that $\gamma_s \cap \omega \neq \emptyset$. It is shown in [28] that (V-GCC) implies that the eigenfunctions of our Schrödinger operator are observable.

Theorem 10 ([28]) *Suppose that ω satisfies (V-GCC). Then, there exists $C > 0$ such that*

$$\|\varphi\|_{L^2(\mathbb{S}^2)} \leq C\|\varphi\|_{L^2(\omega)}$$

uniformly for every solution to

$$(-\Delta_x + V(x))\varphi(x) = \lambda\varphi(x), \quad x \in \mathbb{S}^2. \qquad (3)$$

The strongest obstruction to V-GCC comes from the fact that $\mathcal{I}(V)$ always has critical points:

$$\mathcal{C}(V) = \{\gamma \text{ geodesic } : d\mathcal{I}(V)(\gamma, \dot{\gamma}) = 0\} \neq \emptyset.$$

If $\gamma_0 \in \mathcal{C}(V)$, then $\gamma_s = \gamma_0$ for every $s \in \mathbb{R}$. Therefore, if ω satisfies V-GCC, then necessarily

$$\omega \cap \gamma \neq \emptyset \quad \text{for every } \gamma \in \mathcal{C}(V).$$

In the next section (Proposition 12), we will show that as soon as $\mathcal{C}(V) \neq \mathbb{S}^2$, there exist open sets $\omega \subset \mathbb{S}^2$ such that (V-GCC) holds, but (GCC) does not. Theorem 9 follows from the construction of a V such that, after identification of geodesics with the same image but different orientations, $\#\mathcal{C}(V) = 3$ (which, as we shall see, is the minimal cardinal of a set of critical points on the space of geodesics on the sphere) and the analysis of the corresponding flow ϕ_s^V.

Remark 11 Some remarks are in order.

- Theorem 10 holds for spheres of any dimension, not only the two-dimensional sphere.
- If V is **odd** (meaning $V(x) = -V(-x)$), then $\mathcal{I}(V) = 0$ and (V-GCC) is equivalent to (GCC). However, in [28], it is shown that a similar result holds under a new geometric condition, in which the Radon transform of the potential is replaced by a different nonlinear transform of V, whose expression is a bit more complicated.
- This yields the following question: suppose that ($O_{\mathrm{E}}(\omega)$) holds for every eigenfunction of (3) if and only if ω satisfies (GCC). Does this imply that V is constant?

Let us conclude this section giving some insight on how Theorem 7 essentially follows from Theorem 10. The first step is to use a normal form result due to Weinstein [32], see also [8, 10, 33], in order to write the Laplacian of a Zoll surface $(\mathbb{S}^2, g)$ (whose geodesic flow has period 2π) as

$$-\Delta_x = A^2 + Q,$$

where A and Q are pseudo-differential operators on $\mathbb{S}^2$ of order one and order zero, respectively, such that

- The spectrum of A is contained in $\mathbb{N} + \frac{\alpha}{4}$, with α a fixed integer.
- $[\Delta_x, A] = [\Delta_x, Q] = 0$.

One then reproduces the argument that leads to Theorem 10 replacing $-\Delta_x$ by A^2 and V by Q. One can show that the same result holds in this case, modulo replacing $\mathcal{I}(V)$ by $\sigma_0(Q)$, the principal symbol of Q. Hence, it suffices to show that $\sigma_0(Q)$ is non-constant; this will ensure that ϕ_s^Q is nontrivial and therefore that (Q-GCC) is non-empty and non-equivalent to (GCC).

Fortunately, a formula for the principal symbol of Q involving Jacobi fields has been obtained by Zeldtich [34] for Zoll surfaces of revolution. Although the formula is not completely explicit, it is not difficult to show that $\sigma_0(Q)$ is not constant, and therefore $\mathcal{C}(\sigma_0(Q)) \neq \mathbb{S}^2$. This is enough to prove the theorem, by Proposition 12 below. See [27, 28] for full details.

4 A Geometric Interpretation of (V-GCC) and Proof of Theorem 9

A nice consequence of working in the two-dimensional sphere is that its **space of oriented geodesics** $G(\mathbb{S}^2)$ can be identified to the sphere $\mathbb{S}^2$ itself and the symplectic form on $T\mathbb{S}^2$ induces a symplectic structure on $G(\mathbb{S}^2)$ (which must necessarily be a non-zero multiple of the volume form on $\mathbb{S}^2$). This is due to the fact that any oriented geodesic can be canonically identified to a unique point in $\mathbb{S}^2$: every geodesic γ of $\mathbb{S}^2$ is obtained by intersecting the sphere by a plane that is uniquely determined by γ. The two unit normal vectors of this plane (which define two distinct points in $\mathbb{S}^2$) are then identified to the two orientations of the geodesic. For instance,

$$\gamma = \{x_3 = 0\} \quad \text{is identified to} \quad (0, 0, 1) \text{ and } (0, 0, -1).$$

The set of all geodesics issued from the same point $x_0 \in \mathbb{S}^2$ is then identified to the geodesic in $\mathbb{S}^2$ that lies in the plane through the origin that is orthogonal to x_0.

With this identification in mind, the Radon transform can be identified to an operator:

$$\mathcal{I} : C^\infty(\mathbb{S}^2) \longrightarrow C^\infty(\mathbb{S}^2),$$

and, see [15],

$$\ker \mathcal{I} = C^\infty_{\text{odd}}(\mathbb{S}^2) := \{u \in C^\infty(\mathbb{S}^2) : u(-x) = -u(x), \ \forall x \in \mathbb{S}^2\},$$

whereas

$$\mathcal{I}(C^\infty(\mathbb{S}^2)) = C^\infty_{\text{even}}(\mathbb{S}^2) := \{u \in C^\infty(\mathbb{S}^2) : u(-x) = u(x), \ \forall x \in \mathbb{S}^2\}.$$

Therefore,

$$\mathcal{I} : C^\infty_{\text{even}}(\mathbb{S}^2) \longrightarrow C^\infty_{\text{even}}(\mathbb{S}^2) \text{ is bijective.} \tag{4}$$

Analogously, the Hamiltonian vector field $X_{\mathcal{I}(V)}$ can be identified to a vector field on $\mathbb{S}^2$ that is Hamiltonian with respect to the new symplectic form. In particular,

$$\phi_s^V : \mathbb{S}^2 \longrightarrow \mathbb{S}^2, \quad \text{and} \quad \mathcal{I}(V) \circ \phi_s^V = \mathcal{I}(V), \quad \forall s \in \mathbb{R}. \tag{5}$$

Once this has been established, we are able to prove the following result.

Proposition 12 *Suppose that $\mathcal{I}(V)$ is not identically constant. Then, there exists an open set $\omega \subset \mathbb{S}^2$ that satisfies (V-GCC) but for which (GCC) fails.*

Proof Suppose that $\mathcal{I}(V)$ does not have critical points in a neighborhood of $\gamma_0 \in G(\mathbb{S}^2)$. Given $\epsilon > 0$, write

$$\omega_\epsilon := \left\{ x \in \mathbb{S}^2 \,:\, \mathrm{dist}(x, \gamma_0) > \epsilon \right\};$$

these sets are open and do not satisfy (GCC).

