Advances in Mathematics Education

Rolf Biehler · Michael Liebendörfer · Ghislaine Gueudet · Chris Rasmussen · Carl Winsløw *Editors*

Practice-Oriented Research in Tertiary Mathematics Education

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Advances in Mathematics Education

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Chapter 1
Practice-Oriented Research in Tertiary Mathematics Education – An Introduction

Rolf Biehler, Michael Liebendörfer, Ghislaine Gueudet, Chris Rasmussen, and Carl Winsløw

Abstract This chapter first outlines the genesis of the book. We briefly describe the development of university mathematics education as an international field of research and development. This includes the role of the khdm ("Kompetenzzentrum Hochschuldidaktik Mathematik"; Centre for Higher Mathematics Education) and one of its founding directors, Reinhard Hochmuth, to whom this book is dedicated.

We then consider five practice-oriented topics for research in university mathematics education: the secondary-tertiary transition, university students' mathematical practices, teaching and curriculum design, university students' mathematical inquiry, and mathematics for non-specialists. These topics represent main areas of recent research and development.

The five topics appear in the book as sections, each with several chapters. The sections and their contents are introduced in the final parts of this first chapter. For each section, we sketch the connection between its chapters and the specific field of research. We further provide a brief description of each chapter in terms of theoretical and methodological approaches, as well as of the results presented.

Keywords Secondary-tertiary transition · University students' mathematical practices · Teaching and curriculum design · University students' mathematical inquiry · Mathematics for non-specialists

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1.1 Context of This Book

This book is devoted to current practice-oriented research in tertiary mathematics education. The birth of every book has its occasion, its reasons, and its history. The occasion of this book consists of two anniversaries: 10 years of work at the “Kompetenzzentrum Hochschuldidaktik Mathematik” (khdm; Centre for Higher Mathematics Education) starting in the fall of 2010, and the 60th birthday of Reinhard Hochmuth, one of the two founding directors of the khdm, in March 2021. We, as editors invited international colleagues, including persons from the khdm, Reinhard’s and the khdm’s current international scientific network, and beyond, to contribute to this volume. Transcending these occasions, the editorial team discussed the specific orientation of the book, which should show a panorama of current research that is practice-oriented and relevant for both university mathematics teachers and scholars in the growing field of tertiary mathematics education research.

In keeping with this orientation, the book presents practice-oriented research covering a broad range of research topics in tertiary mathematics education. This reflects vital international activities of a quickly developing field. It is meant to help both researchers and practitioners to get inspiration on what good teaching and learning may mean, and how it may happen. The forward-looking nature of this volume has strong potential to influence the further development of the field. The book is organized into the following five sections:

Section 1: Research on the secondary-tertiary transition
Section 2: Research on university students’ mathematical practices
Section 3: Research on teaching and curriculum design
Section 4: Research on university students’ mathematical inquiry
Section 5: Research on mathematics for non-specialists.

Tertiary mathematics education has been thought about for as long as university mathematics exists, and individuals have published research on issues in tertiary mathematics education for decades. A crucial initial compilation of early work in university mathematics education research is the edited volume on advanced mathematical thinking (Tall, 1991) and the result of an ICMI study (Holton & Artigue, 2001). However, when the khdm started its work in late 2010, the field of university mathematics education as a large research community was still in its infancy in most countries. An exception was the RUME community in the United States, whose annual conferences started in the late 1990s.

Since 2010, national and international structures have emerged in which scholarly exchange occurs. In Europe, working groups on tertiary mathematics education have been present at the CERME conferences (held by ERME, The European Society for Research in Mathematics Education) since 2005 (see Winsløw et al., 2018 for a more detailed outline). In 2016, the International Network for Didactic Research in University Mathematics (INDRUM), which is closely associated with ERME, held its first conference devoted exclusively to the didactics of university mathematics.

Since 2015, the *International Journal of Research in Undergraduate Mathematics Education* (IJRUME) has been published. It is the first journal with a focus on undergraduate mathematics education research.

In addition to these international activities, several national centers for tertiary mathematics education have been established. In the United Kingdom, the SIGMA Centre for Excellence in Teaching and Learning was founded in 2005 (https://www.sigma-network.ac.uk/). In early 2000 the Mathematical Association of America (MAA) formed the Special Interest Group of the MAA on Research in Undergraduate Mathematics Education (SIGMAA on RUME). The MatRIC Center for Research, Innovation, and Coordination of Mathematics Teaching was founded in Norway in 2014.

The "Kompetenzzentrum Hochschuldidaktik Mathematik" (khdm; Center for Higher Mathematics Education) was founded by Rolf Biehler and Reinhard Hochmuth at the universities of Kassel and Paderborn in Germany in late 2010 by winning a competition for grants from the Volkswagenstiftung and the Stiftung Mercator. Today it is a joint scientific institution of the universities of Hannover, Kassel, and Paderborn. One mission of the khdm was to establish a national research network, relate it to international developments, and contribute to the development of "Didactics of Mathematics in Higher Education as a Scientific Discipline," as was the title of the 2015 international khdm conference. The khdm organized national conferences in Kassel 2011 (Bausch et al., 2014) and Paderborn 2013 (Hoppenbrock et al., 2016) on tertiary mathematics education, and international conferences in Oberwolfach 2014 (Biehler et al., 2014; Biehler & Hochmuth, 2017) and Hannover-Herrenhausen 2015 (Göller et al., 2017).

Reinhard Hochmuth, to whom we dedicate this book on the occasion of his 60th birthday, co-directed the khdm from the beginning. Reinhard, who also holds a degree in psychology, was a professor of mathematics at Kassel University. Since 2014, he is a professor of mathematics education at the Leibniz University Hannover. He is currently co-directing the khdm with Andreas Eichler (Kassel University) and Michael Liebendörfer (Paderborn University), who took over from Rolf Biehler in 2020. Michael was the first doctoral student of Reinhard in the field of tertiary mathematics education. Since its foundation, Reinhard Hochmuth has been one of the khdm's managing directors and has been intensely involved in the further development of higher mathematics education and its national and international structures, particularly through the activities of the khdm, mentioned above. In addition, he has been involved in international conferences and networks in various forms. For instance, he is the host for the INDRUM meeting 2022 at Hannover (Germany).

Reinhard represents the collaboration of mathematics and mathematics education research needed for the fruitful development of tertiary mathematics education. He contributed valuable perspectives to the field, for example, on higher mathematics beyond the first year of study (Hochmuth & Schreiber, 2016) and the role of society for mathematics (Hochmuth & Schreiber, 2015): mathematics always takes place in social structures and whoever deals with mathematics also always contributes a piece to the reproduction or change of these structures. Reflecting on this relationship to society is one of his particular perspectives (and will be reflected in Chap. 3 of this book).

Thus, his activities and this book aim in the same direction. The book's five sections also reflect parts of Reinhard's research contribution to the field. We point to some exemplary contributions. Transition problems (section 1), especially the design of mathematical bridging courses, were his early focus before the khdm was founded (Biehler et al., 2012). Recently, the WiGeMath project, whose coordinating director was Reinhard, made a comprehensive study of transition problems and possible remedies by bridging courses, transition lectures, mathematics support centers, and additional supporting measures such as e-learning elements (Hochmuth et al., 2022). Students' mathematical practices (section 2), in particular from the perspective of the Anthropological Theory of the Didactic (ATD), have become a major research topic of Reinhard (Hochmuth & Peters, 2021; Bosch et al., 2021). Research on teaching and curriculum design (section 3) were in the center of the LIMA and KLIMAGS project, where an innovative course for primary and secondary teacher students was developed with a specific focus on qualifying mathematical tutors (Biehler et al., 2018; Hochmuth et al., 2021). Research on university students' mathematical inquiry (section 4) was a focus of the Platinum Project that Reinhard has been co-directing (Gómez-Chacón et al., 2021). Reinhard also did research on mathematics for non-specialists (section 5): He co-directed the Kom@ING project that analyzed the required and attained mathematical competencies in courses for engineering students (Peters et al., 2017), where he was particularly interested in the field of signal theory (Hochmuth et al., 2014, Hochmuth & Peters, 2020, 2021).

1.2 Overall Structure of the Book

The questions and methods of tertiary mathematics education are manifold. For a long time now, the transition from school to university has been a major topic, often experienced as particularly challenging in mathematics. For example, based on psychological theories, researchers analyze how students experience and act in the transition process. Universities have set up various support structures such as bridging courses or learning support centers. A growing body of knowledge theoretically underpins the design of such structures and supplies instruments for evaluating their success.

In the transition and also later, another part of the research focuses on how to support students in becoming an active part of mathematical practices. In addition to focusing on mathematical concepts and mathematical theories, research focuses on mathematical practices such as proving, which play a role both in the transition and in later studies. While such research mainly analyzes existing courses and practices, also many studies constructively design teaching scenarios and material and study the learning processes that they initiate. These studies result in scientifically based courses or curriculum conceptions. Inquiry-based approaches have been developed that often challenge traditional teaching and learning models. We find experiments that show us how we could radically transform traditional teaching. While current research is mainly oriented toward future mathematicians or mathematics teachers, research is also making remarkable progress in mathematics for non-specialists. At many universities, the STEM (science, technology, engineering, and mathematics) domains and economics are the domains where improved teaching and learning processes can reach a much larger number of students than in the courses for mathematics majors. The design of these courses poses new challenges because of possible links to the workplace and different mathematical practices in the mathematics courses and the engineering and economics courses.

1.2.1 Section 1: Research on the Secondary-Tertiary Transition

The transition from school to university involves many disruptions, such as study goals, course designs, working techniques, the didactical contract, and the nature of mathematics (Gueudet & Thomas, 2020). The problems only become visible at the university, where innovative measures are often taken. To understand these transitional difficulties, however, we need to look at both sides, school, and university.

This section starts with two chapters that deepen our understanding of the students experiencing the secondary-tertiary transition. Göller and Rück (Chap. 2 of this book) examine first-year students' achievement emotions and their interaction with self-regulated learning. To this end, they first integrate both aspects into one model showing the constant interaction of emotions and activity and their relation to both mastery of the content and personal well-being. Based on 21 interviews with first-year mathematics students, Göller and Rück illustrate the ways students experience important emotions like joy or hopelessness and their connections to self-regulated learning. The chapter highlights the importance of students perceiving control over their learning and valuing the new mathematics for not only mastery but also well-being. Given the disruptions during the secondary-tertiary transition, it seems typical to struggle with both points. The model can thus explain why students start coping and adjust their own goals based on their emotional experiences. Göller and Rück therefore open up a new and promising perspective on the secondary-tertiary transition.

Ruge (Chap. 3 of this book) takes a critical perspective on the secondary-tertiary transition for student teachers. She first outlines the subject-scientific approach (Hochmuth, 2018; Holzkamp, 2013) that puts a strong emphasis on the societal dimension of individual experiences and behavior. This approach is then used to construct a perspective on students' feelings about mathematics, society and the teaching profession. Ruge draws on selected interview data with teacher students to illustrate their unease with being identified as mathematicians and the current state of academic mathematics. She also reports how the research community reacted to her observations. This unease reveals a great learning potential and has been studied primarily in belief research to date. Ruge argues for a different perspective based on a subject-scientific reinterpretation of beliefs-research, conceiving beliefs less as an individual trait and taking social and societal dimensions into account. She extends the research of Skott (2019) and advocates the common goal of student teachers and mathematics education scholars to promote a humane mathematics-society relationship. This is discussed for universities regarded as both teacher education institutions and research institutions on mathematics education.

The two other chapters focus on teaching in the secondary-tertiary transition. Corriveau (Chap. 4 of this book) presents findings from an inter-level community formed by mathematics teachers from secondary and post-secondary levels. The community headed for developing practices that smoothen the transition. This allowed comparing teachers' ways of doing using an ethnomethodological approach. Firstly, the findings highlight two very distinct territories concerning the use of contexts in secondary and post-secondary teaching. Secondary mathematics is strongly contextualized, whereas post-secondary mathematics is only illustrated in contexts. This affects both meaning and reasoning. It adds a new perspective on differences in the ways mathematics is taught. The chapter further demonstrates the power of cross-institutional collaborations to both understand and improve the transition. Teachers from both territories could explicate the way they do mathematics and reflect on the others' ways. This led to a collaboration that tackles the transition problem from both sides. The chapter thus also provides a good way of implementing a fruitful collaboration across the institutions.

Finally, Liebendörfer et al. (Chap. 5 of this book) present a framework developed to describe the goals of support measures in the transition between school and university. The underlying observation is that recent research has brought up various measures like bridging courses, redesigned lectures, and mathematics learning support centers, which all address the transition problems in their own ways. A first step to structuring and understanding this field is reconstructing the goals, which are often implicit as staff may focus on their concrete actions in teaching and support. After describing the framework, Liebendörfer et al. illustrate the goals of several pre-university bridging courses, redesigned lectures, and mathematics learning support centers. This may clarify both the specific roles that these kinds of support measures have and the variability within these categories. The framework thus helps to understand the particular directions of support measures and compare or evaluate them. It was developed in the WiGeMath project led by Reinhard Hochmuth.

1.2.2 Section 2: Research on University Students' Mathematical Practices

The section deals with research studies that focus on mathematical practices. Practices can be understood as closely related to mathematical processes such as proving and defining. A more comprehensive theoretical framework for studying practices in mathematical institutions has been developed by the Anthropological Theory of the

Didactic (ATD). Its wider notion of praxeology includes the components theory, technology, technique, and task. This and other theoretical frameworks are used in this section.

The first set of papers is concerned with reconstructing (problematic) practices of students that have evolved in an institutional context and explaining these practices by the implicit didactical contract of a lecture that may also be specified in expectations of lectures and tasks assigned to the students. The second set of papers focuses on the practices of proving and defining as characteristic processes of university mathematics.

The section has epistemological, interventional, and observational studies that characterize, reconstruct, observe or intervene to elaborate the participation in advanced practices in institutional settings. Some of the studies are design studies that address learning challenges and opportunities, while others are theoretical-conceptual studies that present an a priori analysis of certain mathematical practices.

The first set of papers is concerned with reconstructing institutional practices of mathematics that are relevant for developing students' practices. Bašić and Šipuš (Chap. 6 of this book) reconstruct the practices expected from students and the didactical contract in a lecture on multivariable calculus from the perspectives of ATD, the Theory of Didactical Situations, and the theory of the didactical contract, studying students' problem-solving results and related reflections on their difficulties and problems of learning and understanding. Moreover, the authors conducted related interviews with the lecturers. The implicit didactical contract is reconstructed from both sources, and suggestions for improving the teaching-learning system are developed. Broley and Hardy (Chap. 7 of this book) take a similar theoretical stance using ATD and reconstruct students' practices in a course on real analysis, using various sources of observation, including assessment. They found a substantial number of so-called non-mathematical practices related to students' orientation to minimal requirements for success and considering only superficial properties of previous exercises. Discovering these practices is essential for improving teaching and learning toward a deeper mathematical understanding.

The second set of papers is concerned with proving and defining the interrelation between these mathematical practices and various educational levels: secondary level graduates as beginning university students, students in their transition process in the first semester, and practices relevant for more advanced students. Lew, Fukawa-Connelly, and Weber (Chap. 8 of this book) argue in a theoretical chapter that when mathematicians lecture, they not only cover mathematical content but also model how students should learn mathematics. They analyze a corpus of 11 lectures in various advanced mathematics courses to investigate how mathematicians present the definitions of concepts and gain insight into how mathematicians may expect students to learn from lectures. They highlight how the instructors modeled what it means to study a concept and its definition and argue that students are expected to engage in independent study outside of class.

Ostsieker and Biehler (Chap. 9 of this book) focus on the concept of convergence, where in a design study, students are supported in re-inventing this concept and its definition. Meta-knowledge on defining is needed, and a substantial and rich concept image of the convergence of a sequence. The research is situated on the transition between school and university. The learning environment consists of examples and non-examples of convergent sequences, a task, and expected obstacles with prepared supports for each expected obstacle. The learning environment was developed in the Design-Based Research paradigm, conducted twice, and analyzed and refined each time. In this chapter, the analysis focuses on the changes in the formulation of the initial task for the students, that were made based on the results of the analysis of the first two implementations.

Kempen (Chap. 10 of this book) investigates the practice and meta-knowledge on proof and proving of 12 high-school graduates, with a view towards what proof conceptions were developed in school mathematics and have to be taken into account when beginning students at university are being introduced to the proving practices at the university level. The author conducted task-based interviews focusing on learners' usage and assigned meaning of statements with regard to their embeddedness in a local deductive organization, their epistemic values, and their respective effects on the conclusion's modal qualifier. While all graduates accept definitions and rules for term manipulation, there is no consensus concerning the statements involved. Furthermore, the individuals' epistemic values concerning the statements involved affect their usage in a chain of arguments and the individuals' evaluation of the conclusion.

Whereas the preceding papers focus on defining or proving, a new analysis of the interrelation between proving and defining on a general level in university mathematical practices is the central focus of Durand-Guerrier's (Chap. 11 of this book) paper. The main goal of this chapter is to underline, from an epistemological point of view, the relevance of engaging university students in intertwined proving and defining practices. The chosen examples are *real numbers* and *infinity*. The intertwined practices of proving and defining are taken from the case of the construction of irrational numbers by Dedekind (1872) and Cantor (1874). Next, the author presents an example of a situation involving R-completeness versus Q-incompleteness that has the potential to foster students' engagement in intertwined proving and defining practices. These intertwined relationships are further explored from a didactical point of view concerning the relation between practices of enumeration, the definition of infinite sets, and diagonal proofs that the set of rational numbers is denumerable while the set of irrational numbers is not.

1.2.3 Section 3: Research on Teaching and Curriculum Design

Teaching at the university level is increasingly studied by researchers in mathematics education (Biza et al., 2016). The first three chapters in this section present research contributing to new understandings of university teachers' practices and professional development, drawing on a great variety of theoretical approaches and associated methods.

Jaworski and Potari (Chap. 12 of this book) use Activity Theory (Leont'ev, 1979) to analyze the interactions between a tutor and her students in terms of tensions and contradictions arising when the tutor tries to engage students in mathematics meaning-making. The authors video-recorded and transcribed tutorial sessions for first-year mathematics students in a university in the UK. The teacher was the first author of the chapter (Jaworski) and was actively involved in the data analysis, and this was essential for identifying her goals. This analysis leads to the observation of different objectives of the teacher's activity concerning desired students' actions (e.g., listen to each other and build on what another person expresses) or the teacher's own activity (e.g., listen to the students and discern meaning from what they say). The authors observed positive outcomes for the students (e.g., expressing their informal ideas or valuing the collaboration with peers). They also identified tensions and related contradictions, e.g., between the teacher's guidance (needed for achieving the task) and students' autonomy. These contradictions make the development of inquiry-oriented practices challenging; the authors claim that facing this challenge requires a continuous professional reflection of the teacher.

Mesa (Chap. 13 of this book) studies another aspect of university teachers' professional activity: their lesson planning activity (in the context of their ordinary professional activity for 'traditional' courses). The theoretical framework in this chapter is the documentational approach to didactics (DAD, Trouche et al., 2020), which introduces a difference between resources used by the teacher and documents developed along with their activity. Mesa focuses in this chapter on the "lesson notes" document. Twenty-one post-secondary teachers in 15 different universities in the United States were interviewed about their use of resources for their lesson planning activity (for calculus, linear algebra, or abstract algebra). During the interviews, the teachers were asked to draw maps representing their resource system; the author also collected resources they used and designed. Analyzing this data, the author noted that the textbook played a significant role in the teacher's lesson planning activity; nevertheless, many other resources (material or non-material) intervened in developing the 'lecture notes' document. While the resources influence the teachers' design of their lecture notes ('instrumentation process,' according to DAD), the teacher's operational invariants (propositions they consider true) also influence their activity. Some teachers prioritize students' meaning-making, and search elements in the resources that can support it. Mesa suggests that this could be an interesting orientation for textbooks authors.

Gabel and Dreyfus (Chap. 14 of this book) study a particular aspect of university teachers' practices, namely their teaching of proof. They introduce an original theoretical concept, "the flow of proof," building on an argumentation theory, "the New Rhetoric" (Perelman & Olbrechts-Tyteca, 1969). The "flow of proof" comprises the logical structure of the proof and informal considerations about the proving process within the presentation of a proof in a teaching context. The authors illustrate the use of this theoretical construct by analyzing an introductory course on set theory for first year prospective mathematics teachers in Israel. They observe the lessons, audio-recorded and transcribed, and interview the teacher after each lesson. This data is analyzed with a specific method, building on theoretical elements coming from the "New Rhetoric." In this chapter, the authors focus on an episode where the teacher uses a cognitive conflict. Investigating the effects of this conflict, they observe that it fosters the involvement of the students in the proof process. The students experience a conflict (union and intersection for sets are analogous to addition and multiplication for numbers; nevertheless, the distributivity properties differ) which emphasizes the need for a proof as a tool for dissociating what is the truth and what is an opinion. Using this conflict and solving it through this dissociation, the teacher created a shared "basis of agreement" with her students. Gabel and Dreyfus claim that the "flow of proof" is not only a theory and an associated methodology that can be used by researchers in mathematics education; it can also be a pedagogical concept useful for teachers who want to develop their proof teaching practices.

The other theme in section 3 is curriculum design (closely connected with the previous theme since curriculum design is often informed by research results about teaching). The next four chapters address this theme.

Nardi and Biza (Chap. 15 of this book) present the design, implementation (by the two authors, in their university in the UK), and assessment of two courses on "Research in Mathematics Education," one for education students and one for mathematics students. Transitioning to Mathematics Education is a challenge for these two kinds of students – and supporting this transition is a challenge for the teachers. Nardi and Biza designed the courses and their assessment by drawing on research literature about this transition, on the commognitive approach (Sfard, 2008), and a teacher education program (MathTASK). In this program, the authors use what they call '*mathtasks*': descriptions of classroom situations, including a mathematical problem, answers from students, and reactions from the teacher. The authors introduce four characteristics of the students' discourse about the *mathtasks* proposed in the courses: consistency, specificity, the reification of RME discourse, and the reification of mathematical discourse. These four characteristics frame the assessment of the two systems; more generally, they constitute tools for the teachers to formatively evaluate their students' progress. Examples of *mathtasks* and the assessment frame are presented in the chapter. Nardi and Biza contend that such courses provide opportunities for linking the communities of Mathematics, Education, and RME.

The other chapters in section 3 concern curriculum design in mathematics. Internationally, there is a growing interest in teaching practices fostering students' active engagement. Research in mathematics education can support them by designing relevant curricula, their dissemination, and associated professional development. Nevertheless, contributing to the evolution of teaching practices at scale at the university level remains challenging, and researchers also investigate levers for this instructional change (Smith et al., 2021). These chapters reflect these tendencies; the studies presented in these chapters propose and evaluate different kinds of research-supported changes, with a common aim of students' active learning.

Wawro, Andrews-Larson, Zandieh, and Plaxco (Chap. 16 of this book) present design-based research in the context of the Inquiry-Oriented Linear Algebra (IOLA) project in the United States. Drawing on Realistic Mathematics Education (RME, Freudenthal, 1991), they propose a "design-based research spiral," encompassing five phases of design, implementation, and dissemination: Design; Paired Teaching Experiment; Classroom Teaching Experiment; Online Work Group; and Web. Along with the five phases, an increasing number of teachers (and students) get involved in implementing the inquiry-oriented material designed; this enlargement of the user's group is essential in an instructional change perspective. The use of this "design-based research spiral" is illustrated in the chapter through the example of the IOLA unit on the concept of determinants. The authors argue that the "Online Working Group" in particular plays an essential role in productively connecting the three theories informing their project: RME, Inquiry-Oriented Instruction, and Instructional Change. Indeed, this stage allows to consider teachers' goals and orientations (which can differ from the RME principles that informed the initial design). This is crucial for the dissemination phase since it increases the potential of adoption of the material designed within the project by teachers who were not involved in its design and subsequent evolutions of their classroom practices.

Wessel and Leuders (Chap. 17 of this book) analyze the design of a curriculum in abstract algebra in a pre-service mathematics teacher education program in Germany. Adopting a Didactical Design Research perspective, they combine different theoretical elements to guide their design. The authors draw on categories introduced by Prediger (2019) for answering what (related to the content) and how (associated with the professional development course) questions about the theoretical elements needed. For their course concerning abstract algebra and addressing prospective teachers, the authors combine theoretical elements concerning teacher knowledge (Ball et al., 2008) and results of previous research about abstract algebra (Larsen et al., 2013). In the chapter, the authors present two successive design cycles implemented. They constructed a course associating successive situations of 'guided reinvention' with different approaches to abstract algebra. Inquiry-oriented tasks were central in the course, particularly using the software GeoGebra and Cinderella. The course was specifically tailored for prospective primary or secondary teachers – the authors call it a 'profession-specific' course. The first implementation of the course led them to observe that the inquiry-based tasks were too challenging for some students. For the second design cycle, the authors made their expectations more explicit and put an emphasis on the connections between abstract algebra and school algebra. Wessel and Leuders foreground the contribution of their study in terms of design principles for curricula at universities specific for prospective teachers' courses. Using such principles can foster the design of mathematics courses relevant for their future school teaching experience.

Smith, Voigt, Martinez, Rasmussen, Funk, Webb, and Ström (Chap. 18 of this book) pursue an objective of evolutions of the teaching practices toward active learning and equity. They investigate changes at the level of mathematics departments that can contribute to improving calculus programs. Using a theory of change perspective (Reinholz & Andrews, 2020), the authors focus on the drivers and strategies related to this improvement objective. Smith et al. study in this chapter the cases of three universities participating in the "Student Engagement in Mathematics through an Institutional Network for Active Learning" (SEMINAL) network in the United States. They visited each site and collected different kinds of data: interviews with different actors, observation data, and visit reports in particular. The analysis of this data allowed the authors to draw a "driver-strategy diagram" for each university, related to the improvement aims, the drivers for change, and the strategies used. Comparing the three cases, they observe that different strategies, depending on local conditions, can contribute to the changes. Nevertheless, these changes always require the involvement of many different actors: teachers, but also administrators, and students. The authors identify the collective work within Networked Improvement Communities (NICs, Bryk et al., 2015) as a crucial lever for change.

All the chapters in this section present original practice-oriented research: research about teaching practices and teachers' work and design-based research where the practice informs the design of innovative curricula. This section is closely connected with section 4 since most chapters investigate inquiry-based courses or the use of inquiry-oriented tasks.

1.2.4 Section 4: Research on University Students' Mathematical Inquiry

Several studies of university mathematics education suggest that standard forms of teaching at this level leave too little initiative to students and give them far too few and limited experiences of mathematics as a creative endeavor. This is also a concern of many university mathematics teachers, who are typically also researchers with many such experiences from their own scholarly work. Indeed, what students meet in most of their courses, is often tightly packed lectures, in which they learn results and methods from mathematical research that was typically done several decades, if not centuries, ago, along with more or less closed exercises of application. As observed by Burton (2004, p. 198), there is thus a considerable "gap" between the perspectives that learners and mathematicians may get on mathematics. Similar gaps between students' experiences of mathematics at university, and the needs they will face in professions outside of the university after their studies, have been identified by various scholars (e.g., Bergsten et al., 2015; Klein, 2016). When it comes to both future mathematicians and future members of such professions, a general hypothesis is that students need a more creative, autonomous, and conceptually oriented relationship with mathematics than what is produced by classical coursework.

In view of these and other calls for reform in university mathematics education, various methods to provide students with lively and inquiry-based approaches to mathematics, even in large main core courses, are being experimented with and implemented in universities worldwide. This section presents seven rigorous studies of such efforts in Canada, Denmark, Finland, France, New Zealand, Spain, and the United States.

Two chapters relate to inquiry in mathematics courses for students with specific professional aims outside of the university; these are both based on the anthropological theory of the didactic. The chapter by Bosch et al. (Chap. 19 of this book) investigates the use of study and research paths in the teaching of statistics for business students, as well as in teaching elasticity for engineering students; both areas involve mathematical elements, but they are taught here, with specific applications and professional needs in mind. Study and research paths begin with a "generating question" posed in an initial situation, which stages the work and may also provide information such as data or sources to consider. This situation and question form the basis of a longer inquiry process for the students. It turns out that the professional character of the initial situation, with its staging of both the question and the forms in which students are to deliver their answers, can be very crucial to the dynamics of students' inquiry, even with the initial question being the same.

Meanwhile, Barquero and Winsløw's chapter (Chap. 25 of this book) studies efforts to make students revisit more advanced aspects of the real numbers in the context of a capstone course for future high school teachers. Here, students have completed a sequence of standard bachelor's courses in mathematics, and the course aims to teach students how to use the mathematics learned there as a resource to investigate high school mathematics, for instance, by interpreting surprising or misleading graphs produced by a computer tool which is commonly used there. The authors observe that even with careful task design, some students fail, as they misread the requirements: either they consider that informal, "high school like" methods suffice (while they do not), or they try to apply irrelevant advanced methods. Others, indeed, succeed.

The remaining five chapters concern inquiry-type instruction in pure mathematics courses that are not specifically directed towards certain professions. These chapters employ a variety of theoretical frameworks for didactical design and for analyzing its outcomes. Hausberger (Chap. 20 of this book) presents a didactical engineering study carried out in the context of undergraduate abstract algebra. Didactical engineering refers here to the French tradition of design-based research, dating back to the 1980s, and often relying (as this chapter does in part) on the theory of didactical situations. Students explore a problematique about seatings at a banquet. With only elements of group theory as prerequisites, they are able to engage in creative algebraic thinking, with a fertile interplay between intuitive and formal moves. Also, more specific actions, proper to mathematical structuralism, such as classifying, generalizing, and identifying, are developed by the students. In these and other ways, they experience forms of interaction and thinking close to that of the mathematical researcher.

With the chapter by Larsen, Elizondo, and Brown (Chap. 21 of this book), we move to an inquiry-oriented instructional design in basic real analysis, drawing on ideas and methods from Realistic Mathematics Education (RME). Students get to reinvent fundamental ideas behind a classical proof of the intermediate value theorem while developing, on the way, related results such as convergence of monotone bounded sequences. Special attention is given to the generation of conjectures and sharing more or less correct proposals towards a proof. RME principles are used to allow a classroom community to engage in a collective, authentic mathematical activity in which classical proofs emerge from students' exploration of carefully designed questions.

The Extreme Apprenticeship model focuses on developing generic skills through inquiry-based university mathematics teaching. Rämö, Häsä, and Tuononen (Chap. 22 of this book) begin their chapter by explaining how generic mathematical skills and competencies have been theorized by previous research and how Extreme Apprenticeship has been experimented with at the University of Helsinki as a proposal for how to integrate their development in standard undergraduate and graduate mathematics courses, ranging from linear algebra to advanced courses on abstract algebra. They explain how the details of this proposal were developed over several years in close connection with more sophisticated descriptions of generic skills, which (when built into the curriculum) can provide more connectivity and progression in a study program in mathematics.

Within the context of Calculus, Kondratieva (Chap. 23 of this book) explores different levels of inquiry called for by student assignments, drawing on theoretical ideas like the Herbartian schema and praxeologies from the Anthropological theory of the Didactic. Taking as starting points specific tasks proposed in standard and reform Calculus texts, she demonstrates how more advanced forms of guided student inquiry can be generated through careful design of different kinds of activities which, while based on standard tasks, leave progressively more room for students' development of advanced forms of mathematical inquiry, such as posing derived questions and investigating hypotheses that emerge from the initial question.

Finally, Kontorovich, L'Italien-Bruneau and Greenwood (Chap. 24 of this book) analyze cases of students' proving activity in an experimental graduate course on topology based on the commognitive framework. The cases considered unfolding more or less according to the "proving at the board" approach proposed by Texan topologist Robert Lee Moore. The authors demonstrate how two individual students present proofs in contexts such as the finite intersection property of a collection of closed subsets of a compact set, and also how their commognitive actions in front of the class reveal subtle differences in their relationship with proof narratives, which should convince not only the prover but also a more or less concrete audience of the validity of a given proposition.

To sum up the contributions of this section, we are presented with a wide variety of cases and theoretical approaches to the highly complex notion of inquiry, as it is currently conceived in the context of university mathematics education. While classical "talk and chalk" lectures continue to be important in this context, the chapters provide different kinds of evidence that lectures can be supplemented or even replaced by more demanding forms of engaging students in mathematical inquiry – as participants, rather than mere spectators.

1.2.5 Section 5: Research on Mathematics for Non-specialists

While mathematics for non-specialists has long been problematized, it is only within the past two decades (or less) that the field has turned to systematically inquiring into the curriculum, teaching, and learning of mathematics for non-specialists, with a more focused examination of the mathematics service courses taken by engineering students and the mathematics that students encounter in their engineering courses. In the last several years alone, we have witnessed an increased interest in both of these areas of interest. Hochmuth recently surveyed some of the recent literature on *Service-Courses in University Mathematics Education* in the *Encyclopedia of Mathematics Education* (Hochmuth, 2020), and in 2021 the *International Journal of Research in Undergraduate Mathematics* published a special issue on mathematics in engineering education (Pepin et al., 2021). The chapters in this volume further contribute to these lines of inquiry, covering both mathematics courses for engineering students and mathematics in engineering or science courses. As a whole, the chapters in this section provide theoretically grounded insights into pedagogical, curricular, epistemological challenges, and frameworks for analysis.

The chapter by Romo Vázquez and Artigue (Chap. 26 of this book) provides an overview of the field of engineering education by surveying the literature with the goal of identifying how challenges have been dealt with over time and how they are produced and re-produced alongside scientific and technological advances, societal evolution, and emerging concerns. Their historical review includes the case of the *École Polytechnique* in France and three International Commission for Mathematical Instruction studies. Grounded in this historical overview, the authors then tender examples selected from recent research and development work, illustrating the progression of theoretical approaches, especially that of ATD, and the opportunities for future research.

Following nicely from the ATD overview provided by Romo Vázquez and Artigue, four chapters provide detailed ATD analyses of the praxeologies and challenges that learners encounter in their mathematics, engineering, and science courses as well as in the workplace.

In their chapter, González-Martín, Barquero, and Gueudet (Chap. 27 of this book) demonstrate how ATD praxeological analyses can uncover differences in the way mathematical tools are used in mathematics courses and engineering courses. They tender two examples of the outcomes of transposition processes by examining praxeologies involving mathematics in engineering courses. In the first example, they review a study that analyzed and compared the use of the Laplace transform in a mathematics course and in two control theory courses. In the second example, they highlight the use of integrals in reference textbooks and teaching practices in two engineering courses, on the strength of materials and electricity and magnetism. Their chapter concludes with examples of the ATD innovative instructional approach, study and research paths, aimed at reducing the gap between educational and professional practices with respect to mathematics for engineers.

The chapter by Peters (Chap. 28 of this book) uses the context of a mathematics service course for engineers with a focus on complex numbers. In contrast to the more standard approaches to make mathematics service courses more salient for engineering students (i.e., approaches that introduce engineering applications or approaches that make use of innovative instructional strategies such as study and research paths or project work), Peters presents a third approach, one that takes mathematical exercises and focuses on establishing and promoting connections to electrical engineering discourses within mathematical discourses. This is done without also introducing the engineering context. At the core of this approach is the ATD concept of institutional dependence of knowledge.

Rønning (Chap. 29 of this book), in his chapter, reports on the redesign of a basic course in mathematics for first-year students that is taught in close connection with a course on electronic system design and analysis. Rønning uses ATD to interrogate the discourses that develop with the aim of unpacking how the praxeologies in mathematics and engineering influence and interact with each other. In particular, he uses electric circuits as a concrete example to demonstrate how one can shift the emphasis of specific topics as well as change the sequencing of the topics to better meet the needs and interests of engineering students. Interviews with teachers in both mathematics and electronics design course reveal the challenges and tensions that both teachers face.

In contrast to the chapters by González-Martín et al., Peters, and Ronning, all of which examine praxeologies in either mathematics service courses for engineers or the mathematics in engineering or science courses, Castela and Romo Vázquez (Chap. 30 of this book) focus on the opportunities and challenges of designing teaching sequences based on authentic professional workplace situations. They present an ATD analysis of praxeologies in three different professional workplace situations: land surveying, automotive industry, and computer science. One of the challenges they identify is the felt need to teach students more sophisticated mathematics than those employed in normal professional practices. Reasons for this include the potential usefulness for career advance and the possible need to adapt to future changes in professional practice.

Three additional papers report on empirical investigations that examine students' understanding of mathematics that they encounter in their engineering courses. Each of these papers offers fresh insights into the nuances of student thinking, and each offers innovative frameworks of considerable potential for future research.

Hjalmarson, Nelson, Buck, and Wage (Chap. 31 of this book) examine students' reasoning with conceptual problems in signals and systems, a subfield of electrical engineering. Using student interview data from two different institutions, they investigate how students interpret, describe, and reason with graphical representations of signals and systems problems. To do so, they adapt a framework originally developed to interpret student understanding of derivatives that leverages the constructs of concept image and process-object pairs. Their analysis highlights both students' challenges and successes in thinking about and translating among multiple representations of the same signal and opens up possibilities for further adaption of the use of this framework in other contexts.

Kortemeyer and Biehler (Chap. 32 of this book), in their chapter, examine the issue of how mathematics and what kind of mathematics is used and needed when students are asked to solve problems in their engineering course. As a case study, they analyze tasks on an end-of-year examination in a fundamentals of electrical engineering course. Their analysis includes interviews with experts to identify the expected competencies, both implicit and explicit, which were then used together with theoretically informed approaches to develop an a priori student-expert solution. This analysis culminated in a transferable framework for the analyses of exercises, problem-solving strategies in engineering exercises, and typical sources of errors.

The penultimate chapter in this section by Jablonka and Bergsten (Chap. 33 of this book) examines the social and cultural conditions and the institutional context using student interview data from first-year engineering students. Their analysis, which is grounded in Bourdieu's notion of habitus and the dialectic between experiences and perceptions, contributes to a deeper understanding of students' appreciation of specificities of mathematical discourse encountered in the core mathematics, students' perceptions of the usefulness of mathematics, and their experiences of studying mathematics as compared to other subjects. Their findings reflect that success in the service courses depends on recognizing the criteria of pure mathematics as opposed to mathematical applications or modeling. Their work also contributes to a theoretically and empirically informed framing of four different student-perceived modes of the usefulness of mathematics.

The final chapter by Fredriksen et al. (Chap. 34 of this book) is unique from the previous chapters in that it provides an overview of several dissertations that have been carried out under the Norwegian Centre for Excellence in Education (MatRIC). The common focus across these dissertations is the teaching and learning of mathematics as a service subject. Indeed, the improvement of student success in these courses across Norway is a foundational mission of MaTRIC. The research projects highlighted in this chapter adopt a variety of approaches to address concerns about teaching development (flipped classroom and blended learning approaches), shortcomings in students' prior knowledge, use of digital technology in learning, mathematical modeling, and exposing causal relationships between learning approaches and outcomes. Taken as a whole, the chapter sets forth a strong foundation for continued research that aims to improve student success both in mathematics courses for non-specialists and for the teaching and learning of mathematics within engineering and science courses.

As the brief overview and highlights of the chapters in this section reveal, the body of research focusing on mathematics for non-specialists is a rich and growing domain. While much has been learned, there is clearly a continued need for further theoretically informed innovations that build on the advances to date and address the thorny and persistent epistemological, pedagogical, and curricular challenges.

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Part I
Research on the Secondary-Tertiary Transition

Chapter 2
Emotions in Self-Regulated Learning of First-Year Mathematics Students

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Abstract This contribution aims at describing and explaining relations between emotions and self-regulated learning of first-year university mathematics students. To this end, models of self-regulated learning are discussed considering empirical findings on studying mathematics at university, emotions are introduced from the perspective of control-value theory, and these two approaches are integrated in a joint model of achievement emotions in self-regulated learning. Empirical findings on the basis of problem-centered interviews with 21 first-year university mathematics students emphasize the importance of perceived control, but also the perceived value of mathematical content and exercises for the arousal of emotions as well as self-regulated learning. These findings contribute to new perspectives for interpreting some phenomena of university mathematics education as well as for evaluating the relevance and further development of already existing mathematics support activities.

Keywords Self-regulated learning · Emotions · Control-value theory · University mathematics education · Mathematics teacher education

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2.1 Introduction: The Transition from School to University in Mathematics

The transition from school to university in mathematics has become an increasingly popular subject for research projects in recent years (e.g., Gueudet & Thomas, 2020, for a short overview). In this context, particular attention is paid to differences between mathematics at school and at university and the cognitive demands and, in some cases, the motivational effects associated with these (Gueudet & Thomas, 2020; Liebendörfer & Schukajlow, 2017; Rach & Heinze, 2017; Rach & Ufer, 2020). Besides, it must be considered that the institutionally intended learning of mathematics differs significantly between school and university. For example, the time allocated to self-study is significantly higher at university than at school. According to study regulations at German universities, about 60% of the study time scheduled for mathematics courses is assigned to self-study.

In such institutional settings, the relevance of self-regulated learning is even higher than at school. Self-regulated learning can be defined as “an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and constrained by their goals and the contextual features in the environment” (Pintrich, 2000, p. 453). Accordingly, goals, strategies, and the regulation of these strategies with respect to goals and contextual conditions are essential components of self-regulated learning that are elaborated differently in different theoretical models (e.g., Panadero, 2017 for an overview). The relevance of emotions for self-regulated learning is included in most of these models, but is often not explicitly detailed (see Muis et al., 2018 for an exception).

With regard to the learning of mathematics at university, in addition to the goals of learning mathematical content or solving tasks, exam-related goals and goals concerning students’ well-being can be identified, which in turn influence self-regulated learning (Anastasakis et al., 2017; Göller, 2020). Moreover, research in undergraduate mathematics education has provided important insights into students’ (and experts’) strategies, e.g., by investigating their problem-solving strategies (e.g., Pólya, 1945; Schoenfeld, 1985; Selden & Selden, 2013; Sommerhoff et al., 2021), their activities learning new mathematics (Wilkerson-Jerde & Wilensky, 2011), or different types of students’ engagement with proofs (Selden & Selden, 2017), including students’ proof-reading strategies in detail using task-based interviews (e.g., Weber, 2015) or eye-movement studies (e.g., Panse et al., 2018).

However, the goal to solve the problem, to learn new mathematics, or to comprehend a proof is given in these studies by study design. Accordingly, it is not clear to what extent such strategies are used in “everyday” self-study, or whether they may be displaced due to competing other goals. For example, there is evidence that students do not only use such learning and problem-solving strategies, but also, especially when solving exercises, work in groups, search for solutions on the internet or in books, or copy solutions from other students (Göller, 2020; Gueudet & Pepin, 2017, 2018; Haak et al., 2020; Kock & Pepin, 2018; Liebendörfer & Göller, 2016; Stadler et al., 2013).

In addition, there is evidence that the transition from school to university in mathematics also challenges students from an emotional perspective (Hailikari et al., 2016; Martínez-Sierra & García-González, 2016), for example, being perceived as frustrating (Liebendörfer & Hochmuth, 2015; Sierpinska et al., 2008), and that such emotions have an impact on students' self-regulated learning (Göller & Gildehaus, 2021). In the present chapter, we will examine such interconnections between emotions and self-regulated learning of first-year mathematics students.

2.2 Theory

In this paper, we aim to theoretically and qualitatively empirically identify relations between emotions and self-regulated learning in undergraduate mathematics. To this end, theoretical models of self-regulated learning are discussed in consideration of empirical findings on studying mathematics at university (Sect. 2.2.1), emotions are introduced from the perspective of control-value theory (Sect. 2.2.2), and these two approaches are integrated in a joint model of achievement emotions in self-regulated learning (Sect. 2.2.3).

2.2.1 Self-Regulated Learning in Undergraduate Mathematics

Self-regulated learning is often described via cyclical phase models. Within a *forethought phase*, students first construct a perception of a given learning occasion which defines the task (Muis et al., 2018; Winne & Hadwin, 1998). This task definition is influenced by context, behavior, cognition, motivation, and also emotions (Muis et al., 2018). As a second step within the forethought phase, students set goals and plan their strategies according to this task definition, which are influenced by their motivational beliefs such as self-efficacy, outcome expectations, task interest and value, and goal orientation (Zimmerman & Moylan, 2009). Within a *performance phase*, these strategies are enacted, monitored and adapted if necessary (Muis et al., 2018; Winne & Hadwin, 1998; Zimmerman & Moylan, 2009). Finally, the outcomes of the preceding phases are evaluated in a *self-reflection phase*. These evaluations in turn influence the task definition, goal setting and strategic planning of the forethought phase (Muis et al., 2018; Winne & Hadwin, 1998; Zimmerman & Moylan, 2009).

The dual processing self-regulation model of Boekaerts (2007, 2011) posits three goals of self-regulated learning: Expanding one's knowledge and skills, maintaining well-being by preventing threat to self, and protecting one's commitment to the learning task. Students constantly appraise learning contents and learning tasks in terms of these goals. If learning contents or tasks are in line with students' values, needs, and goals, students are on the "mastery path" (cf. Fig. 2.1), focus on expanding their knowledge and skills, and use strategies to achieve their learning goals. If the learning situation is appraised as threatening to well-being (e.g., because the task is perceived as too complex or associated with negative emotions), there is a mismatch between the task-intended learning goals and students' values, needs, and goals (e.g., feeling safe, respected, self-confident, protected), students' focus is on preventing threat to self, and they will use coping strategies like working harder, focusing on the positive, seeking social support, avoidance, denial, giving up, or distraction (Frydenberg, 2004) to maintain or restore well-being ("well-being path"). Students on the mastery path may also appraise a task as threatening, which causes them to switch towards the well-being path. In this case they can use volition strategies (e.g., prioritizing goals, time and resource management, Boekaerts & Corno, 2005) to protect their commitment to the learning task and switch back onto the mastery path. Also students on the well-being path can use volition strategies to get on the mastery path (see Boekaerts, 2007, 2011, for a detailed description of the dual processing theory).

While the mentioned phase models provide a suitable framework for categorizing and describing students' task-related beliefs, goals, strategies, and evaluations on the mastery path, Boekaerts dual processing model enables a broader view on goals and strategies which do not need to be exclusively related to learning new content or solving a task. With regard to the study of mathematics, it can be seen as mentioned above (Sect. 2.1) that, in addition to learning and problem-solving strategies, students also use coping strategies, such as searching for exercise solutions on the internet or copying exercise solutions from others (Göller, 2020; Gueudet & Pepin, 2018; Liebendörfer & Göller, 2016). In addition to these strategies, corresponding goals can be identified, which are more accurately described in Boekaerts' dual than in the phase models described previously (Göller, 2020). Furthermore, there is some evidence for the importance of emotions for self-regulated learning of mathematics at university (Göller & Gildehaus, 2021), which will be considered in more detail in this contribution.

2.2.2 *Achievement Emotions and Control-Value Theory*

The role of emotions in self-regulated learning has received little attention in research on university mathematics. Emotions are defined as affective episodes which constantly mediate between changing events, social contexts and the reactions and experiences of individuals (Mulligan & Scherer, 2012). Such emotion episodes comprise various components including appraisals of the situation, action preparation, physiological responses, expressive behavior, and subjective feelings (Scherer & Moors, 2019).

Emotions can be characterized by their valence (positive – negative) and degree of activation (Boekaerts & Pekrun, 2015): Examples of activating positive emotions are joy and hope, whereas relief and relaxation are deactivating positive emotions.

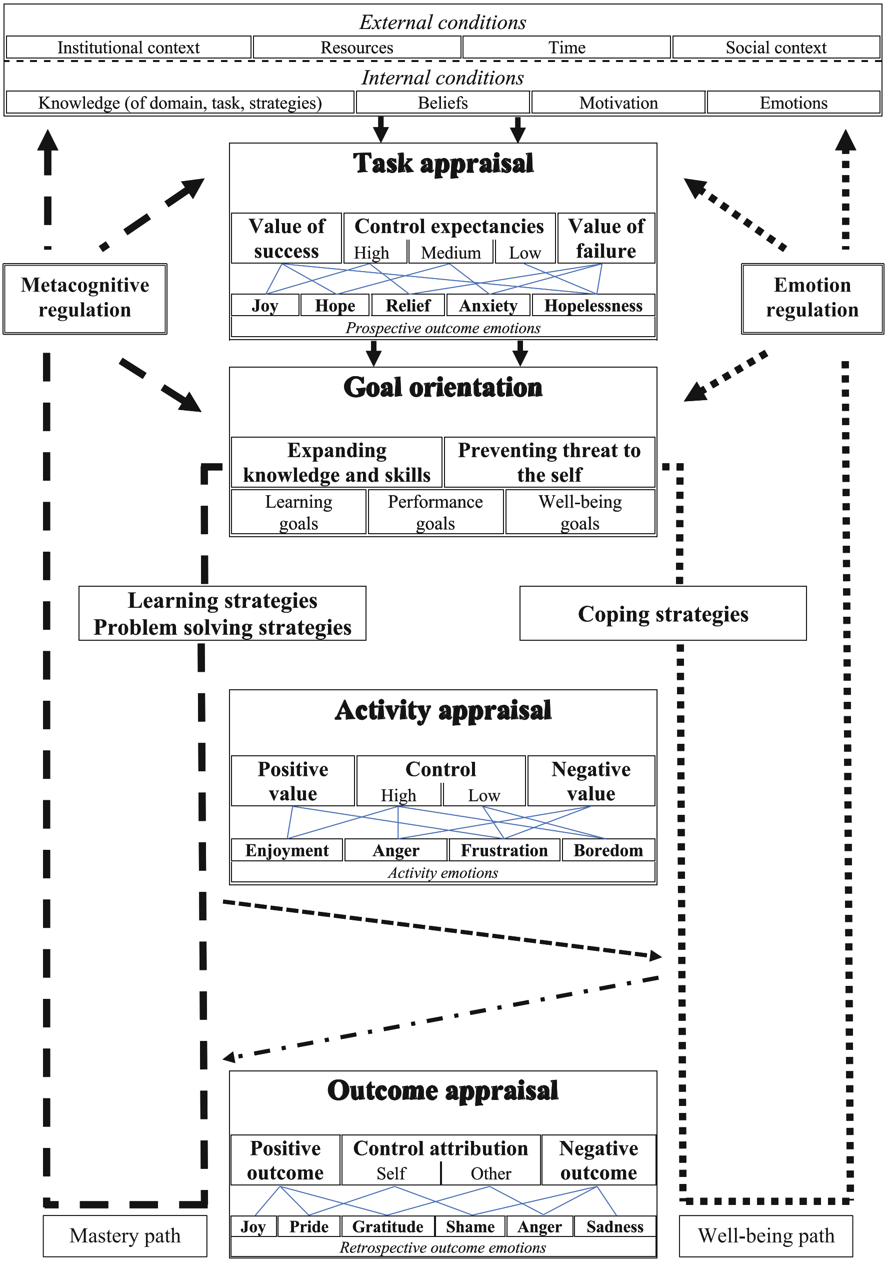

Fig. 2.1 An integrated model of achievement emotions and self-regulated learning

Examples of negative activating emotions are anger and anxiety, whereas hopelessness and boredom are deactivating negative emotions. In this contribution, we focus on joy, relief, anxiety, and hopelessness.

There is some evidence that positive activating emotions (e.g., joy) induce intrinsic motivation, focus attention on the learning task, and support flexible learning strategies and academic performance, whereas positive deactivating emotions (e.g., relief) can have diverse motivational effects, may lead to superficial learning strategies, and thus variable academic performances (Boekaerts & Pekrun, 2015). Conversely, negative activating emotions (e.g., anxiety) undermine intrinsic motivation, but can stimulate motivation to avoid failure and thus more rigid learning strategies which may have different effects on academic performance. Finally, negative deactivating emotions undermine motivation, are related to task irrelevant thinking, shallow learning strategies, and thus weaker academic performances (Boekaerts & Pekrun, 2015).

Emotions which refer to academic achievement activities (e.g., learning or studying) or achievement outcomes (e.g., grades) are called *achievement emotions*. Joy, relief, anxiety, and hopelessness are achievement emotions. The *control-value theory of achievement emotions* (Pekrun, 2006) proposes that achievement emotions are a multiplicative function of two groups of appraisals: (1) The subjectively perceived *control* over achievement activities and their outcomes and (2) the subjective *values* of these activities and outcomes. Both, subjective control and subjective values can refer prospectively and retrospectively to outcomes as well as to activities and accordingly result in *prospective outcome emotions, retrospective outcome emotions*, and *activity emotions*. Control describes the prospective, concurrent (in action), or retrospective subjective appraisal of how outcomes depend on actions that can be or have been autonomously initiated and executed (Pekrun, 2006; Weiner, 1985).

In control value theory, intrinsic and extrinsic values are distinguished. Intrinsic values refer to the value of an outcome or activity per se, while extrinsic values refer to the instrumental utility of outcomes or activities for achieving other goals. Outcomes and activities in control-value theory can be negatively valued, e.g., in form of the subjective value (respectively cost) of an outcome that is appraised as failure, or when the effort required by an activity is experienced as unpleasant (Pekrun, 2006).

According to control value theory, joy as prospective outcome emotion is aroused when subjective control and the value of success is high. As retrospective outcome emotion, joy is the result of a positively valued outcome (independent of control, see Fig. 2.1). If the value of failure is high (e.g., because a negative outcome threatens passing an examination or (respectively, and thus) well-being), high control arouses relief, medium control arouses anxiety, and low control arouses hopelessness. Hopelessness is also aroused, when control is low, and the value of success is high (Pekrun, 2006, or Fig. 2.1).

2.2.3 An Integrated Model of Achievement Emotions and Self-Regulated Learning

Figure 2.1 shows a model of self-regulated learning which integrates Boekaerts' (2007, 2011) dual processing self-regulation model and Pekrun's (2006) control-value theory of achievement emotions. The model can also be seen as a cyclical phase model (cf. Muis et al., 2018; Winne & Hadwin, 1998; Zimmerman & Moylan, 2009) with a forethought phase, in which students define their task and set their goals according to external and internal conditions as well as their appraisal of the task, a performance phase, in which learning, problem-solving and coping strategies are enacted, and a self-reflection phase, in which outcomes are appraised.

If learning content or tasks correspond to students' values, needs, and goals, they follow these phases on the mastery path and use learning or problem-solving strategies to expand their knowledge and skills. In line with Boekaerts dual processing self-regulation model, the model includes well-being goals which gain primacy when a task is appraised to threaten well-being. In this case students use coping-strategies to restore well-being (well-being path) rather than learning or problem-solving strategies. Students can use volition strategies to stay or get on the mastery path (◄· ━) although a task is appraised to threaten well-being (━━►).

Control-value theory (Pekrun, 2006) enables the inclusion of achievement emotions in this model by looking at perceived control and values of outcomes and activities. For example, it is hypothesized that a student who expects the (successful) outcome of a given task to be highly controllable and of high value will experience (anticipatory) joy (as prospective outcome emotion) and aim at expanding her knowledge and skills by completing the task. To do so, she will use learning or problem-solving strategies and monitor her progress. While working on the task, the task outcome may turn out to be more difficult to control (e.g., due to unexpected problems) or she might lose interest (negative value). She will experience anger, frustration, or boredom (respectively, see Fig. 2.1, for details) as activity emotion, which may lead her to focus on reducing these negative emotions rather than completing the task. She may use coping-strategies (e.g., seek social support) to reduce her negative emotions (well-being path) or use volition strategies to stay on the mastery path. In case of a positive (respectively negative) perceived outcome she will experience joy (respectively sadness) as well as pride or gratitude (respectively shame or anger) as retrospective outcome emotion, depending on whether she attributes control to herself or to others (e.g., her social support). Such retrospective outcome emotions are in turn assumed to influence internal conditions and thus the appraisal of new tasks.

Conversely, for example, a student who expects the outcome of a given task to be uncontrollable is assumed to experience hopelessness as a prospective outcome emotion, both if the value of success or the value of failure is high. In this case, the student will rather not try to solve the task using learning or problem-solving strategies (since he does not expect to control the outcome autonomously with these), but rather use coping strategies (e.g., seek social support, or give up) to prevent threat to the self. Yet, the subsequent phases can proceed as in the first example.

These two examples serve to provide a brief impression of the possible connections between emotions and self-regulated learning. As Fig. 2.1 shows, many other pathways are feasible. In particular, empirical research is needed to determine how, e.g., anxiety (as activating negative prospective outcome emotion) or relief (as deactivating positive prospective outcome emotion) influences self-regulated learning. Hence, in the following we will qualitatively examine how joy, relief, anxiety, and hopelessness interfere with self-regulated learning of mathematics students within their first year of study.

2.3 Research Interest and Research Questions

The purpose of the present study is to examine interconnections of the achievement emotions joy, relief, anxiety, and hopelessness with self-regulated learning of university mathematics. For this purpose, the following research questions are investigated:

RQ 1. In which contexts do mathematics students experience joy, relief, anxiety, or hopelessness within their first year of study?

RQ 2. What roles do perceived control and subjective values play in the emergence of joy, relief, anxiety, and hopelessness?

RQ 3. How do joy, relief, anxiety, and hopelessness interfere with self-regulated learning of mathematics students within their first year of study?

2.4 Methods and Research Design

Since it is assumed that the institutional context influences achievement emotions and self-regulated learning, the institutional settings of the study are presented first, then the research design is outlined, and finally the data analysis methods are described.

2.4.1 Institutional Context of the Study

The study was conducted at a German university. Like in many German universities, the mathematics modules of the first semesters contained lectures and related exercises, for several study programs (e.g., mathematics, physics, pre-service teachers). The lectures introduce mathematical theory (e.g., Analysis like in Rudin, 2007), i.e., definitions, examples, theorems and their proofs are presented. Exercises are handed out every week and have to be worked on by students in self-study. Students' solutions are submitted, get corrected and graded, and are discussed in a separate lesson. In order to pass such a module, a certain number of exercises (often 50% of all exercises) has to be solved correctly and a written exam has to be passed.

2.4.2 Data Collection

Empirical basis for the results of the present study are problem-centered interviews (Witzel, 2000) on self-regulated learning with a total of 21 students (14 of whom were female). Ten interviewees (9 female) were enrolled in a degree program for mathematics teachers at upper secondary level (gymnasiales Lehramt), seven (3 female) in the degree program Mathematics B.Sc., two (1 female) in the degree program Physics B.Sc., and two (1 female) in a degree program for business education with a minor in mathematics (no further socio-demographic data were collected).

All interviewees were in their first year of study in their respective programs and were interviewed up to four times. A first interview period was about 2 weeks before the first lecture, a second about 4 weeks after the start of lectures, a third just before the end of the first semester, and a fourth in the middle of the second semester. The respective interviews had a duration of about 45 minutes. The interviewees were asked about their strategies, goals, beliefs, and evaluations regarding their study of mathematics (cf. Göller, 2020, for a detailed description). Emotions were not explicitly addressed by the interviewer, but they were nevertheless reported by the interviewees at some points. The interviews were conducted with two cohorts in two consecutive years between October 2013 and June 2015.

2.4.3 Data Analysis

The interviews were audio-recorded and transcribed completely. To investigate the research questions listed above, the transcripts were searched for word fragments in order to identify text passages addressing joy, relief, anxiety, and hopelessness: Joy was operationalized by the word fragments "freue/freut/freud" (German freuen/Freude = to enjoy/joy) and "Spaß" (=fun). Anxiety was operationalized by the word fragments "angst/ängstl" (German Angst/ängstlich = anxiety/anxious), "panik" (= panic) and "sorge" (German sich sorgen = to worry). Hopelessness was operationalized by the word fragments "hilfl" (German hilflos = helpless), "verzw" (German verzweifeln/Verzweiflung = (to) despair), "depri" (German deprimiert/deprimierend = depressed/depressing), and "frust" (=frustration). Relief was operationalized by the word fragments "erleichter" (German Erleichterung/erleichtert = relief(ed)), as well as by passages were students reported a reduction of anxiety, stress, or hopelessness.

The transcript passages (between two interview questions) that contained these words were then analyzed in more detail using a mixed approach of concept-driven and data-driven development of codes (Kuckartz, 2019). In this way, subcategories of joy, relief, anxiety, and hopelessness were developed, with particular attention to the context of these passages and connections to statements related to control and value. To illustrate possible further steps in coding, some of the passages cited here were coded deductively with these codes, resulting in codings that reached beyond the keywords mentioned above. Finally, to approach RQ 3), overlaps of coping and volition strategies given in the literature (Boekaerts, 2011; Boekaerts & Corno, 2005; Frydenberg, 2004) and found in previous analyses of the data (Göller, 2020, 2021, 2022; Göller & Gildehaus, 2021) were considered and discussed with regard to theoretical implications and explanations by means of the integrated model of achievement emotions and self-regulated learning (Sect. 2.2.3). The interview excerpts presented here were translated from German by the authors.

2.5 Results

2.5.1 Joy, Relief, Anxiety, and Hopelessness in the First Year of Study

To answer RQ 1), Table 2.1 provides an overview of the contexts in which joy, relief, anxiety, and hopelessness were mainly reported by the interviewees.

Joy (as a retrospective outcome emotion) was primarily experienced when students were able to solve an exercise task autonomously. Although joy was also experienced in the context of evaluations by others, such as feedback on the exercises (e.g., "*I am always happy when I have more than 50 %.*") or the exam (e.g., *"I passed both exams last semester. I was very happy about that."*), both the number of reports and the intensity experienced with regard to the joy of having solved an exercise task successfully on one's own were higher. In addition, students seemed to have certain preferences in terms of their enjoyment of certain content or tasks (cf. Sect. 2.5.2).

Relief was predominantly experienced in the second semester and was reported in various forms by all students who participated in the interviews at that time. One reason for this was that in the first semester, the **anxiety** of not being able to solve 50% of the exercises or not passing the exams was a prevalent emotion. At the same time, many students experienced **hopelessness** when they feel unable to solve the exercises autonomously (low control regarding the outcome of the exercises) and did not know what to do to understand the content, or when solving the exercises and understanding the content demanded too much time and other resources (overload).

A major factor for (exam) anxiety was the uncertainty of not being able to determine which contents and skills are essential for the exam and further studies, which makes everything seem important (overload). This can also explain the relief in the second semester, after having experienced an exam. The following interview

Table 2.1 Codes and example quotations of contexts joy, relief, anxiety, and hopelessness was reported (searched keywords **bold**)

Code	*Example quotations*
Joy success solving a task	*When I get something done, I'm* ***happy****, I cheer.* *And then, bam, I had solved this task! And I was so* ***happy*** *that I got it right! That's cool, I really* ***enjoy*** *that.*
Joy certain tasks/ content	*For example, with these ranks, with the matrices and so on. I found that really exciting. I had a lot of* ***fun*** *with it.*
Relief exam experience	*After seeing the first exams, I thought to myself, okay,* ***don't stress*** *so much. So, these exams have really had a very big impact on me.*
Relief strategies	*At the moment, I'm* ***not afraid*** *of not getting the 50 percent. If necessary, I would copy [solutions], I have to admit. That's clear. But I think now, it should work out.*
Anxiety exam	*I'm also pretty* ***scared*** *of the exam because I don't know what to expect…*
Anxiety exercises	*And now I've done maybe half of the tasks in three days. That's when you start to* ***panic*** *a bit.*
Hopelessness low control	*… and then at some point you get* ***frustrated****, because it's just stuff that you don't understand anymore.* *It's* ***depressing*** *when you're about halfway somewhere in the middle of nowhere and you don't know what to do.*
Hopelessness overload	*And just this inner voice that keeps saying: "Yes, you still have a whole math exercise sheet, which takes forever." that […] somehow already completely destroys me psychically, knowing that I still have such a huge amount of work ahead of me that I somehow still have to do. So somehow that already makes me psychologically totally unstable, unhappy, I don't know. It* ***frustrates*** *me.*

excerpt provides good insight into this shift from anxiety to relief to joy by a relatively successful (female mathematics major) student[1]:

> *Before the first exam, I didn't know what I should learn. I didn't know how I should prepare myself, because somehow, I didn't quite get to grips with proofs and how to come up with such a proof.*[2] *I was totally afraid that there would be a proof, because somehow, I didn't know at all what to expect.*[3] *[…] And then I saw the first one [exam] and saw that I only had to calculate.*[4] *I had only one/ uh two proofs. […] And when learning, I just took the old exercise sheets and calculated them again, so the calculation tasks, where you could imagine that they would come up. [...] And then I also looked at the smaller proofs. And, yes, that went quite well. My proof was also completely correct. That was totally awesome.*[5] *[…] That was quite funny, because I spontaneously started to prove it. And I looked afterwards, and I had it just right, I thought: Yes! That's how it should be. I was totally happy about that.*[6]

[1] Associated codes are given in the footnote. See ESM for the original in German.

[2] Hopelessness low control.

[3] Anxiety exam.

[4] Relief exam experience.

[5] Joy success solving a task.

[6] Joy success solving a task.

The relief in the second semester can also be explained by the fact that students apparently develop learning, problem-solving and coping strategies in order to be able to cope with the demands more effectively (cf. Sect. 2.5.3). The following interview excerpt illustrates the emotional roller coaster ride from the first to the second semester:

*It was an absolute torture for me [in the first semester]. It was just, you had to do these f*** exercises, and nothing worked.*[7] *And now all of a sudden, it's flying, and life is a lot easier too.*[8] *It's no joke. [Before] You just handed in your homework on Thursdays, and you were happy. You were happy as cheese.*[9] *And then the next one came around.*[10] *And now it's just that you think to yourself like this: Yes, good, the next exercise is here. Then I'll start on it soon. But it doesn't weigh so heavily on your shoulders if you allow yourself a day to do something other than homework.*[11] *Before it was like this, I always had in the back of my mind: Ha, I still have to do these stupid exercises. I still have a week, but it always takes me so long*[12] *and I can't do it.*[13] *And what if they are even harder than the last time?*[14] *Because it doesn't get easier and stuff. And I couldn't sleep. I slept a maximum of three hours at night, it was like that for weeks, before Christmas and even after Christmas, that I just, I couldn't sleep. [. . .] I always had everything in my head and yet at the same time I had nothing in my head. I have always forgotten everything. [. . .] And now that I know that I'll probably be able to do it, I think to myself, okay. Then I'll take three or four days for it, I'll manage that. [. . .] Then I can look forward to a weekend and just chill out with my boyfriend and so on.*[15] *Then I just start on Monday. And no more Saturday and Sunday evenings where I sit there with my computer and stuff.*[16] *That was really terrible. So that was really disgraceful. And now that you know so much more and also remember it, it's really fun when you get a homework done. That's really cool, yeah. That's really good.*[17]

2.5.2 The Roles of Perceived Control and Subjective Values in the Emergence of Joy, Relief, Anxiety, and Hopelessness

The previous subsection (Sect. 2.5.1) already demonstrated the importance of perceived control for the emergence of joy, relief, anxiety, and hopelessness: Joy was primarily aroused when a task was solved autonomously (positive outcome, and

[7] Hopelessness low control.
[8] Relief strategies.
[9] Joy success solving a task.
[10] Hopelessness overload.
[11] Relief strategies.
[12] Hopelessness overload.
[13] Hopelessness low control.
[14] Anxiety exercises.
[15] Relief strategies.
[16] Hopelessness overload.
[17] Joy success solving a task.

attribution of control to the self). Exam-related and exercise-related anxiety, as well as hopelessness when students feel unable to solve the exercises autonomously or do not know what to do to understand the content can likewise be seen as a lack of perceived control regarding outcomes or strategies to influence these outcomes positively. In the second semester, students felt more able to determine what content and skills are essential (e.g., based on exam experience) and developed strategies to better manage the demands of studying mathematics, which increased perceived control and could explain relief. These patterns are in line with control-value theory, since it can be assumed that passing exams and exercises has a high (extrinsic) value for the students (value of success for joy, value of failure for relief, anxiety, and hopelessness).

In addition to these extrinsic values which are induced by the institutional setting, at least three other values were found to influence the total value appraisal of mathematical content and tasks: (1) the perceived difficulty, (2) the perceived utility value, and (3) the intrinsic value of content and tasks.

The perceived difficulty of content and exercises in turn was influenced by the social context.

> *Well, because you're usually not the only one or the only one in math, who despairs, it is actually okay. So as long as you have some kind of contact with your peers, it works. I mean, you're usually not alone. If an exercise is completely difficult, then there are at least five other people you know who can't do it either. And that's always a little comfort. But of course, it's depressing when you see people who can do it easily.*

In this social context, the difficulty of the content and exercises was considered rather high, which increases the value of success and decreases the value of failure, which in turn may explain relief (comfort) here. However, the fact that others could solve exercises easily decreases the value of success and increases the value of failure, which explains hopelessness (depressing) here. The generally perceived high difficulty allows small successes to be valued highly, which increases joy:

> *And that, I think, is very difficult. Especially for students who have always been the best [at school]. And always had good grades. And then when you come here, and you've only got 50 % on your exercise sheet and everyone jumps for joy when they've got the 50 %.*

The perceived utility value of mathematics content and tasks mattered to some student teachers, who often considered it rather low when they did not see many similarities between school and university mathematics (e.g., *"I don't see where I could use this in school"*).

The intrinsic value of content and tasks depended on individual preferences. Evidently, there seem to exist differences in student preferences that do not change much during the first year of study. Some students enjoyed calculations (and mathematics that is familiar to them and similar to what they did in school) and tended to dislike proofs:

> *What we're doing right now is at least a little bit more fun because it's the math that I remember from school. You get your instruction, you get your rule, and then you calculate according to it. That's what I like about math. That's why I wanted to study math, because I enjoy it. Because once you can do it, you can do it. And then you can do the calculations. But this proving, uh, I'm going crazy with it, really.*

Others rather preferred proofs and new content:

> *What I found really fun were some of the proofs. I liked them very, very much. I really enjoyed them. I sat on one of those for quite a long time, because I really wanted to complete it. I also had some problems, but I was happy. At the latest at the moment when you write q.e.d. on it, you are just happy. Because then you have proven it, on your own, you have thought about it, you have just proven something. And if it's correct in the end, then it's just wonderful.*

In addition, these quotes show that perceived control is also important for intrinsic values.

2.5.3 Joy, Relief, Anxiety, Hopelessness, and Self-Regulated Learning

As we have seen in the previous subsections (Sects. 2.5.1 and 2.5.2), perceived control is a major concern in university mathematics education in terms of achievement emotions. In case of low perceived control, the exercises can be seen as a major threat for students' goals in the given institutional setting: both from an emotional point of view (as anxiety and hopelessness are aroused) and with regard to the success of the study as a whole (e.g., the long-term goal of passing the module or study program). Accordingly, if perceived control is low, coping strategies (such as working harder, focusing on the positive, seeking social support, avoidance, or denial) or volition strategies (such as prioritizing goals, time and resource management), are theoretically expected to be used to stay (at least partly) on the mastery path, and protect the self and long-term goals. The interviews support this assumption (cf. Table 2.2).

In terms of goals, this means: The institutionally predetermined goals (passing the exam, achieving 50% of the exercise points) had the highest priority among the students surveyed here (prioritizing goals). Thus, if perceived control is low and these goals are threatened, subordinate goals (e.g., understanding lecture content, solving exercises autonomously) are adjusted (e.g., to understanding only certain parts, comprehending solutions to exercises).

In terms of strategies, this means: If perceived control is low and, for example, the goal of achieving 50% of the exercise points is threatened, students will try to complete exercises by working harder, seeking social support (which ranges from working collaboratively with others on the exercises to copying exercise solutions from others), searching for ideas or solutions in books, internet, etc., or (in most cases) using all of these strategies together. The following interview excerpt provides a good illustration of this process:

> *Tuesday at noon we have two and a half hours off. All from my math crew. And then we always sit together and try to solve the algebra exercises. After the two and a half hours we usually realize that we have solved maybe one and a half problems and have no idea about the rest. Since the submission deadline for the exercises is on Thursday [. . .], the panic breaks out at the latest on Thursday morning, when we sit together for another two and a*

Table 2.2 Codes and example quotations of some coping and volition strategies

Code	*Example quotations*
Working harder	*Monday I was really working on the exercise sheet until half past three in the morning, because I had to hand it in on Tuesday.*
Seeking social support	*Sometimes I also asked other students if I didn't understand something.*
Working together	*And then we talk about it and come up with the solution. Together.*
Copying	*Because we have a study group now, and somebody always has the solution. And most of the time I just copy them to get my admission [50%].*
Resource management	*If I can't get any further, I read in the book or on the internet and try to find a solution.*
Adjusting goals	*And I think that's just the problem at the beginning, that you have to really overpower yourself, but also accept that sometimes it doesn't get better than poor.*
Focusing on the positive	*And then I think to myself, okay, I haven't given up on it so far. That's pretty good.*
Avoidance	*[trying to] sit down, understand everything again. I think that would just frustrate me for the whole semester.*

half hours. Because still no one has a clue about two tasks. And then we talk or write to other math people, where most of them have just as little idea as we do, and then at some point we just send the solutions back and forth, copy them and hand them in.

Although such coping strategies help to achieve institutionally predetermined goals, they do not seem to be supportive of solving exercises autonomously, which in turn is the main reason for joy (cf. Sect. 2.5.1). In this way, such strategies reflect back on perceived control and thus on achievement emotions.

At this point, the importance of values for achievement emotions and self-regulated learning becomes apparent: With a focus on the value of failure (e.g., not achieving 50%, or not passing the exam), most of the students interviewed here had developed strategies for the second semester which increased their perceived control over not reaching 50% of the exercise points or failing the exam. Accordingly, there was a shift from anxiety and hopelessness towards relief (cf. Sect. 2.5.1). With a focus on the value of success (e.g., understanding lecture content, solving exercises autonomously), however, these strategies were not necessarily suitable to control these positive outcomes. Additionally, some students rather did not value lecture contents and exercises (cf. Sect. 2.5.2). As a result, some students did not enjoy their mathematics study which impacted their motivation:

I can't do anything, I don't understand anything, it's totally screwed up. I don't have any motivation anymore, it's just gone down the drain, already last semester.

Others valued content and exercises and had developed strategies to control their success:

This precision, however, is exactly why I started studying math, and now I see that it can be done much more precisely. And I'm enjoying it even more. [. . .] I actually think Christmas was one of those clicking moments. Before Christmas, I was so incredibly in despair. [. . .] That was the moment when you realized: You are no longer in school here. When you *realized that this hasn't much to do with school mathematics anymore, you should really get rid of all that. And I think that was the moment when I could best get involved in the mathematics that is here at the university. [. . .] And then all of a sudden, I started to gradually understand more and more. And somehow also got better at the exercises and so on. That has somehow given such a push forward.*

But even if students were not able to solve exercises autonomously, they could still achieve positive emotions towards their mathematics studies by setting more controllable goals. The following interview excerpt shows that this is possible even for students who rather prefer calculations to proofs (if they are able to define goals that they can value and control):

As I said, calculations with such things that you have to find somehow. That is, not to prove or not to show, but to find. I've always enjoyed that. Or in general, that I still sit in the lecture, although the lecture hall consists of 25 people, meaning all the others are no longer there. And you sat in the first lectures, you were so desperate yourself. And then your friend sits next to you and she's writing with someone who's also studying math, and she's just writing: "Yes, I've just dropped math, I'm doing a bachelor's degree in biology now." Where you can think, oh God! Actually, you don't understand anything too. But I'm still somewhat proud of the fact that you're doing what not everyone would do, that you've still kept it up until now. I also know a lot of people who somehow learn with us or something like that, who just copy. So, they really just sit with us. And then they wait until we've got something worked out, written it down. Then they just copy it to get points again. And hand that in. And they know that too. I don't know how they want to write the exams. Because I am somehow still proud of myself that I still understand everything that I have calculated so far. I didn't come up with it myself, but at least I understood the procedure. That's why this already gives me some joy somehow. [. . .] I still enjoy studying math, even though I don't keep up very well or can't do everything myself.

Such emotions and adapted goals interfere with corresponding strategies.

2.6 Discussion

2.6.1 Discussion of Results

The results show that mathematics students experience achievement emotions in sometimes very pronounced forms and that these emotions interact with self-regulated learning in a way that should not be underestimated. They also show the predominant importance of perceived control, but also of a positive valuation of content and exercises for the arousal of such emotions as well as self-regulated learning. The relationships found between perceived control, values, and achievement emotions are in line with control value theory (Pekrun, 2006).

The results also demonstrate the importance of considering volitional and coping strategies (as, e.g., described in Boekaerts' (2011) dual processing self-regulation model), in addition to learning and problem-solving strategies, for a thorough description and understanding of students' strategies in their engagement with first-year university mathematics. In summary, the model proposed in Sect. 2.2.3

(cf. Figure 2.1) seems to be well suited to describe the complex relationships between perceived control, values, achievement emotions, and self-regulated learning in higher mathematics education. In particular, it can help to explain some phenomena of university mathematics learning, such as:

1. *Seeking social support,* which may range from solving exercises collaboratively in study groups to copying exercise solutions from others (which are well-known strategies used by mathematics students, Göller, 2020; Haak et al., 2020; Liebendörfer & Göller, 2016; Stadler et al., 2013), can be seen as a consequence of perceived too low control and thus as a coping strategy to reduce anxiety and hopelessness. The same applies to the *use of external resources*, such as searching for solutions in books or on the internet (Göller, 2020, 2021; Gueudet & Pepin, 2017, 2018; Kock & Pepin, 2018)
2. Conversely, while these coping strategies support the accomplishment of institutional requirements, they do not necessarily support perceived control with respect to autonomous solving of exercises or understanding lecture content. In line with control-value theory, the results indicate that such coping strategies are useful to reduce anxiety and hopelessness but rather not to increase joy. Given the known difficulties of mathematics students in the transition from school to university (Gueudet & Thomas, 2020), such results may contribute to the explanation of the often observed *decline of interest and motivation for mathematics in the first semester* (e.g., Kolter et al., 2016; Liebendörfer, 2018; Rach, 2014) and the sometimes reported *dissatisfaction of (especially teacher) mathematics students with the mathematical content* of their studies (e.g., Gildehaus & Liebendörfer, 2021; Göller, 2020; Mischau & Blunck, 2006).
3. Joy is primarily experienced when students are able to solve exercises autonomously. A high value of autonomous exercise solutions (e.g., due to a high intrinsic value, or because the difficulty of exercises is considered high the social environment) provides a high level of joy when exercises can be solved autonomously. Thus, interventions ideally focus on improving both perceived control and intrinsic value (cf. Sect. 2.6.2).
4. If exercises cannot be solved autonomously (low control), an *adjustment of goals* (e.g., understanding exercise solutions instead of solving exercises oneself) enables a positive valuation of outcomes that are easier to control. From the theoretical perspective of this chapter such adjustments of goals can be interpreted as emotion regulation strategies for a participation in university mathematics that enable an arousal of joy and a reduction of frustration (Göller & Gildehaus, 2021). In addition, from this perspective, a *devaluation of university mathematics* (e.g., because of a perceived low utility value for future teachers, which is sometimes reported, Gildehaus & Liebendörfer, 2021; Göller, 2020; Wenzl et al., 2018) can be interpreted as an emotion regulation strategy to reduce frustration (Göller & Gildehaus, 2021). Of course, such explanations cannot be exhaustive, since, for example, personal values, goals, and sense of belonging, the social environment, cultural expectations, and the like will also have an influence (Bergey, 2021; Gildehaus & Liebendörfer, 2021; Lahdenperä & Nieminen, 2020; Solomon, 2007).

2.6.2 Implications, Limitations, and Outlook

Difficulties of mathematics students at the transition from school to university are well documented (Gueudet & Thomas, 2020) with particular attention to differences between mathematics at school and at university and the cognitive demands or, in some cases, the motivational effects associated with these (Gueudet & Thomas, 2020; Liebendörfer & Schukajlow, 2017; Rach & Heinze, 2017; Rach & Ufer, 2020). The results of this chapter indicate that investigating emotions associated with these difficulties can contribute to a better understanding of transition problems.

Implications of this work are, first, to raise awareness of the importance of emotions in learning mathematics at university. From this perspective, strengthening perceived control, which has to be newly established due to the difficulties in the transition from school to university, would be a first starting point for interventions. Possibly, this can be addressed in the design of courses by a stronger focus on opportunities to increase perceived control or to enable experiences of success (e.g., by offering practice exams, or at least partially also exercises that can be solved by many students). In addition, individual support, like it is offered in mathematics support centers (e.g., Lawson et al., 2020; Schürmann et al., 2021), seems to be a promising approach to promote the development of mathematics-specific learning and problem-solving strategies, and thereby improve students' control over their learning progress by providing them with agency.

