Chapter 7
University Students' Development of (Non-) Mathematical Practices: The Case of a First Analysis Course

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Abstract In this chapter, we present a study that investigated the nature of the task solving practices developed by students in a first Analysis course at a North American university, and how these practices may be shaped by the evaluations (assignments and exams) given in the course. Task-based interviews with 15 students after their successful completion of the course revealed that students' practices could vary in nature, being more or less "mathematical," i.e., more or less reflective of mathematicians' practices. As suggested by previous research on Calculus courses, we also found that the practices students develop in this Analysis course are likely shaped by the minimal requirements for success. To try to make sense of this, we introduce the theoretical notion of "path to a practice" and a characterization of three ways in which students' practices may reveal themselves to be "non-mathematical."

Keywords Mathematical practices · Task solving · Praxeology · Institutional perspective · Real analysis · Functions

7.1 Introduction

Previous research has investigated what students learn in Calculus courses and documented its potentially rote procedural nature. Orton (1983) interviewed 110 Calculus students and found that many were operating according to *rules without reasons*: When it came to performing integral calculations, they knew *what* to do, but did not know *why* they were doing it. Shortly after, similar results concerning a variety of Calculus topics were published by other researchers (e.g., Artigue et al., 1990, in France; Cox, 1994, in Britain; Selden et al., 1994, in the United States; White & Mitchelmore, 1996, in Australia), some of whom began to look more systematically into why students may learn rules without reasons. Cox's (1994) discussions with Calculus teachers and students revealed that they may tailor their

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teaching and learning to typical exam questions. Selden et al. (1994) also pointed to the potential impact of emphasizing routine tasks in instruction and evaluation: Tests administered to students who received good passing grades in Calculus courses showed that they could solve routine tasks quite well, but lacked the conceptual understanding needed to solve only moderately nonroutine tasks. Later research by Lithner and colleagues echoed these findings and worked on characterizing the nature of the reasoning underlying students' solving of routine tasks. For example, in observing students' task solving, Lithner (2000, 2003) saw how some students explained their strategies based on established experiences from their learning environment or superficial features of similar-looking tasks (rather than "mathematical" reasoning). Subsequent studies made sense of this through systematic analyses of the tasks that are typically posed in Calculus textbooks and final exams, which were found to not *require* students to go beyond superficial mimicry or the basic recall of algorithms based on properties of the tasks that are not relevant from a "mathematical" point of view (e.g., Bergqvist, 2007; Brandes & Hardy, 2018; Hardy, 2009; Lithner, 2004; Tallman et al., 2016).

It seems reasonable to expect that as students move beyond Calculus courses, the nature of what they know and learn would be required to change. Theoretically speaking, it has been proposed that curricula of more advanced courses in Analysis invite students to deepen their understanding of mathematical concepts and theories underlying procedures learned in Calculus, and to develop formal proof practices that require the use of mathematical reasoning (Winsløw, 2006). On a practical level, however, scholars (a) doubt that students naturally make productive connections between what they learn in Analysis and what they learned in Calculus (e.g., Kondratieva & Winsløw, 2018; Winsløw et al., 2014), (b) pinpoint epistemic, cognitive, and didactic obstacles to students' learning of formal proof practices (e.g., Bergé, 2008; Maciejewski & Merchant, 2016; Raman, 2002, 2004; Sfard, 1991; Tall, 1992; Timmermann, 2005), and (c) point to the possibility of students successfully completing[1] Analysis courses by memorizing a particular subset of definitions, theorems, and proofs (e.g., Darlington, 2014), or by learning new kinds of superficial (non-mathematically relevant) and algorithmic task solving practices (e.g., Weber, 2005a, b).

To contribute to the literature outlined above we conducted an exploratory study (Broley, 2020) of an Analysis course at a North American university. In this chapter, we deepen our analysis of a refined subset of results and extend our reflections concerning two general research questions we explored:

1. What is the nature of the practices developed by students in a first Analysis course?
2. How might these practices be shaped by the nature of the tasks offered to students in the course?

[1] Both in those studies and in ours, successful completion of a course means obtaining a passing grade.

Responses to these questions could have practical implications for teachers or curriculum developers involved in designing the tasks offered in university mathematics courses. In what follows, we introduce how we framed our questions (Sect. 7.2), describe our methodology and the more specific objectives it addressed (Sect. 7.3), and present and discuss some results (Sects. 7.4 and 7.5).

7.2 Theoretical Framework

To frame our research questions, we first specify how we think about "practices" and their "nature" (Sect. 7.2.1). Then we elaborate our perspective on how practices may be "shaped by the nature of the tasks offered to students" in a course (Sect. 7.2.2).

7.2.1 Mathematical and Non-Mathematical Practices

To help us think about the nature of students' practices, we turned to theoretical tools within the Anthropological Theory of the Didactic (ATD; Chevallard, 1985, 1991, 1992, 1999).[2]

In the ATD, *practices* refer to regularized and purposeful human actions, which can be *personal* (developed by an individual) or *institutional* (created, encouraged, and enforced in a particular institution). An *institution* is understood in a broad sense as a relatively stable structural element of a society that has been established to organize human (inter)actions and orient them towards certain outcomes. Any profession (pure mathematics research, actuarial science, engineering, etc.) or form of organized education (school mathematics, university mathematics, etc.) can be thought of as an institution (called *professional* or *didactic institutions*, respectively). An individual is said to have developed an institutional practice if they have developed a personal practice that is judged to be acceptable and worthwhile within that institution.

With his theory of didactic transposition, Chevallard (1985) brought to light the transformation of practices as they migrate from a professional institution into a didactic institution, which serves to exemplify the institutional relativity of practices.[3] In particular, the ATD acknowledges that what is considered "mathematics" or "mathematical" may change from one institution to the next. We nevertheless claim that one overall aim of *university mathematics* is to support students' eventual development of *mathematicians' practices*, by which we mean the practices

[2]To learn more about the ATD and its use in mathematics education, see Bosch et al. (2020) for a recent comprehensive description and Winsløw et al. (2014) for an overview specific to the university level.