Now, if γ_0 is identified to $\pm x_0 \in \mathbb{S}^2$, then the set of those geodesics that are contained in $\mathbb{S}^2 \setminus \omega_\epsilon$ is identified to a neighborhood U_ϵ of $\pm x_0$ in $\mathbb{S}^2$. By choosing ϵ small enough, we can ensure that $\mathcal{I}(V)$ has no critical points in a neighborhood that is slightly bigger than U_ϵ. Then, the Poincaré–Bendixon theorem implies that given any $x \in U_\epsilon$, there exist $t > 0$ such that the Hamiltonian flow ϕ_t^V maps x to some point in $\mathbb{S}^2 \setminus U_\epsilon$. Therefore, ($V$-GCC) is satisfied, whereas (GCC) is not. □

We conclude this section by proving Theorem 9. We first define the class of potentials $\mathcal{T}$. Let

$$Q_{(a,b,c)}(x) = a x_1^2 + b x_2^2 + c x_3^2, \quad x \in \mathbb{R}^3.$$

Then, $Q_{(a,b,c,}|_{\mathbb{S}^2} \in \mathcal{C}^\infty_{\mathrm{even}}(\mathbb{S}^2; \mathbb{R})$, and we define, using (4),

$$\mathcal{T} := \mathcal{I}^{-1}(\{Q_{(a,b,c)}|_{\mathbb{S}^2} \,:\, 0 < a < b < c\}).$$

For any $V \in \mathcal{T}$, the function $\mathcal{I}(V) = Q_{(a,b,c,}|_{\mathbb{S}^2} \in \mathcal{C}^\infty_{\mathrm{even}}(\mathbb{S}^2; \mathbb{R})$ has exactly six (nondegenerate) critical points:

$$\mathcal{C}(V) = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\},$$

and

$$\min_{\mathbb{S}^2} \mathcal{I}(V) = a, \quad \max_{\mathbb{S}^2} \mathcal{I}(V) = c.$$

The orbits of ϕ_t^V are contained in the connected components of the level sets $\mathcal{I}(V)^{-1}(E)$, $E \in [a, c]$, by (5). These orbits are equilibria when $E = a, c$, closed simple orbits around equilibrium points when $E \notin \{a, b, c\}$. When $E = b$, the level set consists of two equilibrium points and four heteroclinic orbits.

The geodesic corresponding to the critical points of $\mathcal{I}(V)$ is

$$\gamma_i = \{x_i = 0\}, \quad i = 1, 2, 3.$$

Suppose that $\omega \subseteq \mathbb{S}^2$ is an open set that intersects these geodesics at points p_1, p_2, and p_3, respectively. This means that the set of all geodesics that intersect ω contains all geodesics issued from p_i, $i = 1, 2, 3$. The set of these geodesics corresponds, via our identification, to the three geodesics

$$\mathbb{S}^2 \cap p_i^\perp, \quad i = 1, 2, 3,$$

whose union has non-empty intersection with all the orbits of ϕ_t^V. Therefore, (V-GCC) is satisfied, and the result follows from Theorem 10.

Let us mention that six is the least number of critical points an even Morse function on $\mathbb{S}^2$ may have. This is due to the fact that any such function induces a Morse function on the projective plane $\mathbb{P}$. Since the Euler characteristic of $\mathbb{P}$ is equal to one, the Poincaré–Hopf theorem implies

$$1 = \chi(\mathbb{P}) = \sum_{j=0}^{2}(-1)^j \#\{\gamma \in \mathcal{C}(V) \,:\, \gamma \text{ has index } j\}.$$

There are at least one critical points of index zero and index two, and therefore one must have also at least one saddle point. The number of critical points of $\mathcal{I}(V)$ when viewed as a function of $\mathbb{P}$ must be at least three, hence the claim.

5 On the Proof of Theorem 10

The proof is based on an argument by contradiction involving semiclassical defect measures that goes back to [21]. We make an extensive use of properties of semiclassical pseudo-differential operators; the reader may consult [9, 12, 30, 36] for background on the theory. Introductory accounts on semiclassical defect measures can be found in [11, 14, 26].

Suppose that ω satisfies (V-GCC) but that ($O_{\mathrm{E}}(\omega)$) fails. This means that there exist sequences $\lambda_n \to \infty$ and φ_n such that

$$(-\Delta_x + V)\varphi_n = \lambda_n \varphi_n, \quad \|\varphi_n\|_{L^2(\mathbb{S}^2)} = 1, \tag{6}$$

and

$$\lim_{n\to\infty} \|\varphi_n\|_{L^2(\omega)} = 0.$$

Along a subsequence, which we do no relabel, one has, for the weak-$*$ topology on the set of Radon measures on $\mathbb{S}^2$,

$$|\varphi_n|^2 dx \overset{*}{\rightharpoonup} \nu, \quad n \to \infty,$$

where, since $\mathbb{S}^2$ is compact, ν is a **probability measure** on $\mathbb{S}^2$ and

$$\nu(\omega) = 0.$$

The goal now is to show that $\nu = 0$ in order to obtain a contradiction. To this aim, we are going to prove that ν enjoys additional regularity properties that come from the fact that φ_n solves (6). This is achieved by replacing position densities $|\varphi_n|^2 dx$ by phase-space densities (distribution on $T^*\mathbb{S}^2$):

$$\mathcal{P}(\mathbb{S}^2) \ni |\varphi_n|^2 \quad \rightsquigarrow \quad w_{\varphi_n} \in \mathcal{D}'(T^*\mathbb{S}^2).$$

We lift the positive density

$$\phi \longmapsto \int_{\mathbb{S}^2} \phi(x)|\varphi_n|^2(x)dx = (\phi\,\varphi_n \mid \varphi_n)_{L^2(\mathbb{S}^2)}$$

to the Schwartz distribution

$$w_{\varphi_n} : a \longmapsto (\mathrm{Op}_{h_n}(a)\varphi_n \mid \varphi_n)_{L^2(\mathbb{S}^2)},$$

with $a \in \mathcal{C}_c^\infty(T^*\mathbb{S}^2)$, $h_n = 1/\sqrt{\lambda_n}$, and $\mathrm{Op}_h(a)$ stands for the Weyl semiclassical pseudo-differential operator of symbol a. The distribution w_{φ_n} is called the **Wigner distribution** of φ_n. Note that we use the cotangent $T^*\mathbb{S}^2$ as our phase-space instead of $T\mathbb{S}^2$, this is because the (principal) symbol of a pseudo-differential operator is defined intrinsically as a function on $T^*\mathbb{S}^2$.

One can consider Wigner distributions of arbitrary L^2-functions u_h. The choice of h is then dictated by the frequency behavior of u_h. This is easier to understand in $\mathbb{R}^d$. Consider a **wave-packet** or **coherent state**:

$$u_h := \frac{1}{h^{d/4}}\rho\left(\frac{x-x_0}{\sqrt{h}}\right)e^{i\frac{\xi_0}{h}\cdot x}.$$

Then,

$$w_{u_h} \overset{*}{\rightharpoonup} \delta_{x_0} \otimes \delta_{\xi_0}, \quad h \to 0^+, \quad \text{in } \mathcal{D}'(T^*\mathbb{S}^2).$$

Accumulation points μ of (w_{φ_n}) are always **probability measures** on $T^*\mathbb{S}^2$ [13, 14, 22]. They are called **semiclassical measures**. In addition, for every $\phi \in \mathcal{C}(\mathbb{S}^2)$,

$$\int_{\mathbb{S}^2} \phi(x)\nu(dx) = \int_{\mathbb{S}^2}\int_{T_x^*\mathbb{S}^2} \phi(x)\mu(dx, d\xi),$$

which means that μ is a lift of ν, and therefore $\nu(\omega) = 0$ translates into

$$\mu\left(\bigcup_{x\in\omega} T_x^*\mathbb{S}^2\right) = 0.$$

By noting that the eigenvalue equation can be rewritten as

$$-h_n^2\Delta_x\varphi_n + h_n^2 V\varphi_n = \varphi_n, \quad h_n = 1/\sqrt{\lambda_n},$$

one can use the symbolic calculus of semiclassical pseudo-differential operators to conclude that, see for instance [13, 26],

$$(\|\xi\|_x^2 - 1)\mu(dx, d\xi) = 0 \implies \operatorname{supp}\mu \subseteq \{\|\xi\|_x = 1\} =: S^*\mathbb{S}^2,$$

and μ is **invariant by the geodesic flow**: for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{S}^2)$,

$$\int_{T^*\mathbb{S}^2} \{a, \|\xi\|^2\}\mu(dx, d\xi) = 0 \implies (\phi_t)_*\mu = \mu, \ \forall t$$

that is, μ can be identified to a measure on the space of geodesics of $\mathbb{S}^2$. All the properties mentioned so far hold on any compact Riemannian manifold.

The main step of the proof is the following fact.

Proposition 13 *The measure μ is also invariant by the Hamiltonian flow ϕ_s^V associated with $\mathcal{I}(V)$. This means that μ is constant along orbits of ϕ_s^V. In particular,*

$$\nu(\gamma_s) = \nu(\gamma_0), \quad \forall s \in \mathbb{R}.$$

If one combines this property with:

- ω satisfies V-GCC.
- $\mu\left(\bigcup_{x\in\omega} S_x^*\mathbb{S}^2\right) = 0.$

one concludes that $\mu = 0$ and therefore $\nu = 0$, which is a contradiction.

