Chapter 1
Practice-Oriented Research in Tertiary Mathematics Education – An Introduction

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Abstract This chapter first outlines the genesis of the book. We briefly describe the development of university mathematics education as an international field of research and development. This includes the role of the khdm ("Kompetenzzentrum Hochschuldidaktik Mathematik"; Centre for Higher Mathematics Education) and one of its founding directors, Reinhard Hochmuth, to whom this book is dedicated.

We then consider five practice-oriented topics for research in university mathematics education: the secondary-tertiary transition, university students' mathematical practices, teaching and curriculum design, university students' mathematical inquiry, and mathematics for non-specialists. These topics represent main areas of recent research and development.

The five topics appear in the book as sections, each with several chapters. The sections and their contents are introduced in the final parts of this first chapter. For each section, we sketch the connection between its chapters and the specific field of research. We further provide a brief description of each chapter in terms of theoretical and methodological approaches, as well as of the results presented.

Keywords Secondary-tertiary transition · University students' mathematical practices · Teaching and curriculum design · University students' mathematical inquiry · Mathematics for non-specialists

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1.1 Context of This Book

This book is devoted to current practice-oriented research in tertiary mathematics education. The birth of every book has its occasion, its reasons, and its history. The occasion of this book consists of two anniversaries: 10 years of work at the "Kompetenzzentrum Hochschuldidaktik Mathematik" (khdm; Centre for Higher Mathematics Education) starting in the fall of 2010, and the 60th birthday of Reinhard Hochmuth, one of the two founding directors of the khdm, in March 2021. We, as editors invited international colleagues, including persons from the khdm, Reinhard's and the khdm's current international scientific network, and beyond, to contribute to this volume. Transcending these occasions, the editorial team discussed the specific orientation of the book, which should show a panorama of current research that is practice-oriented and relevant for both university mathematics teachers and scholars in the growing field of tertiary mathematics education research.

In keeping with this orientation, the book presents practice-oriented research covering a broad range of research topics in tertiary mathematics education. This reflects vital international activities of a quickly developing field. It is meant to help both researchers and practitioners to get inspiration on what good teaching and learning may mean, and how it may happen. The forward-looking nature of this volume has strong potential to influence the further development of the field. The book is organized into the following five sections:

Section 1: Research on the secondary-tertiary transition
Section 2: Research on university students' mathematical practices
Section 3: Research on teaching and curriculum design
Section 4: Research on university students' mathematical inquiry
Section 5: Research on mathematics for non-specialists.

Tertiary mathematics education has been thought about for as long as university mathematics exists, and individuals have published research on issues in tertiary mathematics education for decades. A crucial initial compilation of early work in university mathematics education research is the edited volume on advanced mathematical thinking (Tall, 1991) and the result of an ICMI study (Holton & Artigue, 2001). However, when the khdm started its work in late 2010, the field of university mathematics education as a large research community was still in its infancy in most countries. An exception was the RUME community in the United States, whose annual conferences started in the late 1990s.

Since 2010, national and international structures have emerged in which scholarly exchange occurs. In Europe, working groups on tertiary mathematics education have been present at the CERME conferences (held by ERME, The European Society for Research in Mathematics Education) since 2005 (see Winsløw et al., 2018 for a more detailed outline). In 2016, the International Network for Didactic Research in University Mathematics (INDRUM), which is closely associated with ERME, held its first conference devoted exclusively to the didactics of university mathematics.

Since 2015, the *International Journal of Research in Undergraduate Mathematics Education* (IJRUME) has been published. It is the first journal with a focus on undergraduate mathematics education research.

In addition to these international activities, several national centers for tertiary mathematics education have been established. In the United Kingdom, the SIGMA Centre for Excellence in Teaching and Learning was founded in 2005 (https://www.sigma-network.ac.uk/). In early 2000 the Mathematical Association of America (MAA) formed the Special Interest Group of the MAA on Research in Undergraduate Mathematics Education (SIGMAA on RUME). The MatRIC Center for Research, Innovation, and Coordination of Mathematics Teaching was founded in Norway in 2014.

The "Kompetenzzentrum Hochschuldidaktik Mathematik" (khdm; Center for Higher Mathematics Education) was founded by Rolf Biehler and Reinhard Hochmuth at the universities of Kassel and Paderborn in Germany in late 2010 by winning a competition for grants from the Volkswagenstiftung and the Stiftung Mercator. Today it is a joint scientific institution of the universities of Hannover, Kassel, and Paderborn. One mission of the khdm was to establish a national research network, relate it to international developments, and contribute to the development of "Didactics of Mathematics in Higher Education as a Scientific Discipline," as was the title of the 2015 international khdm conference. The khdm organized national conferences in Kassel 2011 (Bausch et al., 2014) and Paderborn 2013 (Hoppenbrock et al., 2016) on tertiary mathematics education, and international conferences in Oberwolfach 2014 (Biehler et al., 2014; Biehler & Hochmuth, 2017) and Hannover-Herrenhausen 2015 (Göller et al., 2017).