[3]One should pause and reflect on the relationship between the terms "practice" and "knowledge" from an ATD perspective. We let this hang in the subtext of our chapter, to be addressed in further discussion and subsequent theoretical research.

produced and used by mathematicians in the broad professional institution referred to as *scholarly mathematics*. Thus, in our work, we use mathematicians' practices as a reference with which to compare the practices of university mathematics students, and we use the term *mathematical practices* (and *non-mathematical practices*, in contrast) in a particular way: to refer to practices that would be considered acceptable and worthwhile (or not acceptable or not worthwhile, in contrast) within the scholarly mathematics institution.

Chevallard (1999) offers the notion of *praxeology* as a way of modelling practices as they exist across institutions and individuals; any practice can be represented by a quadruplet $[T, \tau, \theta, \Theta]$ – called a "praxeology" – involving four interconnected, essential components:

- a *type of task, T*, to be accomplished;
- a corresponding collection of *techniques, τ*, to accomplish T;
- the *technology*, θ, i.e., discourses to describe, justify, explain, and produce the techniques; and
- the *theory, Θ*, that serves as a foundation of θ.

This representation of a practice recognizes both a practical part (the know-how), $[T, \tau]$, called the *praxis*, and a theoretical part (the know-why), $[\theta, \Theta]$, called the *logos*.

The notion of praxeology gives us a way to think about the nature of students' practices, which in turn allows us to reflect on whether and in what ways the practices are mathematical or not (in the sense posed above). As we consider a praxeology to be a static model of a practice, and inspired by previous work (e.g., Lithner's, 2008, task solving framework), we say that an individual *enacts a mathematical practice* if they carry out the action of solving a given task by

- identifying the task as belonging to a mathematical type of task;
- selecting and implementing a mathematical technique to accomplish the task;
- describing, in a mathematical discourse, how and why the technique works; and
- acknowledging a mathematical theory that supports the discourse[4];

where, as explained above, "mathematical" is used in a particular way, to describe a component (type of task, technique, etc.) as acceptable and worthwhile within the scholarly mathematics institution.[5] If, conversely, some component would not be

[4]It is possible that an individual will not explicitly engage in each of these actions when solving a task. Following the example of Chevallard (1999), we take the position that "having a practice" means being able to engage in four actions reflecting the four components of a praxeology. For example, if an individual has a practice, they would be able to give some description of why their chosen technique works. This description need not be "mathematical": e.g., "I know the technique works because my teacher told me to do it that way."

[5]We are assuming that there are some uniform, implicit ideas among mathematicians of what is (or is not) acceptable and worthwhile. We also acknowledge that there could be pertinent differences between mathematicians' judgements depending, for example, on the specific area of mathematics in which they work (mathematical physics, numerical analysis, algebraic topology, . . .), which could warrant a definition of mathematical practice that depends on a specified area of mathematics. We did not consider such differences in this research.

considered acceptable and worthwhile according to scholarly mathematics, we say that the individual *enacts a non-mathematical practice*. Certainly, enacting a mathematical practice cannot be equated with having developed one.[6] Nevertheless, in our work, we assume that an Analysis student who enacts a non-mathematical practice has not developed a mathematical practice – we say that these students have developed non-mathematical practices.

As an example, we could expect a mathematician faced with finding $\lim_{x \to 1} \frac{x-1}{x^2+x}$ to identify the task as belonging to the type *find the limit of a rational function at a point* and to solve the task by direct substitution. If prompted to describe how and why the technique solves the task, we could expect them to acknowledge certain theoretical elements such as theorems, laws, and definitions. In contrast, when asked to find $\lim_{x \to 1} \frac{x-1}{x^2+x}$, many of the Calculus students in Hardy's (2009) study seemed to identify the task with a type characterized by an easily factorable expression, which necessitates some sort of algebraic technique: 20 out of 28 students tried factoring, seven of which did direct substitution first. Furthermore, the students' discourses were of the sort: "We do this because that's what our teacher showed us, and that's what we normally do for this kind of problem." Hardy (2009) concluded that the students learned to behave "normally" rather than "mathematically." In the context of our study, we would say that the students were enacting non-mathematical practices. In the following, we propose one way of thinking about how the students may have developed such non-mathematical practices.

7.2.2 *The Progressive Development of Practices*

In our work, and in line with previous research (e.g., Bergqvist, 2007; Cox, 1994; Hardy, 2009; Lithner, 2004; Selden et al., 1994), we conjecture that in university mathematics courses, students encounter numerous tasks that progressively determine the practices they develop. The tasks may occur in lectures, recommended exercises, assessments, and students' independently driven work. To model how the nature of such tasks might contribute to moulding students' practices, we introduce a distinction between *isolated tasks* and *tasks forming a path to a practice* (building on Broley & Hardy, 2018).

The tasks in Table 7.1 were offered by teachers to students in the Analysis course we studied, along with written solutions. The written solutions for the tasks on the right of Table 7.1 use the Intermediate Value Theorem (IVT). These tasks are meant to help students identify a particular type of task – that of showing that a function has

[6] A student who enacts a mathematical practice could simply be mimicking behaviour. But the focus of our work is the development of non-mathematical practices. Given our task-based interview approach (see Sect. 7.3), we are convinced that the students we interviewed who enacted non-mathematical practices had not developed mathematical practices; they had developed non-mathematical practices.

Table 7.1 Examples of tasks found in assessment documents in the Analysis course we studied

Consider $Z_3 = \{0,1,2\}$ with the usual operations $+$ and $\cdot$. Check that $\{Z_3, +, \cdot\}$ satisfies the axioms of a field.	Show that the polynomial $P(x) = x^5 - 3x + 1$ has a zero in the interval $(0,1)$.
Prove $[ax = a \text{ and } a \neq 0] \Rightarrow x = 1$ using the axioms given in class.	Show that $f(x) = e^{-x^2}$ has a fixed point in the interval $(0,1)$.
Prove that if $a < b$, then there is a $q \in Q$ such that $a < \sqrt{2}q < b$.	Prove that the function $f(x) = e^x - 100x$ has exactly one zero in the interval $[0,1]$.
Does the subset $\{(x,y) \in \mathbb{R} \times \mathbb{R}: x^3 = y^2\}$ define a function?	Prove that the equation $\cos(x) = 5x(1-x)$ has exactly two solutions in $[0,1]$.
Find the intervals on which the function $f(x) = \frac{x^2+1}{x^4+1}$ is monotonic.	Let $f: [0,1] \to [0,1]$ be continuous. Prove that the equation $f(x) = x^2$ has a solution in $[0,1]$ (you may use the Intermediate Value Theorem).
	Show that the equation $e^x = 3x^2$ has at least two positive solutions.
	How many zeros does $f(x) = 4x^3 - 32x^2 + 79x - 60$ have in the interval $[0,5]$?
	…
"ISOLATED TASKS"	TASKS FORMING A "PATH TO A PRACTICE"

a certain number of zeros – and to master a particular technique – one afforded by the IVT. Teachers' solutions propose to students discourses to describe how and why the technique works: "by the Intermediate Value Theorem".