Proof of Proposition 13 This is based on the **Quantum Averaging Method**, which goes back to Weinstein [32]. Write

$$-\Delta = A^2 - \frac{1}{4}$$

so that the spectrum of A equals $\mathbb{N} + 1/2$. Then,

$$e^{2i\pi A} = e^{i\pi}\mathrm{Id}. \tag{7}$$

Given a in $\mathcal{C}_c^\infty(T^*\mathbb{S}^d \setminus \{0\})$, write $h = 1/\sqrt{\lambda}$ and, by analogy with the Radon transform of functions, we define the quantum average of the operator $\mathrm{Op}_h(a)$:

$$\mathcal{I}_{\mathrm{qu}}(\mathrm{Op}_h(a)) := \frac{1}{2\pi}\int_0^{2\pi} e^{-isA}\,\mathrm{Op}_h(a)e^{isA}ds.$$

Then, it follows from (7) that

$$\left[\mathcal{I}_{\mathrm{qu}}(\mathrm{Op}_h(a)), A\right] = \left[\mathcal{I}_{\mathrm{qu}}(\mathrm{Op}_h(a)), \Delta\right] = 0.$$

Moreover, Egorov's theorem [9, 12, 36] implies

$$\mathcal{I}_{\text{qu}}(\text{Op}_h(a)) = \text{Op}_h(\mathcal{I}(a)) + \mathcal{O}(h).$$

With these identities in mind, we deduce that if

$$(-h^2\Delta + h^2 V)\psi_h = \psi_h, \quad \|\psi_h\|_{L^2(\mathbb{S}^2)} = 1,$$

then

$$0 = \left(\left[-h^2\Delta + h^2 V, \mathcal{I}_{\text{qu}}(\text{Op}_h(a))\right] \psi_h | \psi_h\right) = h^2\left(\left[V, \mathcal{I}_{\text{qu}}(\text{Op}_h(a))\right] \psi_h | \psi_h\right)$$

and

$$0 = \left(\left[V, \mathcal{I}_{\text{qu}}(\text{Op}_h(a))\right] \psi_h | \psi_h\right) = \frac{h}{i}\left(\text{Op}_h\left(\{V, \mathcal{I}(a)\}\right) \psi_h | \psi_h\right) + \mathcal{O}(h^2).$$

Let μ be the semiclassical measure of the sequence (ψ_h). Then, after letting h go to 0, one finds that

$$\lim_{h \to 0^+} \left(\text{Op}_h\left(\{V, \mathcal{I}(a)\}\right) \psi_h | \psi_h\right) = \int_{S^*\mathbb{S}^2} \{V, \mathcal{I}(a)\}(x, \xi)\mu(dx, d\xi) = 0.$$

Applying the invariance by the geodesic flow twice, one finally gets that

$$\int_{S^*\mathbb{S}^2} \{\mathcal{I}(V), a\}(x, \xi)\mu(dx, d\xi) = \int_{S^*\mathbb{S}^2} \{\mathcal{I}(V), \mathcal{I}(a)\}(x, \xi)\mu(dx, d\xi) = 0,$$

which implies that μ is invariant by the Hamiltonian flow associated with $\mathcal{I}(V)$. □

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Stability of a Graph of Strings with Local Kelvin–Voigt Damping

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1 Introduction

Viscoelastic materials, as their name suggests, combine two different properties: viscosity and elasticity. They are used for isolating vibration, dampening noise, and absorbing shock. They are intended to dissipate mechanical energy from vibrations or noises, to limit their propagation in structures, they have a decisive impact on the fatigue of these structures and on our comfort.

Viscoelastic materials have applications in all fields of engineering and mechanical systems, from the automotive to civil engineering, from space to home appliances (engine and machine mounts and supports, transmission seals and belts, glazing edges and fixing of subsystems, damping of metal plates and shells, parts of seats and interior of cabs, tire and wheels, tuned damping systems) [7, 15, 24, 40].

Since the 1980s, the development of modern technologies has required the use of innovative materials with high mechanical properties, suitable for their use, and having low densities. A composite material meets most of these requirements; it is a kind of mixture of different materials whose properties are superior to each of its components taken separately. These materials were first developed and used in the 1940s in the aeronautical field (essentially for military airplanes and helicopters) and are today in automobile construction, in shipbuilding, and in buildings. But

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these materials are excellent transmitters of mechanical and acoustic vibrations, which can affect the integrity of the entire system. Also, thanks to these composite materials it is possible to reduce the number of parts of a structure, there would then be less frictions at connections between elements. It is, therefore, imperative to associate with these materials effective damping techniques. One solution is to add full or partial layers of viscoelastic materials, glued on (or incarnated between) the parts. A viscoelastic product can be integrated into the composite material [28, 36].

In this context we have chosen to study a network of elastic and viscoelastic materials; More precisely, we investigate the asymptotic stability of a graph of elastic strings with local Kelvin–Voigt damping.

Models of the transient behavior of some or all of the state variables describing the motion of flexible structures have been of great interest in recent years, for more details about physical motivation for the models, see also [23, 29], and the references therein. Mathematical analysis of transmission partial differential equations is detailed in [29]. For the feedback stabilization problem for the wave or Schrödinger equations (in networks, in particular), we refer the readers to references [3–6, 8–13, 29].

A wave equation on a (single) string of length ℓ, with (local) Kelvin–Voigt damping is modeled by the following equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + a(x)\,\frac{\partial^2 u}{\partial x \partial t}\right) = 0 \quad \text{in} \quad (0,\ell)\times(0,\infty), \tag{1}$$

where $a(x), x \in [0, \ell]$ is a nonnegative function.

As boundary conditions, we often associate the Dirichlet conditions:

$$u(0,t) = u(\ell,t) = 0.$$

From a mathematical point of view, the Kelvin–Voigt damping model (1) has been studied by several authors. let us recall some results in the literature,

- Huang proved in 1988 [27] that when the damping is global (i.e., distributed over the entire domain), the corresponding semigroup is not only exponentially stable but also analytic. Thus, the Kelvin–Voigt damping is much stronger than the viscous damping (i.e., the damping term is replaced by $-a(x)\,\frac{\partial u}{\partial t}$), where the corresponding semigroup is only exponentially stable and not analytic (see, e.g., [21] and [18]).
  Such a comparison is not valid anymore if the damping is localized:
- Chen et al. [21] proved in 1991 that in the case of localized viscous damping, the associated semigroup is exponentially stable no matter the size or the location of the subinterval where the damping is effective, and even if the damping coefficient function has a jump discontinuity at the interface.

However, the local Kelvin-Voigt damping does not follow the same analogue.

- It was first proved in 1998 by S. Chen et al. [30] that, when the viscoelastic damping is locally distributed (precisely, they took $a(x) = a_0\chi_{(\alpha,\beta)}$, with $a_0 > 0$), the associated semigroup is not exponentially stable.
- In 2002, K. Liu and Z. Liu [31] proved that if $a \in \mathcal{C}^2[0,\ell]$, and $\int_0^\ell a(x)dx > 0$, then the system is exponentially stable: the asymptotic behavior depends on the regularity of the damping coefficient.

The works cited below consider the domain $[-1, 1]$ instead of $[0, \ell]$ and suppose that $a(x) = 0$ on $[-1, 0)$ and $a(x) = b(x)$ on $(0, 1]$.