Reinhard Hochmuth, to whom we dedicate this book on the occasion of his 60th birthday, co-directed the khdm from the beginning. Reinhard, who also holds a degree in psychology, was a professor of mathematics at Kassel University. Since 2014, he is a professor of mathematics education at the Leibniz University Hannover. He is currently co-directing the khdm with Andreas Eichler (Kassel University) and Michael Liebendörfer (Paderborn University), who took over from Rolf Biehler in 2020. Michael was the first doctoral student of Reinhard in the field of tertiary mathematics education. Since its foundation, Reinhard Hochmuth has been one of the khdm's managing directors and has been intensely involved in the further development of higher mathematics education and its national and international structures, particularly through the activities of the khdm, mentioned above. In addition, he has been involved in international conferences and networks in various forms. For instance, he is the host for the INDRUM meeting 2022 at Hannover (Germany).

Reinhard represents the collaboration of mathematics and mathematics education research needed for the fruitful development of tertiary mathematics education. He contributed valuable perspectives to the field, for example, on higher mathematics beyond the first year of study (Hochmuth & Schreiber, 2016) and the role of society for mathematics (Hochmuth & Schreiber, 2015): mathematics always takes place in social structures and whoever deals with mathematics also always contributes a piece to the reproduction or change of these structures. Reflecting on this relationship to society is one of his particular perspectives (and will be reflected in Chap. 3 of this book).

Thus, his activities and this book aim in the same direction. The book's five sections also reflect parts of Reinhard's research contribution to the field. We point to some exemplary contributions. Transition problems (section 1), especially the design of mathematical bridging courses, were his early focus before the khdm was founded (Biehler et al., 2012). Recently, the WiGeMath project, whose coordinating director was Reinhard, made a comprehensive study of transition problems and possible remedies by bridging courses, transition lectures, mathematics support centers, and additional supporting measures such as e-learning elements (Hochmuth et al., 2022). Students' mathematical practices (section 2), in particular from the perspective of the Anthropological Theory of the Didactic (ATD), have become a major research topic of Reinhard (Hochmuth & Peters, 2021; Bosch et al., 2021). Research on teaching and curriculum design (section 3) were in the center of the LIMA and KLIMAGS project, where an innovative course for primary and secondary teacher students was developed with a specific focus on qualifying mathematical tutors (Biehler et al., 2018; Hochmuth et al., 2021). Research on university students' mathematical inquiry (section 4) was a focus of the Platinum Project that Reinhard has been co-directing (Gómez-Chacón et al., 2021). Reinhard also did research on mathematics for non-specialists (section 5): He co-directed the Kom@ING project that analyzed the required and attained mathematical competencies in courses for engineering students (Peters et al., 2017), where he was particularly interested in the field of signal theory (Hochmuth et al., 2014, Hochmuth & Peters, 2020, 2021).

1.2 Overall Structure of the Book

The questions and methods of tertiary mathematics education are manifold. For a long time now, the transition from school to university has been a major topic, often experienced as particularly challenging in mathematics. For example, based on psychological theories, researchers analyze how students experience and act in the transition process. Universities have set up various support structures such as bridging courses or learning support centers. A growing body of knowledge theoretically underpins the design of such structures and supplies instruments for evaluating their success.

In the transition and also later, another part of the research focuses on how to support students in becoming an active part of mathematical practices. In addition to focusing on mathematical concepts and mathematical theories, research focuses on mathematical practices such as proving, which play a role both in the transition and in later studies. While such research mainly analyzes existing courses and practices, also many studies constructively design teaching scenarios and material and study the learning processes that they initiate. These studies result in scientifically based courses or curriculum conceptions. Inquiry-based approaches have been developed that often challenge traditional teaching and learning models. We find experiments that show us how we could radically transform traditional teaching. While current research is mainly oriented toward future mathematicians or mathematics teachers, research is also making remarkable progress in mathematics for non-specialists. At many universities, the STEM (science, technology, engineering, and mathematics) domains and economics are the domains where improved teaching and learning processes can reach a much larger number of students than in the courses for mathematics majors. The design of these courses poses new challenges because of possible links to the workplace and different mathematical practices in the mathematics courses and the engineering and economics courses.

1.2.1 Section 1: Research on the Secondary-Tertiary Transition

The transition from school to university involves many disruptions, such as study goals, course designs, working techniques, the didactical contract, and the nature of mathematics (Gueudet & Thomas, 2020). The problems only become visible at the university, where innovative measures are often taken. To understand these transitional difficulties, however, we need to look at both sides, school, and university.

This section starts with two chapters that deepen our understanding of the students experiencing the secondary-tertiary transition. Göller and Rück (Chap. 2 of this book) examine first-year students' achievement emotions and their interaction with self-regulated learning. To this end, they first integrate both aspects into one model showing the constant interaction of emotions and activity and their relation to both mastery of the content and personal well-being. Based on 21 interviews with first-year mathematics students, Göller and Rück illustrate the ways students experience important emotions like joy or hopelessness and their connections to self-regulated learning. The chapter highlights the importance of students perceiving control over their learning and valuing the new mathematics for not only mastery but also well-being. Given the disruptions during the secondary-tertiary transition, it seems typical to struggle with both points. The model can thus explain why students start coping and adjust their own goals based on their emotional experiences. Göller and Rück therefore open up a new and promising perspective on the secondary-tertiary transition.