We say that tasks that relate to the same type of task and exist in relatively high quantity, including in situations that are relevant to a student's success, form a *path to a practice*: they communicate to the student that some kind of practice should be developed. In contrast, certain types of tasks may be encountered by students only in disconnected, rare, or seemingly non-relevant (e.g., non-tested) situations. The action of accomplishing the related tasks may hence remain isolated and particular, not contributing to the development of a practice.[7] We say that such tasks are *isolated* (as opposed to forming a path to a practice).

[7]We consider the important distinction between the action of solving a task for which one has developed a practice and the action of solving a task for which one has not developed a practice. For instance, an individual may engage in the tasks of cooking a meal or hammering a nail without having developed practices for doing so. In contrast, professional chefs or carpenters are typically required to develop practices to ensure the regular and suitable accomplishment of those tasks.

The nature of the *practice suggested by a path* may depend on different elements: for instance, the nature of the tasks forming the path (e.g., the way the tasks are phrased or the kinds of objects they concern); the context within which the different tasks take place (e.g., tasks occurring on past exams may have a greater influence than tasks occurring in assignments); what is made explicit about the tasks (e.g., steps and discourses present or absent in teachers' solutions); or who is observing the tasks (e.g., a researcher, a teacher, or a student). It is possible that different students see different paths or develop different practices when engaging in the same given tasks (we return to this idea in Sects. 7.4 and 7.5). Observing from our perspective as researchers, we found the idea of a path to a practice helpful in framing the way we collected and analysed data.

7.3 Methodology

Our study focussed on a first Analysis course (A1) offered at a large North American university. A1 is a mandatory course for mathematics programs leading to graduate work (e.g., in statistics or pure mathematics). It is typically preceded by courses in single variable and multivariable Calculus, and followed by a second course in Analysis (A2). Together, A1 and A2 form an introduction to Analysis of single variable real-valued functions. Most topics are identical to those in single variable Calculus courses (e.g., limits, continuity, derivatives). The difference is the expectation (explicit in curricular documents) that the courses will introduce students to mathematical rigour and proofs.

A1 is an institution in the ATD sense (Sect. 7.2.1). The teacher (typically a full-time mathematics professor engaging in teaching and research) provides 3 h of lecture per week; students are evaluated through weekly assignments, a midterm, and a final exam, with their successful completion of the course significantly determined by their final exam grade[8]; and there is a course examiner who is responsible for ensuring consistency in evaluations across teachers and terms.

To address our two research questions, we used a task-based interview approach (Goldin, 1997, 2000), founded on an a priori analysis of some of the tasks typically offered in A1. Hence, our study proceeded in two stages. The focus of the first stage was an analysis of tasks proposed to students in A1 and the solutions teachers made available to students for studying. The objective of this first stage was two-fold. On the one hand, we expected to test our capacity to predict, based on previous research and on the tasks and solutions proposed to students, what (non-)mathematical[9] practices students would develop (based on our analysis of the tasks, we would model practices we expected students to develop – for further clarity, see footnote

[8] At the time of our study, students were evaluated by taking the best of two possible distributions: 10% assignments, 30% midterm, 60% final exam or 10% assignments, 90% final exam.

[9] We use "(non-)mathematical" to mean "non-mathematical or mathematical."

10). On the other hand, the analysis of tasks offered to students in A1 was key to the creation of task-based interviews that would elicit students' developed practices. The objective of the second stage of our study – the creation, implementation, and analysis of task-based interviews – was to build models of practices actually developed by students and to reflect on the (non-)mathematical nature of these practices. We present relevant details of our methodology below (for more details, including a thorough description of the methodological approach and illustrative examples, see Chap. 4 in Broley, 2020).

In the first stage of our study (described in detail in 4.1 in Broley, 2020), we analyzed over 200 tasks listed in assessment documents provided to students in A1, including the weekly assignments, midterm, and final exam posed in a particular iteration of the course, as well as midterms and final exams from previous iterations that students were given to guide their studying. While our research questions refer to the practices developed by students in A1, our analysis was focussed exclusively on the tasks presented in the documents listed above (as opposed to considering all the tasks offered to students, including, e.g., in lectures). We considered these tasks sufficient for our objectives for several reasons: Past research has shown the potentially strong influence of assessments on the practices students develop (e.g., Cox 1994; Hardy 2009); and in the course we studied, tasks that will be tested appear with high frequency in assessment documents (assignments, midterms, and final exams) and study guides (the textbook and solutions to tasks provided to students), which, we conjecture, drives students towards the development of practices that will be tested.

In our analysis of the tasks, we sought to identify those that relate to the same type of task and exist in relatively high quantity – the tasks in a path to a practice (Sect. 7.2.2). If tasks relating to a certain type of task occurred in low quantity and only on assignments, we considered them to be isolated; otherwise, we considered them as forming paths to practices. To characterize the practices that we expected students to develop (what we will refer to as the "suggested practices," from our perspective as researchers),[10] we built praxeological models, including specific characteristics of the tasks and teachers' solutions that we conjectured (based on previous research) might have shaped students' practices (e.g., we recorded whether tasks concerned particular kinds of objects and which theoretical elements were explicit in teachers' solutions). We then selected a subset of paths to practices on which to base our interview tasks, for different reasons: e.g., we chose a variety of paths (in terms of topic) to explore patterns or differences in students' practices; our interest in the evolution of students' practices from Calculus to Analysis also led us to favour paths (and eventually tasks) that students may link to practices developed in prior Calculus courses. Our six interview tasks can be found in Appendix A in Broley (2020).

[10]To be clear, we are not referring to the expectations that the institution or the teacher may have, which may well be that students learn mathematical practices. We are referring to our expectations as researchers critical of the tasks being proposed. Based on previous research, we expected students to develop some non-mathematical practices (e.g., focusing on superficial, non-mathematically relevant features of highly frequent tasks).