- In 2004, Renardy [41] supposed that $a(x) = 0$ on $[-1, 0]$ and $a(x) > 0$ on $(0, 1]$ and he assumed that

$$\lim_{x\to 0^+} \frac{a'(x)}{x^\alpha} = k > 0 \quad \text{for some} \quad \alpha > 0, \tag{2}$$

  then the eigenvalues of the system (1) are such that the decay rate tends to infinity with frequency.
- Z. liu and B. Rao [32], 2005, and M. Alves et al. [2], 2014, proved that if $b(x) \geq c > 0$ on $(0, 1)$ and $b \in \mathcal{C}(0, 1)$. The associated semigroup is polynomially stable of order 2.
- In 2010, Q. Zhang [43] improved the result in [32]: the author took $a \in \mathcal{C}^1[-1, 1]$, $b(0) = b'(0) = 0$ and supposed the existence of a positive constant c such that $\int_0^x \frac{|b'(s)|^2}{b(s)}ds \leq c|b'(x)|$ for all $x \in [0, 1]$, (for example, $b(x) = x^\alpha,\ \ \alpha > 1$).
- In 2016 Z. Liu and Q. Liu [35] took over the condition (2) of Renardy. Precisely they took $a \in L^\infty(-1, 1)$, $b(x) > 0$ on $(0, 1]$ and $b(0) = 0$; $b', b'' \in L^\infty(0, 1)$, and supposed that $\lim\limits_{x\to 0^+} \frac{a(x)}{x^\alpha} = k > 0$. Then the system (1) is exponentially stable for $\alpha = 1$ and polynomially, nonexponentially stable for $0 \leq \alpha < 1$.
- It is proved [33] in 2017 that if $a \in \mathcal{C}^1[-1, 1]$ and satisfies conditions in the last point, then the system (1) remains exponentially stable for $\alpha > 1$.

In this work we study a more general case, it is about a network of strings with local Kelvin–Voigt damping.

We first introduce some notations needed to formulate the problem under consideration (as introduced in [1, 37] or [7]. Let $\mathcal{G}$ be a planar connected graph embedded in $\mathbb{R}^3$, with N edges $e_1, \ldots, e_N$, $N \geq 1$ and p vertices $s_1, \ldots, s_p$, $p \geq 2$. By degree of a vertex of $\mathcal{G}$ we mean the number of edges incident at the vertex. If the degree is equal to one, the vertex is called exterior; otherwise, it is said to be interior. We denote by I_{int} and I_{ext}, respectively, the sets of indices of interior and exterior vertices, then $I := I_{int} \cup I_{ext}$ is the set of indices of all vertices. Finally, we define $J := \{1, \cdots, N\}$ and for $k \in I$, we will denote by J_k the set of indices of edges adjacent to the vertex s_k. If $k \in I_{ext}$, then the unique element of J_k will be denoted by j_k.

The length of the edge e_j is denoted by ℓ_j. Then, e_j may be parametrized by its arc length by means of the functions $\pi_j : [0, \ell_j] \longrightarrow e_j, \;\; x \longmapsto \pi_j(x)$. But sometimes, we identify e_j with the interval $(0, \ell_j)$.

For a function $\underline{u} : \mathcal{G} \longrightarrow \mathbb{C}$ we set $u_j = \underline{u} \circ \pi_j$ its restriction to the edge e_j. For simplicity, we will write $\underline{u} = (u_1, \ldots, u_N)$ and we will denote $u_j(x) = u_j(\pi_j(x))$ for any $x \in (0, \ell_j)$.

The incidence matrix $D = (d_{kj})_{p\times N}$ is defined by,

$$d_{kj} = \begin{cases} 1 \text{ if } \pi_j(\ell_j) = s_k, \\ -1 \text{ if } \pi_j(0) = s_k, \\ 0 \text{ otherwise.} \end{cases}$$

Suppose that the equilibrium position of our network of elastic strings coincides with the graph $\mathcal{G}$. Then, we consider the following initial and boundary value problem (Fig. 1):

$$\frac{\partial^2 u_j}{\partial t^2}(x,t) - \frac{\partial}{\partial x}\left(\frac{\partial u_j}{\partial x} + a_j(x)\,\frac{\partial^2 u_j}{\partial x \partial t}\right)(x,t) = 0, \quad 0 < x < \ell_j,\ t > 0,\ j \in J, \tag{3}$$

$$u_{j_k}(s_k, t) = 0,\ k \in I_{ext},\ t > 0, \tag{4}$$

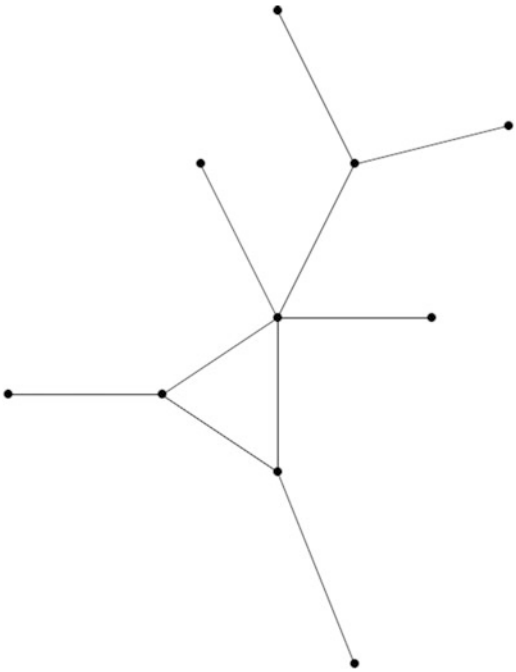

Fig. 1 A Graph

$$u_j(s_k,t) = u_l(s_k,t), \quad t>0,\ j,l \in J_k, \quad k \in I_{int}, \tag{5}$$

$$\sum_{j\in J_k} d_{kj}\left(\frac{\partial u_j}{\partial x}(s_k,t) + a_j(s_k)\frac{\partial^2 u_j}{\partial x \partial t}(s_k,t)\right) = 0, \quad t>0,\ k\in I_{int}, \tag{6}$$

$$u_j(x,0) = u_j^0(x),\ \frac{\partial u_j}{\partial t}(x,0) = u_j^1(x), \quad 0<x<\ell_j,\ j\in J, \tag{7}$$

where $u_j : [0,\ell_j]\times(0,+\infty)\to\mathbb{R}$, $j\in J$, be the transverse displacement in e_j, $a_j \in L^\infty(0,\ell_j)$ and, either a_j is zero, that is, e_j is a purely elastic edge, or there exists a subinterval w_j of $(0,\ell_j)$, nonreduced to a singleton, such that $a_j(x)>0$, a.e. on w_j. Such edge will be called a K-V edge.

We assume that $\mathcal{G}$ contains at least one K-V edge and contain at least one external node (i.e., $I_{ext} \neq \varnothing$). Furthermore, we suppose that every maximal subgraph of purely elastic edges is a tree, whose leaves are attached to K-V edges.

Our aim is to prove, under some assumptions on damping coefficients a_j, $j\in J$, exponential and polynomial stability results for the system (3)–(7).

We define the natural energy $E(t)$ of a solution $\underline{u} = (u_j)_{j\in J}$ of (3)–(7) by

$$E(t) = \frac{1}{2}\sum_{j\in J}\int_0^{\ell_j}\left(\left|\frac{\partial u_j}{\partial t}(x,t)\right|^2 + \left|\frac{\partial u_j}{\partial x}(x,t)\right|^2\right)dx. \tag{8}$$

It is straightforward to check that every sufficiently smooth solution of (3)–(7) satisfies the following dissipation law

$$\frac{d}{dt}E(t) = -\sum_{j\in J}\int_0^{\ell_j} a_j(x)\left|\frac{\partial^2 u_j}{\partial x\partial t}(x,t)\right|^2 dx \le 0, \tag{9}$$

and; therefore, the energy is a nonincreasing function of the time variable t.

The main results of this paper then concern the precise asymptotic behavior of the solutions of (3)–(7). Our technique is a special frequency domain analysis of the corresponding operator.

This work is organized as follows: In Sect. 2, we give the proper functional setting for system (3)–(7)and prove that the system is well-posed. In Sect. 3, we analyze the resolvent of the wave operator associated with the dissipative system (3)–(7) and prove the asymptotic behavior of the corresponding semigroup. For more details in the proofs, see [14].