Ruge (Chap. 3 of this book) takes a critical perspective on the secondary-tertiary transition for student teachers. She first outlines the subject-scientific approach (Hochmuth, 2018; Holzkamp, 2013) that puts a strong emphasis on the societal dimension of individual experiences and behavior. This approach is then used to construct a perspective on students' feelings about mathematics, society and the teaching profession. Ruge draws on selected interview data with teacher students to illustrate their unease with being identified as mathematicians and the current state of academic mathematics. She also reports how the research community reacted to her observations. This unease reveals a great learning potential and has been studied primarily in belief research to date. Ruge argues for a different perspective based on a subject-scientific reinterpretation of beliefs-research, conceiving beliefs less as an individual trait and taking social and societal dimensions into account. She extends the research of Skott (2019) and advocates the common goal of student teachers and mathematics education scholars to promote a humane mathematics-society relationship. This is discussed for universities regarded as both teacher education institutions and research institutions on mathematics education.

The two other chapters focus on teaching in the secondary-tertiary transition. Corriveau (Chap. 4 of this book) presents findings from an inter-level community formed by mathematics teachers from secondary and post-secondary levels. The community headed for developing practices that smoothen the transition. This allowed comparing teachers' ways of doing using an ethnomethodological approach. Firstly, the findings highlight two very distinct territories concerning the use of contexts in secondary and post-secondary teaching. Secondary mathematics is strongly contextualized, whereas post-secondary mathematics is only illustrated in contexts. This affects both meaning and reasoning. It adds a new perspective on differences in the ways mathematics is taught. The chapter further demonstrates the power of cross-institutional collaborations to both understand and improve the transition. Teachers from both territories could explicate the way they do mathematics and reflect on the others' ways. This led to a collaboration that tackles the transition problem from both sides. The chapter thus also provides a good way of implementing a fruitful collaboration across the institutions.

Finally, Liebendörfer et al. (Chap. 5 of this book) present a framework developed to describe the goals of support measures in the transition between school and university. The underlying observation is that recent research has brought up various measures like bridging courses, redesigned lectures, and mathematics learning support centers, which all address the transition problems in their own ways. A first step to structuring and understanding this field is reconstructing the goals, which are often implicit as staff may focus on their concrete actions in teaching and support. After describing the framework, Liebendörfer et al. illustrate the goals of several pre-university bridging courses, redesigned lectures, and mathematics learning support centers. This may clarify both the specific roles that these kinds of support measures have and the variability within these categories. The framework thus helps to understand the particular directions of support measures and compare or evaluate them. It was developed in the WiGeMath project led by Reinhard Hochmuth.

1.2.2 Section 2: Research on University Students' Mathematical Practices

The section deals with research studies that focus on mathematical practices. Practices can be understood as closely related to mathematical processes such as proving and defining. A more comprehensive theoretical framework for studying practices in mathematical institutions has been developed by the Anthropological Theory of the

Didactic (ATD). Its wider notion of praxeology includes the components theory, technology, technique, and task. This and other theoretical frameworks are used in this section.

The first set of papers is concerned with reconstructing (problematic) practices of students that have evolved in an institutional context and explaining these practices by the implicit didactical contract of a lecture that may also be specified in expectations of lectures and tasks assigned to the students. The second set of papers focuses on the practices of proving and defining as characteristic processes of university mathematics.

The section has epistemological, interventional, and observational studies that characterize, reconstruct, observe or intervene to elaborate the participation in advanced practices in institutional settings. Some of the studies are design studies that address learning challenges and opportunities, while others are theoretical-conceptual studies that present an a priori analysis of certain mathematical practices.

The first set of papers is concerned with reconstructing institutional practices of mathematics that are relevant for developing students' practices. Bašić and Šipuš (Chap. 6 of this book) reconstruct the practices expected from students and the didactical contract in a lecture on multivariable calculus from the perspectives of ATD, the Theory of Didactical Situations, and the theory of the didactical contract, studying students' problem-solving results and related reflections on their difficulties and problems of learning and understanding. Moreover, the authors conducted related interviews with the lecturers. The implicit didactical contract is reconstructed from both sources, and suggestions for improving the teaching-learning system are developed. Broley and Hardy (Chap. 7 of this book) take a similar theoretical stance using ATD and reconstruct students' practices in a course on real analysis, using various sources of observation, including assessment. They found a substantial number of so-called non-mathematical practices related to students' orientation to minimal requirements for success and considering only superficial properties of previous exercises. Discovering these practices is essential for improving teaching and learning toward a deeper mathematical understanding.