Complete results of this stage of our study can be found in Chap. 6 (6.i.1, i = 1 to 6) in Broley (2020). In Sect. 7.4.1, we present the suggested practice associated with our second interview task (T2): Show that the function $f(x) = e^x - 100(x - 1)(2 - x)$ has 2 zeros. We selected this task as the focus of this chapter since its results illustrate well our approach and the different kinds of (non-)mathematical practices we found students may develop.

The second stage of our study (described in detail in 4.2 in Broley, 2020) focussed on the design, implementation, and analysis of our task-based interview. This kind of interview was fitting for our objectives since it could allow us to observe students as they enact practices to solve given tasks. We designed the interview tasks and protocol with the goal of eliciting students' practices and revealing their nature. Key to the design was achieving *recognizability* and *deception*: students needed to recognize the interview tasks as being solvable using practices they had developed in A1; they also needed to be potentially deceived by some element of the task so that any non-mathematical nature of their practices would be revealed. Generally speaking, we chose interview tasks that mirrored, but also differed in some significant way, from tasks within the paths selected from the first stage of our study (in Sect. 7.4.1, we give the example of T2). Once the tasks were chosen, following Goldin's (1997, 2000) principles, we created a protocol (Appendix A in Broley, 2020), which outlined the rules of interaction between the interviewer, an interviewee (a successful A1 student), and the tasks. After receiving a task (printed on the top of a blank sheet of paper), an interviewee had as much time as possible[11] to engage in independent task solving, thinking aloud and using the tools made available to them (paper, a pencil, and a scientific calculator). If the interviewee struggled to engage with a task, the interviewer offered heuristic suggestions that became progressively more directive as needed (potential suggestive questions were created for each task and can be found in the protocol). At the end of an interviewee's task solving attempt, the interviewer asked follow-up questions with our objectives in mind (e.g., it was not important for the interviewee to develop a final polished solution; but they were encouraged to clarify the approach they took or would take for solving the given task, and why, which was crucial for modelling their practices). We conducted two- to three-hour interviews with 15 students (S1 to S15) after they successfully completed A1.

Our analysis proceeded in several steps. First, audio recordings of the 15 interviews were combined with participants' written work to create verbatim transcripts. Second, for each participant and each task, we created a table with three rows, where we recorded observations from the participant's transcript that would help us infer the different components of their practice(s): the type(s) of task(s) identified, the technique(s) selected and implemented, and the discourses used to describe how and why the technique(s) work, including any acknowledgement of underlying theory. For example, in the row corresponding to technique(s), we synthesized the steps the

[11] The time available for solving a given task was constrained by the planned duration of the interview (2 h) and the priority of observing a participant formulate at least one approach, and a reason for the approach, for each of the six interview tasks.

participant took to solve the task. Third, for each task, we then used the tables to categorize participants according to criteria that emerged as we read the tables and thought about our objectives. Criteria varied across tasks and were not limited to "task(s) solved," "technique(s) considered," or "technologies/theories referred to": e.g., for T2, the criteria also included "the first thing a participant spoke about or did upon receiving the task" and "how they chose which x values to plug in (to locate sign changes in f)." Using this categorization, we engaged in a fourth step, writing about patterns in participants' task solving: i.e., how they identified types of tasks, selected and implemented techniques, and described how and why those techniques worked. Finally, we created models of the practices enacted by the students, which we used to reflect on their nature (using the lens elaborated in Sect. 7.2.1), and how they may have been shaped by the tasks offered in the course (by comparing them to our models of suggested practices). In Sect. 7.4.2, we present some of the results of this analysis, exemplified in relation to T2.

Before presenting the results, it is important to address the fact that we model students' practices – regularized and purposeful actions (Sect. 7.2.1) – based on a solution to *one* task of a certain type. Since we interviewed students at the end of the course, we expected them to have developed regularized and purposeful actions for solving potentially evaluated tasks. Our interview tasks were designed to trigger such practices, and the deceptive nature of the tasks meant that when a practice did not work (to solve the task), the student was forced to explain it. Moreover, a student would often exhibit specific cues that their behaviour was indicative of a practice (see footnote 7). For instance, they would describe their approach in a general sense (i.e., not specific to the given task), or they would say things like "I am going to use the method learned in class," "I've repeated this approach so often," or "I usually do it this way" (as exemplified in Sect. 7.4.2). This said, we recognize that there may have been times where identified "practices" were "potential" and could have been more "practices in development, in adaptation, or in evolution." This is a complex issue and an interesting direction for future work.

7.4 Results

In our analysis, we found that participants' practices were (non-)mathematical in different ways. We also observed variability in the ways in which participants' practices could be linked to our models of suggested practices[12] (and, by extension, the assessment tasks that had been offered in A1). The next sections exemplify these results using our second interview task (T2): Show that the function $f(x) = e^x - 100(x - 1)(2 - x)$ has 2 zeros. We first present our model of the suggested practice associated with T2 (Sect. 7.4.1). Then we present our analysis of a selection of practices enacted by participants for solving T2, and their links to the suggested practice (Sect. 7.4.2).

[12] See Sect. 7.3 for the meaning of *suggested practice* in the context of this study.

7.4.1 Suggested Practice Associated with T2

The first stage of our methodology (Sect. 7.3) was an analysis of assessment tasks and teachers' solutions to those tasks offered to students in A1, which involved an identification of paths of tasks and a characterization of practices suggested by those paths. Table 7.2 depicts our model of a practice suggested by one of the paths we identified. The model is founded on the type of task: T, Prove that a function $f(x)$ has exactly n zeros on an interval I. Examples of tasks belonging to the path are shown in Table 7.1. Teachers' solutions to those tasks (which they made available to students for studying purposes) suggested that T be split into two sub-tasks: T_a, prove that $f(x)$ has *at least n* zeros on I and T_b, prove that $f(x)$ has *at most n* zeros on I.