2 Well-Posedness of the System

In order to study system (3)–(7) we need a proper functional setting. We define the following space

$$\mathcal{H} = V \times H,$$

where $H = \prod_{j\in J} L^2(0, \ell_j)$ and $V = \left\{ \underline{u} \in \prod_{j\in J} H^1(0, \ell_j) \,:\, u_{j_k}(s_k) = 0,\ k \in I_{ext},\ \text{satisfies (10)} \right\}$

$$u_j(s_k) = u_l(s_k) := \underline{u}(s_k),\ \ k \in I_{int}, \quad j, l \in J_k, \tag{10}$$

and equipped with the inner products

$$< (\underline{u}, \underline{v}, (\underline{\tilde{u}}, \underline{\tilde{v}}) >_{\mathcal{H}} = \sum_{j\in J} \int_0^{\ell_j} \left(v_j(x)\, \bar{\tilde{v}}_j(x) + u'_j(x)\, \bar{\tilde{u}}'_j(x) \right) dx. \tag{11}$$

System (3)–(7) can be rewritten as the first order evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t} \begin{pmatrix} \underline{u} \\ \frac{\partial \underline{u}}{\partial t} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \underline{u} \\ \frac{\partial \underline{u}}{\partial t} \end{pmatrix}, \\ \underline{u}(0) = \underline{u}^0,\ \dfrac{\partial \underline{u}}{\partial t} = \underline{u}^1 \end{cases} \tag{12}$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A} \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} := \begin{pmatrix} \underline{v} \\ (\underline{u'} + \underline{a} * \underline{v'})' \end{pmatrix},$$

with

$$\underline{a} := (a_j)_{j\in J} \ \text{ and } \ \underline{a} * \underline{v'} := (a_j v'_j)_{j\in J},$$

and

$$\mathcal{D}(\mathcal{A}) := \left\{ (\underline{u}, \underline{v}) \in \mathcal{H},\ \underline{v} \in V, (\underline{u'} + \underline{a} * \underline{v'}) \in \prod_{j\in J} H^1(0, \ell_j) \,:\, (\underline{u}, \underline{v}) \text{ satisfies (13)} \right\},$$

$$\sum_{j\in J_k} d_{kj} \left(u'_j(s_k) + a_j(s_k) v'_j(s_k) \right), \qquad t > 0,\ k \in I_{int}. \tag{13}$$

Lemma 2.1 *The operator $\mathcal{A}$ is dissipative,* $0 \in \rho(\mathcal{A})$: *the resolvent set of* $\mathcal{A}$.

Proof For $(\underline{u}, \underline{v}) \in \mathcal{D}(\mathcal{A})$, we have

$$Re(\langle \mathcal{A}(\underline{u}, \underline{v}), (\underline{u}, \underline{v})\rangle_{\mathcal{H}}) = Re \sum_{j \in J} \left(\int_0^{\ell_j} v'_{\alpha} \overline{u'_j} dx + \int_0^{\ell_j} (u'_j + a_j v'_j)' \overline{v_j} dx \right).$$

Performing integration by parts and using transmission and boundary conditions, a straightforward calculations leads to

$$Re(\langle \mathcal{A}(\underline{u}, \underline{v}), (\underline{u}, \underline{v})\rangle_{\mathcal{H}}) = -\sum_{j \in J} \int_0^{\ell_j} a_j(x) \left| v'_j(x) \right|^2 dx \le 0$$

which proves the dissipativeness of the operator $\mathcal{A}$ in $\mathcal{H}$.

Next, using Lax–Milgram's lemma, we prove that $0 \in \rho(\mathcal{A})$. For this, let $(f, g) \in \mathcal{H}$ and we look for $(\underline{u}, \underline{v}) \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}(\underline{u}, \underline{v}) = (\underline{f}, \underline{g})$$

which can be written as

$$v_j = f_j, \;\; j \in J, \tag{14}$$

$$(u'_j + a_j v'_j)' = g_j, \;\; j \in J. \tag{15}$$

$\underline{v}$ is completely determined by (14). Let $\underline{w} \in V$; multiplying (15) by w_j, then summing over $j \in J$, we obtain, using transmission and boundary conditions,

$$\sum_{j \in J} \int_0^{\ell_j} \left(u'_j + a_j v'_j \right) \overline{w'_j} dx = -\sum_{j \in J} \int_0^{\ell_j} g_j \overline{w_j} dx. \tag{16}$$

Replacing v_j in the last equality by (14), we get

$$\varphi(\underline{u}, \underline{w}) = \psi(\underline{w}), \tag{17}$$

where

$$\varphi(\underline{u}, \underline{w}) = \sum_{j \in J} \int_0^{\ell_j} u'_j \overline{w'_j}$$

and

$$\psi(\underline{w}) = -\sum_{j \in J} \left(\int_0^{\ell_j} g_j \overline{w_j} dx + \int_0^{\ell_j} a_j f'_j \overline{w'_j} dx \right).$$

The function φ is a continuous sesquilinear form on $V \times V$ and ψ is a continuous anti-linear form on V; here V is equipped with the inner product

$$\left\langle \underline{f}, \underline{g} \right\rangle = \sum_{j \in I} \int_0^{\ell_j} u_j' \overline{w_j'}.$$

Since φ is coercive on V, by the Lax–Milgram lemma, equation (17) has a unique solution $\underline{u} \in V$. Then taking $\underline{w} \in \prod_{j \in J} \mathcal{D}(0, \ell_j)$ in (17) and integrating by parts, we deduce that $(\underline{u}' + \underline{a} * \underline{v}') \in \prod_{j \in J} H^1(0, \ell_j)$ and $(\underline{u}, \underline{v})$ satisfies (15). Moreover $(\underline{u}, \underline{v})$ satisfies (13).

Return back to the Lax–Milgram lemma, $(\underline{u}, \underline{v})$ verifies

$$\left\| (\underline{u}, \underline{v}) \right\|_{\mathcal{H}} \leq \left\| (\underline{f}, \underline{g}) \right\|_{\mathcal{H}}.$$

In conclusion $(\underline{u}, \underline{v}) \in \mathcal{A}$ and $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$, which assert that $0 \in \rho(\mathcal{A})$. □

By the Lumer–Phillip's theorem (see [38, 42]), we have the following proposition.

Proposition 2.2 *The operator $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup of contraction $(S_d(t))_{t \geq 0}$ on the Hilbert space $\mathcal{H}$.*

Hence, for an initial datum $(\underline{u}^0, \underline{u}^1) \in \mathcal{H}$, there exists a unique solution $\left(\underline{u}, \frac{\partial \underline{u}}{\partial t}\right) \in C([0, +\infty), \mathcal{H})$ to problem (12). Moreover, if $(\underline{u}^0, \underline{u}^1) \in \mathcal{D}(\mathcal{A})$, then

$$\left(\underline{u}, \frac{\partial \underline{u}}{\partial t}\right) \in C([0, +\infty), \mathcal{D}(\mathcal{A})).$$

Furthermore, the solution $(\underline{u}, \frac{\partial \underline{u}}{\partial t})$ of (3)–(7)with initial datum in $\mathcal{D}(\mathcal{A})$ satisfies (9). Therefore, the energy is decreasing.

3 Asymptotic Behavior

In order to analyze the asymptotic behavior of system (3)–(7), we shall use the following characterizations for exponential and polynomial stability of a $\mathcal{C}_0$-semigroup of contraction:

Lemma 3.1 ([26, 39]) *A $\mathcal{C}_0$-semigroup of contraction $(e^{t\mathcal{B}})_{t \geq 0}$ defined on the Hilbert space $\mathcal{H}$ and such that*

$$\mathbf{i}\mathbb{R} \subset \rho(\mathcal{B}) \tag{18}$$

is exponentially stable if and only if

$$\limsup_{\beta\in\mathbb{R},|\beta|\to+\infty} \left\|(\mathbf{i}\beta\mathcal{I}-\mathcal{B})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{19}$$

Lemma 3.2 ([19]) *A $\mathcal{C}_0$-semigroup of contraction $(e^{t\mathcal{B}})_{t\geq 0}$ on the Hilbert space $\mathcal{H}$ such that $i\mathbb{R}\subset\rho(\mathcal{B})$ satisfies*

$$\left\|e^{t\mathcal{B}}\right\|_{\mathcal{L}(\mathcal{D}(\mathcal{B}),\mathcal{H})} \leq \frac{C}{t^{\frac{1}{\alpha}}}$$

for some constant $C > 0$ and for $\alpha > 0$ if and only if

$$\limsup_{\beta\in\mathbb{R},|\beta|\to+\infty} \frac{1}{|\beta|^{\alpha}} \left\|(\mathbf{i}\beta\mathcal{I}-\mathcal{B})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{20}$$

Lemma 3.3 (Asymptotic Stability) *The operator $\mathcal{A}$ verifies (18) and then the associated semigroup $(S(t))_{t\geq 0}$ is asymptotically stable on $\mathcal{H}$.*

Proof Since $0\in\rho(\mathcal{A})$ we only need here to prove that $(i\beta\mathcal{I}-\mathcal{A})$ is a one-to-one correspondence in the energy space $\mathcal{H}$ for all $\beta\in\mathbb{R}^*$. The proof will be done in two steps: in the first step we will prove the injective property of $(i\beta\mathcal{I}-\mathcal{A})$ and in the second step we will prove the surjective property of the same operator.