The second set of papers is concerned with proving and defining the interrelation between these mathematical practices and various educational levels: secondary level graduates as beginning university students, students in their transition process in the first semester, and practices relevant for more advanced students. Lew, Fukawa-Connelly, and Weber (Chap. 8 of this book) argue in a theoretical chapter that when mathematicians lecture, they not only cover mathematical content but also model how students should learn mathematics. They analyze a corpus of 11 lectures in various advanced mathematics courses to investigate how mathematicians present the definitions of concepts and gain insight into how mathematicians may expect students to learn from lectures. They highlight how the instructors modeled what it means to study a concept and its definition and argue that students are expected to engage in independent study outside of class.

Ostsieker and Biehler (Chap. 9 of this book) focus on the concept of convergence, where in a design study, students are supported in re-inventing this concept and its definition. Meta-knowledge on defining is needed, and a substantial and rich concept image of the convergence of a sequence. The research is situated on the transition between school and university. The learning environment consists of examples and non-examples of convergent sequences, a task, and expected obstacles with prepared supports for each expected obstacle. The learning environment was developed in the Design-Based Research paradigm, conducted twice, and analyzed and refined each time. In this chapter, the analysis focuses on the changes in the formulation of the initial task for the students, that were made based on the results of the analysis of the first two implementations.

Kempen (Chap. 10 of this book) investigates the practice and meta-knowledge on proof and proving of 12 high-school graduates, with a view towards what proof conceptions were developed in school mathematics and have to be taken into account when beginning students at university are being introduced to the proving practices at the university level. The author conducted task-based interviews focusing on learners' usage and assigned meaning of statements with regard to their embeddedness in a local deductive organization, their epistemic values, and their respective effects on the conclusion's modal qualifier. While all graduates accept definitions and rules for term manipulation, there is no consensus concerning the statements involved. Furthermore, the individuals' epistemic values concerning the statements involved affect their usage in a chain of arguments and the individuals' evaluation of the conclusion.

Whereas the preceding papers focus on defining or proving, a new analysis of the interrelation between proving and defining on a general level in university mathematical practices is the central focus of Durand-Guerrier's (Chap. 11 of this book) paper. The main goal of this chapter is to underline, from an epistemological point of view, the relevance of engaging university students in intertwined proving and defining practices. The chosen examples are *real numbers* and *infinity*. The intertwined practices of proving and defining are taken from the case of the construction of irrational numbers by Dedekind (1872) and Cantor (1874). Next, the author presents an example of a situation involving R-completeness versus Q-incompleteness that has the potential to foster students' engagement in intertwined proving and defining practices. These intertwined relationships are further explored from a didactical point of view concerning the relation between practices of enumeration, the definition of infinite sets, and diagonal proofs that the set of rational numbers is denumerable while the set of irrational numbers is not.

1.2.3 Section 3: Research on Teaching and Curriculum Design

Teaching at the university level is increasingly studied by researchers in mathematics education (Biza et al., 2016). The first three chapters in this section present research contributing to new understandings of university teachers' practices and professional development, drawing on a great variety of theoretical approaches and associated methods.

Jaworski and Potari (Chap. 12 of this book) use Activity Theory (Leont'ev, 1979) to analyze the interactions between a tutor and her students in terms of tensions and contradictions arising when the tutor tries to engage students in mathematics meaning-making. The authors video-recorded and transcribed tutorial sessions for first-year mathematics students in a university in the UK. The teacher was the first author of the chapter (Jaworski) and was actively involved in the data analysis, and this was essential for identifying her goals. This analysis leads to the observation of different objectives of the teacher's activity concerning desired students' actions (e.g., listen to each other and build on what another person expresses) or the teacher's own activity (e.g., listen to the students and discern meaning from what they say). The authors observed positive outcomes for the students (e.g., expressing their informal ideas or valuing the collaboration with peers). They also identified tensions and related contradictions, e.g., between the teacher's guidance (needed for achieving the task) and students' autonomy. These contradictions make the development of inquiry-oriented practices challenging; the authors claim that facing this challenge requires a continuous professional reflection of the teacher.

Mesa (Chap. 13 of this book) studies another aspect of university teachers' professional activity: their lesson planning activity (in the context of their ordinary professional activity for 'traditional' courses). The theoretical framework in this chapter is the documentational approach to didactics (DAD, Trouche et al., 2020), which introduces a difference between resources used by the teacher and documents developed along with their activity. Mesa focuses in this chapter on the "lesson notes" document. Twenty-one post-secondary teachers in 15 different universities in the United States were interviewed about their use of resources for their lesson planning activity (for calculus, linear algebra, or abstract algebra). During the interviews, the teachers were asked to draw maps representing their resource system; the author also collected resources they used and designed. Analyzing this data, the author noted that the textbook played a significant role in the teacher's lesson planning activity; nevertheless, many other resources (material or non-material) intervened in developing the 'lecture notes' document. While the resources influence the teachers' design of their lecture notes ('instrumentation process,' according to DAD), the teacher's operational invariants (propositions they consider true) also influence their activity. Some teachers prioritize students' meaning-making, and search elements in the resources that can support it. Mesa suggests that this could be an interesting orientation for textbooks authors.