In teachers' solutions, the most common technique for showing that f has *at least n* zeros was to locate n sign changes (τ_a). Teachers' solutions did not consistently include justifications beyond "by the Intermediate Value Theorem" (θ_a). None of such solutions commented on the usefulness of the IVT (e.g., "because the zeros of f cannot be found analytically") or on how the IVT works (e.g., "if f is a continuous function on an interval $[a, b]$ and $d \in (f(a), f(b))$, then there is some number $c \in (a, b)$ such that $f(c) = d$"). Also, the continuity condition necessary for applying the IVT was not always mentioned or justified in teachers' solutions. Moreover, these solutions did not elaborate on how students should look for sign changes (only listing the values of $f(x)$ that proved the existence of the sign changes), and the intervals and functions were always of a type such that sign changes could be easily found (by plugging in the endpoints of the interval, normally integers, and possibly some points in between, normally also integers and/or the midpoint of the interval). Accordingly, we wondered if students would have developed a non-mathematical practice and we constructed T2 in attempt to reveal this. We did not specify an interval, and we constructed $f(x) = e^x - 100(x - 1)(2 - x)$ so that plugging in integer values for x would lead to only positive values for $f(x)$ and, thus, would not be enough to locate sign changes (this was part of the deceptive nature of the task; see Sect. 7.3). In the absence of an interval, we expected students to work with the domain of definition of the function (i.e., to assume $I = (-\infty, \infty)$).

Note that with the way we phrased T2, we expected the participants of our study to identify it with T_a and for the interviewer to pose a follow-up question asking if participants' approaches would be different if they needed to show that the function has exactly two zeros. There was potential for a variety of responses. Indeed, in contrast with T_a, and as portrayed in Table 7.2, there were several techniques ($\tau_{b_1}, \tau_{b_2}, \tau_{b_3}$) illustrated in teachers' solutions for showing that f has *at most n* zeros. These techniques were illustrated on different subsets of T_b: e.g., when $n = 2$, as in T2, teachers' solutions suggested that students should argue by contradiction, assuming the function has 3 zeros, applying Rolle's Theorem twice to find that f'' should have a zero, and then calculating f'' to find that it actually has none.

Finally, within the path of assessment tasks related to T, we identified three equivalent task types (Table 7.1 shows some related tasks):

Table 7.2 Our model of a practice suggested by a path of assessment tasks offered in A1, based on the nature of the tasks themselves, as well as the techniques illustrated and technologies made explicit in teachers' solutions. The notation used aligns with that in the concept of praxeology: T for types of tasks, τ for techniques, and θ for technologies

Three equivalent types of tasks, to be solved by transforming them into a task of type T_2:

$\dot{T}$: Prove that $g(x)$ has exactly n fixed points on an interval I.

$\widetilde{T}$: Prove that $g(x) = h(x)$ has exactly n solutions on an interval I.

$\widehat{T}$: Prove that g and h intersect exactly n times on an interval I.

The main type of task, to be solved by solving two sub-tasks, T_{2_a} and T_{2_b}:

$\boldsymbol{T}$: Prove that a function $f(x)$ has exactly n zeros on an interval I.

Typically: $n \in \mathbb{N}$ is small (1, 2, 3, or 4) and I is of the form (a, b) or $[a, b]$ with $a, b \in \mathbb{Z}$.

The main technique and technology for solving sub-task T_{2_a}:

$\boldsymbol{T_a}$: Prove that $f(x)$ has at least n zeros on I.

$\boldsymbol{\tau_a}$: Find n sign changes of $f(x)$ on I.

Typically: calculate $f(a)$ and $f(b)$, and maybe $f(c)$ for c equal to integers in (a, b) or midpoints between integers.

$\boldsymbol{\theta_a}$: "By the Intermediate Value Theorem."

The three main techniques and technologies for solving sub-task T_{2_b}:

$\boldsymbol{T_b}$: Prove that $f(x)$ has at most n zeros on I.

$\boldsymbol{\tau_{b_1}}$: Show that f' is strictly positive (or negative) on n intervals I_i that form a partition of I. Illustrated for $n = 1$: Show that f' is strictly positive (or negative) on I.	$\boldsymbol{\tau_{b_2}}$: Assume that f has $n + 1$ zeros and derive a contradiction. More specifically, argue that f' has n zeros, f'' has $n - 1$ zeros, …, and f^n has 1 zero; and show f^n has no zeros. Illustrated for $n = 2$: Assume that f has 3 zeros, whereby f'' has 1. Show that f'' has no zeros.	$\boldsymbol{\tau_{b_3}}$: Illustrated for f a polynomial: Note that the degree (or order) of f is n.
$\boldsymbol{\theta_{b_1}}$: "If $f' > 0$ (or < 0) on an interval I, then f is strictly increasing (or decreasing) on I and can cross the line $y = 0$ at at most one point."	$\boldsymbol{\theta_{b_2}}$: "By Rolle's Theorem and by contradiction."	$\boldsymbol{\theta_{b_3}}$: "If f is a polynomial of order n, then f has at most n zeros."

1. $\dot{T}$, Prove that a function $g(x)$ has (exactly) n fixed points on an interval I.
2. $\tilde{T}$, Prove that an equation $g(x) = h(x)$ has (exactly) n solutions on an interval I.
3. $\widehat{T}$, Prove that two functions, g and h, intersect (exactly) n times on an interval I.

In teachers' solutions, tasks of these types were solved by transforming them into a task of type T (see the first and second rows of Table 7.2). For example, to prove that $e^x = 100(x - 1)(2 - x)$ has exactly 2 solutions, students were shown to introduce a new function, $f(x) = e^x - 100(x - 1)(2 - x)$, and to argue that f has exactly 2 zeros using the techniques mentioned above. None of the teachers' solutions leveraged the equivalence in the other direction (e.g., thinking about the intersections of $g(x) = e^x$ and $h(x) = 100(x - 1)(2 - x)$ could lead to a graphical solution for T2). Hence, we did not expect students to spontaneously construct such a solution.

7.4.2 *Practices Enacted by Participants for Solving T2*

The second stage of our methodology (Sect. 7.3) involved the implementation of a task-based interview, including T2 (Show that the function $f(x) = e^x - 100(x - 1)(2 - x)$ has 2 zeros.), with 15 students who had successfully completed A1. In what follows, we provide selected examples of our analyses of the interview data, to illustrate three different ways in which students' practices revealed themselves to be (non-)mathematical: how students identified T2 with a type of task and technique (Sect. 7.4.2.1), how students implemented their chosen technique for accomplishing T2 (Sect. 7.4.2.2), and how students explained their chosen technique for accomplishing T2 (Sect. 7.4.2.3).