- Suppose that there exists $\beta\in\mathbb{R}^*$ such that $Ker(\mathbf{i}\beta\mathcal{I}-\mathcal{A})\neq\{0\}$. So $\lambda=\mathbf{i}\beta$ is an eigenvalue of $\mathcal{A}$, then let $(\underline{u}, \underline{v})$ an eigenvector of $\mathcal{D}(\mathcal{A})$ associated with λ. For every j in J we have

$$v_j = \mathbf{i}\beta u_j, \tag{21}$$

$$(u_j' + a_j v_j')' = \mathbf{i}\beta v_j. \tag{22}$$

We have

$$\left\langle \mathcal{A}(\underline{u}, \underline{v}), (\underline{u}, \underline{v})\right\rangle_{\mathcal{H}} = \sum_{j\in J}\int_0^{\ell_j} a_j \left|v_j'\right|^2 dx = 0.$$

Then $a_j v_j' = 0$ a.e. on $(0, \ell_j)$.

Let e_j a K-V edge. According to (21) and the fact that $a_j v_j' = 0$ a.e. on $(0, \ell_j)$, we have $u_j' = 0$ a.e. on ω_j. Using (22), we deduce that $v_j = 0$ on ω_j. Return back to (21), we conclude that $u_j = 0$ on ω_j.

Putting $y = u_j' + a_j v_j' = (1+\mathbf{i}\beta a_j)u_j'$, we have $y\in H^2(0,\ell_j)$ and $y' = -\beta^2 u_j$. Hence y satisfies the Cauchy problem

$$y'' + \frac{\beta^2}{1+\mathbf{i}\beta a_j} y = 0, \quad y(z_0) = 0, \quad y'(z_0) = 0$$

for some z_0 in ω_j. Then y is zero on $(0, \ell_j)$ and hence u'_j and u_j are zero on $(0, \ell_j)$. Moreover u_j and $u'_j + a_j v'_j$ vanish at 0 and at ℓ_j.

If e_j is a purely elastic edge attached to a K-V edge at one of its ends, denoted by x_j, then $u_j(x_j) = 0$, $u'_{\overline{\alpha}}(x_j) = 0$. Again, by the same way we can deduce that u'_j and u_j are zero in $L^2(0, \ell_j)$ and at both ends of e_j. We iterate such procedure on every maximal subgraph of purely elastic edges of $\mathcal{G}$ (from leaves to the root), to obtain finally that $(\underline{u}, \underline{v}) = 0$ in $\mathcal{D}(\mathcal{A})$, which is in contradiction with the choice of $(\underline{u}, \underline{v})$.

- Now given $(\underline{f}, \underline{g}) \in \mathcal{H}$, we solve the equation

$$(\mathbf{i}\beta\mathcal{I} - \mathcal{A})(\underline{u}, \underline{v}) = (\underline{f}, \underline{g})$$

or equivalently,

$$\begin{cases} \underline{v} = \mathbf{i}\beta\underline{u} - \underline{f} \\ \beta^2\underline{u} + \underline{u}'' + \mathbf{i}\beta\, (\underline{a} * \underline{u}')' = (\underline{a} * \underline{f}')' - \mathbf{i}\beta\underline{f} - \underline{g}. \end{cases} \tag{23}$$

Let us define the operator

$$A\underline{u} = -\underline{u}'' - \mathbf{i}\beta\, (\underline{a} * \underline{u}')', \quad \forall\, \underline{u} \in V.$$

It is easy to show that A is an isomorphism from V onto V' (where V' is the dual space of V obtained by means of the inner product in H). Then the second line of (23) can be written as follows

$$\underline{u} - \beta^2 A^{-1}\underline{u} = A^{-1}\left(\underline{g} + \mathbf{i}\beta\underline{f} - (\underline{a} * \underline{f}')'\right). \tag{24}$$

If $\underline{u} \in \mathrm{Ker}(\mathcal{I} - \beta^2 A^{-1})$, then $\beta^2\underline{u} - A\underline{u} = 0$. It follows that

$$\beta^2\underline{u} + \underline{u}'' + \mathbf{i}\beta(\underline{a} * \underline{u}')' = 0. \tag{25}$$

Multiplying (25) by $\overline{\underline{u}}$ and integrating over $\mathcal{T}$, then by Green's formula we obtain

$$\beta^2 \sum_{j\in J} \int_0^{\ell_j} |u_j(x)|^2\, \mathrm{d}x - \sum_{j\in J} \int_0^{\ell_j} |u'_j(x)|^2\, \mathrm{d}x - \mathbf{i}\beta \sum_{j\in J} \int_0^{\ell_j} a_j(x)\, |u'_j(x)|^2\, \mathrm{d}x = 0.$$

This shows that

$$\sum_{j\in J}\int_0^{\ell_j} a_j(x)\,|u_j'(x)|^2\,\mathrm{d}x = 0,$$

which imply that $\underline{a} * \underline{u}' = 0$ in $\mathcal{G}$.
Inserting this last equation into (25) we get

$$\beta^2\underline{u} + \underline{u}'' = 0, \qquad \text{in } \mathcal{G}.$$

According to the first step, we have that $\mathrm{Ker}(\mathcal{I} - \beta^2 A^{-1}) = \{0\}$. On the other hand, thanks to the compact embeddings $V \hookrightarrow H$ and $H \hookrightarrow V'$ we see that A^{-1} is a compact operator in V. Now thanks to Fredholm's alternative, the operator $(\mathcal{I} - \beta^2 A^{-1})$ is bijective in V, hence the Eq. (24) have a unique solution in V, which yields that the operator $(\mathbf{i}\beta\mathcal{I} - \mathcal{A})$ is surjective in the energy space $\mathcal{H}$. The proof is thus complete.

□

Before stating the main result, we define a property (P) on $\underline{a}$ as follows

$$(P) \quad \forall j \in J,\ \ a_j', a_j'' \in L^\infty(0,\ell_j) \ \text{ and } \quad \forall k \in I_{\mathcal{M}},\ \sum_{j\in J_k} d_{kj} a_j'(s_k) \le 0.$$

Theorem 3.4 *Suppose that the function $\underline{a}$ satisfies property (P), then*

(i) If $\underline{a}$ is continuous at every inner node of $\mathcal{T}$, then $(S_d(t))_{t\ge 0}$ is exponentially stable on $\mathcal{H}$.
(ii) If $\underline{a}$ is not continuous at least at an inner node of $\mathcal{T}$, then $(S_d(t))_{t\ge 0}$ is polynomially stable on $\mathcal{H}$, in particular, there exists $C > 0$ such that for all $t > 0$ we have

$$\left\| e^{\mathcal{A}t}(\underline{u}^0, \underline{u}^1)\right\|_{\mathcal{H}} \le \frac{C}{t^2}\left\|(\underline{u}^0, \underline{u}^1)\right\|_{\mathcal{D}(\mathcal{A})},\ \forall\,(\underline{u}^0, \underline{u}^1) \in \mathcal{D}(\mathcal{A}).$$

Proof According to Lemmas 3.1, 3.2, and 3.3, it suffices to prove that for $\gamma = 0$, when $\underline{a}$ is continuous at every inner node, or $\gamma = 1/2$, when $\underline{a}$ is not continuous at an inner node, there exists $r > 0$ such that

$$\inf_{\|(\underline{u},\underline{v})\|_{\mathcal{H}},\beta\in\mathbb{R}} \beta^\gamma \left\|(\mathbf{i}\beta\mathcal{I} - \mathcal{A})(\underline{u}, \underline{v})\right\|_{\mathcal{H}} \ge r. \tag{26}$$

Suppose that (26) fails. Then there exists a sequence of real numbers β_n, with $\beta_n \to \infty$ (without loss of generality, we suppose that $\beta_n > 0$), and a sequence of vectors $(\underline{u}_n, \underline{v}_n)$ in $\mathcal{D}(\mathcal{A})$ with $\left\|(\underline{u}_n, \underline{v}_n)\right\|_{\mathcal{H}} = 1$ such that

$$\beta_n^\gamma \left\|(\mathbf{i}\beta_n\mathcal{I} - \mathcal{A})(\underline{u}_n, \underline{v}_n)\right\|_{\mathcal{H}} \to 0. \tag{27}$$

We shall prove that $\left\|(\underline{u}_n, \underline{v}_n)\right\|_{\mathcal{H}} = o(1)$, which contradict the hypotheses on $(\underline{u}_n, \underline{v}_n)$.