Gabel and Dreyfus (Chap. 14 of this book) study a particular aspect of university teachers' practices, namely their teaching of proof. They introduce an original theoretical concept, "the flow of proof," building on an argumentation theory, "the New Rhetoric" (Perelman & Olbrechts-Tyteca, 1969). The "flow of proof" comprises the logical structure of the proof and informal considerations about the proving process within the presentation of a proof in a teaching context. The authors illustrate the use of this theoretical construct by analyzing an introductory course on set theory for first year prospective mathematics teachers in Israel. They observe the lessons, audio-recorded and transcribed, and interview the teacher after each lesson. This data is analyzed with a specific method, building on theoretical elements coming from the "New Rhetoric." In this chapter, the authors focus on an episode where the teacher uses a cognitive conflict. Investigating the effects of this conflict, they observe that it fosters the involvement of the students in the proof process. The students experience a conflict (union and intersection for sets are analogous to addition and multiplication for numbers; nevertheless, the distributivity properties differ) which emphasizes the need for a proof as a tool for dissociating what is the truth and what is an opinion. Using this conflict and solving it through this dissociation, the teacher created a shared "basis of agreement" with her students. Gabel and Dreyfus claim that the "flow of proof" is not only a theory and an associated methodology that can be used by researchers in mathematics education; it can also be a pedagogical concept useful for teachers who want to develop their proof teaching practices.

The other theme in section 3 is curriculum design (closely connected with the previous theme since curriculum design is often informed by research results about teaching). The next four chapters address this theme.

Nardi and Biza (Chap. 15 of this book) present the design, implementation (by the two authors, in their university in the UK), and assessment of two courses on "Research in Mathematics Education," one for education students and one for mathematics students. Transitioning to Mathematics Education is a challenge for these two kinds of students – and supporting this transition is a challenge for the teachers. Nardi and Biza designed the courses and their assessment by drawing on research literature about this transition, on the commognitive approach (Sfard, 2008), and a teacher education program (MathTASK). In this program, the authors use what they call '*mathtasks*': descriptions of classroom situations, including a mathematical problem, answers from students, and reactions from the teacher. The authors introduce four characteristics of the students' discourse about the *mathtasks* proposed in the courses: consistency, specificity, the reification of RME discourse, and the reification of mathematical discourse. These four characteristics frame the assessment of the two systems; more generally, they constitute tools for the teachers to formatively evaluate their students' progress. Examples of *mathtasks* and the assessment frame are presented in the chapter. Nardi and Biza contend that such courses provide opportunities for linking the communities of Mathematics, Education, and RME.

The other chapters in section 3 concern curriculum design in mathematics. Internationally, there is a growing interest in teaching practices fostering students' active engagement. Research in mathematics education can support them by designing relevant curricula, their dissemination, and associated professional development. Nevertheless, contributing to the evolution of teaching practices at scale at the university level remains challenging, and researchers also investigate levers for this instructional change (Smith et al., 2021). These chapters reflect these tendencies; the studies presented in these chapters propose and evaluate different kinds of research-supported changes, with a common aim of students' active learning.

Wawro, Andrews-Larson, Zandieh, and Plaxco (Chap. 16 of this book) present design-based research in the context of the Inquiry-Oriented Linear Algebra (IOLA) project in the United States. Drawing on Realistic Mathematics Education (RME, Freudenthal, 1991), they propose a "design-based research spiral," encompassing five phases of design, implementation, and dissemination: Design; Paired Teaching Experiment; Classroom Teaching Experiment; Online Work Group; and Web. Along with the five phases, an increasing number of teachers (and students) get involved in implementing the inquiry-oriented material designed; this enlargement of the user's group is essential in an instructional change perspective. The use of this "design-based research spiral" is illustrated in the chapter through the example of the IOLA unit on the concept of determinants. The authors argue that the "Online Working Group" in particular plays an essential role in productively connecting the three theories informing their project: RME, Inquiry-Oriented Instruction, and Instructional Change. Indeed, this stage allows to consider teachers' goals and orientations (which can differ from the RME principles that informed the initial design). This is crucial for the dissemination phase since it increases the potential of adoption of the material designed within the project by teachers who were not involved in its design and subsequent evolutions of their classroom practices.

Wessel and Leuders (Chap. 17 of this book) analyze the design of a curriculum in abstract algebra in a pre-service mathematics teacher education program in Germany. Adopting a Didactical Design Research perspective, they combine different theoretical elements to guide their design. The authors draw on categories introduced by Prediger (2019) for answering what (related to the content) and how (associated with the professional development course) questions about the theoretical elements needed. For their course concerning abstract algebra and addressing prospective teachers, the authors combine theoretical elements concerning teacher knowledge (Ball et al., 2008) and results of previous research about abstract algebra (Larsen et al., 2013). In the chapter, the authors present two successive design cycles implemented. They constructed a course associating successive situations of 'guided reinvention' with different approaches to abstract algebra. Inquiry-oriented tasks were central in the course, particularly using the software GeoGebra and Cinderella. The course was specifically tailored for prospective primary or secondary teachers – the authors call it a 'profession-specific' course. The first implementation of the course led them to observe that the inquiry-based tasks were too challenging for some students. For the second design cycle, the authors made their expectations more explicit and put an emphasis on the connections between abstract algebra and school algebra. Wessel and Leuders foreground the contribution of their study in terms of design principles for curricula at universities specific for prospective teachers' courses. Using such principles can foster the design of mathematics courses relevant for their future school teaching experience.