7.4.2.1 The Identification of T2 with a Type of Task and Technique

When presented with T2, eleven[13] out of 15 participants almost immediately indicated that they would use the IVT. For example, S4's first words after receiving the task were: "Ok. I remember this being with the Intermediate Value Theorem." In our analysis, we found examples of participants who seemed to be drawn to the word "zeros" as the way of identifying T2 with a type of task necessitating the use of the IVT. For instance, S6 explained:

> Show that it has zeros is IVT for sure. [...] like if it's a continuous function, [. . .] you plug in some values, you get a negative, then positive, then negative, it must cross the [the x-axis], at some point, it does have a zero.

[13] Of the four participants who did not immediately speak of using the IVT, one (S15) spoke about needing to use a "theorem" but could not remember which one, one (S2) immediately took the derivative of f, and the other two were S9 and S3 mentioned below.

S8's actions and utterings also suggested that to show a function has zeros, one applies the IVT: "I understand that the IVT works like that. [. . .] If I find one that's positive and one that's negative, I [can] find a zero." S11 explained similarly:

> My logic with finding zeros is finding a value before and finding a value after that point at which it's equal zero that are alternating signs. And the only theorem that we have that talks about that is [. . .] the Intermediate Value Theorem. So that's why I instantly thought of that.

These students did not stop to reflect on the properties of the function in T2 to inform their decision to use the IVT, their focus seeming to be exclusively on the task being about "zeros." It is in this sense, or for this reason, that we consider that they were enacting a non-mathematical practice. We note, importantly, that in the context of the assessment tasks given in A1, all tasks about "zeros" could be easily solved using the IVT (Sect. 7.4.1).

In comparison, there were examples of participants who considered the nature of the function $f(x) = e^x - 100(x - 1)(2 - x)$ in T2 to support their choice of an IVT-inspired technique and did not focus solely on the fact that it was a task about "zeros." After being triggered to use the IVT, S1 stopped to note about f: "if e^x wasn't here, it'd be pretty easy to find the two zeros. But since there's [e^x], we have to do the non-high school way," meaning, as S9 did, that one could not algebraically solve the equation $f(x) = 0$. S9 said: "It's not as simple as just isolating x. [. . .] So, in this case, we have to use one of those theorems we saw in [A1]." These participants showed some awareness that choosing an IVT-based technique is appropriate for tasks involving a function whose zeros cannot be found using other, simpler, analytical or algebraic techniques. These students were considering the task in its entirety and not focusing exclusively on the fact that it was about the zeros of a function. We consider their identification of the task to be mathematical (as opposed to the students referred to in the previous paragraph[14]).

S3 also considered the nature of the function in T2: "a classic example where you cannot use [. . .] easy things to find the root." The difference with S3, when compared to all other participants, is that he transformed T2 into the equivalent tasks "show that $e^x = 100(x - 1)(2 - x)$ has two solutions" and "show that the graphs of e^x and $100(x - 1)(2 - x)$ have two intersections"; and he developed an unexpected (see Sect. 7.4.1) solution based on proving the properties shown in his sketch (Fig. 7.1). We infer that S3 identified T2 with three equivalent types of tasks (about zeros of functions, about solutions of equations, and about intersections of graphs) and chose a technique based on essential mathematical properties of $f(x) = e^x - 100(x - 1)(2 - x)$; namely, that it is the sum of two functions whose graphical properties are known (to A1 students). This, we concluded, was indicative of the development of a mathematical practice.

[14] Our judgement is that their identification of the task exclusively on the fact that it is about the zeros of a function is not worthwhile from the perspective of scholarly mathematics.

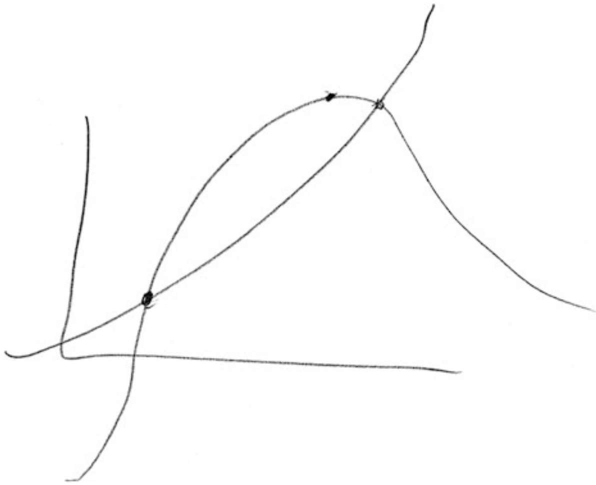

Fig. 7.1 S3's reinterpretation and solution of T2 as a task about intersections of graphs

7.4.2.2 The Implementation of a Technique to Accomplish T2

Of the twelve participants who tried to implement an IVT-based technique (i.e., finding two sign changes in $f(x)$), eleven eventually struggled to complete the task, facing the expected challenge (described in Sect. 7.4.1) of finding only (or mainly[15]) positive values for $f(x)$. All eleven of these participants seemed to, at some point, choose x values "at random" (perhaps considering ease of calculation or variance in chosen values), with several explicitly indicating taking this approach. In our analysis, we found examples of participants who seemed to be choosing their next step in carrying out their technique simply by trying to remember what had worked when solving tasks from A1. After checking the limits of f at infinity,[16] S1 said: "I'd like to see if there's a negative. [. . .] So, I'd just try random numbers." He used his calculator to do so (e.g., calculating $f(0)$, $f(100)$, $f(-5)$), finding only positive values, and explained his choice to go "at random" by saying that he "forgot the better way." As another example, S11 used a calculator to find $f(0)$, $f(1)$, and $f(-1)$, and explained: "usually what we saw in [A1] was that. . . [. . .] the interval in which [the function] is alternating between negative and positive [values] is like somewhere in a close range of zero." We infer that S11 was selecting x values, not by reasoning about the mathematical properties of the given function f, but based on his memory of the kinds of x values (close to zero) that had resulted in sought-after sign changes when solving tasks in A1. S11 later described a more specific list of steps he would have expected to work had T2 included the specification of an interval (like in the assessment tasks from A1; see Sect. 7.4.1):

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[15] Some participants found negative values for f(x) due to calculation errors.

[16] This is not something we had anticipated based on our model of a suggested practice and so we do not know where this first step came from. Since S1 was not the only one to do it, perhaps it was shown to students in lectures.