A second starting point, from this perspective, are the values related to university mathematics. The institutional conditions existing in this study, in particular the achievement of 50% of the exercises and the exams, act as a clear driver of students' strategies and emotions in this study. They induce a high extrinsic value, which, however, tends to emphasize the value of failure and thus (especially in the case of low perceived control) rather arouses anxiety and hopelessness (cf. Sects. 2.2.2 and 2.2.3). There is some research on tasks to bridge the gap between school and university mathematics and demonstrate connections, especially for pre-service teachers (e.g., Bauer, 2013; Eichler & Isaev, 2017; Hoffmann, 2021; Neuhaus & Rach, 2021). From the perspective of expectancy-value theory of motivation (Wigfield et al., 2017), however, such tasks probably primarily address the utility and attainment value of university mathematics, while their significance for intrinsic values needs to be investigated in more detail. Overall, it is unclear how a positive (intrinsic) valuation of university mathematics content can be achieved, as preferences appear to be relatively stable. However, perceived control apparently plays an important role here as well (respectively the closely related concepts of perceived competence and autonomy, Liebendörfer, 2018; Ryan & Deci, 2017).

When interpreting the results, the small sample size, the institutional setting, and the theoretical framework must be taken into account. As was pointed out at several places in the chapter, the institutional setting in which the study took place, and in particular the institutionally set conditions (50% limit, exams, lectures, exercises, etc.), but also the cultural context of the study, with explicit and implicit messages of students' school and university instructors, peers, the interviewer, and the broader culture they all act in, probably had a significant influence on the results of the study. Moreover, by searching for word fragments, the analysis for RQ 1 and RQ 2 focuses on only part of the data, while the answers to RQ 3 proposed here are composed of theoretical considerations and results of previous studies, which should be more systematically verified and grounded in the data in future studies. The theoretical perspective proposed here is, of course, only one of many possible ones for describing the phenomena presented here. Accordingly, further studies are desirable that investigate relationships between achievement emotions and self-regulated learning of undergraduate mathematics students in different institutional settings as well as from different theoretical perspectives to reveal overarching relationships and possible approaches for the design of innovative mathematics courses or institutionalized individual support at university. In general, as the present study indicates, a stronger inclusion of emotions in research on undergraduate mathematics education seems to be a promising approach for explaining and understanding student learning and performance of mathematics at university.

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Chapter 3
The Unease About the Mathematics-Society Relation as Learning Potential

Johanna Ruge

Abstract In this contribution, I present an inquiry that was prompted by an empirical observation that emerged in interviews with student-teachers: Speaking about their beliefs about mathematics and its teaching and learning was disrupted by expressions of *unease* about popular myths related to their future profession and the current status of the relation between mathematics and society. Based on a theoretical position of the subject-scientific approach, that also vague feelings – such as unease – entail the potential to gain further insights into the object at stake, I analysed its potential for learning. Since the unease is related to beliefs, I take a closer look at, and formulate a critique of, current trends in belief research and their practical implications. Instead of repeatedly designing more teaching interventions for students to align with certain beliefs along the way, I propose to understand the unease as a starting point for an intentional and collaborative learning process of mathematics education scholars and students.

Keywords Mathematics teacher education · Subject-scientific approach · Critique of belief-research · Reciprocal learning · Shared struggle · Mathematics-society relation

> *[T]hinking is essentially the possibility of reproducing real contradictions in a contradiction-free reasoning so that they can be recognized as aspects of reality and be overcome in practice. [. . .], we ourselves are part of the society which we have to reproduce in thinking. At first glance, this implies a kind of circle, but it is one that can be overcome by epistemic distance* (Holzkamp 2013a, p. 22).

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3.1 Prelude

For my research on learning experiences of pre-service mathematics teachers at the university, I conducted interviews with student-teachers. In Germany, the initial education of future teachers is split into two phases: The first phase is situated at the university and the second phase takes place in school and in specific seminars outside the university. In contrast to the second phase, the first phase does not contain immediate vocational requirements and stands for the academic part of the professional development: the socialisation into the respective academic field (s) (Blömeke, 2001; Wenzl et al., 2018). My work is situated within the university part of mathematics teacher education (the first phase) and addresses the socialisation of student-teachers into the academic field of mathematics education. This socialisation has to go beyond forming a solid understanding of the mathematics present in school, as teachers are also expected to be representatives of mathematics as an academic subject. However, academic socialisation into mathematics education is specific, and differs from socialisation into the two separate academic disciplines of mathematics and education: the structure, rationale and interconnectedness of the selected topics are of greater importance than their level of detail, compared with studying a single discipline (Blömeke, 2001).

One important aspect of the socialisation of student-teachers of mathematics is the negotiation of attitudes and beliefs towards the subject matter: mathematics and its teaching and learning.[1] In this chapter, I present an inquiry of the popular research strand that focuses on attitudes and beliefs, known as belief research. This inquiry was prompted by an empirical observation, which led to a theoretical reflection: Student-teachers' talk about beliefs about mathematics and its teaching and learning was disrupted by expressions of *unease* about popular myths related to their future profession, and linked to current struggles in forming the relation between mathematics and society. My attempts to understand the implications of this unease for the practice of mathematics teacher education have led me to pursue the following question:

> *How can the unease with the relation between mathematics and society [in short: mathematics-society relation] be understood as a learning object that holds a potential [in short: learning potential] for, both, the further development of mathematics teacher education and the academic discipline of mathematics education?*

My pursuit of this issue led me to formulate a critique of belief research that goes beyond an inner-scientific debate of adequate definitions and suggestions of refinement to a critique of the practical implications of belief research. At the core of my critique is the learning theory underlying current belief research, which forecloses possibilities for, both, student-teachers' learning and the development of the

[1] Because of the importance of interconnectedness for mathematics education, I consider both as common subject matter.

academic discipline. In the discussion, I propose an alternative that is based on understanding the unease as a shared learning potential.

The chapter draws on empirical work and theoretical inquiry, which is reflected in an alternative-to-the-usual text structure:

First, I introduce my theoretical positioning: the subject-scientific[2] approach [Subjektwissenschaft] (Sect. 3.2) and its categories (Sect. 3.2.1). This approach offers an understanding of learning (Sect. 3.2.2) that integrates the societal dimension (Hochmuth, 2018; Holzkamp, 1995) and allows to relate learning to ongoing social struggles, via the learning object at stake. Second, I present the empirical observation of unease (see above). The unease – an initially language-less and rather vague feeling – points to a societal dimension that relates to students' learning conditions and popular myths about mathematics (Sect. 3.3.1) and mathematical ability (Sect. 3.3.2). Reactions at conferences regarding this empirical observation (Sect. 3.3.3) have led to the third topic, an inquiry of the current belief research (Sect. 3.4). In Germany, belief research is often related to the double discontinuity (Klein, 1908) and linked to programmes situated at the beginning of university studies that seek to foster a change in future teachers' beliefs (Sect. 3.4.1). Internationally, prospective teachers' participation in their own professional development is of a greater concern (Sect. 3.4.2). As a fourth topic, I interrogate the learning theory underlying current belief research. A reformulation of the concern of belief research allows me to understand the unease as common interest of belief research endeavours and student-teachers (Sect. 3.5). Finally, this allows me to discuss practical implications of understanding the unease as a learning potential, even though these implications clash with current institutional learning conditions (Sect. 3.6).

3.2 Subject-Scientific Approach

The subject-scientific approach originated from a Marxist-oriented critique of psychology and provides categories for analysing and understanding human actions within capitalist societies (see Holzkamp, 1972, 1985; Markard, 2010).[3] These categories also allow to explicate current restrictions within societal and institutional arrangements, e.g., university teacher education programmes, and conceptualisations within theories, e.g., underlying learning theories. Furthermore, they provide the potential to think beyond current restrictions by including a well-formulated conception of humanity and social theory, which can only be hinted at in the following.

[2] Here, 'subject' refers to human beings and their subjective perspectives, not to the subject of mathematics.

[3] Introduction in Spanish: Vollmer and Holzkamp (2015); English introduction: Tolman (2013); Selected writings in English: Schraube and Osterkamp (2013).

3.2.1 Fundamental Assumptions and Subject-Scientific Categories

A fundamental assumption is that “human beings [are] producers of the life conditions to which they are simultaneously subject” (Holzkamp, 2013a, p. 20). Thus, the relation between the individual and societal conditions is regarded as being dialectical. All human actions are understood as societal-mediated, which opposes a view that assumes that the individual is determined by the structure of society, or by immediate circumstances. This approach calls into question an external control over individuals and criticises research endeavours that seek to provide knowledge for the manipulation, management or operating of individuals or their psychological processes (Markard, 2010), which is partly also present within belief research (Sect. 3.5). Rather, the reasons for and meanings of human practices – such as learning, teaching, thinking, doing mathematics, etc. – are of interest for a practice-oriented research. Nissen (2012) summarises the importance of seeing practice in relation to societal conditions: “praxis is the general process of production in which we humans provide conditions for the reproduction and development of our cultural modes of life” (Nissen, 2012, p. 110), in which human practice is understood as “the reflexive and anticipatory productive provision of conditions of life that accumulates [. . .] into a contradictory cultural development and recreates ourselves” (Nissen, 2012, p. 37). The analytical movement of researching practice does not take practices and their arrangements for granted as a neutral field but always implies to look behind superficial appearances (Nissen, 2012, p. 110). Practice in general, and thus also knowledge (re-)production, in particular, are conceptualized as containing a transformative creativity (Nissen, 2012, pp. 117–118). Knowledge can be regarded as shared objects of/within practice: formerly subjective activities that have been externalised and generalised and which by now are *generalised societal action possibilities* [verallgemeinerte Handlungsmöglichkeiten]. This perspective on knowledge highlights its political nature. Knowledge, in general, and learning objects in particular, always refer to the ambiguous, conflictual and contradictory nature of social/societal reality (Marvakis & Schraube, 2016, p. 205). They are thus always embedded in contradictory societal meaning structures and their endless networks of meaning. For example, mathematics is credited a high relevance for social development because it makes everyday practices calculable, and it is even ascribed to mathematics that it holds the ability to provide unambiguous solutions to practical problems. The progress of mathematics, on the other hand, is based on a detachment from the ambiguities of practice (for further elaborations of the complexities of current mathematics-society relation, see e.g., Nickel et al., 2018).

Besides relating knowledge to its societal-mediatedness, the subject-scientific approach seeks to include subjective perspectives. Including subjective perspectives into, resp. addressing subjectivity in research, does not aim to classify nor to evaluate individuals but to gain an understanding of their actions from their specific standpoint – e.g., as learners, as teachers, as researchers, as persons identified to be mathematically able, etc. This entails going beyond a mere reference to immediate human experience and “includes the notion that humans do not just live their life in a world, but are actively engaged in the making of the societal world on the basis of their experience and action, which in turn is re-making themselves" (Schraube & Højholt, 2019, p. 9) and the shared practice in which they participate. From this perspective, socialisation is not a one-way affair, but a reciprocal task. Socialisation does not only involve an acquisition of current or desired practices, but also responsibility for working towards (necessary) changes in the professional field.

A central characteristic in all categories of the subject-scientific approach is the *twofold possibility* [doppelte Möglichkeit] to either reproduce restrictive conditions or the (however small) possibility to extend established practices and alter societal conditions. This is also reflected in the analytical core-category *action potence* [Handlungsfähigkeit] that describes the individual's opportunities and constraints to act from her or his specific position within societal conditions (Holzkamp, 1985), in the analytical split of the category into two alternatives: "First, *restricted action potence* [restriktive Handlungsfähigkeit] stands for a modality of alignment with or subjection to given power structures describes the safekeeping of one's own action potence at the cost of the (re-)production of restrictive conditions. Second, *generalised action potence* [verallgemeinerte Handlungsfähigkeit] is directed towards extending one's own control over restrictive conditions, and thus entails the possibility of overcoming existing power relations, in alliance with others." (Ruge et al., 2019a, p. 752).

Psychological processes – such as thinking and emotions – are grounded in the particular individual's concrete life situations that are always embedded in social reality. Emotions and affects, also negative ones, are not considered as hindrances for gaining knowledge per se, rather they may provide valuable insights about the respective social reality (Holzkamp, 1985, 2013a; Osterkamp, 1978) – also if they are initially language-less and cannot be formulated in a precise manner. Thus, a feeling of unease can be a starting point for gaining knowledge.

3.2.2 Subject-Scientific Understanding of Learning

Learning can be analytically[4] split into a learning process and its directionality (Holzkamp, 1995). On the one hand, the terms incidental learning, or co-learning, denotes learning that takes place in parallel with other processes, through participation in a particular practice. Characteristic of such a learning process is that we are sometimes not even aware that we are learning something and thus have no conscious control[5] over the learning process. On the other hand, we have intentional learning processes that are particularly important in relation to educational institutions. They can, in turn, be analytically differentiated on the basis of the

[4] The distinction is made analytically, in real life situations it is not distinctly classifiable.

[5] This does not imply that it is not possible to become aware, or to consciously unlearn something in retrospect that has been initially learned along the way.

directionality of the learning process. A distinction is made between defensive and expansive reasons for learning (Holzkamp, 1995, 2013b). Learning can be grounded in trying to avert an experienced or anticipated threat to one's action potence. In this case, the learning process is not primarily directed towards a deep understanding of the subject matter, but towards dealing with this threat. This general pattern of reasoning regarding learning activities is described by the term defensive learning. Expansive learning, on the other hand, is characterised by a deeper processing of the object of learning, which transcends one's own immediate experience and looks beyond the superficial appearance of a phenomenon (or empirical observation), trying to understand it in its societal-mediatedness. This focus on societal mediation is derived from the premise that a learning object is not a neutral object but always embedded in a social/societal reality with all its contradictions (Sect. 3.2.1). Practice-oriented research can be regarded as an intentional learning process, in this sense. An emancipatory expansion includes questioning one's own immediate experiences and dominant patterns of thinking. During the learning process, this can in turn be accompanied by resistance and defensive affects (e.g., feeling inadequate, questioning learning effort or usefulness of anticipated learning outcomes) (Ruge et al., 2019b).

The social practice of mathematics teacher education is located in specific institutional arrangements. Thus, the learning process and its directionality have to be balanced with the scope of the specific institution (Dreier, 2003, 2008). In order to coordinate their learning in all their contradictions, individuals need to develop personal stances: "Stances develop and sustain an orientation for subjects in the structures of their complex, ongoing, personal social practice. This concept emphasizes the practical anchoring and consequences of personal reflections. [. . .] stances guide persons in their transitions between diverse contexts" (Dreier, 1999, pp. 15–16). Institutions combine contradictory demands and struggles take place. Within the academic discipline of mathematics education – an institutional arrangement that student-teachers are confronted with and have to relate to – struggles for a desirable mathematics-society relation take place. Struggles manifest e.g. in teaching-learning conditions and related reform movements. The object at stake are – inherently ambiguous, conflictual and contradictory (Sect. 3.2.1) – aspects of the cultural practice mathematics and its (re-)production in its current state, necessary further developments, and possible transformations. An intentional learning process entails a deliberate engagement with these aspects.

3.3 Unease to Be Identified as a Mathematician (Only)

In the following, I situate the emergence of the empirical observation by sketching my method for analysing interview transcripts and broader empirical observations of my overall study on learning experiences of pre-service mathematics teachers at university, before referring to two examples. The empirical observations reported below are the starting point for the following theoretical reflection. The theoretical reflection strives for making an initially language-less and vague feeling tangible in the first place and accessible to further reflection.

I analysed the interview transcripts using a combination of grounded theory (Bryant & Charmaz, 2007; Strauss & Corbin, 1990) and objective hermeneutics (Wernet, 2009, 2012). The analysis process within the grounded theory approach is based on a constant contrasting comparison of the available data material. The researcher detaches him−/herself from a specific interview text by making cross-case comparisons which shift his/her focus towards phenomena in its spectrum. Interpretations that refer to a single case thus always entail the spectra of the phenomena referred to. Objective hermeneutics makes it possible to grasp latent meanings of an interview text. The detachment from a single case takes place by a concentration on the transcript's sequentiality and ruptures within articulations. These ruptures are not attributed to the interviewee's personality or ability, but linked to current social practices and underlying societal structures. Both techniques allow for linking individual expressions to societal meaning structures and their network of meaning (Sect. 3.2.1).

The students I interviewed aimed to be mathematics teachers at different school levels,[6] but all had in common that they choose to study mathematics as one of two school subjects they want to teach in the future. In the interviews, students described their motives to become mathematics teachers, their conceptions of professionalisation and ideals of teaching-learning-relations in the subject of mathematics (Ruge, 2017; Ruge & Hochmuth, 2015a, b, 2017). My interview partners displayed various levels of comfort[7] with mathematics at university, ranging from doubts about their ability to keep up in the mathematics courses to referring to mathematics as their personal strength. Some students relate to their mathematical abilities in a contradictory way by describing mathematics as a personal strength and the need for gaining confidence in their mathematical skills at the same time (for a more detailed description, see Ruge & Hochmuth, 2015a). Within the interviews, a variety of beliefs about mathematics and ideals of mathematics learning and teaching were expressed. Expressed beliefs about mathematics ranged from a positive relation with mathematics as a subject that provides definite solutions (Ruge & Hochmuth, 2017) to seeing mathematics as a shared activity that everyone can participate in. Ideals of learning and teaching ranged from describing a thoroughly structured guidance (Ruge & Hochmuth, 2015b, 2017) to allowing for participation on diverse levels of action. Interestingly, independently of their expressed comfort level or ideal pictures of teaching and learning mathematics, the students voiced an unease to be identified as mathematicians, resp. with a perceived current state of the academic discipline of mathematics. *Unease* denotes a still vague feeling of discomfort, that is

[6]The study programmes vary in their structure and focus. In primary and lower secondary teacher education programmes more weight is put on pedagogical subjects, while in secondary school teacher education programmes, which prepare for teaching at lower and upper secondary level, the mathematical requirements are higher.

[7]Arranging qualities in a continuum is an analytical strategy of grounded theory.

also associated with a lack of ease in social relations (Merriam-Webster Thesaurus Online, n.d.). Student-teachers distanced themselves, for example, from the idea of becoming a constricted one-track-specialist (Sect. 3.3.1) or from being just mathematically able (Sect. 3.3.2). They essentially distanced themselves from being identified as mathematicians. I detail the above-mentioned unease with reference to two interview partners: Bianca and Georg.[8] Both felt comfortable with mathematics within their study context. Afterwards, I briefly describe reactions that my observations have provoked in the research community (Sect. 3.3.3), which have directed my further inquiry.

3.3.1 Against Being Identified as Becoming a Constricted One-Track-Specialist

Talking about her choice of the study programme, Bianca stepped away from her initial choice of becoming an upper secondary school teacher and enrolled in a programme for becoming an elementary and lower secondary school teacher. She justifies her decision with a reference to the structure of the respective study programmes: *"… For me it was always first and foremost, that I wanted to become a teacher WITH math, but not first a mathematician and then a teacher"*. She states on a manifest level, that she did not want to exclusively become a mathematician. Instead, she wants to combine mathematics and the educational profession and study them in an integrated manner: *"… because I would like to do something with people, but also especially math, which was always my favourite at school and where I really see my strengths. To really bring that together, to bring them together"*. Throughout the interview, she substantiates her argument against a pathway into the teaching profession that follows the ideal of socialising into the discipline of mathematics first. She interprets this pathway as standing in opposition to a mathematics in which all, the "super many others", can participate, e.g.:

> *So I definitely have a negative role model, where I had a math teacher at secondary school who, um, partly proved a lot, where maybe I didn't have difficulties with it, but super many others had difficulties with it. And where it really crystallised that he did that in his studies and that kind of his whole studies, where there was so much proving, that he wanted to pass on to the pupils somehow.*

In popular myths[9] about mathematics (Kogelman & Warren, 1978), mathematics is said not to be accessible to everyone. It is imagined as a purely mental activity for which the use of the body is a hindrance. The purpose of mathematical activities is defined by their outcome: it is important to always be able to present an exact solution. Mathematics is characterised by hard work and doing mathematics is

[8] Names are changed.

[9] Popular myths condense views regarding mathematics to images, these are often counter to desirable views from a mathematics education perspective.

seen as an uncreative activity. How someone could master mathematics seems inexplicable to many (Dowling 1998; Kogelman & Warren, 1978). Within her articulations, Bianca answers to such myths. Her unease against becoming a constricted specialist in mathematics, who is unable to make it accessible to others, shows that her conflict goes beyond a mere matter of a personal identification with mathematics. It is based in a wish to care for others and to oppose an exclusive, resp. excluding understanding of mathematics.

For her, mathematics is more than just academic mathematics and only acquires its relevance for all members of society through joint action. Bianca does not only express her own personal beliefs about mathematics teaching, she interlinks them with the process that the profession of mathematics teaching heads towards: *"So, in my opinion, I see the/especially the math teacher in a state of flux, or just the whole . . . training, which is somehow different or newer and that with these competences, so to speak, we, a bit, have to carry it into the school first"*. In her affirmation with a competence-oriented teaching, she relates professionalisation into the teaching profession with the social struggle for inclusive mathematics, being humane in the sense that all can participate in doing mathematics.

3.3.2 Against Being Identified as Just Being Mathematically Able

Georg presents his mathematical abilities in the interview in a twofold manner. He describes doing *"math-gimmicks"* as a leisure activity in a hesitant and cautious way – he *"likes to do that too"*. Throughout the interview, he vividly describes topics that caught his interest, he in contrast to others, wants to understand *"why it works"*. This display of interest in mathematics is interlaced with segments where Georg presents himself as *"lazy"* when it comes to mathematics, as well as also having to work for it and having the drive to dig deeper. In the interview, he cannot straightforwardly present himself as being good at and interested in mathematics. Such displays are always interlaced with retracting segments.

Popular myths about people who are mathematically able include the idea that mathematics supposedly comes naturally to these people. They are said to be successful in mathematics without much effort. Thus, this 'talent' distinguishes them from the others, those who are *not* mathematically able. However, the more mathematically gifted a person is considered to be, the more strongly it is assumed that this giftedness comes at the expense of other abilities, such as social skills (Kogelman & Warren, 1978; Roodal Persad, 2014). Georg's retracting moves relate to struggles of not identifying himself with such myths. The unease of being identified as a mathematically able person remains on a latent level throughout the whole interview.

3.3.3 Interlude

In summary, by retracing the unease to popular myths and social struggles for a socially relevant and inclusive mathematics, it can be related to the contested and political nature of the practice of mathematics teaching and learning and the struggle for a humane mathematics-society relation (Sect. 3.2.2). Therefore, the unease addresses not just an individual difficulty in the socialisation process, but an indication of an important issue for the further development of mathematics teacher education, which Bianca explicitly relates to developments in the field (Sect. 3.3.1). Inspired by the subject-scientific insight that also feelings entail the potential to provide valuable insights (Sect. 3.2.1), this led me to the question:

> *How can the unease with the mathematics-society relation be understood as a learning potential for, both the further development of mathematics teacher education and the academic discipline of mathematics education?*

Before I can formulate first ideas, a digression is necessary: when I presented my observations at conferences, as a first reaction, scholars, who identified themselves with the academic discipline of mathematics, deprecated my interview partners for their supposedly "inadequate" beliefs or for not adhering to the role as representatives of the academic discipline in an unbrokenly positive manner: e.g. scholars voiced doubts about the student-teachers' capabilities, or their right to pursue a career in the teaching profession and to represent mathematics. These scholars expressed a need for changes, for changing the beliefs of the student-teachers. This led me to take a closer look at belief research, which I summarize with respect to its critique and developments in the following.

3.4 Belief Research

The concept of beliefs belongs to a research strand that focuses on mathematics-related affect. Beliefs are categorised in different ways (Goldin et al., 2016). According to Hannula (2012), concepts and theories concerning the affective domain can be mapped in a three-dimensional space. The first dimension consists of three broad categories to describe affect: motivation, emotions, and beliefs. Beliefs stand out, because they are often conceptualised as including cognitive and affective components and thus go beyond mere affect (e.g., Törner & Grigutsch, 1994). The second dimension describes a continuum between state and trait. Mostly, beliefs are regarded as a trait-type affect, i.e., as being more stable. To what extent stability is seen as a defining quality of beliefs is in debate (Liljedahl et al., 2012). The third dimension concerns the theorising level of the broad categories. In Hannula's model, this level distinguishes between the physiological (embodied), psychological (individual), and social level. Mathematics-related affects in general have mostly been studied on the psychological level (Goldin et al., 2016) and beliefs are usually studied and perceived as a concept that refers to the individual level.

There is no internationally accepted definition of the term beliefs, rather different characteristics are highlighted, such that beliefs are “subjective and hidden” (Furinghetti & Pehkonen, 2002), or it is proposed that “beliefs might be thought of as lenses that affect one’s view of some aspect of the world or as dispositions toward action” (Philipp, 2007, p. 259). Following this description, popular myths can be considered as beliefs.

An influential belief construct has been developed by Törner and Grigutsch (1994). They differentiate between four different views regarding the nature of mathematics, that can be summarised in the two following perspectives (Grigutsch et al., 1998; Törner & Grigutsch, 1994): the first, conceptualisation of mathematics as a static science includes the formalism-related view[10] and the scheme-related view.[11] The second, conceptualisation of mathematics as a dynamic process includes the process-related view[12] and the application-related view.[13] All these views provide a description of a specific way of thinking about the relation between mathematics and society. In contrast to prevailing myths, these views are compatible with philosophical considerations about the nature of mathematics. Myths, in contrast to the proposed belief-categories are more specific concerning assumptions about those members of society that are mathematically able.

In teacher-belief research, two major trends can be identified: Change of – supposedly “inadequate” – teacher beliefs (Sect. 3.4.1) and in−/consistencies between teacher beliefs and teacher practices (Sect. 3.4.2). Both are related to a proclamation that mathematics education should be more learner-centred and that the process character of mathematics should be given greater prominence. This can be understood as a countermeasure to popular myths (see Sects. 3.3.1 and 3.3.2). Improving the quality of mathematics teaching in this sense is a concern shared in the international mathematics education community and also by my interview partners. Skott (2004) summarises measures taken to reach this ideal *the reform*. Reform efforts within the German context are articulated as a demand for “competence orientation” (Ruge, 2017), which also Bianca refers to in taking a stance for an inclusive mathematics education (Sects. 3.3.1). The reform discourse goes hand in hand with regarding certain beliefs, e.g., seeing mathematics as a dynamic process, as desirable outcomes of teacher education.

3.4.1 Change of Teacher Beliefs

Within the context of teacher education programmes, several research studies aim at the promotion of belief change (Goldin et al., 2016). Within the German discourse

[10] Mathematics is an exact science on an axiomatic basis and is further developed by deduction.

[11] Mathematics is a collection of terms, rules and formulae.

[12] Mathematics concerned with problem-solving and the discovery of structure and regularities.

[13] Mathematics is presented as a science relevant to society and life.

on university mathematics education, affect in general and beliefs in specific are often linked to the idea of a double discontinuity, originally introduced by Klein (1908), and recently discussed in connection to beliefs (e.g., Hoppenbrock et al., 2016; Isaev & Eichler, 2017; Roth et al., 2015). It is assumed that there is a discontinuity specific to the subject of mathematics during the transition from school to university and from university back to school, which leads to students having difficulties in relating university and school mathematics to each other.

International studies, e.g., TEDS (Tatto & Senk, 2011), regularly proclaim beliefs as important factors in teachers' professional development and their success in teaching mathematics. Within the German discourse especially large-scale quantitative studies on (prospective) teachers' beliefs are important references in the debate, e.g., the TEDS-studies (Blömeke, et al., 2010, 2013; Tatto & Senk, 2011), and the COACTIV-study (Krauss et al., 2008; Kunter, et al., 2011), which relate beliefs to professional teaching competence. Most studies refer to an influential model by Baumert and Kunter (2006), that considers epistemological beliefs concerning mathematical knowledge and subjective theories about teaching and learning mathematics as essential components of the belief and value structure of the individual teacher. The concept of beliefs has become established for the part of professional competence that is regarded as not purely cognitive (Schwarz, 2013, pp. 49–50).

The need for changing student-teachers' beliefs is justified by its supposed beneficial effects on the acquisition of mathematical knowledge and on the success of mathematics teaching. Existing beliefs of student-teachers are seen as influencing the acquisition of new knowledge and skills. Beliefs are said to control the perception and handling of the learning content by acting as filters. A fit between a person's existing beliefs and the learning content is regarded beneficial for the learning outcomes of this particular person (e.g., Blömeke, 2004; Schwarz, 2013). For example, a construction view is thought to be more beneficial for student-teachers' learning than a transmission view (Blömeke, et al., 2010), and is judged an "adequate" mathematical world view. "Inadequate" mathematical world views are used to explain deficits in building up content knowledge (Blömeke, et al., 2010).

In the design of teaching innovations, beliefs are seen as agents of change to sustainably adjust the practice of teachers (e.g., Bernack-Schüler, 2018; Bernack et al., 2011; Holzäpfel et al., 2012). For example, Süss-Stepancik and George (2016) suggest to consider epistemological beliefs concerning mathematics knowledge in the didactical design of mathematics and mathematics education courses for first-year students to counteract this filter function of beliefs. They suggest measures such as gradually introducing students to the exact way of speaking and thinking about mathematics – a typical measure discussed within the discourse on softening the first transition (e.g., Hoppenbrock et al. 2016; Roth et al., 2015). Despite that the adjustment of beliefs is claimed to be "nearly impossible" (Pajares, 1992, p. 323), efforts are directed at the design of teaching interventions that aim at influencing beliefs – resp. counteracting (undesired) beliefs – of student-teachers (e.g. Schwarz, 2013; Süss-Stepancik & George, 2016).

So-called "affective factors" behind teaching behaviour in general have been an increasing area of interest in mathematics education research (Goldin et al., 2016).

This trend fits with Fenstermacher's (1979) prognosis that teacher effectiveness research would focus more on the study of beliefs. Roth and Walshaw (2019) criticise the way emotions and affect are conceptualised. They criticise a double movement of affect being "intellectualized and approached as external to and separate from intellect[14]" (Roth & Walshaw, 2019, p. 111). Concepts such as beliefs, attitudes, engagement and motivation, that are marked as "affective traits" have been "developed as means of capturing the non-rational aspects of subjectivity" (Roth & Walshaw, 2019, p. 112). These concepts are thus seen as allowing researchers to keep up a "cause-effect logic in descriptions of the performance of affected individuals." (Roth & Walshaw, 2019, p. 112).

3.4.2 In—/Consistencies Between Teacher Beliefs and Teaching Practices

Within research focusing on the relationship between beliefs and teaching practices, both consistencies and inconsistencies have been found between teacher beliefs and teaching practices (Goldin et al., 2016). The observed differences between espoused and enacted beliefs have led to a shift of focus on context and also on teachers' identity construction in the international mathematics education research community. Skott's (2009, 2013, 2019) research addresses the discrepancy between expressed beliefs and enacted beliefs in the teaching practice of (new) teachers. He shifts the focus to context in general, and more specifically to the question of how (prospective) teachers participate in their development of a professional identity (2013, 2019). For Skott, existing concepts from mathematics education research, such as knowledge, beliefs and teacher identity, are not able to grasp the relationship between integration in a specific practice context and personal development. Within Skott's patterns-of-participation-approach, different institutional contexts that play a role in teacher education and its practice structures are included. Practice is always understood as embedded within a community, within which rationales of action are negotiated. Socialisation is understood as more than just an adaptation of (prospective) teachers to the respective community and existing practice structures, but also as the possibility to get involved in it and contribute to a change. This aligns well with the subject-scientific perspective (Sect. 3.2.1). Skott conceptualises the personal development, resp. the socialisation process, as an integration movement from an initially peripheral to a more inclusive participation, based on the idea of situated learning (Lave & Wenger, 1991). Coming back to the double discontinuity (Sect. 3.4.1): the conscious engagement with meanings and rationales with different communities can be seen as a potential for personal and institutional development. To this end, the observed difficulties of making connections between the

[14] Regarding something as separate from intellect usually leads to consider it as not being part of intentional learning processes.

institutional[15] meaning constructions must be tackled *with*, and not *for*, the prospective teachers.

Skott's work is explicitly directed at *the reform*[16] (see above) and builds on his analysis that it is required from teachers to adopt a reflective stance, which is grounded in an apparent contradiction: a simultaneous shift in emphasis from teaching to learning and a greater emphasis on the role of the teacher. This shift entails that the teacher must be a linking element between the priorities set by the subject and the priorities that prevail due to the specific school context. For Skott, this "new" teacher role raises the question of how development processes in teacher education can be scientifically recorded and described, and thus, how they can be modified on the basis of research into the conditions for development. This leads to thinking in terms of a management of conditions (instead of people). Still, his approach has a strong focus on immediate context situations and does not include broader societal learning conditions and the social embeddedness of the communities in broader structures.

3.5 Current Trends in Belief Research from a Subject-Scientific Perspective

From the perspective of the subject-scientific approach, the rationale behind trying to change student-teachers' beliefs (Sect. 3.4.1) can be summarised as based on a unidirectional scheme: Perceiving mathematics in a certain way is seen as beneficial for understanding mathematics at university (Sect. 3.4.1). Because of this, beneficial beliefs need to be taken up and transported by the future teachers into school. It is about an external shaping of what mathematics shall be for people and what kind of mathematics they shall identify themselves with. This view can be criticised for its direction to the management of individuals, and attempts towards controlling their stance towards the object at stake (Sect. 3.2.1). This unidirectional scheme can be linked to a common one-sidedness in learning theories that dissociates learning objects from any conflictual and contradictory meanings and negotiates them as supposedly neutral semantics (Marvakis & Schraube, 2016, pp. 196–197). This is criticised by Schraube and Marvakis (2019) to be a "bisected learning", that can be found within the cause-effect descriptions in belief research. Beyond Roth and Walshaw's (2019) critique of dichotomization of affect and knowledge (Sect. 3.4.1), a dialectical relationship between the learner and the object is denied. The object, resp. the learning outcome, is fixated in its meaning, and only the learner shall change. Fixating the object leads to a restricted relationship between the learning

[15] School and university.

[16] His analysis is based on policies and research concerning the school level, but it could be argued that also on university level comparable contradictory requirements for university teachers prevail. For an example of contradictions that university teachers face, see Ruge et al. (2021).

process and knowledge: acquisition in the manner of an internalisation of a given authoritative knowledge (Marvakis & Schraube, 2016, p. 197) – knowledge that is restricted to stay within set limits. This confinement only allows for a quite restricted form of understanding the relationship between the learner and knowledge: "an obedient relation of subordination" (Marvakis & Schraube, 2016, p. 197, translated by the author), and encourages defensive learning (Sect. 3.2.2). Strategies for fixating the object and pushing aside entailed social struggles can be found in curricula (e.g., delegating it to another course, demarking it as "not a mathematics education problem"), and the individual (e.g., relegating it to later or another context, or even suppression (Brown & McNamara, 2011)). But, when approaching an object to figure out its meaning, resp. generalised societal action possibilities, this entails the activity of (re-)connecting the object with social practices (Nissen, 2012, pp. 117–118). Within a learning practice thus also lies an expansive potential (see expansive learning, Sect. 3.2.2) of directing learning towards a change of not only oneself, but also changing restrictive practices (Marvakis & Schraube, 2016, pp. 196–197) – practices behind the unease.

Within the prominent German model of mathematics professional competences of teachers (Baumert & Kunter, 2006), beliefs and values are acknowledged as playing a crucial role, but "[d]ifferent elements of mathematics that seem contradictory are split into different aspects and marked as mathematical world views." (Ruge, 2017, p. 831). This model fragments[17] the learning object at stake: the mathematics-society relation. Debates of educational philosophy about what constitutes mathematics (Sect. 3.4), is redefined as an individual trait and is even "degraded[18] to an affective-motivational conditional factor of learning" (Ruge, 2017, p. 830) that a student must face in order to become a successful learner. The affective-motivational sphere is here not seen as a starting point for gaining insights (Sect. 3.2.1), but merely "as an agent of change[19] that has to be manipulated to a specific standpoint." (Ruge, 2017, p. 831), a standpoint that shall be learned along the way (see co-learning, Sect. 3.2.2), instead of conceptualising deliberate engagement to consciously develop a stance based on reflections. The design of teaching interventions takes up the idea of beliefs as "agent of change" and thus implement the fragmentation of the theoretical model into practice. They rely on gradual acquisition with the help of a change in the respective mathematics-related beliefs (Sect. 3.4.1). The question of the relationship between mathematics and society is learned along the way. This separates the learning object into academic knowledge, which is to be processed by the students through intentional learning, and a co-learning (Sect. 3.2.2) of its societal embeddedness. This separation obstructs a dialogue between teacher educators, which in Germany are mathematics education scholars, and student-teachers about existing contradictions within the mathematics-society

[17] A detailed argumentation can be found in Ruge (2017).

[18] Degraded in the sense that commonly the cognitive is referred to as higher function.

[19] Interestingly, it is not the student who is the agent of change but a scientific construct.

relation and thus obstruct a deliberate engagement with inherently ambiguous, conflictual and contradictory aspects of the learning object.

Theories based on the idea of situated learning (Lave & Wenger, 1991), such as the patterns-of-participation-approach (Skott, 2013; Sect. 3.4.2) allow for a conceptualisation of the learner as being actively involved in shaping practice. However, the focus on anchoring the practice of learning in everyday life is quite often understood in terms of immediate demands. This disambiguates the social practices in which the learning process takes place. However, social practices in which learning is situated, are always complex and contested (Dreier, 2003, 2008; Marvakis & Schraube, 2016, p. 205). By reducing learning objects and/or the learning context to neutral means or technical univocacies, the social struggle embodied in the specific institutional arrangements and its knowledge is denied.

At first, I thought that I had reached a dead end and I could not find any inspiration in belief research. But then I also considered belief research and the associated attempts to change mathematics teacher education as a specific practice to be interpreted in relation to current social struggles regarding mathematics education. From this perspective, the ever-increasing interest in belief research can be interpreted as an unease of mathematics education scholars with the current state of the mathematics-society relation. The eagerness to find ways of changing teacher beliefs, and through this mathematics teaching to a mathematics teaching that emphasises a process-orientation and speaks against popular myths (Sect. 3.4.1), can be interpreted as answering to this unease in a certain manner. In this sense, I consider the unease to be shared by student-teachers (Sects. 3.3.1 and 3.3.2) and mathematics education scholars[20]: both groups care for working towards a mathematics education that aims to promote a humane mathematics-society relation. The shared concern for the object at stake, can be regarded as a basis for an alliance with each other (Sect. 3.2.1) for a learning process, resp. a further development directed to this aim.

3.6 Discussion

In summary, the unease expressed by the students in the interviews (Sects. 3.3, 3.3.1 and 3.3.2), which I have also recognized in research projects (Sect. 3.5), refers to struggles within society. Even if these are not explicitly addressed on a manifest level (Sect. 3.3.2), there is a learning potential within this unease. A potential for reflection on the mathematics-society relation and thus to look beneath the surface (as aimed at in a practice-oriented research, Sect. 3.2.1) and further develop mathematics education theory. A one-sided conceptualisation of beliefs as individuals' state of mind or traits denies the reference to the social and the societal level. A

[20]My belief in this interpretation was strengthened by the experiences I gathered at conferences (Sect. 3.3.3).

management of this unease which aims at making students adapt to beliefs that are supposedly conducive to learning, bears the danger that engagement with these societal struggles will fall off the agenda (Sect. 3.5). It furthermore forecloses taking up the unease about the current status of the mathematics-society relation and making it a shared learning endeavour of scholars that identify with mathematics and future teachers of mathematics. If one considers what is addressed in belief research not as individual beliefs, but as an unease actually shared by scholars and student-teachers, this opens up possibilities to think about the importance of beliefs in teacher education beyond a focus on the management of individuals to take up stances that fit to currently prevailing reform movements. It makes it possible to understand the unease as a shared learning potential: The unease can be a starting point for a learning process, which not only has a potential for the development of teachers but can also be seen as a starting point for a shared inquiry of researchers and (future) teachers.

What happens at university is not only important for the learning of the individuals that participate in a study programme. The university is not only an educational institution but also a research institution. As a research institution, the university is also the place where knowledge in mathematics and mathematics education that serves as an important reference for the professional knowledge base of the teaching profession is produced. Thus, the university is also a place of further development of the professional discipline of mathematics education, which Bianca explicitly addresses in her reference to recent competence-oriented reform efforts (Sect. 3.3.1). However, this further development always refers to more than just developing individuals (Ruge & Peters, 2021). The double discontinuity can be understood as bringing to the fore meaning-relations which are difficult to relate to each other under the current status quo of the respective institutions. A deliberate engagement with the prevailing discontinuity between school and university mathematics can bring forward a deeper understanding of both knowledges, school and university mathematics, but a "softening" (Sect. 3.4.1) that relies upon a non-conscious co-learning of "adequate" beliefs, bears the danger of not realising the potential, or even channelling difficulties with prevailing meanings to the latent level (e.g. identifying as mathematically able, and therefore with current roles and images of mathematics, Sect. 3.3.2). Thus, trying to support an acquisition of university mathematics might come at the expense of closing off the transformative creativity of the socialisation process. Instead, the idea of "unity of research and education"[21] allows for heeding the stances that students bring along (Sect. 3.3.) and to struggle together at university for a humane mathematics-society relation. Such an approach offers the potential for generalisable insights and goes beyond the attempt to govern (psychological processes of) individuals (for the purpose of gaining control over the provision of mathematics teaching). Taking seriously the idea of a reciprocal learning process (Sect. 3.2.2), as Skott (2013, 2019; Sect. 3.4.2) pointed out, the participation of student-teachers not only lead to an alignment of the student-

[21] Central idea about the institution university in Germany.

teachers, but also always holds the potential of change within the institution. If "situated" is understood as integrating social fields of struggles and co-articulating social, political and epistemological locations (Marvakis & Schraube, 2016, p. 205), then patterns-of-participation hint at structural implications and typical modes of manoeuvring between different institutional contexts and, thus, the scope of the current status of the institutional arrangements and hindrances and possibilities for learning.

To consciously take up the struggle for a humane mathematics-society relation as a shared responsibility of scholars and student-teachers would also entail rethinking the prevailing learning conditions. It requires a specific learning context: A shared and conscious engagement with the unease always requires to a certain extent of oneself to reveal a stance towards the mathematics-society relation, a stance towards struggles, in order to even have a starting point for a dialogue. A mutual and reciprocal caring and mattering for the concerns of the other person is needed. However, this reciprocal caring and mattering requires refraining from common practices of institutional teaching-learning processes (e.g., presenting oneself as in alignment with the scope of the institution, fixed student and teacher positions, assessment standard that favour the transmission of sanctioned knowledge etc.).[22] The implementation of such an understanding of a shared learning process clashes with current institutional arrangements, requirements and expectations of university teaching (see also Nissen & Mørck, 2019).

The extent to which a realisation of such a learning process is feasible under prevailing institutional conditions can certainly be doubted, but to start out such a shared learning process might have the potential to change restrictive practices. To acknowledge the struggle can be the first step.

To what extent such a shared learning process changes the academic discipline of mathematics education remains an open question.

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[22] Especially those practices that serve the allocation and selection function of educational institutions run counter to caring and mattering.

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Chapter 4
Collaboration Between Secondary and Post-secondary Teachers About Their Ways of Doing Mathematics Using Contexts

Claudia Corriveau

Abstract The transition from secondary to postsecondary mathematics has been studied from various angles. Students encounter difficulties in this transition, but the investigation should not only be restricted to their perspective. In our study, we address the transition with teachers from both levels, fostering a dialog between them. This entry allows to tackle the transition from implicit part of teaching. Indeed, we focus on teachers' ways of doing mathematics using contexts at each level. We drew on ethnomethodology to conceptualize the object "teachers' ways of doing." Adopting a collaborative approach and research-practice partnership principles, we established a study in two phases. Results are presented according to these two phases. From the initial phase arise the reconstitution of ways of doing mathematics using contexts at each level, revealing two different "territories." The second phase exposes a certain rapprochement of the two levels.

Keywords Secondary-to-tertiary transition · Collaboration between teachers · Etnomethodology · Use of contexts in mathematics · Rapprochement perspective · Contextualisation and application comparison

4.1 Introduction

Our research springs from the desire to help students in mathematics during secondary-postsecondary transition, by ensuring that teachers from both level work side-by-side. We are seeking to establish inter-level collaboration by which teachers can discuss issues of transition, their respective ways of doing mathematics and also difficulties faced by their students. In doing so, these teachers work jointly to bridge the gap between the two levels.

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While drawing upon studies from other countries, this investigation occurs in a specific context. It takes place in Québec (Canada) where postsecondary education begins with cégep.[1] This follows secondary level and is prior to university. Cégep lasts 3 years for career programs and 2 years for pre-university programs (equivalent to Grades 12 and 13). Not all students are required to do mathematics, however, there are three mandatory mathematics courses in science pre-university programs: Differential Calculus, Integral Calculus, and Linear Algebra. Cégep mathematics teachers generally hold a master's degree in mathematics. The university culture of mathematics constitutes an important reference for these teachers (Mathieu-Soucy, 2020). As Gueudet (2008a) stresses, academic institutions differ from one country to another. Nonetheless, we hope that what we report here can inspire further research on transition.

In this chapter, we offer a brief description of studies concerning the secondary-postsecondary transition in mathematics and we explain the orientation of our research. The angle adopted is that of a dialogue between the teachers, and the theme is the use of contexts in mathematics. After presenting our theoretical foundations in the second section, we will describe the methodology and analytical process in the third section. The results are presented in two phases. The initial phase allows for the reconstitution of the ways of using contexts in mathematics at each level. The second phase reveals a certain rapprochement of the two levels.

4.2 The Secondary to Postsecondary Transition in Mathematics

Many researchers stress that the passage from secondary to postsecondary level is challenging for students who are not always able to grasp the new expectations (e.g. Liebendörfer & Hochmuth, 2013; McPhail, 2015). Certain students, who encounter difficulties when starting university, say that, in some way, they appreciate these more demanding aspects (Hernandez-Martinez et al., 2011). They say that these make them more responsible, and spur them to redouble their efforts to overcome their difficulties. Others, in contrast, drop their mathematics courses and impute their difficulties to the important differences between the levels (Di Martino & Gregorio, 2019). Indeed, a number of researchers have identified mathematical difficulties experienced by students entering university (e.g. Gueudet, 2008b; Vandebrouck, 2011; De Vleeschouwer, 2010).

[1] A French acronym referred to in English as General and Vocational College.

4.2.1 A Need for Dialogue Between Secondary and Postsecondary Teachers

However, students are not the only ones affected by these difficulties associated with the transition. At the postsecondary level, teachers expect a certain uniformity with respect to their students' knowledge, but soon become aware that this is rarely the case. Teachers are facing students who have either not all seen the same content, or who have not been presented with it in the same way (Corriveau, 2017; Stadler, 2011). Indeed, the transition marks a profound change of cultures in mathematics (Artigue, 2004). Teaching approaches and assessment strategies are different (Thomas & Klymchuk, 2012); there is a rupture in the didactic contract (Pepin, 2014), new kinds of mathematical organizations (Bosch et al., 2004; Gueudet, 2004; Winsløw, 2007), a shift in the discourse (Thoma & Nardi, 2018) and different ways of doing mathematics (Corriveau, 2017; Corriveau & Bednarz, 2017). The students must adapt to all these differences, about which the teachers are not well informed. There are "lacks of understanding of the issues involved in the transition from the other's perspective, and there is a need for improved communication between the two sectors" (Thomas & Klymchuk, 2012, p. 298).

Secondary and postsecondary teachers have received different training, do not share the same curriculum, and work in separate institutions. Furthermore, they rarely have the opportunity to discuss with each other what they are doing in mathematics. This compartmentalization engenders mutual misunderstandings (Emerson et al., 2015) and leads certain teachers to think that their students were poorly prepared at the lower level (Corriveau et al., 2020). Yet, much of what is done in teaching is not explicit, and this leads to this type of misreading of the other level.

In light of the above, on the one hand, there is a necessity to consider teachers' role in understanding the transition phenomenon, especially how mathematics is done at each level. On the other hand, there is a need for dialogue. Indeed, the lack of collaboration between the levels appears important when it comes to transition. This collaboration would allow for enhanced awareness of the other level situation, and be conducive to the development of ways to better support students during the transition.

4.2.2 The Use of Contexts in Mathematics and the Secondary to Postsecondary Transition

In the teaching of mathematics, the use of context is often seen either as a way of introducing mathematics or else, as a way of exemplifying abstract mathematics in its field of application. On the question of the secondary-postsecondary transition, these two different manners to use contexts are meaningful. Indeed, researchers such

Table 4.1 A discussion between teachers from both levels

Patricia:	There's **problem solving**, but we're much more inclined to start with **the whole theory and then, afterwards, apply it**. Sometimes, I use a problem, and I try it [with students]. Sometimes, I tell myself, "I haven't done enough," but **I like to start with the general**. I say to myself (slight pause) **I won't start with the context**.
Sam:	**That's the whole secondary program (starting with the context).**
Patricia:	That's right because everything is context. The thing is that I want to show them how to evaluate a limit, how to come up with the derivative. I establish all the rules for derivatives, I really do all the proofs, not using epsilon delta, but we start with the definition of slope and go from there to obtain the derivative, so... I have a lot of theory to communicate, so I lack the time to start with the context**. My context will come into play after. Then, there are books with a bunch of problems; there's no shortage of problems.**
Sandra:	At the same time, in secondary, there are still places where there is no context.
Sergio:	This is not exclusively in context; **it's that our explanations are often contextualized. We convey the subject with the context.**
Patricia:	I agree; that's true.
Sergio:	**Something without context, that often comes after.**
Patricia:	**We do it first without context and then, after, we work within a context.**
Peter:	But that's what we were being told… **we're not surprised to hear that.**
Patricia:	But it scares the students…

as Artigue (2004) and Leviatan (2008) point to this "cultural gap" as adding further complexity to the problems of the transition. The knowledge of secondary students is undermined by its contextual nature.

In research we conducted with secondary and postsecondary teachers, the teachers were convinced of this "cultural gap" (Corriveau, 2013). For these teachers, on one side, there would be a tradition by which doing mathematics consists of acquiring the tools through rigorous processes (definitions, theorems, and demonstrations) which then allow for the resolution of problems in context (postsecondary). On the other side, the context would be used to introduce and develop mathematics, to explain it, and then to move on to mathematics with no context (secondary). Table 4.1 presents an excerpt of the discussion (names beginning respectively by S and P relates to secondary and postsecondary teachers).

In the discussion, used as an illustration, the teachers bring up different meanings for the notion of context:

- application, problem solving for postsecondary teachers;
- contextualized introduction and explanation, given some background for mathematical concepts for secondary teachers.

Yet, the difference raised by the teachers is in terms of "before and after." However, these overall tendencies, well-known as Peter reminds us, might conceal more complex aspects. Thus, it seems important to better understand this *a priori* and to specify, for each of the levels, the ways of using contexts when doing mathematics.

4.2.3 Research Questions

Our objective is to develop and establish a collaboration between the levels which: (a) allows for dialogue between the teachers; (b) contributes to enhanced understanding of how mathematics is done at each level; and (c) envisages ways of easing the transition for students. In particular, we are working on the use of context and, in so doing, we address the following questions.

1. What are the specificities of the ways of doing mathematics using contexts at each level?
2. How can a rapprochement be established between these ways of doing mathematics using contexts?

4.3 The Theoretical Perspective

The theoretical foundations of ethnomethodology,[2] originated by Garfinkel (1967) in sociology, allowed to finely broach this object – teacher's ways of doing mathematics (here when using contexts) – and to configure the various dimensions (see Corriveau & Bednarz, 2013, 2017). From this perspective, actors produce knowledge and "ways of doing things" in a form of incessant practical investigation. The actors' point of view is therefore pivotal since it is in assigning meaning to what surrounds them that they construct their social world or any socially organized activity such as teaching mathematics at a specific level.

The concept of *ethnomethods*, these methods used by actors in the pursuit of their everyday activities (professional or other), provides an angle to approach teachers' ways of doing. This concept is described through other concepts related to actors' actions and actors' interpretations (Fig. 4.1).

In our specific case, that means that while teachers are engaged in their daily teaching activities (planning their teaching, choosing tasks, explaining a concept, solving a mathematical problem with students, assessing, etc.) they constitute and make visible (reflexivity/accountability) how we do mathematics at their specific level. They are the ones who know best familiar ways of using contexts at their level (interpretative procedures/membership). These ethnomethodological foundations will provide insights to the methodological approach.

[2]Some researchers have drawn on ethnomethodology to study different aspect of mathematics education, especially classroom interactions (see Ingram, 2018; Krummheuer, 2020).

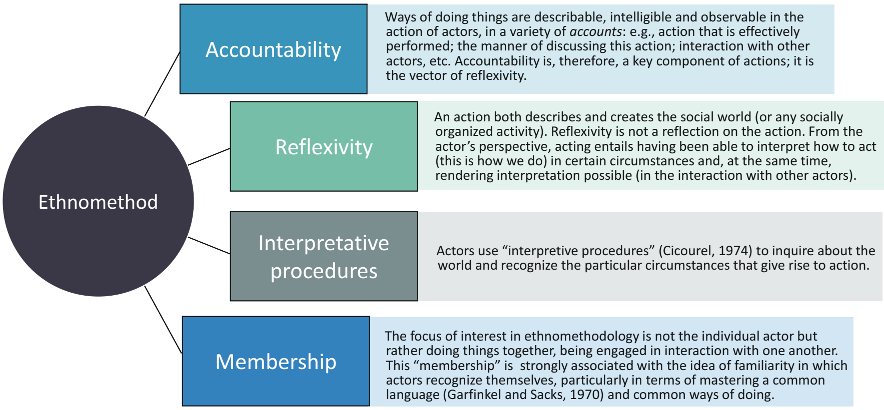

Fig. 4.1 Ethnomethod defined through a group of interrelated concepts

4.4 Methodology

Our research adopts a collaborative approach (e.g. Bednarz, 2004) and draws upon the basis of *research-practice partnership* (Coburn & Penuel, 2016): (1) it takes place over the long term, rather than focusing on a single study; and (2) the work is negotiated jointly and the leadership is shared. The results that we present stem from a two-phase investigation.

4.4.1 An Investigation in Two Phases

The first phase is aimed at better understanding each level's way of doing mathematics. This initial phase, which started in 2011, brought together six teachers, three from each level. Six encounters, lasting a full day (9 am to 4 pm), were spread out over a year. They were structured around different mathematical themes such as functions, symbolism, contexts, proofs, etc. The results of this first phase led us to develop a second phase.

The second phase (2017–2019)[3] brought together 13 teachers, six postsecondary and seven secondary teachers. To ensure continuity between the two phases, and to distribute the leadership as a way to facilitate collaboration (Heo et al., 2011), two teachers who had participated in the first phase also participated in the second phase. Yet, this time, they worked in conjunction with the researcher to organize seven

[3]Three major themes emerged from phase 1: functions and their representations; the use of contexts and symbolism. Between the two phases presented here, from 2014 to 2017, the question of transitions was brought further by the researcher from the angle of symbolism (see Corriveau & Bednarz, 2016, 2017).

meetings with the other teachers. Also, while the new participants were likely to bring up new ideas, these two teachers could share reflections coming up from phase 1 and changes made in their practice.

4.4.2 The Dialogue Organized Around a Reflexive Activity

Reflexive activity in collaborative research is based on ethnomethodology (Desgagné, 2001). It means that teachers must be in action so that they account for and constitute their ways of doing. This was organized in various *situations serving as a basis for discussion*. We used familiar situations which are meaningful in teachers' daily activities such as commenting on teaching situations, establishing the way to make use of a problem, giving meaning to a student's solution, etc.

The researcher's role in mediating the discussion amongst the teachers was central since teachers from both levels are not used to working together. Here, it was a matter of encouraging and contributing to clarifying the ways of doing at a given level, thus, adopting the role of interpreter (Davis, 2005). Also, with the aim of engendering a rapprochement, the researcher was engaged, with the teachers, in indicating areas that still needed to be clarified and in being on the lookout for any opportunities to reveal issues associated with the transition.

4.4.3 The Overall Analytical Process

All the meetings were filmed. They constitute the core material of the research. We relistened to the recordings, eliminating parts where the discussions were not directly linked to the use of context. For Phase 1, after having retraced all the transcripts about context, we separated those related to secondary from those related to postsecondary. We reinvested the concept of accountability to divide the transcripts in different types of account. Then, at a first level, we described, remaining very close to what the teachers said, their ways of doing mathematics when using contexts. At a second level, we proceeded to an analysis of these ways of doing, pointing at their specificities, bringing forward what we call a "territory"[4] at each level. For Phase 2, we retraced, chronologically, the different episodes related to context in order to proceed to the analysis of a rapprochement, still from the angle of teachers' ways of doing.

[4]We use this metaphor to illustrate that during the research, teachers of a specific level consistently organize a familiar territory about their ways of using contexts in which they recognize themselves (as members). We see it as a space that is continuously undergoing organization.

4.5 Results

Here we present the results according to the phases described above.

4.5.1 Phase 1: Two Territories Established Around the Use of Contexts

The ways of doing related to the use of contexts are related through three types of accounts.

1. Through "action": a mathematical task taken from a textbook was submitted to the teachers and was spontaneously explored by them. They worked in teams of teachers from the same level. This was a matter of interpreting graphs in context (anomalous behaviour of water).
2. Through storytelling: in discussions, without context being the main subject, teachers ended up sharing certain aspects of their practice in illustrating their way of using contexts.
3. In discussing how to make use of a teaching situation: a teaching situation presenting the graph of the reproductive behaviour over time of two populations of bacteria was given to teachers. Teachers were asked whether this was the type of task they worked on with their students and whether they could (how they would) use it in class.

The analysis led to our nuancing the teachers' initial position: the idea of "before and after" and reconstitute two territories around the use of context. To describe each of these territories, we drew upon the theoretical elements proposed by Janvier (1990, 1991) and Douady (1986, 1991).

4.5.1.1 Secondary Level Territory: Contextual Mathematics

The territory as constituted by secondary teachers who participated in Phase 1 of the research is characterized by what we called contextual mathematics. Before presenting the global specificities, we first present excerpts (Tables 4.2 and 4.3) that exemplify our analysis (from the second type of account).

In this example, we see that secondary teachers are accustomed to introducing the rational function in a context of a bus rental. This context is reworked to allow, amongst other things, the illustration of some characteristics of the rational function:

The fact of adding a guide and counting or not a driver as describe by the secondary teacher Scott relays mathematical intentions, such as contextualizing the impossibility of dividing by zero, contextualizing parameters in the equations, etc. Thus, the context evolves and this work is guided by the mathematics at play.

Table 4.2 Excerpt of Scott explaining his way of introducing the rational function

Scott:	Me, I always start with that for the rational function, without having seen anything else; it's a bus that you rent for $800. This is an organized group and there is a guide who doesn't pay anything. Once at the destination, it's a museum visit and that's $25 per person, and then it's a matter of finding out, and there are a number of questions, but how does one calculate the cost per person according to the number of participants.
Researcher:	So then, what do you come up with?
Scott:	Well, obviously, the students make mistakes. They work in teams, asking questions: "Well, it's $800 for the bus, there's the guide, plus $25 for the visit there". . . in this problem, there are the two asymptotes in a plausible context. Clearly, they very often forget the guide, and there, there are questions, if there are so many people, 2 or 3, who participate in this, how much will that cost? And there, let's suppose that the bus holds 50 people; we could put in as many as we want; it's a Harry potter bus! And there, it's a question of seeing what happens and is it possible just to have one person in the bus? "Well, there's going to be a driver!" [. . .] in the different versions, at a certain point, we said that the driver doesn't count! And there's also the guide who pays nothing, but takes a place in the bus. Obviously in that, there is a way to say that there cannot be only a single participant, because there will not then be an organized trip. There have to be at least two people, the guide and a participant. That's it, the x, it's the number of people in the bus.

Table 4.3 Excerpt of Sam adding to Scott's explanation

Sam:	Because it's costing me a lot, I'm going to invite some friends. So, when we're two friends, that will cost us $2500 each. And then, the two of us, we're going to look for more friends. The price diminishes more and more. . .

Another secondary teacher Sam (and colleagues) also **plays with the context and transforms it in connection with the mathematics at play.**[5] Indeed, Sam could identify with Scott's remarks. The context of the bus rental reveals, according to Sam, that what varies in the rational function is "what we divide by". Thus, Sam and Scott, in the bus rental situation (at $5000 in Sam's example), are demonstrating how the division varies in context:

In other words, a certain generalization occurs in context (**to generalize in context**). The secondary teachers are not trying to model the situation and are considering the possibility of having 2,000,000 people in the bus (an example given by Sam). This example serves instead to indicate that, regardless of the number of people on the bus, even if it goes up and up, there will be a steadily declining cost, without ever arriving at zero. The teacher tries to **provide an image, using the context, of the concept** of asymptote.

All these ways of doing were, highlighted by the first layer of analysis, allowed to characterize the secondary territory.

[5] We emphasize in bold the teachers' ways of doing using context.

Three Specificities to Characterize the Territory at the Secondary Level

– Mathematics, inseparable from the context

When it is a question of contextual mathematics, it is difficult to separate mathematics and context. In other words, teachers are speaking of mathematics and of context at the same time. They make the context evolve at the same time as mathematics, and vice versa. Furthermore, when the teachers play with the context, they also reveal, in the background, a progression of mathematics to be worked on. We can readily see the underlying mathematical intentions but, in reality, it is the context which is being discussed. (**Teachers play with the context and transform it in connection with the mathematics at play.)**

This relates to the work of Janvier (1990). The latter emphasizes that a key characteristic of contextual mathematics consists of proceeding in arithmetic with specific measures or amounts rather than abstract numbers. In the example presented above, the secondary teachers stress that functions are appreciated when they link quantities and allow for the interpretation of phenomena. The function does not lose its contextual connotation; it is the representation of a phenomenon (anomalous behaviour of water, the cost of bus rentals, etc.).

– Oral Mathematics

All this work, by which contextual mathematics occurs, is principally communicated through the spoken language, and mathematics is discussed in terms of the context. For example, tracking a point on a graph in the context means, for secondary teachers, saying, “at 4 degrees Celsius, this is the temperature when the volume of water is equal to 1.” The oral approach is in the forefront. Indeed, what emerges from our analysis of teachers’ ways of doing, through all the types of accounts is: **speaking about phenomena**, **verbalizing in context**, **using terms related to the context**, **and talking about mathematical concepts in context**.

– Imagery Mathematics

Along the same lines, doing mathematics in context entails speaking about mathematics in this context and engaging in eloquent discourse. Within this mode of communication, one uses images (in the broad sense) to “make mathematics speak.” Similar to what Artigue said (2004), it is difficult to revive the memory of a group which does not share the same story. Now, for a teacher, **evoking images** is, in some ways, a means to remedy this situation. This requires teachers to **start from mathematical concepts and find ways to discuss them which evoke images**. As Sam mentions in the example below (Table 4.4), it is not necessary to explain further since the image speaks for itself:

Table 4.4 Excerpt about "evoking images"

Sam:	When we left here after the last meeting, I worked on contexts in class and I asked myself: "what am I doing there; why am I doing that?" [. . .] A boat that follows a wave. When you say that the boat is following a wave, the students know what a boat is, what a wave is. It rises and falls [mimics the movement]. You don't need to explain further; it's already in the context.
Scott:	That's right, instead of saying, "a sinus function, it's something which is periodic, with a maximum and minimum." [he looks confused, imitating a student.]

Table 4.5 Excerpt about seeing mathematics as a prerequisite for science

Patricia:	What I want to say is that postsecondary students, they need to get out of their bubble...
Sam:	I completely agree with you. . .
Patricia:	You're in natural sciences, what I want to show you, it's that differential and integral calculus will be useful in engineering... Because there are links to be made with chemistry, there are links to be made with physics. That's what I want to say. . . you must get out of...[your bubble]. We can do fun things.
Peter	. . . I think, as you say, we prepare them to sciences. Personally, I don't prepare them as if they were in a math program. I prepare them to study sciences. If they do chemistry or physics, they should be able to do some math. . . I want to prepare them. I don't want them to be "mathematicians". They are not doing a bachelor in mathematics as we did. They are learning other stuff. But they need to be good enough in math so they can use it if they need it. "I want to be a doctor". Well, logic will help you a lot. They don't have a choice. They need to do math, they are in a math class.
Paula:	They will not necessarily use this math in their daily life, but we train them as if they would have a scientific career. This is our goal, the aiming. . . we don't want to lie and say, this is going to be very useful in your daily life!

4.5.1.2 The Territory of Postsecondary Mathematics: Illustrated Mathematics

Postsecondary teachers interpret the usage of context at secondary level as inadequate for their students. Their ways of using contexts assign a particular role in mathematics that of a prerequisite for science (see Table 4.5).

In our analysis, we noticed that the postsecondary teachers, Patricia, Peter and Paula, associate contexts with problem solving (which is not the case for secondary teachers). We chose "illustrated mathematics" to characterize the specificities of their ways of doing mathematics using contexts. *Illustrated* means exemplified, because the mathematical notions are applied within problems for illustrative purposes. Then we also used this term because *illustrated* (from *lustre*) brings us back to *illuminating* and, at the postsecondary level, even in context, light is shed on mathematics.

Four Specificities to Characterize the Territory at the Postsecondary Level

– An unequivocal correspondence between elements of the problem and mathematical elements

The postsecondary teachers in Phase 1 make a correspondence between mathematical and contextual elements of a problem. Their work consists of **"injecting" usable mathematics into the problem.** For example, Patricia and Paula found the interpretation of the graph of the anomalous behaviour of water quite disconcerting. They become more at ease when they could "inject" considerations of the limits and rates of variation into the required task, and then they could envisage using this with students. The finding of usable mathematics is ensured by the teacher. He or she attempts to find, in a context, a mathematical task to be done, and in a problem, an algorithm to perform, results to invoke or a process to implement.

– Mathematics as Tools

Teachers also refer to mathematical notions as tools. This echoes Douady's work (1986). For teachers, there seems to be interplay between work which is purely mathematical and work with tools which mathematics provides. For postsecondary teachers, a reference to the mathematician's work is important. For Douady, a major part of mathematicians' activity consists of problem solving. To do so, they are led to create conceptual tools which are subsequently decontextualized and formulated in the most general way possible. Thus, the generalized concept-tool acquires the status of an object. From the teaching perspective, one could refer to progressive reification, that is, beginning by working with notions as tools but gradually objectifying them, progressively, as the work progresses. Yet, for postsecondary teachers, it seems that this shift between tool and object operates in the opposite direction, in an application approach. Peter says: "we give them problems to know whether they are capable of applying what we have shown them."

As Patricia mentions, one needs tools to solve problems. Thus, teachers **introduce concepts/decontextualized objects and then use them as tools**, amongst other things, for problems of applications in contexts.

According to Janvier (1991), when the mathematical perspective is one of application, the mathematical concept has a status in its own right and is not context-sensitive. Facing contextual problems, teachers must **put the light on the mathematics**. Once this association has been established, the idea is, for postsecondary teachers, to **move away from the context to carry out the mathematics**.

– Objectified Mathematics

Patricia, Peter and Paula consider that the notions they teach, and which they use as tools, were developed by mathematicians: e.g. "we're not the ones who invented it" (Peter). In their teaching, what they present to students is, in some ways, already reified and objectified (by mathematicians). This does not signify that postsecondary teachers think that mathematics does not evolve, but that, the mathematics they work on with students in their specific course does not. In context, they use mathematics as a tool, but the status of mathematical objects is unchanging. In this sense, from this perspective, clearly the role of context will be less important.

– Written and Symbolized Mathematics

The mathematics worked by postsecondary teachers in contexts is communicated through writing. When one seeks to apply tools, put into action formulas, operate in problems, the preferred mode is that of written and symbolized mathematics. As the teachers mentioned themselves, the context is incidental, it serves little or no purpose. **Problem solving is done mathematically, without referring to the context**.

4.5.2 Phase 2: A Process of Rapprochement Between Levels

The first part of the analysis highlights notable differences. In the second phase, we proposed situations so as to encourage collaboration, and to see whether a rapprochement was viable. Before presenting the elements characterizing the rapprochement, we present types of account used in the meetings:

1. Discussion on the way to manage a teaching situation (Meeting 2): we utilize a situation from Phase 1 (a comparison of the reproductive behaviour of two populations of bacteria over time). The same questions were asked to the teachers. The idea was to determine whether these new teachers reported the same ways of doing things in context as their colleagues. In addition, this initial joint reflection seemed necessary in order to go further.
2. A study of two lesson plans at the beginning of postsecondary level (Meeting 5): the first lesson plans were based on observations made at the postsecondary level and the second inspired by work in Phase 1. The teachers at each level discussed them to determine what was a recall of prior knowledge and what constituted new content.
3. The syntheses of the use of contexts (Meeting 6): this synthesis, led by the researcher, was rich in terms of discussions. The teachers clarified a certain rationale with respect to their use of contexts.
4. Exploration of the intuitive meaning of mathematical concepts evoked by a situation in context (Meeting 6): we used a contextual task to determine which concepts from postsecondary were intuitively mobilized.
5. Joint planning (Meeting 7): at the request of a postsecondary teacher, in a sub-group (with two teachers from each level), we planned the first lessons of Differential calculus course for students arriving from secondary level.

Through these means, the teachers account for some ways of doing raised in Phase 1. The analysis of the rapprochement draws upon these accounts. We distinguish four key moments.

4.5.2.1 Moment 1: Explaining Their Respective Ways of Doing

The discussion around the reproductive behaviour of two populations of bacteria over time brought out comparable elements as in Phase 1. For example, the secondary teachers indicate that this task could be done with their students. They see it as a work of interpretation of a graph or of a mathematical model in terms of context; mathematical concepts retain a contextual connotation (the interval is spoken of in terms of duration, and the increases in terms of the growth and decline of the population). At the postsecondary level, the teachers make it clear that for the task to be assigned at their level, the equation of functions, and not of curves, would be required (injecting usable mathematics): “You must have the rule. If you’ve seen the derivative concept, you, you want the second derivative, an inflection point” (Philip). He makes elements of the problem correspond to the mathematics to be tackled. This situation does not necessarily entail a rapprochement. Nevertheless, this allows the researcher to confirm that teachers place themselves in their respective territory, to take action that elucidates their respective ways of doing things, and to base the work ahead.

The study of two different lesson plans (the first lessons of Differential calculus course) brings to light some reciprocal misunderstandings. The one familiar to secondary teachers presents new elements for postsecondary teachers, and vice versa. This finding, of which the teachers gradually become aware in the discussion, highlights the relevance of a dialogue between teachers. Secondary teachers are sensitive to the way mathematics is approached in the lesson plans. While the first plan (chosen by the postsecondary teachers) goes over methods of factorization and the introduction of the concept of limit and its properties (work on the definitions and properties), in the second scenario (chosen by the secondary teachers), students are led to explore two activities, the second of which is contextualized. In the discussion, the teachers made their respective territories clear (which mainly comes back to what is outlined in Phase 1).

4.5.2.2 Moment 2: The Establishment of Common Elements

While the teachers discussed their respective ways of using contexts, it seemed important to pursue this explanation, but through the underlying rationale. Thus, a synthesis of elements agreed upon is presented to the teachers. (See Fig. 4.2 and Table 4.6) In light of this synthesis, the teachers interpret the elements and bring up certain limitations of their use of context.

In this discussion, Sophie brings to light the limitations of this way of using contexts. It can pose difficulties for students. This issue, raised by Sophie, spurs Patricia to mention that, at the postsecondary level, there are also challenges: students have done “the construction of a concept, but the application is impossible, since they have not sufficiently mastered the concept” (corroborated by other postsecondary teachers). There are limitations in the ways of using context for both levels.

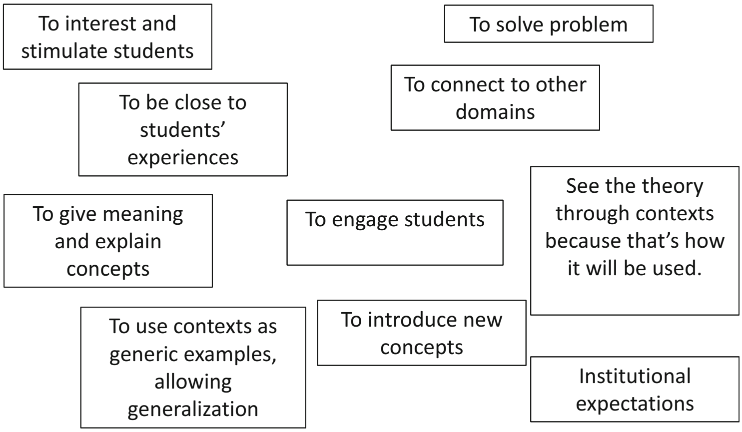

Fig. 4.2 Synthesis of teachers' reasons for using contexts

Table 4.6 Excerpt of a discussion from the synthesis

Sophie:	We want so much to contextualize [at secondary level] that sometimes it's far-fetched.
The other secondary teachers agree and give some examples from ministerial exams.	
[...]	
Sarah:	At secondary level, for certain functions, to understand them, it's interesting to have a context at the outset. . . the exponential, it's super-relevant to use examples to get there. . . they [students] end up finding an equation that there is an exponential function associated with that....You know, the rational [function]....You can start with a context in which you have a linear divided by another linear... They see what this gives; they analyze the function...It becomes more representative and, after that, you can go further into the abstract.
Sophie:	There, where I've got a problem sometimes, is that the contexts end up confusing them. Especially when we're into continuous variables and discrete variables. They mix up the contexts from 1 year to another and even in the same year. [She gives an example of the same context treated as a discrete context and then as a continuous context, and the other teachers corroborate this.]

However, the choice of context seems important for teachers at both levels: "everything depends on the context" (Sarah); "when the context is well chosen. . ., not too easy, not too difficult" (Scott); "not too complicated" (Piero); "there are some cool contexts to motivate students" (Piero); "some classic contexts which are great when one is introducing a concept" (Petra), etc. In so doing, teachers agree on some reasons for their use, while still remaining in their respective territory. Teachers agree that contexts help "give meaning," "motivate," "interest," "stimulate," etc. Despite the different ways of using context, the goals concur.

4.5.2.3 Moment 3: Revisiting One's Territory in Light of the Other

Playing the role of the moderator of the interaction among the teachers, the researcher charts the outline of Fig. 4.3 below[6] and proposes the hypothesis that the issues are represented by the dotted arrows.

The teachers add (Table 4.7).

The researcher then suggests reflecting on the dotted Arrow A (see Fig. 4.3) starting from key postsecondary concepts. She proposes considering the task in Fig. 4.4. This task seems to her at the juncture between secondary and postsecondary. The situation evokes concepts from secondary level but there are also a number of interrelated elements.

At the outset, the teachers place themselves in their respective territory: those from the postsecondary level say that this is a complex task which should appear at the end of the calculus course, and remark that the equation is not already included in the task (objectified mathematics). Those from secondary also see a complex, but accessible, task: "I would have them do it" (Sarah); "problems like that, they get lots of them in secondary; this one is a bit different, but familiar" (Sophie, supported by Scott). After explaining the particular features of secondary level (e.g. the work surrounding the rational function, with the canonical form of equation) and of postsecondary (e.g. the notion of limits and of indeterminate forms), a discussion emerges (Table 4.8).

In the previous, bridges are built. Postsecondary teachers, in collaboration with secondary teachers, revisit their territory, with that of secondary teachers as a horizon (intuitively approaching the limit, in context, starting from familiar elements from the secondary level). Petra reveals that this work belongs to the postsecondary territory. This joint questioning invites postsecondary teachers into the territory of secondary teachers and opens up an extension of the postsecondary territory. In doing so, "intermediate zones" are created which contribute to the rapprochement of the two levels.

4.5.2.4 Moment 4: Joint Planning

In August 2019, Petra asked us to set up a meeting (the 7th) in a subgroup before postsecondary students came back to cégep. The organization of the initial lessons of the calculus course was discussed. Amongst the various suggestions, the task "overhanging roof" came back as a means to introduce the notion of limit. Then, the question raised was how to work on this in class and how to use this contextualized task to arrive at the concept of limit.

[6]This schema was also utilized in Phase 1 of the project with a goal of rapprochement, but was handled differently. Throughout the discussions, the teachers remained in their own territory and engaged in dialogue without necessarily entering the territory of the other (Corriveau, 2013).

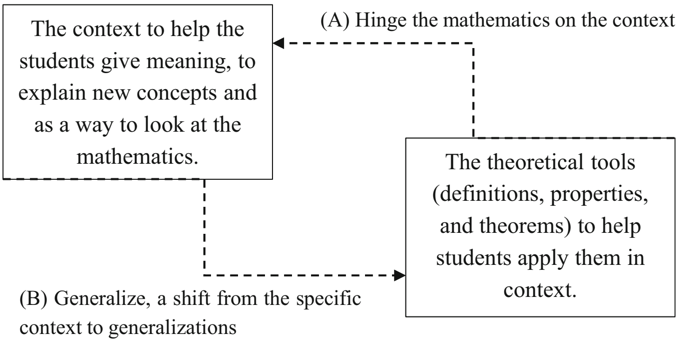

Fig. 4.3 Schema used as a basis for a rapprochement

Table 4.7 Discussion arisen from the schema

Sophie:	Finally, being able to do both. . .
Petra:	Well, it's finding the balance, I think. . .
Scott:	. . .starting from the general to come back to context (he laughs).
Sophie:	Being able to take concepts to apply them in context and being able to take the contexts to come up with generalities. Me, I like that a lot. You've got to do both.

An overhanging roof

A inclining roof lies on the walls MN and AB. These walls are 3 m high and 4 m apart from each other. The roof can be more or less inclined. How does the height h of the ridge change when the distance of x approaches 2 m?

- How does h vary when x gets bigger and bigger?
- How are these previous observations interpreted in the graph representing h as a function of x?

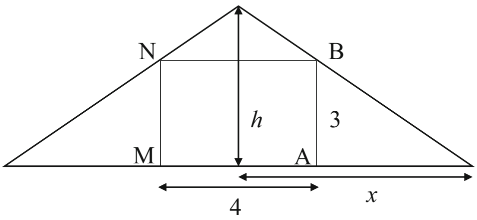

Fig. 4.4 Context used for exploration of the postsecondary concept. (Taken from *groupe A. H. A.*, 1999, our translation)

Everyone agrees with keeping the task as is, but Patricia writes on the board a more complex function ($f(x) = \sqrt{\frac{x^3+1}{x-1}}$). The group then debates the progression from the situation of the roof to the function proposed by Patricia. In other words, the idea is to work on the dotted arrow B (see Fig. 4.2). The synthesis of the progression is presented in Table 4.9.

Table 4.8 Discussion revealing a rapprochement

Sophie:	It seems to me that if one was able to make this link... [she's talking about the link between the limit (intuitively) and the infinite and the different forms of the rule (homographic and canonical form of equation)].
Petra:	**But we're the ones who should be doing it!** We, we know what you're working on. A lot with the canonical form of equation. There, we tell them [students]: "we're a bit short of time because we have infinity over infinity. We're going to have some tools, but is there anything else that you have [with reference to the canonical form of equation]?"
Sophie:	They're going to do it!
Piero:	[surprised] It's true that, that means we can avoid here going through an indeterminate form!
Sophie:	For the introduction at least and that's going to allow to make a link to something that they already know.
Petra:	There, that's okay, they'll be able to do it, but they'll see that they need other things after, tools to be developed. We are creating a need.

4.6 Conclusion

What can we conclude from this analysis? First, it leads to nuancing teachers' initial position, presented in Sect. 4.2.2: the idea of "before and after" when it comes to using contexts. The ethnomethodology perspective via teachers and their ways of doing mathematics using contexts has allowed us to report what usually is implicit. We perceive here a first contribution as these ways of doing could not be as visible from an analysis of the curriculum and textbooks. This allowed us to (re)constitute a certain territory at each level: from contextual mathematics at the secondary level to illustrated mathematics at the postsecondary level. The teachers' discussions revealed a broader vision of mathematics (in relation to the use of context), a way of conceiving it. Moreover, we can affirm that, with reference to the use of contexts, secondary and postsecondary teachers who participated in the research "live" in two very distinct territories. Thus, it is not surprising that students arriving at the postsecondary level have difficulties decoding the rules of the game.