Smith, Voigt, Martinez, Rasmussen, Funk, Webb, and Ström (Chap. 18 of this book) pursue an objective of evolutions of the teaching practices toward active learning and equity. They investigate changes at the level of mathematics departments that can contribute to improving calculus programs. Using a theory of change perspective (Reinholz & Andrews, 2020), the authors focus on the drivers and strategies related to this improvement objective. Smith et al. study in this chapter the cases of three universities participating in the "Student Engagement in Mathematics through an Institutional Network for Active Learning" (SEMINAL) network in the United States. They visited each site and collected different kinds of data: interviews with different actors, observation data, and visit reports in particular. The analysis of this data allowed the authors to draw a "driver-strategy diagram" for each university, related to the improvement aims, the drivers for change, and the strategies used. Comparing the three cases, they observe that different strategies, depending on local conditions, can contribute to the changes. Nevertheless, these changes always require the involvement of many different actors: teachers, but also administrators, and students. The authors identify the collective work within Networked Improvement Communities (NICs, Bryk et al., 2015) as a crucial lever for change.

All the chapters in this section present original practice-oriented research: research about teaching practices and teachers' work and design-based research where the practice informs the design of innovative curricula. This section is closely connected with section 4 since most chapters investigate inquiry-based courses or the use of inquiry-oriented tasks.

1.2.4 Section 4: Research on University Students' Mathematical Inquiry

Several studies of university mathematics education suggest that standard forms of teaching at this level leave too little initiative to students and give them far too few and limited experiences of mathematics as a creative endeavor. This is also a concern of many university mathematics teachers, who are typically also researchers with many such experiences from their own scholarly work. Indeed, what students meet in most of their courses, is often tightly packed lectures, in which they learn results and methods from mathematical research that was typically done several decades, if not centuries, ago, along with more or less closed exercises of application. As observed by Burton (2004, p. 198), there is thus a considerable "gap" between the perspectives that learners and mathematicians may get on mathematics. Similar gaps between students' experiences of mathematics at university, and the needs they will face in professions outside of the university after their studies, have been identified by various scholars (e.g., Bergsten et al., 2015; Klein, 2016). When it comes to both future mathematicians and future members of such professions, a general hypothesis is that students need a more creative, autonomous, and conceptually oriented relationship with mathematics than what is produced by classical coursework.

In view of these and other calls for reform in university mathematics education, various methods to provide students with lively and inquiry-based approaches to mathematics, even in large main core courses, are being experimented with and implemented in universities worldwide. This section presents seven rigorous studies of such efforts in Canada, Denmark, Finland, France, New Zealand, Spain, and the United States.

Two chapters relate to inquiry in mathematics courses for students with specific professional aims outside of the university; these are both based on the anthropological theory of the didactic. The chapter by Bosch et al. (Chap. 19 of this book) investigates the use of study and research paths in the teaching of statistics for business students, as well as in teaching elasticity for engineering students; both areas involve mathematical elements, but they are taught here, with specific applications and professional needs in mind. Study and research paths begin with a "generating question" posed in an initial situation, which stages the work and may also provide information such as data or sources to consider. This situation and question form the basis of a longer inquiry process for the students. It turns out that the professional character of the initial situation, with its staging of both the question and the forms in which students are to deliver their answers, can be very crucial to the dynamics of students' inquiry, even with the initial question being the same.

Meanwhile, Barquero and Winsløw's chapter (Chap. 25 of this book) studies efforts to make students revisit more advanced aspects of the real numbers in the context of a capstone course for future high school teachers. Here, students have completed a sequence of standard bachelor's courses in mathematics, and the course aims to teach students how to use the mathematics learned there as a resource to investigate high school mathematics, for instance, by interpreting surprising or misleading graphs produced by a computer tool which is commonly used there. The authors observe that even with careful task design, some students fail, as they misread the requirements: either they consider that informal, "high school like" methods suffice (while they do not), or they try to apply irrelevant advanced methods. Others, indeed, succeed.

The remaining five chapters concern inquiry-type instruction in pure mathematics courses that are not specifically directed towards certain professions. These chapters employ a variety of theoretical frameworks for didactical design and for analyzing its outcomes. Hausberger (Chap. 20 of this book) presents a didactical engineering study carried out in the context of undergraduate abstract algebra. Didactical engineering refers here to the French tradition of design-based research, dating back to the 1980s, and often relying (as this chapter does in part) on the theory of didactical situations. Students explore a problematique about seatings at a banquet. With only elements of group theory as prerequisites, they are able to engage in creative algebraic thinking, with a fertile interplay between intuitive and formal moves. Also, more specific actions, proper to mathematical structuralism, such as classifying, generalizing, and identifying, are developed by the students. In these and other ways, they experience forms of interaction and thinking close to that of the mathematical researcher.