> Like if you tell me [. . .] it's not this function, it's another function,[17] and you tell me [the interval is] zero to five [writing [0,5]], then at that point you can just plug in the values [...] Zero, one, two, three, four, five. [. . .] And you'll see which one alternates between negative and positive. And you'll figure out how many zeros you have.

The steps S11 described would have worked to solve assessment tasks in A1, but they do not make sense from a mathematical point of view in the context of T2. Implementing the IVT in this way illustrates another way in which practices can be non-mathematical.

In response to their struggle, nine participants (including S11) eventually indicated a (possible) change of approach to looking for sign changes in *f*, based on reasoning about mathematical properties of the given function (Table 7.3).[18] Still, there are interesting differences in the nature of these approaches. For example, (1) and (2) in Table 7.3 rely on local studies of the function's monotonicity to make predictions about whether it will change sign somewhere nearby (they involve a quantitative study of *f* that does not take advantage of its essential features). In comparison, (3), (4), and (5) are based on a qualitative study of *f* to try to understand its global behaviour, although (3) (like (1) and (2)) still includes a degree of arbitrariness in the choice of *x*. Only one participant (S12 – see (4) in Table 7.3) implemented the IVT-based technique solely by performing a qualitative mathematical study of *f* (i.e., by reasoning mathematically[19]). This, we concluded, was indicative of the development of a mathematical practice.

7.4.2.3 The Explanation of a Technique for Accomplishing T2

In our analysis, we found examples of students who seemed to explain their IVT-inspired technique based solely on the technique being a normal part of what occurred in A1 (what we refer to as "established experiences" from the learning environment, following Lithner, 2000). In reference to his use of the IVT for solving T2, S1 explained: "I know that [the IVT is] applicable in this situation.[20] [. . .] Why do I know? Well, I'm cheating. Cause I know that that's how we used to solve it [in A1]. [. . .] Cause we did it in class." S4 said similarly: "It's just having repeated it so often, whether it be assignments, class, practice, . . ." From this, and the interactions that occurred during the interviews, we interpret that S1, S4, and other participants did not actually know why the IVT-inspired technique solves T2:

[17] S11 made the specification that "it's not this function, it's another function" when giving the example of the interval [0, 5] because he had already tried plugging in $x = 0, 1, 2, 3, 4, 5$ and had not found the two zeros. This said, the zeros for $f(x)$ do indeed occur on [0, 5].

[18] This may be an example where students were exhibiting "practices in development" (see the last paragraph of Sect. 7.3).

[19] As in the use of the adjective "mathematical" in this study, "mathematically" here refers to a way of reasoning that is acceptable and worthwhile by the institution of scholarly mathematics.

[20] S1 was one of six participants who did not mention the continuity condition required for applying the IVT during his solving of T2.

Table 7.3 Models of participants' approaches and reasoning for choosing x values to find sign changes in $f(x) = g(x) - h(x)$, where $g(x) = e^x$ and $h(x) = 100(x - 1)(2 - x)$. * indicates the participant only described (did not try) the approach. Bold indicates the participant successfully solved the task using the approach

Approach	Participants	Examples of Reasoning (after finding only positive values for $f(x)$)
(1) Pay attention to how the value of $f(x)$ is changing as x changes.	S4, S5, **S6**	S6: If f is continuous and I see that $f(x)$ is getting closer to zero as I change x, then I'm getting closer to finding a sign change.
(2) Study the sign of $f'(x)$ for particular x.	S10, S11*	S11: If you take the derivative at different points, you can see if the slope of the tangent line is negative. Then you know the function is decreasing at that point. And you might want to check the intervals around that.
(3) Compare the growth of $g(x)$ and $-h(x)$.	**S8**, S11*, S14*	S8: Since $g(x) = e^x$ grows much quicker than any polynomial (like $-h(x)$), the sign changes will occur for small values of x.
(4) Compare the signs of $g(x)$ and $-h(x)$.	S4, **S5, S9, S12**	Since $g(x) = e^x$ is always positive, we need to look for where $-h(x)$ is negative: • S4: Expanding it to $100x^2 - 300x + 200$ shows that x needs to be positive. • S5: Looking at $-h$, we see that if x is more than 1 and less than 2, the negative signs won't cancel out. • S9, S12: $-h$ is an upward-facing parabola with roots at 1 and 2, and minimum (or most negative point) at 1.5.
(5) Graph g and h.	**S7**	Sign changes will occur where the graphs of g and h cross. In this case, the graphs are known: It is just e^x and a downward-facing parabola with roots at 1 and 2, and maximum at 1.5.

They know it is a task about "zeros," they know one applies the IVT in that case, and they follow learned steps to implement the technique. Their logos is of non-mathematical nature, hence illustrating another way in which students' practices may be non-mathematical.

While solving T2, twelve participants eventually considered the task of showing that $f(x) = e^x - 100(x - 1)(2 - x)$ has *at most* two zeros (T_b). Several techniques were exhibited, reflecting the diversity in the suggested practice (Table 7.2). This contributed to enrich the collection of examples of what we deemed (non-) mathematical explanations of selected techniques. For instance, to solve T_b, five participants (S1, S2, S7, S13, and S15) considered using Rolle's Theorem (RT) or exhibited a technique based on it (somewhat, though not exactly, reflecting τ_{b_2} in Table 7.2), for which the underlying explanation seemed to be limited to citing the theorem (void of understanding what the theorem says or how it can be used to afford a technique). S15 recalled the complete statement of RT, but could not see how to use it to produce a technique for solving T2. In comparison, S1, S2, S7, and S13 chose to show that $f'(x) = e^x + 200x - 300$ has exactly one zero, based on "a theorem." According to S1, the theorem says that "if the derivative [function] has one [zero], [. . .] the [function] has at most two [zeros]." According to S2: "It says that if you have n zeros for $f(x)$, then [. . .] you have $n - 1$ zeros for the derivative." No participant provided a mathematical explanation connecting these two statements (i.e., why, mathematically speaking, RT – or a generalized version of it – produces the technique). We infer that the students' references to RT (or "a theorem") were disconnected acknowledgements of a piece of theory, which remained a static part of their practice that they were not able to use. This is what we mean by explaining a technique based on inert knowledge; another example of how an explanation for a technique may be non-mathematical. As in the example above, this kind of explanation ("by a theorem") aligns with our model of the suggested practice in relation to T2 (Table 7.2).