Janvier's work (1990, 1991) in which he contrasts the notion of application and that of contextualization is evocative to compare the two levels. For Janvier, from an application perspective (illustrated mathematics in our case), mathematics is considered general knowledge. The field in which this mathematics is significant and where it can be used does not change, and does not affect the nature of this mathematics. For example, mathematical work done on equations will not relate to the underlying context that enabled them to be established.

From the contextual mathematics perspective, the context which frames mathematical activity provides support for the reasoning. According to Janvier (1990, 1991), contextual mathematical reasoning develops at the intersection of two domains. The context contains specific elements; so, it is normal to expect these particular elements to play a role in the way of broaching mathematics. The characteristics of the context contribute to the development of a mathematical

Table 4.9 A four-step progression

Intermediate zone (the "overhanging roof" task seen in Fig. 4.4**)**
According to teachers, secondary students have enough knowledge to complete the task. Nevertheless, they will find the rule $h = 3x/(x\text{-}2)$ a form to which they are relatively unaccustomed. The secondary teachers expect students to shift to the canonical form: $h = (6/(x\text{-}2)) + 3$. The postsecondary teachers plan to explore the notion of limit in this context, but also to connect the two forms of equation (homographic and canonical) in relation to the intuitive notion of limit. A secondary teacher also proposes discussing with the students what happens with the height (h) when the base of the roof comes closer and closer to the wall at x = 2. The context being here central in the explanations.
New way of doing at the postsecondary level: Gradually moving away from the context ($f(x) = \frac{x^3-3x}{x-3}$)
A postsecondary teacher proposes intuitively pursuing the exploration of the limit according to a certain progression. She suggests the function $f(x) = \frac{x^3-3x}{x-3}$. What happens around $x = 3$? The secondary students have not worked with "holes". This is an initial detachment from context, but the work is not yet generalized. Instead, it is based on an example close to that which was done in context (the division of polynomials), but liable to unsettle students' conceptions (the rational expression means asymptote). As Sarah mentions: "They (students) operationalize elsewhere what they have just seen before" (Sarah).
Establishing a clear border ($f(x) = \sqrt{\frac{x^3+1}{x-1}}$) to enter postsecondary level
With this function, according to the postsecondary teachers, we arrive at the boundary of what students can accomplish intuitively. Therefore, they propose an introduction to the concept of limit and properties. This is the objective at the postsecondary level.
New way of doing at the postsecondary level: Drawing upon the previous work in context
While introducing the limit (definition and properties), it is proposed to come back to the initial function (from the "overhanging roof" task) to exemplify the concept of limit and certain properties in context.

meaning. Contextual reasoning aims to reconcile the mathematics carried out with the characteristics of the context, through the support defined by the context: graphs, verbal description, images, gestural. Figure 4.5, which distinguishes contextualization from application, is inspired by that of Janvier (1991).

Our research is not about finding the best ways to do mathematics using contexts, but rather understanding how mathematics is done at each level. However, as mentioned by Biza et al. (2016), there is an increased interest about the role of mathematics in other disciplines at tertiary level. They recall Harris et al. (2015) study in which they suggest that the students "need to get insight of the real use of mathematics in their discipline." (Biza et al., 2016, p. 6). The applicationist understanding of the use of mathematics in other disciplines (as shown by postsecondary teachers participating in our research) neglect, according to Hochmuth (2020), the dialectic relation between mathematics and the context of its use. What would contextual mathematics brought by secondary[7] teachers here mean at postsecondary level? What elements of context from postsecondary scientific or professional

[7] And also from research conducted in situated perspectives (Please see, in particular, Lave, 1988, 1996; Noss, 2002; Noss et al., 1999; Nunès et al., 1993; Traoré & Bednarz, 2009).

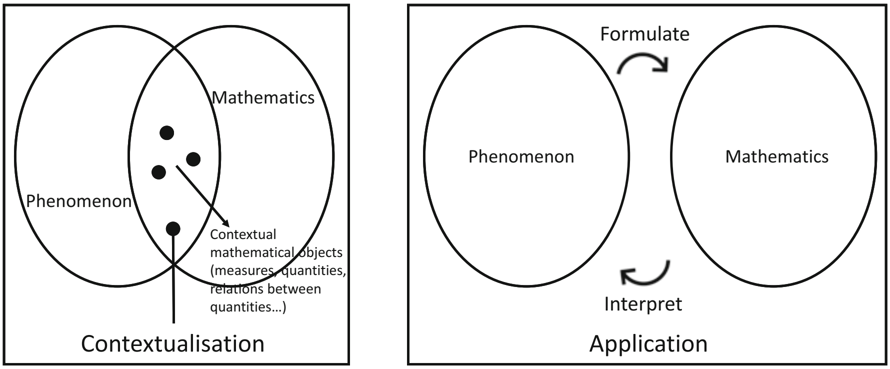

Fig. 4.5 Schematization of the differences between secondary and postsecondary territories based on Janvier's comparison of contextualisation and application (1991, p. 146)

disciplines (e.g., engineering, economy, etc.) could help understand the mathematics and vice versa?

Also, the back and forth between contextualized and decontextualized (dotted Arrows A and B from Fig. 4.2) become important aspects to work on when teaching to non-mathematics students at postsecondary level.[8] Even if the operationalization of these reflections remains to be formulated, these suggestions suppose, in our view, an important change in postsecondary mathematics teaching.

Finally, the goal of a rapprochement was achieved in a progression of four key moments: explication of respective ways of doing mathematics, establishment of common elements, revisit one's territory in light of the other, joint planning. This rapprochement is not a way to ensure that each level adopts the same ways of doing mathematics using contexts. As it was developed within the group, it reveals the need to take into account what is done at the other level and how it is done. That occurs through a progressive familiarization with the territory of the other level. Facilitating inter-levels collaboration enables the understanding of the other and new ways of doing to accompany the students in the transition.

[8]The reflection would be completely different for mathematics students. New requirements in terms of formalism, proofs, rigor and abstract mathematics in mathematics major programmes have been an important focus to understand the transition and remain very relevant for those students. We chose to focus here on non-mathematics students and on the relationship between mathematics and other disciplines at tertiary level. Indeed, there is an increased interest about the role of mathematics in other disciplines at tertiary level as pointed out by Biza et al. (2016), however one can ask what is the role of other disciplines – and more broadly of contexts – in mathematics courses?

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Chapter 5
Framing Goals of Mathematics Support Measures

Michael Liebendörfer, Christiane Büdenbender-Kuklinski, Elisa Lankeit, Mirko Schürmann, Rolf Biehler, and Niclas Schaper

Abstract Tertiary mathematics education has produced a multitude of measures in recent years, all of which aim at improving the learning of mathematics at universities. Such support measures pursue a diversity of goals that have hardly been explicitly captured and compared in the literature so far. This chapter takes a step in this direction by presenting a framework that was developed within the WiGeMath project. The WiGeMath (Wirkung und Gelingensbedingungen von Unterstützungsmaßnahmen für mathematikbezogenes Lernen in der Studieneingangsphase; Effects and success conditions of mathematics learning support in the introductory study phase) project was funded by the German Federal Ministry of Education and Research (BMBF, grant identifiers 01PB14015A and 01PB14015B) to compare innovative support measures in Germany. Based on concrete measures, a differentiated category system for goals was developed. One benefit of the framework is illustrated in the second part of this chapter where several pre-university bridging courses, redesigned lectures and mathematics learning support centres are compared regarding their targeted goal categories. The results show both the variance within measures of a similar type and variance between these types. We discuss how the framework can contribute in making goals more explicitly visible for the comparison of measures, but also for their change or redesign.

Keywords Educational objectives · Mathematics support · Taxonomy · Bridging courses · Redesigned lectures · Teacher goals

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5.1 Supporting Students in the Secondary-Tertiary Transition and the WiGeMath Project

The secondary-tertiary transition in mathematics is difficult for students and sometimes disappointing for teachers (Gueudet, 2008). Internationally, similar difficulties are reported, such as students' decline in motivation (e.g. Daskalogianni & Simpson, 2002) and problems in coping with the new requirements. In Germany, as in some other countries, drop-out rates are particularly high in mathematics and mathematics-related study programs (Heublein, 2014). Internationally, new approaches to improving this transition have existed for a long time but have recently gained momentum (e.g. Durand-Guerrier et al., 2021). Traditional teaching can both be enriched and changed. We speak of mathematics support measures and mean different kinds of measures designed to help ease the secondary-tertiary transition for students. In Germany, many such measures have been initiated in recent years (cf. Göller et al., 2017, for an overview).

This development raises the questions of which aims such measures pursue and which ones of those they actually achieve. The WiGeMath and WiGeMath-Transfer projects, led by Reinhard Hochmuth, Rolf Biehler and Niclas Schaper, have tried to answer both questions. The overall aim was to evaluate and compare different support measures in terms of their implementation, necessary conditions for success and impact. In the WiGeMath project, three types of measures were investigated in depth: pre-university bridging courses, redesigned lectures and mathematics learning support centres (MLSC). Pre-university bridging courses are courses offered by universities shortly before the start of the regular first semester in order to teach mathematics-related competencies, lasting several days to weeks. In most cases, participation is optional and free of charge. Redesigned lectures are regular compulsory lectures introducing students to university mathematics in a non-traditional way to ease their transition from school to university. MLSCs offer low-threshold individual support in a specific site on campus (not an office) in addition to the curriculum. They have longer been offered in English-speaking countries (Cronin et al., 2016; Lawson, 2015; Rylands & Shearman, 2018) and are becoming more and more common in Germany (Schürmann et al., 2021). In MLSCs, advice and support are offered to students on mathematical topics and tasks.

As a prerequisite for the communication about measures' goals and frame conditions, we created a comparative framework which includes goals, frame conditions and measure characteristics. In this paper we focus on the goals. Making measures' varying goals visible can have a benefit beyond the WiGeMath project, because the question of how we can design and evaluate such measures is relevant internationally. This contribution is based on preliminary work (Hochmuth et al., 2018; Liebendörfer et al., 2017).

In the first part of this chapter, we describe how we reconstructed the goals of different support measures and integrated them in one model, which is then presented. In the second part, we illustrate the benefit of the model and the diversity of goals that recent support measures in Germany had via a description of three types of measures based on categories from our model. Finally, we discuss the use of this model and further steps for research.

5.2 Development of the Goal Categories in the WiGeMath Framework

5.2.1 The Underlying Concept of Theory-Driven Evaluation

The WiGeMath project followed the approach of theory-driven evaluation by Chen (1990, 2012) which is central to understand the development of goal categories in this chapter. A main concept of Chen's 'Theory-driven Evaluation' is the program theory, which is "a set of explicit or implicit assumptions by stakeholders about what action is required to solve a social, educational or health problem and why the problem will respond to this action" (Chen, 2012, p. 17). These assumptions may be both descriptive and prescriptive. Chen distinguishes six domains of program theories that need to be considered for a holistic and comprehensive evaluation of intervention approaches such as support measures in our case. The first three program theory domains are so-called normative theories, i.e. they describe the predefined or assumed structure of the measure and its context. The first domain describes the theory of outcomes, i.e. the goals of the measure. In the second domain, the procedural theory is developed, e.g. activities to be carried out, material to be used. The third domain, the theory about the implementation environment, describes the frame conditions under which the measure is to be realised (e.g. characteristics of the participants, competences of the implementers). The remaining three theory domains refer to the causal relationships between input and output of the program. In this chapter, we regard the first domain of goals only.

Goals describe the intended outcomes of a measure. When goals are stated explicitly, they may guide the activities in a measure and be used to assess a measure's effectiveness. We should acknowledge, however, that goals may sometimes be present only implicitly in stakeholders' beliefs and views. Chen (1990) therefore introduces the goal revelation evaluation to uncover such goals.

The initiators of redesigned lectures in the WiGeMath project, for example, had clear ideas about topics or procedures in the lectures, such as allowing more time for collaborative problem solving. These are part of the procedural theory. Partly, however, they had not documented related goals. It would be wrong to state that such measures had no goals. Instead, it was important to jointly reconstruct what was to be achieved with such processes. This might include strengthening students' self-efficacy beliefs or enculturation into university mathematics.

5.2.2 The Purpose of a Framework Model for Goal Categories

We started developing the framework model because we lacked a common basis for the theory-based study of mathematics support measures addressing the secondary-tertiary transition. In WiGeMath, goals of support measures addressing mathematics students, mathematics teacher education and mathematics in engineering students should be classified and evaluated across different universities. Such comparisons are not part of the original theory-driven evaluation approach. It was thus not only necessary to reveal and describe the goals of single measures but to create a consistent model that would serve to compare goals of different measures. We call this the framework model. The framework model should offer descriptive categories for different aspects of the support measures in a structured form.

The description categories should be both comprehensive and relevant across measures. Constructing the model called for reflections of the overarching relevance of categories that initially only appeared to be relevant for specific measures and for standardization via abstractions from the descriptions of the single measures.

5.2.3 Main Steps in Developing the Model

The development of the framework model required work on two levels. On the lower level, program theories of single measures were collected. According to Chen (1990), a variety of sources can be used like the evaluation of documents on the goals of a measure or interviews with those who initiated or implemented the measure. On the higher level, the framework model was established through a synthesis of the various program theories of single measures. In this process, goals from the program theories that were considered relevant for the transition problem were first selected and grouped. Categories were then derived through abstraction rendering individual theoretical elements of the program theories as manifestations of corresponding categories.

An initial descriptive grid for the support measures was developed based on theoretical descriptions of teaching and learning (e.g. Wildt, 2002; Winteler & Bartscherer, 2008). We did not only include variables that represent independent goals but also some that can serve to clarify effects (e.g. learning strategies). Parallel to this deductive procedure, we used an inductive procedure to create further goal categories by reconstructing program theories of selected measures of different types: We carried out a case-related, qualitative analysis of documents and analysed documents from 28 measures representing 13 institutions, namely our own two universities, ten other German universities and one Norwegian university. At this point, we did not only focus on pre-university bridging courses, MLSCs and

Table 5.1 Numbers of measures with either rich or sparse documents on their goals

	Pre-university bridging courses	Mathematics learning support centres	Redesigned lectures	Other measures	Total
Rich material	7	3	4	6	20
Sparse material	2	2	1	3	8
Total	9	5	5	9	28

redesigned lectures but also other measures offering special teaching, material or consulting to complement regular teaching. Details on the numbers of different measures are displayed in Table 5.1.

The documents greatly varied in their quality and quantity. Thus, we classified the material: rich material included documents that more or less explicitly named goals of a measure like brochures for teaching staff, proposals for external funding, academic publications including books on some measures or project reports. Sparse Material included documents that reflected goals without explicitly naming them like advertising material for students, teaching material including tasks or lecture notes, short instructions for teaching staff and informal emails.

All documents were reviewed and the contents were assigned to the goals, implementation and implementation conditions according to Chen's (1990) approach. We followed the typical steps of a document analysis: skimming the documents for relevant passages, reading them and interpreting them in an iterative process (Bowen, 2009). Relevant categories and their underlying document excerpts were then discussed in the project team. We reconstructed goals that were not mentioned explicitly but seemed plausible. If, for example, a document stated a measure should present challenging problems to the students, we did not take this as a goal (because it does not describe an outcome) but noted the learning of problem-solving skills as a possible goal.

In this process, the expected limitations of a document analysis became apparent. Documents are not always accessible, often incomplete, they cannot be interrogated in depth and they have been created in a specific social context for a certain purpose (Bowen, 2009; Prior, 2016). Most significant, however, was the limitation that the existing documents often only described the measures very roughly. Yet, the information was enough to set up possible goal categories and not a complete description of the single measures' goals in this step.

The two approaches were then merged. Often, the elements of the program theories from the inductive approach could be classified as expressions of existing or easily supplemented categories in the deductively obtained model.

We tested the framework model through a total of ten measure-specific guideline-based expert interviews (Helfferich, 2014) with responsible staff for various of the measures of the earlier inductive approach. The weaknesses of the document analysis were well compensated for by interview questions, especially follow-up questions and joint discussions between the interviewers and the experts. The evaluation of the interviews followed a theory-based qualitative content analysis (Mayring, 2015). Every goal mentioned by the interview partners needed to correspond to a category in the model. If necessary, the model was modified or extended so that all goals, procedures and framework conditions of the measures according to interviewees or researchers could be located in the framework model.

The program theories of the single measures were then summarised. Some goals in the documents, in particular those relating to psychological aspects like beliefs, had been described in diverse and sometimes vague terms including metaphors. We unified the categories and in doing so, we based the categories on concepts from mathematics education, psychology or similar disciplines where it seemed reasonable, especially on concepts that had already been used to formulate categories in the first, deductive step. As there is no standardized methodology for this step, we simply discussed theoretical relations based on their recent use in mathematics education and their fit to the goal categories in our team.

Both in the document analysis and interviews, our compiled descriptions focused on the procedural theory of the measures and these descriptions were often detailed. Goals (as well as conditions), on the other hand, were not always explicitly stated although they always seemed present. It was clear for bridging courses, for example, that prospective students should be prepared for their studies. Yet, there were sometimes no more precise goal statements and staff might answer questions for goals giving details on the procedure to be implemented. A closer look at the measures revealed a large variance, for example with regard to their repetitive treatment of school mathematics or their propaedeutic covering of university mathematics and also with regard to possible secondary goals such as promoting the formation of learning groups. Such goals could be derived more or less clearly from the procedural theory for all types of measures, although it was not always possible to clarify to what extent the goals here could be assigned to the measure or to individual teachers. In accordance with the evaluation approach by Chen (1990) they were nevertheless collected and taken as relevant for the framework model.

Next, the framework model was validated and further developed in an expert workshop with 21 WiGeMath partners representing all 13 institutions from step 1. Some partners represented more than one measure. After a presentation of the framework model in the current version, results for the specification of the framework model, considering the measure-specific characteristics, were presented in measure-type-specific working groups and discussed for further refinement and optimisation of the model categories. In addition, for each category of the framework model, the participants rated the perceived relevance of this category for describing their type of measure on a scale (1–4). Corresponding assessments by the project team were given in advance and were also available so that deviating assessments could be discussed. Subsequently and based on the expert workshop, some minor modifications were made to the framework model for advanced clarity.

To ensure that all relevant goals of a measure could be classified in the framework model in terms of its scope and the clarity of its categories, a second round of expert interviews was then conducted with representatives of two pre-university bridging courses, three MLSCs and two redesigned lectures that had participated in the prior steps to locate their measures in the framework model. The resulting changes were minimal. Overall, the framework model seemed to cover all goals that we or the partners considered as relevant. However, these last analyses also showed that not all categories are useful for every measure or can be collected appropriately. Finally, the revised framework model was presented to the WiGeMath partners in a second workshop and was approved by them.

To ensure the model's connectivity to mathematics education research, the part of the model referring to goals was presented at CERME 10 (Liebendörfer et al., 2017). The discussion there underlined the versatile purposes but also its limitations. In particular, the formulation of system-related goals (cf. the descriptions below) was emphasised as a profitable approach.

5.2.4 Presentation of the Goal Categories

The framework model consists of a hierarchically structured list of categories with short descriptions, in this case categories of different goals. The categories are formulated as part of the program theory from the perspective of the designers. The framework model is intended to classify the respective program theory of one or more measures. Because of the multitude of aspects, we chose to structure the framework model in several levels. The goals are subdivided into educational goals, which map individual students' changes, system-related goals, which refer to the functioning of the higher education system, and goal qualities, which classify these goals on a meta-level (e.g., if they are SMART; cf. Lawlor & Hornyak, 2012). In this chapter we focus on the categories of educational goals and system-related goals.

In line with the development of the model and its objectives, now follows a brief description of each category in terms of what it encompasses, why or for what purpose it was relevant in the WiGeMath project, and to what extent it is linked to existing literature on university mathematics education. Category names are italicized. A compact presentation of the framework model with all goal categories and short descriptions is given in the appendix of this chapter.

5.2.4.1 Educational Goals

Educational goals refer to targeted changes in knowledge, actions and attitudes of the measure participants. They should be the starting point for the support measures' didactic design of the teaching/learning environment and the learning process. All educational goals are linked to the difficulties in the secondary-tertiary transition in mathematics. Naturally, all measures aimed for educational goals.

Knowledge goals refer to both the declarative ("knowledge, that") and procedural knowledge ("knowledge, how"; Renkl, 2015, p. 4) that is developed through the measure. The category of *improvement of school mathematics knowledge and abilities* includes all content and techniques that are or were taught in school mathematics lessons. Students may not actually have covered this content in their lessons at secondary level, though. For example, this category also includes topics from the area of trigonometry which used to be in the curriculum but have since been dropped in many places. The classification as school mathematical knowledge suggests that such additions are treated as school mathematics, e.g. through predominantly descriptive concept formation. The need for a demarcation from university mathematics knowledge was made clear, for example, by various bridging courses that clearly had different emphases in this respect. School mathematics knowledge is deemed necessary for mathematical studies by many German university teachers, including aspects like trigonometric functions that are hardly covered in recent syllabi (Neumann et al., 2017). In recent studies, prior school knowledge has proven to be by far the most significant predictor of academic success (Halverscheid & Pustelnik, 2013; Rach & Ufer, 2020). *Improvement of higher mathematics knowledge and abilities*, on the other hand, concerns the content taught in regular mathematics courses of the degree program. This content is diverse and specific to the target group, so that a standardised differentiation did not seem useful. However, special attention is paid to *promote learning of the language of mathematics*, which includes symbols (e.g. the sum sign), abbreviations (e.g. like OBdA, corresponding to the English abbreviation "w.l.o.g." for "without loss of generality") and basic technical terms (e.g. "injective"). The language of mathematics is not linked to specific subject areas or courses but refers to overarching elements. Such university mathematics knowledge is taught in some bridging courses and redesigned lectures. Dealing with the language of mathematics regularly poses difficulties for students (Corriveau & Bednarz, 2017) and is addressed in separate literature on study support (Beutelspacher, 2004; Houston, 2010; Vivaldi, 2014).

Action-oriented learning goals refer to skills of mathematical working and learning as well as the concrete design of learning processes. Such goals were pursued by all types of measures, albeit with a different focus. *Enhancing mathematical modes of operation* concerns an enhancement of activities for working out mathematical content and solving mathematical problems. Mathematical modes of operation include problem-solving skills such as the use of heuristics. These have long been described as important for mathematics (Polya, 1945; Schoenfeld, 1985), especially in proving mathematically at university (Weber, 2005). Modes of operation also include (local) defining, working out examples and counterexamples, making conjectures and proving, as well as approaches to exercises. Such modes of operation are described in advice literature (Alcock, 2013a, b; Houston, 2009; Mason et al., 2010) and became a central content in some redesigned lectures. In contrast, *enhancing university modes of operation* concerns subject-unspecific aspects such as time management, self-organisation, self-regulation or taking and organising notes, which can be optimised especially at the beginning of university studies (Dehling et al., 2014). *Promoting learning strategies* includes the promotion of activities that serve to build mathematical knowledge like summarising important content, planning, monitoring and evaluating learning, or practising and memorising. Such learning strategies can also explain performance in mathematics-related studies (Griese, 2017; Liebendörfer et al., 2020). While the learning goals mentioned so far refer to a development of skills or routines, the category *support of learning and working conduct* concerns changes in the actual exercise. This category concerns the learning rhythm (when learning takes place), learning effort (how much is learned), learning material (what is learned with), learning environment (where and with whom learning takes place) and use of offers. These objectives were mainly related to MLSCs.

Attitudinal goals refer to a change in attitude towards mathematics. A positive attitude towards mathematics of some kind was pursued by all types of measures. Here, attitude is defined more broadly than is usually the case in psychology. For example, *change in beliefs*, i.e. mathematical world views, is included (Goldin et al., 2009; Grigutsch & Törner, 1998; Törner & Pehkonen, 1996). Typical views describe mathematics as a collection of procedures (toolbox beliefs) or (also) a game for exploring and (re)inventing structures (process beliefs). Such worldviews can influence how students perceive mathematics and how they work mathematically. They are closely related to motivational development and drop-out (Geisler & Rolka, 2020; Liebendörfer & Schukajlow, 2017). *Change in affective features* describes a change in emotional attitudes towards mathematics. This includes interest in mathematics, which has been addressed by recent research particularly in Germany (Kosiol et al., 2019; Liebendörfer, 2018; Rach, 2014; Ufer et al., 2016). Especially in the case of motivational variables, recourse to theoretically elaborated concepts is helpful because they can have different roles in the learning process (Marsh et al., 2019). Referring to more extrinsic motivation, both a *perception of relevance for the future job* and a *perception of relevance for future studies* are relevant goal aspects of this attitudinal category. The latter may be given when mathematical content is seen as the basis for further courses. Both aspects can be discerned and found important (Hernandez-Martinez & Vos, 2018). *Mathematical enculturation* describes the introduction into a community in the sense of socio-cultural theories (e.g. Wenger, 1998). This refers, among other things, to the willing participation in “authentic” activities of the new, university mathematical culture. Enculturation involves the adoption of values, goals, ways of doing things and an adjustment of one’s identity. It is thus closely related to beliefs, problem solving and proof (Perrenet & Taconis, 2009), but also aspects of one’s identity (Kaspersen et al., 2017). Enculturation was addressed in some but not all pre-university bridging courses and even stronger in redesigned lectures.

5.2.4.2 System-Related Goals

System-related goals are not oriented towards individual students, but towards the functioning of the university system. This type of objective seemed necessary because the approach of various measures showed that a measure’s success cannot necessarily be assessed by looking at individual students and their development only. Whereas with regard to individuals, for example, any promotion of knowledge is helpful and any avoidance of drop-out makes sense, at the system level the special promotion of disadvantaged groups or the achievement of a seemingly acceptable level of drop-out can be targeted. The term "system-related" shall express that these goals relate to the functioning of the system as a whole and not to individuals. System-related goals are therefore formulated at the level of groups and from the perspective of institutions.

The goal category "*creation of prerequisites for knowledge/abilities*" refers to the fact that future courses can assume certain knowledge or skills. This goal is mainly pursued by bridging courses. It involves giving students the opportunity to learn this content and ensuring that a certain proportion of students has actually achieved this goal so that this content can be assumed as shared knowledge. This concerns the *improvement of school knowledge and abilities as a prerequisite for university studies*, such as calculating with fractions, sine, cosine and solving systems of equations, especially if it is known that larger proportions of the student body do not have such knowledge. To a lesser extent, it also concerns the *creation of requirements for lectures that exceed school knowledge* in redesigned lectures, for example logical and set-theoretical basics or the teaching of proof techniques like mathematical induction. These topics were mostly compulsory or optional content in school curricula at some time. Both categories show that measures may be designed to help to maintain the previous structure and content of study programs when cohorts with changed abilities begin their studies. They thus refer to an institutional disruption in the secondary-tertiary transition. Universities need to adjust to changes in schools and sometimes seem to try to maintain the traditional content of lectures but add measures addressing the emerging gaps.

The category "*improvement of formal study success*" refers to objectively measurable study success criteria such as a *reduction of the dropout rate*, i.e. the proportion of students originally enrolled at the beginning of the semester who have dropped out after the semester. It also includes an *increasement of passing rates/achievements*, i.e. the proportion of students who have passed a certain module and the distribution of students' grades.

In addition, the system-related goals include categories that address the framework conditions for students' self-directed learning. The goal category "*improvement of feedback quality*" means that students should receive qualitatively better feedback (Hattie & Timperley, 2007), which is a goal of MLSCs in particular. In addition, measures can aim at a "*promotion of social contacts and connections relevant for studies*", e.g., the formation of learning groups (MacBean et al., 2004). This holds especially for bridging courses. The student groups mentioned in the category "*supporting certain student groups*" could for example be women, who are often underrepresented in mathematics-related degree programs (Gildehaus & Liebendörfer, 2021), but also students who finance their studies with part-time jobs, have children or have limited language skills. *Making university study demands transparent* means that measures provide an insight into university requirements, especially regarding the subject-related prerequisites and requirements in the further course of studies. This is an objective of some bridging courses. In addition, some measures' goal can be the *improvement of teaching quality*, which refers primarily to evaluations by students. These categories mirror also substantially different demands for students' self-regulated learning in school and university. Students' need to rapidly adjust to the new demands during the transition (e.g., Göller & Rück, this volume).

The system-related goals are also found in a similar way as learning goals and as such often appear in the literature. The classification reflects that almost all measures were conceived as a supplement and support to ensure that one can continue to use the already existing teaching system for the further course of studies.

5.3 Using the Goal Categories of the Framework to Compare Measures

To illustrate how we have used the framework and how different current measures can be described, we compare measures from the WiGeMath-Transfer project below.

5.3.1 Background and Methods

Given the restricted space, we limit ourselves to measures that are aimed at mathematics majors and preservice teachers. Measures for engineering mathematics would often have different emphases. In explicit, we focus on

- bridging courses that were implemented as in-class courses (as opposed to online courses) and aimed at future mathematics students, preservice teachers for mathematics and in some cases additionally future computer science students,
- redesigned lectures that focus more on the way mathematics is done at university than on the mathematical content itself (cf. Grieser, 2018, for a concept of such a "redesign") and address preservice teachers but may also be attended by mathematics students
- and MLSCs.

The ratings on which the following presentation is based were given by those responsible for or carrying out the measures. For some measures, this was a single person, for some there were teams of up to four people. The model was first explained to them in workshops which were carried out in the WiGeMath-Transfer project.[1] The responsible persons indicated on a four-point scale whether the

[1] The WiGeMath-Transfer project aimed at a transfer of the results of the WiGeMath project (including the framework model and its application) to a wider community.

objectives for their measure were a main goal (4), important goal (3), subordinate goal (2) or no goal (1). If there was more than one person responsible for a measure, all responsible persons discussed which rating they would give and decided on one together. Some categories were not considered useful for certain types of measures and were therefore excluded.

5.3.2 Pre-University Bridging Courses

Our sample includes four bridging courses B1, B2, B3, and B4 held at three different German universities (B3 and B4 being held at the same university but at different points in time by different lecturers) in the weeks before the start of winter term. The courses lasted 7–10 days. B3 and B4 were mandatory for future mathematics students and preservice teachers at this particular university and also recommended to computer science students. B1 and B2 were optional for the students. Each course had between 100 and 180 participants. The courses included lectures in which mathematical content was presented and tutorial group meetings in which the future students worked in groups or individually on problems applying lecture content and methods. Course B1 was organized and held completely by students. In B2, B3 and B4, lectures were given by university staff and student tutors supervised the tutorial group meetings. All four courses focused on basic topics in higher mathematics such as naïve set theory, functions, logic, methods of proof and elementary number theory. While the content and the overall structure is similar, differences and similarities in the lecturers' goals can be observed. We will first focus on *educational goals* (cf. Fig. 5.1).

Regarding *knowledge goals*, only B1 aimed at an *improvement of school mathematics knowledge and abilities* and only as a subordinate, less important goal. This is not typical for mathematical bridging courses in general since many mathematical bridging courses – especially those aimed at engineering students – cover school mathematics as a main topic. Instead, B1 and B2 regarded *improvement of higher mathematics knowledge and abilities* as a main goal. B3 and B4 regarded this only as a subordinate goal. This might seem surprising given the similar content in all of the courses. But it can be explained by different perspectives of the lecturers: In the courses B3 and B4, the content was only used as a means for illustrating mathematical modes of operation. To *promote learning of the language of mathematics* was important for B1 and a main goal for the three other courses, making this the most important knowledge goal of the bridging courses in this study.

Regarding *action-oriented goals*, all courses rated *enhancing mathematical modes of operation* as a main goal. The views on the other goals differed, with *support of learning and working conduct* being the least important goal.

None of the *attitudinal goals* were rated as a main goal in any of the bridging courses, showing that the focus is on knowledge and action-oriented goals. However, some of the attitudinal goals were seen as important in some courses but there was no consensus as to which. The least important attitudinal goals seemed to be

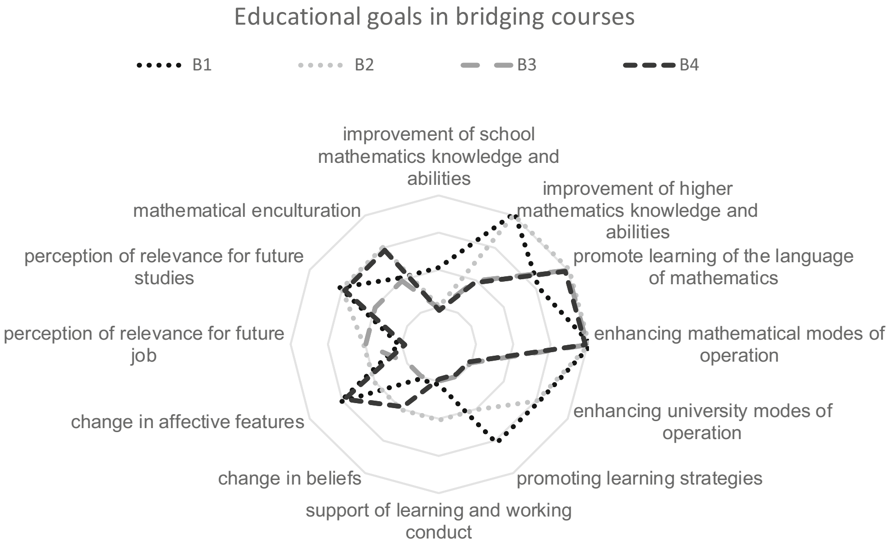

Fig. 5.1 Educational goals in bridging courses B1-B4

change in beliefs and *perception of relevance for future jobs*. *Perception of relevance for future studies* was a subordinate goal in B3 and an important goal in B1, B2 and B4.

System-related goals (cf. Fig. 5.2) were rated variably. The *promotion of social contacts and connections relevant for studies, making university study demands transparent* and the *improvement of teaching quality* were the most important system-related goals in these four bridging courses. None of the courses aimed at a *creation of requirements for following lectures that exceed school knowledge* or at *supporting certain study groups*. Only B1 aimed at the *improvement of school knowledge and abilities*, but only as a minor goal. This is consistent with the rating of the educational goal *school mathematics knowledge and abilities*. The improvement of study success was regarded differently in the different courses (ranging from no goal to important goal), with *reduction of the dropout rates* being more important than *increasement of passing rates/ achievements* which was not a goal in three of the courses.

5.3.3 Redesigned Lectures

We will look at three redesigned lectures R1, R2 and R3 held at different German universities. R1 was designed for preservice teachers only, R2 was compulsory for preservice teachers and mathematics students alike and R3 was compulsory for preservice teachers and elective for mathematics students. Between 100 and 200 students attended each lecture per semester. All three lectures were given by different

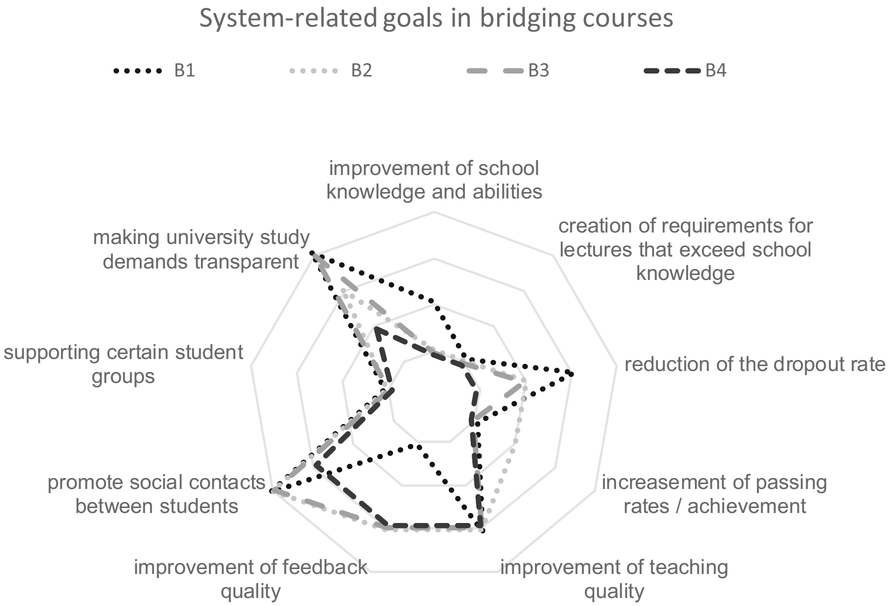

Fig. 5.2 System-related goals in bridging courses B1-B4

lecturers in different semesters and the lecturers had some freedom in selecting the learning content. The learning culture in all three redesigned lectures was characterized by a lot of activity from the students' part. Students were encouraged to work on problems themselves and try out new methods during lecture time (Kuklinski et al., 2019).

There were some differences in how the three redesigned lectures set their focus concerning educational goals and system-related goals. We will first consider the *educational goals*, starting with *knowledge goals* (cf. Fig. 5.3).

None of the three redesigned lectures aimed at an *improvement of knowledge of school mathematics and abilities* in their students. Meanwhile, they all focussed on an *improvement of higher mathematics knowledge and abilities* and R3 set an even stronger focus here than the other two lectures. To *promote learning of the language of mathematics* was less important for R3 while it was just as important for R1 and R2. Altogether, R1 and R2 seemed to set a very similar focus concerning the knowledge goals but R3 differed somewhat.

Concerning the categories of *action-oriented goals*, all three redesigned lectures had the goal of *enhancing mathematical modes of operation* and to a somewhat smaller degree also *promoting learning strategies*, whereas an *enhancement of university modes of operation* formed a subordinate goal only. Especially the strong focus on mathematical modes of operation was very characteristic for redesigned lectures for preservice teachers. None of the redesigned lectures aimed at a *support of the learning and working conduct* of the students.

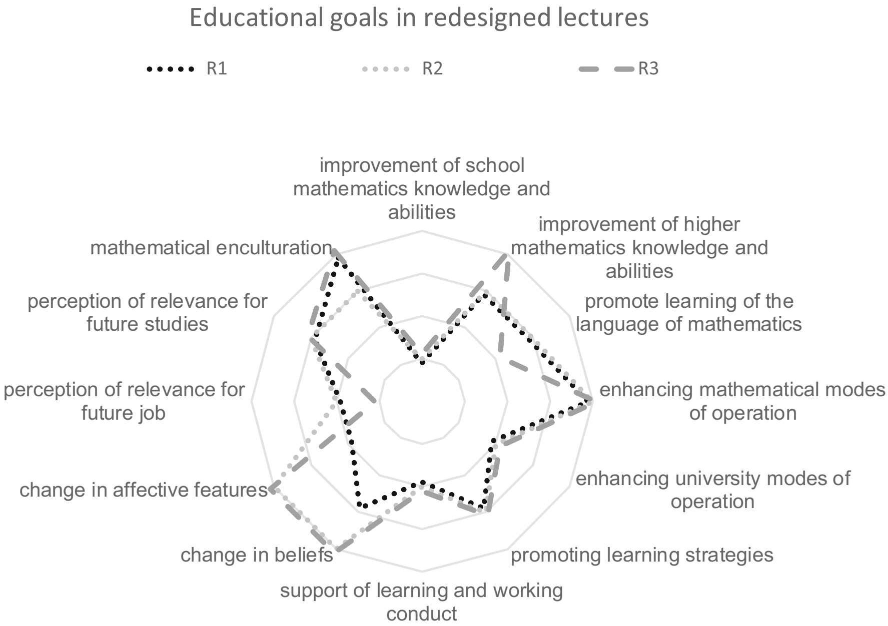

Fig. 5.3 Educational goals in redesigned lectures R1-R3

There was some more variation between the three redesigned lectures concerning the *attitudinal goals*. Causing a *change in beliefs* in students was a main goal in R2 and R3 and a little less important in R1. The gap widens for *change in affective features* which was also in the focus of R2 and R3 but not very important in R1. That students gain a *perception of relevance for their future job* was no goal in R3 and a subordinate goal in R1 and R2 but in all three redesigned lectures it was more important that students gain a *perception of relevance for their future studies*. A *mathematical enculturation* was a main goal in R1 and R3 but less important in R2. We see that the goal that students make learning progress in using the language of mathematics was rated higher for R1 than for R3 and the goals that students gain more knowledge in higher mathematics and develop positive affective features were rated lower for R1 than for R3. Thus, we conclude that the different redesigned lectures pursued different paths in reaching their focused target of a mathematical enculturation of the students.

All in all, the educational goals of redesigned lectures showed similar profiles but they set their focus a little differently, especially concerning the category of attitudinal goals. We will now look at the *system-related goals* (cf. Fig. 5.4).

While none of the three lectures aimed for an *improvement of school knowledge and abilities as a prerequisite for university studies*, the *creation of requirements for lectures that exceed school knowledge* were rated much more important for R1 and R2. R3 did not aim at this goal had the *reduction of the dropout rate* and the *increasement of the passing rates* as main goals. Even though with less emphasis

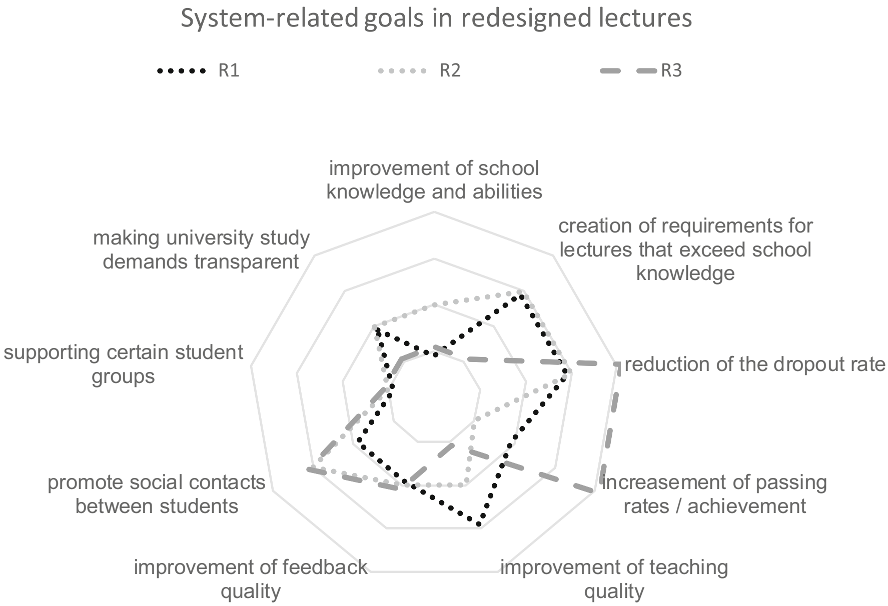

Fig. 5.4 System-related goals in redesigned lectures R1-R3

than R3, R1 and R2 also aimed at reducing dropout rates but improving passing rates was not on their agenda. An *improvement of teaching quality* was more important to R1 than it was to R2 or R3 whereas *improvement of feedback quality* was a subordinate goal for all three redesigned lectures. Even less focus did they set on *supporting certain student groups*. A *promotion of social contacts and connections relevant for the studies* and *making university study demands transparent* were rather not pursued by R1 while R2 and R3 did find it important to promote social contacts.

5.3.4 Mathematics Learning Support Centres

The six MLSCs are characterised by a special place in the university that students can attend to work on mathematical topics and receive support. The respective rooms or spaces in the universities varied greatly in capacity. MLSCs that used former seminar or teaching rooms had capacities of between 35 and 60 students. In some cases, several rooms or adjoining areas were used in these MLSCs and the rooms were exclusively used by them. MLSCs with larger capacities used places in the university only for certain times of the day and that were used alternatively at other times. Five MLSCs (1–5) had a focus on (pure) mathematics support. An additional one (MLSC6) focused on mathematical support for teacher students with an

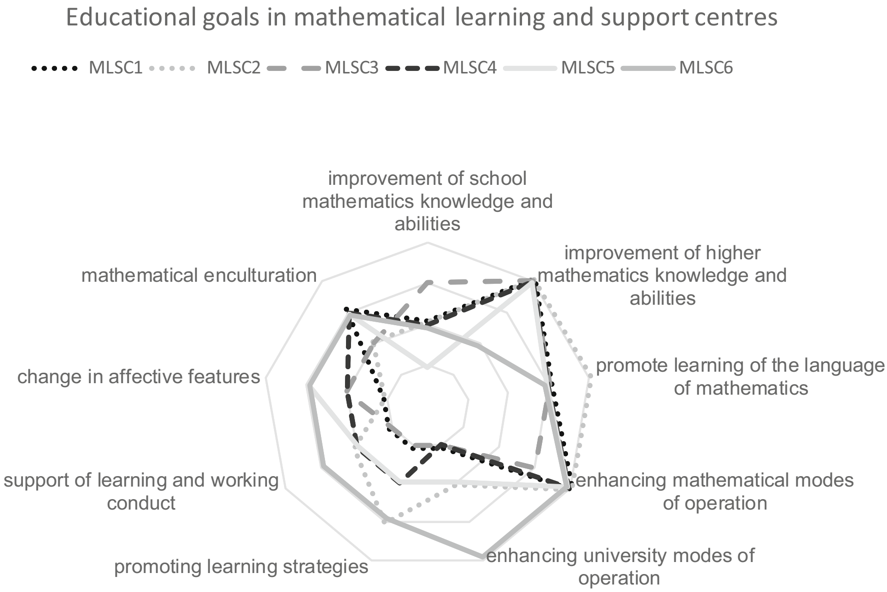

Fig. 5.5 Educational goals in mathematics learning and support centres MLSC1-MLSC6

additional focus on didactical considerations and background theories. It helped the secondary-tertiary transition by supporting reflections on the differences.

Support or additional advice in the MLSCs was mainly provided by tutors, who were sometimes also involved in tutorials for courses. They were mainly qualified through site-specific training by their MLSC. However, the qualification measures for the tutors varied in scope from several hours to several days of training. If support was given by staff members, instructions for staff was provided (e.g. by learning centre managers or persons in charge) about how the counselling should generally be conducted, but no specific training was given to these staff members.

With regard to the assessed *educational goals* (cf. Fig. 5.5), all respondents stated the *improvement of knowledge and abilities in higher mathematics* as the main goal, except MLSC6, which had its focus on mathematics didactics and therefore did not have a priority in the support of students if they had problems in higher mathematics. *Enhancing mathematical modes of operation* and *promoting learning of the language of mathematics* were chosen as other main and important goals by all six MLSCs. Further important goals for at least two of the six MLSCs were the promotion of *mathematical enculturation* (MLSC1, 4, 5 & 6), *changes in affective features* of students (MLSC5 & 6) and *promoting learnings strategies* (MLSC2 & 6). Some of the other goals were rated as subordinate or as no goal by most MLSCs. Overall, the assessments reflected individual priorities of the learning centres, which were also due to the different focus on the students' degree programs the MLSCs gave support to.

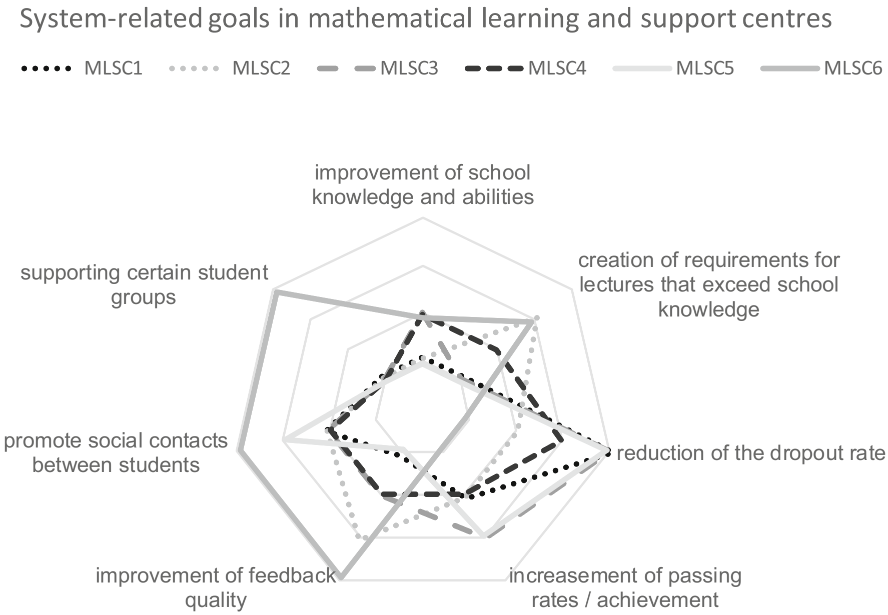

Fig. 5.6 System-related goals in mathematics learning and support centres MLSC1-MLSC6

The *system-related goals* (cf. Fig. 5.6) of the participating MLSCs varied slightly and also indicated the individual priorities of the centres. For example, the *reduction of dropout rates* was rated as a main goal (MLSC1, 3 & 5), an important goal (MLSC4), a subordinate goal (MLSC2) or as no goal (MLSC6). The *improvement of feedback quality* was no goal for MLSC1 and MLSC5, a subordinate goal for others (MLSC3 & 4), an important goal for yet another MLSC (MLSC2) and a main goal for MLSC6. Most respondents rated the *support of certain student groups* and the *improvement of school knowledge and abilities* as no goal or only a subordinate goal. In sum, MLSCs 1 to 5 gave more similar ratings in terms of the presented goals compared to MLSC6, which again reflected the emphasis of the latter on mathematics didactics support in the context of mathematics teacher education study programs.

5.3.5 Comparing Different Types of Measures

Across all measures (aimed at mathematics students and preservice teachers), the focus was on teaching both higher mathematics and mathematical modes of operation. These goals form the central aspects of mathematical competence and seem therefore important for each subject-specific support measure.

The diagrams also show different emphases between the types of measures. Bridging courses particularly emphasized social contacts and making study requirements transparent. Both are considered prerequisites for a good start to university studies. Redesigned lectures had a focus on enculturation, beliefs and affective variables. They provide an answer to known problems of disaffected and distanced students that are not targeted by the other types of measures. Redesigned lectures may change the relationship with mathematics in a profound way that relatively short bridging courses or voluntary visits to an MLSC cannot. MLSCs also focussed on goals that were less closely related to mathematics, such as university modes of operation, learning strategies, and learning and working conduct. Possibly, these problems are best tackled individually. In addition, they set a focus on reducing drop-out and on improving feedback quality. This is another strength of MSLCs, where individual cases can be discussed and reflected. At least one MLSC, in contrast to the other measures, also focused on specific student groups.

5.4 Discussion

We presented a model of goals that mathematics support measures for the secondary-tertiary transition may have. It has an empirical basis given by diverse analyses, yet it may be incomplete and other research could derive other goals. We further compared three different types of measures, showing that the types differ in their goals and that there is also variation within the types.

5.4.1 The Framework Model

Despite its limited empirical basis, the model shows a variety of goals that are hardly discussed in the literature as a whole system of goals. Learning goals focusing on mathematical content have been discussed for a while (e.g., Speer et al., 2015). In discussing prerequisites for studying mathematics, mathematical knowledge has been supplemented by affective variables, beliefs and abilities to employ mathematical processes (Deeken et al., 2020). Such abilities like problem solving have also been highlighted as learning goals in the first semester (Alpers, 2014; Rensaa et al., 2020). Our new contributions are the focus on goals pursued by recent support measures, systematization of these goals and comparisons of different measures concerning their goals.

This chapter illustrates that the secondary-tertiary transition in mathematics is a multi-faceted affair and support can look diverse. A new approach lies in the category of “system-related goals”. They demonstrate that not every progress by individual students is equal to an improvement of the functioning of universities.

This point is particularly important in the secondary-tertiary transition in mathematics: The gap between school and university mathematics and teaching is not only mirrored in students' problems to succeed in their first semester but also in institutions running smoothly. In Germany, drop-out is particularly high in the first semesters (Geisler & Rolka, 2020; Heublein, 2014) which may have negative consequences for the institutions. The system-related goals reveal that universities try to help students to succeed with their studies without changing classical lectures too much. To some extent, the support measures seem to address gaps that arise when the school system changes. However, it might be necessary to adjust classical lectures as well. This calls for a more intense dialogue between school teachers and university teachers (Corriveau, this volume).

Measures of the same type often had similar goals. However, this result cannot yet be generalized to the particular type of measure. For this, a more representative sample of further measures would have to be systematically described. By focusing on mathematics majors and preservice teachers, we further restricted the scope of results. If, for example, measures for engineering mathematics were also included, which was realized in the WiGeMath project, a strong focus on the repetition of school mathematics would become visible. In addition, it must be considered that measures can also have an effect on unintended goals. Bridging courses, for example, may improve passing rates but this was not a goal for many respondents. Possibly this goal is too distal.

5.4.2 Using the Framework Model to Evaluate Measures

In the WiGeMath project, we used the framework model to derive quantitative instruments to evaluate whether targeted changes took place (for a documentation of the instruments, cf. Hochmuth et al., 2022). The achievement of the most important goals in the bridging courses was measured by self-assessment of the students (cf. Lankeit & Biehler, 2022). Students reported that they met fellow students and felt well integrated socially, showing that the promotion of social contacts worked well. Additionally, they reported that they had learned new mathematical content, gotten good insight into university teaching and university mathematics. In terms of dealing with mathematical texts, the students felt they had learnt that at least to some extent. Instruments for testing whether students reached these goals objectively and not only based on self-reports (especially regarding mathematical modes of operation and the language of mathematics) are yet to be designed.

Concerning redesigned lectures, results of the WiGeMath evaluations indicated that these lectures seem to be successful in supporting students to develop and maintain rather positive affective features and that students find these lectures very helpful (cf. Kuklinski et al., 2018, 2019; Liebendörfer et al., 2018).

Evaluations of the MLSCs showed overall positive results of the centers' quality, especially concerning the respective framework conditions and the individual support services in particular. MLSCs thus seem to be a support measure which is used with high satisfaction and which is positively evaluated due to the low-threshold support and easy access. Comparative analyses between users and non-users showed that MLSCs promote students who have a higher need for support which indicates that this measure is successful in supporting certain student groups (Hochmuth et al., 2018).

5.4.3 Further Use of the Framework Model

Notwithstanding its main purpose of evaluation, the framework model and especially its application in the area of goals can also provide a good basis for an advanced or new conception of support measures. On the basis of clearly defined goals, measures can be better designed and their components can be described in advance in the sense of impact hypotheses. Thus, results to be achieved can be defined and verified in accordance with the theory by Chen (1990, 2012).

Finally, with the help of the model, university teachers can often better reflect which goals they have and which of these are or should be essential for their work. This became evident in the first phase of the WiGeMath project. Knowing the categories, some descriptions of measures in the interviews became even richer or ideas for changes emerged. In addition, the explicit listing of goals can help with trade-offs, such as whether to focus on certain student groups (e.g., low- or high-achieving students) more strongly at the expense of other groups. Therefore, the framework model can also be helpful for the planning of one's own measure, both for the identification of essential aspects of one's own conception and, for example, in the search for measures already described that can serve as a model for certain aspects.

Appendix

Educational Goals

1.1 Knowledge Goals *Knowledge-related goals refer to both the declarative and procedural knowledge that is imparted by the measure. It is a matter of determining what kind of knowledge is to be imparted by the measure.*	1.1.1 Improvement of school mathematics knowledge and abilities	*School mathematics knowledge and abilities include all content and techniques taught in school mathematics classes. Are these repeated, rounded off or completed in the measure?*
	1.1.2 Improvement of higher mathematics knowledge and abilities	*Higher mathematics knowledge and abilities comprise the content taught in the regular courses in mathematics.*
	1.1.3 Promote learning of the language of mathematics	*By language of mathematics we understand symbols, abbreviations and technical expressions of mathematics.*
1.2 Action-oriented goals *Action-oriented goals refer to skills of mathematical working and learning as well as the concrete design of learning and working processes.*	1.2.1 Enhancing mathematical modes of operation	*Mathematical modes of operation describe activities for the development of mathematical content and the solution of mathematical problems.*
	1.2.2 Enhancing university modes of operation	*University modes of operation include, for example, time management, self-organization, self-regulation, or note-taking and organization.*
	1.2.3 Promoting learning strategies	*Learning strategies include activities that are essential for building mathematical knowledge, such as summarizing important content, planning, monitoring, and evaluating learning, and using visualization.*
	1.2.4 Support of learning and working conduct	*Learning and working conduct refers to the learning rhythm (when is learned), learning effort (how much is learned), learning material (with what is learned), learning environment (where and with whom is learned) and use of offers.*
1.3 Attitudinal Goals *Attitudinal goals refer to a change in attitude toward math. Here, attitude is defined more broadly than the usual psycholo-gical definition.*	1.3.1 Change in beliefs	*We only study beliefs about the nature of mathematics, also called mathematical worldviews.*
	1.3.2 Change in affective features	*Affective features describe emotional attitudes toward mathematics (e.g., math anxiety, self-efficacy expectations, or interest).*
	1.3.3 Perception of relevance for the future job	*The measure provides an insight into the relevance of mathematics for the profession to be practiced later.*
	1.3.4 Perception of relevance for future studies	*In the measure, it is clear that the content and strategies taught are relevant to future studies.*
	1.3.5 Mathematical enculturation	*This means the voluntary participation in "authentic" activities of the new, university mathematics culture.*

System-related Goals

2.1 Creation of prerequisites for knowledge/ abilities *This refers to enabling students to follow the advanced courses after the introductory phase of study.*	2.1.1 Improvement of school knowledge and abilities as a prerequisite for university studies *We mean areas and methods that are not taught in the beginners' lectures at the university, but should have been taught at school.*
	2.1.2 Creation of requirements for lectures that exceed school knowledge *For example, systematic outsourcing of topics such as groups, rings, or fields that would otherwise be addressed in follow-up or parallel events.*
2.2 Improvement of formal study success *This category refers to the objectively measurable following study success criteria.*	2.2.1 Reduction of the dropout rate *Rate of how many of the students originally enrolled at the beginning of the semester dropped out after the semester.*
	2.2.2 Increasement of passing rates/ achievements *Rate of how many of the students registered for a module have reached the passing mark; also the distribution of the students' grades.*
2.3 Improvement of feedback quality *This category refers to the fact that students should receive feedback of a better quality.*	
2.4 Promotion of social contacts and connections relevant for studies *Promote social exchange, conversation, professional assistance, encourage learning groups, etc.*	
2.5 Supporting certain student groups *Special support is given to certain groups of students.*	
2.6 Making university study demands transparent *The measure provides an insight into the university requirements, especially anticipating the professional prerequisites and requirements in higher semesters / in the further course of studies.*	
2.7 Improvement of teaching quality *The aim of the institution is that the quality of events is improved and that they are better evaluated by students.*	

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Part II
Research on University Students' Mathematical Practices

Chapter 6
"It Is Easy to See": Tacit Expectations in Teaching the Implicit Function Theorem

Matija Bašić and Željka Milin Šipuš

Abstract We investigate the teaching and learning of curves and surfaces within multivariable calculus courses in university mathematics education, focusing on certain tacit expectations of teachers. More specifically, our aim is to describe the position of students and teachers with respect to knowledge based on the Implicit Function Theorem with applications to spatial geometry and nonlinear problems. The student perspective was analysed based on their interaction with the designed tasks, in which they were asked to reflect on their acquired knowledge. The teacher perspective was analysed based on interviews, with the focus on the knowledge taught and their actual expectations of students. Our findings shed light on the particular type of a didactic contract, in which we highlight the tacit expectations concerning the use of non-routine algebraic manipulations and geometric interpretation of equations, and the students' perception of their responsibility for learning, stating that "*they did not study enough*".

Keywords Multivariable calculus · Implicit function theorem · Didactic contract · Tacit expectations

6.1 Introduction

University mathematics education is characterised by a fast pace and dense exposition of knowledge, along with a high level of student responsibility for learning the required content. In addition, mathematics knowledge is complex, abstract, and advanced, and it is mainly compartmentalized and prone to the influence of many discontinuities and breaks in teaching (Artigue, 1999). In their cumulative nature, many concepts evolve from the basic concepts defined at the beginning of university education to advanced concepts for specialised purposes, while in this process, their value or purpose, their *raison d'être* (Bosch & Gascón, 2014) often

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remains hidden. Therefore, depending on the professional choices of a particular student, many of the concepts the student faces may seem to be the *ends of education* (Gascón & Nicolás, 2019).

Here we investigate an interconnected network of discontinuities that students in university mathematics education are confronted within core mathematical subjects, and which may be seen as part of the interplay between the two main mathematics domains, algebra, and geometry. Formulated in the domain of algebra, we discuss problems that involve systems of nonlinear equations in terms of the existence of a solution and possible methods for solving and describing the solution set, including its geometrical interpretation. The topic is primarily relevant in multivariable calculus in the context of the Implicit Function Theorem (IFT), finding constrained extrema by Lagrange multipliers, or when graphing and determining boundaries of double and triple integrals, or curve and surface integrals. It also appears in elementary mathematics, linear algebra, differential geometry, as well as in applications in physics, and can be perceived as the further development of techniques learned within linear algebra, however, with a theory behind it being undefined or unanswered.

Hochmuth (2020) considers certain similar aspects of concepts in nonlinear analysis as *bridging and extending concepts* within and across the domain of mathematical analysis. These concepts are to be followed in the transition from basic lectures in analysis to advanced specialised courses in nonlinear approximation. He suggests that analysis of mathematical and didactic practices of these concepts might help overcome compartmentalisation of mathematical knowledge at university.

In this paper, we question the position of this mathematical knowledge on the Implicit Function Theorem, especially whether there are tacit teacher expectations of student work. We aim to reconstruct students' mathematical practices required for multivariable calculus courses by inspecting their work and by generally taking an institutional perspective.

This study is a continuation of our previous research on student practices concerning the interplay of geometry and algebra in multivariable calculus and the subsequent course (differential geometry). Both authors of this study are mathematicians and researchers in mathematics education. The first author was involved in teaching courses as a teaching assistant, and his research in mathematics focuses on algebraic topology. The research interests of the second author focus on differential geometry. In mathematics education, both authors study the teaching and learning of mathematical concepts in multivariable calculus, especially concerning the interplay of geometry and algebra. This study is motivated by earlier observations while teaching these courses where student difficulties in working with equations (mostly nonlinear) for geometric objects (curves and surfaces) arose as an issue. In the first study (Bašić & Milin Šipuš, 2019), we observed persistent difficulties emerging already in linear algebra; for instance, with false generalisations from 2D to 3D. In 2D, a straight line or a curve, in general, is represented by a single implicit equation, whereas in 3D, a single implicit equation represents a surface, even if it is an equation without the z-variable (e.g., $x^2 + y^2 = 1$ in 3D represents a circular cylinder). We further observed difficulties in changing the form of representation between parametric and implicit forms, which is often assumed to be a prerequisite in multivariable calculus. However, the study also pointed to the influence of the didactic contract, which may prevent students from reasoning "geometrically". In our second study (Bašić & Milin Šipuš, 2021), we investigated student work on two tasks designed to appraise their learning of the notion of a curve in 3D space. The tasks were designed to support the students' independent work, i.e., as having (theoretically) adidactic, linking, and deepening potentials (Gravesen et al., 2017). The realisation of these potentials was observed during a tutorial given by the authors with six students who volunteered to solve designed tasks, in which we observed specific values of the theoretical potentials, but also student difficulties in switching from 2D to 3D.

6.2 Theoretical Framework and Research Questions

This research is carried out from the perspective of the Anthropological Theory of the Didactic (ATD) (Bosch & Gascón, 2006, 2014; Chevallard, 1985) and the Theory of Didactic Situations (TDS) (Brousseau, 1997). The framework of ATD highlights the institutional dimension of mathematical and didactic activities and thus analyses the process of didactic transposition of knowledge in relation to an institution. It postulates that knowledge is taken from the scholarly knowledge built by mathematicians or other scientists at the university level, for instance, and is transposed and identified as knowledge to be taught. In the next step, it is transposed by teachers into knowledge actually taught in the classroom. To describe the relevant behaviours with respect to the didactics of a particular piece of knowledge, ATD proposes an analysis of their *economic*, *ecological,* and *epistemological* dimensions (Gascón & Nicolás, 2019). The economic dimension of a didactic phenomenon refers to the rules and principles that govern the system in its current behaviour; the ecological dimension refers to the set of conditions and constraints that allow the evolution of a system and facilitate or hinder its modification, while the epistemological dimension describes various aspects directly related to a particular piece of (mathematical) knowledge. The three dimensions are interwoven, and "there is a hierarchy among them: if a question is included in the ecological dimension (e.g., how to change the condition for a study of a certain piece of knowledge), one needs answers from the economic dimension (what is the current state with this piece of knowledge), which in turn are based on the (perhaps implicit) assumption of an epistemological point of view, e.g., a certain description of the piece of knowledge at stake" (Gascón & Nicolás, 2019, p. 7).

In addition to analysing the epistemology of the mathematical knowledge in question, in this study, we also describe the economy- and ecology-governing behaviours conditioned by the institution and the mathematical subject by using results from student questionnaires and informal interviews with university teachers of a multivariable calculus course while examining a recent modification of the course. In this way, we analyse the teachers' relationship to the knowledge to be taught in the observed cases.

As a further theoretical basis for the description of the didactical problem in the present study, we consider the notion of the didactic contract as a core of TDS. This is an implicit set of rules and expectations that shape the interactions between the teacher and students within a particular institution, in this context, the university mathematics department. TDS as a theoretical foundation in a university setting is used, for example, in a study on teaching differential equations by Artigue (1999), a study of cases from calculus and proofs by González-Martín et al. (2014), and concerning resources used by students by Gueudet & Pepin (2018).

Much of what is expected from students in university mathematics education is not taught directly during lectures, and teachers often assume that students would "cope somehow" with the expectations. Every mathematics student will likely recall the phrase "it is easy to see, and we'll leave it as an exercise", which indicates the need for an "increasing amount of personal work [. . .], and the importance of personal initiative" (Rogalski, 1998, cited in González-Martín et al., 2014, p. 122). For the purpose of this analysis, we consider the teachers' knowledge and beliefs that are not expressed explicitly and the teachers' expectations of students concerning this knowledge as *tacit*. In the literature (e.g., McGrath et al., 2019; Nonaka, 1994), explicit knowledge is addressed as the knowledge that is transmittable in formal, systematic language. Tacit (sometimes implicit) knowledge is the knowledge that is difficult to formalise and communicate, that is rooted in action, or maybe described simply as "the idea that we know more than we can tell". In education, the presence of tacit knowledge invokes different didactic issues; among others, it is observed that students "can face a lack of transparency in what is expected in assignments" (McGrath et al., 2019, p. 836), i.e., what the teachers' expectations are.

The transition process from knowledge to be taught to knowledge (actually) taught is shaped by the university mathematics teacher, who is very often a research mathematician. Their decisions and actions seem to be shaped by their mathematical knowledge and subject matter considerations on what is essential or important (Schoenfeld et al., 2016); through a combination of knowledge about mathematics and pedagogical considerations, their skills and experience, the goals of the course and understanding of how students learn and what they have to learn, and moreover by their academic fields of research in mathematics or mathematics education (Weber, 2004, cited in Karavi et al., 2020). This wider context of university teacher behaviour undoubtedly governs the teaching process. This study, however, concerns the description of teachers' tacit knowledge of a chosen specific piece of mathematical knowledge. Therefore, our research questions can be formulated as follows: *What are the rules and principles (economy) of the didactic contract of teaching and learning the Implicit Function Theorem, the didactic conditions and constraints (ecology) governing the contract, and which parts of it can be recognized as tacit?*

6.3 Context of the Study

The department in this study is characterised by large numbers of students and a strong tradition of dividing courses into “lectures” and “exercise classes”, with a very traditional (ex-cathedra) approach to teaching in both. With teaching in the scope of *visiting works*, which indicates addressing mathematical works *as ends* of education (Bosch et al., 2018; Gascón & Nicolás, 2019), it often tends not to analyse the students’ relationship to a particular piece of mathematical knowledge.

The mathematical context of this research, related to systems of nonlinear equations as the common piece of knowledge, is relevant in multivariable calculus in the context of introductory work with curves and surfaces, such as determining the tangent line or tangent plane when determining boundaries of double and triple integrals, calculating curve and surface integrals, or when finding constrained extrema by Lagrange multipliers. We base our study on the analysis of two compulsory courses, one on multivariable differential and one on integral calculus, taught in the undergraduate mathematics study program at a university mathematics department in Croatia, each for a period of one semester. The content of the first course covers standard topics in the calculus of functions of several variables, their continuity, and differentiability, followed by foundational theorems, such as Schwarz’s theorem, Taylor’s theorem, and Implicit and Inverse Function Theorems. The subsequent course in the following semester deals with integrability (Riemann integral of functions of two variables, multiple integrals, Fubini’s theorem, curve and surface integrals).

The knowledge relevant for this study begins with a chapter on the Implicit Function Theorem (IFT), followed by the introductory chapter on curves and surfaces. This is addressed towards the end of the first course, usually within a time constraint, and therefore with very few worked-out examples. However, future work in the mentioned mathematical contexts requires students to be familiar with flexible, non-routine algebraic manipulations and to have an understanding of the nature of the solution set in terms of passing between different representations in algebra and geometry, which are not explicitly discussed or built during the course. Official, detailed lecture notes are available in Croatian that provide precise definitions, examples, statements, and proofs. On the other hand, there are no introductory texts or informal discussions, though it should be stated that pictures for visualisation are presented during lectures. The content of the courses allows for certain modifications, though the content is usually stable over many years. A minor modification of the first course was made in 2019, prior to this study. The course teachers expanded certain sections of the content, particularly in the mentioned chapters with the mathematical content in question.

The lectures provide some examples illustrating the mathematical theory. More examples are given in exercise classes (mostly taken from previous written exams), for instance, on the application of the Lagrange multiplier theorem and the Implicit Function Theorem. Typical tasks involving nonlinear equations within the scope of the mentioned courses are of the following type:

1. Find the tangent line at the point (1,1) of the curve $y^2 = x + \ln \frac{y}{x}$.
2. Show that there is a unique C^1-class function $g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$x + y + g(x,y) = e^{-(x+y+g(x,y))}$$

   holds for each $(x, y) \in \mathbb{R}^2$.
3. Determine the global extrema of the function $f(x, y, z) = x + y + z$ on the set

$$D = \{(x,\ y,\ z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 1\}.$$

The first task requires implicit derivation of y considered as a function of x. The second task requires invoking IFT to show that g is smooth. The last task requires the geometric interpretation of the equations that gives meaning to the position of different surfaces in order to apply Lagrange's method and obtain nonlinear systems whose solutions are candidates for the global extrema. Similar equations appear in solving integration problems (e.g., to integrate a function f over the set D).

In these contexts, the intertwining of algebra and geometry cannot be seen as limited to one or a few institutions but is generally seen in many sources. For example, in the introduction to his textbook, Edwards (1995, p. ix) discusses that "modern conceptual treatment of multivariable calculus emphasizes the interplay of geometry and analysis via linear algebra and the approximation of nonlinear mappings by linear ones". Regarding the Implicit Function Theorem, we point out that it addresses a question concerning functions of several variables but can be interpreted from two additional aspects: one of nonlinearity, where it considers the question of finding solutions of a system of nonlinear equations, and a second one of geometry, where it addresses the question of the geometric structure of a solution set. Zaldivar (2013), in his review of Krantz and Parks (2003), writes that "the implicit function theorem in its various guises (the inverse function theorem or the rank theorem) is a *gem of geometry*, taking this term in its broadest sense, encompassing analysis, both real and complex, differential geometry and topology, algebraic and analytic geometry".

We illustrate these aspects with two exemplary problems in which the three viewpoints (algebraic, geometric, and functional) intertwine:

1. Which curve is the intersection of the surfaces given by the equations:

$$x^2 + y^2 + z^2 = 1,$$
$$x + y + z = 1?$$

   Find the tangent line in a point of that curve with $z = \frac{1}{2}$.

2. One solution of the system

$$x^3y - z = 1,$$
$$x + y^2 + z^3 = 6$$

is $(1, 2, 1)$. Estimate the x and y for $z = 1.1$.

These problems rely on the procedure of implicit derivation, which in the first problem provides the direction of the tangent line, and the second problem gives the coefficient for linear approximation of the variables x and y depending on z. The procedure is justified by the IFT as it implies that, under certain conditions, we may consider some of the variables as functions of the other and consequently find derivatives of these functions. We point out that these types of tasks were not included in the mentioned courses before this study but were intended to be included in the second course after this study, following the suggestions made by the authors.

6.4 Mathematical Analysis of Student Tasks in the Exercise Class

In this section, we present the tasks designed by the authors for independent student work and implemented in a previous study 1 year earlier on a smaller scale with a group of volunteers (Bašić & Milin Šipuš, 2021). The previous study offered the authors specific insights into the didactic potentials of the tasks and the difficulties students encountered in solving them. The tasks presented in Fig. 6.1 are used in this study as a basis for students’ reflections on their difficulties that may indicate tacit expectations from the teachers in the course on multivariable calculus.

Mathematical knowledge required for solving the tasks embraces the knowledge in question: algebraic manipulation of equations and their interpretations in geometric contexts (in 3D in particular), the use of different forms of representation of

1. An ellipse is given by the implicit equation $2x^2 + y^2 = 6$.
   a) Determine its tangent line at the point (1, 2) using implicit differentiation.
   b) Parametrise the ellipse.
   c) Determine the tangent line at the point (1, 2) using parametric differentiation.
   d) Which theoretical results connects two ways of calculating the derivatives?
2. A curve is given as the intersection of the elliptic paraboloid $x^2 + y^2 = 3z$ and the plane $4x + 4y + 3z = 1$.
   a) Which curve is given?
   b) What does the equation $4x + 4x + x^2 + y^2 = 1$ represent?
   c) Determine the tangent line to the curve at the point (−2, 1, 5/3).

Fig. 6.1 Tasks for independent student work geometric objects (parametric and implicit), and referencing to the underlying theory. The tasks explicitly refer to geometric objects, meaning they are not completely set up algebraically.

In the first task, the slope of the tangent line is given by the derivative at a point, calculated either in the implicit form by assuming that a curve is locally described (as, e.g., $y = y(x)$ as provided by IFT under the conditions that the theorem requires), or parametrically (as $x = x(t)$, $y = y(t)$ which may be invoked by using trigonometric functions or by explicitly expressing y as a function of x, $y(x) = \pm\sqrt{6 - 2x^2}$). It is expected that students invoke IFT to justify the use of the implicit derivation $y'(x) = -\frac{\partial_x F}{\partial_y F}$ for the implicit function $F(x, y) = 2x^2 + y^2 - 6$, where $\partial_x F$, $\partial_y F$ stand for the partial derivatives of F. The chain rule for differentiation of a composite function $y'(t) = y(x(t))$ (under the assumption that x is a strictly monotonous function of t) implies that $y'(x) = \frac{y'(t)}{x'(t)}$, thus providing the connection between the two approaches.

In the second task, the intersection is obviously planar since the second equation represents a plane in space. It can be recognised (e.g., by geometric arguments) as an ellipse. The equation in subtask (b) is obtained by algebraic manipulation from the first two equations given and, as seen in our previous study, referred to as "a solution" by students, i.e., as an equation representing the intersection curve. However, this is an equation of a new surface (circular cylinder) upon which the intersection curve lies and whose projection onto the xy-plane is a circle with the given equation. We call this way of reasoning the "intersection-projection misconception". Determining the derivative in subtask (c) requires the use of either the implicit or parametric form of the intersection curve. In the first approach, a more subtle use of IFT is needed for local curve description in expressing the two variables in the system of equations $F_1(x, y, z) = 0$, $F_2(x, y, z) = 0$ given in the task by one variable, e.g., as $y = y(x)$, $z = z(x)$, which can be achieved if the matrix formed by partial derivatives of F_1, F_2 with respect to y, z is regular.

6.5 Methodology

Our study uses three sources of data: lecture notes from the courses on multivariable calculus (described in the section on Context), students' productions and reflections obtained during the exercise class based on the interaction with the two tasks in Fig. 6.1, and informal interviews with the teachers of these courses.

In total, 87 students participated during the first exercise class in the second course in multivariable calculus, after which their productions were collected and analysed. Each student received two sheets, one task per sheet, with empty space for their solutions. The tasks were previously presented to teachers who confirmed that they corresponded to the theory in lectures and agreed to include them in the exercise class. The parametric and implicit differentiation in the case of curves in 2D is covered in single variable calculus, so our assumption is that the structure of the task would serve as a scaffold for students to recall and link the two techniques with the underlying theory given at the end of the first course. The second task had not been previously covered in the exercise class, but it was expected that students would reason by analogy. The theory of parametrized curves and surfaces was covered in the previous course, and the analogous linear situations of the intersection of two planes were treated in the first year courses, so these pieces of knowledge would support solving tasks (2a) and (2b), while students’ solutions to task (2c) would show their ability to apply IFT by following the strategy outlined by the first task. Students’ difficulties would direct our further investigation of the interaction between the students, teachers, and the knowledge at stake.

After solving the first task on their own for about 20 min, students were asked to discuss their solutions in pairs and then to write reflections about their work on the same paper. They were asked to be explicit about what they learned from their discussion with peers and what remained unknown or unclear, and what they were not certain about. After that, the teaching assistant provided the solution on the blackboard, and the whole process was repeated with the second task.

The main assumption was that by asking students to reflect upon and explain their own difficulties, we would be able to recognize parts of mathematical knowledge that indicate discontinuities and breaks in the didactic process and hence form expectations that remain tacit. This would be the starting point for the interviews with the teachers in which we would aim to identify whether and under which conditions these discontinuities arise.

Students’ (anonymous) written work on both tasks was collected and analysed. It included the students’ mathematical solutions of the task and their reflections on learning this topic at the end of the first course.

In the mathematical analysis of the students’ solutions, we consider the use of implicit and parametric differentiation, the arguments that students give about their connection, the possible occurrence of the “intersection-projection misconception,” and the analogies the students draw between the two tasks. We have presented the answers to each subtask according to our a-priori analysis.

For the reflections, students’ answers were short and very similar, falling into several categories, so in the end, we decided to present them by examples and indicate their count. Following that, we interpreted the results and formulated hypotheses with which we entered the interviews with the teachers.

The two courses on multivariable calculus relevant for this study are taught by two research mathematicians and university teachers, in two parallel classes of approximately 65 students. One of the teachers (teacher A) has long experience (15 years) in teaching the course and is also a researcher in applied mathematics. The second teacher (teacher B) is a young university mathematician with research interests in functional analysis and with teaching experience in differential geometry.

The main goal of the interview was to collect further data that would enable us to understand the didactic contract and the evolution of the didactic system that leads to students’ difficulties and practice. The form of the interview was chosen primarily because the intention is to understand the conditions and certain teacher decisions that are related to tacit expectations appearing in the course. As mathematicians and researchers in mathematics education, we ourselves are interested in this topic and have the same experience with it as these teachers, so the intersubjective aspect of the interview emphasises "the social situatedness of the research" and the interview provides "space for spontaneity, complete answers, responses about complex and deep issues" (Cohen et al., 2011, p. 349).

Based on the analysis of the course materials, students' solutions and reflections, and our previous research and experience, we have hypothesized that it is demanding to cover the course material within the allotted time (two lectures of 3 h) given for the two chapters (IFT and curves and surfaces), so there are many decisions that the teachers make regarding the level of details of the theory and examples on three levels: presented in the lecture notes, presented during the lecture, and expected from the students. The aim of the interview was to understand the latter two, which we could not reach otherwise.

Based on these, we formulated the interview questions that provided us with the structure of the interview and its subsequent analysis. Teachers were asked to read the selected students' answers to the two tasks in advance. Each teacher was interviewed independently by both authors, and each conversation lasted about 60 min. The structure of the interview was approached through the following questions:

1. How do you comment on the students' work, and what do you think about the learning potential of the activity in general?
2. What have you changed in the course materials in recent years, and why?
3. How do you see the position of geometry in the course?
4. How do you see the position of tasks involving systems of nonlinear equations in the course?
5. Which parts and aspects of IFT do you emphasise?
6. How do you deliver the material from lecture notes? Do you cover all examples?
7. Which expectations from students do you make explicit?
8. What do you expect from students, concerning the previous content, during your assessment?

Questions 1 and 2 are aimed at teachers' attitudes towards changes and decision making concerning the use of time and organization of teaching – teachers' beliefs and conditions directing the didactic process. Questions 3, 4, and 5 aim at the extent to which the pieces of knowledge are made explicit, while Question 6 aims at the delivery of the material during the lectures. Question 7, in addition allows the teachers to identify other expectations that remain tacit in the didactic contract that we might overlook, while question 8, and in some parts question 1, is related to assessment and the expectations about the knowledge actually taught at the end of the course. We expected teachers to highlight what was omitted and provide reasons for particular student difficulties as a consequence of delivery. Furthermore, we hoped that they would describe constraints that govern their decisions about the course and that they would also make some expectations explicit and state or become aware that they are tacit for students.