With the chapter by Larsen, Elizondo, and Brown (Chap. 21 of this book), we move to an inquiry-oriented instructional design in basic real analysis, drawing on ideas and methods from Realistic Mathematics Education (RME). Students get to reinvent fundamental ideas behind a classical proof of the intermediate value theorem while developing, on the way, related results such as convergence of monotone bounded sequences. Special attention is given to the generation of conjectures and sharing more or less correct proposals towards a proof. RME principles are used to allow a classroom community to engage in a collective, authentic mathematical activity in which classical proofs emerge from students' exploration of carefully designed questions.

The Extreme Apprenticeship model focuses on developing generic skills through inquiry-based university mathematics teaching. Rämö, Häsä, and Tuononen (Chap. 22 of this book) begin their chapter by explaining how generic mathematical skills and competencies have been theorized by previous research and how Extreme Apprenticeship has been experimented with at the University of Helsinki as a proposal for how to integrate their development in standard undergraduate and graduate mathematics courses, ranging from linear algebra to advanced courses on abstract algebra. They explain how the details of this proposal were developed over several years in close connection with more sophisticated descriptions of generic skills, which (when built into the curriculum) can provide more connectivity and progression in a study program in mathematics.

Within the context of Calculus, Kondratieva (Chap. 23 of this book) explores different levels of inquiry called for by student assignments, drawing on theoretical ideas like the Herbartian schema and praxeologies from the Anthropological theory of the Didactic. Taking as starting points specific tasks proposed in standard and reform Calculus texts, she demonstrates how more advanced forms of guided student inquiry can be generated through careful design of different kinds of activities which, while based on standard tasks, leave progressively more room for students' development of advanced forms of mathematical inquiry, such as posing derived questions and investigating hypotheses that emerge from the initial question.

Finally, Kontorovich, L'Italien-Bruneau and Greenwood (Chap. 24 of this book) analyze cases of students' proving activity in an experimental graduate course on topology based on the commognitive framework. The cases considered unfolding more or less according to the "proving at the board" approach proposed by Texan topologist Robert Lee Moore. The authors demonstrate how two individual students present proofs in contexts such as the finite intersection property of a collection of closed subsets of a compact set, and also how their commognitive actions in front of the class reveal subtle differences in their relationship with proof narratives, which should convince not only the prover but also a more or less concrete audience of the validity of a given proposition.

To sum up the contributions of this section, we are presented with a wide variety of cases and theoretical approaches to the highly complex notion of inquiry, as it is currently conceived in the context of university mathematics education. While classical "talk and chalk" lectures continue to be important in this context, the chapters provide different kinds of evidence that lectures can be supplemented or even replaced by more demanding forms of engaging students in mathematical inquiry – as participants, rather than mere spectators.

1.2.5 Section 5: Research on Mathematics for Non-specialists

While mathematics for non-specialists has long been problematized, it is only within the past two decades (or less) that the field has turned to systematically inquiring into the curriculum, teaching, and learning of mathematics for non-specialists, with a more focused examination of the mathematics service courses taken by engineering students and the mathematics that students encounter in their engineering courses. In the last several years alone, we have witnessed an increased interest in both of these areas of interest. Hochmuth recently surveyed some of the recent literature on *Service-Courses in University Mathematics Education* in the *Encyclopedia of Mathematics Education* (Hochmuth, 2020), and in 2021 the *International Journal of Research in Undergraduate Mathematics* published a special issue on mathematics in engineering education (Pepin et al., 2021). The chapters in this volume further contribute to these lines of inquiry, covering both mathematics courses for engineering students and mathematics in engineering or science courses. As a whole, the chapters in this section provide theoretically grounded insights into pedagogical, curricular, epistemological challenges, and frameworks for analysis.

The chapter by Romo Vázquez and Artigue (Chap. 26 of this book) provides an overview of the field of engineering education by surveying the literature with the goal of identifying how challenges have been dealt with over time and how they are produced and re-produced alongside scientific and technological advances, societal evolution, and emerging concerns. Their historical review includes the case of the *École Polytechnique* in France and three International Commission for Mathematical Instruction studies. Grounded in this historical overview, the authors then tender examples selected from recent research and development work, illustrating the progression of theoretical approaches, especially that of ATD, and the opportunities for future research.

Following nicely from the ATD overview provided by Romo Vázquez and Artigue, four chapters provide detailed ATD analyses of the praxeologies and challenges that learners encounter in their mathematics, engineering, and science courses as well as in the workplace.

In their chapter, González-Martín, Barquero, and Gueudet (Chap. 27 of this book) demonstrate how ATD praxeological analyses can uncover differences in the way mathematical tools are used in mathematics courses and engineering courses. They tender two examples of the outcomes of transposition processes by examining praxeologies involving mathematics in engineering courses. In the first example, they review a study that analyzed and compared the use of the Laplace transform in a mathematics course and in two control theory courses. In the second example, they highlight the use of integrals in reference textbooks and teaching practices in two engineering courses, on the strength of materials and electricity and magnetism. Their chapter concludes with examples of the ATD innovative instructional approach, study and research paths, aimed at reducing the gap between educational and professional practices with respect to mathematics for engineers.