In contrast, we found examples of participants who seemed to understand and use elements of mathematical theory to produce and explain a technique for solving T_b. Five participants seemed to solve T_b by implicitly or explicitly turning to theorems about what f' or f'' tell us about the shape of f's graph. S12, for example, devised a technique reflecting τ_{b_1} (Table 7.2): Expecting f to have a global minimum (based on his previous work, including a sketch of f), S12 planned to locate the minimum by finding x_m such that $f'(x_m) = 0$; and then argue that $f'(x) < 0$ (f is strictly decreasing) on $(-\infty, x_m)$ and $f'(x) > 0$ (f is strictly increasing) on (x_m, ∞). S12 got stuck implementing his technique when he realized he could not analytically solve $f'(x_m) = 0$. This said, he gave the following mathematical explanation for how and why the technique worked:

> If a function is [. . .] increasing strictly, it means that [. . .] if I have two points, a and b, where $a < b$, then [. . .] $f(a) < f(b)$. So, if I have some point that is a zero, [say b], [the value of f at] any point that is greater than b is going to have to be greater than zero. So that shows that no value c greater than b is actually going to give something that's a zero in our function. Similarly, no value less than b will give us a value of zero. [. . .] The same is true for decreasing functions.

In this argument, S12 does not rely on his personal understanding alone; rather his understanding seems to be shaped by the mathematical theory of functions (e.g., the definitions of increasing or decreasing functions and the definition of a zero of a function). This is an example of what we mean by clarifying, questioning, and verifying one's own understanding with mathematical theory, which we see as one way in which students' practices may be mathematical.

7.5 Discussion

At the beginning of this chapter, we posed two research questions:

1. What is the nature of the practices developed by students in a first Analysis course?
2. How might these practices be shaped by the nature of the tasks offered to students in the course?

Considering the above results, we discuss possible elements of response, critically reflect on our study (its contribution and limitations), and propose some directions for future work.

7.5.1 Answer to the Research Questions and Contribution of the Study to Research in University Mathematics Education

Our study involved task-based interviews (see Sect. 7.3) with 15 students after they successfully completed a first Analysis course. In our analysis of these students' task solving, we found examples of practices that were not mathematical (see Sects. 7.2 and 7.4.2). These kinds of practices have been identified in research on Calculus courses (e.g., Hardy, 2009; Lithner, 2000; Orton, 1983; Selden et al., 1994). Given the procedural focus of those courses, the development of non-mathematical practices is perhaps not surprising. It is surprising, however, that students may still be developing such practices in more advanced theoretical courses such as A1, often taken in the second-last year of mathematics programs leading to graduate work. Some studies (e.g., Weber 2005a, b) have hinted at this possibility; our study contributes a focused theoretical and empirical exploration of this issue. Using the notion of "path to a practice" (see Sect. 7.2) contributed by our study, we conjecture that the development of non-mathematical practices may be permitted and encouraged (for any student) by paths of tasks that do not help students to identify relevant mathematical features of the tasks, and where it is not necessary to learn how to mathematically explain a technique for a mathematical type of task (e.g., the path described in Sect. 7.4.1).

This said, we also found that some students enacted practices that, while non-mathematical from the perspective of this study, could be considered mathematical in some way (e.g., the student is paying attention to mathematically relevant aspects of the task to choose a technique, but does not have mathematically sound discourses; or vice versa). This could be empirical evidence of students going through the expected shift (Winsløw, 2006), from a more procedural focus (encouraged in Calculus) to a more theoretical focus (encouraged in Analysis).

The differences we found in the nature of students' practices also seemed to reflect different ways in which students' practices may be linked to (or influenced by) the assessment tasks given in A1 (as suggested by comparing results from Sects. 7.4.1 and 7.4.2). This may further reflect our expected differences in "a practice suggested by a path" depending on the observer (see the last paragraph of Sect. 7.2.2); that is, it is possible that different students abstracted different practices from the paths that we identified, or that they made different connections among tasks than we did (forming different kinds of paths). We have begun trying to make sense of this by characterizing different general ways in which students may position themselves within their courses (e.g., Broley, 2021).

7.5.2 *Limitations and Directions for Future Research*

It is interesting to note that the notion of "path to a practice" arose within our study context, which followed the paradigm of "visiting works" (Chevallard, 2015). As evidenced by Section 4 of this book, this paradigm is being challenged by innovative approaches such as inquiry-based mathematics education (Artigue & Blomhøj, 2013). Recently arising in the ATD is a variation that proposes to organize learning around another kind of "path": *study and research paths*, which start with an open-ended question that the teacher and students seek to answer through studying existing works and researching new questions (e.g., Florensa et al., 2019). One direction for future work could be to analyse the nature of the practices students develop while engaging in such paths and to theoretically reflect on how they relate to the notion of "path to a practice."

Our study had limitations, which point to other future directions. For instance, we analysed only some of the tasks offered to students: Future work could look at the role, if any, of the tasks students encounter outside assessment or teachers' lectures in the paths they identify for themselves, or the influence of tasks (or paths) from other courses. Another limitation is that our task-based interview was not designed to distinguish between different practices' *states*; e.g., "practices in development," "practices in adaptation," or "practices in evolution," which could be the focus of future work.

Based on the conclusions of our study, namely that students develop non-mathematical practices, another future direction could be design-based research to create and evaluate learning experiences for the development of mathematical practices. The examples in this study – of students' non-mathematical practices – could inform the design of tasks for that purpose. More detailed analyses of how students form paths and abstract practices could also provide interesting and important empirical and theoretical insights.

Taking an institutional point of view reminds us of the complex web of constraints faced by teachers and students (examination procedures, time limitations, curricular expectations). One participant from our study gave a poignant reflection, highlighting how the larger context may encourage the development of non-mathematical practices:

> I feel that we're grinded to do so many questions really quickly. So, we need to associate problems to a solution [...] really fast. [...] Cause I don't really have the time to analyze the problem and try different things during an exam. So, I grind problems at home. And when I get in an exam, I see the problem and I say, "Ok, that's exactly the kind of problem. . .it goes down to this."

Although we claim that one overall aim of university mathematics is the eventual development of practices reflecting the aimed (mathematical) profession, a pertinent question raised by our study, and many others before ours (some cited here), is: To what extent is this aim achievable under existing constraints?

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