The order of questions was decided during the interview, but all planned questions were posed to both teachers. For each interview, a transcript was produced, the answers to questions identified, and then compared between the teachers for each topic. If the answers provided the same information, then we report on these responses as joint answers of the teachers, whereas if differences were detected, the answers of each teacher are given separately.

6.6 Results – Students' Solutions and Reflections

For task (1a) we have observed that 22 out of 87 students obtained the equation of the tangent line by implicit differentiation using the procedure from first year course, 10 students obtained it by expressing y explicitly as a function of x, while 7 students used the gradient of the function $f(x, y) = 2x^2 + y^2$, which we can interpret as imitating the procedure for obtaining the tangent plane of a surface taught in the course. Further 34 students calculated partial derivatives, but without any sense how to obtain the tangent line. Finally, 14 students did not write anything relevant to this task (blank).

In (1b) 44 students wrote down the trigonometric parametrisation and 10 of them the explicit parametrisation. Three students only divided the equation by 6, while 30 students skipped this task. In (1c) only 13 students obtained the equation of the tangent line, while 19 students calculated the derivative of their parametrisation, but did not know how to continue (either by not knowing which t to use, or how to use the calculation to obtain coefficients in the equation of the tangent line). We have 42 students who skipped this task and 13 with unclear strategies. Finally, in (1d) we have 42 students who mention IFT, 5 students mention chain rule, 4 mention that the derivative of a function is the slope of the tangent at its graph, while 34 students did not answer the question. Only 2 students have given an argument connecting the two approaches.

In (2a) and (2b), we have 26 students showing the projection-intersection misconception. Typical answers showing this are:

> The intersection is a circle, the equation $4x + 4y + x^2 + y^2 = 1$ is the equation of that circle.
>
> After elimination of z we obtain the equation $(x + 2)^2 + (y + 2)^2 = 9$ which is a circle of radius 3 with centre $(-2, -2)$, the second equation is the same as above.

In (2c), 48 students applied the procedure of implicit differentiation (shown as a solution for the first task) to the equation in (2b) instead of applying it to the system of equations given in the task. No student has given a correct solution assuming that x and y are a function of z. There were 5 students that calculated the tangent plane to the paraboloid and its intersection with the given plane. In 9 cases, a student parametrized x and y from the equation $(x + 2)^2 + (y + 2)^2 = 9$ and expressed $z = \frac{1}{3}(x^2 + y^2)$ or $z = \frac{1}{3}(1 - 4x - 4y)$, but they either made calculational errors or did not use the derivation of the parametrization.

Students' reflections can be organized into several categories. There were 32 students that did not write any reflection or wrote that they did not learn anything from the discussion in pair. There were 21 students who wrote that they do not know how to obtain a parametrisation or what parametric or implicit differentiation is, e.g.

> We lack basic knowledge about the parametrisation of a surface and parametric differentiation.

> We concluded that we do not know what parametric and implicit differentiation is. I know the definition of parametrisation, but I do not understand it.

Furthermore, 6 students wrote that the notion of parametrisation was not covered in enough detail in the previous courses:

> We talked about the parametrisation, and we thought that it could be better explained (in general, its purpose and meaning).

> Is something unclear? Yes, but I don't even know how to phrase the question. Did we miss any lectures last week?

On the other hand, 11 students wrote that they did not learn enough:

> Not much is clear, we have forgotten it all.

> I have not studied enough the subject matter about curves.

Two students wrote that they did not know how to solve it because that chapter was assessed only for higher grades.

> After the discussion, I see that we do not understand the theoretical background of the problem, likely because it was only required for a higher mark.

Five students wrote that they had learned something from a peer, three students that they realized they had used the explicit equation, and three students had mathematical comments (derivative is the slope of the tangent; the curve is independent of its parametrisation). One student provided a full solution and provided an alternative approach as a reflection. Three students wrote that they are re-taking the course and forgot the material of the previous course because of the year break.

The main conclusion from the mathematical solutions is that students lack procedural fluency in the methods and that the hypothesis that all required pieces of knowledge for the suggested task were covered in the previous courses. We have a variety of students, ranging from those familiar with everything to those who did not answer a single question. The majority of students have difficulties connecting the derivative of the parametrisation to the equation of the tangent line, which can be explained by inspecting the lecture notes and noting that there is no example of that type. Furthermore, the majority of students have difficulties in interpreting equations in three variables, which cannot be assumed to be easily adapted from linear algebra. The overall conclusion from these results is that it is not straightforward for the students to connect the fragmented pieces of knowledge even if they had encounters with all of the pieces in their previous education.

From students' solutions and reflections, we see that they do not tend to communicate verbally about mathematics or learning mathematics – their main mathematical discourse is symbolic. This may be related to the orientation toward procedures as we see indications of techniques that are applied in the wrong contexts (as in 2c) or parts of techniques (calculation of derivatives) without a clear strategy on how to use the results (to obtain the tangent line). Another related aspect might be a part of the contract to *learn the theorems* (or to be aware of them) even when techniques are not present in the knowledge. We find from the high number of students mentioning IFT as justification even when the tangent line was not obtained that these responses confirm the hypothesis that the formal treatment of IFT does not equip students to apply implicit differentiation in concrete situations.

Finally, some students note that in the previous courses, they did not see the needed techniques, while some feel that they are responsible for not learning more to be able to solve the given tasks. We find this point subtle and that it might be relevant in other university programs with larger numbers of students and dense lectures. We interpret that these comments indicate students’ uncertainty about the expectations and the opportunities they have had to get prepared for these expectations.

6.7 Results – Interviews with Teachers

Concerning students’ productions (question 1), none of the teachers were surprised by the variety in the answers and the difficulties which appeared in them. Teacher A emphasised that he was aware that the subject was difficult but expressed that he has realistic expectations.

> Look, this course has a standard issue: we do too few examples, and most examples are theoretical. We teach this part at the very end of the semester. There is a lot to cover. The other parts of the theory are more important, and we are not left with much time. All the details are written in the lecture notes.

The comments of teacher B were more outcome-oriented, as he pointed out that “IFT requires a certain maturity”. He emphasised that the audience is large and diverse, which sometimes makes the choices very difficult. He was aware that students would not approach the material in-depth:

> It depends on a student’s interest whether they will develop the geometric viewpoint. Many students just want to pass the course. If they hear that this will not be part of the exam, and we explicitly say that we do not ask that at the oral exam, they do not learn it.

A modification of the first course (question 2) was implemented before this study, following a departmental decision to include an extra hour per week for teaching it. The decision of teachers was to expand the lecture notes (and lectures) for the chapter on curves and surfaces, given their previous teaching experience that “*they were going too fast*”, while “*students had a vague understanding of the topic*”. The notes were expanded by teacher B based on his experience teaching a course *Differential geometry,* and he closely followed a foreign textbook. Before the modification, IFT was only stated and proved in the context of functions; the modification included the geometric aspect of the theorem (questions 3 and 5). Teacher B stated:

> Now we have examples where we use geometric interpretation. We consider the intersection of two spheres given by their equations. After checking the analytical conditions of IFT, we give a geometrical explanation for the singularity… Geometry is neglected in our department. I think differential geometry should be a compulsory course because it is reachable, and students can see the applications of the analytical tools that we have developed.

Both teachers agreed that geometric intuition is very helpful but difficult to learn and teach. Teacher A gave a broader discussion of the complexity of the subject:

> Intuition is accumulated experience. Geometric intuition is useful to understand what something looks like. When I see an algebraic equation (sometimes), I see the set of solutions, but the student does not see anything. Geometry in 2D and 3D is very close to my scientific interests. Interpretations of any kind are helpful, but they are neither easy for students to grasp nor for us to show. When I teach, geometric interpretation is my starting point, but in the oral exam, I only ask this part of the subject at the theoretical level (and only for a high grade). It does not surprise me that this is hard for students.

Nonlinear systems of equations (question 4) are seen only as an extension of the linear. Those that appear in examples or in exam problems (question 8) mostly have an exact solution since they include second-degree equations. Both teachers agreed that they need to cover a great deal of theoretical materials, and there is no time to deal with techniques systematically, or even tricks, for solving nonlinear systems. These techniques are often of a higher level of algebraic manipulation, and since there is more important basic knowledge, they are not in focus. By working only with second-degree equations, teachers rely on students' knowledge of high school algebra and put more emphasis on the conceptual interpretation of the procedures. Teacher A considered that solving a nonlinear system is not an application of IFT. In practice, nonlinear systems are usually treated by numerical methods, and an exact solution cannot be found. The case of nonlinearity is further discussed from various points of view:

> As a student, I was not aware of the problem of nonlinearity; only as a researcher could I see how little we know about nonlinear problems. A problem may very easily become too hard for an exam. Those systems that could be solved require a trick, and that is not what we assess at the exam. [...] I think we should make it clear to students that we usually encounter nonlinear problems in real life. [...] We should change the whole system in order to enable students to "get their hands dirty"; this would require project teaching, and we have 130 students.

Regarding course organisation, we can summarise that the teachers agreed that IFT is an important theorem but that there are more important topics in the course (e.g., continuity and differentiability). They felt that there was not enough time to cover all results they would like to during the lecture and that they delivered only the basic ideas (e.g., basic steps in the proof of IFT) and skip many examples (question 6).

Regarding expectations (questions 7 and 8) on student work, teacher A showed awareness that the topic is difficult and tried to make his expectation explicit by pointing out to students what is required for a particular grade. He also concluded that students know that one exam question concerns the application of IFT and that it is up to the students to decide whether or not to prepare for it. Teacher B emphasised, "I try to repeat and explain what I notice that the students do not know," but also mentioned that students struggle even with basic concepts in more than two dimensions, that they do not ask questions, they did not acquire skills of algebraic manipulation in high school, or that parametric differentiation should be understood from the single variable calculus.

From these comments, we infer that the described didactical situation is governed mainly by the lecturing tradition of teaching and that for many students, this is a constraint to establishing two-way communication. The teachers have mentioned time, the number of students, and complexity of the subject as constraints for in-depth teaching. They expect that students have mastered specific techniques from high school algebra (solving second-degree equations and systems of equations) and concepts from differential calculus (e.g., parametric differentiation), but at the same time consider the geometric reasoning with corresponding curves and surfaces complex for teaching and learning. We interpret that all of these constitute tacit knowledge that teachers find either hard to transfer or that the students must have already acquired by the second-year courses on multivariable calculus.

These observations point to the fact that teachers show awareness of tacit moments in their teaching where they do not explicitly address certain pieces of mathematical knowledge. At the same time, they express limiting expectations from students, which is not known to students. They aim to provide reasonable challenges in mathematical problems regarding nonlinear manipulation in assessment, and they do not expect fluent geometric reasoning from students since it is not the focus of their teaching.

By taking into account students' solutions, we conclude that students might be aware of the theory but lack more opportunities to build the techniques and connect the techniques to the corresponding theoretical discourse. By considering the lecture notes and the choices for modifications, we may also notice the tendency to "cover all the theory" as an obligation that a university teacher might feel when following a textbook and hence limiting their own freedom in determining the scope and the level of the details. The teachers, therefore, provide lecture materials that are very detailed and cover more content than presented in the lectures. Also, in the assessment they require only a fragment of that content. They know that the techniques which are needed in the second course will be revisited during the exercise classes, but at the moment when students are learning IFT this is not clearly articulated, and students might be facing a choice of how much to learn under these circumstances.

6.8 Conclusions and Further Perspectives

In this study, we aim to describe the didactic contract in which teaching and learning of the Implicit Function Theorem and its applications in spatial geometry take place, and predominantly whether there are certain tacit knowledge and teacher expectations related to it. We assumed that from the students' reflections about their own difficulties and doubts in solving two tasks, we would find indications of discontinuities in the didactic process causing parts of knowledge to remain tacit and that these would then be confirmed by the teachers' own description of their teaching process.

By analysing students' reflections, we found that students do not tend to express themselves in many words when discussing (learning) mathematics. Many of them have named the relevant theorem, even in the situations where they did not write the techniques that should be justified, indicating that they follow the rule to learn the theorems only to reproduce their statements. We have observed their difficulties with procedures related to underlying theory (Implicit Function Theorem), that is, with the application of the theory to a particular task and change of representation (parametric-implicit, with difficulties in establishing parametrisation). The intersection-projection misunderstanding as a difficulty concerning algebraic manipulation and the geometric interpretation of equations was also observed. At the same time, some students expressed a sense of accountability for their own learning, having the impression that it was expected of them that the given tasks were within their reach but that they have "*not studied enough*" or "*is it possible that we missed a lecture*". However, the interviews with teachers revealed that not much time or effort was devoted to mathematical contents regarding Implicit Function Theorem or spatial analytic geometry, which allows us to consider this knowledge as tacit.

The teacher interviews gave a further description from their side regarding the course organisation, decision-making process on mathematical content, and expectations from students. The interviewed teachers reported that they always try to provide connections of different parts of mathematics while teaching, but that interpretation is an expert ability and is difficult to teach. They do not expect the same level of mathematical fluency from students, as this can be, even with the best of intentions, made explicit only to a limited extent and even reason that "*intuition is accumulated experience*". This *limited expectation,* as well as limited possibilities of transferring certain aspects of knowledge, remain tacit for the students.

There are a few crucial aspects that govern the decisions made by the teachers. Lack of time, a large number of students, and extensive subject material are seen as the main constraints to providing students with the opportunity "*to get their hands dirty*". While the course material is detailed, teachers have to select only a part of the subject matter to present in the lecture. The Implicit Function Theorem forms "*a grain of knowledge that everyone should see, but only a few would go in its depths*". Both teachers considered that this theorem is not the most important part of the course, and hence it would not be justified to spend more lecture time on it. On the other hand, we could say that for mathematicians and university teachers, there is an understandable sense of discomfort regarding what would happen "*if we do not cover all the theory*" and what remains for the students to rely on in that case. Much of the content is required in assessment only for a high mark, which also adds to the ambiguity of what the students should learn and what they actually learn. The situation is balanced by the teachers' awareness that some parts will remain unclear or tacit after the first course and will need to be repeated or solidified during the second course.

As we know that "the tacit knowledge is the knowledge that comes from *personal experience*, and it is a trait of an expert" (McGrath et al., 2019), we could only speculate about the improvements that might stem from possible out-of-the-box orientation toward peer and inquiry learning that would yield more adidactical potential. At the level of university mathematics education, given the large numbers of students and strong tradition of lecturing, we can certainly not expect colossal changes in the teaching paradigm, or as Hochmuth (2020) points out, "it is not clear to what extent the potentials can be realised under the currently dominant teaching and learning conditions". However, as we have seen in the presented case, teachers themselves have already introduced certain modifications, were willing to consider other suggestions (e.g., by including the exercises suggested by the authors of this study), and to participate in the interviews to discuss and reflect upon their own decisions, so we might perhaps see other changes in the future leading to a more efficient didactic process.

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Chapter 7
University Students' Development of (Non-) Mathematical Practices: The Case of a First Analysis Course

Laura Broley and Nadia Hardy

Abstract In this chapter, we present a study that investigated the nature of the task solving practices developed by students in a first Analysis course at a North American university, and how these practices may be shaped by the evaluations (assignments and exams) given in the course. Task-based interviews with 15 students after their successful completion of the course revealed that students' practices could vary in nature, being more or less "mathematical," i.e., more or less reflective of mathematicians' practices. As suggested by previous research on Calculus courses, we also found that the practices students develop in this Analysis course are likely shaped by the minimal requirements for success. To try to make sense of this, we introduce the theoretical notion of "path to a practice" and a characterization of three ways in which students' practices may reveal themselves to be "non-mathematical."

Keywords Mathematical practices · Task solving · Praxeology · Institutional perspective · Real analysis · Functions

7.1 Introduction

Previous research has investigated what students learn in Calculus courses and documented its potentially rote procedural nature. Orton (1983) interviewed 110 Calculus students and found that many were operating according to *rules without reasons*: When it came to performing integral calculations, they knew *what* to do, but did not know *why* they were doing it. Shortly after, similar results concerning a variety of Calculus topics were published by other researchers (e.g., Artigue et al., 1990, in France; Cox, 1994, in Britain; Selden et al., 1994, in the United States; White & Mitchelmore, 1996, in Australia), some of whom began to look more systematically into why students may learn rules without reasons. Cox's (1994) discussions with Calculus teachers and students revealed that they may tailor their

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teaching and learning to typical exam questions. Selden et al. (1994) also pointed to the potential impact of emphasizing routine tasks in instruction and evaluation: Tests administered to students who received good passing grades in Calculus courses showed that they could solve routine tasks quite well, but lacked the conceptual understanding needed to solve only moderately nonroutine tasks. Later research by Lithner and colleagues echoed these findings and worked on characterizing the nature of the reasoning underlying students' solving of routine tasks. For example, in observing students' task solving, Lithner (2000, 2003) saw how some students explained their strategies based on established experiences from their learning environment or superficial features of similar-looking tasks (rather than "mathematical" reasoning). Subsequent studies made sense of this through systematic analyses of the tasks that are typically posed in Calculus textbooks and final exams, which were found to not *require* students to go beyond superficial mimicry or the basic recall of algorithms based on properties of the tasks that are not relevant from a "mathematical" point of view (e.g., Bergqvist, 2007; Brandes & Hardy, 2018; Hardy, 2009; Lithner, 2004; Tallman et al., 2016).

It seems reasonable to expect that as students move beyond Calculus courses, the nature of what they know and learn would be required to change. Theoretically speaking, it has been proposed that curricula of more advanced courses in Analysis invite students to deepen their understanding of mathematical concepts and theories underlying procedures learned in Calculus, and to develop formal proof practices that require the use of mathematical reasoning (Winsløw, 2006). On a practical level, however, scholars (a) doubt that students naturally make productive connections between what they learn in Analysis and what they learned in Calculus (e.g., Kondratieva & Winsløw, 2018; Winsløw et al., 2014), (b) pinpoint epistemic, cognitive, and didactic obstacles to students' learning of formal proof practices (e.g., Bergé, 2008; Maciejewski & Merchant, 2016; Raman, 2002, 2004; Sfard, 1991; Tall, 1992; Timmermann, 2005), and (c) point to the possibility of students successfully completing[1] Analysis courses by memorizing a particular subset of definitions, theorems, and proofs (e.g., Darlington, 2014), or by learning new kinds of superficial (non-mathematically relevant) and algorithmic task solving practices (e.g., Weber, 2005a, b).

To contribute to the literature outlined above we conducted an exploratory study (Broley, 2020) of an Analysis course at a North American university. In this chapter, we deepen our analysis of a refined subset of results and extend our reflections concerning two general research questions we explored:

1. What is the nature of the practices developed by students in a first Analysis course?
2. How might these practices be shaped by the nature of the tasks offered to students in the course?

[1] Both in those studies and in ours, successful completion of a course means obtaining a passing grade.

Responses to these questions could have practical implications for teachers or curriculum developers involved in designing the tasks offered in university mathematics courses. In what follows, we introduce how we framed our questions (Sect. 7.2), describe our methodology and the more specific objectives it addressed (Sect. 7.3), and present and discuss some results (Sects. 7.4 and 7.5).

7.2 Theoretical Framework

To frame our research questions, we first specify how we think about "practices" and their "nature" (Sect. 7.2.1). Then we elaborate our perspective on how practices may be "shaped by the nature of the tasks offered to students" in a course (Sect. 7.2.2).

7.2.1 Mathematical and Non-Mathematical Practices

To help us think about the nature of students' practices, we turned to theoretical tools within the Anthropological Theory of the Didactic (ATD; Chevallard, 1985, 1991, 1992, 1999).[2]

In the ATD, *practices* refer to regularized and purposeful human actions, which can be *personal* (developed by an individual) or *institutional* (created, encouraged, and enforced in a particular institution). An *institution* is understood in a broad sense as a relatively stable structural element of a society that has been established to organize human (inter)actions and orient them towards certain outcomes. Any profession (pure mathematics research, actuarial science, engineering, etc.) or form of organized education (school mathematics, university mathematics, etc.) can be thought of as an institution (called *professional* or *didactic institutions*, respectively). An individual is said to have developed an institutional practice if they have developed a personal practice that is judged to be acceptable and worthwhile within that institution.

With his theory of didactic transposition, Chevallard (1985) brought to light the transformation of practices as they migrate from a professional institution into a didactic institution, which serves to exemplify the institutional relativity of practices.[3] In particular, the ATD acknowledges that what is considered "mathematics" or "mathematical" may change from one institution to the next. We nevertheless claim that one overall aim of *university mathematics* is to support students' eventual development of *mathematicians' practices*, by which we mean the practices

[2]To learn more about the ATD and its use in mathematics education, see Bosch et al. (2020) for a recent comprehensive description and Winsløw et al. (2014) for an overview specific to the university level.

[3]One should pause and reflect on the relationship between the terms "practice" and "knowledge" from an ATD perspective. We let this hang in the subtext of our chapter, to be addressed in further discussion and subsequent theoretical research.

produced and used by mathematicians in the broad professional institution referred to as *scholarly mathematics*. Thus, in our work, we use mathematicians' practices as a reference with which to compare the practices of university mathematics students, and we use the term *mathematical practices* (and *non-mathematical practices*, in contrast) in a particular way: to refer to practices that would be considered acceptable and worthwhile (or not acceptable or not worthwhile, in contrast) within the scholarly mathematics institution.

Chevallard (1999) offers the notion of *praxeology* as a way of modelling practices as they exist across institutions and individuals; any practice can be represented by a quadruplet $[T, \tau, \theta, \Theta]$ – called a "praxeology" – involving four interconnected, essential components:

- a *type of task, T*, to be accomplished;
- a corresponding collection of *techniques, τ*, to accomplish T;
- the *technology*, θ, i.e., discourses to describe, justify, explain, and produce the techniques; and
- the *theory, Θ*, that serves as a foundation of θ.

This representation of a practice recognizes both a practical part (the know-how), $[T, \tau]$, called the *praxis*, and a theoretical part (the know-why), $[\theta, \Theta]$, called the *logos*.

The notion of praxeology gives us a way to think about the nature of students' practices, which in turn allows us to reflect on whether and in what ways the practices are mathematical or not (in the sense posed above). As we consider a praxeology to be a static model of a practice, and inspired by previous work (e.g., Lithner's, 2008, task solving framework), we say that an individual *enacts a mathematical practice* if they carry out the action of solving a given task by

- identifying the task as belonging to a mathematical type of task;
- selecting and implementing a mathematical technique to accomplish the task;
- describing, in a mathematical discourse, how and why the technique works; and
- acknowledging a mathematical theory that supports the discourse[4];

where, as explained above, "mathematical" is used in a particular way, to describe a component (type of task, technique, etc.) as acceptable and worthwhile within the scholarly mathematics institution.[5] If, conversely, some component would not be

[4] It is possible that an individual will not explicitly engage in each of these actions when solving a task. Following the example of Chevallard (1999), we take the position that "having a practice" means being able to engage in four actions reflecting the four components of a praxeology. For example, if an individual has a practice, they would be able to give some description of why their chosen technique works. This description need not be "mathematical": e.g., "I know the technique works because my teacher told me to do it that way."

[5] We are assuming that there are some uniform, implicit ideas among mathematicians of what is (or is not) acceptable and worthwhile. We also acknowledge that there could be pertinent differences between mathematicians' judgements depending, for example, on the specific area of mathematics in which they work (mathematical physics, numerical analysis, algebraic topology, . . .), which could warrant a definition of mathematical practice that depends on a specified area of mathematics. We did not consider such differences in this research.

considered acceptable and worthwhile according to scholarly mathematics, we say that the individual *enacts a non-mathematical practice*. Certainly, enacting a mathematical practice cannot be equated with having developed one.[6] Nevertheless, in our work, we assume that an Analysis student who enacts a non-mathematical practice has not developed a mathematical practice – we say that these students have developed non-mathematical practices.

As an example, we could expect a mathematician faced with finding $\lim_{x\to 1}\frac{x-1}{x^2+x}$ to identify the task as belonging to the type *find the limit of a rational function at a point* and to solve the task by direct substitution. If prompted to describe how and why the technique solves the task, we could expect them to acknowledge certain theoretical elements such as theorems, laws, and definitions. In contrast, when asked to find $\lim_{x\to 1}\frac{x-1}{x^2+x}$, many of the Calculus students in Hardy's (2009) study seemed to identify the task with a type characterized by an easily factorable expression, which necessitates some sort of algebraic technique: 20 out of 28 students tried factoring, seven of which did direct substitution first. Furthermore, the students' discourses were of the sort: "We do this because that's what our teacher showed us, and that's what we normally do for this kind of problem." Hardy (2009) concluded that the students learned to behave "normally" rather than "mathematically." In the context of our study, we would say that the students were enacting non-mathematical practices. In the following, we propose one way of thinking about how the students may have developed such non-mathematical practices.

7.2.2 *The Progressive Development of Practices*

In our work, and in line with previous research (e.g., Bergqvist, 2007; Cox, 1994; Hardy, 2009; Lithner, 2004; Selden et al., 1994), we conjecture that in university mathematics courses, students encounter numerous tasks that progressively determine the practices they develop. The tasks may occur in lectures, recommended exercises, assessments, and students' independently driven work. To model how the nature of such tasks might contribute to moulding students' practices, we introduce a distinction between *isolated tasks* and *tasks forming a path to a practice* (building on Broley & Hardy, 2018).

The tasks in Table 7.1 were offered by teachers to students in the Analysis course we studied, along with written solutions. The written solutions for the tasks on the right of Table 7.1 use the Intermediate Value Theorem (IVT). These tasks are meant to help students identify a particular type of task – that of showing that a function has

[6] A student who enacts a mathematical practice could simply be mimicking behaviour. But the focus of our work is the development of non-mathematical practices. Given our task-based interview approach (see Sect. 7.3), we are convinced that the students we interviewed who enacted non-mathematical practices had not developed mathematical practices; they had developed non-mathematical practices.

Table 7.1 Examples of tasks found in assessment documents in the Analysis course we studied

Consider $Z_3 = \{0,1,2\}$ with the usual operations + and ·. Check that $\{Z_3, +, \cdot\}$ satisfies the axioms of a field.	Show that the polynomial $P(x) = x^5 - 3x + 1$ has a zero in the interval (0,1).
Prove $[ax = a \text{ and } a \neq 0] \Rightarrow x = 1$ using the axioms given in class.	Show that $f(x) = e^{-x^2}$ has a fixed point in the interval (0,1).
Prove that if $a < b$, then there is a $q \in Q$ such that $a < \sqrt{2}q < b$.	Prove that the function $f(x) = e^x - 100x$ has exactly one zero in the interval [0,1].
Does the subset $\{(x, y) \in \mathbb{R} \times \mathbb{R}: x^3 = y^2\}$ define a function?	Prove that the equation $\cos(x) = 5x(1 - x)$ has exactly two solutions in [0,1].
Find the intervals on which the function $f(x) = \frac{x^2+1}{x^4+1}$ is monotonic.	Let $f: [0,1] \to [0,1]$ be continuous. Prove that the equation $f(x) = x^2$ has a solution in [0,1] (you may use the Intermediate Value Theorem).
	Show that the equation $e^x = 3x^2$ has at least two positive solutions.
	How many zeros does $f(x) = 4x^3 - 32x^2 + 79x - 60$ have in the interval [0,5]?
	…
"ISOLATED TASKS"	**TASKS FORMING A "PATH TO A PRACTICE"**

a certain number of zeros – and to master a particular technique – one afforded by the IVT. Teachers' solutions propose to students discourses to describe how and why the technique works: "by the Intermediate Value Theorem".

We say that tasks that relate to the same type of task and exist in relatively high quantity, including in situations that are relevant to a student's success, form a *path to a practice*: they communicate to the student that some kind of practice should be developed. In contrast, certain types of tasks may be encountered by students only in disconnected, rare, or seemingly non-relevant (e.g., non-tested) situations. The action of accomplishing the related tasks may hence remain isolated and particular, not contributing to the development of a practice.[7] We say that such tasks are *isolated* (as opposed to forming a path to a practice).

[7] We consider the important distinction between the action of solving a task for which one has developed a practice and the action of solving a task for which one has not developed a practice. For instance, an individual may engage in the tasks of cooking a meal or hammering a nail without having developed practices for doing so. In contrast, professional chefs or carpenters are typically required to develop practices to ensure the regular and suitable accomplishment of those tasks.

The nature of the *practice suggested by a path* may depend on different elements: for instance, the nature of the tasks forming the path (e.g., the way the tasks are phrased or the kinds of objects they concern); the context within which the different tasks take place (e.g., tasks occurring on past exams may have a greater influence than tasks occurring in assignments); what is made explicit about the tasks (e.g., steps and discourses present or absent in teachers' solutions); or who is observing the tasks (e.g., a researcher, a teacher, or a student). It is possible that different students see different paths or develop different practices when engaging in the same given tasks (we return to this idea in Sects. 7.4 and 7.5). Observing from our perspective as researchers, we found the idea of a path to a practice helpful in framing the way we collected and analysed data.

7.3 Methodology

Our study focussed on a first Analysis course (A1) offered at a large North American university. A1 is a mandatory course for mathematics programs leading to graduate work (e.g., in statistics or pure mathematics). It is typically preceded by courses in single variable and multivariable Calculus, and followed by a second course in Analysis (A2). Together, A1 and A2 form an introduction to Analysis of single variable real-valued functions. Most topics are identical to those in single variable Calculus courses (e.g., limits, continuity, derivatives). The difference is the expectation (explicit in curricular documents) that the courses will introduce students to mathematical rigour and proofs.

A1 is an institution in the ATD sense (Sect. 7.2.1). The teacher (typically a full-time mathematics professor engaging in teaching and research) provides 3 h of lecture per week; students are evaluated through weekly assignments, a midterm, and a final exam, with their successful completion of the course significantly determined by their final exam grade[8]; and there is a course examiner who is responsible for ensuring consistency in evaluations across teachers and terms.

To address our two research questions, we used a task-based interview approach (Goldin, 1997, 2000), founded on an a priori analysis of some of the tasks typically offered in A1. Hence, our study proceeded in two stages. The focus of the first stage was an analysis of tasks proposed to students in A1 and the solutions teachers made available to students for studying. The objective of this first stage was two-fold. On the one hand, we expected to test our capacity to predict, based on previous research and on the tasks and solutions proposed to students, what (non-)mathematical[9] practices students would develop (based on our analysis of the tasks, we would model practices we expected students to develop – for further clarity, see footnote

[8] At the time of our study, students were evaluated by taking the best of two possible distributions: 10% assignments, 30% midterm, 60% final exam or 10% assignments, 90% final exam.

[9] We use "(non-)mathematical" to mean "non-mathematical or mathematical."

10). On the other hand, the analysis of tasks offered to students in A1 was key to the creation of task-based interviews that would elicit students' developed practices. The objective of the second stage of our study – the creation, implementation, and analysis of task-based interviews – was to build models of practices actually developed by students and to reflect on the (non-)mathematical nature of these practices. We present relevant details of our methodology below (for more details, including a thorough description of the methodological approach and illustrative examples, see Chap. 4 in Broley, 2020).

In the first stage of our study (described in detail in 4.1 in Broley, 2020), we analyzed over 200 tasks listed in assessment documents provided to students in A1, including the weekly assignments, midterm, and final exam posed in a particular iteration of the course, as well as midterms and final exams from previous iterations that students were given to guide their studying. While our research questions refer to the practices developed by students in A1, our analysis was focussed exclusively on the tasks presented in the documents listed above (as opposed to considering all the tasks offered to students, including, e.g., in lectures). We considered these tasks sufficient for our objectives for several reasons: Past research has shown the potentially strong influence of assessments on the practices students develop (e.g., Cox 1994; Hardy 2009); and in the course we studied, tasks that will be tested appear with high frequency in assessment documents (assignments, midterms, and final exams) and study guides (the textbook and solutions to tasks provided to students), which, we conjecture, drives students towards the development of practices that will be tested.

In our analysis of the tasks, we sought to identify those that relate to the same type of task and exist in relatively high quantity – the tasks in a path to a practice (Sect. 7.2.2). If tasks relating to a certain type of task occurred in low quantity and only on assignments, we considered them to be isolated; otherwise, we considered them as forming paths to practices. To characterize the practices that we expected students to develop (what we will refer to as the "suggested practices," from our perspective as researchers),[10] we built praxeological models, including specific characteristics of the tasks and teachers' solutions that we conjectured (based on previous research) might have shaped students' practices (e.g., we recorded whether tasks concerned particular kinds of objects and which theoretical elements were explicit in teachers' solutions). We then selected a subset of paths to practices on which to base our interview tasks, for different reasons: e.g., we chose a variety of paths (in terms of topic) to explore patterns or differences in students' practices; our interest in the evolution of students' practices from Calculus to Analysis also led us to favour paths (and eventually tasks) that students may link to practices developed in prior Calculus courses. Our six interview tasks can be found in Appendix A in Broley (2020).

[10]To be clear, we are not referring to the expectations that the institution or the teacher may have, which may well be that students learn mathematical practices. We are referring to our expectations as researchers critical of the tasks being proposed. Based on previous research, we expected students to develop some non-mathematical practices (e.g., focusing on superficial, non-mathematically relevant features of highly frequent tasks).

Complete results of this stage of our study can be found in Chap. 6 (6.i.1, i = 1 to 6) in Broley (2020). In Sect. 7.4.1, we present the suggested practice associated with our second interview task (T2): Show that the function $f(x) = e^x - 100(x - 1)(2 - x)$ has 2 zeros. We selected this task as the focus of this chapter since its results illustrate well our approach and the different kinds of (non-)mathematical practices we found students may develop.

The second stage of our study (described in detail in 4.2 in Broley, 2020) focussed on the design, implementation, and analysis of our task-based interview. This kind of interview was fitting for our objectives since it could allow us to observe students as they enact practices to solve given tasks. We designed the interview tasks and protocol with the goal of eliciting students' practices and revealing their nature. Key to the design was achieving *recognizability* and *deception*: students needed to recognize the interview tasks as being solvable using practices they had developed in A1; they also needed to be potentially deceived by some element of the task so that any non-mathematical nature of their practices would be revealed. Generally speaking, we chose interview tasks that mirrored, but also differed in some significant way, from tasks within the paths selected from the first stage of our study (in Sect. 7.4.1, we give the example of T2). Once the tasks were chosen, following Goldin's (1997, 2000) principles, we created a protocol (Appendix A in Broley, 2020), which outlined the rules of interaction between the interviewer, an interviewee (a successful A1 student), and the tasks. After receiving a task (printed on the top of a blank sheet of paper), an interviewee had as much time as possible[11] to engage in independent task solving, thinking aloud and using the tools made available to them (paper, a pencil, and a scientific calculator). If the interviewee struggled to engage with a task, the interviewer offered heuristic suggestions that became progressively more directive as needed (potential suggestive questions were created for each task and can be found in the protocol). At the end of an interviewee's task solving attempt, the interviewer asked follow-up questions with our objectives in mind (e.g., it was not important for the interviewee to develop a final polished solution; but they were encouraged to clarify the approach they took or would take for solving the given task, and why, which was crucial for modelling their practices). We conducted two- to three-hour interviews with 15 students (S1 to S15) after they successfully completed A1.

Our analysis proceeded in several steps. First, audio recordings of the 15 interviews were combined with participants' written work to create verbatim transcripts. Second, for each participant and each task, we created a table with three rows, where we recorded observations from the participant's transcript that would help us infer the different components of their practice(s): the type(s) of task(s) identified, the technique(s) selected and implemented, and the discourses used to describe how and why the technique(s) work, including any acknowledgement of underlying theory. For example, in the row corresponding to technique(s), we synthesized the steps the

[11] The time available for solving a given task was constrained by the planned duration of the interview (2 h) and the priority of observing a participant formulate at least one approach, and a reason for the approach, for each of the six interview tasks.

participant took to solve the task. Third, for each task, we then used the tables to categorize participants according to criteria that emerged as we read the tables and thought about our objectives. Criteria varied across tasks and were not limited to "task(s) solved," "technique(s) considered," or "technologies/theories referred to": e.g., for T2, the criteria also included "the first thing a participant spoke about or did upon receiving the task" and "how they chose which x values to plug in (to locate sign changes in f)." Using this categorization, we engaged in a fourth step, writing about patterns in participants' task solving: i.e., how they identified types of tasks, selected and implemented techniques, and described how and why those techniques worked. Finally, we created models of the practices enacted by the students, which we used to reflect on their nature (using the lens elaborated in Sect. 7.2.1), and how they may have been shaped by the tasks offered in the course (by comparing them to our models of suggested practices). In Sect. 7.4.2, we present some of the results of this analysis, exemplified in relation to T2.

Before presenting the results, it is important to address the fact that we model students' practices – regularized and purposeful actions (Sect. 7.2.1) – based on a solution to *one* task of a certain type. Since we interviewed students at the end of the course, we expected them to have developed regularized and purposeful actions for solving potentially evaluated tasks. Our interview tasks were designed to trigger such practices, and the deceptive nature of the tasks meant that when a practice did not work (to solve the task), the student was forced to explain it. Moreover, a student would often exhibit specific cues that their behaviour was indicative of a practice (see footnote 7). For instance, they would describe their approach in a general sense (i.e., not specific to the given task), or they would say things like "I am going to use the method learned in class," "I've repeated this approach so often," or "I usually do it this way" (as exemplified in Sect. 7.4.2). This said, we recognize that there may have been times where identified "practices" were "potential" and could have been more "practices in development, in adaptation, or in evolution." This is a complex issue and an interesting direction for future work.

7.4 Results

In our analysis, we found that participants' practices were (non-)mathematical in different ways. We also observed variability in the ways in which participants' practices could be linked to our models of suggested practices[12] (and, by extension, the assessment tasks that had been offered in A1). The next sections exemplify these results using our second interview task (T2): Show that the function $f(x) = e^x - 100(x - 1)(2 - x)$ has 2 zeros. We first present our model of the suggested practice associated with T2 (Sect. 7.4.1). Then we present our analysis of a selection of practices enacted by participants for solving T2, and their links to the suggested practice (Sect. 7.4.2).

[12] See Sect. 7.3 for the meaning of *suggested practice* in the context of this study.

7.4.1 Suggested Practice Associated with T2

The first stage of our methodology (Sect. 7.3) was an analysis of assessment tasks and teachers' solutions to those tasks offered to students in A1, which involved an identification of paths of tasks and a characterization of practices suggested by those paths. Table 7.2 depicts our model of a practice suggested by one of the paths we identified. The model is founded on the type of task: T, Prove that a function $f(x)$ has exactly n zeros on an interval I. Examples of tasks belonging to the path are shown in Table 7.1. Teachers' solutions to those tasks (which they made available to students for studying purposes) suggested that T be split into two sub-tasks: T_a, prove that $f(x)$ has *at least n* zeros on I and T_b, prove that $f(x)$ has *at most n* zeros on I.

In teachers' solutions, the most common technique for showing that f has *at least n* zeros was to locate n sign changes (τ_a). Teachers' solutions did not consistently include justifications beyond "by the Intermediate Value Theorem" (θ_a). None of such solutions commented on the usefulness of the IVT (e.g., "because the zeros of f cannot be found analytically") or on how the IVT works (e.g., "if f is a continuous function on an interval $[a, b]$ and $d \in (f(a), f(b))$, then there is some number $c \in (a, b)$ such that $f(c) = d$"). Also, the continuity condition necessary for applying the IVT was not always mentioned or justified in teachers' solutions. Moreover, these solutions did not elaborate on how students should look for sign changes (only listing the values of $f(x)$ that proved the existence of the sign changes), and the intervals and functions were always of a type such that sign changes could be easily found (by plugging in the endpoints of the interval, normally integers, and possibly some points in between, normally also integers and/or the midpoint of the interval). Accordingly, we wondered if students would have developed a non-mathematical practice and we constructed T2 in attempt to reveal this. We did not specify an interval, and we constructed $f(x) = e^x - 100(x - 1)(2 - x)$ so that plugging in integer values for x would lead to only positive values for $f(x)$ and, thus, would not be enough to locate sign changes (this was part of the deceptive nature of the task; see Sect. 7.3). In the absence of an interval, we expected students to work with the domain of definition of the function (i.e., to assume $I = (-\infty, \infty)$).

Note that with the way we phrased T2, we expected the participants of our study to identify it with T_a and for the interviewer to pose a follow-up question asking if participants' approaches would be different if they needed to show that the function has exactly two zeros. There was potential for a variety of responses. Indeed, in contrast with T_a, and as portrayed in Table 7.2, there were several techniques ($\tau_{b_1}, \tau_{b_2}, \tau_{b_3}$) illustrated in teachers' solutions for showing that f has *at most n* zeros. These techniques were illustrated on different subsets of T_b: e.g., when $n = 2$, as in T2, teachers' solutions suggested that students should argue by contradiction, assuming the function has 3 zeros, applying Rolle's Theorem twice to find that f'' should have a zero, and then calculating f'' to find that it actually has none.

Finally, within the path of assessment tasks related to T, we identified three equivalent task types (Table 7.1 shows some related tasks):

Table 7.2 Our model of a practice suggested by a path of assessment tasks offered in A1, based on the nature of the tasks themselves, as well as the techniques illustrated and technologies made explicit in teachers' solutions. The notation used aligns with that in the concept of praxeology: T for types of tasks, τ for techniques, and θ for technologies

Three equivalent types of tasks, to be solved by transforming them into a task of type T_2:

$\dot{T}$: Prove that $g(x)$ has exactly n fixed points on an interval I.

$\widetilde{T}$: Prove that $g(x) = h(x)$ has exactly n solutions on an interval I.

$\widehat{T}$: Prove that g and h intersect exactly n times on an interval I.

The main type of task, to be solved by solving two sub-tasks, T_{2_a} and T_{2_b}:

$\boldsymbol{T}$: Prove that a function $f(x)$ has exactly n zeros on an interval I.

Typically: $n \in \mathbb{N}$ is small (1, 2, 3, or 4) and I is of the form (a, b) or $[a, b]$ with $a, b \in \mathbb{Z}$.

The main technique and technology for solving sub-task T_{2_a}:

$\boldsymbol{T_a}$: Prove that $f(x)$ has at least n zeros on I.

$\boldsymbol{\tau_a}$: Find n sign changes of $f(x)$ on I.

Typically: calculate $f(a)$ and $f(b)$, and maybe $f(c)$ for c equal to integers in (a, b) or midpoints between integers.

$\boldsymbol{\theta_a}$: "By the Intermediate Value Theorem."

The three main techniques and technologies for solving sub-task T_{2_b}:

$\boldsymbol{T_b}$: Prove that $f(x)$ has at most n zeros on I.

$\boldsymbol{\tau_{b_1}}$: Show that f' is strictly positive (or negative) on n intervals I_i that form a partition of I. Illustrated for $n = 1$: Show that f' is strictly positive (or negative) on I.	$\boldsymbol{\tau_{b_2}}$: Assume that f has $n + 1$ zeros and derive a contradiction. More specifically, argue that f' has n zeros, f'' has $n - 1$ zeros, …, and f^n has 1 zero; and show f^n has no zeros. Illustrated for $n = 2$: Assume that f has 3 zeros, whereby f'' has 1. Show that f'' has no zeros.	$\boldsymbol{\tau_{b_3}}$: Illustrated for f a polynomial: Note that the degree (or order) of f is n.
$\boldsymbol{\theta_{b_1}}$: "If $f' > 0$ (or < 0) on an interval I, then f is strictly increasing (or decreasing) on I and can cross the line $y = 0$ at at most one point."	$\boldsymbol{\theta_{b_2}}$: "By Rolle's Theorem and by contradiction."	$\boldsymbol{\theta_{b_3}}$: "If f is a polynomial of order n, then f has at most n zeros."

1. $\dot{T}$, Prove that a function $g(x)$ has (exactly) n fixed points on an interval I.
2. $\tilde{T}$, Prove that an equation $g(x) = h(x)$ has (exactly) n solutions on an interval I.
3. $\widehat{T}$, Prove that two functions, g and h, intersect (exactly) n times on an interval I.

In teachers' solutions, tasks of these types were solved by transforming them into a task of type T (see the first and second rows of Table 7.2). For example, to prove that $e^x = 100(x - 1)(2 - x)$ has exactly 2 solutions, students were shown to introduce a new function, $f(x) = e^x - 100(x - 1)(2 - x)$, and to argue that f has exactly 2 zeros using the techniques mentioned above. None of the teachers' solutions leveraged the equivalence in the other direction (e.g., thinking about the intersections of $g(x) = e^x$ and $h(x) = 100(x - 1)(2 - x)$ could lead to a graphical solution for T2). Hence, we did not expect students to spontaneously construct such a solution.

7.4.2 *Practices Enacted by Participants for Solving T2*

The second stage of our methodology (Sect. 7.3) involved the implementation of a task-based interview, including T2 (Show that the function $f(x) = e^x - 100(x - 1)(2 - x)$ has 2 zeros.), with 15 students who had successfully completed A1. In what follows, we provide selected examples of our analyses of the interview data, to illustrate three different ways in which students' practices revealed themselves to be (non-)mathematical: how students identified T2 with a type of task and technique (Sect. 7.4.2.1), how students implemented their chosen technique for accomplishing T2 (Sect. 7.4.2.2), and how students explained their chosen technique for accomplishing T2 (Sect. 7.4.2.3).

7.4.2.1 The Identification of T2 with a Type of Task and Technique

When presented with T2, eleven[13] out of 15 participants almost immediately indicated that they would use the IVT. For example, S4's first words after receiving the task were: "Ok. I remember this being with the Intermediate Value Theorem." In our analysis, we found examples of participants who seemed to be drawn to the word "zeros" as the way of identifying T2 with a type of task necessitating the use of the IVT. For instance, S6 explained:

> Show that it has zeros is IVT for sure. [...] like if it's a continuous function, [. . .] you plug in some values, you get a negative, then positive, then negative, it must cross the [the x-axis], at some point, it does have a zero.

[13] Of the four participants who did not immediately speak of using the IVT, one (S15) spoke about needing to use a "theorem" but could not remember which one, one (S2) immediately took the derivative of f, and the other two were S9 and S3 mentioned below.

S8's actions and utterings also suggested that to show a function has zeros, one applies the IVT: "I understand that the IVT works like that. [. . .] If I find one that's positive and one that's negative, I [can] find a zero." S11 explained similarly:

> My logic with finding zeros is finding a value before and finding a value after that point at which it's equal zero that are alternating signs. And the only theorem that we have that talks about that is [. . .] the Intermediate Value Theorem. So that's why I instantly thought of that.

These students did not stop to reflect on the properties of the function in T2 to inform their decision to use the IVT, their focus seeming to be exclusively on the task being about "zeros." It is in this sense, or for this reason, that we consider that they were enacting a non-mathematical practice. We note, importantly, that in the context of the assessment tasks given in A1, all tasks about "zeros" could be easily solved using the IVT (Sect. 7.4.1).

In comparison, there were examples of participants who considered the nature of the function $f(x) = e^x - 100(x - 1)(2 - x)$ in T2 to support their choice of an IVT-inspired technique and did not focus solely on the fact that it was a task about "zeros." After being triggered to use the IVT, S1 stopped to note about f: "if e^x wasn't here, it'd be pretty easy to find the two zeros. But since there's [e^x], we have to do the non-high school way," meaning, as S9 did, that one could not algebraically solve the equation $f(x) = 0$. S9 said: "It's not as simple as just isolating x. [. . .] So, in this case, we have to use one of those theorems we saw in [A1]." These participants showed some awareness that choosing an IVT-based technique is appropriate for tasks involving a function whose zeros cannot be found using other, simpler, analytical or algebraic techniques. These students were considering the task in its entirety and not focusing exclusively on the fact that it was about the zeros of a function. We consider their identification of the task to be mathematical (as opposed to the students referred to in the previous paragraph[14]).

S3 also considered the nature of the function in T2: "a classic example where you cannot use [. . .] easy things to find the root." The difference with S3, when compared to all other participants, is that he transformed T2 into the equivalent tasks "show that $e^x = 100(x - 1)(2 - x)$ has two solutions" and "show that the graphs of e^x and $100(x - 1)(2 - x)$ have two intersections"; and he developed an unexpected (see Sect. 7.4.1) solution based on proving the properties shown in his sketch (Fig. 7.1). We infer that S3 identified T2 with three equivalent types of tasks (about zeros of functions, about solutions of equations, and about intersections of graphs) and chose a technique based on essential mathematical properties of $f(x) = e^x - 100(x - 1)(2 - x)$; namely, that it is the sum of two functions whose graphical properties are known (to A1 students). This, we concluded, was indicative of the development of a mathematical practice.

[14]Our judgement is that their identification of the task exclusively on the fact that it is about the zeros of a function is not worthwhile from the perspective of scholarly mathematics.

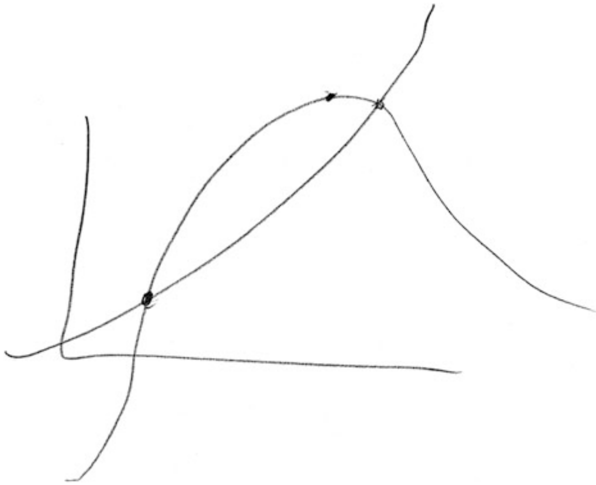

Fig. 7.1 S3's reinterpretation and solution of T2 as a task about intersections of graphs

7.4.2.2 The Implementation of a Technique to Accomplish T2

Of the twelve participants who tried to implement an IVT-based technique (i.e., finding two sign changes in $f(x)$), eleven eventually struggled to complete the task, facing the expected challenge (described in Sect. 7.4.1) of finding only (or mainly[15]) positive values for $f(x)$. All eleven of these participants seemed to, at some point, choose x values "at random" (perhaps considering ease of calculation or variance in chosen values), with several explicitly indicating taking this approach. In our analysis, we found examples of participants who seemed to be choosing their next step in carrying out their technique simply by trying to remember what had worked when solving tasks from A1. After checking the limits of f at infinity,[16] S1 said: "I'd like to see if there's a negative. [. . .] So, I'd just try random numbers." He used his calculator to do so (e.g., calculating $f(0)$, $f(100)$, $f(-5)$), finding only positive values, and explained his choice to go "at random" by saying that he "forgot the better way." As another example, S11 used a calculator to find $f(0)$, $f(1)$, and $f(-1)$, and explained: "usually what we saw in [A1] was that. . . [. . .] the interval in which [the function] is alternating between negative and positive [values] is like somewhere in a close range of zero." We infer that S11 was selecting x values, not by reasoning about the mathematical properties of the given function f, but based on his memory of the kinds of x values (close to zero) that had resulted in sought-after sign changes when solving tasks in A1. S11 later described a more specific list of steps he would have expected to work had T2 included the specification of an interval (like in the assessment tasks from A1; see Sect. 7.4.1):

[15] Some participants found negative values for f(x) due to calculation errors.

[16] This is not something we had anticipated based on our model of a suggested practice and so we do not know where this first step came from. Since S1 was not the only one to do it, perhaps it was shown to students in lectures.

> Like if you tell me [. . .] it's not this function, it's another function,[17] and you tell me [the interval is] zero to five [writing [0,5]], then at that point you can just plug in the values [...] Zero, one, two, three, four, five. [. . .] And you'll see which one alternates between negative and positive. And you'll figure out how many zeros you have.

The steps S11 described would have worked to solve assessment tasks in A1, but they do not make sense from a mathematical point of view in the context of T2. Implementing the IVT in this way illustrates another way in which practices can be non-mathematical.

In response to their struggle, nine participants (including S11) eventually indicated a (possible) change of approach to looking for sign changes in *f*, based on reasoning about mathematical properties of the given function (Table 7.3).[18] Still, there are interesting differences in the nature of these approaches. For example, (1) and (2) in Table 7.3 rely on local studies of the function's monotonicity to make predictions about whether it will change sign somewhere nearby (they involve a quantitative study of *f* that does not take advantage of its essential features). In comparison, (3), (4), and (5) are based on a qualitative study of *f* to try to understand its global behaviour, although (3) (like (1) and (2)) still includes a degree of arbitrariness in the choice of *x*. Only one participant (S12 – see (4) in Table 7.3) implemented the IVT-based technique solely by performing a qualitative mathematical study of *f* (i.e., by reasoning mathematically[19]). This, we concluded, was indicative of the development of a mathematical practice.

7.4.2.3 The Explanation of a Technique for Accomplishing T2

In our analysis, we found examples of students who seemed to explain their IVT-inspired technique based solely on the technique being a normal part of what occurred in A1 (what we refer to as "established experiences" from the learning environment, following Lithner, 2000). In reference to his use of the IVT for solving T2, S1 explained: "I know that [the IVT is] applicable in this situation.[20] [. . .] Why do I know? Well, I'm cheating. Cause I know that that's how we used to solve it [in A1]. [. . .] Cause we did it in class." S4 said similarly: "It's just having repeated it so often, whether it be assignments, class, practice, . . ." From this, and the interactions that occurred during the interviews, we interpret that S1, S4, and other participants did not actually know why the IVT-inspired technique solves T2:

[17] S11 made the specification that "it's not this function, it's another function" when giving the example of the interval [0, 5] because he had already tried plugging in $x = 0, 1, 2, 3, 4, 5$ and had not found the two zeros. This said, the zeros for $f(x)$ do indeed occur on [0, 5].

[18] This may be an example where students were exhibiting "practices in development" (see the last paragraph of Sect. 7.3).

[19] As in the use of the adjective "mathematical" in this study, "mathematically" here refers to a way of reasoning that is acceptable and worthwhile by the institution of scholarly mathematics.

[20] S1 was one of six participants who did not mention the continuity condition required for applying the IVT during his solving of T2.

Table 7.3 Models of participants' approaches and reasoning for choosing x values to find sign changes in $f(x) = g(x) - h(x)$, where $g(x) = e^x$ and $h(x) = 100(x - 1)(2 - x)$. * indicates the participant only described (did not try) the approach. Bold indicates the participant successfully solved the task using the approach

Approach	Participants	Examples of Reasoning (after finding only positive values for $f(x)$)
(1) Pay attention to how the value of $f(x)$ is changing as x changes.	S4, S5, **S6**	S6: If f is continuous and I see that $f(x)$ is getting closer to zero as I change x, then I'm getting closer to finding a sign change.
(2) Study the sign of $f'(x)$ for particular x.	S10, S11*	S11: If you take the derivative at different points, you can see if the slope of the tangent line is negative. Then you know the function is decreasing at that point. And you might want to check the intervals around that.
(3) Compare the growth of $g(x)$ and $-h(x)$.	**S8**, S11*, S14*	S8: Since $g(x) = e^x$ grows much quicker than any polynomial (like $-h(x)$), the sign changes will occur for small values of x.
(4) Compare the signs of $g(x)$ and $-h(x)$.	S4, **S5, S9, S12**	Since $g(x) = e^x$ is always positive, we need to look for where $-h(x)$ is negative: • S4: Expanding it to $100x^2 - 300x + 200$ shows that x needs to be positive. • S5: Looking at $-h$, we see that if x is more than 1 and less than 2, the negative signs won't cancel out. • S9, S12: $-h$ is an upward-facing parabola with roots at 1 and 2, and minimum (or most negative point) at 1.5.
(5) Graph g and h.	**S7**	Sign changes will occur where the graphs of g and h cross. In this case, the graphs are known: It is just e^x and a downward-facing parabola with roots at 1 and 2, and maximum at 1.5.

They know it is a task about "zeros," they know one applies the IVT in that case, and they follow learned steps to implement the technique. Their logos is of non-mathematical nature, hence illustrating another way in which students' practices may be non-mathematical.

While solving T2, twelve participants eventually considered the task of showing that $f(x) = e^x - 100(x - 1)(2 - x)$ has *at most* two zeros (T_b). Several techniques were exhibited, reflecting the diversity in the suggested practice (Table 7.2). This contributed to enrich the collection of examples of what we deemed (non-) mathematical explanations of selected techniques. For instance, to solve T_b, five participants (S1, S2, S7, S13, and S15) considered using Rolle's Theorem (RT) or exhibited a technique based on it (somewhat, though not exactly, reflecting τ_{b_2} in Table 7.2), for which the underlying explanation seemed to be limited to citing the theorem (void of understanding what the theorem says or how it can be used to afford a technique). S15 recalled the complete statement of RT, but could not see how to use it to produce a technique for solving T2. In comparison, S1, S2, S7, and S13 chose to show that $f'(x) = e^x + 200x - 300$ has exactly one zero, based on "a theorem." According to S1, the theorem says that "if the derivative [function] has one [zero], [. . .] the [function] has at most two [zeros]." According to S2: "It says that if you have n zeros for $f(x)$, then [. . .] you have $n - 1$ zeros for the derivative." No participant provided a mathematical explanation connecting these two statements (i.e., why, mathematically speaking, RT – or a generalized version of it – produces the technique). We infer that the students' references to RT (or "a theorem") were disconnected acknowledgements of a piece of theory, which remained a static part of their practice that they were not able to use. This is what we mean by explaining a technique based on inert knowledge; another example of how an explanation for a technique may be non-mathematical. As in the example above, this kind of explanation ("by a theorem") aligns with our model of the suggested practice in relation to T2 (Table 7.2).

In contrast, we found examples of participants who seemed to understand and use elements of mathematical theory to produce and explain a technique for solving T_b. Five participants seemed to solve T_b by implicitly or explicitly turning to theorems about what f' or f'' tell us about the shape of f's graph. S12, for example, devised a technique reflecting τ_{b_1} (Table 7.2): Expecting f to have a global minimum (based on his previous work, including a sketch of f), S12 planned to locate the minimum by finding x_m such that $f'(x_m) = 0$; and then argue that $f'(x) < 0$ (f is strictly decreasing) on $(-\infty, x_m)$ and $f'(x) > 0$ (f is strictly increasing) on (x_m, ∞). S12 got stuck implementing his technique when he realized he could not analytically solve $f'(x_m) = 0$. This said, he gave the following mathematical explanation for how and why the technique worked:

> If a function is [. . .] increasing strictly, it means that [. . .] if I have two points, a and b, where $a < b$, then [. . .] $f(a) < f(b)$. So, if I have some point that is a zero, [say b], [the value of f at] any point that is greater than b is going to have to be greater than zero. So that shows that no value c greater than b is actually going to give something that's a zero in our function. Similarly, no value less than b will give us a value of zero. [. . .] The same is true for decreasing functions.

In this argument, S12 does not rely on his personal understanding alone; rather his understanding seems to be shaped by the mathematical theory of functions (e.g., the definitions of increasing or decreasing functions and the definition of a zero of a function). This is an example of what we mean by clarifying, questioning, and verifying one's own understanding with mathematical theory, which we see as one way in which students' practices may be mathematical.

7.5 Discussion

At the beginning of this chapter, we posed two research questions:

1. What is the nature of the practices developed by students in a first Analysis course?
2. How might these practices be shaped by the nature of the tasks offered to students in the course?

Considering the above results, we discuss possible elements of response, critically reflect on our study (its contribution and limitations), and propose some directions for future work.

7.5.1 Answer to the Research Questions and Contribution of the Study to Research in University Mathematics Education

Our study involved task-based interviews (see Sect. 7.3) with 15 students after they successfully completed a first Analysis course. In our analysis of these students' task solving, we found examples of practices that were not mathematical (see Sects. 7.2 and 7.4.2). These kinds of practices have been identified in research on Calculus courses (e.g., Hardy, 2009; Lithner, 2000; Orton, 1983; Selden et al., 1994). Given the procedural focus of those courses, the development of non-mathematical practices is perhaps not surprising. It is surprising, however, that students may still be developing such practices in more advanced theoretical courses such as A1, often taken in the second-last year of mathematics programs leading to graduate work. Some studies (e.g., Weber 2005a, b) have hinted at this possibility; our study contributes a focused theoretical and empirical exploration of this issue. Using the notion of "path to a practice" (see Sect. 7.2) contributed by our study, we conjecture that the development of non-mathematical practices may be permitted and encouraged (for any student) by paths of tasks that do not help students to identify relevant mathematical features of the tasks, and where it is not necessary to learn how to mathematically explain a technique for a mathematical type of task (e.g., the path described in Sect. 7.4.1).

This said, we also found that some students enacted practices that, while non-mathematical from the perspective of this study, could be considered mathematical in some way (e.g., the student is paying attention to mathematically relevant aspects of the task to choose a technique, but does not have mathematically sound discourses; or vice versa). This could be empirical evidence of students going through the expected shift (Winsløw, 2006), from a more procedural focus (encouraged in Calculus) to a more theoretical focus (encouraged in Analysis).

The differences we found in the nature of students' practices also seemed to reflect different ways in which students' practices may be linked to (or influenced by) the assessment tasks given in A1 (as suggested by comparing results from Sects. 7.4.1 and 7.4.2). This may further reflect our expected differences in "a practice suggested by a path" depending on the observer (see the last paragraph of Sect. 7.2.2); that is, it is possible that different students abstracted different practices from the paths that we identified, or that they made different connections among tasks than we did (forming different kinds of paths). We have begun trying to make sense of this by characterizing different general ways in which students may position themselves within their courses (e.g., Broley, 2021).

7.5.2 Limitations and Directions for Future Research

It is interesting to note that the notion of "path to a practice" arose within our study context, which followed the paradigm of "visiting works" (Chevallard, 2015). As evidenced by Section 4 of this book, this paradigm is being challenged by innovative approaches such as inquiry-based mathematics education (Artigue & Blomhøj, 2013). Recently arising in the ATD is a variation that proposes to organize learning around another kind of "path": *study and research paths*, which start with an open-ended question that the teacher and students seek to answer through studying existing works and researching new questions (e.g., Florensa et al., 2019). One direction for future work could be to analyse the nature of the practices students develop while engaging in such paths and to theoretically reflect on how they relate to the notion of "path to a practice."

Our study had limitations, which point to other future directions. For instance, we analysed only some of the tasks offered to students: Future work could look at the role, if any, of the tasks students encounter outside assessment or teachers' lectures in the paths they identify for themselves, or the influence of tasks (or paths) from other courses. Another limitation is that our task-based interview was not designed to distinguish between different practices' *states*; e.g., "practices in development," "practices in adaptation," or "practices in evolution," which could be the focus of future work.

Based on the conclusions of our study, namely that students develop non-mathematical practices, another future direction could be design-based research to create and evaluate learning experiences for the development of mathematical practices. The examples in this study – of students' non-mathematical practices – could inform the design of tasks for that purpose. More detailed analyses of how students form paths and abstract practices could also provide interesting and important empirical and theoretical insights.

Taking an institutional point of view reminds us of the complex web of constraints faced by teachers and students (examination procedures, time limitations, curricular expectations). One participant from our study gave a poignant reflection, highlighting how the larger context may encourage the development of non-mathematical practices:

> I feel that we're grinded to do so many questions really quickly. So, we need to associate problems to a solution [...] really fast. [...] Cause I don't really have the time to analyze the problem and try different things during an exam. So, I grind problems at home. And when I get in an exam, I see the problem and I say, "Ok, that's exactly the kind of problem. . .it goes down to this."

Although we claim that one overall aim of university mathematics is the eventual development of practices reflecting the aimed (mathematical) profession, a pertinent question raised by our study, and many others before ours (some cited here), is: To what extent is this aim achievable under existing constraints?

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Chapter 8
The Mathematical Practice of Learning from Lectures: Preliminary Hypotheses on How Students Learn to Understand Definitions

Kristen Lew, Timothy Fukawa-Connelly, and Keith Weber

Abstract In this theoretical chapter, we argue that when mathematicians lecture, they not only cover mathematical content, but also model how student should learn mathematics. We analyze a corpus of eleven lectures in a variety of advanced mathematics courses to investigate ways in which mathematicians present the definitions of concepts and gain insight into how mathematicians may expect students to learn from lectures. We highlight how the instructors modeled what it means to study a concept and its definition and argue that students are expected to engage in independent study outside of class.

Keywords Proof · Lecture-based instruction · Mathematical behaviors · Mathematical practice

8.1 Introduction

Imagine a mathematician about to enter a new area of research. This might be a doctoral student who is beginning her dissertation research or she might be an established faculty member changing her research focus. What sort of activities might this individual engage in to learn about this area of research? If she is like many mathematicians, she would attend a workshop. Mathematical workshops

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typically consist of a series of coordinated lectures that survey the latest developments in important areas of research. These workshops typically have social purposes (such as fostering collaboration), but they also have a cognitive purpose: the participants expect to come to understand some key ideas of the research area better.

A mathematician might also engage in other activities to learn about a new domain of research, such as attending colloquia or auditing a relevant doctoral-level course. Again, the expectation is that the mathematician can learn about a new mathematical domain by attending these lectures. Of course, a mathematician does not learn mathematics *only* by attending lectures. Other activities, such as studying research papers and talking with colleagues who work in the area, are essential as well. We are not claiming that attending lectures is sufficient to master a new mathematical domain, only that mathematicians regularly use lectures as a key resource for learning new mathematics. Learning from lectures is an important mathematical practice.

Some readers may be surprised to find a chapter on lecturing in an edited volume on practice-oriented research in tertiary mathematics. After all, lecture-based instruction is sometimes viewed as the antithesis of instruction that engages students in authentic mathematical practices. There is good reason for this sentiment. In a typical lecture-based course, students are given little opportunity to engage in core mathematical practices such as problem solving and proving during their course meetings. Indeed, some practices such as conjecturing and defining are given scant attention at all (c.f. Johnson et al., 2018). We agree in tertiary mathematics instruction, there is an overemphasis on learning from lectures, while too little attention is paid to other mathematical practices.

Despite the weaknesses of teaching exclusively via lecture, we advocate for more mathematics education research on the practice of learning from lectures for three reasons. First, as noted, learning from lectures is an important mathematical practice and one we believe is poorly understood. Second, lectures are potentially a valuable resource that students can use to learn mathematics. As Larsen (2017) noted, lectures are appropriate in some contexts, particularly when "students have developed to the point of knowing what it means to do mathematics" (p. 245) and are capable of doing the hard work of studying the ideas from lecture outside of class. That is, lectures might be useful to students who know how to engage in the practice of learning from lectures. Third, for better or worse, most tertiary proof-oriented courses are taught via lecture and we anticipate this will be the case for the foreseeable future. Even if lecturing is limited pedagogically, tertiary mathematics students will experience lecturing frequently in their mathematical development.

In this theoretical chapter we advance the following (speculative) argument: When mathematicians lecture, they not only cover content, but also model how students should learn mathematics; some even give explicit suggestions about how to learn mathematics. We claim by studying how mathematicians model engaging in mathematics, we can gain insight into how mathematicians expect students to learn from lectures, with much of the learning occurring outside of class. We use a corpus of 11 lectures in a variety of advanced mathematics courses, focusing on how mathematicians present the definitions of concepts. We highlight how the instructors modeled what it means to study a concept and its definition, where we argue students are expected to engage in similar study habits outside of class. We conclude the chapter by proposing open questions and future directions for how mathematics educators might investigate the practice of learning from lectures.

8.2 Literature Review

8.2.1 What Do We Mean by Learning from Lectures?

Asiala et al. (1997) defined a students' mathematical knowledge as her ability to productively respond to the mathematical situations that she may encounter. To us, these situations, such as those where students are asked to prove a mathematical conjecture, require having an understanding of the relevant mathematical concepts at play as well as a facility with the mechanics of proving. However, we maintain that productively responding to these situations also involves the development of productive dispositions (e.g., that one should try to prove or disprove open conjectures) and an internalization of sociomathematical norms (e.g., how proofs should be written). Learning mathematics therefore involves developing the skills, understanding, and dispositions and internalizing the sociomathematical norms needed to respond effectively to mathematical situations. Students have many resources with which they can interact to do this, including textbooks to read and colleagues with whom they can interact. An important resource students have is the lectures their teachers provide.

8.2.2 Research on Lecturing in Advanced Mathematics

In the last 10 years, there has been tremendous growth in research on lecturing in advanced mathematics courses. For the sake of brevity, we point the reader to the recent surveys of Gabel (2019) and Melhuish et al. (2022). Below, we highlight five findings relevant to this chapter.

First, although mathematics educators have developed innovative student-centered curricula for teaching advanced mathematics (e.g., Larsen, 2013; Leron & Dubinsky, 1995), most advanced mathematics courses are still taught by lecture. This research finding is supported by both surveys with mathematicians (Fukawa-Connelly et al., 2016) and classroom observations (Artemeva & Fox, 2011; Fukaway-Connelly et al., 2017). Second, advanced mathematics lectures are not strictly formal affairs in which mathematicians present rigorous proofs of theorems. Instead, mathematicians attempt to convey informal content that mathematics educators value, such as providing examples of concepts (e.g., Fukawa-Connelly & Newton, 2014; Mills, 2014), informal representations of concepts, and heuristics for writing proofs (e.g., Fukawa-Connelly et al., 2017). Third, mathematicians are able to provide sensible and nuanced rationales for their pedagogical decisions. This finding is supported both by studies in which mathematicians are asked to discuss specific pedagogical actions they take in their lectures (e.g., Lew et al., 2016; Pinto, 2019; Weber, 2004) and are interviewed more generally about their teaching (e.g., Nardi, 2008). Fourth, there are typically limited opportunities for students to contribute mathematical ideas or engage in authentic mathematical practices during lectures. Paoletti et al. (2018) found that although mathematicians frequently ask questions during their lectures, the questions usually ask students to recall a specific fact or to state the next step in the proof and students are usually provided less than 5 s of wait time to provide an answer even for questions asking for more. Students are rarely given the opportunity to engage in practices like conjecturing or defining during a lecture.

Finally, students learn less than mathematicians and mathematics educators would like in their lecture-based advanced mathematics classes. Students typically emerge from these classes with an impoverished understanding of core mathematics concepts (e.g., Dubinsky et al., 1994; see Rasmussen & Wawro, 2017 for a recent review) and an inability to write proofs (e.g., Mejía-Ramos & Weber, 2019; see Stylianides, Stylianides, & Weber, 2017 for a recent review), which are amongst the primary learning goals of these courses.

8.2.3 The Inadequacy of a Transmission Model of Learning

While mathematics educators have developed a solid understanding of how and why mathematicians lecture in advanced mathematics and amassed extensive evidence that students fail to learn what their instructors desire from their lectures, much less is known about how students interpret lectures and why students fail to learn the desired content in these environments (Melhuish et al., 2022).

Some mathematics educators believe lectures are undergirded by a transmission model of teaching and learning (e.g., Jaworski et al., 2017). According to this pedagogical model, learning occurs via the following progression: First, the lecturer thinks carefully about what content she would like the students to learn and what understandings she would like the students to possess; she then articulates the content in a clear and engaging manner in her class meetings. If the lecturer is effective, the students will interpret the content as intended and thereby acquire the desired understandings.

Some mathematics educators (ourselves included) have claimed lectures do not work in advanced mathematics because the transmission model is not viable. For instance, Leron and Dubinsky (1995) declared “the teaching of abstract algebra is a disaster” (p. 217) due to the prevalence of lecturing. According to Leron and Dubinsky, lecturing is bound to fail because “*telling* students about mathematical processes, objects, and relations is not sufficient to induce meaningful learning (hence the sorry state of affairs even with the best of lecturers)” (p. 241).

Our own published accounts of why lectures fail also implicitly blame the inadequacy of a transmission model. For instance, in Lew et al. (2016), we examined a particular lecture by a professor, Dr. A. We interviewed Dr. A about what mathematical content he hoped to convey and six students about what they learned from that lecture. By exposing the mismatch between what Dr. A was saying and what the students heard, we believed we were exposing the limitations of lectures.[1]

Mathematics educators widely reject a transmission model of learning for teaching students complex mathematical content. However, mathematicians also reject a transmission model. Consider Wu's (1999) remarks on his defense of lecturing:

> Learning mathematics is a long and arduous process, and no matter how one defines 'learning', it is not possible to learn all the required material of any mathematics course in 45 hours of discussion. To make any kind of teaching possible, professors and students must enter into a contract . . . the professor gives an outline of what and how much students should learn, and students do the work on their own outside of the 45 hours of class meetings. Lecturing is one way to implement this contract. It is an efficient way for the professor to dictate the pace and convey his vision to the students, on the condition that students would do their share of groping and staggering towards the goal on their own. It should be clear that without this understanding, lectures would be of no value whatsoever to the students.

Based on this passage, if Wu read Leron and Dubinsky's (1995) claim that telling students about mathematics is not sufficient to induce learning, he would surely agree. However, to Wu, this does not imply that lecture is not a viable mode of instruction. Rather, the lecture is one valuable resource for helping students come to understand the material, but attending lectures alone is insufficient for learning. Wu's viewpoint seems to be shared by most mathematicians. For instance, Weber (2012) interviewed nine mathematicians about why they presented proofs in the classroom. He found mathematicians expected students to spend a substantial amount of time studying the proofs outside of class. More broadly, Fukawa-Connelly et al. (2016) found through a survey that even though most mathematicians believed lecturing was the best way to teach, most mathematicians also believed students learned best when they did work in addition to attending lectures.

In summary, both mathematics educators and mathematicians believe lecture alone is inadequate for students to learn mathematics. Students need to struggle with the mathematical content if they are to develop appropriate understandings. There are certainly practices students might engage in during lectures that would support learning, such as taking notes (c.f. Fukawa-Connelly et al., 2017; Iannone & Miller, 2019) and thinking about the content (e.g., Mueller & Oppenheimer, 2014). These during-lecture practices might be thought of as supporting the outside-of-class struggle with ideas of mathematics Wu sees as central to the pedagogical construct of lectures. At the same time, there is strong evidence that students are not learning the content of lectures. Wu (1999) attributes this failure to students not honoring the didactic contract. We have a more generous account for students: maybe they do not know *how* to honor the contract.

[1] See Krupnik et al. (2018) for a similar study undergirded by a similar methodological logic.

8.2.4 The Importance of Modeling During Lectures

When mathematicians lecture, they are not only doing so to convey mathematical content. They are also modeling how students can successfully do mathematics is a primary reason that they choose to lecture. Consider one mathematician's comment below:

> I do believe that modeling is really important. That is, the instructor should be doing *what the students are going to be expected to do*. And modeling that in real time rather than relying on a text to be the model of 'this is how you write the math, this is how you actually write the mathematics, *I'm just gonna give you the big pictures and here's how you actually do the details*'. That modeling, thinking about the details in real time is important. So I think that is a good thing about the old-fashioned way that we work (Woods & Weber, 2020, p. 7, italics are our emphasis).

This passage illustrates the importance that mathematicians ascribe to modeling during lectures. We use it to highlight three other key points. First, this mathematician expects students to continue working on the mathematics that was presented in the lecture outside of class—the mathematician gives the big picture and the students "do the details" at a later time. Second, mathematicians claim that the purpose of modeling is to illustrate *how* students should actually do the details and, more generally, to make clear what students are expected to do (although the extent that this is done and how students perceive this modeling are open research questions). Third, following Pinto (2019), we view the modeling described above as achieving a meta-level learning goal in which students are taught how to fill in the details missing from lectures. We might say this mathematician wants to make explicit the didactic contract Wu (1999) described. Our claim is that mathematicians model productive practices students might use to develop some types of understanding of content, not that the practices are sufficient for developing complete understanding.

In summary, we conjecture that mathematicians are not only modeling how to do mathematics; they are modeling how students are supposed to learn from lectures. Moreover, mathematicians are modeling the types of activities they want students to engage in outside of class. To learn from lectures then would involve students apprehending and adopting the modeled behaviors. By our definition, "learning from lectures" means students are able to interact with the lectures in such a way that, by engaging in the modeled practices, students should both develop better content understandings and be able to more effectively learn from future lectures.

8.2.5 Goals of This Chapter

In this chapter, we analyze how mathematicians present concepts and definitions in their lectures, focusing on how they model the process of coming to understand these concepts and definitions. The purpose of this analysis is threefold. First, we illustrate how mathematicians are modeling how to learn mathematics. Second, we use these mathematicians' comments to form initial speculations on the mathematical practice of learning from lectures. Finally, we use the insights that we gained from our analysis to pose open questions and propose new research directions.

8.3 Data and Analysis

8.3.1 A Data Corpus of Lectures in Advanced Mathematics

The individual lectures we discuss in this chapter come from a corpus of data we used to analyze the types of mathematical content that mathematicians convey in lectures. A detailed account of the data collection is provided in Fukawa-Connelly et al. (2017); a brief summary of the procedure is provided here.

The data came from eleven 80-min lectures we observed at three large PhD granting universities in the northeastern United States. These courses were all proof-oriented university courses intended for third-year and fourth-year mathematics students from a variety of mathematical subjects, such as real analysis, abstract algebra, linear algebra, geometry, number theory, and set theory. There were between 7 and 25 students attending each lecture with an average of about 17 students per lecture. Each lecturer allowed a member of our research team to audio-record the lecture. A member of our research team also used a Smart Pen to transcribe everything the lecturer wrote on the blackboard. We then created a verbatim transcription of the audio and coordinated it with the board transcription to create a final transcript that included both.

In previous reports, we analyzed the types of formal and informal mathematical content that mathematicians conveyed in their lectures (Fukawa-Connelly et al., 2017), what types of teacher questioning were used (Paoletti et al., 2018), and what metaphors were used (Olsen et al., 2020). In this chapter, and other papers, we identify the lecturers as L#. The numbering scheme is consistent across the published work.

8.3.2 Analysis

The analysis in this chapter is qualitative and interpretive. The purpose of this analysis is exploratory and generative. We will present episodes illustrating the ways lecturers modeled the practice of learning definitions and we will use these to generate research questions and directions for future research in the conclusion of the chapter.

To find the episodes, we looked in our corpus of lectures for instances of mathematicians informing students what they should be doing outside of class when they encounter a new definition. To seek out these instances, we identified cases where the lecturer expressed words or phrases indicating a duty or responsibility, such as “need”, “should”, “have to”, and “obligation”. We then interpreted whether the text surrounding these words or phrases indicated things students ought to be doing when they encountered a definition.

We believe if students noticed and followed their lecturer’s counsel, they would indeed be learning mathematics. We speculate that if the students engaged in the practices emphasized by their lecturers, they would develop a better understanding of the definitions being presented. More importantly, they would be better prepared to respond productively to future lectures they attended. Overall, they would be (developing the mathematical practice of) learning from lectures.

The next section reports some of the responsibilities students had in learning mathematics, according to the mathematicians we observed. Although there were also episodes about understanding propositions and proving, we have chosen to focus on the learning of definition because they allow us to illustrate the phenomena of interest. We do not claim these are representative of all mathematics lectures, or even the other lectures in our corpus. Rather, they allow us to illustrate and analyze how *some* professors model how students might learn definitions.

8.4 Results

8.4.1 When Learning a Definition, One Should Justify Why the Definition Has Desirable Attributes

The first lecture we consider is L1’s lecture on set theory, in which he set the stage for presenting the definition of the cardinality of a set. Early in the lecture, he clearly states the goal of what he will discuss:

> [1] So what I want to discuss here is not so much the definition but as the desire to have a definition which will come later.
>
> [2] So what we want, alright, let’s put it like this.
>
> [3] The relation of equinumerosity is defined using bijections but we want it to mean, we know what we want it to mean.
>
> [4] We want it to mean that A and B have the same size.
>
> [5] So ideally we would like to say what do we mean by the size and then just define this as saying it’s the same.

In the next lines he noted they will replace the term size with cardinality and claimed they would not define how to assign a cardinality in this class. There are several noteworthy comments in this passage. In [1], L1 is quite clear that his learning goal is not to present a definition, but the desire to have a definition. In [4] and [5], L1 states the broad goal that the definition should satisfy: “we would like to say what do we mean by size”. L1 is not setting expectations on how students should learn mathematics, but implicitly suggests some epistemological views on what should be learned and how mathematics is done. According to L1, mathematicians create definitions to capture important intuitive notions.