Writing (27) in terms of its components, we get for every $j \in J$,

$$\beta_n^{\gamma} (\mathbf{i}\beta_n u_{j,n} - v_{j,n}) =: f_{j,n} = o(1) \quad \text{in } H^1(0, \ell_j), \tag{28}$$

$$\beta_n^{\gamma} (\mathbf{i}\beta_n v_{j,n} - (u'_{j,n} + a_j v'_{j,n})') =: g_{j,n} = o(1) \quad \text{in } L^2(0, \ell_j). \tag{29}$$

Note that

$$\beta_n^{\gamma} \sum_{j \in J} \int_0^{\ell_j} a_j(x) \left|v'_j(x)\right|^2 dx = Re\left(\left\langle \beta_n^{\gamma} (\mathbf{i}\beta_n \mathcal{I} - \mathcal{A}_d)(\underline{u}_n, \underline{v}_n), (\underline{u}_n, \underline{v}_n)\right\rangle_{\mathcal{H}}\right) = o(1).$$

Hence, for every $j \in J$

$$\beta_n^{\frac{\gamma}{2}} \left\|a_j^{\frac{1}{2}} v'_{j,n}\right\|_{L^2(0,\ell_j)} = o(1). \tag{30}$$

Then from (28), we get that

$$\beta_n^{\frac{\gamma}{2}} \left\|a_j^{\frac{1}{2}} \beta_n u'_{j,n}\right\|_{L^2(0,\ell_j)} = o(1). \tag{31}$$

Define $T_{j,n} = (u'_{j,n} + a_j v'_{j,n})$ and multiplying (29) by $\beta_n^{-\gamma} q T_{j,n}$ where q is any real function in $H^2(0, \ell_j)$, we get, using (28) and some integrations by parts,

$$\begin{aligned} &\frac{1}{2}\int_0^{\ell_j} q' \left|v_{j,n}\right|^2 dx + \frac{1}{2}\int_0^{\ell_j} q' \left|T_{j,n}\right|^2 dx - Im \int_0^{\ell_j} q a_j \beta_n v_{j,n} \overline{v'_{j,n}} dx \\ &-\frac{1}{2}\left(\left[q(x)\left|v_{j,n}(x)\right|^2\right]_0^{\ell_j} + \left[q(x)\left|T_{j,n}(x)\right|^2\right]_0^{\ell_j}\right) = o(1). \end{aligned} \tag{32}$$

□

Lemma 3.5 *The following property holds*

$$Im \int_0^{\ell_j} q a_j \beta_n v_{j,n} \overline{v'_{j,n}} dx = o(1). \tag{33}$$

Proof Since $\beta_n^{\frac{\gamma}{2}} a_j^{\frac{1}{2}} v'_{j,n} \to 0$ in $L^2(0, \ell_j)$ and $q \in L^\infty(0, \ell_j)$, it suffices to prove that

$$\beta_n^{1-\frac{\gamma}{2}} \left\|a_j^{\frac{1}{2}} v_{j,n}\right\|_{L^2(0,\ell_j)} = O(1). \tag{34}$$

For this, taking the inner product of (29) by $\mathbf{i}\beta_n^{1-2\gamma} a_j v_{j,n}$ leads to

$$\beta_n^{2-\gamma}\left\|a_j^{\frac{1}{2}} v_{j,n}\right\|^2_{L^2(0,\ell_j)} = -\mathbf{i}\beta_n^{1-\gamma}\int_0^{\ell_j} T'_{j,n} a_j \overline{v_{j,n}} dx - \mathbf{i}\beta_n^{1-2\gamma}\int_0^{\ell_j} g_{j,n} a_j \overline{v_{j,n}} dx. \tag{35}$$

Since $a_j \in L^\infty(0,\ell_j)$ and $g_{\bar{\alpha},n} \to 0$ in $L^2(0,\ell_j)$ we can deduce the inequality

$$-Re(\mathbf{i}\beta_n^{1-2\gamma}\int_0^{\ell_j} g_{\bar{\alpha},n} a_j \overline{v_{j,n}} dx) \leq \frac{1}{4}\beta_n^{2-\gamma}\left\|a_j^{\frac{1}{2}} v_{j,n}\right\|^2_{L^2(\omega_j)} + o(1). \tag{36}$$

On the other hand, we have [14]

$$\begin{aligned} -Re(\mathbf{i}\beta_n^{1-\gamma}\int_0^{\ell_j} T'_{j,n} a \overline{v_{j,n}} dx) &\leq -Re\left[\mathbf{i}\beta_n^{1-\gamma} T_{j,n}(x) a_j(x)\overline{v_{j,n}}(x)\right]_0^{\ell_j} \\ &+\frac{1}{2}\left[\beta_n^{-\gamma} a'_j(x)\left|v_{j,n}(x)\right|^2\right]_0^{\ell_j} + \frac{1}{4}\beta_n^{2-\gamma}\left\|a_j^{\frac{1}{2}} v_{j,n}\right\|^2_{L^2(0,\ell_j)} + O(1). \end{aligned} \tag{37}$$

Note that in the proof of (37) we have used that a'_j and a''_j belong to $L^\infty(0,\ell_j)$.

Thus, substituting (36) and (37) into (35) leads to

$$\begin{aligned} \frac{1}{2}\beta_n^{2-\gamma}\left\|a_j^{\frac{1}{2}} v_{j,n}\right\|^2_{L^2(0,\ell_j)} &\leq -Re\left[\mathbf{i}\beta_n^{1-\gamma} T_{j,n}(x) a_j(x)\overline{v_{j,n}}(x)\right]_0^{\ell_j} \\ &+\frac{1}{2}\left[\beta_n^{-\gamma} a'_j(x)\left|v_{j,n}(x)\right|^2\right]_0^{\ell_j} + O(1). \end{aligned} \tag{38}$$

Summing over $j \in J$,

$$\begin{aligned} \sum_{j\in J}\beta_n^2\left\|a_j^{\frac{1}{2}} v_{j,n}\right\|^2_{L^2(0,\ell_j)} &\leq -2\sum_{k\in I_{int}} Re\left(\mathbf{i}\beta_n^{1-\gamma}\underline{v}_n(s_k)\sum_{j\in J_k} d_{kj} a_{j_k}(s_k) T_{j_k,n}(s_k)\right) \\ &+\beta_n^{-\gamma}\sum_{k\in I_{int}}\left|\underline{v}_n(s_k)\right|^2\sum_{j\in J_k} d_{kj} a'_{j_k}(s_k) + O(1). \end{aligned} \tag{39}$$

We have used the continuity condition of $\underline{v}_n$ and the compatibility condition (7) at inner nodes and the Dirichlet condition of $\underline{u}$ and $\underline{v}$ at external nodes.