The chapter by Peters (Chap. 28 of this book) uses the context of a mathematics service course for engineers with a focus on complex numbers. In contrast to the more standard approaches to make mathematics service courses more salient for engineering students (i.e., approaches that introduce engineering applications or approaches that make use of innovative instructional strategies such as study and research paths or project work), Peters presents a third approach, one that takes mathematical exercises and focuses on establishing and promoting connections to electrical engineering discourses within mathematical discourses. This is done without also introducing the engineering context. At the core of this approach is the ATD concept of institutional dependence of knowledge.

Rønning (Chap. 29 of this book), in his chapter, reports on the redesign of a basic course in mathematics for first-year students that is taught in close connection with a course on electronic system design and analysis. Rønning uses ATD to interrogate the discourses that develop with the aim of unpacking how the praxeologies in mathematics and engineering influence and interact with each other. In particular, he uses electric circuits as a concrete example to demonstrate how one can shift the emphasis of specific topics as well as change the sequencing of the topics to better meet the needs and interests of engineering students. Interviews with teachers in both mathematics and electronics design course reveal the challenges and tensions that both teachers face.

In contrast to the chapters by González-Martín et al., Peters, and Ronning, all of which examine praxeologies in either mathematics service courses for engineers or the mathematics in engineering or science courses, Castela and Romo Vázquez (Chap. 30 of this book) focus on the opportunities and challenges of designing teaching sequences based on authentic professional workplace situations. They present an ATD analysis of praxeologies in three different professional workplace situations: land surveying, automotive industry, and computer science. One of the challenges they identify is the felt need to teach students more sophisticated mathematics than those employed in normal professional practices. Reasons for this include the potential usefulness for career advance and the possible need to adapt to future changes in professional practice.

Three additional papers report on empirical investigations that examine students' understanding of mathematics that they encounter in their engineering courses. Each of these papers offers fresh insights into the nuances of student thinking, and each offers innovative frameworks of considerable potential for future research.

Hjalmarson, Nelson, Buck, and Wage (Chap. 31 of this book) examine students' reasoning with conceptual problems in signals and systems, a subfield of electrical engineering. Using student interview data from two different institutions, they investigate how students interpret, describe, and reason with graphical representations of signals and systems problems. To do so, they adapt a framework originally developed to interpret student understanding of derivatives that leverages the constructs of concept image and process-object pairs. Their analysis highlights both students' challenges and successes in thinking about and translating among multiple representations of the same signal and opens up possibilities for further adaption of the use of this framework in other contexts.

Kortemeyer and Biehler (Chap. 32 of this book), in their chapter, examine the issue of how mathematics and what kind of mathematics is used and needed when students are asked to solve problems in their engineering course. As a case study, they analyze tasks on an end-of-year examination in a fundamentals of electrical engineering course. Their analysis includes interviews with experts to identify the expected competencies, both implicit and explicit, which were then used together with theoretically informed approaches to develop an a priori student-expert solution. This analysis culminated in a transferable framework for the analyses of exercises, problem-solving strategies in engineering exercises, and typical sources of errors.

The penultimate chapter in this section by Jablonka and Bergsten (Chap. 33 of this book) examines the social and cultural conditions and the institutional context using student interview data from first-year engineering students. Their analysis, which is grounded in Bourdieu's notion of habitus and the dialectic between experiences and perceptions, contributes to a deeper understanding of students' appreciation of specificities of mathematical discourse encountered in the core mathematics, students' perceptions of the usefulness of mathematics, and their experiences of studying mathematics as compared to other subjects. Their findings reflect that success in the service courses depends on recognizing the criteria of pure mathematics as opposed to mathematical applications or modeling. Their work also contributes to a theoretically and empirically informed framing of four different student-perceived modes of the usefulness of mathematics.

The final chapter by Fredriksen et al. (Chap. 34 of this book) is unique from the previous chapters in that it provides an overview of several dissertations that have been carried out under the Norwegian Centre for Excellence in Education (MatRIC). The common focus across these dissertations is the teaching and learning of mathematics as a service subject. Indeed, the improvement of student success in these courses across Norway is a foundational mission of MaTRIC. The research projects highlighted in this chapter adopt a variety of approaches to address concerns about teaching development (flipped classroom and blended learning approaches), shortcomings in students' prior knowledge, use of digital technology in learning, mathematical modeling, and exposing causal relationships between learning approaches and outcomes. Taken as a whole, the chapter sets forth a strong foundation for continued research that aims to improve student success both in mathematics courses for non-specialists and for the teaching and learning of mathematics within engineering and science courses.

As the brief overview and highlights of the chapters in this section reveal, the body of research focusing on mathematics for non-specialists is a rich and growing domain. While much has been learned, there is clearly a continued need for further theoretically informed innovations that build on the advances to date and address the thorny and persistent epistemological, pedagogical, and curricular challenges.

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