Later, in his presentation of the definition of addition of cardinal numbers, L1 models the mathematical practice of learning definitions by considering the purposes of various aspects of the definitions and verifying they are accomplished. We present an extended segment of transcript to illustrate this point:

[6] Alright so this is definition 18.6.
Board text: Def. 18.6: $\kappa + \lambda = \ card\ (\kappa \cup \lambda)$
Where $card(\kappa) = k$, $card(\lambda) = l$
And $\kappa \cap \lambda = \emptyset$.
Well-defined: If $\kappa \cong \kappa'$ and $\lambda \cong \lambda'$, where $\kappa \cap \lambda = \emptyset$ and $\kappa' \cap \lambda' = \ \emptyset$,
then, $\kappa \cup \lambda \cong \kappa' \cup \lambda'$

[7] And it's a definition of kappa plus lambda.

[8] And as usual, there will be something to prove in order to justify the definition.

[9] So all these definitions will go something like this.

[10] Kappa is the cardinality of some set, let's say *K*. Lambda is the cardinality of some set, let's say *L*.

[11] What shall we do to *K* and to *L* to get kappa plus lambda?

The key passage of text we wish to highlight here is [8]. After presenting the definition, L1 notes that "*as usual*, there will be something to prove in order to justify the definition."[2] We interpret this as L1 conveying two expectations. The first is that definitions need to be justified. The second is that this might be done by proving something or exploring whether it adequately captures an intuitive concept. The "as usual" may also have been L1 referring to the fact that *he* usually performed such justifications in his lectures. We did not have access to L1's prior or subsequent lectures, so we cannot determine if this is what L1 meant. If he did have this interpretation, this would still corroborate our point that this is a common expectation.

L1 immediately continued:

[12] And you think something like union.

[13] Yeah. So union [inaudible].

[14] So then there is something to think about as a result.

[15] So I'll start out as if I'm being extremely careless.

[16] Kappa plus lambda is defined as the cardinality of *K* union *L* where, you know, I have to write all this stuff down.

[17] This is basically the rules of the game at this point. [writes $\kappa + \lambda = card\ (\kappa \cup \lambda)$, where $card(\kappa) = k$, $\text{card}(\lambda) = l$, and $\kappa \cap \lambda = \emptyset$]

[18] I actually have two small obligations at this point to prove that we're actually making any sense at all.

[19] I guess the obvious obligation is this. I mean, why are we doing this.

[20] If they weren't disjoint, we could get different answers, depending on how many [inaudible] were together.

[21] So what we have to prove is that now that we decided that they should be disjoint, the answer no longer depends on anything.

[22] So what we need to know is, what we need to know is if we make this computation twice using *K* and *L* and then *K*' and *L*' also disjoint, we will get the same answer.

[2] In this passage, and throughout this section, italics will always be our emphasis.

[23] And there's something else we need to check, which is probably different, you can probably think of it.

[24] But in any case, the thing that we obviously need to check is this, so I'll start with that.

For our purposes, the key lines in this transcript occur in [18] and [19]. L1 refers to *obligations* one has when introducing definitions referring to representatives of equivalence classes. We interpret L1's word choice as identifying the need for himself as a learner (and possibly producer) of mathematics to verify the definition captures its intended purpose. As L1 views the intended purpose of cardinality is to capture the notion of the size of sets (see lines [4] and [5] of the transcript presented earlier) and the definition of adding cardinals κ and λ makes reference to sets of size κ and λ, it should not matter which representatives of size κ and λ are chosen. L1 summarizes this idea succinctly in [22], essentially explaining what it means for the operation of cardinal addition to be well-defined in this context. Finally, in [24], L1 again expresses the notion of obligation: "the thing that we *obviously need* to check is this".

A few minutes later, L1 focuses the discussion on the other obligation that he referenced in [23]:

[25] What was the other thing that I didn't pay attention to?

[26] Well, it's kind of silly but in order for this definition to have any meaning at all, I need to verify the following small claim.

[27] When I have two cardinals that I can [with emphasis] take two representatives that are disjoint.

[28] In order to do this, I have to take the two sets and if they're not disjoint, I have to replace them with two sets that are disjoint.

[. . .]

[29] Given a kappa, lambda, there are, I would call them representatives *K, L* with, I mean I was taking this for granted a second ago and it's one of those things that seems kind of obvious.

[30] And it is.

[31] But you have to think of some definite way of explaining it.

In this passage, we again see L1 emphasizing the need to verify the definition is accomplishing its goal. In [25], L1 says "in order for the definition *to have any meaning* at all I *need* to verify the following small claim". The point here is subtle. As expressed in [27] and [28], if the definition of cardinal addition is based on choosing arbitrary representative disjoint sets of given cardinalities, one needs to check that such disjoint sets exist (otherwise the operation would not be defined for certain cardinals). Although it might seem obvious this can be done ([28, 29]), L1 again expresses an obligation in [30]: "You *have to* think of some definite way of explaining it". We note the subject is "you", which may suggest a shared obligation in which students must take responsibility.

In summary, we view L1 as modeling how students should learn a mathematical definition. When a definition is offered, they have the obligation of showing how the definition captures its intended meaning, which usually involves proving the definition satisfies some desirable properties. That proofs *can* serve this purpose has been noted in the mathematics education research literature (see Weber's, 2002, proofs that justify the use of definitions and more generally, deVillier's, 1990, proofs that systematize theories). We interpret L1's work as extending these notions to convey that a learner of mathematics has the obligation to seek out or produce such proofs.

8.4.2 When a New Definition Is Proposed, One Should Actively Explore the Definition

The next lecture we consider is a geometry lecture by L7, in which L7 introduces a formal definition of dilation. Like L1's treatment of cardinality, L7 begins with some introductory remarks about what to expect from her definition before introducing it:

> [1] What do we need in order to define a dilation?
> [2] The scale, a proportion, the constant of proportionality and what else?
> [3] You're probably not thinking of what else because you're always used to thinking of dilations from the origin, but it doesn't have to be from the origin, right?
> [4] You could dilate from any point. So, we need a center of our dilation and we need some constant of proportionality.

Like L1, we see L7 emphasizing obligations. The obligations that L7 describes are different from the ones L1 presents, in the sense that L7's obligations refer to what needs to be contained in a definition, rather than what a lecturer or a student needs to do to justify that a definition is good. In [1] and [2], L7 says that to define a dilation, "we *need*... the scale, a proportion, [and] the constant of proportionality" and in [4], "we *need* a center of our dilation". Rather than offering a completed definition at the outset, we believe L7 is highlighting to her students a definition must offer a complete description of a concept. We suggest this has the following corollary: When students encounter a new concept, they should consider what properties the definition needs to include to capture the intuitive sense of the concept and to be a complete description.

L7 then offers the formal definition: "Let O be a fixed point and K be a real number. The homothety (or dilation) $H_{O,\ K}$ maps O to O and any other point P to a point P' such that $O,\ P,\ P'$ are collinear and $OP' = K(OP)$".[3] Next, L7 asks and sometimes answers a sequence of rhetorical questions about the presented definition:

> [5] So here's my center of dilation, *O*, I have some point *P*.
> [6] Where does the image of *P* go? What do I know about it?
> [7] Ah. It might get closer to *O*, it might get further from *O*, but in general it must be on the same line, right?
> [8] Such that *O*, *P*, and *P'* must be co-linear, and what's the connection between *OP* and *OP'*? It's exactly that *K*.

[3]Here, we include the verbatim definition provided and wonder if K should be a positive real number.

We interpret the second sentence in [8] to indicate that the length of *OP* and *OP'* are different by a factor of *K*. We interpret the rhetorical questions expressed in [6] as modeling the natural questions one might ask to understand the definition of a particular mapping, and could be generalized to a heuristic of 'ensure you understand what the parts of the definition mean'. If so, we might view these as the types of questions students should ask when presented with a concept.

Immediately following this discussion, L7 has the students draw sketches to investigate the impact the value of *K* has on the dilation:

> [9] So now I would like us to do some sketches.
> [10] I almost brought in rulers and I must say after this fiasco I'm really sorry I didn't bring in rulers, however what we're drawing right now it's not quite this complicated.
> [11] So there's some cases to think about here.
> [12] This isn't really enough yet to get some sense of how they work.
> [13] What are our cases that we should think about?
> [14] What are gonna be some key differences? Different kinds of problems? Is every dilation the same? No, it depends a lot on...? [A student says "*K*"]
> [15] *K*, right? Ok, so how can they be different? How does *K* make them different?
> [16] What are some really different, so obviously, every *K* is a different dilation, but what are some really important cases to think about?

In her launch of the sketching task, in [12], L7 motivates the activity indicating the definition itself "isn't really enough yet to get some sense of how they work." This conveys the expectation that understanding a definition involves making sense of how the ideas work. L7 then asks a series of rhetorical questions in [13–16]. Note in [13], L7 expresses one of these rhetorical questions in terms of an obligation: "What are our cases that *we should think about?*" In asking these rhetorical questions, we interpret L7 as modeling mathematical behaviors on two levels. At the broadest level, she is modeling for her students how questions can be used to make sense of the definitions they encounter. Second, she is suggesting, in a situation like this, they *should* consider paradigmatic cases and she is asking students to think about what makes a case paradigmatic. At this point (and at multiple points during the rest of the lecture), L7 laments her decision to not bring rulers to aid her students' drawing. We believe this suggests the importance to L7 that her students engage in developing this example space of dilations with varying *K* values (although we concede this is speculative). The class spends 6 min with the students drawing sketches and discussing the sketches before L7 leads the class discussion to question what is invariant under a dilation.

Based on these interactions, we believe L7 is modeling how to make sense of definitions to her students. It is clear she wants her students to engage in example generation to help them gain a better conceptual understanding of the definition. In the mathematics education literature, Watson and Mason (2005) and others (e.g., Fukawa-Connelly & Newton, 2014) have advocated for students engaging in exemplifying to understand new concepts better. We see L7 as encouraging students to do this in various ways, such as by asking them "what are the cases *we should* think about" and having them generate examples of those cases during lectures. Like L1, L7 is not only teaching about the specific concept of dilations, but also preparing students to productively respond to other definitions they will encounter.

8.4.3 When a New Definition Is Provided, One Should Exemplify this Definition in Many Ways

Our final illustration of a lecturer modeling how to learn is from L8's abstract algebra lecture in which the definition of ideals is introduced. As L8 presents and explains the definition of ideals, she makes a series of declarative statements about her own activities and makes claims for student behavior. She does not frame example generation or questioning the possible meanings of terms generally or state that students should engage in such behaviors when they encounter a new definition. Hence, while L8 is certainly modeling how she gains an understanding of a definition, it is unclear as to whether she is setting expectations for how students should learn outside of class. However, at several points, L8 does explicitly tell students they should consider definitions on their own time.

After presenting the definition of ideals, L8 presents several examples. The examples begin with multiples of five ($I = (5) = \{5k | k \in \mathbb{Z}\}$), the polynomial ring with multiples of $x^2 + 1$ ($I = (x^2 + 1) = \{f(x)(x^2 + 1) | f(x) \in \mathbb{R}[x]\}$), and the set containing zero ($I = \{0\}$). L8 then says: "We did these yesterday... at least that's what you guys *should have been checking* in the recitation, yes?" Later, she goes on to say:

[1] All right, let me write down an example that's slightly different.

[2] All right, and maybe, it's not going to be of critical importance to us, so I'm not going to write out all of the details for you.

[3] I'm going to tell you some things about this example that will hopefully convince you that it's something legitimately different from these others.

[4] Okay, all right?

[5] Life can be deceiving when all of your examples are of a certain type, you think everything is of that type.

[...]

[6] There are a lot more examples in the book and you should read them.

[7] That's a good idea, it will give you... the richer your set of examples, the better you'll understand the concept.

There are two noteworthy features of this passage of text we wish to highlight. First, in [2], L8 says: "I'm not going to write out all of the details for you". She had previously told the students they *should have* been working out the details themselves in recitation. Outside of class, the students have the responsibility of working out the details that are not of "critical importance" during lecture. Second, in [6], L8 conveys a specific obligation: Students *should read* the examples in the textbook. In [7], L8 indicates this is a general expectation: "the richer your set of examples, the better you'll understand the concept". This complements her previous warning in [5] that "life can be deceiving when all of your examples are of a certain type". Thus, we see L8 indicating directly that her students should engage in considering, comparing, and contrasting across examples to help them understand and make sense of definitions outside of the classroom.

In the literature, Alcock (2004) and others (e.g., Watson & Mason, 2005) have claimed that students need to see a wide range of examples to understand a concept properly. Focusing on a narrow range of examples could lead students to overgeneralize and infer that concepts possess properties that do not generally hold. We see L8 urging students to read a wide range of examples outside of class for exactly this purpose.

8.4.4 How Should Students Study New Definitions That Are Presented in Lectures?

The above episodes offer insight into the ways in which mathematicians model how students should work to understand a definition. This was not always done in the lectures that we observed, although we cannot speculate as to whether a discussion of how to learn and make sense of definitions occurred in lectures other than those observed.

In the following excerpt, L4 provides the definition of a general eigenvector by writing and reading the definition verbatim, offering no commentary on what to do to understand it:

> Let T be a linear transformation of a finite dimensional vector space. Let there be an LT on an F (inaudible). And let λ be an eigenvalue of T. Then the generalized eigenvector of T relative to λ, is a not zero vector not in the null space of $T - \lambda(I)$ but in the null space of some power of $T - \lambda(I)$ for some k greater than 0.

Immediately after completing this definition, he says: "That's good enough." Nothing about this introduction of the definition provides evidence of L4 modeling any particular learning behavior beyond learning the statement of the definition. Following the definition, L4 does offer a discussion about the null space of the powers of the linear transformation versus the null space of the transformation itself; but provides no further examples or discussion to provide insight into the definition itself or its various components or conditions.

Eigenvectors are a difficult concept. Most students probably cannot come to understand eigenvectors by simply copying the definition and unpacking the meaning of each word outside of class. How can the preceding episodes advise students in this lecture to study this definition outside of class?

First, following L1, students may try to think about what properties the definition of general eigenvector is trying to capture and verify that such properties are indeed captured. Second, following L7, students might consider what mathematical objects needed to be considered in posing the definition and how they were represented in the definition. Third, following L8, students may want to consider a wide range of examples of matrices, eigenvalues, and their generalized eigenvectors. We cannot be sure of L4's intentions, but if L4 is like Wu (1999), he may have reasoned that he could not cover a sufficient amount of linear algebra if he modeled all these things in the lectures (and perhaps especially not if students engaged in these explorations themselves). The expectation may have been that students work on the material themselves outside of class.

8.5 Discussion

In the previous section, we presented episodes that we interpreted as mathematicians conveying to students how they are expected to work to understand the definitions they encounter. L1 believed that some definitions were meant to capture certain intuitive ideas (like size) and that individuals should justify that the definitions actually capture this intuition. L1 said that this is sometimes accomplished by writing basic proofs. L7 asked students to consider what features a definition should have and invited them to explore how a definition works with examples. L8 also considered examples, stating students should consider a wide range of examples and that this consideration should occur outside of class. In the beginning of this chapter, we noted Larsen's (2017) claim that students can learn some mathematics from lectures if they know what it means to really do mathematics. We view L1, L7, and L8 as each explaining some aspect of what it means to do mathematics when they encounter a new definition.

We use these observations to raise several questions and pose some directions for research. At the broadest level, how do mathematicians expect students to learn from lectures? What do mathematicians expect students to do outside of class? Earlier in this chapter, we cited Wu (1999) as saying that lecturing (as well as any other form of university mathematics instruction) only works if students honor a contract in which they work hard to understand the content outside of class. Do mathematicians agree with Wu? If so, what exactly is the nature of this didactic contract that students need to honor?

We offer two ways of addressing these broad issues. The first is to conduct interviews and surveys with mathematicians about their expectations of students' behavior, both inside and outside of lectures. Preliminary work has been conducted; we know mathematicians agree with Wu that learning mathematics is an arduous process requiring students to take responsibility in their own learning (e.g., Fukawa-Connelly et al., 2016; Weber, 2012). However, we know little about the specifics of what mathematicians expect students to do. An alternative approach is to conduct studies on how *mathematicians* learn mathematics by attending lectures in their own professional practice. As Burton (1999) noted, investigating mathematicians qua enquirers not only has pedagogical implications, but also can shed light on hidden aspects of the nature of mathematics as a social enterprise.

Assuming there are (broadly speaking) shared expectations of how students will study outside of the mathematics classroom, how do mathematicians convey these expectations to students? We know from interviews that *some* mathematicians (claim to) do so via modeling (Woods & Weber, 2020), although we emphasize we have no good evidence that these findings generalize to the broader population of mathematicians. We've illustrated in this chapter that we can interpret mathematicians as conveying expectations via modeling in their lectures. Is this method of teaching typical? What are some other things that mathematicians might do?

Our final topic of discussion concerns why lecturing in advanced mathematics seems to be ineffective. As we noted earlier, a robust finding from tertiary mathematics education research is that students emerge from their advanced mathematics courses with impoverished understanding and limited ability to engage in core mathematical competencies. Why is this the case? We've noted that mathematics educators have argued that teaching by telling does not work because of limitations of the transmission model of pedagogy, but we countered that this was not mathematicians' rationale for lecturing. Is it possible for students to learn from lectures if they knew what to do? Even though Larsen (2017) does not endorse teaching primarily by lecture, he suggested well-prepared students *can* learn from lectures, based on his personal anecdotal experience.

We can use our episode with L4 to contextualize our broader interests. L4 simply presented the definition of eigenvectors. How would his students study that definition outside of class? If L7 or L8 had simply presented a definition as L4 did, would their students (know to) consider the types of examples that L7 and L8 implored students to do in the episodes we highlighted? More generally, if students are not doing the right things outside of class, why aren't they? Would students interpret the episodes we presented in this chapter in the same way we did? Or would they not see the mathematicians as describing their responsibilities? Is there a way that expectations could be explained so students might satisfy them? Do students need to experience in-class active learning if they are to engage in studying mathematics outside of class in desirable ways?

We end this chapter by considering a performance model that is often used to make sense of lectures and to debate their viability. Krantz (2015) initially used a violin metaphor to justify why lecturing was a viable practice, in spite of the poor learning outcomes associated with it. Imagine you heard two players enter a lecture hall and butcher a violin performance. Then you heard a third violin player enter the lecture hall and provide a marvelous performance. Would you conclude the violin is a terrible instrument because it frequently leads to poor performances? Or would you conclude the violin can be a magnificent instrument in the right hands? Presumably the latter. Krantz argued we should view lecturing similarly. Lectures often are not successful, but this should be viewed as an indication of poor lecturers, not a poor pedagogical method. In a critique of Krantz's argument, Larsen (2017) again used a violin metaphor. Larsen asks whether we could ever hope to play the violin competently simply by listening to masterful performances in a lecture hall. Presumably not. But then, Larsen asks, how can we expect our students to be competent at *doing* mathematics simply by *watching* mathematicians give virtuoso performances?

We conclude by suggesting a third violin analogy. What if mathematicians are not the violinists intending to give a concert performance, but are instead the music teachers who are showing students how to practice? What if mathematicians are not playing the great songs but playing practice scales? Of course, there is no guarantee that one can prepare a competent violinist only by showing her how to play the violin when she only does so outside of class, just as there is no guarantee we can prepare a competent mathematician by exemplifying how she should learn mathematics outside of class. But if we are going to understand why lectures fail or speculate on how they might succeed, this might be the right metaphor to use.

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Chapter 9
Supporting Students in Developing Adequate Concept Images and Definitions at University: The Case of the Convergence of Sequences

Laura Ostsieker and Rolf Biehler

Abstract This chapter presents a learning environment in which students are guided to reinvent the definition of convergence of a sequence. This learning environment consists of a set of examples and non-examples of convergent sequences, a task, and expected obstacles with prepared supports for each expected obstacle. The learning environment was developed in the Design-Based Research paradigm, conducted twice, and analyzed and refined each time. In this chapter, we focus our analysis on the changes, especially to the task, that were made based on the results of the first two implementations.

Keywords Convergence of sequences · Reinvention of definitions · Learning Environment · Design Research

9.1 Introduction and Overview

The purpose of this chapter is twofold, first, explore the difficulties students face in reinventing the definition of the convergence of sequences. Second, to develop a learning environment ("workshop") that supports students in overcoming these difficulties. The learning environment for reinventing the definition of sequence convergence was a workshop developed for the target group of mathematics students in a proof-oriented Analysis 1 course at a German university, comparable to Real Analysis courses in other countries. This workshop was an optional add-on course offered before the concept of convergence was covered in the associated lecture. The

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aspects presented in this chapter are part of a broader study (Ostsieker, 2020) done in the design-based-research paradigm (Gravemeijer & Cobb, 2006). The research study was carried out in three different, cyclically repeating phases. In the experiment's preparation phase, especially the learners' starting points, the design of the activities, the targeted endpoints, and theories about the learning processes are formulated in the form of a Hypothetical Learning Trajectory (Bakker & van Eerde, 2015). After the experiment was conducted, a retrospective analysis followed. Based on in-depth analyses of transcripts of student discussions during the workshops, both the learning environment and the local theory of how students learn the limit concept was revised.

In this chapter, we will report on how the workshop's design and, in particular, the task that asks students to reinvent the concept of limit changed in the successive cycles of the study.

9.2 Theoretical Background and Literature Review

Tall and Vinner (1981) introduced the terms concept image and concept definition to deal with mathematical concepts. Under the notion of concept image, they summarize the entire cognitive structure of an individual to a concept. In contrast, they use the term concept definition for an individual's definition for a concept. This personal definition may differ from the generally accepted definition. Suppose learners – as is often the case with the limit concept – already have some ideas about a concept and later learn a formal definition. In that case, there are three possible implications for the concept image and concept definition. First, it would be desirable that the concept image is adapted coherently to the new concept definition. However, it can also happen that the concept image remains unchanged and co-exists with the formal concept definition for some time. Both conceptions can be activated in problem-solving. Furthermore, the concept image and concept definition may remain unchanged despite having seen the formal definition.

The role of definitions changes significantly with students' transition from school to college (Engelbrecht, 2010). Zaslavsky and Shir (2005), in a study with twelfth graders, found that students have different views about what property a mathematical definition must satisfy. For example, there was disagreement about whether a definition may contain superfluous conditions and whether there can be multiple definitions for a term. Since it is rarely made explicit to learners what functions and properties mathematical definitions have, the differing views are not surprising. Edwards and Ward (2004) suggest that the notion of a definition should be explicitly addressed at some point. In one study, they had students complete tasks in which the use of a definition would have been adequate. Although they had the written definition available when they worked on the task, they did not necessarily use it. It happens that they work on such tasks solely based on their concept image. Edwards and Ward believe it is helpful for students to participate in the process of defining. It was an open question for us to which extent, and when, metaknowledge about mathematical definitions is needed in the process of defining.

The learning environment presented here has a two-fold challenge: developing a concept image of convergence coherent with the usual formal definition and developing a personal concept definition equivalent to the standard formal definition. First, typical misconceptions about sequence convergence should be actively overcome by starting with a broad set of examples, including some that are – based on misconceptions – not considered as convergent. For example, a limited conception is that the distance of the sequence elements to the limit must be strictly monotonically decreasing when convergent (Roh, 2005). Moreover, (strictly) monotone convergent sequences are often viewed as prototypical of convergent sequences (Alcock & Simpson, 2004; Cornu, 1991; Davis & Vinner, 1986; Robert, 1982). Another misconception is that no sequence element may be equal to the limit of the sequence (Davis & Vinner, 1986; Roh, 2005; Szydlik, 2000, Tall & Schwarzenberger, 1978; Williams, 1991). This misconception can be distinguished from the misconception that the limit is an upper or lower bound that may not be passed (Cornu, 1991; Davis & Vinner, 1986; Robert, 1982; Szydlik, 2000).

Roh (2005, 2007, 2008, 2009, 2010a, b; Roh & Lee, 2011, 2017) has investigated the connection between the intuitive understanding of the limit concept and the level of so-called reverse thinking in several studies with different target groups. For example, suppose the convergence of a sequence against a value with the ε- N-definition is to be proven. In that case, a suitable index N must be chosen for an arbitrary but fixed ε so that from this index on all further sequence elements deviate from the value by less than ε. However, students often proceed intuitively in a different way:

> *students typically first choose an index number, and next determine how close the term corresponding to the index is to a certain value* (Roh, 2005, p. 7)

Students would have to proceed in a reverse way compared to this intuitive approach, which Roh refers to as reverse thinking in the context of the limit of a sequence:

> *reversibility in the context of the limit of a sequence (...) means the ability to think of the infinite process in defining the limit in terms of the index and simultaneously to reverse the process by finding an appropriate index in terms of an arbitrarily chosen error bound.* (Roh, 2005, p. 20)

To encourage this reverse thinking in the context of the concept of limit, she has proposed an ε-strip activity (Roh, 2010c). Students work with transparent strips of different widths in this activity, with the center marked by a horizontal line. This line was placed at the potential limit in graphical representations of different sequences. The students were asked to answer how many sequence members were inside and outside the ε-strips, respectively. From this, it could be discovered that all but finitely many sequence members are within each strip for convergent sequences. Thus, the goal for students was to understand the relationship between ε and N in the formal definition.

Przenioslo (2005) has proposed a series of lessons to introduce the notion of convergence in school to counteract the various restricted conceptions. The main task consists of presenting students with eleven different sequences convergent to 1 and one sequence with two cluster points at 1 and 2 (named b_n) and asking the students:

> *What common property not shared by the sequence* (b_n)*, do the infinite sequences* (a_n) *mentioned below have?* (Przenioslo, 2005, p. 76)

She speaks of infinitely many sequences because the defining equation of one of the sequences contains a parameter. She has carefully selected the example sequences. For each of the typically restricted conceptions, there are some sequences that would be considered non-convergent if the particularly restricted conception would be applied. In addition, the author has developed several "fictitious discussions" that can be used at various stages of the reinvention process to focus learners' attention on particular aspects. Przenioslo reports she has successfully implemented this series of lessons several times but has not followed up with an empirical study.

In our study, we used her general approach, most of the examples, and some of the fictitious discussions as a starting point of our research and development project. The individual elements adopted and the modifications in the first and the subsequent cycles will be discussed in the presentation of the design of the learning environment.

An empirical study of the guided reinvention of the notion of sequence convergence was carried out by Oehrtman and colleagues (Oehrtman et al., 2011, 2014). At the beginning of the teaching experiment, the observed pairs of students were asked to collect as many different examples as possible of sequences that have and do not have the limit 5, respectively. Afterward, students were asked to complete the sentence "a sequence converges to 5 as $n \to \infty$provided…" (Oehrtman et al., 2011, p. 328) so that the statement was true for all examples and not true for all non-examples. A cyclical process followed in which students proposed formulations of the definition, discussed it, and revised it as necessary. Two teachers were present and assisted them in this process by asking specific questions. They evaluated the reinvention process as helpful for the students and showed that the self-generated ideas are retained in the long term based on a study conducted with the same students 6 months later (Martin et al., 2012).

Other publications have addressed the guided reinvention of other concepts such as convergence of functions, series, and pointwise convergence (Martin et al., 2011; Swinyard, 2008, 2011; Swinyard & Larsen, 2012).

The study was part of the Ph.D. project of the first author, supervised by the second author (Ostsieker, 2020). Schüler-Meyer (2018, 2020), whose study was published after finishing the Ph.D. project, also addresses the convergence of series on the object level of convergence and on the meta-level of defining with a different theoretical approach in the context of supporting secondary students.

Concerning the role of definitions in tertiary mathematics, the learning environment we describe is limited because it ends with formulating a definition. However, the role of definitions in proving and creating a deductively organized mathematical theory is essential in tertiary mathematics education, and activities have to be designed to highlight this role (see Pinto & Tall, 1999). Sierpinska (2000) also emphasizes this aspect under the role of theoretical thinking with concepts. In Ostsieker (2020), a second learning environment was developed addressing these aspects after the formal definition, and some theorems concerning convergence had been introduced in the lecture. We will not report on this "follow-up" learning environment in this chapter.

9.3 Research Questions

What difficulties arise for students when they are asked to reinvent the notion of convergence of sequences in a form commonly used in university mathematics?

How can overcoming these difficulties be supported through an adequate learning environment?

9.4 Context of the Study

A non-compulsory 4-h workshop for students of an Analysis 1 course that should support them in constructing the formal definition of convergence of a sequence in an attempt of guided reinvention. The workshop was held some weeks after starting the Analysis 1 course, just before the lecture introduced the formal definition. It was offered in two cycles to all students of the course. We took a random selection from the volunteers to create a non-biased control group. The results of a quantitative study comparing participants with non-participants are reported in Ostsieker (2020). In the first round, 16 students participated; in the second round, 12 students. After the workshop, the notion of convergence was introduced in the lecture, and a second workshop was offered to deepen the conceptual knowledge of the participating students. This second workshop is not part of the discussion in this chapter.

9.5 The Design of the Initial Learning Environment

The study builds on work by Roh (Roh, 2010a, b; Roh & Lee, 2017) and Oehrtman (Oehrtman et al., 2011, 2014). However, the main idea was to empirically test and adapt the theoretical approach of Przenioslo (2005).

Her idea was to present the students a set of examples and non-examples of series being convergent to 1 and asks that are called to be "convergent" and others that are called non-convergent and ask the students to construct a definition of "convergence" where the examples are examples of, respectively non-examples are not. The students are supported by referring to "fictitious discussions" between fictitious students when they got stuck. The set of examples was chosen to avoid an inadequate concept image from the beginning and base the definition on this set.

Our learning environment has three components:

1. The set of examples and non-examples and its anticipated use
2. The task formulation
3. Anticipated obstacles: The prepared prompts when students got stuck (including fictitious discussions and visualizations)
4. Spontaneous teacher interventions if needed

9.5.1 The Set of Examples and Non-examples and Its Anticipated Use

The initial set of examples and non-examples is shown in Fig. 9.1. The sequence (a_n) is the prototype of a monotonically convergent sequence that never reaches its limit. The sequences (b_n) and (c_n) stand for sequences that reach their limit after some n and are "finally constant." Whereas (b_n) is monotone, (c_n) is not. (d_n) is convergent but not monotonic. The sequence (e_n) is a well-known case where many students have misconceptions. Still, it is crucial to learn that the limit of this sequence is also 1, (f_n) has an "exception" of the rule at $n = 10$ and stands for alternating sequences. The sequence (x_n) is an example of a sequence with two accumulation points. The sequences (b_n), (c_n), and (f_n) stand for sequences with "exception" at a finite number of points (which should be regarded as irrelevant for convergence). (d_n) and (x_n) stand for sequences with an infinite number of "exceptions," which can matter.

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{n+1}{n}\right)_{n\in\mathbb{N}}$$

$$(b_n)_{n\in\mathbb{N}} \text{ with } b_n = \begin{cases} 1-\frac{1}{n} & n \le 125 \\ 1 & n > 125 \end{cases}$$

$$(c_n)_{n\in\mathbb{N}} \text{ with } c_n = \begin{cases} -3 & 200000 \le n \le 500000 \\ 1 & else \end{cases}$$

$$(d_n)_{n\in\mathbb{N}} \text{ with } d_n = \begin{cases} 1 & if\ n\ is\ a\ multiple\ of\ 10 \\ 1+\frac{1}{n} & else \end{cases}$$

$$(e_n)_{n\in\mathbb{N}} \text{ with } e_n = 0,\underbrace{9\ldots9}_{n}\ \forall n \in \mathbb{N}$$

$$(f_n)_{n\in\mathbb{N}} \text{ with } f_n = \begin{cases} 2 & n = 10 \\ 1+\left(-\frac{1}{2}\right)^n & else \end{cases}$$

$$(x_n)_{n\in\mathbb{N}} \text{ with } x_n = \begin{cases} 2 & if\ n\ is\ a\ multiple\ of\ 10 \\ 1+\frac{1}{n} & else \end{cases}$$

Fig. 9.1 The set of sequences $(a_n)\ldots(f_n)$ are examples for the concept to be defined, (x_n) is a non-example. (Ostsieker, 2020, p. 96; our translation)

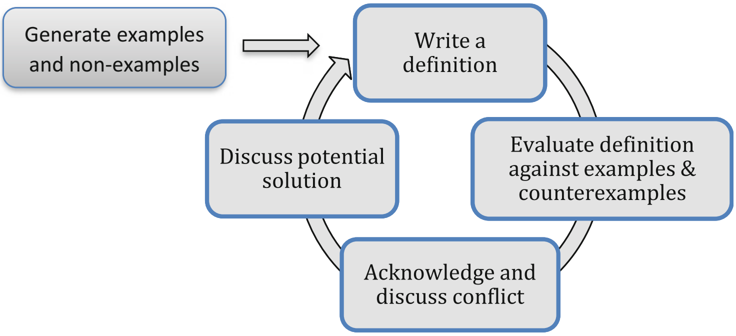

Fig. 9.2 Iterative Refinement in the process of guided reinvention, figure redrawn similar to Oertman et al. (2014, p. 135)

In contrast to Przenioslo's approach, we gave the example sequences different names to make it easier for the students to talk about them. We also reduced the number of examples to not overwhelm students by dealing with too many sequences at once. The sequences (a_n) to (d_n) were taken from Przenioslo, and only the "exception" at $n = 10$ was changed in the sequence (f_n). We added the sequence (e_n). The unusual presentation of (e_n) was chosen to avoid additional difficulties due to the summation sign. The example sequences we have chosen are still very diverse and cover the various typical restricted conceptions.

All the examples were, in a sense, generic examples, exemplifying a specific type of sequence. We expect that students will not take the examples too literally. For instance, we hoped they would recognize that the concrete numbers in (c_n) do not matter, as long as the number of "exceptions" is final.

In the a priori analysis creating a hypothetical learning trajectory – which is an important first step in frameworks for design research but also in the tradition of didactical engineering in the French tradition – we formulated the hypothesis that the students will use the examples in an iterative process of refining their definition similar to what is shown in the diagram of Oehrtman (Fig. 9.2).

However, we prepared a meta-cognitive prompt directed to checking whether a preliminary definition holds for all the examples in case students did not use this meta-cognitive strategy.

9.5.2 *The Initial Task Formulation*

The formulation of the task (Fig. 9.3) is slightly different from that of Przenioslo. First, the term "convergent" was deliberately used. Without knowing the formal definition, students who had met the term before could otherwise have answered that the common property was convergence to 1, and the task would have been solved in

> *"The sequences (a_n) ... (f_n) are called convergent to the limit 1; the sequence (x_n) is not convergent. Describe, as best as possible, the property that the sequences (a_n) ... (f_n) share but which the sequence (x_n) does not have."*

Fig. 9.3 Initial task formulation. (Ostsieker, 2020, p. 86; our translation)

their eyes. Second, the wording "describe as best as possible" was intended to convey that merely describing a property is insufficient. At the same time, we consciously decided against explicitly asking for a definition because we had the initial hypothesis that this may distract students from starting to focus on the content level first and refine the formulation of the property later.

9.5.3 Anticipated Obstacles and Prepared Support

From analyzing the hypothetical learning trajectory, we expected five types of obstacles and prepared support interventions for this. First, we used "fictitious interactive discussions" between fictitious students that include questions for the readers instead of just giving hints by the teacher for supporting self-regulated learning in several parallel groups and empirically testing the fictitious discussions approach. Moreover, unlike Przenioslo, we identified several clear triggering situations in which support would be given to the students. Finally, the numbering of the supports means that the second or third support was only provided when needed (graduated aids).

Situation	Description	Support
A	Problems with starting	1. "Find a common property for a subset of examples and check this with the other examples." 2. First fictitious discussion, where students exemplify the strategy to focus on a subset first to find a common property, taken from Przenioslo (2005, p. 80)
B	Students stop with a formulation that is not sufficient	1. The teacher selects an example, which does not fit the description
C	Students formulate the shared property as a disjunction of several separate properties that characterize subsets of the examples.	1. Encourage students to formulate one property for all examples 2. Second fictitious discussion (from Przenioslo 2005, p. 81): Idea to focus mentally on a strip around the limit 1 and search for n_0, so that all further elements are inside the strip
D	The formulated property is too vague, so that a decision whether the property is true of one of the sequences or not is not possible.	1. "Check whether you can get decisions for all sequences of the task." 2. Several ε-strips are handed out as material, and students are encouraged to

(continued)

Situation	Description	Support
		find a more precise characterization based on focusing on the distance between the elements and the limit, respectively the limit's neighborhood (adapted from Roh, 2005, evoking "reverse thinking")
E	The property is true for all examples and not true for the non-example, but not equivalent to the standard definition of convergence.	1. Students are praised for a correct solution. Afterward, they are given a further ad hoc example that does not fulfill their condition but which is convergent to 1

9.6 Design of the Study, Sample, Collected Data, Methods of Data Analysis

9.6.1 Instructional Design of the Workshop

After a brief introduction to the task, the participants of the workshop were divided into small groups. They were then presented with the set of example sequences and the non-example, as shown above. They were also given graphical representations of the sequences they were first asked to match. After a brief discussion in the plenary, the reinvention of the notion of convergence began in small group work. The teacher acted as a facilitator, observing the groups' collaborative processes and providing the prepared aids as needed. The discussions of each small group were audio-recorded, and in addition, a video recording of the whole classroom was made. The small group discussions were later retrospectively analyzed. This analysis was done following methods from interpretive classroom research (Krummheuer & Naujok, 1999). The reconstructed real learning trajectories were compared to the hypothetical epistemological trajectories formulated in advance.

9.6.2 Iterative Analysis from the Perspective of Design Research

These three components of the learning environment and how they evolved throughout the study are discussed in more detail in the results section. As part of the comprehensive study, the first version of a Hypothetical Learning Trajectory (with several possible hypothetical reinvention processes) was developed, and the learning environment was implemented. In the retrospective analysis, the actual reinvention processes of all observed small groups were reconstructed and compared with the different variants of the previously formulated hypothetical processes. In particular, the extent to which the set of examples and non-examples, the task, the triggering situations, and the supports were helpful and sufficient for the reinvention was examined. The Hypothetical Learning Trajectory, including these four components, was revised based on the results. This second version was also carried out, analyzed and thus, a third version of the Hypothetical Learning Trajectory was developed.

9.7 Results

9.7.1 Changes in the Set of Examples/Non-examples and the Anticipated Use

In the second cycle, some notational adaptations (for instance, for (a_n) were made. However, the example (e_n) turned out to be causing difficulties that were not productive for the concept development. On the other hand, a constant sequence was not among the examples, but only finally constant sequences, which caused further difficulties. As the constant sequence is often not part of the concept image of convergence, it was added as the new sequence (e_n).

In the second cycle, an unanticipated event occurred. A group of students characterized convergence as follows: Either an element of the sequence equals the limit or the distance of the element to the limit is less than the distance of the previous element to the limit that was identical to the limit. If the students had added that this distance has to become arbitrarily small, their characterization would be valid for a real subset of all convergent sequences. However, there are convergent sequences that do not have this property, such as

$$(g_n)_{n\in\mathbb{N}} \text{ with } g_n = \begin{cases} 1+\frac{1}{2^n} & \textit{if n is odd} \\ 1+\frac{1}{n} & \textit{if n is even} \end{cases}$$

Therefore, this example was added to the list of examples. We also discovered that the anticipated process of iterative refinement of definitions using examples often did not occur, so we added a suitable recommendation to the task formulation (see Sect. 9.7.2).

9.7.2 Changes in the Prepared Support for the Second Cycle Based on Retrospective Analysis of Cycle 1

Situation	Description	Support (changes in italics)
A	Problems with starting	1. “Find a common property for a subset of examples and check this with the other examples.” 2. First fictitious discussion about common properties, a *new version from first cycle discussions*
B	Students stop with a formulation that is not sufficient	1. *“Check whether your property is true for all examples and not true for non-example* 2. The teacher selects an example, which does not fit the description
C1 new	Students formulate that elements “approach 1 or reach 1”	1. “Is this property formulated in a way to decide whether it is true for any possible sequence?” 2. Second fictitious discussion about common properties, *a new version from first cycle discussions*
C2 new	Students formulate that the distance of elements to 1 gets small or is 0	1. “Is this property formulated in a way to decide whether it is true for any possible sequence?” 2. Third fictitious discussion about common properties, *a new version from first cycle discussions*
C*3*	Students formulate the shared property as a disjunction of several separate properties which characterize subsets of the examples.	1. Encourage students to formulate one property for all examples 2. ~~Second fictitious discussion (from Przenioslo): Idea to focus on a strip around the limit 1 and search for~~ n_0~~, so that all further elements are inside the strip~~
D1 new	Students describe the property as an approximation to 1.	1. *“Is this property formulated in a way to decide whether it is true for any possible sequence?”* 2. *Fourth fictitious discussion about common properties, new from first cycle discussions*
D2 new	The students characterize the property by the fact that the distance of the sequence elements to 1 becomes small.	1. “Is this property formulated in a way to decide whether it is true for any possible sequence?” 2. *Fifth fictitious discussion about common properties, new from first cycle discussions*

(continued)

Situation	Description	Support (changes in italics)
D3 new	Students characterize the property by the fact that the distance of the elements to 1 becomes arbitrarily small for large n.	1. "Is this property formulated in a way to decide whether it is true for any possible sequence?" 2. Sixth fictitious discussion about common properties, *new from first cycle discussions*
D*4*	The formulated property is too vague, so a definite decision whether the property is true of one of the sequences is not possible.	1. "Check whether you can get definite decisions for all sequences of the task." 2. Several ε-strips are handed out as material, and students are encouraged to find a more precise characterization based on focusing on the distance between the elements and the limit, respectively the limit's neighborhood (adapted from Roh, 2005, evoking "reversed thinking")
E	The property is true for all examples and not true for the non-example, but not equivalent to the standard definition of convergence.	1. Students are praised for a correct solution. Afterward, they are given a further ad hoc example that does not fulfill their condition but which is convergent to 1

A significant change is that the triggering problem situations were refined based on the problem situations that occurred in the first cycle. Situations (A) and (E) did not happen at all in the first cycle, while situations (B), (C), and (D) each occurred in two of the four groups. In problem situation (B), the support provided worked well and quickly. On the other hand, the groups spent the most time dealing with problem situations (C) and (D). To provide more specific support to the students, situation (C) was broken down into (C1), (C2), and (C3), and (D) was refined into (D1), (D2), (D3), and (D4).

For example, the following situation occurred in cycle 1:

For n towards infinity, a_n goes towards a fixed value or is a fixed value.

This situation was taken into account by the new triggering situation C1.

Also, the fictitious discussions on the refined problem situations now begin with a statement that roughly corresponds to the characterization last formulated by the students.

Moreover, the fictitious discussions were revised by using excerpts from the actual discussions from the first cycle. This choice should make the fictitious discussions more authentic and closer to the language used by the students. Potentially, this could also lead to students finding the discussions more understandable. This point was a point of criticism with some small groups in the first cycle, as shown in the following transcript excerpt from a group's conversation as they were working through a fictional discussion:

462 S: What does she mean?
463 N: Yes, I don't understand that either, because first, she says you can disregard some, but then she says we should consider all of them.
464 S: It is somehow illogical.

9.7.3 Changes in the Prepared Support for the Third Cycle Based on Retrospective Analysis of Cycle 2

The following revisions of triggering situations and support were created due to the retrospective analysis of cycle 2.

Situation	Description	Support (changes in italics)
1.1 (A)	Problems with starting	1. "Find a common property for a subset of examples and check this with the other examples." 2. First fictitious discussion about common properties
1.2 (B)	Students stop with a formulation that is not sufficient	1. "Check whether your property is true for all examples and not true for non-example." 2. Teacher selects example, which does not fit the description
2.1 (C1)	Students formulate that elements are 1 or "approach 1"	1. "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" 2. Second fictitious discussion about common properties 3. "*Write down the common property.*"
2.2 (C2)	Students formulate that the distance of elements to 1 gets small or is 0	1. "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" *2. Several ε-strips are handed out with specific questions* 3. Third fictitious discussion about common properties 4. "*Write down the common property.*"
2.3 (C3)	Students formulate the shared property as a disjunction of several separate properties which characterize subsets of the examples.	*1. "Is this property formulated in a way to objectively decide and demonstrate whether it is true for any possible sequence?"* *2. Several ε-strips are handed out with specific questions* 3. "*Write down the common property.*"

(continued)

Situation	Description	Support (changes in italics)
3.1 (D1)	Students describe the property as an approximation to 1.	1. "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" *2. Several ε-strips are handed out with specific questions* 3. Fourth fictitious discussion about common properties 4. "*Write down the common property.*"
3.2 (D2)	The students characterize the property by the fact that the distance of the sequence elements to 1 becomes small.	1. "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" *2. Several ε-strips are handed out with specific questions* 3. Fifth fictitious discussion about common properties 4. "*Write down the common property.*"
3.3 (D3)	Students characterize the property by the fact that the distance of the elements to 1 becomes arbitrarily small for large n.	1. "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" *2. Several ε-strips are handed out with specific questions* 3. Sixth fictitious discussion about common properties 4. "*Write down the common property.*"
3.4 (D4)	The formulated property is too vague, so a definite decision whether the property is true of one of the sequences is not possible.	1. "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" *2. Several ε-strips are handed out with specific questions* 3. "*Write down the common property.*"
4.1 *new*	The students have formulated that the condition should be valid for decreasing or "infinitely small" epsilon.	1. "*How would you prove that a sequence has the property you formulated?*"
4.2 *new*	The students have formulated that the condition should hold "from a particular n onwards" but cannot formalize this.	1. "*Does the condition have to be valid for every strip or epsilon starting from the same n? Must the condition hold for every sequence starting from the same n?*" (for becoming aware that the respective n may depend on epsilon, i.e. on the strip).
0 (E)	The property is true for all examples and not true for the non-example, but not equivalent to the standard definition of convergence.	1. Students are praised for a correct solution. Afterward, they are given a further ad hoc example that does not fulfill their condition but which is convergent to 1

Reasons for defining new obstacle situations and redesigning the support:

During the retrospective analysis of the second cycle, it was possible to distinguish four common phases in the students' learning trajectories were identified in which the reinvention process usually takes place:

1. Dealing with the examples and conceptualization of characteristic features
2. Finding a single characterization that applies to all examples
3. Optimizing the formulation so that it is accurate enough that it could be used as a decision rule
4. Writing down the final characterization in formal mathematical language

The triggering situations were assigned to these phases. In the process, the obstacle situations were renamed so that it is evident in which stage they are to be expected. For example, situations (1.1) and (1.2) usually occur in phase 1, and so on. Apart from this renaming, other changes were made to the obstacle situations and the associated supports.

A significant change is the prompt "*Write down the common property.*" which was added to all triggering situations in phases 2 and 3. In cycle 2, when spontaneously made by the tutor, this intervention led to a more precise and formal characterization of the property or definition, so we hypothesize that the request to move from oral discussion to a written formulation as per se, a significant strategic impact. Similarly, we constantly changed the formulation to "Is this property formulated in a way to *objectively* decide *and demonstrate* whether it is true for any possible sequence?" as a new strategic intervention. This change is to stimulate students to reflect on their formulation from different persons' points of view and apply it to fictitious new sequences, which had not been done spontaneously.

One change is that in situations (2.1, 2.2, and 2.3), the impulse is no longer to express the property by a single condition. This change is because several students did not consider a formulation by two conditions connected by "or" as a deficiency and therefore questioned this impulse. Therefore, this requirement had already been included in the assignment for the third cycle. Secondly, the formulations that led to a classification in these situations were not precise enough to be used as a decision rule. Therefore, the first prompt now accurately points this out. When revising the characterization from this point of view, it was often automatically expressed by a single condition in cycle 2. So, the hypothesis is that it suffices to include the "or" problem in the task formulation.

Another revision relates to the use of the ε-strips. It has become apparent that students need more guidance in this process. Therefore, specific questions and prompts have been added to the ε-strips to guide the work with the strips to help students transition to a characterization that can be used as a decision rule. These are different questions and impulses than proposed by Roh (2010c). First, she asks students questions about the number of sequence members inside and outside the ε-strips, respectively. Then they have to evaluate the validity of two "ε-strip definitions." One of the two ε-strip definitions is that a sequence converges to a value if there are infinitely many sequence members within each strip around that value. This characterization applies not only to limits but also to cluster points.

The other ε-strip definition, on the other hand, characterizes limits by saying that there are only finitely many sequence members outside each ε-strip around the value. In our case, the first instruction was to start with one of the ε-strips and then address how it could be placed on the graphic representations of the sequences. Differences between the example sequences and the non-example should then be observed. Finally, if necessary, the question is asked, similar to Roh, how many sequence members are outside the ε-strips in each case. There are several specific goals to be pursued with the ε-strips. One is that students might get the idea of looking at the distance of the sequence members from 1 or neighborhoods of 1. Another goal might be to discover that in each of the given examples, as opposed to the non-example, there exists an N such that all sequence members with larger index are inside the ε-strips. In using them, students might also discover that the particular N depends on both the specific sequence and the width of the strip. A similar alternative discovery would be that in each of the example sequences, only finitely many elements of the sequence are outside each ε-strip. In contrast, in the case of the non-example, some ε-strips have infinitely many elements outside the strips. Closely related to this discovery is the step to reverse thinking (Roh, 2005).

During the first cycle, the impulse was sometimes spontaneously given to write down the verbally formulated characterization. It turned out that the transition to a written characterization always led to a more precise formulation. It seems that the students have lower demands on an oral formulation regarding precision than on a written formulation. This impulse was therefore added to all obstacle situations where a specification is necessary.

Finally, the two obstacle situations (4.1) and (4.2) were added. These situations can occur when writing down the final characterization in formal mathematical language.

9.7.4 Changes in the Task Formulation

In general, the task formulations were expanded, and aspects were added that had been used in the first cycle within the intervention phase (Fig. 9.4). We had the hypothesis that this may lead to two improvements: First, assuming that students read and comprehend the task formulation, they may need fewer interventions in the process. And secondly: instead of introducing a new precision of the task for the students in the process itself, their attention may be drawn to the task formulation as a strategic intervention, instead of making the task more precise in the process, to which students may react negatively if they think that the task was changed "in the process."

The formulation in 1. explicitly asks for a mathematical definition, not only for a "property." When students present formulations of a property that does not qualify for a definition, it is possible to refer to socio-mathematical norms for definitions. A "definition in university mathematics" was added because some students had entered the debate with a school-mathematical concept of "convergence," according to

Task formulation in the third cycle

The third formulation resulted from a profound retrospective analysis after the second trial, but it was not yet tested empirically in a third cycle.

1. *"We are going to discover the definition (in university mathematics) of the convergence of a sequence to 1. The concept must be defined so that the sequences (a_n) … (g_n) are convergent, and the sequence (x_n) is not.*
2. *Aim at formulating a shared property of the sequences (a_n) … (g_n) that the sequence (x_n) does not possess as one single condition.*
3. *This condition has to be formulated so that somebody to whom this formulation is presented can objectively decide and argue for every arbitrary sequence, whether this sequence has the property or not. Every person to whom this formulation is presented should come to the same conclusion about every specific sequence.*
4. *If you believe that you have formulated an adequate property, check whether all the example sequences possess this property and check whether (x_n) does not have it. If some of these checks are negative, revise your formulation."*

Fig. 9.4 Final task formulation after two iterations of the workshop. (Ostsieker, 2020, p. 542, our translation. The numbering was added to make references in the following text clearer)

which some examples were not convergent that were claimed to be "convergent" by the task. This event happened when students had a school-mathematical convergence concept that calls sequences convergent only when the elements are approaching a limit (monotonically) but never reach it. We did not expect that some students did not accept the rules of our "game," so we had to make it explicit that we have to distinguish possible previous concepts by name from the new concept that is to be defined. This reformulation may help; however, self-confident students may not be willing to give up a concept they found helpful in the past. This observation may point to limitations of the approach that Ostsieker (2020) and Przenioslo (2005) have chosen. The approach does not provide problems that motivate a change of previous definitions because of theoretical or practical reasons but forces the students to accept that there is some hidden good reason in the choice of the provided examples. In the sense of Freudenthal's didactical phenomenology, no actual problem situation is presented that motivates the creation of the new concept (Freudenthal, 2002).

The formulation in 2 ("one single condition") reflects situations that occurred in the students' debates. Some groups adequately characterized several subgroups of the examples and then formulated the definition by several statements connected by "or." For instance, "convergence" can mean monotonically approaching a number with arbitrary closeness but not reaching it" or "being equal to this number from a certain n onwards" or "alternating around that number with . . ." or doing one of these things with a finite number of exceptions" etc. This solution – even if it fulfills all criteria of rigor --, would be considered as "inelegant" in mathematics. This phenomenon reveals another norm that mathematical definitions have to meet.

Students in the first two workshops were correct in claiming that this requirement was not specified in advance. That is why it was added.

Formulation 3 reacts to the situation that some students were satisfied with problematic formulations. Pointing to potential “readers” of the definition should initiate a reflection of the preciseness and understandability of the definition and motivate students to check their definition with further self-created examples or use other students’ views. This extension of the task formulation may be of practical value. However, students may need to have more opportunities to reflect on criteria that a mathematical definition must fulfill (Edwards & Ward, 2004; Ouvrier-Buffet, 2006; Zaslavsky & Shir, 2005). Moreover, the formulation bears another problem. A mathematician would say that a mathematical definition must “in principle” allow one to decide whether an object fulfills it or not. It will, of course, often happen that determining whether a concrete object satisfies a definition may be considered a severe mathematical problem to be solved in the future. Moreover, the formulation “every person should come to the same conclusion”, is of limited practical value: how can this be validated?

Formulation 4 reflects the following observation. In the anticipated learning trajectories, it was assumed that the students work on a successive improvement of their formulations of a definition, creating a first version of the definition from a limited set of examples, then systematically testing this formulation on all initial examples and also on new ones, then revising the definition and so on. However, such a systematic approach was seldom observed. Therefore, the formulation was added to foster such an approach. The relatively unsystematic approach towards concept definition may not only be due to limited experiences in creating mathematical concept definitions, but it is plausible that students may have seldom met concept definition tasks or situations requiring new concept definitions in their school or university life. In sum, these results also support the need to make implicit norms explicit and provide more extensive experiences for students to actively and reflectively participate in the new culture, where a more formal way of reasoning about limits is required.

9.8 Discussion

In retrospect, it is not surprising that the first and second versions of the set of examples and non-examples, assignment, triggering situations, and supports were insufficient to initiate a self-regulatory reinvention process of consequence convergence. The students have never worked on such a task before. They are unaware of the necessary strategies and requirements for formulating a definition, and therefore, these need to be made explicit in the task formulation. An alternative might be to explicitly discuss the functions and characteristics of mathematical definitions with students before conducting the workshop. This option would fit the recommendation of Edwards and Ward (2004). Reverse thinking, identified by Roh (2005) as a crucial step in understanding the ε- N-definition, also emerged as a critical obstacle in our study, where, in contrast to Roh's studies, students reinvented the definition themselves with guidance. The ε-strips suggested by Roh can help overcome this obstacle, but most students need to be guided to use them to move to reverse thinking. The "ε-strip definitions" proposed by Roh were not suitable for the reinvention of the definition.

Swinyard and Larsen (2012) refer to a similar step in the context of the ε- δ-definition of the limit of functions from moving from an "x-first" perspective to a "y-first" perspective and have also called this as one of the main difficulties in reinventing the definition. We also encountered the second central problem they mentioned, the operationalization of expressions of the "infinitely close" type. Moreover, similar obstacles emerged in our study, such as expressions like "for large n."

Unlike other studies, we precisely worked out what kind of support could optimize the reinvention process. Another critical problem that emerged in this study is the lack of systematicity among students in the reinvention process. For example, students did not systematically check the characterizations they formulated on all the given example sequences. As a result, in some cases, they did not notice when they formulated a property that did not apply to all given example sequences or applied to the given non-example. In Oehrtman et al. (2011, 2014), such problematic formulations that students are not aware of were referred to as *problematic issues*, different from problems that students are aware of. However, problematic issues include not only formulations that do not apply to all given example sequences or that apply to the given non-example, but also, for example, imprecise formulations. In this respect, a major problematic issue was added by our study with the students' lack of systematicity in reinventing the definition.

Moreover, our study confirms the need to support the students on a meta-level concerning requirements for a mathematical definition. It is compatible with similar results that Schüler-Meyer (2018, 2020) found in a different setting for supporting students' reinvention processes of the limit concept definition.

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Chapter 10
Investigating High School Graduates' Basis for Argumentation: Considering Local Organisation, Epistemic Value, and Modal Qualifier When Analysing Proof Constructions

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Abstract The present study is about high school graduates' basis for argumentation in elementary arithmetic. Besides knowing the elements of the basis for argumentation, the question arises in how far individual understandings of these components differ. We conducted task-based interviews focussing on learners' usage and meaning of statements in terms of their embeddedness in a local organisation, the epistemic values assigned to them, and respective effects on the conclusion's modal qualifier. We want to highlight the following results: While all graduates accept definitions and rules for term manipulation, there is no consensus concerning the statements involved. Furthermore, the individuals' epistemic values concerning the statements involved affect their usage in a chain of arguments and the individuals' evaluation of the conclusion. Although the assessments of a local organisation of mathematical content differ, the epistemic values seem to be decisive for the individual evaluation of the conclusion. Thus, we extend the existing theory by investigating the meaning of epistemic value in the context of the basis for argumentation and its effects on the individual's proof constructions. For practice-oriented research, we contribute to the ongoing discussion about the learning of proof in school mathematics by investigating the basis for argumentation of high school graduates in arithmetic.

Keywords Basis for argumentation · Local organisation · Epistemic value · Toulmin model

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10.1 Introduction

The lack of the global axiomatic-deductive structure in school mathematics causes the question concerning the choice of arguments when proving a theorem. Accordingly, when dealing with proof, the question arises, which parts of the mathematical theory might be used for verifying a given claim? This question has been discussed theoretically in terms of “set of accepted statements” (Stylianides, 2007) and “basis for argumentation” (Bürger, 1979). The (individual) set of accepted statements consists of statements and forms of reasoning that the student subjectively believes to be true. Theoretically, such knowledge is considered shared knowledge in a classroom community. These accepted statements are meant to be embedded or organised in a “local organisation” (Freudenthal, 1973), displaying a small part of the wider mathematical theory in school mathematics. It is an open question: What elements do constitute the basis for argumentation of today’s high school graduates and the extent to which corresponding knowledge might be considered shared knowledge?

In Germany, the role of mathematical proof in school has declined in the last decades. In the German national standards for mathematics in school, proof is subsumed in the context of the competence “mathematical reasoning” covering other ways of reasoning (like operative proofs, preformal proofs, and plausible reasoning), too. However, various studies have shown that German school students tend to have rather basic competencies in this area (cf. Brunner 2014, p. 82 ff.). Brunner (2014, p. 2; author’s translation), therefore, concludes: “It can therefore be assumed that in the topic of ‘proving’ there is a greater discrepancy between the claim, as manifested for example in the educational standards, and the reality, realised as an everyday practice of mathematics teaching [...]”. This chapter aims at investigating the basis for argumentation of high school graduates in the context of elementary arithmetic in Germany. Apart from what elements this basis for argumentation contains, the question arises to what extent the knowledge can be considered shared knowledge among the graduates. However, this question does not only deal with the availability of knowledge. Moreover, the embeddedness of knowledge in a local organisation may vary and also the so-called epistemic value of the mathematical statements involved may differ individually: A statement considered a valid theorem by one student might be viewed as a (still unproven) conjecture by another. This yields the question of to what extent the epistemic value assigned to the statements affects their usage in argumentation and therefore the conclusion. Since proof is considered a main hurdle in the transition to university, it seems necessary and promising to investigate high school graduates’ understanding of mathematical proof to get more profound insights into the problematic issue of teaching proof at university, too.

10.2 Theoretical Background

10.2.1 Set of Accepted Statements, Local Organisation, and the Basis for Argumentation

In his conceptualization of the meaning of proof in school mathematics, Stylianides (2007, p. 291) stresses the relevance of the *set of accepted statements*:

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; [. . .]

Following Stylianides' (2007) conceptualization, the term proof can be used to denote various deductive arguments. In such arguments, statements and forms of reasoning might be used that are accepted and available in an educational setting. Also, the way of communicating proof depends on this setting. For example, a proof can be expressed in a narrative manner (narrative proof), by explicating the argument with reference to (generic) examples (generic proof), or by using mathematical symbolic language (sometimes called formal proof in the context of school mathematics).

Stylianides refers to Kitcher (1984, p. 178) when explicating on the set of accepted statements: "the set of sentences, formulated in the mathematical language of the time, to which an omnivorous and alert reader of the current texts, journals, and research papers would assent". In this sense, both the meaning and the acceptance of proof within a class community depend on what is accepted and thus known or conceptually accessible at a given time. There is no axiomatic-deductive construction of the mathematical content in school mathematics (at least in Germany). This lack of a global theory can lead to problems when proving in the context of school mathematics:

> The principal problem here is the arbitrariness of the choice of initial statements and the different status of statements depending on the choice of initial statements. It is difficult for the student to understand that some propositions may be regarded as correct while others have to be proved. (Tietze et al., 1997, p. 160; author's translation).

To counteract this problem and to provide the idea of a deductive theory to the students, the idea of "local organisation" (Freudenthal, 1973) was developed in the context of geometry. In a local organisation, the deductive connections between constituting elements (definitions, statements, etc.) are highlighted. Accordingly, a small part of the wider mathematical theory becomes visible. Moreover, by stressing the deductive connections between the statements, the statements' truth is established. This way, a set of accepted and justified statements might be constructed. A local organisation in the context of odd and even numbers is provided (see Fig. 10.1).

The lack of the global deductive structure in school mathematics causes the question about the choice of (valid) arguments when trying to prove a theorem,

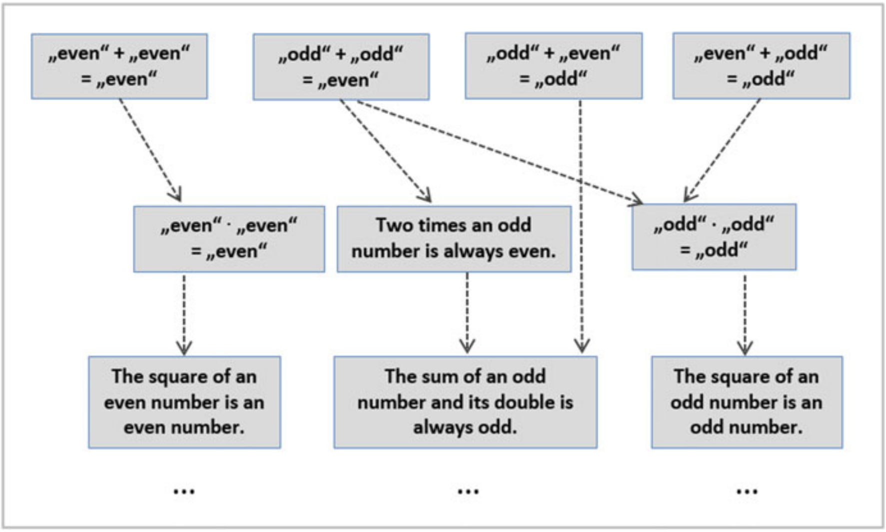

Fig. 10.1 A local organisation of statements concerning odd and even natural numbers.

which the phenomenon of the local organisation might partly answer. However, the idea of the "set of accepted statements" can be elaborated by referring to the concept of "basis for argumentation" by Bürger (1979). This author uses the term to summarize a student's mathematical basis for proving theorems. In addition to Styliandes' "set of accepted statements", the basis for argumentation consists of statements, definitions, and forms of reasoning that the student subjectively considers correct. Such statements may include, among others, propositions that appear to be intuitively evident or plausible (see Tietze et al. (1997, p. 160 ff.) for a more detailed description). Besides, one should note that the various contents of the person's basis of argumentation do not have to be (explicitly) conscious but can also be implicit (ibid.). In the following, we refer to the basis for argumentation because this concept seems more appropriate for discussing high school graduates' knowledge and handling of mathematical content in a proving context.

The elements of the basis for argumentation are considered shared knowledge in a given community. However, Stylianides (2007) stresses that the scope and the individual understanding of the shared knowledge may differ:

> The knowledge that can be considered as shared within the community [. . .] does not necessarily reflect the individual understanding of each student. Accordingly, [. . .] I do not wish to imply that each student in the community understands in the same way the elements of this set. (Stylianides, 2007, p. 293).

It becomes evident that apart from knowing the elements of the basis for argumentation, the individuals' understanding of the different statements may differ, too. Here, the aspect of epistemic value helps to elaborate on one particular part of understanding mathematical statements.

10.2.2 The Epistemic Value of Statements

The individual knowledge and understanding of statements used in a proving context touch upon the aspect of "epistemic value" (Duval, 1991, 2007): "The epistemic value is the degree of certainty or conviction assigned to a statement" (Duval, 1991, p. 254; author's translation). Accordingly, it can take on values as obvious, likely, absurd, necessary, etc., and it is closely connected to the individuals' understanding of the content (Duval, 2007, p. 138). In the axiomatic-deductive theory of mathematics, a statement's theoretical status (definition, axiom, hypothesis, etc.) implies its epistemic value. In school mathematics, as well as in real-life argumentations, the situation is quite different. As Knipping (2003) stresses:

> Students stick to their individual assessment of the reliability of a statement, even if a universal, i.e., universally valid value has been produced in a proof. In addition, different individuals evaluate the same statement as having different degrees of certainty in collective learning processes. Just like the quality of the reasons and their relevance, the epistemic value of statements in general is also judged differently. (ibid., p. 36 f.; author's translation).

It must be noted that even an epistemic value such as "necessary" cannot arise for an individual solely through logical-mathematical reasoning. Duval (2007, p. 147) also names the direct observation (e.g., of a geometric configuration) and "the fact that others agree to its truth" (ibid., p. 148).

Even though students might share a basis for argumentation, the epistemic values linked to the statements involved might differ individually. This, of course, affects the use of statements within an argumentation and the certainty of logical reasoning. Due to the statements' epistemic value, an argument might count as proof for one student and for another not.

10.2.3 Toulmin's Model for Structuring Argumentation

Toulmin (1958) proposed a scheme for structuring argumentations in general. Inglis et al. (2007, p. 4; emphasis in original) summarize this scheme as follows:

> Toulmin's (1985) scheme has six basic types of statement, each of which plays a different role in an argument. The *conclusion* (C) is the statement of which the arguer wishes to convince their audience. The *data* (D) is the foundations on which the argument is based, the relevant evidence for the claim. The *warrant* (W) justifies the connection between data and conclusion by, for example, appealing to a rule, a definition or by making an analogy. The warrant is supported by the *backing* (B) which presents further evidence. The *modal qualifier* (Q) qualifies the conclusion by expressing degrees of confidence; and the *rebuttal* (R) potentially refutes the conclusion by stating the conditions under which it would not hold.

The complete Toulmin scheme is shown in Fig. 10.2.

It has been shown that the Toulmin model can be used for revealing structures of argumentations in proving processes (e.g., Inglis et al., 2007; Knipping, 2008). Inglis et al. (2007) described that the warrants involved might affect the modal qualifier.

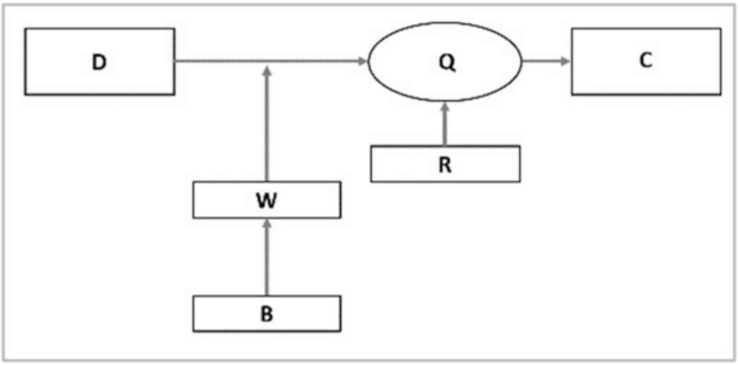

Fig. 10.2 Toulmin's full scheme of argumentation

10.2.4 Basis for Argumentation, Local Organisation, and Epistemic Value

As mentioned above, the basis for argumentation consists of statements and forms of reasoning that the student subjectively believes to be true, including propositions that appear to be intuitive-evident or plausible. From a theoretical point of view, such components' theoretical status is constituted by their localisation in the axiomatic-deductive system of mathematics. In school mathematics, the truth of a statement is inferred in the context of a local organisation. What is considered a plausible proposition in the first stage might become a statement after having proved it. This is why the theoretical status implies the epistemic value from a theoretical point of view. However, it has been argued that the epistemic values assigned to the statements might differ individually in the context of a group of learners.

Speaking in the Toulmin model context, one statement's use as a warrant in an argument is not per se combined with one specific modal qualifier. However, due to the epistemic value that the individual assigns to the statement, modal qualifiers such as *with certainty* or *probably* seem to be possible. This theoretical conclusion based on the previous considerations can only be understood as a hypothesis at this point. It will be a concern of this article to pursue this hypothesis.

10.2.5 Findings from the Literature

Edwards (1998) asked ten first-year high school students (ages 14 to 15) to work on three proving tasks concerning the sum and the product of odd and even numbers. None of these students offered an algebraic proof, even though using algebraic notation to represent and manipulate quantities had been a topic in their math classes for eight months. Only three students gave somehow coherent arguments whose elements could be interpreted in terms of basis for argumentation. Knuth et al. (2002) asked 30 sixth through eighth grade students to generate arguments concerning two statements from elementary arithmetic. Here, about 70% of the responses were based on single examples. The author states that only a minority of students gave general arguments, mentioning the order of odd and even numbers on the number line or the statement that the sum of an even and an odd number is always odd. Coe and Ruthven (1994) investigated the proof practices of 60 students in their final year of high school. They conclude that only a few students were concerned about locating the arguments used within a mathematical system. Instead, students' strategies were predominantly empirical.

Empirical evidence is also a matter for high school graduates and first-year students. In the study by Reiss and Heinze (2000), only 51.9% of 81 German high school graduates indicated that the check of a single example was not sufficient for general verification. Besides, most graduates failed to use deductive reasoning when completing proof tasks on geometry from the TIMS study. In this study, earlier findings from TIMS were confirmed that German students show low performance in mathematical proof even in upper secondary school (Baumert et al., 1998).

Kempen and Biehler (2019a) investigated first-year pre-service teachers' proof competencies when entering university. In this study, only 10% of the 71 students gave a coherent argument when verifying the claim that the sum of any two odd numbers is always even. While 14% of the given answers were purely empirical-inductive, another 32% gave pseudo answers, i.e., rephrasing the respective statement about the sum or naming irrelevant or wrong facts. It turned out that only 4.6% of students used algebraic variables. However, a minority of students made use of statements from elementary arithmetic. Finally, in the study of Kempen and Biehler (2019b), 29.7% of the pre-service teachers in their first semester at university rated a mere check of single examples as correct proof.

To sum up, learners are reported to have rather basal proof skills across ages. Besides, students in these studies did not seem to use algebraic variables to prove general statements. Finally, the use of examples in the context of proving appears to be a delicate matter. Even though some learners might held a misconception that testing single examples would constitute a general valid proof, the use of single number examples does not mean that learners must have corresponding misconceptions. As stressed by Weber et al. (2020), such students do not necessarily believe that an example provides a proof. Maybe, students just start their proving process by testing several examples and then do not know how to continue to produce a deductive argument.

10.3 Research Questions

In this study, we focus on describing and analysing the basis for argumentation of high school graduates in elementary arithmetic.[1] The investigation framing the basis for argumentation presented in this chapter is part of a wider research project focussing on high school graduates' proof productions and respective theoretical issues (e.g., norms and values). Here, we will only focus on the proving task from arithmetic and the parts from the interview concerning the basis for argumentation and the related theoretical issues named in Sect. 10.2.

RQ 1a: What elements form the basis for argumentation of high school graduates in the given context of elementary arithmetic?

When using statements in a proving context, their embeddedness in a (local) theory becomes important, because their validity has to be assured for deriving logical conclusions.

RQ 2: To what extent can the statements of elementary arithmetic be regarded as embedded in a local mathematical theory in the case of high school graduates?

As shown above, apart from knowing the definitions and statements and their position in a local organisation, the individuals' understanding of the different elements in terms of epistemic value has to be considered. The epistemic value of the statements involved might influence their use in the proving context.

RQ 3: To what extent does the epistemic value of the definitions and statements used in the proving context vary among high school graduates?

RQ 4: To what extent does the epistemic value assigned to the statements affect the modal qualifiers of the argument's conclusion?

10.4 Methodology

10.4.1 Research Instruments

We followed the concept of task-based interviews (Goldin, 2000) to make the graduates explain and evaluate their proof productions. Goldin (2000, p. 523) proposes the following four-stage model: (1) posing the question and free work, (2) minimal heuristic suggestions, if the learner struggles with the given task, (3) the guided use of heuristic suggestions, and (4) exploratory, metacognitive questions. Accordingly, the high school graduates were first asked to work on proving tasks [phase (1) and (2)]. Afterwards, an interview was conducted [phase (3) and (4)]. Personal data was collected at the beginning of the study.

[1] We use the term "elementary arithmetic" to summarize the mathematical content covering properties of the natural numbers and divisibility issues.

10.4.1.1 Task Analysis and Expected Solution

The proving task used in this study should be accessible to all high school graduates and allow for different approaches. The following task is taken from Biehler and Kempen (2013): *"Prove that the sum of an odd natural number and its double is always odd"*.

The use of an algebraic representation of an odd number (e.g., $2n - 1$) would lead to the result "$6n - 3 = 2(3n - 1) - 1$" which is an odd number by definition. However, such a solution does not seem appropriate for high school graduates in Germany, because using such algebraic representations is no common strategy or approach for performing reasoning in school mathematics. When starting with one algebraic variable, the summation with its double leads to three times the initial number (e.g., $a + 2a = 3a$). At this point, at the latest, the graduates would have to name further statements to reason why this result has to be an odd number. Of course, the whole argumentation could be formulated as narrative proof, too. Besides, it is possible to use generic examples to point to a general argument giving a generic proof (cf. Biehler & Kempen, 2013). However, in the context of such argumentation, statements like the following seem to be in reach of the graduates and might be used in this context (compare the local organisation given in Fig. 10.1): "Three times an odd number is always odd.", "Two times an odd number is always even.", etc. Finally, also the checking of the claim with several examples is possible. However, this approach does not lead to a general verification.

10.4.1.2 Construction of the Interview Guide

The interview phase was meant to make the high school graduates explain and reflect upon their proof constructions, i.e., answers on the given proving tasks. The interview guide consisted, among other things, of the following components to meet our needs:

1. Prompts to explain one's work (e.g., "Tell me, what did you do?").
2. Questions to stimulate evaluation (e.g., "Are you satisfied with your solution - why (not)?").
3. Questions, if the solution produced is considered being a mathematical proof (e.g., "Is this a proof - why (not)?").

The interview contained some final questions concerning a personal definition of mathematical proof and the previous experiences with proof in school mathematics. However, due to the focus of this chapter, we will not elaborate on these aspects.

10.4.2 Procedure

At the beginning of each investigation, the graduates were informed about the research project's principles (anonymity, voluntariness, and collection and use of data). Then, personal data were collected by a questionnaire. Afterwards, two proving tasks were handed out with the hint of having about 20 minutes to work on the given tasks. The graduates had the opportunity of using several tools (coloured pencils, eraser, set square, compass, ruler, and calculator). This way, we wanted to allow following their individual approaches without suggesting specific ways of coping with the tasks. This first phase lasted until the test persons indicated that they had finished working on both tasks. Then, the interview phase took place.

10.4.3 Piloting the Research Instrument

The whole setting was piloted in the context of a master thesis in 2019 (Krämer, 2019), focusing on high school students' personal meaning of proof. Corresponding results are published in Kempen et al. (2020). The last section of the interview guide was revised based on the pilot study. However, this does not influence the issues discussed in this chapter.

10.4.4 Data Collection

The study was conducted in August 2019. The selected students had passed their high school graduation but had not yet started their university studies. Twelve persons (six female, six male; $m_{age} = 17.92$, $SD = 0.29$; six persons taking an advanced course in mathematics at school) from different high schools from the area of Paderborn (Germany) participated in this study. Participants' final high school graduation mark ($M = 1.83$, $SD = 0.70$),[2] as well as their final high school grade in mathematics was raised ($M = 11.34$, $SD = 2.19$).[3] Participants' proving attempts were collected and scanned for research purposes. All twelve sessions (working on the task and subsequent interview) were video- and audio-taped. The dialogues have been transcribed for further analysis.

[2] Marks are scaled from 1 to 4, 1 being the best mark.

[3] Marks in school subjects are scaled from 0 to 15, 15 being the best mark.

10.4.5 Data Analysis

In a first step, the graduates' whole chain of argument was reconstructed in a Toulmin-scheme. This was done by using the graduates' written proving attempts and their explanations in the subsequent interview. Referring to both resources was necessary because the graduates did not write down all their arguments but elaborated on their proving attempts in the interview. In this sense, the use of the Toulmin-scheme made it possible to summarize the proof construction of a graduate in a clear picture. Moreover, it became possible to organize and structure the different parts and elements of the graduates' proof attempts.

The elements of their basis for argumentation used by the students in this context could thus be read from the respective Toulmin-scheme (being the warrants in the whole chain of argument), even if they had not been explicitly written down beforehand. We coded the warrants based on the different approaches listed in Sect. 10.4.1.1 (algebraic variables and term manipulation, (narrative) arguments, and checking several examples). The modal qualifier in the Toulmin-scheme was obtained by citing or rephrasing respective phrasings from the interview transcript (compare Inglis et al. (2007)). A qualitative content analysis (Mayring, 2014) was conducted on the transcripts to investigate a statement's possible embeddedness in a local organisation. Here, we focused on graduates' explanations of the elements functioning as backings in the Toulmin-scheme. Distinguishing whether and how the arguments were supported by respective backings led to the inductive construction of categories. The "epistemic values" of the statements involved could be inferred from the interview transcript when the graduates explained their proving attempts. This was methodologically done by a deductive/inductive qualitative content analysis, based on Duval's examples (2007, p. 138) (see Sect. 10.2.2). We illustrate this part of the data analysis by reconstructing participant 1's whole chain of argument in a Toulmin-scheme. His work on task one is shown in Fig. 10.3.

In addition, we cite the corresponding lines from the interview transcript (author's translation).

1	I:	What does it mean, these abbreviations?
2	P1:	Odd, even, odd [in german: ungerade, gerade, ungerade]. I thought about how to prove it.
3		Anyway, everybody knows that the sum is an odd number.
4	I:	Tell me. . .
5	P1:	If you double an odd number, it will be even.
6		[. . .]
7	P1:	If you add an even and an odd number, the sum is an odd number.
8		But why is it like that? I cannot say. It's just like that.
9		[. . .]
10	I:	Now, is it a proof?
11	P1:	Proof. . . is that it is true for all examples. No. yes?
12	I:	What do you think?
13	P1:	It is true for all odd numbers.

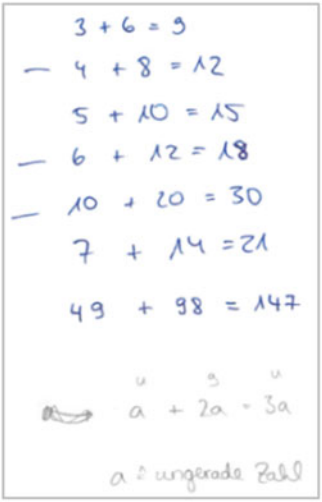

Fig. 10.3 Participant 1's working on task 1. [last line: "a: odd number"]

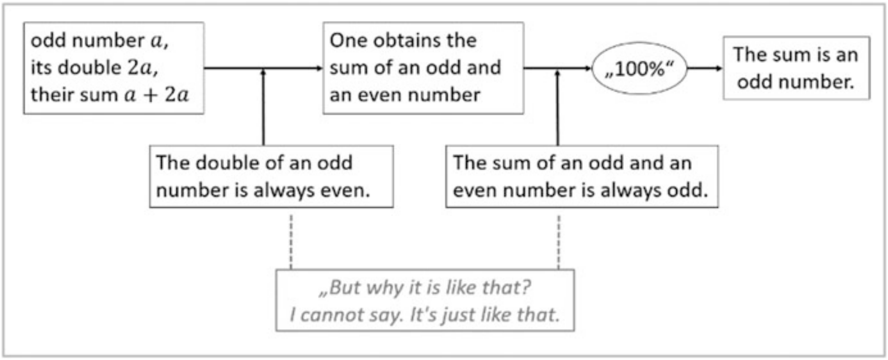

Fig. 10.4 The Toulmin-scheme displaying participant 1's whole chain of argument

Participant 1 is making use of the following statements, functioning as warrants in the Toulmin-scheme: "The double of an odd number is always even" (line 5) and "The sum of an odd and an even number is always odd" (line 7). Participant 1 cannot produce any backing for these warrants (line 8). Accordingly, we have no evidence of any kind of statements' embeddedness in a local organisation. However, the participant is sure about the warrants' validity (line 8) reflecting his epistemic value of these statements as "necessary". Finally, he stresses that the whole chain of argument is valid for all odd numbers (line 13). Accordingly, the modal qualifier can be described as "100%", and no rebuttal can be identified (see Fig. 10.4). In this episode, we gain two statements being part of one student's basis for argumentation. The phenomenon that these statements do not seem to be embedded in a local organisation serves as a starting point for the inductive construction of respective categories.