Notes that from property (P) we have

$$\sum_{k\in I_{\mathcal{M}}}\left|\underline{v}_n(s_k)\right|^2\sum_{j\in J_k} d_{kj} a'_j(s_k) \leq 0, \tag{40}$$

then to conclude, it suffices to estimate

$$\sum_{k\in I_{int}} Re\left(\mathbf{i}\beta_n^{1-\gamma}\underline{v}_n(s_k)\sum_{j\in J_k} d_{kj}a_{j_k}(s_k)T_{j_k,n}(s_k)\right).$$

Case (i), corresponding to $\gamma = 0$: Here $\underline{a}$ is continuous in all nodes. It follows that $\sum_{k\in I_{int}} Re\left(\mathbf{i}\beta_n^{1-\gamma}\underline{v}_n(s_k)\sum_{j\in J_k} d_{kj}a_{j_k}(s_k)T_{j_k,n}(s_k)\right) = 0$.

Then, (39) and (40), yield

$$\beta_n^2 \left\| a_j^{\frac{1}{2}} v_{j,n} \right\|^2_{L^2(0,\ell_j)} = O(1)$$

for every $j \in J$, and the proof of Lemma 3.5 is complete for case (i).

Case (ii), corresponding to $\gamma = \frac{1}{2}$: Recall that here the function $\underline{a}$ is not continuous at some internal nodes. We want estimate the first term in the right hand side of (38). To do this it suffices to estimate $Re(\mathbf{i}\beta_n^{1-\gamma}T_{j,n}(x)a_j(x_j)\overline{v_{j,n}}(x))$ at an inner node $x = x_j$ when $a_j(x_j) \neq 0$. By means of some Gagliardo–Nirenberg inequality [34] we proved in [14] the following estimate

$$-Re(\mathbf{i}\beta_n^{\frac{1}{2}}T_{j,n}(x_j)\overline{v_{j,n}}(x_j)) = o(1).$$

We then conclude that the first term on the right hand side of (39) converges to zero.

Then, again, using (40), we obtain that

$$\sum_{j\in I} \beta_n^{\frac{1}{2}} \left\| a_j^{\frac{1}{2}} \beta_n v_{j,n} \right\|^2_{L^2(0,\ell_j)} = O(1),$$

then

$$\beta_n^{\frac{3}{2}} \left\| a_j^{\frac{1}{2}} v_{j,n} \right\|^2_{L^2(0,\ell_j)} = O(1)$$

for every $j \in I$, and the proof of Lemma 3.5 is complete for case (ii). □

Return back to the proof of Theorem 3.4. Substituting (33) in (32) leads to

$$\frac{1}{2}\int_0^{\ell_j} q' \left|v_{j,n}\right|^2 dx + \frac{1}{2}\int_0^{\ell_j} q' \left|T_{j,n}\right|^2 dx - \frac{1}{2}\left[q(x)\left(\left|v_{j,n}(x)\right|^2 + \left|T_{j,n}(x)\right|^2\right)\right]_0^{\ell_j} = o(1) \quad (41)$$

for every $j \in J$.

Let $j \in J$ such that e_j is a K-V string. First, note that from (34), we deduce that

$$\left\| a_j^{\frac{1}{2}} v_{j,n} \right\|^2_{L^2(0,\ell_j)} = o(1).$$

Then, we take $q(x) = \int_0^x a_j(s)ds$ in (41) to obtain

$$\frac{1}{2}\int_0^{\ell_j} a_j \left|T_{j,n}\right|^2 dx - \frac{1}{2}\left(\int_0^{\ell_j} a_j(s)ds\right)\left(\left|v_{j,n}(\ell_j)\right|^2 + \left|T_{j,n}(\ell_j)\right|^2\right) = o(1). \tag{42}$$

Since $\frac{1}{2}\int_0^{\ell_j} a_j \left|T_{j,n}\right|^2 dx = o(1)$ and $\int_0^{\ell_j} a_j(s)ds > 0$, then (42) implies

$$\left|T_{j,n}(\ell_j)\right|^2 + \left|v_{j,n}(\ell_j)\right|^2 = o(1). \tag{43}$$

Therefore, (41) can be rewritten as

$$\frac{1}{2}\int_0^{\ell_j} q' \left|v_{j,n}\right|^2 dx + \frac{1}{2}\int_0^{\ell_{\bar{\alpha}}} q' \left|T_{j,n}\right|^2 dx$$
$$+\frac{1}{2}\left(q(0)\left|v_{j,n}(0)\right|^2 + q(0)\left|T_{j,n}(0)\right|^2\right) = o(1). \tag{44}$$

By taking $q = x + 1$ in (44) we deduce that

$$\left\|v_{j,n}\right\|_{L^2(0,\ell_j)} = o(1) \text{ and } \left\|u'_{j,n}\right\|_{L^2(0,\ell_j)} = o(1) \tag{45}$$

and moreover

$$v_{j,n}(\ell_j) = o(1) \text{ and } T_{j,n}(\ell_j) = o(1) \tag{46}$$

implies that $\left\|v_{j,n}\right\|_{L^2(0,\ell_j)} = o(1)$ and $\left\|T_{j,n}\right\|_{L^2(0,\ell_j)} = o(1)$.

Moreover, $\left\|u'_{j,n}\right\|_{L^2(0,\ell_j)} = \left\|T_{j,n} - a_j v_{j,n}\right\|_{L^2(0,\ell_j)} = o(1)$. Also we have

$$v_{j,n}(0) = o(1) \text{ and } T_{j,n}(0) = o(1). \tag{47}$$

Finally, notice that (43) signifies that

$$v_{j,n}(\ell_j) = o(1) \text{ and } T_{j,n}(\ell_j) = o(1). \tag{48}$$

To conclude, it suffices to prove that (45) holds. For every $j \in I$ such that e_j is purely elastic. As in the proof of Lemma 3.3, we start by proving (45) for a string e_j attached at one end to only K-V strings. Then we iterate such procedure on each maximally connected subgraph of purely elastic strings (from leaves to the root).

Thus $\left\|(\underline{u}_n, \underline{v}_n)\right\|_{\mathcal{H}} = o(1)$, which contradicts the hypothesis $\left\|(\underline{u}_n, \underline{v}_n)\right\|_{\mathcal{H}} = 1$.

Remark 6

1. If for every $j \in J$, a_j is continuous on $[0, \ell_j]$ and not vanish in such interval, then we do not need the property (P) in the Theorem 3.4.

Indeed (P) is used only to estimate

$$-Re\left(\mathbf{i}\beta_n^{1-\gamma}\int_0^{\ell_j} T'_{j,n}a_j\overline{v_{j,n}}dx\right)$$

in (35), according to $\beta_n^{1-\frac{\gamma}{2}}\left\|a_j^{\frac{1}{2}}v_{j,n}\right\|_{L^2(0,\ell_j)}$.

This is equivalent to estimate

$$-Re\left(\mathbf{i}\beta_n^{1-\gamma}\int_0^{\ell_j} T'_{j,n}\overline{v_{j,n}}dx\right)$$

according to $\beta_n^{1-\frac{\gamma}{2}}\left\|v_{j,n}\right\|_{L^2(0,\ell_j)}$:

$$\begin{aligned}
&-Re\left(\mathbf{i}\beta_n^{1-\gamma}\int_0^{\ell_j} T'_{j,n}\overline{v_{j,n}}dx\right)\\
&\quad=-Re\left[\mathbf{i}\beta_n^{1-\gamma}\,T_{j,n}\,\overline{v_{j,n}}\right]_0^{\ell_j}+Re\left(\mathbf{i}\beta_n^{1-\gamma}\int_0^{\ell_j} T_{j,n}\overline{v'_{j,n}}dx\right)\\
&\quad=-Re\left[\mathbf{i}\beta_n^{1-\gamma}\,T_{j,n}(x)\,\overline{v_{j,n}}(x)\right]_0^{\ell_j}+o(1)
\end{aligned}$$

as in case (ii) (proof of Theorem 3.4) we prove without using (P) that

$$-Re\left[\mathbf{i}\beta_n^{1-\gamma}\,T_{j,n}(x)\,\overline{v_{j,n}}(x)\right]_0^{\ell_j}\le\frac{\beta_n^{2-\gamma}}{4}\left\|v_{j,n}\right\|^2_{L^2(0,\ell_j)}+o(1).$$

2. We find here the particular cases studied in [2, 25, 30, 31, 33]. Note that concerning the result of polynomial stability in [2, 25] the authors proved that the $\frac{1}{t^2}$ decay rate of solution is optimal when the damping coefficient is a characteristic function.

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