Finally, it should be noted that this chapter does not deal with individual in-depth case analyses concerning the different subjects. In the following, the results will be considered in cross-section. Therefore, the focus is on the sum of the total number of phenomena identified, irrespective of the individual case.

10.5 Results

10.5.1 Results concerning the Elements of the Basis for Argumentation used in the Proof Constructions

The elements used by the graduates when working on the proving task are shown in Table 10.1. The following elements have been detected as warrants in the Toulmin-schemes: algebraic variables and term manipulation, (narrative) statements, and the testing of several examples (compare Sect. 10.4.5). Nearly all graduates used algebraic variables and term manipulations. However, no participant proved the whole claim by only relying on algebra. Narrative statements were used by most of the graduates. However, the definitions of odd and even numbers were not stated or used in the context of algebraic representations but in the context of narrative reasoning (e.g., "An odd number means that after division by 2 you have the remainder 1."). Most high school graduates checked at least one example for testing and/or understanding the given claim. Only participant 11 misinterpreted the claim and gave wrong examples. This participant, however, followed his empirical-inductive approach for verifying the given claim. In addition, participants 6 and 12 did not mention any argument exceeding the single test of some examples or the attempt with wrong algebraic representation.

Table 10.1 Elements used in graduates' proving attempts

Elements used in graduates' proof attempts	Participant
Algebraic variables and term manipulation	1, 2, 3, 4, 5, 7, 9, 10, 12 (wrong)
(Narrative) statements from arithmetic:	
Statements about the sum/product of odd and even numbers	1, 2, 3, 4, 5, 7, 8, 10
A number is odd iff the last digit in the number is odd.	9
Definition of odd and even numbers	4, 8
Check of examples	1, 2, 3, 4, 5, 6, 7, 9, 11 (wrong), 12

10.5.2 Results concerning the Embeddedness of the Statements used in a Local Organisation

Here, we focus on if and how the statements use by the graduates to prove the given claims can be considered being embedded in a local organisation. The qualitative content analysis led to the four categories shown in Table 10.2. In the first category, the statements used are considered true. However, it is at least an open question for these graduates if such statements might be used in a proving context. Respective statements have not been adequately validated in school mathematics for these learners. Here, the existence of a local organisation is not seen or not considered sufficient. In the second category, the phenomenon of local organisation becomes evident when graduates explain the validity of the statements used by referring to former statements or definitions. In contrast, the statements are mere facts not embedded in a local organisation in the third category. Finally, the fourth category is about considering the statements as conjectures or possible conclusions. Accordingly, there are no (valid) statements in a mathematical sense that might be embedded in a local organisation.

10.5.3 Results concerning the Epistemic Value Assigned to the Statements, Rules, and Definitions used

In the course of the qualitative content analysis, it became clear that the epistemic values elaborated for the statements used could be described in line with the categories regarding statements' embeddedness in a local organisation. Even though we name the statements' epistemic value "necessary" in the context of categories [1], [2], and [3], there are remarkable differences concerning the way the awareness of necessity has been reached. In the context of the first category, the statements are considered true. However, the necessity has not been derived by mathematical proof. The need for proving them thus refers to a (somehow rigorous) understanding of mathematical proof and not to a minor degree of certainty or conviction. In category [2], the necessity has been reached by making references to definitions or former proved statements in the sense of local organisation. The awareness of necessity in category [3] seems to align with Duval's description of relying on others' agreement. Here, the statements appear as mere facts that have been learned before. In this sense, the awareness of necessity is combined with normative regulations of classroom interactions. Finally, some graduates developed these statements as conjectures or possible conclusions while working on the proving task (category [4]). Thus, an epistemic value of "probably" is assigned.

The graduates did not explicate the rules for term manipulation. However, the syntactic results of such term manipulations were not questioned in any case. Accordingly, an epistemic value as "necessary" can be assigned to these rules. This is also true for the use of definitions of odd and even numbers.

Table 10.2 Categories and results concerning the statements' embeddedness in a local organisation and the epistemic values assigned

Category	Explanation	Examples (taken from the interview excerpts; author's translation)	Epistemic value of the corresponding statements
[1] (true) statements with missing embeddedness	The statements are considered true. However, their use in a proof must be discussed, as the statements have not yet been proven.	"Let's say that it is proof if one accepts these basic statements. [...] the question is, what do you refer to? So whether you say you are satisfied with them or [...] you would also like to prove it." [P10]	Necessary
[2] (true) statements embedded in a (local) theory	The statements are considered true. Their validity can be backed up by making references to former statements or definitions.	"The sum of three odd numbers is odd. In any case, I said that an odd number is odd, because it gives a remainder of one when dividing it by two. That means, if one adds another odd number, it has twice this remainder of one, which adds up to two. And thereby, it is even again, because two is even, it has the remainder zero. If you then add a third number, you add another remainder one, and it is then three. These remainders of all three [numbers] taken together and that is then odd again, because it again has the remainder of one after dividing it by two." [P8]	Necessary
[3] statements considered as facts, not embedded in a theory	Here, the statements are considered facts that one has learned before. There is no reference to former statements or definitions.	"If you add an even and an odd number, the sum is an odd number. But why is it like that? I cannot say. It's just like that." [P1]	Necessary
[4] statements appearing as conjectures	The statements appear as conjectures or possible conclusions. Accordingly, there are no statements that might be embedded in a local organisation.	"If one looks at this observation a little more in detail [. . .] when the summands are of the same kind, if both summands are even or odd, the sum is even. [. . .] that is first of all the conclusion which I have drawn from the observations." [P2]	Probably

10.5.4 Results on the Effects of Epistemic Values on the Conclusion's Modal Qualifier

Those graduates considering the statements from arithmetic as being totally reliable (i.e., assigning an epistemic value of "necessary") did also completely trust their conclusion's validity. In these cases, a modal qualifier like "100%" is considered (see Fig. 10.4 for an example). However, the same statements also occurred as conjectures or claims, and their use as warrants led to a modal qualifier like "It seems to be like that" and thus to an uncertain conclusion since the existence of a counterexample is not excluded (see Fig. 10.5 for an example). Therefore, we can note that there seems to be a strong connection between the epistemic values attributed to the warrants used and the conclusions' modal qualifier.

10.6 Discussion

10.6.1 Elements of the Basis for Argumentation used in the Proof Constructions

Answer to research question 1: In this study, the basis for argumentation of the high school graduates comprised the use of algebraic variables and rules for term manipulation, the definitions of odd and even numbers (as natural numbers leaving a reminder of one respective zero when being divided by two), rules for term manipulation, and statements from elementary arithmetic concerning the sum and the product of odd and even numbers. Besides, the statement was used that the last digit in the number determines whether the number is even or odd.

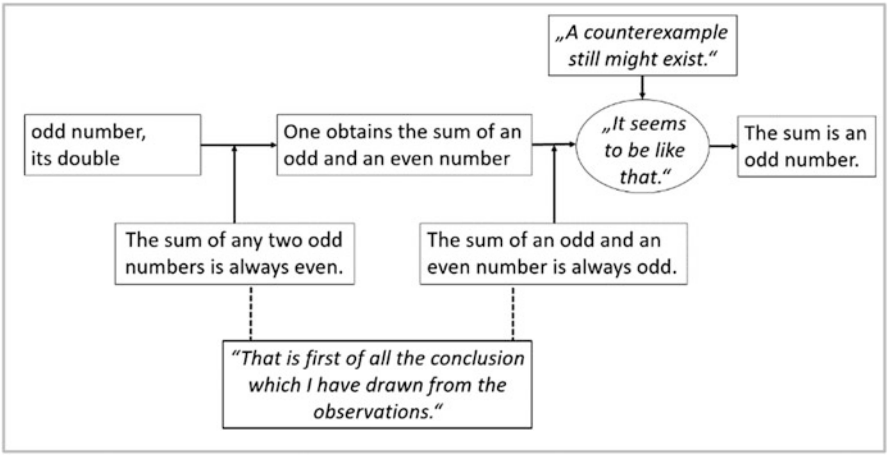

Fig. 10.5 The Toulmin-scheme displaying participant 2's whole chain of argument

The mathematical content from middle school (algebra, definitions of odd and even numbers, and statements from elementary arithmetic) can only be considered shared knowledge to a certain degree. Three participants did not succeed in starting a proof attempt by making at least use of one of such components. Besides, the graduates differed concerning the usage of the examples. While nearly all graduates explicitly stated that testing one or more examples is insufficient to validate the given claim (about all odd natural numbers), one graduate used an empirical-inductive generalisation to 'prove' this claim. These results can be considered a first glance of individual differences concerning this group's shared knowledge.

Graduates' algebraic skills were insufficient to prove the given assertion on a purely syntactic level. One graduate did not succeed in providing an appropriate algebraic representation of the given claim. It has already been mentioned in Sect. 10.4.1 that this phenomenon has been anticipated and can be explained by the minor status of proof and mathematical argumentation in school mathematics in Germany and also by a minor emphasis on the use of the algebraic symbolic language in this context in particular. This result is in line with those mentioned by Edwards (1998) and Kempen and Biehler (2019b).

10.6.2 Statements Embedded in a Local Organisation

Answer to research question 2: The analysis conceptualized four different views regarding the statements 'embeddedness in a local organisation. In the first category, the graduates explicitly miss respective embeddedness in a mathematical theory that would allow using such statements in proving contexts. Due to this perceived lack, we interpret this result as expressing a theoretical dissatisfaction with the way statements are validated in school mathematics. These graduates seem to ask for validation in the sense of a global mathematics theory, where statements are proved by referring to former statements until basic axioms are touched. The second category seems to be in line with the concept of the local organisation. The graduates state references to former statements or definitions to verify a given statements. In contrast to this phenomenon of local organisation, other graduates consider such statements as mere facts that have been learned and are not explicitly embedded in a (local) theory. Finally, some graduates perceived respective statements as mere conjectures. Accordingly, the question of local organisation was not touched here.

The idea of a local organisation of the arithmetical content became implicitly visible in some graduates' proof constructions and reflections. However, this local organisation of school mathematics seems to be insufficient for some graduates who seek somehow more profound mathematical theory (P7: "Since we never did anything like that in math class, so real proofs."; P10: "The fact is, I am relying on these basis statements to prove the given claim. The only question is, what should be based on what? So, whether you say, you are satisfied with that now, or you say, but I really want to get to the core of the whole thing and also prove that again."). Finally, there was also the phenomenon of viewing the statements as mere facts juxtaposed without further connections.

It becomes evident that the concept of a local organisation is at least not a dominant principle in graduates' notion on the mathematical theory. Nor is the validation of the school mathematics statements considered sufficient from a mathematical perspective by all the graduates (compare category [1] and the quotes from graduates number seven and ten above). Interestingly, the different forms of statements' embeddedness in a local organisation (see categories [1], [2], and [3]) did not affect the modal qualifier of the final conlcusion. However, it has been shown, that the modal qualifier changed due to the epistemic value of the statements involved (see below).

10.6.3 The Epistemic Value Assigned to the Statements and Definitions used

Answer to research question 3: It has been shown that an epistemic value of "necessary" regarding rules for term manipulation and definitions of odd and even numbers was not questioned in any case. However, the epistemic value assigned to the statements from elementary arithmetic varied throughout the students. While some graduates considered the statements as being valid mathematical statements and thus implicitly assigned an epistemic value as "necessary", respective statements occurred as mere conjectures for others (epistemic value "probably"). Accordingly, even when sharing the same statements in one's basis for argumentation in the sense of shared knowledge, the epistemic value assigned to these statements might differ individually and affect their use in a proving context. This means that, despite the existing knowledge, the individual basis for argumentation might vary in its perceived certainty because of different epistemic values. However, it became evident that not only the epistemic value of a statement predicts its usage in a proving context: While some graduates used valid statements quite naturally combined in their chain or argument, others denied their usage due to their conception of mathematical proof (see category [1] above). These graduates did not feel that these valid statements had been sufficiently or adequately proved to be used in a mathematical proof.

10.6.4 Effects of Epistemic Values on the Conclusions' Modal Qualifier

Answer to research question 4: In this study, the epistemic value assigned to the statements used as warrants in the chain or arguments affected the respective conclusions' modal qualifier. For example, a statement assigned with an epistemic value of "necessary" led to a modal qualifier like "100%" and thus to a conclusion that follows with certainty. The same statements, considered a conjecture, led to a conclusions' modal qualifier like "it seems to be like this". Besides, one graduate used an inductive warrant but assigned the modal qualifier "100%" for the conclusion.

To sum up, one statement can occur in argumentation in different ways (valid statement, conjecture, etc.) due to the person's knowledge. In this sense, the modal qualifier of the conclusion varies because of the statements' epistemic values.

10.6.5 Limitations

The results presented are based on one study with 12 high school graduates. Due to the voluntary nature of participation in this study, some positive selection can initially be assumed. The rather good grades in graduation also support this assumption. However, it must also be emphasized that two of the graduates did not cope well with the given proof task and did not find a solution independently. Therefore, a certain range of mathematical competencies can also be found in this sample. Finally, the presented study has to be understood as a case study aiming to show phenomena. Therefore, the results obtained in this study must also be confirmed in further studies or discussed concerning their quantitative significance. Another limitation is that we only investigated the named focus in the context of one proving task taken from elementary arithmetic. This raises the question of how corresponding results would turn out in other areas of mathematics or even concerning content in the upper grades.

10.6.6 Conclusions

To sum up, various conclusions can be drawn. Starting from the idea of shared knowledge in school mathematics, the question in how far a common basis for argumentation can be detected among high-school graduates was focused on in this study. However, the individual basis for argumentations varied in terms of the mere knowledge of the elements and the epistemic values assigned to the statements involved. Accordingly, the idea of shared knowledge could only be detected up to a certain degree. In this study, the different degrees of a statement's embeddedness in a local organisation did not affect its usage in the proving context. What mattered was the degree of conviction about the truth of the statements, i.e., the epistemic value assigned by the individuals. This phenomenon contradicts the theoretical mathematical view, where the theoretical status of a statement implies its epistemic value. In this sense, a major difference between school mathematics and tertiary mathematics becomes apparent.

Following these results, the difficulty level of a proving task has to be discussed by taking into account both the knowledge needed to verify the given claim and the epistemic values assigned to the statements needed. Due to the epistemic values assigned to such statements, the difficulty level of a proving task differs individually. Since the phenomenon of differing epistemic values primarily occurs in the context of narrative arguments, this issue is particularly concerning with the construction of narrative proofs or generic proofs. Besides, this phenomenon of differing epistemic values also touches on proof validation. For example, due to the epistemic values assigned to the statements used as warrants in a given proof, the whole proof might be accepted as correct or not.

Even though the results presented were obtained in the context of the German school system, the phenomena mentioned might be transferred to the situation in other countries, too. Since we cannot use the axiomatic-deductive mathematics system at school, the idea of local organisation must be considered an international matter. However, corresponding studies should also be carried out in other countries to show cultural differences concerning proof and proving in school mathematics. Furthermore, respective results have to be discussed in the context of different theoretical issues (national educational standards, ideas, and concepts of school education, science propaedeutics, related socio-mathematical norms, etc.). These issues widen our international perspective on the current state of mathematical proof in school mathematics.

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Chapter 11
Proving and Defining in Mathematics Two Intertwined Mathematical Practices

The Cases of Real Numbers and Infinity

Viviane Durand-Guerrier

Abstract The main goal of this chapter is to underline from an epistemological point of view the relevance of engaging university students in intertwined proving and defining practices. The chosen examples are *real numbers* and *infinity,* both concepts for which didactic research is still needed. In the first part of the paper, we illustrate the intertwined practices of proving and defining in the case of the construction of irrational numbers by Dedekind (1872) and Cantor (1872), recalling that for both authors, the need for these constructions emerged from proving issues. Next, we present an example of a situation involving R-completeness versus Q-incompleteness that has the potential to foster students' engagement in intertwined proving and defining practices. In the second part of the paper, we explore the intertwined relationships between practices of enumeration, the definition of infinite sets, and diagonal proofs that the set of rational numbers is denumerable while the set of irrational numbers is not. We finish by addressing didactic implications.

Keywords Didactics of university mathematics · Mathematical practices · Defining and proving in mathematics · Irrational numbers · Infinite sets

11.1 Introduction

University students' practices have raised increasing attention in the last decade, as evidenced by the number of papers in the INDRUM conferences and synthesized in Rasmussen et al. (2021). A matter of interest is the possibility of more closely aligning the mathematical practices of university students with those of their expert mathematician teachers. In their study comparing the ways mathematicians and graduate students learn an unfamiliar proof, Wilkerson-Jerde and Wilensky (2011)

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found “experts are more likely to refer to definitions when questioning or explaining some aspect of the focal mathematical idea” (p. 21). Assuming that defining and proving are practices at the core of mathematical work (Ouvrier-Buffet, 2011; Zandieh & Rasmussen, 2010), we consider addressing the intertwining, between proving and defining at university, a promising research avenue. By intertwining between proving and defining, we refer to mathematical practices in which the proving process calls for defining new objects, properties, or relationships that will be used further for proving new results in an iterative process.

In previous research, we have shown the relevance of adopting a semantic perspective on proof and proving in mathematics education. In such a perspective, the roles played by objects, properties, and relationships in the proving process appear to be crucial (Durand-Guerrier, 2008; Weber & Alcock, 2004). Nevertheless, it is rather common that this is hidden in usual teaching practises. This is for instance the case for real numbers. Indeed, it is common, at least in France, to define the *set of real numbers* by a list of axioms, including an axiom guaranteeing completeness; here we would avoid explicitly addressing issues related to the incompleteness of $\mathbb{Q}$ which, in the second half of nineteenth century, motivated the *creation of real numbers*. For example, a classical way of stating the Intermediate Value Theorem (IVT) can be found in the Concise Oxford Dictionary of Mathematics (Clapham & Nicholson, 2009):

> If the real function f is continuous on the closed interval [a, b] and η is a real number between $f(a)$ and $f(b)$ then, for some c in [a, b] $f(c) = \eta$.

Since a usual definition of a real function is “a function on the set (or a subset of the set) of real numbers,” the domain of a real function might be an incomplete subset of the set of real numbers (e.g. an interval on the set of rational numbers). This is hidden in the way the IVT is shown above. Indeed, as is generally the case, the fact that the considered interval is real is implicit. Consequently, a discussion that the IVT does not hold if the interval is, for example, an interval on the set of rational numbers is unlikely to appear. We note that addressing the condition on the interval could engage the search for counterexamples and therefore consider real numbers as objects.

One could suppose that, at the secondary/tertiary transition, it is enough to work in the set of real numbers without addressing these advanced questions. Nevertheless, research findings show that students entering universities may still have difficulty recognizing the nature of numbers based on their representations (e.g., considering that a number can be both finite decimal because it is written with a decimal point and irrational because it is written under a root – e.g. $\sqrt{13{,}21}$) (Durand-Guerrier, 2016, p. 344). A didactic research question is therefore “how is it possible to improve students’ conceptualisation of real numbers?”. We address this question on two features of real numbers – the $\mathbb{R}$-completeness *versus* the $\mathbb{Q}$-incompleteness, and potential infinity *versus* actual infinity – through the lens of the intertwining between proving and defining.

In the first part of the paper, we illustrate the intertwining of proving and defining in the case of the construction of irrational numbers by Dedekind (1872) and Cantor

(1872), recalling that for both authors the need for these constructions emerged from proving issues. We first briefly present the epistemological paths. Next, we present an example of a situation involving the ℝ-completeness/ ℚ-incompleteness that has the potential to foster students' engagement in intertwined proving and defining practices. In the second part of the paper, we explore the intertwined relationships between practices of enumeration, the definition of infinite sets, and diagonal proofs that the set of rational numbers is denumerable while the set of irrational numbers is not. We suggest that such an approach might help students to overcome the difficulties they faced for what concerns infinity (e.g., Tsamir, 2001).

11.2 Defining to be Able to Prove – The Case of Irrational Numbers

For a long time, real numbers have been used in mathematical activity without being formally defined. It was enough to know, from Eudoxus, that the rational numbers fail to solve certain problems in mathematics, mainly with regards to magnitude and measure. Relying on geometrical arguments to prove theorems of analysis has long been a common practice. Bolzano was one of the first to address this issue in his memoir on the Intermediate Value Theorem (Bolzano, 1817). Until the beginning of the nineteenth century, the proof relied on geometrical considerations that he considered inadmissible, as he claimed in the preface to the memoir. Despite this strong concern, Bolzano did not give a full definition of real numbers. It was only in the second half of the nineteenth century that Dedekind and Cantor provided formal definitions of the real numbers in very different ways and with different motivations which we recall below.

11.2.1 Defining Irrational Numbers by Cuts (Dedekind, 1872)

In 1872, R. Dedekind published an essay entitled: *Stetigkeit und irrationale Zahlen*.[1]

In this short text, Dedekind proposed a completion of the set of rational numbers relying on the notion of cuts. He refers explicitly to an analogy with the intuitive continuity of the graphic line that he characterized by the following "axiom":

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions (Dedekind, 1963, p. 11).

[1] We use here the English translation by W. W. Beman in Dedekind (1963).

This inspired him regarding how to complete the set of rational numbers. He defines a cut of the set of rational numbers with the standard order as a partition into two subsets A and B such that every element of A is smaller than or equal to every element of B. He then proves that there exist infinitely many cuts which do not correspond to a rational number, and claims that: "In this property that not all cuts are produced by a rational number consists the incompleteness or discontinuity of the domain R of all rational numbers" (Dedekind, 1963, p. 15). Then, whenever a cut is not produced by a rational number, he creates a new, irrational, number, Finally, Dedekind proves that the new set, which consists of all rational and irrational numbers, is complete for the cut procedure (i.e., in the new set, every cut is operated by one and only one element of the set), that means, in other words, that it is continuous.

In the preface of the text, Dedekind asserts that he felt the need for a truly scientific foundation for arithmetic to overcome geometric intuition which he did not consider a sound foundation for differential calculus. This concern echoes that of Bolzano regarding the use of geometry for proofs of the Intermediate Value Theorem. Such proofs are based on the idea that as soon as two straight lines intersect, there is a point corresponding to the intersection. Dedekind proved that this is not true if we consider the line of rational numbers. The creation of irrational numbers allowed him to create a set such that the associated number line has this property of continuity. Consequently, given two axes with a common origin and a unit on each, the intersection of two geometric lines corresponds to a pair of real numbers. This provides a *theoretical foundation* for our *geometrical intuition*, and a solid reference for going back and forth between *algebraic* and *graphic* registers (e.g., conjecturing the existence of solutions in the graphic register and then using the Intermediate Value Theorem to prove it).

11.2.2 Defining Rational Numbers as Fundamental Sequences (Cantor, 1872)

While working on trigonometric series, Cantor considered the need for developing a theory of real numbers to be able to prove the uniqueness of the development of a given function into a trigonometric series. He exposed it in the first paragraph of the essay of 1872 on trigonometric series. Cantor defines an extension of the notion of *numerical magnitude* that he will name in 1883[2] *fundamental sequence*: the concept corresponds to what we call today "Cauchy sequences," namely sequences of rational numbers such that the difference between two terms of the sequence becomes less than any assigned value beyond a certain rank, formally:

[2]This was a revised version of the initial text of 1872.

Definition: a sequence u of rational numbers is a fundamental sequence *if and only if*:

$$\forall \varepsilon \in \mathbb{Q}^{+*} \exists p \in \mathbb{N}^* \forall (n, m) \in \mathbb{N}^2 (n \geq p \wedge m \geq p \Rightarrow |u_m - u_n| < \varepsilon)$$

($\mathbb{Q}^{+*}$ is the set of strictly positive rational numbers, and $\mathbb{N}^*$ is the set of non-zero natural numbers.)

For Cantor, such fundamental sequences should converge. However, some of these sequences do not converge in the set of rational numbers. We will not develop here the theory by Cantor, but we would like to stress his concern to avoid presuming the existence of a limit for a fundamental sequence of rational numbers before having created the irrational numbers. Once his theory developed, Cantor established the link with the number line by stating that, once an origin and a unit have been chosen, each point is defined by its abscissa. He showed that in the case of an abscissa not being rational, there exists at least one fundamental sequence that determines it, and he added an axiom ascertaining that "for every numerical magnitude, there corresponds a definite point of the line, whose coordinate[3] is equal to this numerical magnitude" (quoted in Kanamori, 2020, p. 225).

11.2.3 Impact of the Way of Defining Real Numbers on Proving

Comparing Dedekind's and Cantor's development of the set of real numbers highlights the intertwining between defining and proving.

The goal for Dedekind was to avoid having to rely on geometric arguments when working in differential calculus. This leads him to take the continuity of the straight line as a reference for its theoretical construction which appears as "a formalization of the intuition conveyed by the continuity of the line" (Durand-Guerrier, 2016, p. 339). In doing so, he identified the incompleteness as gaps in the rational line and he chose a definition that allowed him to eliminate these gaps in one go. He was then able to prove that the domain of real numbers was a complete ordered set. He then proved the theorem:

> If a magnitude x grows continually but not beyond all limits, it approaches a limited value (Dedekind, 1963, p. 24–25).

This corresponds in modern terms to the theorem: "If a function is increasing and bounded, then it admits a limit".

Dedekind stressed that:

> This theorem is equivalent to the principle of continuity, i.e., it loses its validity as soon as we assume a single real number not to be contained in the domain $\mathfrak{R}$ (ibid, p. 25).

[3] The author uses "coordinate" where we have used "abscissa".

Cantor's goal, on the other hand, was to prove a theorem on trigonometric series involving convergence problems that lead him to think of numerical quantities as series with a specific property guaranteeing convergence (e.g., Kanamori, 2020, p. 224). Once done, he was able to prove the uniqueness theorem, which had been the motivation of his construction. Nevertheless, he also returned to the number line, proving in its theory the one-to-one correspondence between the points on the line and the real numbers. Hence both authors make explicit the link with the line through an axiom: for Dedekind by using the analogy with a property of the geometric line, which in his view captures the essence of continuity; for Cantor by introducing an axiom making explicit the one-to-one correspondence between the set of real numbers and the line as soon as an origin and a unit are chosen. Both Cantor and Dedekind have elaborated the set of real numbers as a complete ordered set - if we formulate it in modern terms - through very different ways. In the modern university mathematics curriculum, both ways play a crucial role. The vision by Cantor is linked to the role of fundamental sequences as a path for proving convergence of sequences. The vision by Dedekind is closely linked to the concept of supremum, which is difficult to master even by advanced students as shown by Bergé (2010). While Cantor's construction involves the concept of limit, in Dedekind's approach the central concept is order. In this regard, we hypothesize that offering students activities aimed at promoting an adequate appropriation of the two approaches would allow them to perceive the close relationship between defining and proving for their own mathematical activity. An example of such activities could be developed in the case of the IVT already mentioned. Considering different types of proof, relying either on adjacent sequences or on the supremum, and discussing why such proofs do not hold when working on an interval in $\mathbb{Q}$ will contribute in our view to such appropriation. Another example is presented below.

11.2.4 A Didactic Situation to Address Issues Related to $\mathbb{R}$-Completeness Versus $\mathbb{Q}$-Incompleteness

In Durand-Guerrier (2016), we presented a didactical situation aimed at fostering the conceptualisation of the continuum (the completeness of the set of real numbers) by discussing the existence of a fixed-point for an increasing function depending on the domain of the function (Fig. 11.1).[4] Among other didactic issues, this didactical situation leads us to consider, at least in-action, the intertwining between the types of numbers involved and the proving process. In this chapter, we focus on this aspect. We first recall the outline of the situation and then we focus on this intertwining.

Looking at this statement, we foresee that the nature of the numbers in the different questions will play a role. The first question concerns the initial segment of the set of natural numbers, which are finite discrete sets. Generalisation with the

[4]For a priori and a posteriori analysis of this didactical situation, see Durand-Guerrier, 2016, pp. 349–359.

Let us consider a function f of $\{1, 2, \dots n\}$ into $\{1, 2, \dots n\}$, where n is a nonzero natural number, and where f is supposed to be increasing (including non-strictly increasing functions); show that there exists an integer k such that $f(k) = k$; k is named a fixed-point. Then, study possible generalizations in the following cases, with f an increasing function.

1. $f: \mathbb{D} \cap [0;\ 1] \rightarrow \mathbb{D} \cap [0;\ 1]$, where $\mathbb{D}$ is the set of finite decimal numbers,

2. $f: \mathbb{Q} \cap [0;\ 1] \rightarrow \mathbb{Q} \cap [0;\ 1]$, where $\mathbb{Q}$ is the set of rational numbers,

3. $f: [0;\ 1] \rightarrow [0;\ 1]$,

or any other generalization.

Fig. 11.1 The fixed-point problem given to students

interval on the set of decimal numbers (respectively of rational numbers) concerns an interval of an infinite dense uncomplete ordered set. At the same time, the third question concerns an interval of the set of real numbers (this is implicit), which is a dense complete ordered set.

In the first case where the domain is a starting segment of the set of positive integers, it is possible to construct a proof by reductio ad absurdum that relies on the fact that every integer has a successor, that implies that:

$$\forall (p, q) \in \mathbb{N}^2 (p > q \Rightarrow p \geq q + 1)$$

It is also possible to provide a proof by induction. Still, in both cases, the argument relies on this property, closely related to Peano's axiomatic definition of the set of natural numbers. The students had not learned this definition, but the property was available because it is often used in class.

Once moving to the set $\mathbb{D} \cap [0;\ 1]$ or to the set $\mathbb{Q} \cap [0;\ 1]$ we lose the successor property because $\mathbb{D}$ and $\mathbb{Q}$ with their standard order are dense sets (i.e., for any pair of elements, there exists an element in-between different from both initial elements). Consequently, it is not possible to adapt the proof provided in the case of positive integers. From a geometric point of view, it may seem obvious that any increasing function satisfies the property. But this is not the case, precisely because -- as was said by Dedekind -- there are infinitely many more elements on the number line than in the rational number set (this counts also for the decimal numbers set). As expert mathematicians aware of this incompleteness, we know that the statement is not true for question 1 and 2, which means there are counterexamples. Finding such counterexamples requires our referring to decimal (rational) numbers as objects owning properties (semantic aspect). The fixed-point problem resorts to the following result: given an ordered set E that is a complete lattice (i.e., every non-empty subset has both a greatest lower bound and a least upper bound), every increasing function from E on E has at least a fixed point. (See Appendix). In Durand-Guerrier (2016), we have shown, based on Pontille et al. (1996), the relevance of this problem for raising epistemological questions linked to ordered-completeness (the paper reports on a long-term experimental setting with volunteer students in grade 11 in France). Such questions are crucial for an adequate conceptualisation of real numbers and are nearly never addressed even in undergraduate studies. In the same paper, we also reported that prospective secondary mathematics teachers, having followed an Analysis course during their undergraduate studies, showed behaviours very similar to those of the 11th graders for the three first cases: the initial segment of the set of natural numbers, the interval $\mathbb{D} \bigcap [0; 1]$, and the interval $\mathbb{Q} \bigcap [0; 1]$.

11.3 Enumeration, Infinite Sets, and Diagonal Proofs

In this section, we examine the intertwining between defining and proving in the case of infinite sets. We first discuss *defining infinite sets* focusing on two contrastive ways and considering the impact of the chosen way on proving. Next, we present some issues of comparing the size of infinite sets, enlightening the intertwining between defining and proving, focusing on the diagonal proofs by Cantor that $\mathbb{Q}$ is denumerable while $\mathbb{R}$ is not.

11.3.1 How to Define Infinite Sets?

As is well-known in history and philosophy of science, in the *Dialogue on Two New Sciences*, Galileo Galilei (1638) discussed the following paradox: there are as many squares of positive integers as positive integers, while there are many positive integers that are not squares of positive integers. Such a paradox contradicts the Aristotelian claim that the *whole is greater than each of its proper parts*. This leads Galileo to reject the possibility of comparing infinite quantities, or more precisely, of considering quantities that are potentially infinite as a whole. We learn from this that it is possible to provide an unlimited list of elements in one-to-one correspondence with the unlimited list of integers, in other words, *an enumeration*. However, considering the whole collection of elements of such a list introduces a contradiction with the Aristotelian principle recalled above because of the strict inclusion in the set of natural numbers. Today, modern mathematics relies mainly on set theory, including the definition of infinite sets. We present below two contrastive ways of defining infinite sets.

11.4 Infinite Sets as Non-finite Sets

In his book "How to prove it," Velleman (2006) claims:

> One-to-one correspondence is the key idea behind measuring the size of sets and sets of the form $\{1, 2, \ldots, n\}$ are the standards against which we measure the sizes of finite sets (Velleman, 2006, p. 306).

Then, Velleman relies on this idea to define the relation "to be equinumerous" and provide definition of finite sets and then *infinite* sets, as *non-finite* ones.

> Definition 7.1.1. Suppose A and B are sets. We'll say that A is *equinumerous* with B if there is a function $f\ A \rightarrow B$ that is one-to-one and onto.[5] We'll write $A \sim B$ to indicate that A is equinumerous to B. For each natural number n, let $I_n = \{i \in \mathbb{Z}^+ / i \leq n\}$. A set A is called finite if there is a natural number n such that $I_n \sim A$. Otherwise, A is infinite (ibid., p. 306).

The author notes that for a finite set, there is exactly one natural number n such that $I_n \sim A$, which is named the cardinal of the set.

Thus, the definition of finite sets by Velleman appears as a formalization of the intuitive notion of cardinal of a finite collection, as developed from primary school, which underlies the abstract concept of natural number (e.g., Vergnaud, 2009, p. 85). It is important to note that, for this definition, it is not necessary to refer to the infinite set of natural numbers, which refers to the actual infinity as defined in set theory. Indeed, it is sufficient to consider the unlimited list of natural numbers, which refers to the potential infinity (a dynamic form of infinity, a process that continues endlessly), which, as Fischbein (2001, p. 310) stated, is easier to conceive than the actual infinity.

11.5 Infinite Sets as Violating the Principle the "Whole is Greater Than the Part."

In his essay "Was sind und was sollen die Zahlen," published in 1888,[6] Dedekind defines an *infinite set* as a set that can be put in one-to-one correspondence with a proper subset of itself, and a *finite set* is defined as a *non-infinite* set (Dedekind, 1963, p. 63). In doing so, Dedekind assumes a theoretical principle that violates the Aristotelian postulate, which has long prevented mathematicians and philosophers from accepting the consideration of infinite sets, as for example, C.F. Gauß in a letter to Schumacher on 12 July 1831 (Peters, 1860).

In a way, Dedekind's definition completes the movement initiated by Bolzano (1851) in his short essay "Paradoxien des Unendlichen", where he recognises the property for infinite sets that: *there exists a bijection between the set itself and a proper subset of it* but rejects the conclusion that the two would be equal in term of equinumericity, as this would have contradicted strict inclusion (for a discussion in French, see e.g. Sebestik, 1992, pp. 186–189).

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[5] The author seems to use one-to-one correspondence as synonym of injective rather than bijective; this would explain the precision that the correspondence is also onto (surjective).

[6] English translation by W. W. Beman in Dedekind (1963)

11.5.1 Impact of the Ways of Defining on Proving That a Set Is Infinite

We have seen above two different ways of defining an infinite set, one by Dedekind (1888) the other in a modern textbook by Velleman (2006), which are likely to have an impact on how to prove that a given set is infinite.

Having defined an infinite set as a non-finite one, Velleman (2006) assumes that the definition of *equinumerou*s can also be applied to infinite sets, with results that are sometimes surprising" (op. cit. p. 307). To illustrate this, he proves that $\mathbb{Z}$ and $\mathbb{Z}^+$ on the one hand $\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+$ on the other hand, are equinumerous (pp. 307–308). He takes for granted that $\mathbb{Z}^+$is infinite, which of course, is intuitively true. According to H. Benis-Sinaceur,[7] this was the position of Cantor. As we have seen above, this was not the case for Dedekind, who first defined theoretically infinite sets and then proved that the set of integers is infinite. It is noticeable that the definition by Velleman, of being *infinite* for a set as the negation of the property of being *finite,* might direct one to engage in *reductio ad absurdum* to prove that a given set is infinite. It is assumed that the set is finite; this allows one to consider a finite enumeration (the finite list) of all the elements of the set; then, providing an element of the set that is not in the list shows a contradiction. The conclusion is the rejection of the initial hypothesis.

This way of defining finite set by Velleman is in a certain sense *external* because it needs to refer to the starting segment of the set of positive integers. Consequently, it does not explicitly address the paradox pointed out by Galileo and discussed (among others) by Bolzano. However, asserting that the definition of equinumerous can be applied to infinite sets is a theoretical demand leading to giving up Aristotle's postulate "*the whole is greater than the part*."

In Dedekind's approach, the one-to-one correspondence between a given set and a subset plays the key role as a characteristic of infinite sets. This way of defining *infinite* and *finite* sets is *internal*. This could direct one to prove that a given set is infinite, finding a one-to-one correspondence between the set and one of its proper subsets. This way of acting leads to proving that the set $\mathbb{N}$ of natural numbers is infinite by considering the one-to-one correspondence ϕ between $\mathbb{N}$ and the subset of perfect squares defined by $\phi(n) = n^2$;[8] this solves theoretically the paradox raised by Galileo.[9]

This is a new illustration of the intertwining between defining and proving practices in mathematics activity. In both cases, the aim is to be able to deal with an *infinite set* by "accepting theoretically" that while in a proper subset of a given set A, there are *fewer* elements than in A (some elements in A are not in the subset), it could be considered that there are *as many elements* in both. In both cases, the one-

[7] In a footnote in Dedekind (2008), p. 173 (our translation).

[8] Or other subsets: e.g. the subset of even (odd) numbers.

[9] This is not the way chosen by Dedekind who defined *simply infinite systems* and named the elements of such a system *natural numbers* (Dedekind, 1963, 67–70).

to-one correspondence plays the central role, and it is necessary to *create an object*, but they are of different types: an element that is not in the list in the first case (the proof by contradiction); a function in the second. Nevertheless, it is not always easy to find a relevant object in both cases.

In the same period where Dedekind was defining theoretically the concept of an infinite set, Cantor, taking first for granted that the set of integers was infinite, investigated questions relative to the comparison of the size of infinite sets.

11.6 How Big Is Infinity?

The informal question in the title is known to be at the source of the work by Cantor that led him to define *transfinite* numbers. More precisely, Cantor shared with Dedekind in November 1873 the concern below:

> He asked whether it is possible to assign in a unique way (by this, he meant unique in both directions) the set ('Inbegriff') of the positive whole numbers to the set of positive real numbers. He guessed that this is not the case (Jahnke, 2001, p. 178).

Cantor first paid interest to countable sets. Intuitively, these are sets that can be written down in a list; mathematically, these are sets that can be put in one-to-one correspondence with a subset of the set of natural numbers; a countable set may be either *finite* or *denumerable* (Borowski & Borwein, 2002, p. 125).

As already pointed out above, for finite discrete collections, this conforms to the practice of *enumeration,* which is related to the everyday experience of pointing at objects to be able to determine the number of elements (the cardinal) of the considered collection. After having given the formal definition:

> Definition 7.1.4. A set A is called denumerable if $\mathbb{Z}^{+} \sim A$. It is called *countable* if it is either finite or denumerable. Otherwise, it is uncountable (Velleman, 2006, p. 310).

Velleman comments on it:

> You might think of countable sets as those sets whose elements can be *counted* by pointing to all of them, one by one while naming positive integers in order. If the counting process ends at some point, then the set is finite; and if it never ends, then the set is denumerable (ibid, p. 310).

On the mathematical side, following the definition, a set A is countable iff there is an injective function from A to $\mathbb{Z}^{+}$. If there is also an injective function from $\mathbb{Z}^{+}$ to A, then there exists a bijective function from A to $\mathbb{Z}^{+}$ (Cantor-Schröder-Bernstein-theorem); hence A is denumerable (countable infinite).

The set of perfect squares of positive integers is a countable subset of the set of positive integers, and the process of counting "never ends" by construction. For such a set, the "intuitive" definition above might be enough to be convinced that it is denumerable since there is a relatively natural way of counting, but this is not always the case. Moreover, it seems relatively clear that this definition is more consistent with the idea of potential infinity (unlimited list) than actual infinity (infinite set).

Nevertheless, it suggests the idea of enumeration, a kind of one-to-one correspondence that might be sought when trying to prove that a given set is denumerable. Moreover, this makes explicit that the interest in *countable* sets is related to the possibility of expanding our experience with finite sets to infinite sets. This may have given Cantor the idea of proving that $\mathbb{Q}$ is denumerable by concretely proposing an enumeration.

11.6.1 The Diagonal Proof That $\mathbb{Q}$ Is Denumerable

The idea that $\mathbb{Q}$ is infinite countable (denumerable) is not intuitive (e.g., see Branchetti & Durand-Guerrier, 2018, p. 14) because with its standard order it is a dense set, that makes it impossible to isolate elements as is the case for discrete sets, and that appears to be a condition for enumerability. As underlined by Jahnke (2001, p. 193), this presupposed a complete change of perspective. The classical proof is done in three steps. The first step consists in providing an enumeration of the Cartesian product $\mathbb{N}^* \times \mathbb{N}^*$, then to consider the canonical injection f from $\mathbb{Q}^{+*}$ to $\mathbb{N}^* \times \mathbb{N}^*$ defined by $f\left(\frac{p}{q}\right) = (p, q)$ where *p and q* are relatively prime, and finally, to provide a bijection from $\mathbb{Q}^{+*}$ onto $\mathbb{Q}$. What we learn from this proof is that Cantor relies on our common experience of the finite to develop a method of enumeration, instead of looking directly for an explicit bijective function. In the case of the diagonal enumeration of $\mathbb{N}^* \times \mathbb{N}^*$, it is possible to express the corresponding function as an algebraic one. However, it would be surprising if such a function could be found directly, without relying on an enumeration. The diagonal procedure refers to defining a denumerable set as a set for which it is possible to provide an enumeration. This is a way of providing a one-to-one correspondence. It is noticeable that *enumerable* is sometimes used instead of *denumerable* and that in this context, "one-to-one correspondence" is in general preferred to "bijection" or "bijective function."

11.6.2 The Diagonal Proof That $\mathbb{R}$ Is Not Denumerable

The proof that $\mathbb{Q}$ is denumerable with the diagonal procedure was first written by Cantor in 1873. As we mentioned above, the question *of whether the set of real numbers was denumerable or not* was the question shared by Cantor with Dedekind in November 1873. One year later Cantor published a paper (Cantor, 1874) with a proof that the set of algebraic numbers was also denumerable, while the set of real numbers was not. In 1891, Cantor provided a simpler proof of the result using sequences. In line with the diagonal proof for the set of rational numbers, the question turns on the following: is it possible to provide an enumeration of the set of real numbers? The proof consists of first showing that, for any enumeration of binary sequences, it is possible to give a sequence by a diagonal procedure which is not present in the enumeration. A proof by reductio ad absurdum allows one to prove that the set of binary sequences is not enumerable.

The same *diagonal argument* is then used to prove that the set of real numbers between 0 and 1 is not denumerable. A real number between 0 and 1 can be defined as a sequence of integers between 0 and 9 through its proper decimal expansion. Given an enumeration of real numbers, it is always possible to provide an element not in the enumeration by considering the sequence of terms on the diagonal and to change every term into a different term (e.g., changing to 0 if non 0 or to 1 if 0; or changing to $n + 1$ if n is between 0 and 8, and changing 9 to 0). The new sequence represents a real number but differs from all the sequences in the enumeration. By reductio ad absurdum, this shows that the set of real numbers is not denumerable.

The proving process relies on several ideas: a real number can be represented by a sequence, for example, by enumeration of the digits in its decimal expansion; if a set of sequences is enumerable, it can be listed; if not, given an enumeration of the sequences of the set, it is possible to provide a sequence that is not in the list, which presents a contradiction.

One might wonder why such proof could not work for the set of rational numbers proved to be denumerable. An argument is that in an enumeration of a list of sequences corresponding to rational numbers, there is no assurance that the sequence built from the diagonal would be a periodic one. In fact, it is possible to provide a list of sequences of rational numbers such that the diagonal sequence is not periodic. Hence, we cannot assert that in every such enumeration, there is a sequence representing a rational number out of the list; this, of course, is consistent with the proof that $\mathbb{Q}$ is countable (for a discussion in French and an example with the diagonal sequence representing the number π which is known to be irrational, see Vidal, 2003).

This illustrates once more the intertwining between proving and defining. Indeed, showing that the proof does not hold for the set of rational numbers needs one to consider the characterization of rational numbers as periodic decimal expansions. In contrast, the diagonal proof of denumerability relies on the characterization as a ratio of non-zero integers.

11.7 Didactic Implications

Moving back to our initial question, we will draw what we consider to be the main contributions for addressing the broad research question: “how can we improve students’ conceptualisation of real numbers?”

The report of the experiment of the fixed-point situation. in Pontille et al. (1996) showed its potential for engaging students in intertwined proving and defining practices. Indeed, students were able to address relevant aspects of the nature of the numbers considered: (1) they successfully used a property specific to ordered discrete sets; (2) they raised and solved the question of the possibility that for a function with domain in $\mathbb{D}$ or $\mathbb{Q}$, the graphical representation may show an intersection point that does not correspond to a fixed-point of the function. As we discussed in the introduction, this is an important feature of incompleteness that appears at this level (grade11) as a *concept-in-action* (Vergnaud, 2009); (3) the search for a counterexample in the case of the set of decimal numbers leads them to consider a decimal as a particular case of rational number; (4) the search for a counterexample in the case of the set of rational numbers makes them aware that an affine function with rational coefficients always has a rational fixed point, and that a well-chosen quadratic function could provide a counter-example. Hence, this situation presents on the one hand, the density, and on the other hand, the incompleteness of the set of decimal (rational) numbers. In addition, as reported in Durand-Guerrier (2016), observations made in implementation of a module for prospective secondary teachers show that this fixed-point problem is likely to illuminate the role of completeness in real analysis and the fact that a subset of the set of real numbers is not necessarily complete (Durand-Guerrier, 2016, pp. 357–358). As we saw in the introduction, for the Intermediate Value Theorem (IVT), the completeness of the interval on which the given function is defined is a necessary condition of application, which usually remains implicit in the lectures. In our view, the results that we just mentioned confirm the possibility, on the one hand, and the relevance, of the other hand, of addressing these issues in the secondary/tertiary transition, by designing appropriate didactical situations. We have already mentioned the possibility of addressing the issues of completeness/incompleteness by discussing the conditions of application to the IVT. This could help students to understand why the graphical reading of the coordinates of an intersection point might be misleading in some cases, but not in all cases, thus helping them to have a balanced use of the relationship between the numerical and graphical registers: neither unconditional confidence, nor complete rejection. In addition, the nature of numbers involved would be discussed, including the search for counterexamples, as we mentioned in the analysis of the fixed-point problem.

In the second section, the comparison between the two contrasting ways of defining infinite sets introduces two points of view useful in calculus courses: enumeration on the one side as one-to-one correspondence with the list of natural numbers; bijective function between a set and a proper subset. The diagonal proof by Cantor that $\mathbb{Q}$ is denumerable resorts to the first point of view and emphasizes the fact that the enumeration provides a discrete order on $\mathbb{Q}$. In contrast, $\mathbb{Q}$ with the standard order is dense. As Branchetti and Durand-Guerrier (2018) show, this may not be clear, even for advanced students. The diagonal proof by Cantor that the set of real numbers is not denumerable relies on the representation of real numbers as sequences corresponding to their decimal expansion, and the proof that this diagonal argument does not apply for the set of rational numbers relies on the characteristic of rational numbers as periodic decimal expansions. Examining the epistemological paths suggests working on various aspects of infinity at the secondary-tertiary transition: introducing Dedekind's theoretical definition to solve the Aristotle's paradox and engaging in proofs that some sets are not finite; developing enumeration practices for countable sets, this being related with discrete mathematics and computer science; discussing the relations between approximations based on potential infinity and limits based on actual infinity (Durand-Guerrier & Tanguay, 2018, p. 33). We hypothesize that these are prerequisites for students to understand the scope of the proof by Cantor that the set of real numbers is not denumerable, thus opening for the theory of transfinite numbers at a more advanced level.

An additional comment is a strong need for this kind of work in mathematics teacher training. Indeed, as pointed out by Winsløw and Grønbæk (2014).

> In high school, calculus is one of the most advanced topics, and it is usually taught in a quite informal style, leaving the teacher with delicate choices and tasks of explanation (p. 61).

Due to the role of the back-and-forth between numeric register and graphic register in secondary school, one would expect an adequate understanding of the $\mathbb{Q}$-incompleteness / $\mathbb{R}$-completeness and the link with the corresponding number line by secondary teachers (Branchetti & Durand-Guerrier, 2018). Although in general not mentioned in the curriculum, both potential infinity (e.g., a counting process that never ends) and actual infinity (as defined in set theory) are present in Calculus at the secondary level and are a possible source of difficulties that need to be recognized and addressed by teachers (Fischbein, 2001).

11.8 Conclusion

In this chapter, we explore the relationship between defining and proving, highlighting that these two practices at the core of mathematical activity are closely intertwined. In the first part of the chapter devoted to the creation of irrational numbers by Dedekind and Cantor, respectively, we have shown that the paths chosen by each author to reach a formal definition that satisfies them are closely linked to his search for a proof. In the second part of the chapter, we have discussed two different approaches for defining infinite sets. In the last section, we have discussed some didactic implications. Further investigations are needed to confirm the potentialities suggested by the epistemological study. Emphasising the intertwining between *defining and proving* instead of the relationships between *definitions and proofs* opens new research avenues on student's practices at the secondary/tertiary transition. In our view, this is likely to deepen students' conceptualisation of real numbers and of infinity beyond the mastering of technical skills.

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Appendix

Proof of the existence of a fixed-point for an increasing function *f* from [0; 1] to [0; 1].
Let A be the subset given by A = {x ∈ [0;1]|f(x) ≥ x}.
A is a non-empty subset of the set of real numbers (0∈A) and is bounded above.
Hence, it has a supremum α.
Let us prove that α is a fixed-point for f.
Let x be an element in A. Since x ≤ α and f is increasing, f (x) ≤ f (α);
hence f (α) is an upper-bound for A; since α is the supremum of A, α ≤ f (α) (1).
Since f is increasing, from (1) we infer that f (α) ≤ f (f (α)).
This means that f(α) is an element of A;
as α is the supremum of A, f (α) ≤ α (2).
From (1) and (2), we conclude that f(α) = α, e.g., α is a fixed-point for f.

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Part III
Research on Teaching and Curriculum Design

Chapter 12
Developing Mathematics Teaching in University Tutorials: An Activity Perspective

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Abstract In this chapter, we give an example of practice-oriented research in small group tutorials at university level, focusing on the tutor's pedagogical practice for promoting her students' mathematics meaning-making (MMM) and her own developing knowledge in teaching practice. In particular, we analyse, using grounded theory techniques, episodes from a tutorial in Linear Algebra. We focus on the interactions between the tutor and the students and the tutor's interpretations of students' MMM. Adopting an Activity Theory perspective, we seek relationships between the tutor's actions and goals in the *activity* of tutoring, with emerging tensions related to students' outcomes. Our analysis in the different layers of the activity indicates the complexity of the tutoring, identifying contradictions internal to the activity. These contradictions can be seen as central to practice, as revealed in practice-oriented research, and to a methodology in developing mathematics tutoring.

Keywords University tutorials · Contradictions · Activity theory · Developmental research · Teacher goals · Teacher questioning

12.1 Introduction

Practice-oriented research in university teaching of mathematics in this chapter is developmental research that relates closely:

1. the development of students' understandings of mathematics and mathematics meaning making and

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2. teaching design and practice, and the developing insights of the teacher in interacting with students.

We focus specifically on development of teaching in small group tutorials that inquires into the pedagogical practices focused on students' mathematical meaning-making (henceforth MMM), and the decisions and tensions that the tutor faces. We build on previous research in this area: for example, Nardi, Jaworski and Hegedus studied teaching in traditional Oxford University tutorials, characterising their findings through a "Spectrum of Pedagogical Awareness" (SPA – Nardi et al., 2005) in which knowledge and use of pedagogy were 'measured' alongside the conveyance of mathematical concepts, and comparison of tutors' pedagogic approaches (Jaworski, 2002). More recent reports on tutorial teaching come from Jaworski and Didis (2014) and Abboud et al. (2018) both of which we draw on further below.

It is well documented within university culture that the most common form of teaching at university level is *lecturing*: students are used to listening to their lecturers, copying from the board on which the lecturer writes, and going away to work by themselves on problems set by their lecturer. In a lecture culture, they are unfamiliar with being asked to express their own mathematical understandings: particularly, for them, *speaking mathematics* is extremely uncommon (e.g. Alsina, 2001; Artemeva & Fox, 2011; Pritchard, 2010; Sweeney et al., 2004). Even in tutorials, in an observation study, Jaworski (2002) pointed to the most common form of tutoring being exposition by the tutor; Sweeney et al. (2004), from interviews with students about tutorials, offered several reasons for students' *silence*, one being that students were afraid of being laughed at by their peers; Mali (2016), in observation of 26 tutorials in one university, found a spectrum of teaching approaches and student verbal contribution from students' silence to oral interaction.

Thus, in our study, familiar with this history of university teaching, the tutor's intention was to use a pedagogy focusing on MMM that enabled students to engage with mathematics, to express their understandings of mathematical terms and concepts, *their personal meanings,* and to build on insecure meanings through tutor and peer interactions in tutorial dialogue. Our research focus is *practice-oriented* in so far as we form a collaboration between researchers and the tutor-researcher. The practice here is the practice of tutoring to facilitate students' MMM. The "practice" involves students engaging with and making meanings of the mathematics they encounter; also, the tutor engaging and making meanings of pedagogic approaches and her use of pedagogic tools to engage the students in accord with her goals for their learning and understanding. We analyse data to determine relationships between the tutor's use of these tools and what she was able to learn about students' MMM in linear algebra. In doing so, we refer to the overall 'activity' of the tutorial in all its parts and use the term 'activity' in the sense of *activity theory* as we explain below.

12.2 Practice-Oriented/Close-to-Practice Research

In 2018, BERA, the *British Educational Research Association*, commissioned a study of what they called "Close to Practice Research" (CtP – Wyse et al., 2018), addressing the overarching research question, 'How can high quality close-to-practice research be characterised and enhanced for education in the UK?' In particular, they sought to inform the BERA statement on CtP as well as the national *Research Excellence Framework* (REF – ref.ac.uk), the system for assessing the quality of research in UK higher education institutions. Following a 'rapid evidence assessment' of published research papers that focussed on CtP research, the following definition emerged (Wyse et al., 2018), which also fits very well with our own study of tutorial teaching.

> *Close-to-practice research is research that focusses on aspects defined by practitioners as relevant to their practice, and often involves collaborative work between practitioners and researchers (p. 1).*

Our study involved the tutor-researcher (first author), one researcher (the second author) and two research assistants who supported data collection, the analytical process and associated theorising. With respect to the above definition, the tutor-researcher is a practitioner and also a researcher. As a practitioner she has goals for her students within the practices and culture of the university setting, focused on students' mathematical meanings and their development. As a researcher, together with her colleagues, she seeks to make sense of the ways in which her goals for her teaching and for her students' learning are realised (or not) and the issues and tensions that are a force for influence in achieving her goals. In terms of the SPA (Nardi et al., 2005), we would say that she is working at the fourth level which they describe as "Confident and articulate: involving considered and developed pedagogical approaches designed to address recognised issues; recognition and articulation of students' difficulties with certain well-worked-out teaching strategies for addressing them; recognition of issues and critiquing of practice (p. 293)."

We take further the analysis in Jaworski and Didis (2014) and Abboud et al. (2018). For example, these papers included an extract from a tutorial in which the main pedagogical tool was "questioning". There, the tutor's use of questioning was designed to reveal students' understandings of multivariable functions. Jaworski and Didis (2014) identified two different teaching goals in relation to students' MMM: one was to promote and the other to discern students' MMM. They argued that these two goals are often in tension, and they analysed the tutor's questioning in an attempt to see positive relations between questioning and students' MMM.

The tensions and the implied contradictions through the analysis of the same episode were more elaborated in the paper of Abboud et al. (2018) by using an activity theory perspective. The use of 'activity' there, and here, is the activity of university education in mathematics manifested in the tutorial. The tutor uses different means to mediate mathematical meaning; this creates tensions between the mathematical ideas that the tutor wants the students to understand and the ways she interacts with them and promotes the dialogue. Another tension discussed in the paper of Abboud et al. (2018) relates to the division of labour between the tutor and the students, implying possibly different goals. The tutor aims that the students develop a deep understanding of mathematics, while the students' goals include being successful in the course assessment which, for them, may imply that they need to be able to use more algorithmic procedures.

In this chapter, we analyse tutorial interactions from another tutorial in the same corpus of data, this time, in the context of Linear Algebra. We focus on pedagogical tools in and beyond teacher questioning. Pedagogical practices (including the use of tools) are indicated through the actions and goals of the tutor in her attempts to engage students meaningfully with mathematics. These actions and goals are embedded in the *activity* of tutoring in the context of the tutorials with *object* the students' MMM. We illustrate these practices through selected episodes and point to associated tensions and contradictions in the activity.

12.3 Our Use of Activity Theory

We draw on Leont'ev (e.g., 1979) with attention to *the three layers of activity* (see Fig. 12.1) to link *actions* and *goals* to *activity* and its *motive* or *object*. According to Leont'ev, "human *activity* is the non-additive, molar unit of life . . . a system with its own structure, its own internal transformations, and its own development . . . motivated within the sociocultural and historical processes of human life" (Leont'ev, 1979, p. 46). The actions and goals are central to the activity which is engaged and depend on *operations* and *conditions* in the environment in which activity takes place. According to Leont'ev, *motive* may not be conscious, whereas the *object* and *goals* are always conscious.

In what follows below, we see the *activity* to be the activity of university education in mathematics manifested in the tutorial; *motive/object* to be the desire of the tutor to foster students' MMM; tutor's *goals* to be her declared (to herself and

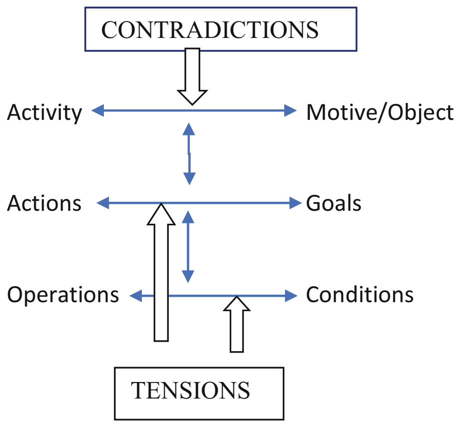

Fig. 12.1 Contradictions and tensions in Leont'ev's three layers of activity

others) intentions for achieving her object of activity; *operations* and *conditions* to be factors relating to the achievement of the object through actions related to goals (examples are the university infrastructure including curriculum and examinations, timetable, lectures and tutorials, as well as levels of culture affecting education, teaching and learning etc.).

The process of achieving the tutor's goals may lead to effective moments in the interaction and overall positive outcomes for the students. For situations in which this is not the case, we seek out the tensions in the interaction between the tutor and the students as well as the underlying contradictions in the activity. These tensions are often embedded in the activity as the participants (the tutor and the students) create their own images of the object of the activity (students' MMM). In our case, the tutor (as subject) interprets the object of the activity as students' meaning making while the students (as subject) may interpret it differently (for example, as being successful in the course assessment). These different perceptions of the object of activity often indicate tensions which appear in the actions of the subject related to difference between outcomes and goals. These tensions imply contradictions at the level of the activity, and as Roth and Lee (2007) argue "refuse to go away" (p. 187). They often indicate tensions between theory and praxis, epistemological and ontological aspects of human development or the disjunction between individual learners with other learners and their social environment. Referring to the work of Il'encov (1977) they see contradictions as accumulated inner contradictions at the level of activity and not "surface expressions of tensions, problems, conflicts and breakdowns" (p. 203). Similar contradictions exist in the way that the subject (here, the tutor) experiences, through division of labour (the different positions and roles of tutor and students), the activity as a whole (e.g., the educational environment, its affordances and constraints) and the particular (e.g., the particular interests of the individuals).

Stouraitis et al. (2017) elaborated further the meaning and the role of these contradictions in mathematics teaching and its development. They classified the contradictions on the basis of their dialectical character (called dialectical oppositions). Examples of dialectical oppositions related to mathematics were the pairs: structure—process, conceptual understanding—procedural fluency, intuition—logic. Dialectical oppositions such as individual–collective, teacher's guidance – student's autonomy and quality – quantity were of pedagogical character. Interpreting why two teachers coped with a particular contradiction (intuition and logic) in different ways, they considered the role of the professional communities in which these two teachers participated and their impact on the way they used curriculum recourses. Barab et al. (2002) consider tensions among the components of an activity system as a way to understand contradictions within the system. Tensions are the manifestations of contradictions; they can be cognitive and emotional emerging at the level of interaction or in the process of self-reflection (Chapman & Heater, 2010). In our study, we identify the emerging tensions in the actual interactions between the tutor and the students and also in the tutor's self-reflection. We consider tensions at the layers of actions and goals and their operations and conditions where the tutor uses specific tools to design the tutorials, interact with the students and reflect on them. Contradictions are the broader interpretations of these tensions at the activity level as we have elaborated in the previous paragraphs.

12.4 Meaning Making

There is a considerable literature on MMM at a range of levels (Kilpatrick et al., 2005). For example, first, Ben-Zvi & Arcavi, 2001 suggest that making meaning in mathematics is a process of "socialisation" into the "culture and values of 'doing mathematics' ('enculturation')" (p. 35). We take this idea further in an activity theory perspective below. Second, we can think of meaning making as *making connections*, both within mathematics and to the world beyond mathematics. Noss et al. (1997) write "[M]athematical meanings derive from connections: intra-mathematical connections which link new mathematical knowledge with old and extra-mathematical meaning derived from contexts and settings which include – though not uniquely – the experiential world" (p. 20). Third, Nardi (2008) refers to students "mediating mathematical meaning through symbolisation, verbalisation and visualisation" (p. 111), suggesting that students experience the tension between the need to *appear to be mathematical* (i.e., using the symbols etc. appropriately), or *to be mathematical* (make sense of, or understand the *meanings* of the concepts). Meaning-making has been studied in the context of interaction between the teacher and the students or the students themselves, usually in the setting of a school classroom. The work of Cobb et al. (e.g., 1990) has been influential in studying the emergence of mathematical meaning in the context of classroom interaction, making links to classroom norms, both social and socio-mathematical, that are established in the classroom. Yackel (2004) writes that "meanings grow out of social interaction, each individual's personal meanings and understandings are formed in and through the process of interpreting that interaction" (p. 5).

Scott (1998), adopting a Vygotskian perspective, has developed a framework that has as a major strand "Supporting student meaning making" that includes the forms of pedagogical intervention of promoting shared meaning and checking student understanding. The authoritative discourse of the teacher's interventions is mainly expressed by the transmissive function of teacher talk while the dialogic discourse is realized through the teacher's attempts to encourage students to express their ideas and debate points of view. Finally, the pedagogical interventions such as scaffolding are related to the zone of proximal development and a gradual withdrawal of assistance from the teacher to give responsibility to the student.

While Cobb and his colleagues designed classroom activity together with the teacher to promote constructivist goals (i.e., teaching designed on the basis of the researchers' theory), Scott's analysis is that of an educator-researcher analysing the practice of teaching and presenting a theoretical model (i.e., researcher generating theory from his observations of teaching practice). In close-to-practice research, we examine teaching in its raw state along with the developmental goals that underpin it where the teacher and the researcher collaborate. The aim of this study is neither to measure the teaching against prescribed teacher activity nor to fit a theoretical frame, but rather to reveal the goals and actions of the activity setting and offer insights to the ways in which activity embodies tensions and contradictions within close-to-practice research.

12.5 Methodology

We take a *close-to-practice* perspective within *developmental research*: that is research which informs ongoing practice as well as studying its development. The practice is that of tutoring. The tutor engages in tutoring, acting according to her goals for her students and developing her practice through actions and reflections. The research process offers a medium for reflection and hence for development of practice.

This practice takes place in a UK university in which first year mathematics students are placed into small groups (5–8) for tutorials (one hour per week for 12 weeks) with a mathematics tutor who is also their personal tutor for the whole of their degree. These students attend lectures with lecturers from the mathematics department who provide example sheets of problems for students to follow up their lectures. Tutors address students' questions about their mathematical work on the topics covered in lectures. We collected data from one semester's tutorials.

Data were collected in the practice setting (the tutorial) by a researcher-observer (one of the research assistants) who recorded and transcribed the words and actions of the tutor and students. Analysis was done jointly by the researchers, including the tutor-researcher who learned from the data analysis to influence future activity with students (Jaworski & Didis, 2014). We use *activity theory* to analyse the developmental process as tutor and students work (and learn) together and the tutor reflects on their interactions. In particular, we recognise issues, tensions and contradictions and analyse their contribution to the goals and outcomes of the practice we study.

We focus here on one tutorial (in week 8 of 12) in which tutor and students addressed questions from the lecturer's example sheet of 10 questions in linear algebra: focusing on questions 1a, b, c, d (see Fig. 12.2).

Our research questions, developed through several stages of analysis, are:

1. What are the tutor's pedagogical practices and their associated goals for promoting the students' MMM?
2. What can we learn from goals and outcomes in the activity and particularly the issues, tensions and contradictions that can be discerned?

In (2) the 'we' includes the tutor, the researchers, and the wider research community who are interested in such close-to-practice research.

1. Find $[\phi]_\beta^\beta$ for each of the following examples. i.e. find the matrix of the linear transformation in the ordered basis β:

(a)

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ y-z \\ x+z \end{pmatrix}$$

with ordered basis $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(b)

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+2y+3z \\ 3x+y+2z \\ 2x+3y+z \end{pmatrix}$$

with the standard basis $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

(c)

$$\phi : \mathbb{R}_3[x] \to \mathbb{R}_3[x],\ p(x) \mapsto p(1-x),$$

with the standard basis $\beta = \{1, x, x^2, x^3\}$ for $\mathbb{R}_3[x]$.

(d)

$$\phi : \mathbb{R}_3[x] \to \mathbb{R}_3[x],\ p(x) \mapsto (1-x^2)\frac{d^2p}{dx^2} - x\frac{dp}{dx},$$

with ordered basis $\beta = \{1, x, 2x^2-1, 4x^3-3x\}$.

Fig. 12.2 Linear algebra questions

12.5.1 Analysis of Data

Through several stages of analysis, we analysed the tutor's pedagogic practices in relation to students' engagement with mathematics, and focused on the issues, tensions and contradictions recognised in this activity.

The tutor-researcher and the research assistants made an introductory pass through the data adding detail to the transcriptions from their recall of the actions in the tutorial (e.g., movement to and from the board, who was writing on the board and what was written). An initial analysis, grounded in this data, then addressed the action of the tutorial and its dialogue, seeking to make sense of the roles and intentions of tutor and students using as a tool the 'Teaching Triad' (Potari & Jaworski, 2002 – not discussed further here). The researchers read from the transcript, turn by turn, asking questions, making comments and forming initial codes. The observer then made another version of the transcript, including the codes and comments, and this formed the basis of another pass, turn by turn, through this enhanced data. During this analysis, the tutor reflected aloud, expressed her intentions for her interactions with the students and was questioned by her two colleagues, revealing the goals presented above. This provided an initial attempt to address tutor's goals in relation to the observed action.

An important part of *close-to-pract*ice research is the involvement of the tutor-researcher. Through reflection, she reveals what her intentions were in working with the students. The analytical dialogue makes it possible to reflect on what occurred in the tutorial and to critique her practice, much as an interview might have achieved. The growth of awareness and understanding for the tutor, as the researchers address the meanings involved in relation to the tutor's actions and goals, are key elements both of analysis and of development of teaching. Tensions are identified when the tutor finds herself acting to satisfy one goal at the expense of another. In development of teaching, it is not a case of removing the tension, but rather of *becoming aware of the choice in the moment* (Mason, 2002), with the option to act differently, or refine goals. Examples follow below.

A later further analysis between the tutor-researcher (first author) and her research colleague (second author) took up overtly the questions relating tutor goals to tutorial observations, starting to note areas of tension and/or contradiction. This later stage of analysis enabled identification of key episodes relating to research questions and a further critique and articulation by the tutor-researcher, prompted by her colleague. From this later analysis, we identified 12 episodes from which we have selected episodes which exemplify *key elements of practice* related to the seven goals and associated tensions/contradictions. The extracts we present below show examples of this analysis, addressing research question 1

1. tutor's attempts to support students' meaning making- unpacking the meaning of words; valuing students' wrong solutions and building on them (Episodes 3 and 4)
2. building on students' solutions to present a general statement, synthesizing/exposing (Episode 5),

We also see *sources of tension* which we discuss below to address research question 2.

12.6 Analysis of Dialogue in Key Episodes

12.6.1 Tutoring for Students' Meaning-Making – Actions and Goals

We have discerned 7 goals arising from the tutor's critical reflections, prompted by her research colleagues in the early stages of analysis and therefore becoming data for the later stage: four concern desired student actions, G1 to G4, i.e., to get students to:

G1
: express what they 'see' (in informal language), their images, their connections, their symbolic awareness, their thinking;

G2
: get used to talking about the mathematical concepts, to express ideas in words, and to link to formal mathematics representation;

G3
: listen to each other and build on what another person expresses;

G4
: feel comfortable about not knowing, and recognise that working together can enable more than they could do alone.

three, G5 to G7, focus on the tutor's own actions:

G5
: to phrase questions in ways to which students can respond;

G6
: to listen to the students and discern meaning from what they say;

G7
: to maintain a focus on the mathematics that is important, without telling, guiding, funnelling in ways that will foster a surface recognition without deeper meaning.

These goals together form a statement about pedagogy. However, goals have to be interpreted into practice and it is ultimately the actions in practice that impact on the students. Here we see an important aspect of CtP research: the research process encourages the tutor to critique her own practice and hence become more conscious of the relationship between goals and actions.

12.6.2 The Practice of Tutoring – Summary of Tutorial – Key Points

Although asked to do so, students have not offered any questions relating to the current week's lectures, so tutor (T) uses the example sheet from the Linear Algebra lectures to choose suitable questions to work on. (Qu 1 a,b,c,d) – they provide opportunity to rehearse key ideas and relationships (basis, vectors, span, vector space, matrix, linear transformation . . .).

T asks students to work first on question 1b and gives time for this – there is a low murmuring of conversation. In the discussion that follows, she uses a dialogic approach, questioning, prompting and guiding students on key ideas and encouraging them to participate. Students seem shy to give responses; it appears some of them can see what is required but do not have the language to express it. They become more willing to offer responses as the tutorial progresses. In the progress of the tutorial, T asks two students (Alex and Julia) to present their writing at the board – choosing them because they hint at understanding – either in what they say or what they have written. What is written on the board becomes a focus for further questions/input from T who offers certain meta-comments telling students what she is doing, or why she does something. Questions 1a and 1d (omitting c due to time pressure) follow in similar style.

12.6.3 The Episodes and the Grounded Analysis

In this section we present key episodes under the headings listed above.

1. Teacher's attempts for meaning making – unpacking the meaning of words (Episodes 3 & 4),

Episode 3, Turns 27–36. **Dialogue with Julia**

27 T: (says) So you start off with and (writes on the board) "The standard basis in R^3". Now, Julia, what is the standard basis in R^3? . . .
28 S: (Julia) the identity matrix?
29 T: Is a basis a matrix?
30 S: (Julia) No
31 T: okay, what is the difference?
32 S: (Julia) Vectors?
33 T: If I asked to write down standard basis in R^3 what would you actually write?
34 S: (Julia) we just write those three as a span [hard to hear – HTH]
35 T: Tell me symbol by symbol how to write the basis
36 S: (Julia) (says and lecturer repeats and writes it on the board) . . . curly bracket and vector 1,0,0 and the vector 0,1,0 (lecturer asks "anything between those?" and student says "comma", lecturer say "right, go on") and 0,0,1.

[Note: Three dots . . . indicates a very short pause or hesitation; HTH stands for 'hard to hear'.]

This episode addresses the meaning of 'basis': what does basis *mean* for the students? In dialogue with Julia, Julia is asked "what is the standard basis in R^3." Her response "the identity matrix" is not correct, but, for T, who interprets what she hears, it suggests a degree of meaning – although a basis is not a matrix, the columns of the matrix can form the basis, so (the tutor asks herself) is this a confusion with terminology or is it conceptual? To find out, T follows up with several questions to prompt the student for more explication so that she can more clearly infer the student's meaning. In turn 35, a more explicit question allows Julia to present what she understands: her responses at turns 32, 34 and 36 suggest that she has confused similar symbolisation and terminology in meanings of basis and matrix. T asks herself whether she could have gone more directly to the confusion by starting with a more precise question such as those at turns 33 and 35. How appropriate were the actions to the goals? Our interpretation is that the actions, the dialogue with the student, reveal some of the confusions that can arise for students between concepts and their terms. This informs T's interpretations and thus contributes to her own development and awareness of students' meanings.

We see the tutor's pedagogy here as asking questions to encourage students to participate and voice their meanings, primitive though these may be (G1/G2). Students should get used to being expected to respond, however imperfectly, and as they respond, T gains a sense of their quality of meaning (G6). As responses are made and terms used, other students can hear and become acculturated into the language and relationships (G3). When T offers explanation, they can hear a correct articulation of a concept (G2). Here there is a tension for the tutor in offering her own explanations to be sure that students have heard a 'correct' version – but they heard a correct version in the lecture, so why is this needed? How does the action fit with the goals? To address this tension she tries to get students to respond to each other, building the relationships collectively (G7) (e.g., as in Episode 4: 55–71). This also creates tensions as we see in Episode 5.

Episode 4 Turns 44–61 **Questions for All**

In a similar style, questioning and prompting students, seeking to involve them all, T emphasises the language of set, components of a set, vectors, vector space R^3 and links between them. In the dialogue following Episode 3 above, students have articulated that a basis is a set – T asks, "a set of what?"

44 S: (Carol) any vector in R^3? (HTH)
45 T: do you agree with that, Alun?
46 S: (Alun) any three component vectors... yeah...
47 T: do you agree with that, Erik?
48 S: (Erik) yes
49 T: what do you agree with?
50 S: (Erik) any set of three components (his response fades)
51 T: any set of three components?
52 S: (Erik? HTH)
53 T: I am not just asking you to repeat what somebody else said. I am trying to find out, do you actually know what we are talking about? (short pause – she points to what is written on the board) What we have here is a set, curly brackets indicate a set, in the set we have three elements separated by commas and each of them vectors. So that is the basis you are given, and, Carol could you tell us again what it was you said?
54 S: (Carol) any vector in set R^3
55 T: Now, what is the condition for such a set of vectors to be a basis?
56 S: (Carol) they are linearly independent
57 T: the vectors must be linearly independent (short pause) and?
58 S: (Alun) Span in ...
59 T: they must span in? ... go on ...
60 S: (Alun) span in R^3
61 T: They must span in R^3, so we are given a basis in the vector space R^3; it is a set of three vectors.

Here T overtly asks students to respond. She interprets students' use of language: some students give evidence of their development of meaning (G5, G6), for example in the meanings of vector, set, basis and span in R^3. At turns 53 and 61, she states her own meanings for the terms (G7). At 53, we see her use of a meta-comment, indicating to students that she wants them to try to articulate for themselves, not just to repeat what someone else offers.

All these elements are part of her pedagogy in practice, where goals are interpreted into actions. There are many ways of interpreting goals in practice. For example, she wonders if perhaps she was too unkind to Erik at turn 53. She is aware that students' responses may result from hesitancy to express what they see or think or indeed a wish to give an acceptable answer and thus escape further questions. Here we see another tension for the tutor as she interprets what students say and decides on an appropriate action. All of this is revealed in the tutor's reflections, first as she responds to students in the tutorial and subsequently as she reads and seeks to interpret critically (post hoc) the tutorial interactions between herself and the students.

12.6.4 Synthesizing/Exposing: Building on Students' Solutions to Present the General Solution Method – (Episode 5)

Episode 5 Turns 61–95 **Building on Julia's Ideas**

We continue here from the dialogue in Episode 4 above. Here we see T probing the meaning of Julia who (it seems to T) has suggested that she sees some relationship between the concept of identity matrix and the set of basis vectors.

61 (cont)T: mmm, somebody said something about the identity matrix a while ago, who was that? Was it you Julia?

62 S: (Julia) it was me.

63 T: Right, when you were thinking about the set of vectors, you mentioned the identity matrix; now what is the relationship between these? This set of vectors is not a matrix. What is the relationship between that set of vectors and the identity matrix?

64 S: (Julia) I don't know.

65 T: But you thought of something didn't you, when you said the identity matrix, and I said, we are not looking for a matrix. But actually, your mention of the identity matrix sort of indicated to me you could see a relationship between the standard basis and the identity matrix.

66 S: (Julia) yeah because the vectors, if you (HTH) there would be a (HTH)

67 T: if you form a matrix from these three vectors, it would be the identity matrix, so what you just expressed is a relationship between the standard basis and the identity matrix

[Note: We terminate this episode here, because of limitations on space, and include a short part of Episode 7 which is part of our analysis of Julia's MMM].

Episode 7118–128 (Section Only)

118

T: [T's words, referring to writing on the board, omitted] Now okay!!

(Tutor cleans what Julia wrote on the board, because it is wrong)

Any thoughts? What will you to do? Perhaps in order to see that, because here what we did worked because ... [again she is pointing at the board]

119

S: (Julia)do we standardize the basis?

120

T: standardize the basis. What do you mean by that?

121

S: (Julia)...(HTH)

122

T: I think you are on the right track. I think you mix two different things – come back to ...

[T continues to compare two solutions to show which of them provides the required transformation.]

Here we see T encouraging/guiding/scaffolding Julia so that she can make sense of the relationship between the set of vectors that forms the basis, and the identity matrix. It could be that the words "relationship between" are unhelpful, perhaps because Julia is not aware of what a 'relationship' implies. T rephrases, seeking to convince Julia that she has expressed something valuable. The words "see the relationship between the standard basis and the identity matrix" seem to encourage Julia to say more about what she sees (G2), convincing T that Julia could indeed see some relationship, even if this is still vague (G6). In any case, T has introduced the word 'relationship' so that students can (she hopes: interpretation again) become more familiar with what it implies mathematically G2). In Episode 5, an emerging tension concerns the guiding/scaffolding, sometimes lengthy exposition by T versus students' autonomous work (individual or in groups).

12.6.5 The Tensions Manifested in the Three Episodes

In these episodes, in contrast with a familiar tutoring style of demonstrating procedures and articulating solutions *for* the students, tutoring can be seen as a series of pedagogic actions that seek to *involve* the students: e.g., time for thinking and writing (parts of) a solution; questioning and asking students to articulate a concept, or some terminology, inviting a student to show a symbolisation on the board. This is a slow and lengthy process, building on incomplete statements, partial reasoning and interpretation of what is said. They could cover more questions from the example sheet more quickly if she demonstrated solutions herself, but this does not fit with her goals and actions. However, at times we see her shift into explanation mode, expressing concepts herself so that the correct solution or reasoning could be heard by the students.

The character of the dialogue can be seen to be controlled by T. She has chosen the questions on which to focus; she gives time for students to work on a question before they discuss it. Experience with the tutor group (this is week 8) demonstrates that these students struggle with concepts, finding it difficult to articulate mathematical relationships. There is some evidence that she is gaining their trust, they are responding even when they do not seem sure how to express what they see. One student, Alex, is confident enough to write a solution on the board twice, and is mostly correct, although he makes little contribution to voicing the mathematical concepts himself. Julia, while more hesitant seem to gain confidence and, ultimately, is willing to write at the board, and to volunteer a relationship (standardising the basis) even when not asked directly (119).

Tensions are manifested by a perceived lack of fit between goals and actions. Actions address tutor's perceptions in practice, whereas goals are more distant from the actions. Several tensions are revealed through the dialogue:

Episode 3: A tension for the tutor lies with the occasions when she answers her own questions, to be sure that students have heard a 'correct' version. The tension is related to the difference between the tutor's goals and (potentially) the students' goals (e.g., conceptual meaning making versus getting the correct answer).

Episode 4: At turns 53 and 61 T offers her own meanings for these terms (e.g., basis is a set of vectors in R^3) She is aware that students may seek a correct building up of a procedure they can reproduce in an assessment, or indeed a wish to give a correct answer and thus escape further questions. The tension for the tutor is related to balancing students' MMM with the procedural fluency required usually in the assessment.

Episode 5: an emerging tension concerns the guiding/scaffolding, sometimes lengthy exposition by T versus students' own articulations The tension involves guidance versus students' autonomy (doing it for themselves).

12.7 Analyzing the Episodes from an Activity Theory Perspective

Through the grounded analysis of the data, we have illustrated the tutor's goals and actions and, to some extent, students' engagement with mathematics; as well as emerging tensions in the teaching in the tutorial context. Reflecting on the level of the activity, the tutoring activity, allows us to go more deeply into relations between actions and goals in the interaction between the students and the tutor and the students themselves and to the achieved outcomes.

We view the tutor's pedagogic practices through her actions and the associated goals related to students' learning and to teaching. Supporting students' MMM involved students in: expressing their informal ideas; talking about mathematics and making formalisations; listening to each other's ideas; and valuing collaboration with their peers. Her goals for teaching included discerning students' meanings, using language accessible to students', avoiding too close guidance that would not allow students' MMM. Specific actions included questioning, prompting, providing input, unpacking the meaning of words, synthesizing students ideas and solutions to more formal results, guiding/scaffolding, valuing students' ideas even when incorrect, meta-commenting, controlling directions, encouraging students' participation. The goals and the corresponding actions were developed to achieve the *object* of the tutoring activity, students' MMM.

To what extent these goals and the corresponding actions achieved the object of the activity and resulted in positive outcomes for the students is not possible to conclude from the data and the analysis of this study. However, we can see that there was an increasing number of students who participated in the class discussion and expressed their ideas to the extent seen in the above episodes, something that it is usually uncommon in this setting. An interesting case is Julia who is very hesitant but, nevertheless, comes to the board and shares her idea; although it is wrong, the tutor addresses it to talk about a relevant mathematical idea that she knows that students confuse. Julia seems to feel more confident and willing to contribute further. Considering the actions and goals at the level of the activity of tutoring allows us to see how these interrelate according to the planned and on the moment decisions of the tutor in the light of students' contributions and meaning making.

As we have indicated in our grounded analysis the process is not smooth, it involves tensions that the tutor experiences that indicate contradictions in the activity and in the way that the tutor and the students understand its object/motive. One tension the tutor experienced was the focus on students' MMM, seeking conceptual awareness, while also aware that their concern for success in the assessment might prefer to be given procedural clarity. This also implies an epistemological contradiction between conceptual understanding and procedural fluency characterized as dialectical opposition in the study of Stouraitis et al. (2017). This dialectical character reflects also on the actions of the tutor who offers in Episode 4 her own meanings of the basic mathematical terms while at other parts of the episode her actions promote and attempt to discern students' conceptual understanding. The synthesis of these poles of the contradiction at the level of the activity allows students' engagement in mathematics and MMM meeting to some extent their goal that it is to be successful in the assessment. Balancing these goals is something that the tutor takes into account. The tension between rigour in mathematics and students' MMM implies also an epistemological contradiction close to what Stouraitis et al. (2017) refer to as 'the dialectical opposition logic versus intuition'. Logic is more related to the rigorous and formal aspects of mathematics while intuition to more informal and meaningful aspects of it. The tutor seems to handle this contradiction by on one hand controlling direction to formalizations and on the other allowing students to give informal and incomplete descriptions. Finally, the tension between the tutor's guidance and the students' autonomy implies also a contradiction referring to pedagogy. This contradiction relates to division of labour in the teaching activity where usually the teacher has the authority to guide students to achieve the object of the activity and the students usually do what the teacher asks them to do. It also has a dialectical character in the tutoring activity as guidance and autonomy are both sides of the same coin in the sense that they are inseparable in the tutor's attempt to achieve students' MMM. The tutor allows students time to work on the questions of the worksheet individually or in groups, express their ideas and share their solutions but she also guides them to make sense of the meaning of basic terms and develop strategies to solve the tasks.

The study that we offer in this chapter, as an example of close-to practice research, shows the complexity of the activity of teaching at the university level: in particular in the tutorials, in the actions and goals of the practitioner, the tutor often brings to the fore contradictions internal to the activity. Handling these contradictions and attempting to achieve positive outcomes for the students requires a continuous development of teacher thinking and consciousness in relation to the activity of teaching. The double role of teacher as practitioner and researcher allows this development and brings changes at the level of the activity and new realisations of its object/motive.

12.8 In Conclusion

Close to practice research here delves into the close relationships of students and tutor as they work together to develop meanings: mathematical meanings for the students, student meanings for the tutor. They learn simultaneously within their own action spheres, acting in relation to their own consciousness within the setting. The tutor controls the action while wishing the students to take control of their mathematics which is difficult for them (therein lies the tensions articulated above). Students, hesitantly, respond to what they see the tutor to want or be asking for, while constrained by their lack of mathematical confidence and fear of being wrong. The tutor has to encourage, probe, prompt, question and challenge the students, and nevertheless articulate the correct version of the concepts involved rather than leave them wondering whether the answer was correct or not. The use of activity theory to address tensions and contradictions, allows some theorisation of the sometimes conflicting challenges the didactic situation encapsulates.

The example of the study we offer as a close-to-practice research at the university level indicates several methodological/developmental issues related to the different levels of analysis (from grounded approaches at the level of interactions to the global level of the activity), and to the role of the researcher as practitioner who reflects on her actions and generates new data for analysis. Our focus also on tensions and contradictions helps us to develop new insights about the complex process of teaching and researching at the university level. Becoming aware of the contradictions and developing ways of handling them seems to be important for the development of mathematics teaching and the professionalisation of the teacher. Extending the research beyond this particular case study, involving more tutors/researchers would bring to the fore more social and institutional issues and a methodology for close to practice research and its contribution to developing mathematics teaching.

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Chapter 13
Lecture Notes Design by Post-secondary Instructors: Resources and Priorities

Vilma Mesa

Abstract Drawing from 21 post-secondary mathematics instructors' descriptions of how they use resources to plan lessons, I reflect on processes of lesson planning and identify three priorities instructors had regarding document design. Using various records instructors created (maps of resources, responses to survey questions, lecture notes) and following the documentational approach to didactics (DAD), I identified resources, features of the lecture and lesson notes, and processes and rationales for their production. As anticipated by DAD, instructors drew from a rich set of resources to develop these documents, but they did so by prioritizing either content, meaning of mathematical ideas, or assessment, as revealed by distinct instrumentation processes. Textbook content and individual preferences might help explain these results.

Keywords Lesson design · Documentational approach to didactics · Resource instrumentation · Teacher work · Lecture notes

Current conceptualizations of instructors' work describe it as intrinsically akin to *design*. The abundance of resources available to instructors to do such design work requires that they engage in creative processes when selecting, curating, and using these resources in order to address their goals and their perceived students' needs (Trouche et al., 2020). This creative process has been studied using the documentational approach to didactics, which acknowledges that instructors may produce and re-produce various types of documents for their work, using a rich system of resources; this system is structured with different components each fulfilling different purposes. Large scale studies of documentational work have been carried out mostly with school teachers (see e.g., Gueudet & Trouche, 2012a) under the aegis of school curriculum reform.

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However, school teachers operate under very different teaching conditions relative to post-secondary instructors. One significant difference is that schoolteachers must conform to external regulations, guidelines, and expectations dictated either nationally or locally and go through a rigorous certification process in order to be allowed to teach. It is also the case that through their careers they have frequent opportunities for continued professional development, which is mandated by their contracts. In addition, at least in the United States, student performance on standardized examinations has been used as an accountability mechanism to establish teacher and school effectiveness. Because of these conditions, there is usually more funding allocated to support and understand work of teachers in schools. Such conditions do not exist in the post-secondary setting; which courses are taught tends to be defined at a program level and their content is primarily determined by textbook availability. Individual departments may choose the textbooks, although it is also common for instructors to make their own decisions about the textbooks and materials they want to use. Instructors teaching in post-secondary settings are typically not required to take any pedagogical training and how well students do in their courses is rarely used as an indicator of their effectiveness. Moreover, funding to study post-secondary teaching is not a priority for agencies that fund educational research. More important for our case, is that distinct from K-12 settings, as courses get more advanced, there are fewer resources available to instructors. This might explain the small number of studies that have investigated this creative design process in university and post-secondary contexts, and the reason why most of these investigations are based on a handful of instructors (Gueudet, 2017; Gueudet & Pepin, 2018; Hammoud, 2012).

We are interested in the creative process involved in teachers' work at the post-secondary level, precisely because of the contextual and institutional differences that surrounds their work. We choose to focus on documents and their production because they can help understand how instructors conceptualize their work and how they use resources—specifically textbooks—given that instructors have autonomy in selecting them, especially in courses that are not departmentally coordinated. Knowledge about post-secondary instructors' work can provide the basis for directing the design of curriculum materials that support lesson design, and eventually teaching, in this context.

A first step in building this knowledge necessitates an understanding of post-secondary instructors' work as they produce documents that support their teaching. Using data from 21 instructors who taught a mathematics course in natural conditions[1] we explore the creation of one document that instructors used to teach a lesson, and that they named "lecture notes." We address the following questions:

- How do instructors describe the process of creating lecture notes for their courses?

[1] The only expectation was for the instructors to use one of the textbooks in the project, a calculus textbook (Boelkins, 2021), a linear algebra textbook (Beezer, 2021), and an abstract algebra textbook (Judson, 2021).

- What resources do instructors use when creating lecture notes?
- How are the resources instrumented to generate the lecture notes?

13.1 Theoretical Tools

We are interested in documentational genesis (Trouche et al., 2020) the process by which a teacher engages in instrumented activity with a set of resources that have been gathered for a specific purpose, a purpose determined by the class of situations that are part of their professional work and that are bounded by a particular context. In our case, a class of situations refers to planning calculus lessons for post-secondary students and the context is determined by the specific goal of planning the teaching of a particular topic—for example, the fundamental theorem of calculus, linear dependence, or groups. We chose the activity of planning lessons over designing assessments because planning is likely to require more resources, as instructors tend to rely on only their textbooks for generating assessments (Leckrone, 2014), and because planning can give us insight into what teaching, the enactment of those plans, might entail.

Using a set of resources involves the dual processes of instrumentalization—directed at the resources and indicating ways in which the teacher takes from the resources to fulfill a goal, and instrumentation—directed at the teacher, and indicating how those resources prompt some changes on the teacher. These process encompass epistemic, pragmatic, interpersonal, and reflexive mediations through which the instructor acquires new knowledge, skills, and practices—even if those are not consciously recognized (Trgalova et al., 2019).

Similar to what happens with teachers in K-12 schools, the documentation in post-secondary teaching includes looking for resources such as textbooks, instructional videos, visualizations, or paradigmatic problems that can be used to illustrate a particular point; discussing ideas with colleagues; attending seminars; or consulting previously generated documents (notes from prior courses, even from graduate school). A main point of departure is that there is more independence about the content that can be included in post-secondary courses, as there is no official body that dictates what should be included in the courses.[2] Universities and other post-secondary settings decide on their own how to organize the individual courses which usually results in individual faculty members making decisions about the textbook and the course materials that they want to use for their courses.

In the documentational approach, a document is the result of the orchestration of multiple resources through particular schemes of use (i.e., aims, rules of use,

[2] In the United States, and for some lower division courses, such as calculus, there are regulations about what content should be covered that are established by the individual states to ensure transferability of courses across institutions (see e.g., http://regents.ohio.gov/transfer/documents/bringing-down-the-silos.pdf and https://www.ohiohighered.org/Ohio-Transfer-36/learning-outcomes for outcomes for Calculus I and Elementary Linear Algebra).

operational invariants, and inference possibilities, see Trouche et al., 2020) of the resources. We principally seek to infer the operational invariants, as those represent rationales that guide the orchestration of the resources.

13.2 Methods

For this exploration we drew data from a large-scale study of use of open-source textbooks in the United States (Beezer et al., 2018). The participants taught with one of three textbooks over one semester: *Abstract Algebra Theory and Applications* (Judson, 2021), *Active Calculus* (Boelkins, 2021), and *A First Course in Linear Algebra* (Beezer, 2021), available in two formats, PDF or HTML. These textbooks were chosen because they represent different levels of content in an undergraduate mathematics curriculum in the United States: calculus is a first year course that is taken by a large number of students as it is nowadays a requirement for many programs; linear algebra is a course intended for mathematics and computer science majors, usually taught in the second semester of the first year or in the second year of the curriculum; abstract algebra is usually a course for math majors (including future secondary teachers). We anticipated that the different audiences of the courses would influence how instructors used their textbooks and other resources for their work. We describe next the textbooks used.

13.2.1 The Textbooks

The textbooks used in this project have been written in PreTeXt (https://pretßextbook.org), a purpose-built 'design neutral' markup language that allows renderings of the textbook in multiple formats, online, and print (including Braille). As open-access and open-source, they can be accessed from any device connected to the internet and modified for personal use. When updates are done, they are nearly immediately available. When viewed in the HTML format, they include features that promote interaction (e.g., automatic and immediate feedback for individual solutions, live computations in Python via Sage cells). Links called *knowls* open boxes when clicked, so the reader gets access to specific content (e.g., definitions, theorems, examples, exercises, etc.) as displayed in Fig. 13.1.

The three textbooks include canonical content in each of the subjects (e.g., derivation, integration in calculus; systems of linear equations, vector spaces, linear transformations in linear algebra; and groups, rings, fields, Galois theory in abstract algebra). The calculus textbook is organized around problems, some of which are to be completed before class (preview activities) and some of which are to be done during class in small groups (activities). The linear algebra and abstract algebra textbooks are organized around definitions, theorems, proofs, and examples, and include *Reading Questions* sections that are to be answered prior to discussing the content in class.

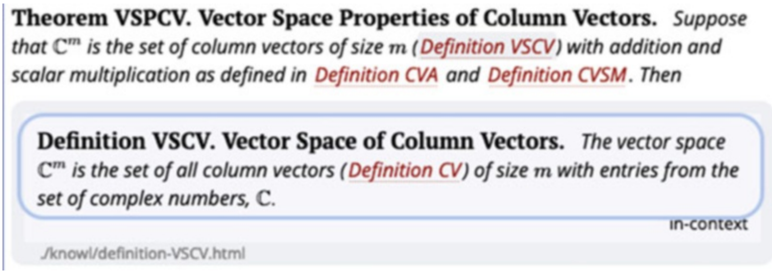


Fig. 13.1 Example of a *knowl* for Definition VSCV, Vector Spaces of Column Vector. The slightly indented text appears when the link in parenthesis is clicked. (Beezer, 2021)

13.2.2 Participants

To date, a total of 50 instructors have participated in the larger study since January 2018. The subset of 21 instructors included in this analysis taught one course over one semester using one of the textbooks, between January 2018 and December 2020 (8 taught linear algebra; 5, abstract algebra; and 8, calculus). They taught in different institutions located in 15 different states in the United States. The selectivity in admissions to these institutions ranged from none (all students who apply are admitted) to very high (less than 10% of students admitted). The instructors reported having between 4 and 40 years of teaching experience (mean 19 years, standard deviation 7 years). The sample included six female instructors (5 taught calculus, 1 abstract algebra).

13.2.3 Data Collected

From the instructors we collected: (1) an initial survey, (2) five short surveys (logs) that were distributed throughout the semester and addressed different aspects of their work, (3) their lecture notes for a specific topic, and (4) their course syllabus. We use the data collected in Log 2 which was devoted to their lesson planning and that asked instructors to upload a map showing the process of creating their lecture notes for a lesson.

The survey that was administered prior to the beginning of the semester was used to obtain contextual information about the course, the students who typically take the course, the institution, and the instructors. We analyzed the responses that all participating faculty provided to seven questions in the second bi-weekly log that sought information about how they created their lecture and lesson notes (see Fig. 13.2) contextualizing and supporting our interpretation with survey data (e.g., goals for the course, beliefs about teaching and learning mathematics, knowledge of students, etc.).

1. How do you create your lecture notes[a] for a class session?
2. What resources are you using to create your lecture notes? (e.g., course textbook, CoCalc, lecture nots from previous years?
3. How do you use your lecture notes during class?
4. Sometimes you may deviate from y our lecture notes; in which cases does this happen and why?
5. Please add other comments you may have on y our lecture notes for this course.
6. We would like for you to create on paper a diagram that showcases the resources you use when planning the course and the lessons (e.g., how they connect to each other, which ones are mostly used.) Please explain your thought process in creating this diagram (e.g., the rationale for each piece in the diagram.)
7. Please create a second diagram that describes the network of people with whom you discuss your Calculus/Linear Algebra/Abstract Algebra course. Please explain your thought process in creating this diagram (e.g., the rationale for each piece in the diagram.)

Fig. 13.2 Log 2 Questions about production of lecture notes. a. We used the expression "lecture notes" because it was the terminology faculty used in our pilot study to refer to what they created to get ready for class. Some instructors said that they did not lecture (e.g., T26), but described, and submitted, what they produced for their lessons

Questions 6 and 7 were inspired by the documentational approach (Hammoud, 2012; Trouche et al., 2020) and generated maps of the resources and processes used to create those lecture notes and the networks of people that were involved in their production. We asked instructors to provide a syllabus of their course and copies of the lecture notes they produced for a specific topic in their course (fundamental theorem of calculus in calculus, linear independence and spanning sets in linear algebra, and groups in abstract algebra). The original purpose of the analysis was to investigate how the textbooks showed up in their lecture notes and in the processes instructors used to create them. Through the analysis, described next, we uncovered a richer set of resources and processes than what we had anticipated.

13.2.4 Analysis

We performed several analyses; first, and following Hammoud (2012), we did a systematic analysis of the representations (hereafter maps) provided in response to Question 6 (Fig. 13.2) that included both resources and processes of creating the lecture notes, which led us to identify three types of maps which we tentatively call, "spike," "interconnected," and "process" based on how the various resources mentioned were represented (see Table 13.1).[3]

These idealizations of the representations were based on surface appearance and contributed to our interpretation of the types of resources called for and how they were being instrumented by the participants. These maps are used to infer the regular pattern those instructors followed when they created the lecture notes, allowing us to envision the rules of action associated with schemes of use of the various resources called for in the creation of the lecture notes.

Next, and using the maps and the responses to Question 2 (Fig. 13.2), we derived a list of resources that we tentatively categorized as material and non-material, attending simply to their tangibility rather than to their function in the process of creating the lecture notes. Material resources include those that are text-based and available in print form, those that are available only in electronic form, and those that are physical objects. The category of electronic resources includes software for distinct purposes, such as mathematical (e.g., Sage), pedagogical (e.g., GeoGebra), and production (e.g., LaTeX); distinct types of repositories: video (e.g., YouTube), reference (e.g., Wolfram alpha, Wikipedia), and files (e.g., GitHub, Google drive); and resources for communication (e.g., Zoom). Resources that did not fit these categories were labeled non-material; these include references to individual cognitive processes (e.g., experience, knowledge), social exchanges (e.g., student questions, discussions with communities), and time, which was only implicitly referred to in the maps.

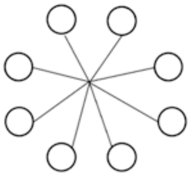
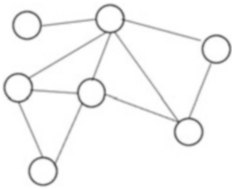
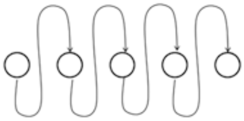
We then reviewed the lecture notes from the instructors who submitted their notes[4] and, through various iterations of coding, identified five potential defining characteristics: their shareability, style, content, means of production, and means of presentation.

- Shareability refers to how shareable the notes are; for example, they may be *personal* notes, indicating that the instructor has no intention to share the notes with the students. Alternatively, the instructor may distribute them ahead of class, either in print, right at the beginning of the class or electronically via a learning

[3]We originally intended to analyze the network maps (Question 7, Fig. 13.2); however all but two of the maps showed only three types of people: the instructor, the students (usually with a bi-directional arrow or a connection between them), and either a colleague or a spouse. The two exceptions named numerous specific communities (e.g., IBL community, project NeXT) suggesting that for most of our instructors, the task of planning lessons is private (to them and their students) and solitary (involving much personal reflection).

[4]Two instructors did not submit their lecture notes, T27 and T31.

Table 13.1 Typology of the representations of the process of generating lecture notes

Typologies of maps	Description
	Spike: The representation shows many resources that are connected to a central (core) resource (e.g., lecture notes, lesson plans, my vast experience). It suggests that the many resources contribute somewhat equally to creating of the document
	Interconnected: The representation shows resources that are combined in particular ways via different connections among them that may contribute differentially to the creation of the document
	Process: The representation shows distinct stages of the creation of the document with various resources intervening at different times

management system. The instructor may project them during class so that the notes can be annotated live in front of the students. Finally, the notes (clean or augmented with annotations) can be shared with the students after class once they are scanned, or through live recording of the lesson, through a learning management system.

- Style refers to the formatting of the content of the notes. They may contain fully or partially written out sentences or text, a table with approximate times for various activities, a template or a structure with blank space that will be filled out in class, or a bulleted list of actions or reminders.
- Content refers to what is included in the notes. It could be definitions, examples, theorems, proofs, list of homework problems or class activities, reminders to self, a narrative that conveys the nature of the argument to be built, or administrative reminders.
- Means of production refers to how are the notes produced, for example handwritten, typed using a word processor (Word, Google doc), or a presentation program (e.g., Beamer, PPT), or more advanced editing programs, such as LaTeX or PreTeXt.
- Means of presentation refers to how the notes are intended to be used in classroom, when that is the case. The instructors may copy the content of the notes on the board or may use a computer or a tablet to project them, or use instead a document camera or Zoom to screen share their own notes.

We next analyzed the responses to Questions 1 and 3 and the accompanying text that instructors provided describing their maps in order to identify instrumentalization and instrumentation processes across all the instructors. Each response was parsed to identify first, phrases that indicated that the teachers were using the various resources and then phrases that suggested that resources had exerted some influence on them. Consider for example, the following response, made by Teacher 15, who was teaching linear algebra:

> My lecture notes tend to follow the text as much as possible. With this course, I find that the vocabulary is very important, so following the definitions in the text helps the students follow the development of the new ideas. At times, I find that there is an example that I prefer to the text, and I slip that in instead. This gives a little more variety to the students, too. The textbook does a good job of highlighting the various definitions, theorems, and examples, and my previous lecture notes help me remember the points that I like to emphasize. (T15, LA)

In this excerpt we found evidence that the textbook provided a structure for the instructor to create the lecture notes ("*tend to follow the text as much as possible*"); that the instructor relied on "*the definitions in the text*" as it "*does a good job of highlighting the various definitions, theorem, and examples*" and that occasionally he brought "*an example that I prefer to the text, and I slip that in instead*" indicating a need to reach for other sources that might fit better his plans; mentioning that his "*previous lecture notes help me remember the points that I like to emphasize*" showed that this resource is a trigger for him that helps in "remembering" what is important to be emphasized, the kernel contents for the lesson. In this case, this resource (prior lecture notes) is a reminder for this instructor that aids his planning. Collectively these sentences illustrate both process of instrumentation and instrumentalization, providing information about the rationales used: supporting the development of ideas, offering more variety to the students in terms of examples that illustrate mathematical notions beyond what the textbook offers, and personal preferences.

The systematic analysis of the instructors' responses was the basis for identifying how the resources were involved in the production of the lecture notes, how instructors were thinking about those resources, and their rationale for action—a process similar to that of identifying schemes of use in the documentational approach to didactics (Gueudet & Trouche, 2012b). These three analyses (map structure, named resources, instrumentation and instrumentalization) were combined in a final step in order to discern the way in which the instructors instrumented the various resources to produce their lecture notes, which led to the identification of three different types of notes that could be distinguished by what seemed to be prioritized in each: either content, meaning, or assessment. The typology emerged through this systematic analysis, done first across instructors (by data source) and then by instructor (using each instructor's data) to corroborate the interpretations.

Finally, we reviewed data from survey questions (e.g., "How do you use your lecture notes during class," "Please list the goals of this course") to augment, corroborate, and provide further support for our classification. T15, for example said the following about the use of the lecture notes during class:

> I present a .pdf of the day's material, which I mark up in real time. I refer to my notes in order to make connections with other theorems, definitions, and examples that we have done previously. I like to use notation that is consistent with the textbook and sometimes use different notation. Generally, I like to inform the students of as many different notations as I can.

This helped us in corroborating that he prioritized content over assessment and meaning. Likewise, in describing the goals for the course, T15 referred only to content that students needed to master:

> This course covers theory and application of the study of systems of linear equations, linear transformations, and characteristics of vector spaces while introducing the student to inner product spaces [organized] into two units: [the first includes] matrix algebra through the study of systems of linear equations, vectors, and properties of vectors [and] sets the foundation for advanced ideas. The second unit uses many of the skills obtained in the first unit to gain a better understanding of general vector spaces, culminating with representations of vectors and matrices in terms of the spectral theorem. By the end of the course, students in [course] will write clear and concise proofs.

13.3 Results

We present the main findings from the analysis of each source of data.

13.3.1 Maps

"Spike" maps had as a core resource, either the lecture notes or the document they were creating or their own experience; "interconnected" maps had many connections between the terms chosen, some indicating loops; the "process" maps denoted the various steps that instructors followed in creating their lecture notes. A spike type of map is shown in Fig. 13.3. The core resource of this spike resource map is "what appears in class on the white board" naming the lecture notes as the outcome of T21's creative process. Indications about the meaning of the arrows state that the most often-used resources are the active learning plan and lecture notes, which this instructor writes out by hand after consulting the textbook. An excel spreadsheet with a schedule of coverage, textbook notes, reading questions and misconceptions from those questions, and homework assignments from the book (as influenced by sage homework) are used often. The sage homework is noted as least often shown on the board. The instructor also notes that student questions start each day without a connecting line to the core resource.

An interconnected type of map is shown in Fig. 13.4. This map shows multiple resources mutually interconnected, including the text, course objectives, the learning management system (Canvas), readings, assessments (quizzes, midterms, homework), the course calendar, and student questions (both those in class and asynchronous). These generate the lecture notes that are stored as a PDF file on a tablet. T15 also mentions the notes being influenced by department objectives.

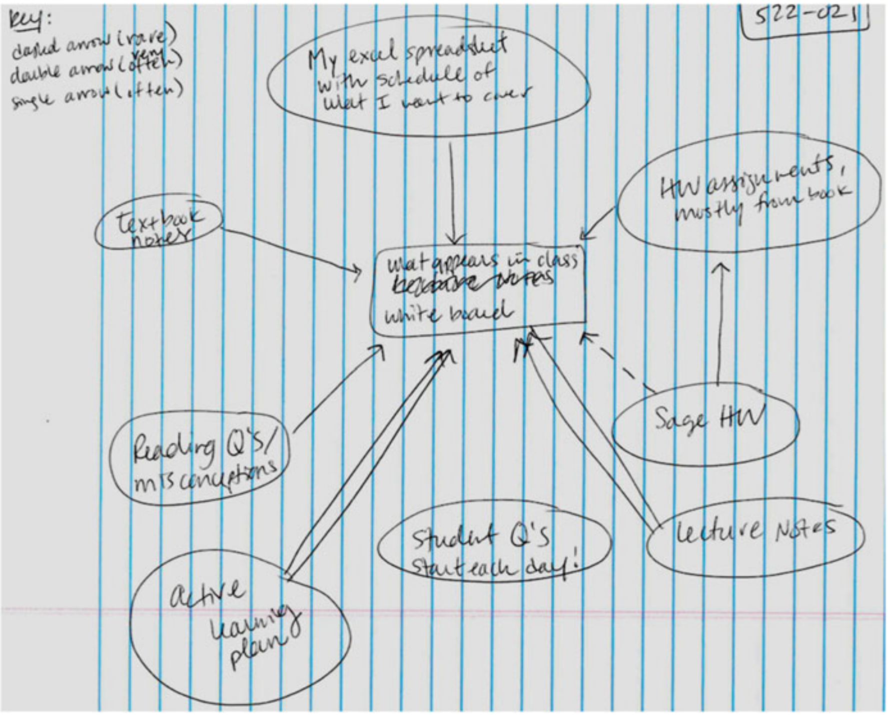


Fig. 13.3 Spike type map by T21 teaching abstract algebra

A process type of map is shown in Fig. 13.5. T25 illustrates a workflow that starts with a class preparation assignment adapted from a past professor, then checking the syllabus to pick topics, then looking over the book to determine which problems to assign and activities to do in class, then looking over old notes to determine what's missing from the current plan, then checking for upcoming labs to see if anything needs to be rearranged. All of this process goes into the outline for class. From there, the instructor uses sage for visualizations, considers whether handouts will be helpful, and determines if students will be working on the board.

Most of the maps submitted were of the spike type (12 maps); four maps were of the process type, and two maps were of the interconnected type; the rest of the maps had combination of a spike and process map, with the process portion devoted to the actual production of the notes (e.g., create in word, scan, project in class) or a spike with some connections between a few terms (e.g., from Desmos to the student activity sheets, and from the student activity sheets to group work). No pattern was evident by type of course.

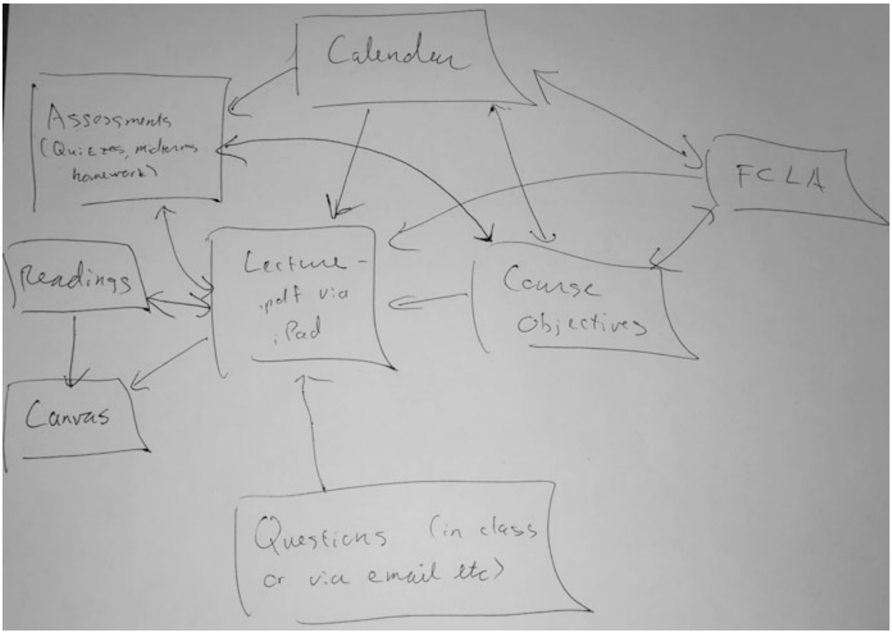


Fig. 13.4 Interconnected type of map by T15 teaching linear algebra

13.3.2 Resources

Figure 13.6 classifies the 170 references to resources made both in maps and in responses to the questions, organized by their nature, material or non-material, and within each category, by how frequently they were mentioned.

Among the material resources that are available in print, the textbook with specific content in the textbook and documents produced in the past were the resources most frequently mentioned by the instructors (66 times out of 83). This is not surprising, as we had instructors who had taught the courses numerous times and instructors who were using the textbooks for the first time. In general, the instructors said that they needed to understand what the textbook author was trying to do so that they could convey that to the students; they checked their prior lecture notes because they wanted to make sure that they were addressing key points or using examples they knew were useful. The instructors also referred to schedules mandated by their departments, as these provided the pacing for the course; two additional resources, the course objectives and the final course assessments were mentioned as providing a lever for making decisions about what to include in the lecture notes.

Mentions of course syllabi, college and departmental competencies, and assessment indicate that instructors attended to institutionally established content goals as they produced their lecture notes, whereas mentions of student productions (e.g.,

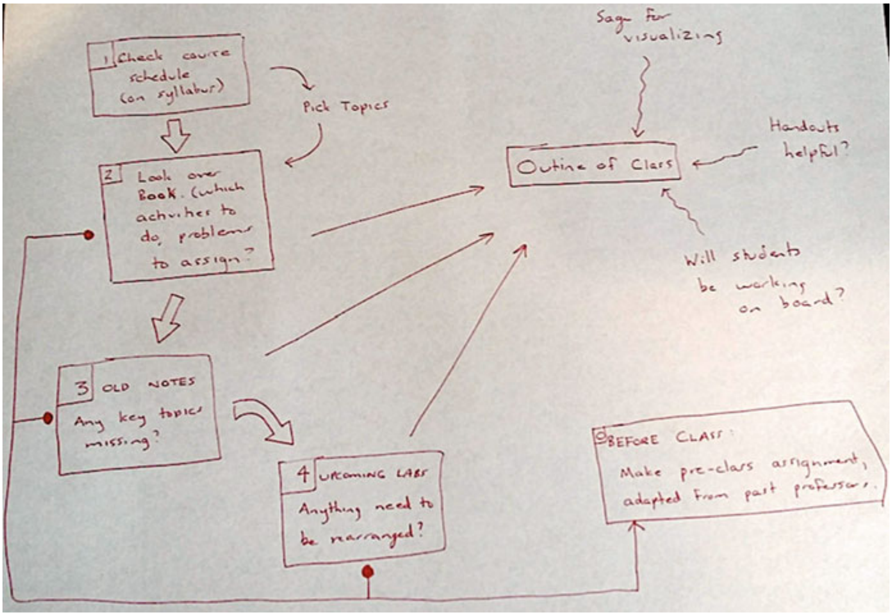


Fig. 13.5 Process type of map by T25 teaching calculus

responses to reading questions or work that had to be turned in before class) suggest that in producing their lecture notes, students and how they were learning the material were important considerations. Resources provided by the authors or by others outside the institution indicate that in some cases the instructors were also oriented outwards to external communities.

Regarding resources that are available only electronically, we identified various categories: mathematical software, software for producing the lecture notes, repositories that are used for storing or sharing resources, the internet, and communication systems—all of these categories fulfilling specific purposes. The math software and the Internet supported pedagogical purposes, to illustrate concepts or to solve complex computational problems; other software supported practical purposes of producing, storing, and sharing the lecture notes. The majority of the physical resources specified supported the production of the notes, although some were mentioned as part of the medium of presentation for the notes. Resources in the non-material category were described as serving a pedagogical purpose: faculty reach to these sources for ideas that will inform what to include or exclude; student questions, formulated either in prior classes or via email, are used to think about additional examples or explanations; personal knowledge and experience, discussions with others—in particular with close relatives, and in one case internal conversations with the “divine”—inform content decisions in the lecture notes.

Time was resource not explicitly mentioned but implicit in the maps (1) in processes that embedded actions spanning minutes (e.g., “quickly glancing at the

Material			Non-material (18%)
Print available (55%)	**Electronic available (18%)**	**Physical (8%)**	
• textbook • specific content (e.g., definition of vector space) or textbook elements (definitions, theorems, proofs, etc.) • documents produced in prior years • course syllabus, college department competencies, including assessments • student work • documents provided by authors (prep assignments, solutions to problems, worksheets) • publications (MAA, research) • other textbooks	• math software (Sage, Desmos, GeoGebra, Mathematica) • software for producing the notes (LaTeX, PreTeXt, Beamer, OneNote, Word, PPT, Google docs) • course management systems • repositories (GitHub, MS OneDrive, Google drive) • Internet (YouTube, Wolfram Alpha, Wikipedia) • communication (Remind, Zoom)	• computer • board • printer • scanner • document projector • tablet	• student questions • own thinking • personal knowledge • experience • discussions with others (students, colleagues, partners, children, IBL/NExT) • "divine" inspiration • (time)

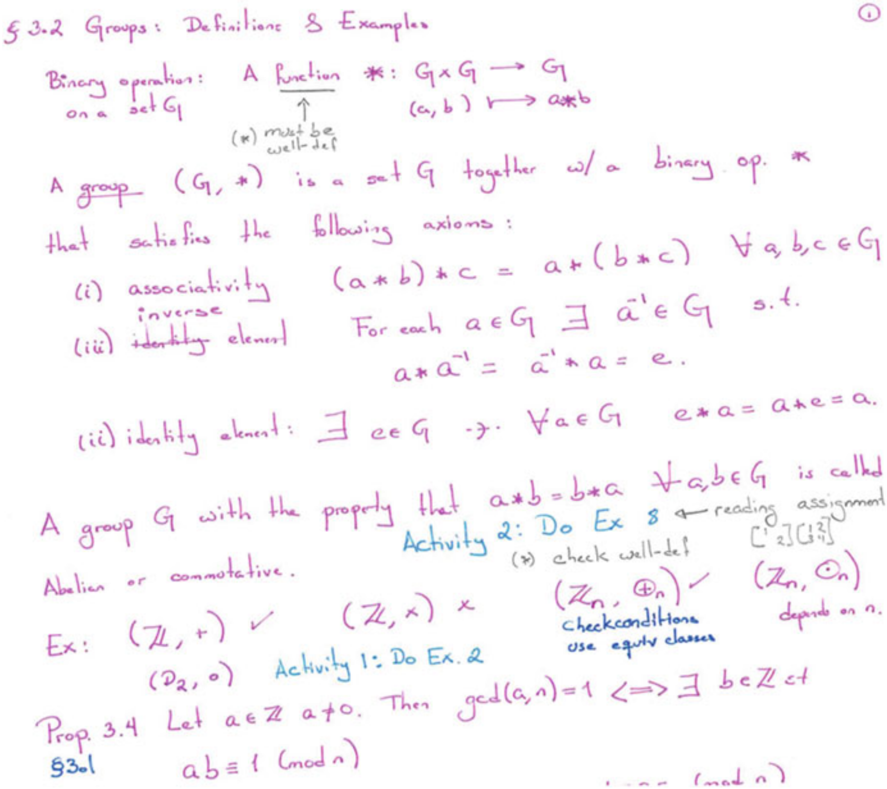
Fig. 13.6 Types of resources mentioned by instructors in their maps (N = 170). Time is noted in parenthesis because it was not explicitly, but implicitly alluded to by the instructors. Percentage calculated from the total number of references

student responses") or hours or days, as instructors perfected their plans; or (2) in the ordering of the processes, some actions occur either before others (e.g., checking the course objectives or assessments occurs before deciding what examples to include) or are cyclical (e.g., "repeat ad nauseum").

13.3.3 Lecture Notes

Figures 13.7 and 13.8 are excerpts of the lecture notes from instructors T20 and T23 that are representative of the variation we saw along shareability, style, content, means of production, and means of presentation. T20's notes are for personal use (not to be shared with students), with mostly fully written out text as style; in terms of content these include definitions, examples, class activities ("Activity 2: do Ex 8"), theorems, proofs, and reminders ("check well-def[ined]"); they are handwritten on paper and with various colored pens; and intended to be written out on the board during class time.

T23's notes are also personal; the text is fully written out and it includes a bulleted list. In terms of content, T23 does not write out definitions or theorems, instead narrating a plan via questions ("What are the subgroups") and directions ("Let's prove that if G is *any* abelian group, then. . ."); the notes are typed in PreTeXt and intended to be shared later with the students.



§ 3.2 Groups: Definitions & Examples ①

Binary operation on a set G: A function ∗: G × G → G, (a, b) ↦ a∗b

(∗) must be well-def

A group (G, ∗) is a set G together w/ a binary op. ∗ that satisfies the following axioms:

(i) associativity (a ∗ b) ∗ c = a ∗ (b ∗ c) ∀ a, b, c ∈ G

(iii) ~~identity~~ inverse element For each a ∈ G ∃ a⁻¹ ∈ G s.t. a ∗ a⁻¹ = a⁻¹ ∗ a = e.

(ii) identity element: ∃ e ∈ G s.t. ∀ a ∈ G e ∗ a = a ∗ e = a.

A group G with the property that a∗b = b∗a ∀ a, b ∈ G is called Abelian or commutative.

Activity 2: Do Ex 8 ← reading assignment

(∗) check well-def

Ex: (ℤ, +) ✓ (ℤ, ×) × (ℤ_n, ⊕_n) ✓ (ℤ_n, ⊙_n)

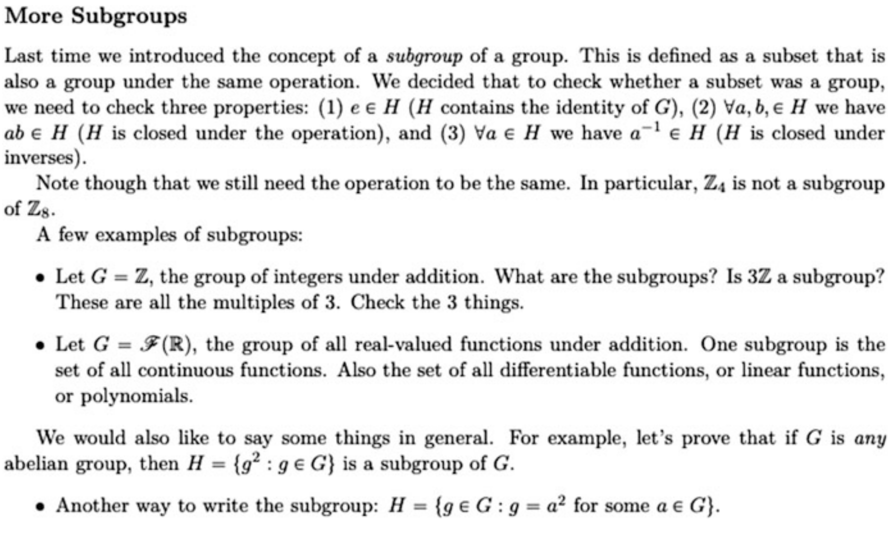
checkconditions use equiv classes

depends on n.

(D₂, ∘) Activity 1: Do Ex. 2

Prop. 3.4 §3.1 Let a ∈ ℤ a ≠ 0. Then gcd(a, n) = 1 ⟺ ∃ b ∈ ℤ st ab ≡ 1 (mod n)

Fig. 13.7 Excerpt of T20's lecture notes on Groups



More Subgroups

Last time we introduced the concept of a *subgroup* of a group. This is defined as a subset that is also a group under the same operation. We decided that to check whether a subset was a group, we need to check three properties: (1) $e \in H$ (H contains the identity of G), (2) $\forall a, b, \in H$ we have $ab \in H$ (H is closed under the operation), and (3) $\forall a \in H$ we have $a^{-1} \in H$ (H is closed under inverses).

Note though that we still need the operation to be the same. In particular, $\mathbb{Z}_4$ is not a subgroup of $\mathbb{Z}_8$.

A few examples of subgroups:

- Let $G = \mathbb{Z}$, the group of integers under addition. What are the subgroups? Is $3\mathbb{Z}$ a subgroup? These are all the multiples of 3. Check the 3 things.

- Let $G = \mathscr{F}(\mathbb{R})$, the group of all real-valued functions under addition. One subgroup is the set of all continuous functions. Also the set of all differentiable functions, or linear functions, or polynomials.

We would also like to say some things in general. For example, let's prove that if G is *any* abelian group, then $H = \{g^2 : g \in G\}$ is a subgroup of G.

- Another way to write the subgroup: $H = \{g \in G : g = a^2 \text{ for some } a \in G\}$.

Fig. 13.8 Excerpt of T23's lecture notes on Subgroups

13.3.4 Instrumentation and Instrumentalization of the Resources

In the documentational approach the use of these resources involves instrumentalization—directed at the resources as the user "shapes the artifacts"—and instrumentation, which constitutes "the schemes of utilization of the artifacts" (Gueudet & Trouche, 2012b, p. 25). These process encompass epistemic, pragmatic, interpersonal, and reflexive mediations through which the instructor acquires new knowledge, skills, and practices (Mesa & Griffiths, 2012). Our analysis suggested three different priorities towards the design of the lecture notes involving the resources, guided by what is privileged in the production of the lecture notes, which in turn guides the instructors' design decisions: prioritizing content, prioritizing meaning, or prioritizing assessment. As we have inferred these as being their implicit reasons for why they create their lecture notes, they are akin to operational invariants in the schemes of use of the resources.

In the cases in which content is prioritized in design, we notice that what matters for the instructors is that the content be rigorously presented. Instructors indicate using the same definitions presented in the textbook, following the same notation, and including full details of definitions and theorems; instructors consider this type of rigor as necessary for building up mathematical ideas. For example, T31 who was teaching calculus wrote in his log: "When multiple notations exist (such as derivative) I will adopt the same notation as the textbook." Fig. 13.9 shows an excerpt of T18's linear algebra lecture notes on Linear Independence and Spanning Sets on "Theorem VRRB" (Vector Representation Relative to a Basis), in which it is possible to see the effort in maintaining the notation and full details of the proof of the theorem, which are almost identical as in the textbook:

The lecture notes in which content is prioritized include full statements of definitions written out; it is as if in the writing of these statements, instructors were also revisiting the ideas contained in the material to make sure of their understanding; fully writing the notes ahead of time allows instructors to rehearse the content included in notes in preparation for what they will do in class ("I often leave them in the office. It's really the preparation of them in the first place that is the important part" T27, LA). Instructors who write the notes during class, do so because they want students to see how mathematics is produced:

> I . . . dive into what's written in my notes, following them more or less verbatim. (. . .) I want to be sure students see an example of type XYZ to better appreciate what the result is saying (T33, AA).

This process of writing and re-writing the notes has an epistemic component; the instructors solidify their knowledge of the content through producing these notes in this way ("Usually, I have copied down everything I need [on the board, without looking at the notes], but sometimes I will pull up the book if I forgot something," T36, LA; "I write and rewrite my notes several times. But then while lecturing, I do not look at my notes," T20, AA).

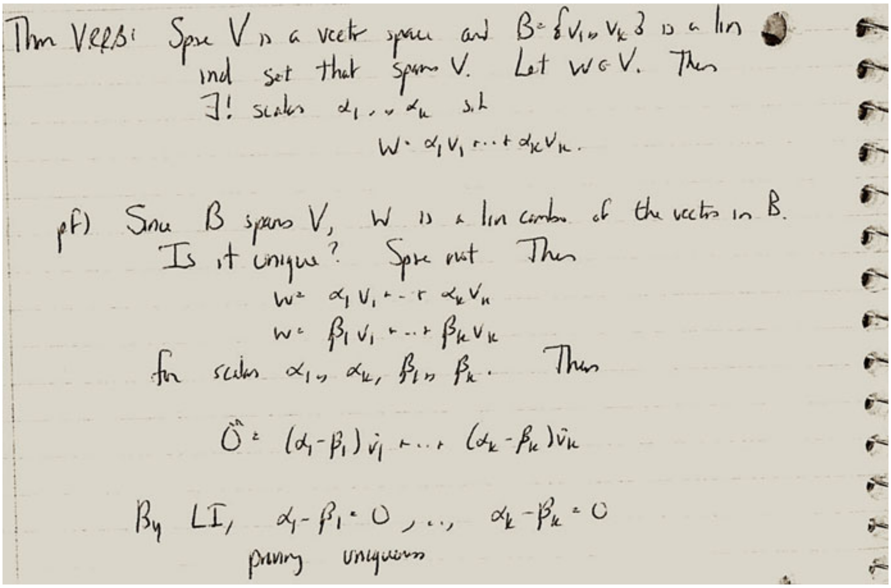


Fig. 13.9 Excerpt from T18's lecture notes on Theorem VRRB (Vector Representation Relative to a Basis, https://books.aimath.org/fcla/section-LISS.html#SKU)

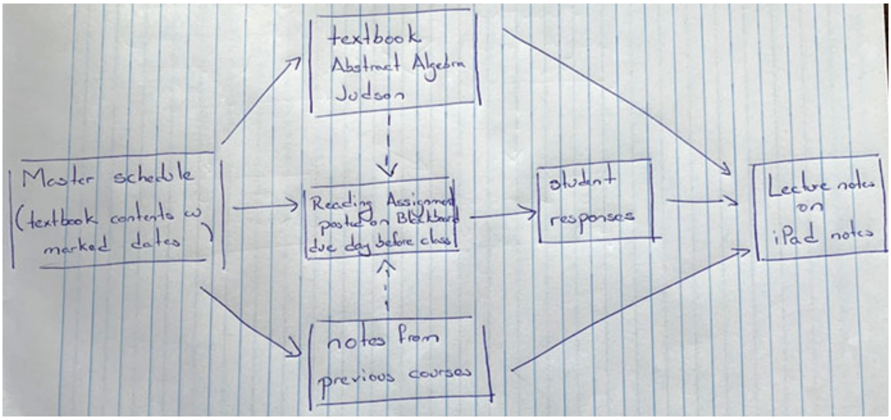


Fig. 13.10 A process type of map by T20 teaching abstract algebra

The notes by T20 (Fig. 13.7) illustrate this priority; this instructor made a process type of map (Fig. 13.10).

The map shows the starting point as the 'master schedule,' a plan, given by the department, for when to cover what textbook content. With this information, the instructor consults the textbook, the notes from prior years, and the reading assignment which informs what students understood from the reading and is posted in the learning management system. All of these will be considered to write the lecture notes. In the notes we see references to "reading assignment" with a note in black on two matrices and other notes about activities that students will be doing, possibly to address what the two matrices signal about student thinking as revealed by their responses to the lecture notes.

In the cases in which meaning is prioritized in designing the lecture notes, instructors seek to highlight what makes the definitions, theorems, and proofs important and where they are coming from. Rather than writing out explicit definitions or theorems, instructors motivate the various elements using a conversational approach ("What are the subgroups?" T23; "Talk about the definition of a limit in whole class, sharing book" T26) and using examples to motivate the need for specific definitions or theorems ("Show that f is graph and f' is a graph – ask about connections. Document camera? Desmos?", T26, C). The notes by T23 (Fig. 13.8) illustrate this priority. These notes have a more conversational style, with the audience being both the instructor and the students; there are directives (e.g., "Check the 3 things." in the first bullet) and questions that are posed to advance the thinking, but with no answer written out. The notes are short statements as invitations to further explore ideas that leave out details that are presumably going to be worked out in the classroom. These notes can be described as 'streams of thought' about the material ("Last time we introduced… We decided to check… We need to check… Note though that… We would also like to say…") suggesting both epistemic and reflexive mediations as the instructor is working out conversationally with themselves the various key elements of the material and envisioning how they will play out during class with the students; precision and rigor, seem to take a backstage role in the notes.

This instructor provided an interconnected type of map Fig. 13.11.

The map has the lecture notes in the center, together with a "Canvas shell" (course management platform); the textbook appears as shaping what goes into the notes and the assessments in the course (quizzes, homework, practice problems). As a counterpoint, the previous materials inform the notes, assessments, and the work that will happen in class.

Finally in cases in which assessment is prioritized in the design of the lecture notes, instructors start with the assignments that students will have to complete for a particular unit and use the problems that are included in them to guide the development of the notes and choose the content from the textbook that will be needed; while content and meaning might be important, these are done at the service of preparing students for the examinations. The notes by T17 (Fig. 13.12) illustrate this priority. These notes have a combination of elements described earlier; the instructor writes notes to self (e.g., "this should be quick, you should ask them to present at the board") as if this self were a third person ("the instructor") and includes fully written out definitions (see "Instructor Notes" in Fig. 13.12) pasted verbatim from the textbook, which is easily done because the textbooks are open source. The lecture notes are structured into parts, signaling the different components of the lesson and possibly the time allocation in class. This lesson had five parts (only two are shown

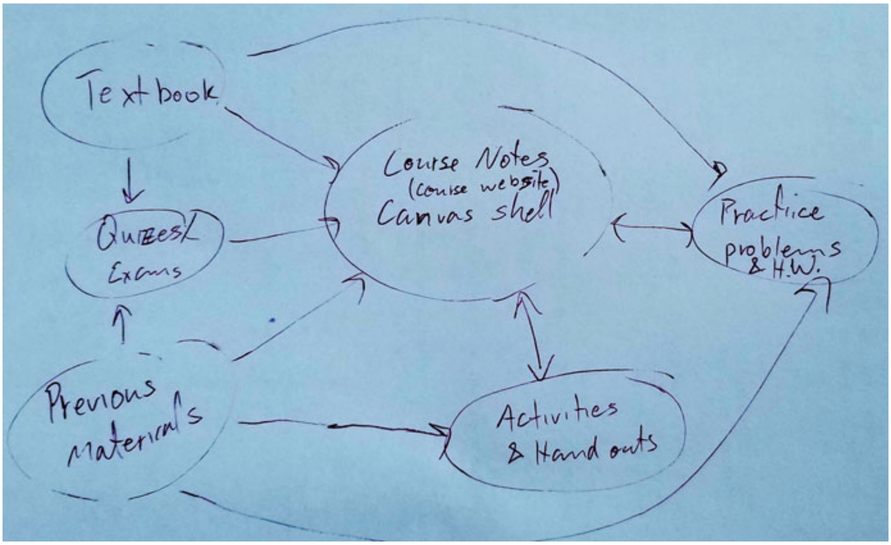


Fig. 13.11 An interconnected type of map by T23 teaching abstract algebra

in Fig. 13.12), three devoted to definitions (general definition of linear independence, spanning sets, and vector representation) and two to student work (on linear independence and spanning sets). The first activity "should be quick" the second activity (not shown) was more complex computationally ("Determine if a set W was a subspace and finding a spanning set for W"). The notes include points to consider in finding the solution. In these notes we see pragmatic and interpersonal mediations in statements about what students and the teacher can collectively and individually do (e.g., "we can define," "we can determine"); the directives to self (e.g., "ask the students," "write on the board," "you can ask them to present at the board") are suggestive of reflexive mediations. These are also directives that are consistent with offering opportunities to assess students informally. The instructor sets out a path for the presentation, and unlike the other cases, it is not clear whether deviations will happen. The rigor of the content is ensured by including and projecting the textbook during class ("show definitions on projector") and by requiring that students work on tasks that will be part of course assessments.

This instructor produced a spike type of map (Fig. 13.13) with homework as a core resource:

> I used the course goals to think about concepts, skills, and tools that I want students to be able to learn and use. These go into *making the homework assignments*. The amount of *material covered per homework assignments* is reflected by how much of the text we have covered, which in turn affects how much of the text I need to present in class. I also *use the homework as a guide* for generating meaningful examples in class so that students can practice certain algorithms and thought processes in class. Those examples I usually write on the board which is on either side of the smart board, which is projecting the text book. (emphasis added)

Part I

General Definition of Linear Independence

- Intro: Ask students: "What is the definition of linear independence." They should say something like Definitions RLDCV and LICV.

(Instructor Notes: The definitions of relation of linear dependence (RLDCV) and linear independence of column vectors (LICV) are as follows: RLDCV said that given a set of vectors $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$, a true statement of the form

$$\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u_2} + \cdots + \alpha_n\mathbf{u_n} = \mathbf{0}$$

is a **relation of linear dependence** on S. A **trivial relation of linear dependence** is a solution when $\alpha_i = 0$ for $1 \leq i \leq n$. LICV said that a set of vectors $S = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ is **linearly dependent** if there is a if there is a non-trivial relation of linear dependence on S. S is **linearly independent** if there is *only* the trivial relation for that set of column vectors.)

- We can now define these definitions more generally since we have a more general definition of Vector Space (Definition VS in Section Vector Spaces). Show definitions on projector (Definitions RLD and LI in Section LISS in Chapter VS).
- We have discussed several vector spaces ($\mathbb{C}^m$,$M_{m\times n}$, P_n, infinite sequences, functions, and Example CVS (crazy vector space)), and now we can determine what it means to be linearly independent in each of these vector spaces.

Part II

Student Work on Linear Independence

- Write on the board: Are $\begin{bmatrix}3\\4\end{bmatrix}$ and $\begin{bmatrix}4\\5\end{bmatrix}$ linearly independent in $\mathbb{C}^2$?
  - This should be quick. You can ask them to present at the board.

Fig. 13.12 Excerpt of T17's lecture notes on Linear Independence

Thus, the mandated course goals set the content for the homework that students will be assigned, but the homework in turn is fundamental to determine what content is included in a lesson, how much of it is presented in class, and what examples will be done; in the map, the goals, the examples and the textbook are connected to this core resource.

Across all the instructors we found 13 who wrote lecture notes that prioritized content, six wrote notes that prioritized meaning, and two whose notes prioritized assessment (one taught linear algebra, T17; the other calculus, T13). Six of the seven instructors teaching linear algebra, prioritized content[5]; we think that there are two reasons for this; one is that the textbook uses a very specific naming convention—it does not use chapter numbers of sections, but rather acronyms that help remember

[5]One instructor, T17, had also elements in the notes that showed attention to content, but assessment guided the production of the notes.

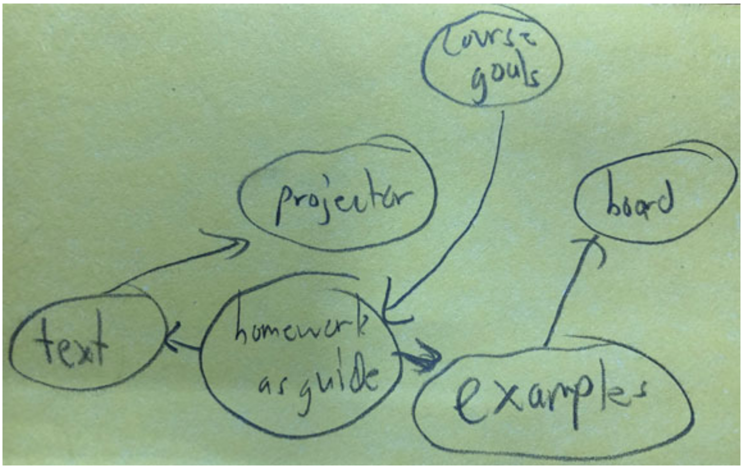


Fig. 13.13 A spike type of map by T17 teaching linear algebra

the content of the definitions and theorems (e.g., SSLE Solving Systems of Linear Equations; Theorem EOPSS. Equation Operations Preserve Solution Sets)—and the textbook is written to serve as an introduction to proofs. Instructors faced with this content have to either use the exact notation in the textbook or translate the language in the textbook using conventions used in other mainstream textbooks. We think that it may be less time consuming to use the textbook as is. The rest of the instructors (13) were teaching abstract algebra or calculus and were evenly split between prioritizing content and meaning; and thus, it seems that personal preferences rather than the specific type of textbook might determine these priorities. Thus, we propose that the textbook and course learning goals (e.g., introducing proofs) as well as personal preference, play a role in instructor's prioritization in designing their lecture notes. Furthermore, the priority and resources influence the process of instrumentalization (instructor using the textbook and other resources to generate lecture notes) and instrumentation because the instructor's epistemic reflections and knowledge gains are part of the process and an outcome of producing notes.

13.4 Discussion and Conclusion

We analyzed various records from post-secondary instructors obtained with the goal of understanding their documentation work, in particular:

- How do instructors view the process of creating lecture notes for their courses?
- What resources do instructors use when creating lecture notes needed to teach a lesson?
- How are the resources instrumented to generate the lecture notes?

We used the participants' definition of lecture notes and the documents that they created to deliver a lesson. The analyses of the various data sources suggest that with goal of creating lecture notes for their lessons, instructors enacted distinct operational invariants that privileged the content that they need to teach, the meaning of the content that students would be learning, or assessment of knowledge and skills that students would need to master to demonstrate competence in the course. The maps were a proxy for how instructors approach their creation of the lecture notes; we found it interesting that 15 of 21 of the maps provided were of the spike type, with all the resources having a single connection to a central element—usually the document they were producing. Does the spike-type of representation chosen for the process indicate that for these instructors the resources used have comparable status and that they contribute equally to the production of the lecture notes? We do not think this is the case. Instead, we think that the specific task that we asked the instructors to perform was novel. Many instructors did mention this request as a difficult one to fulfill; it may be possible that the various decisions that instructors engage in as they prepare their lessons are unconscious, and therefore not easily assailable; it is also possible that these kinds of tasks are foreign as university and post-secondary instructors have not typically received training on pedagogy. The maps showing steps of production or interconnection among terms, are suggestive of a more conscious process as they conveyed temporal relationships and relative importance of the resources. We see this as an area for further investigation.

Instructors mentioned quite rich sets of resources for their teaching. Classifying these is difficult, as Trgalova et al. (2019) noted. The difficulty in classification has consequences for theorization; we chose material and non-material,[6] fully acknowledging that such classification omits the various functions that the resources are helping to fulfil. We also noticed that in their descriptions, time was a ubiquitous but unmentioned resource that played an important role in how instructors described their production of lecture notes. Time is encoded in descriptions of what is done first, second, third, obviously, but also in comments regarding searches for examples or alternative definitions or regarding "going back and forth" about content to include (e.g., 'do the proof?' 'add another activity?'). Time is also implicitly addressed in the cases in which instructors indicate planning on using their notes again, the *next time* they teach—this will *save time*, that they may use to expand other areas of the notes.

The different priorities given to content, meaning, or assessment in the design of the lecture notes show epistemic, pragmatic, interpersonal, and reflexive mediations between the instructors and the resources used (Rabardel & Bourmaud, 2003). Through the analysis of the documents (lecture notes) that participants produced for teaching, we witness their instructional design activity, as instructors write and re-write their notes, identify text inside their textbooks that they can copy or show to the students, search for or create problems and activities that may fit the goals or that will address a student query, annotate their notes to add connections to student

[6]Rudolf Straesser uses immaterial in his foreword to the book, *From Text to 'Lived' Resources* (Gueudet et al., 2012a, b, p. iv).

responses, etc. These different priorities suggest different emphasis towards the material and towards pedagogy; some seem to be guided by the textbook design, as in the case of the linear algebra textbook which emphasized proofs, whereas others seem to be guided by other factors that would merit further investigation.

One question that this investigation has raised for us is how the availability of the textbooks in digital form (HTML or PDF) facilitated or hindered the various aspects of the design process. Besides the convenience of having the text ready to be copied and altered as needed, it is not clear that instructors saw other advantages. In spite of the textbook being available digitally, some instructors planned with a physical copy of their textbook. We were unable to trace major differences because of textbook format. Either the medium (digital/print) has not yet made an impact on these instructors' processes of thinking about their work or there is not much difference in how instructors interact with the digital textbooks and other resources in planning their lessons. Further investigations to ascertain such similarity are warranted.

Prior scholarship has documented that teachers use the textbook primarily as a source of assignments and as a guide for choosing content to use during instruction (Mesa & Griffiths, 2012; Pepin & Haggarty, 2001). This investigation suggests that there is an additional priority for instructors: meaning, making connections with ideas, and wondering about resources that might support those. The findings regarding the subject matter are suggestive: perhaps a linear algebra textbook that is designed differently (e.g., the inquiry-oriented linear algebra, Wawro et al., 2012) might support faculty in prioritizing meaning or a departmental mandate might require that instructors prioritize meaning (Mesa et al., 2019). Having instructors that make meaning a priority might push authors to conceptualize their textbooks differently.

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Chapter 14
Creating a Shared Basis of Agreement by Using a Cognitive Conflict

Mika Gabel and Tommy Dreyfus

Abstract We analyse proof presentation at the tertiary level, using a concept called 'the flow of a proof', which relates to proof classroom presentation of proof. We focus on rhetorical features of the flow of a proof that we analyse using our adaptation of Perelman's 'New Rhetoric'. We present an analysis of an episode from a lesson in Set Theory, given to prospective mathematics teachers, and demonstrate how the lecturer's design of the flow enabled her to create a thought-provoking analogy and to trigger a cognitive conflict. The lecturer's actions created presence of the need for a proof as a tool to solve the conflict, and clarified that analogies may be erroneous; moreover, the discussion highlighted mathematical values that the lecturer wished the students to embrace. The study suggests that mindful planning of the flow of a proof should attend to its rhetorical aspects, so that the flow will promote productive classroom communication that improves proof teaching and learning. Furthermore, a conscious pre-design of the flow may be used by lecturers as a practical pedagogical 'reflective aid' that makes them aware of their own premises. This can be used to support ideas that lecturers perceive as important to convey to students.

Keywords Cognitive conflict · Flow of a proof · Perelman's New Rhetoric · Dissociation · Presence · Basis of agreement

Teaching proof, at all levels, is a difficult instructional task. When mathematicians teach proofs at the tertiary level, they display not only the formal mathematical content of a 'textbook proof' but also meta-proof considerations, effective examples or counterexamples, analogies and diagrams, and consider a variety of contextual

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factors such as the students' previous knowledge, curriculum and time limitations (Lai & Weber, 2014; Weber, 2012); however, eventually students may still not grasp the proof's structure and ideas (Lew et al., 2016). Indeed, the occasional feeling that, no matter what is done or said, the mathematical ideas that are expressed to the students 'go in one ear and out the other' is a frustrating experience for lecturers. Our research tackles this problem and discusses practical rhetorical means that lecturers can employ when they design a classroom presentation of a proof in order to clarify complicated mathematical ideas, engage students in mathematical reflection while proving and raise students' awareness of important ideas. The research involves observing and analyzing actual classroom scenarios using a new theoretical framework that we shall discus shortly.

This triangle of problem-practice-research together with a theory used to mediate between them is in the spirit of Silver and Herbst's (2007) discussion of Mathematics Education Research, in which they state that Mathematics Education Research has "also been done to systematically described practices originated in response to problems" (p. 45). When focusing on theories as mediators between research and practice, Silver and Herbst (2007) argue that research may use theories to describe practices and to explain them. Theories may provide a standard against which one can evaluate practices and conceptualize what can be considered as a desirable practice. In addition, theories may be used to "understand practice by providing rational arguments for why a certain phenomenon should not be surprising, or why it is plausible" (p. 52).

Following this description of the role of theory in practice oriented research, this chapter has two focal points: theoretical and methodological. The theoretical focal point includes an introduction of a theoretical framework, 'The New Rhetoric' (Perelman & Olbrechts-Tyteca, 1969), its adaptation to the context of teaching proof and a concept called 'the flow of a proof' that relates to proof classroom presentation (Gabel & Dreyfus, 2017). We relate in particular to an argument scheme called 'dissociation' and explain how it is used to solve cognitive conflicts and what its possible effects are. The methodological focal point includes demonstrating the use of the theoretical framework to analyze an episode from a lesson in set theory in which the lecturer used a common practice – creating a cognitive conflict. We combine these two focal points to illustrate practice-oriented research in the mathematics classroom.

In Sect. 14.1, we introduce 'the flow of a proof'. In Sect. 14.2, we elaborate on the relation between rhetoric and mathematics, on 'The New Rhetoric' (Perelman & Olbrechts-Tyteca, 1969), and on its adaption; we focus on the use of cognitive conflict and its solution by dissociation as a rhetorical device, for which we provide some relevant background in Sect. 14.3. We then present the study's design (Sect. 14.4) and our findings (Sect. 14.5). Finally, we discuss our findings and formulate some implications in Sect. 14.6.

14.1 The 'Flow of a Proof' and Its Rhetorical Features

The traditional frontal lecture ("chalk talk") still prevails in university mathematics instruction. Yet, even when mathematics lecturers teach "traditionally" their styles may be substantially different (e.g., Pinto, 2013; Weber, 2004). The differences may be a consequence of different attitudes, goals and context. Woods and Weber (2020), for example, interviewed eight mathematicians about the practices they use and their goals, the beliefs that guide their teaching and the relationship between their teaching practices and pedagogical goals. They suggest that mathematics educators should "develop pedagogical recommendations that can be used in a lecture setting and align with mathematicians' current goals and orientations" (p. 15). However, Woods' and Weber's inquiry was based on interviewing the mathematicians outside the classroom setting and not on examining actual pedagogical actions in an actual lesson. Indeed, researchers suggested analyzing classroom observations and discussions with teachers in order to design effective teaching methods (e.g., Speer et al., 2010).

Our inquiry wished to capture the holistic nature of the classroom scenario, taking into account the different types of lecturer actions, students' reactions and the quality of the communication between them. To this end, we offer a conceptual perspective of addressing classroom proof presentation that relates to the personal choices and considerations made by each lecturer, and allows analyzing actual classroom scenarios. We start by presenting a notion called '*the flow of proof*' (Gabel & Dreyfus, 2017), that consists of: (i) the presentation of the logical structure of the proof; (ii) the way informal features and considerations of the proof and proving process are incorporated in the proof's presentation. These two aspects take into account mathematical and instructional contextual factors (e.g., students' previous knowledge and curricular requirements).

The nature of the flow of a proof requires a theory that enables to systematically address a combination of diverse features, such as: formal and informal forms of reasoning, the use of analogies, metaphors, examples, means of illustration, the way that lecturers adjust their presentation to the previous knowledge of the attending students and to various types of classroom scenarios, how lecturers design the interplay between large scale proof modules and single proof arguments, the incorporation of intuitions, meta-proof considerations and explicit and implicit lecturers' values and norms. Many of these features possess an inherent rhetorical character. In order to address these features, we adopted 'The New Rhetoric' (Perelman & Olbrechts-Tyteca, 1969), henceforth denoted PNR (Perelman's New Rhetoric), adapted it to mathematics education, and designed an associated methodology that we used to evaluate rhetorical aspects of the flow of a proof.

14.2 Theoretical Framework – The New Rhetoric

'The New Rhetoric' is a seminal argumentation theory published in 1958 by Perelman and Olbrechts-Tyteca that gained momentum after its English translation (Perelman & Olbrechts-Tyteca, 1969). Perelman envisioned an argumentation theory complementing formal logic that would be able to show how choices, decisions, and actions can be rationally justified, by combining ideas and approaches from formal logic and rhetoric; rhetoric is perceived as a practical discipline that concerns the manner by which a speaker can verbally persuade an audience (van Eemeren et al., 2013). PNR examines "...conditions that allow argumentation to begin and to be developed, as well as the effects produced by this development... it is not concerned with forms of discourse for their ornamental... value but... [as] means of persuasion and... creating 'presence'" (Perelman, 1974, para. 1–3). According to PNR, each argument dimension is tied to a conception of what the arguer believes that the audience will accept (van Eemeren et al., 2013). Arguers who regard premises not accepted by the audience commit an argumentation fallacy, though not necessarily a mistake in formal logic.

Perelman and Olbrechts-Tyteca (1969) discuss two types of premises that may establish a shared basis of agreement: (1) premises relating to the real – facts, truths and presumptions; (2) premises relating to the preferable: values, value hierarchies and loci of the preferable (which are highly abstract constructs that we do not consider). Facts and truths are statements already agreed upon and they are considered to require no further justification and not subject to discussion, where truths stand for connections between facts. Facts or truths might lose their "privileged status" and consequently they can no longer be used as possible starting points, but as conclusions of argumentation. Presumptions are opinions or statements about what is the usual course of events, which need not be proved, although adherence to them can be reinforced, and it is expected that at some point they will be confirmed. Values are normally arranged in hierarchies and they relate to the preference of a particular audience.

Perelman (1982) did not intend to use PNR to account for mathematical argumentation, on the contrary; he states that "... it is inappropriate to be satisfied with merely reasonable arguments from a mathematician as it would be to require scientific proof from an orator" (p. 3). Nevertheless, a few scholars did find PNR suitable to describe mathematical and scientific argumentation (e.g., Dufour, 2013). In addition to mathematical argumentation, we found PNR appropriate to describe the type of informal argumentation employed during a mathematics lesson in which a proof is taught, especially in situations where mathematics lecturers use arguments that are not qualified with 'full certainty', for example: in evaluating conjectures, using analogies, illustrating definitions, giving examples, describing proof structure, or using metaphors. These arguments are an essential part of the proof classroom presentation. In fact, these are the rhetorical features of the flow of the proof mentioned above. For us, this indicates that key notions of PNR, such as audience, basis of agreement, presence and argumentation schemes, may be adapted to the context of proof teaching and more generally to mathematics education.

Table 14.1 Adapting PNR types of premises to a proof teaching context

Premises	Examples
Facts	Axioms, definitions, givens, accepted results, established notations
Truths	Lemmas, theorems, new results
Presumptions	Judgements about using previous knowledge, appropriate examples, useful techniques and proving methods, meta-proof features (e.g., structure, main ideas)
Values (Mathematical and didactical)	The adaptability to a particular audience of a certain proving method or presentation; beliefs about mathematics and its teaching and learning.

A detailed description of PNR's adaptation to teaching proof has been presented by Gabel (2019). Here, we concentrate on the adaptation of PNR's types of premises to the context of proof teaching, as presented in Table 14.1.

The scope and organization and ways of creating presence to elements are two other PNR notions that we utilize in our analysis. The *scope of the argumentation* relates to the choice of the arguments and their order, considering that a discourse that seeks to persuade or convince requires an organization of the selected arguments in an order that will optimally enforce them. The scope of the argumentation also relates to various practical bounds, such as time limits and limits to the attention that an audience can pay (Perelman, 1974). *Presence* is a product of style and delivery, an outcome of the persuasive strategies, which make an audience discriminate and remember ideas, or lines of argument set forth by the arguer (Karon, 1989). Perelman and Olbrechts-Tyteca (1969) claim that presence "…is a psychological datum operative… It is not enough indeed that a thing should exist for a person to feel its presence... one of the preoccupations of a speaker is to make present… [what s/he] considers important…" (pp. 116–117). Perelman (1982) affirms that premises that are not fully accepted by the audience should be reinforced by endowing them with presence through the use of certain rhetorical figures, for example picturesque descriptions, metaphors, analogies and repetitions.

Perelman and Olbrechts-Tyteca assert that acceptance of the audience is established by using various argument schemes, through processes of either association (i.e., establishing a link between two independent entities) or dissociation. We focus on dissociation, since it is particularly relevant for solving cognitive conflicts. Perelman explains that "reasoning by dissociation is characterized… by the opposition of appearance and reality…[it] can be applied to any idea, as soon as one makes use of the adjectives 'apparent' or 'illusory' on one hand, and 'real' or 'true' on the other" (Perelman, 1982, p. 134). The pair 'appearance/reality' is considered a prototype from which other similar pairs can be derived, for example the pairs: 'opinion/truth' and 'subjective/objective'. We will henceforth relate to the pair 'opinion/truth'. Dissociation may be used by a speaker to form a 'contradiction-free' vision of reality (Perelman, 1979). Perelman and Olbrechts-Tyteca explain:

> When a stick is partly immersed in water, it seems curved when one looks at it and straight when one touches it, but *in reality* it cannot be both curved and straight. While appearances

> can be opposed to each other, reality is coherent: the effect of determining reality is to dissociate those appearances that are deceptive from those that correspond to reality. (Perelman & Olbrechts-Tyteca, 1969, p. 416)

Dissociation splits up into separate elements something that the audience previously considered as a whole. The result is one (or more) concept(s) associated with the apparent (i.e., the false aspect of the original single concept) and other associated with the real (i.e., the true aspect of the original single concept). Dissociation involves resolving incompatibilities or conflicts by introducing a value hierarchy or by applying a criterion, a standard that enables to separate between what is apparent and what is real (van Eemeren et al., 2013). In mathematics the standard which is used to dissociate opinion from truth may be definitions or formal proof.

Dissociation involves distinction, because a notion that the audience originally regards as one conceptual unit is split up into two. It also involves a new definition of the original notion. Therefore, dissociation may be used to achieve greater precision and has a clarifying function (van Rees, 2006).

14.3 Cognitive Conflict and Mathematics Education

Albeit the (perhaps natural) inclination to assume that confidence and certainty are preferred over uncertainty and confusion during learning, it appears that there is considerable empirical evidence that confusion is not only prevalent during complex learning but may even be beneficial, provided that it is appropriately regulated and contextually coupled to the learning activity. Impasse-driven theories of learning suggest that impasses that cause confusion provide learning opportunities and invoke cognitive activity that may result in deep cognitive processing and more durable memory representations (D'Mello et al. 2014). Confusion and doubt are often associated with cognitive conflict, which is considered by many as contributing to rational thinking and the genesis of knowledge (Zaslavsky, 2005). The cognitive conflict strategy usually starts by identifying students' current state of knowledge and then, confronting them with contradictory or perplexing information. Although several studies have demonstrated that the use of cognitive conflicts was successful in promoting conceptual change, others have shown that this strategy is not always effective for promoting students' deep understanding of the new information (Limon, 2001).

Limon (2001) offers a possible explanation. She claims that the theoretical models proposed to explain conceptual change should focus not only on the individual's cognitive processes but also take into account other individual characteristics (e.g., motivation, attitudes), the teacher's characteristics (e.g., strategies, training, beliefs) and social factors (e.g., the role of peer collaboration). Furthermore, if an instructor wishes the students to reach a stage of meaningful conflict, the problems introduced to them have to be relevant, so that students need to feel curiosity and to be motivated about the learning activities; students also need to have prior knowledge that will allow them to understand the new information.

The use of cognitive conflicts in the mathematics classroom has long been studied by mathematics education (e.g., Tall 1977). Wijeratne and Zazkis (2015), for example, presented twelve undergraduate students studying a calculus paradox related to indefinite integrals and explored students' attempts at resolving the paradox, the challenges they faced and the mathematical and contextual considerations they relied on. They found that all of the students seemed to be experiencing a cognitive conflict and tried to resolve it, but chose different approaches to deal with the situation. Some students referred to the impossibility of the presented situation, some acknowledged the paradox but accepted it as a part of not yet fully developed mathematics; the majority of students added contextual considerations trying to reduce the level of abstraction whereas a conventional resolution of the paradox involves considering the situation in an abstract mathematical manner.

The study of Zaslavsky (2005) addresses the social aspect that was raised by Limon (2001). Zaslavsky discussed the design and implementation of mathematical tasks that evoke learners' uncertainty. She related to three types of uncertainty associated with tasks: (1) competing claims or beliefs of the learner; (2) exploration tasks and open-ended problems; and (3) lack of confidence regarding the correctness or validity of an outcome. Zaslavsky described in details a doubt provoking task that involved issues related to issues such as validity, multiple proof methods, existence and uniqueness, examples and counterexamples. Most of these issues were initiated by the learners themselves. Zaslavsky's report places social interactions as a central factor in exploiting the potential of such uncertainty evoking tasks, since the rich collection of ideas that was raised by the learners would probably not been as rich if originated from a single learner.

To conclude, cognitive conflict is a strategy that may be used to elicit deep cognitive processes, as well as increase student motivation and curiosity; the effect of using cognitive conflict also depends on social interactions between learners.

14.4 The Study

14.4.1 Objectives

We present part of a larger study conducted in a secondary mathematics teacher training college. In the larger study, rhetorical aspects of the flow of a proof were analyzed using a methodology based on Perelman's New Rhetoric (PNR). A short explanation of the 'PNR analysis' methodology is presented in Sect. 14.4.3; a more detailed explanation about the setting and goals of the larger study, the design of the methodology and the analysis of flow of proof have been presented by Gabel (2019). In the current chapter, we focus on the commonly used practice of creating cognitive conflicts, and use PNR to interpret the effects of creating cognitive conflict and its solution by dissociation on the flow of proof. The objectives of the chapter are:

- to illustrate the use of PNR by analyzing an episode from a tertiary mathematics lesson in which an experienced lecturer creates a cognitive conflict;
- to discuss the findings of the PNR analysis and its affordances in light of existing literature and demonstrate the importance of practice-based research and the type of insights that may be gained.

14.4.2 Setting

The research was performed in a College of Education in Israel, which offers, among others, a degree in mathematics education certifying students to teach grades 7–10. The students take an introductory course in set theory during their first year of training; the course requires no previous knowledge. Class attendance is not mandatory; therefore the number of attending students varied from week to week (34 students were registered and the classroom was usually quite full). The course was taught by Rachel, a very experienced lecturer who volunteered to participate in the study. Together with Rachel, we selected for observation and analysis a few lessons where she taught theorems whose proofs were rich in content and structure. Rachel was not instructed in any way regarding proof presentation before the lessons. The lessons were audio-recorded and transcribed; they were also observed and documented by the first author (blackboard, verbal explanations, lecturer and students comments and students' behavior). Rachel was interviewed after each lesson, and was asked to reflect on the proofs and their presentations. The notes taken during the classroom observation were used during the interview. The interviews were audio-recorded and transcribed.

In this chapter, we present and analyze an episode taken from a lesson in which Rachel taught and proved De Morgan's laws for sets. In this episode, Rachel formulated and proved the distributive laws for sets:

1. Union over intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. Intersection over union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

14.4.3 Analysis

14.4.3.1 Interviews Analysis

The post-lesson lecturer interviews were analyzed using principles of verbal analysis (Chi, 1997). The interviews were transcribed; then, significant excerpts were marked and divided into two categories: excerpts that concern proof teaching in general and excerpts that relate to a specific proof taught in each lesson. These excerpts were then interpreted, summarized and organized in four tables containing: general lecturer considerations regarding proof teaching, lecturer considerations regarding teaching of a specific proof, lecturer's perception of general students' difficulties in learning

Table 14.2 Rachel's considerations

Lecturer consideration	Lecturer element
A. General principles of proof teaching	A1. Exposing the students to different types of proofs
	A2. Refuting a false claim by a counter-example- vs. providing a full proof for a true claim
B. Increasing Students' involvement in the lesson and in the proving process and building students' proving kills	B1. Gradually building students' knowledge to enable involvement
	B2. Drawing the proof from the students through discussion
C. Lesson specific considerations	C1. Justifying the generality of the visual proof

proofs, and lecturer's perception of students' difficulties in learning a specific proof. In Table 14.2, we present only lecturer considerations relevant to the context of this chapter, coded as 'lecturer elements'.

14.4.3.2 PNR Analysis

The PNR analysis starts with analyzing the scope and organization of the proof presentation. The lessons were divided into modules by the main mathematical topic or argument presented in each module. Transitions between modules were placed where the lecturer explicitly stated an intention to start a new topic or argument. The modules, their order, their duration and the inter-relations between them, were organized in a schematic presentation (Fig. 14.2).

Next, each module of the global analysis of the flow of the proof was examined and places where a lecturer element was referred to were marked. An element was categorized as 'endowed with presence' if the lecturer used rhetorical figures (such as repetition or using an analogy) to strengthen its presence. An element was categorized as 'lacking presence' if the lecturer had considered it significant in the interview, but did not endow it with presence in the lecture. When examining the modules in that manner, we found additional elements that had not been mentioned by the lecturer in the interview but were endowed with presence (e.g., elements that were raised by students and were then thoroughly discussed).

The analysis of basis of agreement was performed along several steps but we relate here only to those steps that are directly relevant to the analysis of the episode presented in Sect. 14.5.3.

1. *Identifying potential gaps* between lecturer premises and student premises, according to a list of criteria, mostly based on students' questions during the lesson (for example repeated question or questions that reflected a substantial misunderstanding).

2. *Identifying premises and gaps*: we analyzed excerpts containing potential gaps by using Toulmin schemes (Toulmin, 1958), narrative analysis and other coarse grained schematic representations of the argumentation. We determined relevant lecturer premises and whether they were explicit or implicit. We used students' responses to determine students' premises and their type; if there was a mismatch between student premises and lecturer premises it was categorized as a gap.
3. *Classifying gap source*: If a lecturer attributed a premise a different status then the students (e.g., the lecturer referred to something as '*fact'* but the students referred to it as '*truth*') we determined that the gap source was the *status* of the premise. If a lecturer used a premise unrecognized by students, we determined that the gap source was the *choice* of the premise. If the lecturer used an unfamiliar or unclear notation, we determined that the gap source was the *presentation* of the premise.

14.5 Findings

14.5.1 Findings from the Lecturer Interviews

In Table 14.2 we present Rachel's considerations regarding proof teaching in general and teaching De Morgan's laws in particular. We present only those of Rachel's lecturer elements that we directly refer to in the analysis below. Elements A1, A2, B1, and B2 have a general nature whereas C1 is lesson specific.

14.5.2 Scope and Organization of the Lesson

Rachel opened the lesson by checking homework assignments, which focused on basic concepts and relations in Set Theory, particularly elements, sets, partial sets and the difference between inclusion ($A \subset B$) and elements belonging to a set ($a \in A$). Next Rachel defined the complement of a set A with regards to a set E, $A^C(E)$,and gave examples. She started discussing graphic representations of sets, particularly Euler and Venn diagrams and explained how to represent$A \cup B$, $A \cap B$by diagrams, while reminding the students of the definitions of$\cup$, $\cap$. She then proceeded to representations of$A^C(A \cup B)$, $(A \cap B)^C(A)$ and $(A \cap B)^C(A \cup B)$thus consolidating both the concept of the complement and its graphic representations.

At this point Rachel asked the class what are all the possible reciprocal situations between two sets, represented by diagrams. A classroom discussion led to the diagrams depicted in Fig. 14.1.

This was followed by another classroom discussion, concluding it is enough to use Diagram II in Fig. 14.1 in order to produce a valid proof of a claim relating to properties of two sets. After this general discussion about using diagrams in proofs, Rachel used Venn diagrams (like Diagram II in Fig. 14.1) to prove the associative laws: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

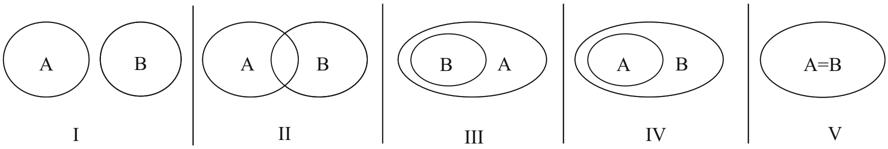

Fig. 14.1 Diagrams of five reciprocal situations between two sets

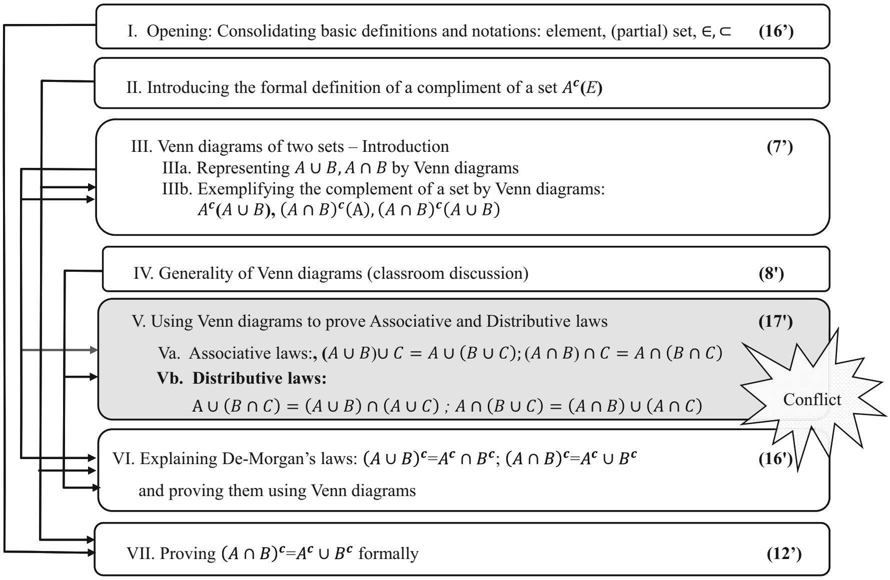

Fig. 14.2 Global flow analysis of the lesson

Then, instead of proceeding directly to proving the distributive laws $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, she started a discussion about the distributive law of multiplication over addition in number sets, suggested a distributive law of addition over multiplication and involved the students in refuting it. Only then, she formulated the distributive laws for sets together with the students and asked if both laws are true. This triggered a vivid discussion involving a cognitive conflict, which will be our focus in Sect. 14.5.3.

Finally, Rachel presented and explained De Morgan's laws and let students prove them on the blackboard using Venn diagrams. One of the laws was also proved formally, using logical notation, after discussing a proving strategy with the students ("double inclusion"). The formal proof of the other law was left as a homework assignment.

Hence, the lesson can be divided into seven modules, as presented in Fig. 14.2, where Module V, which will be the focus of our discussion, is bold.

Figure 14.2 calls for two observations. Firstly, the lesson exhibits complex interrelations between modules (represented by the arrows on the left). Concepts and notations that are introduced in earlier modules are constantly being used in later modules in different contexts and are being consolidated as a result. The different modules are woven into a tangled web of ideas that are introduced, used and re-used throughout the lesson. In that sense the global flow of the lesson is spiral rather than linear. For example, the operations $\cap$, $\cup$ are recalled and illustrated in Module IIIa, exercised in Module IIIb, discussed in Module V and then used in De Morgan's laws in Modules VI-VII. Secondly, all the required elements for understanding and proving De Morgan's laws were not only recalled during the lesson but were thoroughly exercised so that they would be ready to be used when needed; in a way the global flow of the lesson led to the two proofs of De Morgan's laws. Finally, Venn diagrams were treated by Rachel not just as convenient illustrative aid but as a legitimate proving method, and as such considerable time and effort was invested in justifying the use of Venn diagrams.

To summarize, the global flow of the lesson enabled the lecturer to comfortably introduce De Morgan's laws to the students and then to let the students prove the laws by themselves. All the required "ingredients" were at hand and the students only needed to implement them.

Elsewhere we analyze an episode from this lesson, focusing on gaps between lecturer premises and students' premises (Gabel & Dreyfus, 2020, 2022). Here, we focus on the use of a cognitive conflict. Therefore, in the following section we demonstrate a PNR analysis of Module V, as an example of using dissociation to solve a cognitive conflict and consequently create a shared basis of agreement with the students.

14.5.3 Analysis of an Episode from Module V – Cognitive Conflict, and Dissociation

In this section, we discuss the analysis of presence and basis of agreement of an episode in Module V where Rachel discussed and proved the two distributive laws:

1. Union over intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. Intersection over union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

From here on, whenever the terms *fact, truth, presumption*, *value*, *value hierarchy* are used in their PNR sense, they will be written in *italics*.

The relevant episode started as follows:

357	Rachel:	…the question is: Is there a distributive law of union over intersection? What is the distributive law?
358	Student:	What does it mean 'over'?

The student's question caused an elaborate class discussion. Therefore, according to our criteria (Sect. 14.4.3) this module was further inspected for the existence of gaps. However, as will be seen shortly, Rachel was fully aware of a potential gap. She did not only handle it and closed it but rather manipulated the situation to trigger the following discussion:

359	Rachel:	. . .Do you know the distributive law?
360	Student:	Yes
361	Rachel:	Yes. What is it? Generally. . .
362	Student:	A union with. . .
363	Rachel:	No, no, not with union – the law that you are familiar with
364	Student:	$a \cdot (b + c) = \ldots$
365	Rachel:	equals what?...
366	Student:	$a \cdot b + a \cdot c$

Rachel writes $a \cdot (b + c) = a \cdot b + a \cdot c$ on the blackboard and the students agree that they are well familiar with it.

Rachel proceeded and explained the meaning of the word 'over' in the context this law. She verified that she and the students share the same *fact* regarding the distribution of multiplication over addition in numbers and that there is no gap, neither of the status of the *fact*, nor in the use of the word 'over'. In order to verify that, rather than proceeding directly to the distributive laws for sets, she asked the students to formulate a distributive law of addition over multiplication:

369	Rachel:	Now, if I ask: Is there a distributive law of addition over multiplication? How would it look like? What do I need to write?
370	Student:	$a + b \cdot c$
371	Rachel:	Right! $a + b \cdot c$equals what?
372	Student:	[several students] No, it is not equal. . .
373	Rachel:	Is it equal to. . .what?
374	Student:	$(a + b) \cdot (a + c)$
375	Rachel:	[writing on the blackboard] $a + b \cdot c \stackrel{?}{=} (a + b) \cdot (a + c)$ So is it equal or not?
376	Student:	No.
377	Rachel:	No? Why not? Maybe it is equal?

Rachel refused to accept the students' immediate negative answer and asked them to justify what seemed obvious. The students suggested an algebraic proof but Rachel asked for a simpler proof and together they reached the conclusion that in order to refute a claim it is enough to provide a counterexample. She suggested the students to choose small numbers; the students chose $a = 1, b = 2, c = 3$ and refuted the claim. By doing that together with the students, Rachel endowed presence to Element A1 ("Exposing the students to different types of proofs") and Element A2 ("Refuting a false claim by a counter-example") from Table 14.2 in Sect. 14.5.1. This is an efficient example to demonstrate how a shared basis of agreement enables a critical discussion leading to strengthening of new premises of different types: *truths* (the structure of distributive laws in general), *presumptions* (methods of proving), *values* (the need to prove a claim that seems trivial) and *value hierarchies* (preferring a simpler and shorter proof).

Only then Rachel returned to proving the distributive laws for sets, but at this point she had achieved two things: Firstly, the students shared with her premises related to: (i) the structure of distributive laws; (ii) the language ('over'); and (iii) *presumptions* related to proving or refuting the laws. Secondly, she had intentionally planted in the mind of the students the idea that one of the distributive laws is true and the other is false. In fact, Rachel endowed this idea with more presence when she summarized this part and said:

393 Rachel: . . .there is no distributive law of addition over multiplication. However, there *is* a distributive law of multiplication over addition. Now I ask you: what do you think happens with the operations of union and intersection?

We will shortly see how Rachel used this misleading idea in a very sophisticated manner. At that point, together with the students, she formulated the distributive law for union over intersection, taking as an example the distributive law for multiplication over addition and reaching the form: $A \cup (B \cap C) \stackrel{?}{=} (A \cup B) \cap (A \cup C)$, simply by replacing '·' by '∪' and '+' by '∩'.

One of the students still demonstrated the previous gap regarding the word 'over' and Rachel replied:

407 Rachel: Because you did not know what the distributive law was, you did not understand what I meant. Therefore I demonstrated it by using what you all know from before, you all know the distributive law [for numbers], and I ask the same question. Here we saw that the distributive law of multiplication over addition holds, right? So how will you translate it? Instead of multiplication, you take union, and instead of addition take intersection. . .

We see how Rachel strives to maintain a shared basis of agreement with all the students in the class and simultaneously she strengthens the analogy to the distributive law for numbers. She then asks:

409 Rachel: . . . The question is if this law holds or it does not hold. How will we?...

410 Student: [several students] with the circles/ draw circles

411 Rachel: We check it with circles. We found a way to check it. . . .what do you think?

The students, already well acquainted at this point with Venn diagrams suggested to use diagrams to check if the law is true. Then Rachel urged the students to guess and the students guessed it holds. Rachel stated that this guess needs to be checked

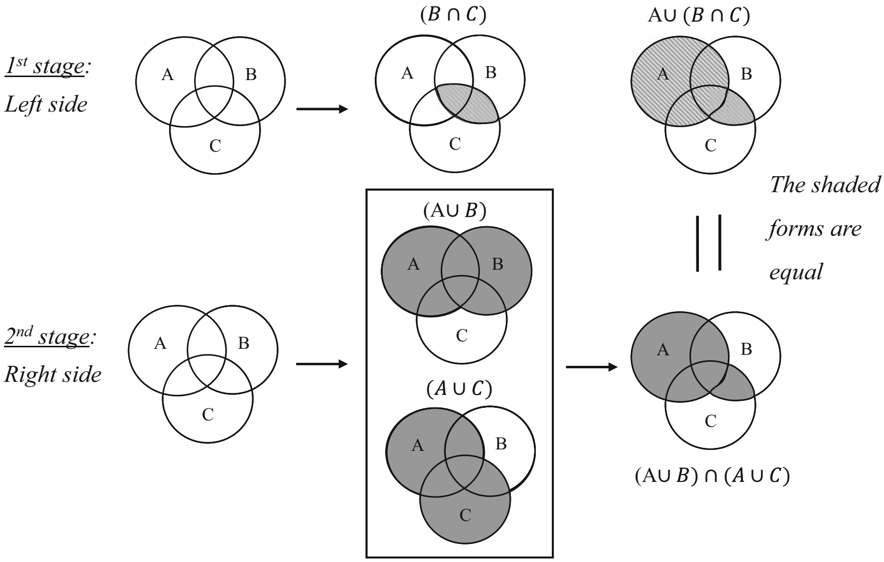

Fig. 14.3 Using Venn diagrams to prove the first distributive law

and proved the law by drawing Venn diagrams on the blackboard as in Fig. 14.3. At the same time, the students are also drawing Venn diagrams in their notebooks. Finally, Rachel summarized that the distributive law of union over intersection holds.

The fact that the students suggested to prove the law by using Venn diagrams indicates that they believe that it is a legitimate way to prove claims about sets. Element C1 from Table 14.2 in Sect. 14.5.1 ("Justifying the generality of the visual proof") was endowed with presence in the previous Module IV thus Rachel now shares this *presumption* with the students.

Next, Rachel turned to the second distributive law of intersection over union. She formulated it together with the students, demonstrating again the shared basis of agreement (in 462 the student is actually dictating character by character):

459	Rachel:	Let us move on. If I want now to write the distributive law of intersection over union, what do I need to write?
460	Student:	*A* intersection with *B* union *C*
461	Rachel:	[writing] $A \cap (B \cup C)$ and I ask...
462	Student:	[student dictating] equal sign, question mark, open parenthesis, *A* intersection *B*, close parenthesis, union, open parenthesis, *A* intersection *B*, close parenthesis.
463	Rachel	$A \cap (B \cup C) \stackrel{?}{=} (A \cap B) \cup (A \cap C)$ [writing on the blackboard]. Good. That is the question. What do you think?

This seems an innocent question, and the students are used to Rachel asking them to make a conjecture before proving (Element B2 in Table 14.2, Sect. 14.5.1). After formulating the first law, there was students' consensus that the law is true. This time the situation is different:

464 Student: No/Yes/Maybe [many voices]
465 Rachel: There is a controversy. One says yes, the other says no. Let us see what the majority thinks. Who thinks that the answer is yes?

The class is split into many voices. Rachel orchestrated this situation deliberately and cleverly, and then took advantage of the shared basis of agreement she had created, in order to create a cognitive conflict among the students. It is tempting to call it a "cognitive trap" for Rachel cunningly manipulated the students and created a confusion that led to a dispute. She says:

469 Rachel: There is a major dispute in the class and we are going to check, we will see who was right, those who said yes or those who said no.

The conflict involved the students in the process and emphasized the need for a proof, not as a way to justify the obvious (as was for them the case with the distributive law for numbers) but as a tool to solve the conflict, and decide if the second law is false (as it was in numbers) or true. The solution of the cognitive conflict involved the use of dissociation as described in Fig. 14.4.

Thus the cognitive conflict and its resolution served as a rhetorical device that endowed presence not only to the two distributive laws for sets but also to the need

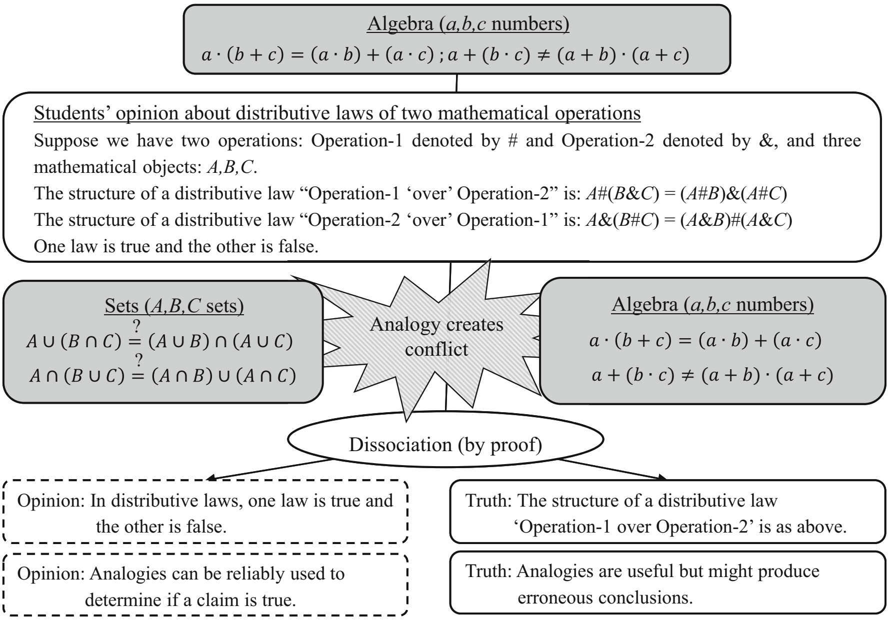

Fig. 14.4 Solving the cognitive conflict by dissociation

for proof as a standard for dissociating opinion from truth, especially when one has no intuition or cannot rely on previous knowledge. In fact, it clarified that one should be very careful with analogies that might produce erroneous conclusions, and that claims may be true in one field but false in another. This is an important message and Rachel stressed it:

505 Rachel: Those who said it was true were right. There is something very surprising here. We saw that in multiplication and addition of numbers the distributive law "works" only in one direction – multiplication over addition, but not in the other direction of addition over multiplication. However, in union and intersection, the distributive law "works" in both directions. There is the distributive law of intersection over union and the distributive law of union over intersection. Therefore, it is really quite surprising. And those who said "no" were inspired by the example of multiplication and addition – but finally the law is true.

In turn 505 Rachel emphasized that: (i) admittedly the situation is not trivial but 'surprising', and (ii) the source of the confusion was the analogy to numbers. She did not embarrass those who were mistaken, because one of her teaching goals is to encourage students' participation and involvement (Elements B1–B2, Table 14.2, Sect. 14.5.1), yet she stressed the need to be careful.

The episode demonstrates how Rachel gradually built a shared basis of agreement with the students, attending to each gap and acknowledging each difficulty. Rachel created confusion. She deliberately confounded the students and accompanied them until they solved the baffle by using proof. The shared premises regarding the structure of the distributive laws and legitimate proving methods (*fact*, *presumption*) enabled a critical discussion leading to strengthening various types of other shared premises: the distributive laws for sets (*truth)*, the danger of using analogies (*presumption*) and the need for proof (*value*). However, the episode also showed that one should be careful not to plant misconceptions in the mind of the students. In this particular case, the lecturer was very experienced, aware and in control. Therefore, she succeeded in using the misconception that she herself created as a cognitive conflict that endowed presence to ideas that she wanted to enforce.

14.6 Discussion and Implications

Our findings demonstrate how the use of concepts and ideas from PNR allows us to investigate a classroom episode, explore the lecturer's pedagogical actions, and interpret their consequences. We presented general and specific lecturer considerations for proof teaching (Sect. 14.5.1), which we related to in our PNR analysis; next, we showed how Rachel endowed her considerations with presence, thus creating a shared basis of agreement with her students. We also presented an analysis of the global flow of the proof (Sect. 14.5.2), a convenient platform to examine the global design of a lesson and to place selected episodes within a mathematical and instructional context; we then elaborated on the use of cognitive conflict as a rhetorical device (Sect. 14.5.3). Thus, our PNR adaptation served as a research-practice mediator; it provided concepts and constructs that allowed us to understand the actions of Rachel – the practice, as well as to evaluate her actions in comparison of the desirable act – designing a flow of a proof such that there exists a shared basis of agreement with the students. As Silver and Herbst (2007) put it: "theories can be languages to encode and read, that is to describe, a practice so that researchers can examine practice according to such reading" (p. 52).

Rachel's pedagogical actions in creating the cognitive conflict reflect the strategy expressed by Limon (2001): She first identifies the current state of knowledge of the students by explaining how the distributive laws are formulated and what the meaning of the word 'over' is; then, she asks the students to formulate themselves the way that a distributive law for addition over multiplication would look like. In PNR terms, she makes sure there will be no gaps in *facts* and *truths*. She then invests some effort in making sure there will be no gaps in *presumptions* and *values* either, by declining the students' immediate denial of a distributive law of addition over multiplication and making them prove their claim (in two ways). This shared basis of agreement serves an important pedagogical goal, since students must have the ability to reason about and evaluate contradictory pieces of evidence, otherwise they will be unlikely to reach a meaningful cognitive conflict (Limon, 2001). In that sense, addressing the distributive laws for numbers was an effective and clever pedagogical move that enabled the students to be engaged and to participate actively in the discussion. However, Rachel does more than that. She 'plants a seed'—the idea that when one of the distributive laws is true, the other is false. The scene is now set for the second stage of the strategy expressed by Limon (2001): confronting students with contradictory or perplexing information.

In this second stage, the students formulate the two distributive laws for sets. After one of the laws has been proved, they are asked if the other one is true. At this point, the analogy to the distributive laws for numbers creates two types of confusion. Firstly, as the students have just experienced, not all distributive laws are true apparently, and this has already created the type of confusion that Zaslavsky (2005) attributes to exploration tasks. Secondly, the seed that was planted by the analogy has evolved and borne fruit: some of the students intuitively responded negatively, some positively and others are uncertain, i.e., there was confusion of the type that Zaslavsky (2005) attributes to competing claims or beliefs of the learner. However, the confusion was not limited to the mind of a single learner, it was expressed explicitly, it split the class, it created a social interaction that felt like a contest, it encouraged student engagement and made them curious to discover what is true. Since the class now shared Rachel's premises, they settled the dispute by proving the second distributive law for sets thus dissociating opinion from truth and devaluing the belief that when one of the distributive laws is true the other is false.

Furthermore, as D'Mello et al. (2014) recommend, the entire instructional process was constantly regulated and supported by Rachel, therefore we surmise that the confusion inflicted on the students by the conflict had positive learning outcomes, yet this is a matter for future research.

A positive outcome of Rachel's use of cognitive conflict is explicating her *presumptions* and *values*. Dawkins and Weber (2017) investigate values and norms of mathematicians regarding proof and suggest that proof instruction must seek to expose the underlying values that guide that practice, otherwise students might find proof and proving confusing and problematic. Their take on the different value systems of mathematicians and students is compatible with PNR ideas about mending gaps and the importance of creating a shared basis of agreement. Furthermore, Dawkins and Weber claim that students are being asked to adopt mathematicians' proof norms although "to students who do not share mathematicians' values, classroom proof norms represent arbitrary solutions, transmitted via imposition, to questions the students never asked and might not even consider meaningful" (p. 133). This echoes D'Mello's et al. (2014) conclusion that confusion is positively related to learning outcomes, if it is contextually coupled to the learning activity.

As we showed above (Sect. 14.5.3) Rachel created a situation that clarified the need for proof as an effective (if not the only) way to solve a conflict. Thus, the rhetorical use of the cognitive conflict endowed presence not only to the considerations Rachel expressed in the post lesson interview (Sect. 14.5.1) but also to a central value discussed by Dawkins and Weber (2017), which Rachel shares: the use of proof as a standard for dissociating opinion from truth; it also endowed presence to a mathematical norm: using analogies but treating them with caution. The instructional setting increased the relevance of the mathematical discussion and made it meaningful for students. This conceptual lens allows a hypothetical interpretation of the findings of Wijeratne and Zazkis (2015) regarding the different approaches that students used in dealing with the conflict (see Sect. 14.3) and Wijeratne's and Zazkis' consequent recommendation to reinforce the way definite and indefinite integrals are defined so that students would be able to rely on the definition and not just apply calculations. In our framework, the outcome of this recommendation would be that the use of definitions would be endowed with presence, so that it will become a *presumption* shared with the students, who will then be able to rely on it as a way to solve the conflict by dissociation; thus, the understanding and working with mathematical definitions will gain significance.

Dawkins' and Weber's (2017) perspective shares with PNR the importance attributed to fostering good communication between mathematicians and their students. This communicational perspective is extensively discussed by Carrascal (2015), who highly recommends that while teaching proofs, lecturers will conduct argumentative dialogues instead of simply presenting proof in what she calls "an authoritarian presentation". Carrascal claims that such dialogues may help students conceive proofs as "constructions built up through an interactive process that looks for the understanding and the acknowledgment of the student, who has to explain all the steps of the inferential process" (p. 317). She attributes other benefits to argumentative dialogues, such as developing competences related to critical reasoning, and states that teachers should have pedagogical and theoretical skills to foster argumentation in the classroom. Moreover, she believes that communicating mathematics may require specific forms of expressions and the use of rhetorical elements to convince the particular audience. Indeed, Rachel has surely considered the characteristics and needs of the particular audience in her lesson, prospective mathematics teachers, as she said in the post lesson interview:

Rachel: . . . a lecturer who teaches this proof at university, does not start all the stories from the beginning but just gives the proof. . . I try to set an example of 'how to teach'. It is an extra. . . a 'bonus'. . . I give them teaching methods, they should think, when something is difficult, how to teach it to students in the clearest way. . . part of my teaching here. . . is to give a personal example of 'how to teach', I believe this is part of our job... our graduates will not be mathematicians but mathematics teachers. . .

We believe that the episode we presented exhibits the type of dialogues that Carrascal endorses. In this chapter we referred to a specific practice, the use of cognitive conflict. However, elsewhere (Gabel & Dreyfus, 2017, 2022; Gabel, 2019) we used the PNR framework to critically examine classroom communication and further elaborate on ways to improve it. Our approach suggests a practical pedagogical concept, the flow of a proof, that if carefully planned may make lecturers aware of important proof elements and of their own premises and promote productive classroom communication.

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Chapter 15
Teaching Mathematics Education to Mathematics and Education Undergraduates

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Abstract Mathematics Education courses are increasingly present in university programmes in Mathematics and in Education. In this chapter, we propose approaches to teaching and assessment which consider and address some of the challenges that university teachers face as they welcome students from diverse communities to Mathematics Education as an academic discipline. To this aim, we draw on our experiences of design, delivery and assessment of two introductory, optional courses in Research in Mathematics Education (RME), one to final-year BA Education students in an Education Department and one to final-year BSc Mathematics students in a Mathematics Department. We aim to discuss how such courses can facilitate students' cross-disciplinary transition (from Mathematics or Education to Mathematics Education). First, we outline the literature and the theoretical underpinnings that ground the design of the two courses and their assessment frame. This frame consists of a typology of four characteristics of engagement with RME discourses which is informed by the Theory of Commognition and has emerged from our prior research with secondary mathematics teachers and university mathematics lecturers: consistency, specificity, reification of RME discourse, reification of mathematical discourse. Subsequently, we outline the two courses and sample one activity and student work from each course to demonstrate the use of our assessment frame and highlight insights emerging from its use (for example, in tracing narratives about mathematics and its pedagogy as students engage with the courses). We conclude with a brief discussion of the pedagogical potential of such activities – and of the two courses more broadly – for undergraduate students' introduction to RME.

Keywords Mathematics Education courses (for Mathematics or Education undergraduates) · Transition (from Mathematics or Education to Mathematics Education) · MathTASK · Formative and summative assessment · Theory of Commognition · Discursive shifts

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15.1 Mathematics Education Courses in the University Curriculum

Courses that introduce Mathematics Education as an academic discipline now feature increasingly in the syllabi of Mathematics undergraduate degrees as well as in the syllabi of Education undergraduate degrees. These courses typically mark students' first encounter with Research in Mathematics Education (RME), whether they are students of Mathematics or of Education. The challenges posed to those in charge of welcoming these two quite different communities of learners – mathematics undergraduates and education undergraduates – into RME are diverse (Nardi, 2015a, b; Biza & Nardi, 2020, 2022). Addressing the challenges present in this first encounter has implications for university-level teaching practice as well as wider implications: teaching these courses is a quintessential opportunity for a much needed rapprochement between the communities of Mathematics, Education and Mathematics Education research (Nardi, 2008, 2016).

In this chapter, we present the design, delivery and assessment of two introductory, optional RME courses, one to final-year BA Education students in an Education Department and the other to final-year BSc Mathematics students in a Mathematics Department. Our account aims to explore how the design, delivery and assessment of such courses can facilitate students' cross-disciplinary transition (from Mathematics or Education to Mathematics Education).

First, we discuss literature which informed the design of the two courses. We start from literature that explores the epistemological differences between research in Education, Mathematics and Mathematics Education (e.g. Boaler et al., 2003). We also discuss challenges which pertain to the transition from studies in Education or Mathematics to Mathematics Education. We then describe the theoretical underpinnings of the design, teaching and assessment of the two courses and we outline how these inform the assessment frame we use for the evaluation of student work in the two courses. This frame consists of a typology of four characteristics (consistency, specificity, reification of RME discourse, reification of mathematical discourse: Biza et al., 2018), is informed by the Theory of Commognition (Sfard, 2008) and emerged from our prior research with secondary mathematics teachers (Biza et al., 2018) and university mathematics lecturers (Nardi, 2008).

Subsequently, we outline the two courses and sample one activity – as well as examples of student work – from each course. Our examples draw on data collected during formative and summative assessments for the two courses. We sample from course activities and from student work in order to demonstrate the use of our assessment frame and illustrate insights emerging from its use into: how students engage with RME and what shifts, if any, in the students' mathematical and pedagogical discourses may need to occur (engineered, encouraged) during said engagement.

We conclude with a brief discussion of the interplay between our research and teaching practice in the way these courses were conceived and continue to grow.

We highlight potentialities of such activities – and of curriculum design and pedagogical practice relating to the two courses more broadly – for undergraduate students' introduction to RME.

15.2 Challenges in the Transition from Studies in Mathematics or Education to Mathematics Education

Welcoming newcomers to the community of RME, typically through optional courses offered to Mathematics and Education undergraduates has become one way in which Mathematics Education is making an increasingly institutionalized presence within Mathematics and Education undergraduate programmes. One motivation for such courses is often to introduce students to RME as they prepare for post-graduate courses for mathematics teaching (in the UK, for example, PGCE, the Post-Graduate Certificate in Education, a qualification that is typically required for granting QTS, Qualified Teacher Status, to primary and secondary teachers).

Often, courses offered to Mathematics undergraduates aim to familiarise mathematics students with the social science of Education and with the new, to them, practices of educational research. These practices are often very different from those for research in Mathematics (Schoenfeld, 2000). For example, in comparison to Mathematics, RME discourses tend to be less absolutist and highly reliant on learning and teaching context – and student responses to a mathematical problem question are typically studied with attention to what may have led the student to producing a particular response (Nardi, 2015b). Knowledge production is rather more "cumulative, moving towards conclusions that can be considered to be beyond a reasonable doubt" (Schoenfeld, 2000, p. 649) – and findings tend to be suggestive, rather than definitive. Such epistemological differences make the navigation across the discourses of Mathematics and RME a challenge that those of us in charge of welcoming newcomers into RME need to actively consider and act upon.

Upon entry into RME, newcomers need to learn how to read, converse, write and conduct research in this largely unfamiliar territory. Working towards membership of the new scholarly community often implies a rethinking of epistemological positions – see, for example, (Nardi, 2008, pp. 257–292) for a discussion of this matter in relation to the experiences and challenges of mathematics educators and university mathematicians engaging with collaborative research.

How to engage with RME literature has emerged in research as one such challenge. For example, Boaler et al. (2003) explore two issues pertaining to the welcoming of newcomers into RME. First, they note, practitioners who embark on research often start off with a view of systematic enquiry as a way to "show, prove, convince" of "what works" (p. 492). A further issue emerges when "new researchers who know they need to become familiar with relevant studies in their field but conceive of this task as a goal in its own sake, or even as a ritual, signalling to other members of the community that they are well read" (p. 492). Boaler et al.'s treatment of these issues – "considering research from the perspective of its practices" (p. 493) – begins to highlight the necessary epistemological shifts that engagement with RME entails, whether the new researcher sets out from an educational practitioner/Education graduate background or from a mathematician/Mathematics graduate background.

As Boaler et al. noted in 2003 – and is still the case – the body of work on the preparation of new mathematics education researchers is "small but expanding" (p. 494). This body of work highlights that, upon entry in the field, newcomers "need at minimum to understand and respect the nature of different research traditions and paradigms" and be "sufficiently open to appreciate the value of different perspectives and frameworks" (p. 496). Their "ability to do so will depend, in part, upon being knowledgeable about the philosophical and epistemological foundations of different paradigms and the nature of evidence they provide." (pp. 496–7). Learning how to exercise "both imagination and discipline" (p. 497) is at the heart of the skills that newcomers to the field need to develop – and the fostering of these skills is at the heart of what their supervision and teaching needs to focus on.

Fostering of said skills was the focus of Nardi's (2015b) study conducted in the context of a postgraduate programme in Mathematics Education that enrols mostly Mathematics – but also some Education – graduates. She proposed activity sets "designed to facilitate incoming students' engagement with the mathematics education research literature" (ibid, p. 135) through gradual familiarisation with the key journals in the field, and through co-engineering, with the students, steps purposefully designed to develop their skills in identifying, reading, summarising, critically reflecting on and contextualising RME literature.

Of particular resonance with the work we present in this chapter is Boaler et al.'s (2003) observation that "the preparation of new mathematics education researchers may be better informed if more explicit attention is given to the work in which they will engage" (p. 497). "Work" here alludes to the "active process of investigation" (p. 497), rather than to static, product-oriented notions of knowledge these researchers need to acquire. Boaler et al.'s (2003) "explicit and purposeful focus on research practices" (p. 517) such as "reading, formulating a research question, using data carefully to make and ground claims, moving from the particular to the general, considering mathematics, and communicating research findings" (p. 497) aligns well with the design of the courses that we focus on here. Our course activities aim to engage newcomers with RME literature in order to: facilitate the newcomers' engagement with diverse and multi-modal readings; to foster their capacity to cope with the complexity of engaging with the arguments and findings of others; to locate their own prior learning and teaching experiences in mathematics, in the realm of these arguments and findings; and, to deploy these insights into analysis and critical reflection on incidents of mathematics teaching and learning that are likely to occur in the classroom.

The course designs that we present here also respond to what Batanero et al. (1994) noted as the need for activity-based RME courses – namely courses designed around negotiating new objects of knowledge that can be applied in prior and future experiences of mathematics learning and teaching (Liljedahl et al., 2013). Our design resonates with works on *engaged pedagogy* that offer a platform for "students' experience of active agency within scholarly communities" (Pyhalto et al., 2009, p. 221). Similarly, *independence, creativity and critical thinking* often described – for example, by Adler and Adler (2005) – as marks of emerging membership of a scholarly community are at the heart of the course design we present here.

Even though the aforementioned works refer mostly to postgraduate courses in RME, and the work we present in this chapter concerns undergraduate provision, our aims align well with the priorities these works highlight.

At this juncture, we note that one underlying assumption of the work presented here is of Mathematics Education as an area of educational research, and therefore a part of the Social Sciences. This is a widely, but not universally, accepted assumption; in fact, it is a culturally dependent assumption. For example, in some countries in continental Europe, didactics of mathematics chairs may be located in science faculties, often alongside those in applied or pure mathematics; and, sometimes, there is a distinction between RME at primary and secondary levels (with the former located in Education departments and the latter in Mathematics departments). In any case, regardless of whether RME is carried out by researchers whose affiliation is in a Mathematics, Education or other department, the epistemological differences between the respective fields can be profound (as documented, for example, in Sierpinska and Kilpatrick, 1998, pp. 445–548; Nardi, 2008, ibid.).

We also note that aforementioned challenges apply for both Mathematics and Education students – and that, for Education students, these may take an additional and different shape. These students typically arrive on RME courses well-versed in the Social Sciences paradigm – but often not having engaged with mathematics since the end of compulsory education. In the UK, this is typically marked by the completion of the Graduate Certificate in Secondary Education (GCSE), at the age of 15–16. These students are often reticent about their mathematical ability and need to revisit their own, sometimes recalled as negative, experiences of learning mathematics. An RME course then becomes a vehicle through which they can overcome their fear of – and trepidation about teaching – mathematics (Nardi, 2015a, 2017).

In this chapter, we report from our work on optional RME courses offered to Mathematics and Education undergraduates that builds on aforementioned studies. We start with outlining the theoretical underpinnings of the design, teaching and assessment of the two courses.

15.3 Theoretical Underpinnings of Undergraduate RME Course Design

The theoretical perspective of this work is discursive and draws on the commognitive framework (Sfard, 2008) that sees Mathematics, Education and Mathematics Education as distinctive discourses and learning as communication acts within these discourses. Our course design aims to attend to discursive differences – and potential conflicts – between Mathematics/Education and RME that our students may experience. We aim towards smooth navigation across these discourses. Specifically, our course design is produced with attention to how students may transform what they know about mathematics from their mathematical studies (respectively, what they know about educational research from their educational studies) and what they learn about RME theory – to which they are introduced during aforementioned courses – into discursive objects that can be used to describe, and reflect upon, the teaching and learning of mathematics. This transformation is the productive discursive activity that Sfard (2008, p. 118) labels as *reification*.

In Nardi's (2015b) study, a set of activities for Masters and doctoral level students were proposed towards facilitating their introduction to RME literature. In these activities, students are asked to read RME literature, to produce summaries of their readings and to write accounts of instances in "their personal and professional experiences that can be narrated in the language of the theoretical perspective" (ibid, p. 151) featured in those readings. These accounts of students' experiences are called *Data Samples*. Engagement with literature, together with the production of Data Samples, are seen as two key ingredients of student engagement with RME literature. Analysis of student work has highlighted students' challenges with coping with the complexity of RME literate discourse and with contextualizing RME literate discourse (ibid). The latter, particularly as triggered by the Data Samples, is one pillar for the course design and activities we focus on in this chapter.

Another pillar comes from our work with pre- and in- service mathematics teachers in the *MathTASK*[1] programme. In MathTASK, we engage teachers with fictional but realistic classroom situations, which we call *mathtasks* (Biza et al., 2018). Mathtasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students. The mathematical problem, the student responses and the teacher reactions are all inspired by mathematics classroom issues that prior research has highlighted as seminal. Respondents are invited to engage with these tasks through reflecting, responding in writing and discussing. At the heart of MathTASK is the claim that discussion related to the teaching and learning of mathematics can be particularly productive when it is situated in specific classroom situations that are likely to occur in actual practice (Biza et al., 2018). Central to mathtasks are pivotal classroom moments in the growth of learners' mathematical thinking presented as brief scenarios following the re-storying techniques deployed in (Nardi, 2008, 2016) and akin to Leatham et al.'s (2015) *Mathematically Significant Pedagogical Opportunities to build on Student Thinking* (MOSTs). Our concern with addressing the complex set of considerations that teachers take into account when they determine their actions also aligns what Patricio Herbst and

[1] "MathTASK" (https://www.uea.ac.uk/groups-and-centres/a-z/mathtask) refers to the overall programme and its principles, whereas *mathtask* refers to a specific task designed in accordance with the principles of the MathTASK programme.

colleagues (e.g. Herbst & Chazan, 2003) label as the *practical rationality of teaching*.

Towards the analysis of the student data we collect during the delivery of the courses – and sample in this chapter – we deploy a typology of four interrelated characteristics that emerged from themes identified as pertinent for mathematics teacher education and professional development in our prior research (see detailed rationale, definitions and examples in Biza et al., 2018; Biza & Nardi, 2019, pp. 46–47) and is tailored to the commognitive underpinnings of our work.

- *Consistency:* how consistent a response to a mathtask is, namely how well-linked the respondent's utterances on stated pedagogical priorities are with their utterances on intended reaction in the teaching situation under consideration.
- *Specificity:* how contextualised and specific a response to a mathtask is, namely how explicitly relevant the respondent's utterances are to the teaching situation under consideration.
- *Reification of RME discourse:* how reified the use of theories and findings from research into the teaching and learning of mathematics – that students are becoming familiar with during the course – appear in a response to a mathtask.
- *Reification of mathematical discourse*: how reified mathematical discourse – that students are familiar with, through prior mathematical studies – appears in a response to a mathtask.

Each one of these four characteristics concern elements that we can trace firmly and concretely in the students' writing. So, for example, for reification of RME discourse we scrutinize the responses in terms of how specific, relevant, reliable and accurate the use of RME terminology is. This scrutiny includes questions such as: Are RME terms used accurately? Is there direct relevance of a reference to the point being made (e.g. is the student referencing a piece of educational research in a very broad manner when a point can be supported more precisely by a specific quotation from an RME piece of research)? Is a reference specific to research into the teaching and learning of the particular mathematical topics at stake in the mathtask? Is the link between the reference and the point explicit? Is the source used credible (e.g. is the student quoting a text of dubious origin in the same breath as peer-reviewed research literature – and not demonstrating explicitly awareness of the difference between the two)? Seeking such evidence in the students' responses not only secures a verifiable (by course moderators and external examiners) route to a student's course mark; it also paves the way for identifying which student narratives about the teaching and learning of mathematics (and RME) subsequent versions of the courses (and research thereof) need to challenge, and, change.

This typology is the basis for the assessment frame deployed towards the formative and summative assessment of the students' work during the two courses. We now describe the two courses and exemplify its use in samples of student work.

15.4 Design, Delivery and Assessment of Two RME Courses

Thereafter, the Mathematics Education course for Mathematics undergraduates is referred to as the BMath course and the Mathematics Education course for Education undergraduates is referred to as the BEd course. Both are offered as optional courses to finalist (Year 3) students, respectively on a Bachelor of Science (BSc Mathematics) and a Bachelor of Arts (BA Education) in a research-intensive university in the UK.[2] The collection and use of the students' productions during the courses for research purposes has been approved by the institution's Research Ethics Committee and consented to by the students.

15.4.1 The BMath Course

The aim of the BMath course (entitled *The Learning and Teaching of Mathematics*) is to introduce Mathematics undergraduates to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum (Biza & Nardi, 2020). The learning objectives of the course include: to become familiar with RME theories; to be able to critically appraise RME literature and use it to compose arguments regarding the learning and teaching of mathematics; to become familiar with the requirements (professional, curricular and other) for teaching mathematics; to engage with findings from research into the use of technology in the learning and teaching of mathematics; and, to practise reading, writing, problem solving and presentation skills with a particular focus on texts that report RME. The BMath course is led by the second author since 2016.

Contact time is 4 h per week (two for lectures and two for seminars) for 12 weeks. Lectures are teacher-led and partly interactive. Seminars are student-led. In these, students: present papers they have been asked to read in advance; identify examples from their experience that resonate with themes in the readings (Data Samples, as per Nardi, 2015b); solve mathematical problems and reflect on these solutions; and, respond to mathtasks (Biza et al., 2018). The students upload weekly seminar contributions in a shared folder. Half-way through the course, a formative assessment activity asks them to produce a response to a mathtask (up to 800 words). Summative assessment at the end of the course is through a Portfolio of Learning Outcomes that involves: nutshell accounts of RME theoretical constructs; reflection on students' own learning experiences in mathematics; solving a mathematical problem and reflecting on the solution; and, responding to mathtasks. An example of a BMath mathtask is in Excerpt 15.1. Examples of coursework and portfolio excerpts are in (Biza & Nardi, 2020, 2022). Students' portfolio entries are marked in accordance with the typology of the four characteristics we outline in the previous

[2]Occasionally, the courses are taken as optional by students with an interest in Mathematics Education enrolled on Natural Sciences and Engineering courses.

Class X is a high attaining group which Ms Jones has taken over at the start of Year 10. So far, Class X has been taught mathematics as a list of rules and they have been practising the application of these rules in a range of examples. These students have learnt to perform well in a competitive classroom environment in which they work on tasks and they are rewarded for the correctness and rapidness of their work. In her teaching, Ms Jones aims to instigate a different approach that includes justifications for the used rules and the relations amongst them.

In a session on the sum of the angles of a polygon, she has asked the students to:

- work with a Dynamic Geometry software in order to sketch polygons with 3, 4, 5, 6, 7, … sides, and
- report the number of sides and the sum of the angles in a table, in order to conclude with a general rule about the sum of the angles of a polygon.

After a couple of trials, the students conclude that the sum of the angles equals 180° multiplied by the number of sides of the polygon minus two. They then verify this rule with trials of a few more polygons. At this point, Ms Jones asks the students to explain why this rule is correct. The dialogue below follows:

Ms Jones: Why is this formula correct? Can you give any explanation?
Student A: It works for all the polygons we tried.
Ms Jones: How do you know that this will work for all polygons?
Student B: It isn't necessary. What we need is a formula that works.
Student C: Yes, we spent so much time playing with the software. If you had given us the formula and a list of problems to work on, by now we would have got more done.
Student A: Practice makes perfect!

Questions

1. Prove that the sum of the angles of a polygon equals 180° multiplied by the number of sides of the polygon minus two.
2. What are the aims of using Dynamic Geometry software in this lesson?
3. What do you think are the issues in students' responses, especially in the use of technology in the class?
4. How would you respond to the students and to the whole class?

Excerpt 15.1 *Polygon* mathtask (2020 BMath Portfolio of Learning Outcomes)

section. Furthermore, in resonance with an institutional requirement that all student work meets certain standards of academic writing, scripts are also marked for clarity, coherence and quality of presentation and referencing.

In (Biza & Nardi, 2020), we illustrate the use of these criteria in the assessment of the mathtask responses of one student, Emily. In (Biza & Nardi, 2022), we report observations that emerged from the assessment of the portfolio responses of an entire student cohort. There, we present evidence that engaging with the portfolio activities reveals, and challenges, deeply rooted, absolutist narratives about mathematics and its pedagogy. We also evidence that students are challenged by the requirement to deploy RME theory and findings towards backing up claims (e.g. about what may constitute pedagogical effectiveness). The RME literature the students choose to reference is often quite generic (namely, their references may lack specificity to the particular issue they are aiming to discuss). Also, these references often appear to be made for the purpose of gaining lecturer approval, and therefore marks, rather than demonstrating explicitly the link between claim and evidence.

We see these students' attempts as engagement with the rituals of RME discourse, as actions taken for the purpose of being accepted as a member of the RME community – what Sfard (2008) calls a "natural, mostly inevitable, stage in routine development" (p. 245). One example of ritualized engagement is the "name dropping" of references to the work of eminent members of the RME community. We stress that we see such ritualized engagement as a potentially productive first step in the shift from making entirely unsubstantiated claims to recognizing the need to back up claims through reference to the work of others. More systematic and robust engagement with the work of others, and thus enculturation in the routines of the RME community may then ensue.

Two further observations concerning said enculturation are: (1) how student responses demonstrate attention to social or institutional aspects of the mathematical activity in the mathtasks (such as group work, student interaction and sociomathematical norms as per Cobb & Yackel, 1996), beyond merely attending to mathematical correctness; and, (2) responses that conflate RME theoretical constructs – intended as interpretive tools in the analysis of learning and teaching situations in mathematics – as tools for pedagogical prescription. Again, we see this as a natural step from the prescriptive and normative position that theory may hold in the Natural Sciences and Mathematics to its more interpretive and reflective role in the Social Sciences. We see evidence in the portfolio entries that the students have become aware of this difference as evidence of "meta-level" learning about RME (namely, learning about "change in the metarules of the discourse", Sfard, 2008 p. 300).

In Excerpt 15.2, we sample responses to the mathtask presented in Excerpt 15.1 that illustrate observations (1) and (2), framed in the terms of the typology of four characteristics we use for assessing students' work.

15.4.2 The BEd Course

The BEd course has been part of the BA Education programme's suite of optional courses since 2012, is entitled *Children, Teachers and Mathematics: Changing Public Perceptions of Mathematics* and is led by the first author. Its structure is similar to that of the BMath course. Its aims are similar to, but also distinct from, those of the BMath course (Nardi, 2017). As about three quarters of the programme's graduates continue into training to become primary teachers, the course is designed to address directly the widely reported reticence of those students towards mathematics and their generally low self-esteem in mathematics. Its aim is to equip these students with the means to tackle the disaffection (Nardi & Steward, 2003) that often tantalises the relationship with mathematics experienced by themselves as well as the young people many of them will soon be preparing to teach.

Lawrence: limited consistency, limited specificity

"Some of the difficulties that some students have with the use of technology is that some of them maybe there are not familiar with the Dynamic Geometry software. Also, a few researchers believe that "overuse lead to students' metacognitive shifts, resulting in a loss of their focus on the original mathematical concept to something else" (Soheila B. Shahmohammadi, 2019). Moreover, the continuously use of technology lead to the loss of mathematical skills and thinking as in nowadays it is obvious that many dynamic software and calculators can do the calculations in a mathematical task with just one click. To conclude, in this example we can see that students use technology to structure many different polygons to find a general formula for the sum of their angles but the dynamic software cannot give you an exact proof of this formula but only that it works for some cases."

Penny: limited reification of RME discourse (RME constructs inaccurate use)

"The student's responses show that they lack the understanding of the concept of relational thinking, especially in their response to the use of technology."

Harry: high reification of RME discourse (RME constructs accurate use), high specificity

"One initial problem is that Mrs Jones has taken this class over from another teacher. Therefore, the class already has established sociomathematical norms with their previous teacher. This sociomathematical norm is that students did not need to provide explanation or reasoning for their given answers, if they could carry out the procedures and rules they had been taught, this was enough."

Nicole: high reification of RME discourse (RME constructs accurate use), high specificity

"[…] The class has been taught instrumentally, however Ms Jones aims for relational understanding. Students B and C show negative attitudes towards using the technology. Pesek and Kirshner (2000) believe initial instrumental teaching can cause cognitive, metacognitive and attitudinal interference with later relational understanding. Students B and C show attitudinal interference, with Student B wanting a 'formula that works' (a preference for an external conviction proof scheme provided by the teacher), and Student C wanting faster solutions."

Excerpt 15.2 A sample of BMath student responses to Question 3 of the *Polygon* mathtask (Excerpt 15.1), with a selection also of their *consistency/specificity/reification of RME and mathematical discourse* characterisations

The course sets out from the assumption that influences on young people's attitudes towards mathematics come from inside school and outside school – and that our role as educators is to optimize all those influences, including these four: while our first priority needs to be with improving students' experience of mathematics within school (1), we need to also be aware of the often stereotypical ways in which mathematics and mathematicians are portrayed outside school, e.g. in the media, popular culture and the arts (2). With such awareness then, within school, we need to develop systematic ways of working against stereotyping and towards engineering more favourable, and accurate, images. Within school, we need to openly address these images: question the inaccurate, undesirable ones, and make the most of the rest (3). Furthermore, outside school (4), we need to work more closely and systematically with the often well-intended, but not always best-equipped, 'outsiders' who create those popular images (e.g. in the press and in the creative industries).

1.	*Mathematics and I* A biographical account of your relationship with mathematics	
2.	*Mathematics in the media* A brief analysis of a mathematics-related media excerpt (paper press or online)	
3.	*School mathematics and I* Reflections on one aspect of the school mathematics curriculum	
4.	*Mathematics over time* A 2-minute *Maths Pitch* from the history of mathematics!	
5.	*Mathematics today* A 2-minute *Maths Pitch* on a current application of mathematics!	
6.	*Mathematics in the classroom* A response to a mathtask (with mathematical, social, affective, meta-mathematical elements)	
7.	*Mathematics in art and popular culture* A brief analysis of a mathematics-related art or popular culture excerpt (film, theatre, literature, arts, music)	
8.	*Mathematical ability on film* A brief analysis of the portrayal of a mathematically able character on film	
9.	*Myths about maths* A brief essay, with evidence, debunking myths about maths (such as Innate, Male, Introvert, Burn Out, Uncreative)	
10.	*Mathematics lesson plan* A plan for a mathematics lesson on a topic of each student's choice	

Excerpt 15.3 The 10 parts of the BEd Portfolio of Learning Outcomes

The inception of the BEd course stems from acknowledging that the preparation of teachers rarely equips them for this complex task – and its 12 weeks of lectures and seminars are organized to address (1)–(4). In the Portfolio of Learning Outcomes, the course's single item of summative assessment, students are asked to: return to ten activities they prepared for in the weekly seminars; study the materials accumulated during the 12 weeks of the course; and, compose a revised contribution to each one of the ten activities, written in the light of what they learnt during those 12 weeks. The headings of the ten activities are presented in Excerpt 15.3.

Students are expected to deploy RME constructs introduced and used throughout the course. Examples of such constructs include: *instrumental and relational understanding* (Skemp, 1976); *social and sociomathematical norms* (Cobb & Yackel, 1996); *commognitive conflict* (Sfard, 2008); *identity and identity work* (Mendick, 2005); *teaching triad* (management of learning, sensitivity to students, mathematical challenge; Jaworski, 1994); *knowledge quartet* (foundation, connection, transformation, contingency; Rowland, 2013). They are also expected to refer to a small number of research papers (and, where needed, other publications such as policy documents, reports or media excerpts) in each part. Excerpt 15.4 shows a mathtask students were asked to respond to in Part 6 of the 2020 portfolio.

Portfolio entries are assessed with the same criteria as for the BMath course. In Excerpt 15.5, we sample the use of these criteria towards the assessment of student responses to the mathtask in Excerpt 15.4.

Ms Jones is about to start a mathematics lesson in a Year 6 class (student age 10-11). As she walks into class, she notices two girls giggling while listening to something in their headset. She asks why they are giggling and she finds out that they are listening to Ariana Grande's new single *34 + 35*. "I am not sure you're even supposed to listen to this – and certainly not during a maths lesson!" she comments. "But, Ms!", says one of the girls, "Ariana is playing a scientist in the video and is doing maths in the song! We too just practised doing sums: 34 plus 35 makes 69!". "Do you even know what this means…?!" exclaims the teacher. The class giggles. Another student says he asked his older sister about this. "She mumbled "something to do with sex" and rushed out of the room…", he says. The class is now roaring with laughter and the teacher interjects: "Ok, everyone! If Ariana is doing maths in her single, so will we in this class, ok?".

Ms Jones: There we go. 69 is the sum of two consecutive numbers, 34 and 35. Can you think of other numbers that are the sum of two consecutive numbers?

Neil: Oh no. This is boring…And what's the point? Can't we just go back to whatever you were planning for today, Ms?

Anna: Well, I can think of some numbers. 49 is the sum of 24 and 25. And 89 is the sum of 44 and 45! I think there is quite a few of them! [*She starts writing down a list: 49, 69, 89…*]

Barack [*a little weary*]**:** Anna, this is taking too long and it's not just the numbers ending in nine: 67 is the sum of 33 and 34, 93 is the sum of 46 and 47 and so on. There is no way you can make a list of all of them. Or, oh, or… maybe you can?! Look! Say I have a number N and then the number after this is N+1. If I add them together, I get N+(N+1). That's 2 times N plus 1 [*he writes:* $N+(N+1) = 2N+1$]. Doesn't that say that the sum is always…a what you call it… that the sum is always an odd number? So, if I have an odd number, I can always break it into two numbers that are next to each other – what did you call them, Ms? – er… consecutive numbers! Look, look, it works: 1005 is an odd number and is the sum of 502 and 503. Wow! I wonder whether this works for three consecutive numbers…! Hm…

Clive [*annoyed*]: Ah, here he goes again with his N this and X that ... Too complicated and boring. I think I am with Ariana on this one: "Math class, never was good"… Well said!

You are the teacher and you just heard what Neil, Anna, Barack and Clive said….

1. Which whole numbers can be written as a sum of two consecutive numbers? Explain your answer.
2. How would you respond to Anna?
3. How would you respond to Barack?
4. How would you respond to Clive?
5. How would you respond to the whole class – also in the light of Neil's initial comment – and conclude the lesson?

Excerpt 15.4 *34 + 35* mathtask (2020 BEd Portfolio of Learning Outcomes)

15.5 The Interplay of Research and Practice in Welcoming Two Different Communities of Learners – from Mathematics and from Education – into RME

In this chapter, we presented briefly the design, delivery and assessment of two introductory RME courses for final-year BA Education and BSc Mathematics students. We also presented examples of mathtasks and the assessment frame used in these two courses. We design and deploy course activities, such as the ones we exemplified here, as tools that can trace, challenge, and potentially shift, students' often deeply rooted mathematical and pedagogical discourses. Especially, for the

Tim (3): limited specificity

"It is clear Barack is comfortable in the classroom environment, which as a teacher is vital to ensure it is friendly (Steward & Nardi, 2002, p. 8). As Barack's teacher it is vital to praise his confidence and process of discovery in order to maintain and grow his interest in the subject."

Tim (4): limited specificity, limited reification of RME discourse (re: use of RME constructs)

"Research suggests that it is likely that, as students proceed to the later years of their schooling, such as in this scenario, often become more disenchanted with the education process (Keys & Fernandes, 1993). As a teacher it is important to ensure that the students, continue to engage with mathematics throughout their education, instead of disengaging with it when it becomes 'complicated and boring'. Therefore, it is important to engage with Clive and talk through the given task in an engaging way, such as listening to the song**"**

Molly (2, 3, 4, 5): high specificity, high reification of RME discourse (re: use of RME constructs)

[Molly articulates her response around the three vertices of Jaworski's (1994) Teaching Triad and Skemp's (1976) relational/instrumental understanding dichotomy across her response to 2-5.]

She deploys the "sensitivity to students' vertex of the Teaching Triad to praise Anna for "for contributing to the class and for thinking for herself" (2). Also in (2), she writes: "I would also give her a "mathematical challenge" (Jaworski 1994:44) […] whilst incorporating sensitivity to Anna by ensuring that this is suitable for her ability. For example, I would ask her, "Is it just numbers that end in nine that can be broken in this way?". In (3) she accurately sees Barack's "discovery of a mathematical pattern" as evidence of his "develop[ing] "relational understanding (Skemp 1976: p.2)" of why the pattern exists" and proposes an "even deeper mathematical investigation by asking him, "Does this work for three and four consecutive numbers?". She also incorporates mathematical and metamathematical elements in (5) when she writes: "To conclude, I would ask the children, "If maths class was never good, then why did Ariana Grande use it make her hit song? Today we were going to learn about fractions, but the nature of mathematics has led us to an investigation from which we have found a very intriguing mathematical pattern!""

Excerpt 15.5 A sample of BEd student responses to Questions 2–5 of the *34 + 35* mathtask (Excerpt 15.4), with a selection also of their *consistency/specificity/reification of RME and mathematical discourse* characterisations

BMath course, activities are inspired by studies that have identified the epistemological differences between practices in Mathematics and Mathematics Education (Schoenfeld, 2000; Boaler et al. 2003; Nardi, 2015b;) and have addressed these differences in the learning of postgraduate students (Nardi, 2015b). Especially for the BEd course, course activities also aim at re-engaging, with Mathematics, students in the Social Sciences who may have been away from it since the end of compulsory education and may also have low self-esteem in mathematics (Nardi, 2017; Nardi & Steward, 2003). The outlined set of activities uses task design principles that contextualize mathematics and its pedagogy – and the use of RME theory and findings – in specific learning situations (mathtasks, Biza et al. 2018). Student responses to these activities are assessed in relation to a typology of four characteristics (consistency, specificity, reification of Mathematics and RME discourses; Biza et al., 2018) informed by discourse analysis (Theory of Commognition; Sfard, 2008).

We see the potency of these activities in the introduction of Mathematics and Education students to RME in how they invite students to contextualise learning about RME theories in their own learning experiences of Mathematics. Finally, we see these activities – and their typology-driven assessment frame – as affording opportunities for nuanced and concrete formative feedback. We stress that we see this feedback as directly and naturally relevant to the students as much as to us, in our role as educators and RME researchers (Nardi, 2015a, b; Biza & Nardi, 2020, 2022) working towards addressing the needs of current and subsequent student cohorts – and welcoming these two quite different communities of learners (Mathematics/Education undergraduates) into RME.

Finally, we also see the potency of these courses – delivered in two departments, Education and Mathematics, that do not often negotiate shared course content – as a quintessential form of much needed rapprochement between the communities of Mathematics, Education and RME. We see the work presented in this chapter as a modest contribution to an exciting area of university mathematics education research and development: welcoming newcomers to RME and searching for an appropriate "research curriculum" (Boaler et al., p. 518) for Mathematics Education studies.

Our work in this area is currently entering its next phase in which more fine-grained analyses of the student data are in progress. The focus of these analyses is on elaborating the discursive shifts – especially regarding much needed "meta-level learning" (Sfard, 2008, p. 300) – that mark how students from diverse disciplinary communities (Mathematics, Education) navigate across discourses governed by distinctly different metarules (Mathematics Education). Through such elaboration, we hope for making these courses – and any template for those that may emerge on the way – an even more enticing invitation of newcomers into RME.

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Chapter 16
Inquiry-Oriented Linear Algebra: Connecting Design-Based Research and Instructional Change Research in Curriculum Design

Megan Wawro, Christine Andrews-Larson, Michelle Zandieh, and David Plaxco

Abstract Bridging curriculum design (theory) and classroom implementation (practice) is a critical issue in tertiary mathematics education. In the Inquiry-Oriented Linear Algebra project, we introduce the Design-Based Research (DBR) spiral as a mechanism to bridge theory and practice. In this chapter, we elaborate our project's DBR spiral informed by Realistic Mathematics Education (RME) instructional design heuristics. The phases of a Design-Based Research spiral are: Design, Paired Teaching Experiment, Classroom Teaching Experiment, Online Working Group, and Web. We explicate these phases to offer insight into the process of conceptualizing and developing an RME instructional sequence focused on determinants. The Online Working Group phase involves work with instructors who were not part of the research project team. This importantly allows us to explicitly connect to research on instructional change as part of our instructional design process. Drawing on data from these instructors' work with the determinants unit, we gain valuable insights into the ways in which differences in instructors' instructional contexts and orientation toward mathematical goals can constrain and afford particular kinds of instructional commitments.

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The Inquiry-Oriented Linear Algebra (IOLA) project develops research-based curricular units for an active learning approach to the teaching and learning of linear algebra at the introductory university level, as well as instructional support materials to help instructors implement IOLA units in their classrooms (Wawro et al., 2013a, b). This research integrates multiple ideologies and theoretical approaches. In terms of instructional design theory, IOLA draws on Realistic Mathematics Education (RME) (Freudenthal, 1991) in conjunction with the ideological orientation of Inquiry-Oriented Instruction (IOI) (Kuster et al., 2018; Rasmussen & Kwon, 2007). IOI is a form of active learning in which students contribute to the reinvention of important mathematical ideas; this contrasts with forms of active learning in which students practice or apply principles that were previously explained or demonstrated by the instructor.

Furthermore, we situate IOLA research more broadly within the design-based research paradigm (Cobb et al., 2003; Prediger et al., 2015), which involves a cyclical process of investigating student reasoning about specific mathematical concepts while concurrently designing and refining task sequences that honor and leverage students' ideas towards desired learning goals (Gravemeijer, 1994; Wawro et al., 2013a, b). Within IOLA, however, we extend this cycle to what we call the *design research spiral*, which includes a new phase that integrates collaboration with mathematicians for classroom-based research in diverse settings across North America. This new phase, which utilizes Online Working Groups (OWG) (Fortune et al., 2020), is informed by research on best practices for sustained propagation of instructional change at scale (e.g., Henderson et al., 2012).

In this chapter, we consider the practical and theoretical question of aligning instructional design theories with research on instructional change. The former informs the researcher and the final product (e.g., cohesive task sequences and instructional materials to support implementation), whereas the latter informs the propagation and establishment of meaningful pedagogical reform across a diverse range of university mathematics classrooms. Throughout the chapter, we illustrate this work and the design research spiral by exemplifying a new unit on an inquiry-oriented approach to the concept of determinants from the IOLA project. An additional goal of this chapter is to put RME and IOI in conversation with instructional change literature. In particular, we consider how these bodies of work can be mutually informative and explore possible points of compatibility or tension, with the goal of connecting research and practice.

16.1 Background Theory and Literature

In this section, we first detail the two main instructional design foundations of IOLA – Realistic Mathematics Education (RME) and Inquiry-Oriented Instruction (IOI). We then summarize literature on instructional change at the university level that have been most influential to the IOLA project.

16.1.1 Realistic Mathematics Education

The main theoretical framework leveraged in IOLA for designing instructional materials is RME (Freudenthal, 1991). In particular, there are three core design heuristics: didactic phenomenology, emergent models, and guided reinvention.

The roots of the *didactic phenomenology* heuristic originated with Freudenthal (1983) and his extension from mathematical phenomenology, "Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world. ... didactical phenomenology [is] a way to show the teacher the places where the learner might step into the learning process of mankind" (p. ix). Categorizing these human creations of mathematical concepts, structures, tools, or ideas broadly as "thought things", Gravemeijer (2020b) elaborates the connection to a task settings and problem situations: "Knowing how certain phenomena are organised by the thought thing under consideration, one can envision how a task setting ... may be used as starting points for a reinvention process" (p. 226). Larsen (2018) unpacks didactical phenomenology as the driving force behind RME-oriented design work in general and in his design work with abstract algebra content at the university level. Interpreting Gravemeijer and Terwel (2000), Larsen explains the 'thought-thing' as the mathematics that the designer wishes the students to learn. In this way Larsen summarizes this heuristic by highlighting that "didactical phenomenology tells the designer that an instructional sequence meant to support the learning of a piece of mathematics should be situated in a context that can be productively organized by students using that piece of mathematics" (p. 25), where 'organize' means to mathematize or make mathematical.

Whereas didactical phenomenology is the means for creating the task setting of the phenomena to be organized, the *emergent models* heuristic describes a process through which students can progress from their less formal understanding of the phenomena to a more formal, more mathematized organization of the phenomena. Gravemeijer (1999) describes four levels of activity that highlight this progression: activity in the task setting (also *situational activity*, cf. Zandieh & Rasmussen, 2010; Gravemeijer, 2020b), *referential activity*, *general activity* and *formal activity*. Students progress from creating a *model of* the initial task setting through activity in which they refer back to this setting while extending the model. The student can then generalize the model beyond the initial task setting. In the end the students are able to use this new model as a *model for* future activity including applying it in a formal, mathematically deductive way towards exploring new phenomena. This mathematical progression is not the work of one class period or one task but extends minimally over several days, and perhaps weeks. The result of this progression is the creation of a *new mathematical reality* for the student (Gravemeijer, 1999, 2020b) "consisting of mathematical objects within a framework of mathematical relations (p. 8)."

Taken together with didactical phenomenology and emergent models, the heuristic of *guided reinvention* is a mechanism by which students can reinvent mathematical ideas guided by the structure of the task sequence and their interactions with the instructor and their peers. Gravemeijer (2020a) points out that "the designer may take both the history of mathematics and students' informal interpretations as sources of inspiration for delineating a tentative, potential route along which reinvention might evolve" (p. 225). Students are expected to generate a variety of solution strategies that can be organized by their instructor in ways that guide students along the path of reinvention. The path is meant to be wide and multifaceted, not a single specific trajectory. Students can discuss and debate various strategies both to better delineate and clarify particular approaches but also to begin to recognize the multiple interconnected paths along the trajectory. These connections make up the framework of mathematical relations that is the new mathematical reality described in the emergent models heuristic.

16.1.2 Inquiry-Oriented Instruction

Inquiry is a form of active learning. Laursen and Rasmussen (2019) detail what they see as the common vision of the two major inquiry traditions in university mathematics instruction in the United States: Inquiry-Oriented Instruction and Inquiry Based Learning. Leveraging the term "Inquiry Based Mathematics Education" (IBME) – first offered by Artigue and Blomhøj (2013) – as an umbrella for both traditions, Laursen and Rasmussen (2019) offer the four pillars of IBME as "student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building" (p. 140). The first pillars highlight the role of the student, and the latter highlight the role of the instructor. Most instructors who seek out the IOLA materials are familiar with at least one of these traditions and thus with some subset of IBME's four pillars.

The IOLA research project works within the Inquiry-Oriented Instruction (IOI) tradition.[1] The defining characteristic of IOI is the central role of RME as an

[1] Inquiry-Oriented Differential Equations (Rasmussen et al., 2018) and Inquiry-Oriented Abstract Algebra (Larsen et al., 2013) serve as the foundational research programs for the IOI movement at the university level within the United States. In addition to IOLA, other IOI-aligned design work exists within other content areas, such as combinatorics (Lockwood & Purdy, 2019), calculus (Oehrtman et al., 2014), ring theory (Cook, 2014), and mathematical logic (Dawkins & Cook, 2017).

instructional design theory (Kuster et al., 2018). In addition to RME framing mathematics as a human activity (Freudenthal, 1991), IOLA leverages the aforementioned core RME design heuristics: didactic phenomenology, emergent models, and guided reinvention (Gravemeijer & Doorman, 1999). These design heuristics inform the development of sequences of tasks that align with a set of learning goals; task sequences are then tested, revised, and retested through repeated teaching experiments. These teaching experiments, which are two components in our Design Research Spiral, entail extensive collection and analysis of data to document student reasoning in the form of classroom video, artifacts of student work, and individual problem-solving interviews with students (Cobb, 2000). As such, the inquiry-oriented task sequences are first theorized according to RME and further built on research in which refinements are informed by the nature of students' mathematical reasoning about the tasks, with the goal of maximizing students' opportunities to engage meaningfully in reinventing important mathematical ideas. The use of such RME-informed task sequences synergistically allows for and facilitates the foundation role of inquiry within the classroom.

Although RME provides the foundation as an instructional design theory for IOI, it does less work in terms of delineating how an IOI classroom functions in actuality. Through analysis of: research on and descriptions of IOI, expert reflections on IOI implementations, and videos of expert and novice implementations of IOI materials, Kuster et al. (2018) articulated four key components that support the successful implementation of inquiry-oriented instruction: generating student reasoning, building on student reasoning, developing a shared understanding, and formalizing language and notation. These components are not only essential for the implementation of an RME task sequence but also provide opportunity for task sequence refinement because of their focus on uncovering and exploring student reasoning. Instructors who already have experience with IOI can be important informants to the development of RME task sequences. Such instructors were recruited for the OWG's described in the design research spiral. In addition, instructors engaged in an OWG have further opportunity to develop their skills at engaging in the key components of successful implementation of IOI by discussing their implementation of new RME units with others. This opportunity to develop as IO instructors contributes to a broader goal of instructional change.

16.1.3 Instructional Change at the University Level

With mounting evidence linking active learning to improved student outcomes (e.g. Freeman et al., 2014), there is evidence of uptake of student-centered instructional approaches to undergraduate STEM in the United States. Indeed, nearly half of STEM faculty responding to a national survey reported incorporating various forms of cooperative learning into their courses (Hurtado et al., 2012). However, Henderson and Dancy (2008) found that faculty who try to implement instructional reforms are often discouraged by the lack of ongoing support from educational researchers. For instance, when shifting to more student-centered pedagogies, instructors are faced with the challenge of deciding what kinds of activities they might engage students in during class time, in addition to questions relating to how to productively facilitate discussion around those activities and assess student learning in ways that align with shifting instructional approaches (Johnson & Larsen, 2012; Speer & Wagner, 2009; Wagner et al., 2007). We argue that it is ideal that instructors have access to instructional materials that are informed by basic research on student learning in content-specific areas.

In a review of literature on instructional change efforts, Henderson et al. (2011) organized these efforts relative to two key dimensions: whether the efforts focused on changing individuals or environments and structures, and whether the intended outcome was prescribed (predetermined by the change agent) or emergent (determined in collaboration with those involved in the change process). They found that effective change strategies align with or seek to change the beliefs of the individuals involved, involve long-term interventions, and are compatible with the institutional context of the university and department. Two common, relatively prescriptive, but frequently ineffective approaches are top-down policies meant to influence instructional practices, and merely making “best practice” curricular materials available to faculty. Our current primary focus in IOLA is instructional change at the individual level, and the OWG phase facilitates an emphasis on emergent outcomes.

Schoenfeld (2010) identified *Resources*, *Orientations*, and *Goals* (ROGs) as foundational for shaping instructors’ behavior, implying they are critical components of instructional change. Within this framework, teachers’ classroom instruction is generally guided by all three of these aspects of their respective circumstances, in tandem, as enacted from their own perspective. Schoenfeld describes *Resources* as all knowledge, technology, curricular materials, and infrastructure that a teacher might use in their instruction. *Orientations* are composed of the values, beliefs, dispositions, and opinions that a teacher may hold regarding learning, their students’ abilities, mathematics, communication, or any other aspect of teacher-student interactions. *Goals* can be articulated at several grain sizes and are often an amalgam of what the instructor feels are the content objectives of the course, important knowledge and skills that students should develop, and meaningful ways of reasoning about the content. None of these three facets of the framework is independent of the others and, indeed, each is necessarily mutually informed by the others. We find this framework to be a helpful means for describing OWG instructors’ adoption of the IOLA materials and approach to using the instructional support materials. As we work with instructors, we have relied on this framework as a lens for making sense of instructors’ adoption and implementation of the new IOLA materials.

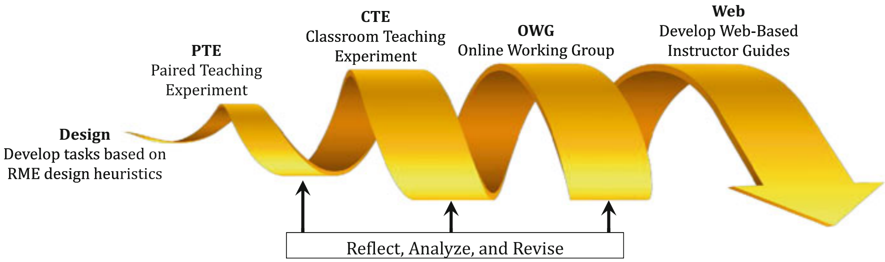


Fig. 16.1 Graphical depiction of our Design Research Spiral

16.2 The Design Research Spiral

The particular approach to design research we currently utilize in the IOLA project is what we refer to as the *Design Research Spiral*. There are five main phases in the Design Research Spiral: *Design*, *Paired Teaching Experiment* (*PTE*), *Classroom Teaching Experiment* (*CTE*), *Online Work Group* (*OWG*), and *Web* (see Fig. 16.1). Briefly stated, the phases are as follows:

- Design Phase: The project creates a first draft of the unit tasks and learning goals
- PTE Phase: The team tries the unit tasks with student pairs in a modified instructional setting
- CTE Phase: The unit tasks are tested in a classroom setting with a team member as the teacher-researcher
- OWG Phase: The unit tasks are tested in classroom settings with experienced IOLA users
- Web Phase: The research team creates the finalized version of the unit tasks and instructor support materials and adds them to the IOLA website for dissemination

These phases align with the crosscutting features of design experiments in that (Cobb et al., 2003) the broad goal is to develop domain-specific learning theories that are accountable to design, and that this goal is achieved by developing conjectures, testing them in highly interventionist studies, and iteratively refining them in a cyclical way. Much like a design research cycle (e.g., Cobb et al., 2003; Wawro et al., 2013a, b), the design research spiral includes and relies on revisions that occur between each of the five phases (see Fig. 16.1). Throughout this iterative process, we reflect on and analyze student reasoning in service of revising our developing curricular materials as well as our research-based insight into student thinking. At each transition along the spiral, the number of researchers and instructors providing feedback on the task development increases until the materials are disseminated to implementing instructors through the IOLA website (Web Phase). Furthermore, we note that our notion of the Design Research Spiral contributes to the field of design research by integrating the Online Work Groups as a key phase.

In the following sections, we exemplify the Design Research Spiral with the Determinants unit from the IOLA project. We draw particular attention to RME and

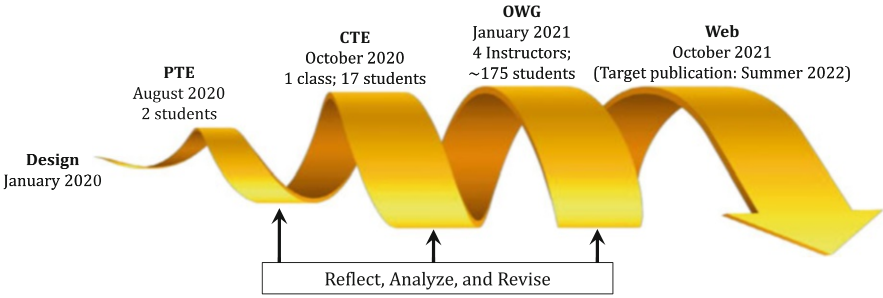


Fig. 16.2 Implementation of the design research spiral for the determinants unit

IOI, as well as bodies of work on instructional change, emphasizing the extent to which the various theoretical foundations are foregrounded in different phases of the Design Research Spiral. Substantially more detail is given to the Design and OWG phases because those more directly put RME, IOI, and Instructional Change Research Theory in conversation with each other.

By its conclusion, the execution of the design research spiral for the Determinants unit will have lasted approximately 2 years. In Fig. 16.2, we indicate the month and year in which each phase began. In the Design Research Spiral, each new phase incorporates input from more researchers and instructors and also from implementation of the materials with more students. The Design phase involved two lead researchers from the project team and four student research assistants. One of the project team's lead researchers conducted a paired teaching experiment with two students, and a different lead researcher from a different university conducted a classroom teaching experiment with a whole class of 17 students. That researcher led and organized the OWG, which was made up of four instructors – two participant instructors who are not members of the research team and two lead researchers who were implementing the unit as participant instructors. The participant instructors were chosen based on their familiarity with and prior experience using the existing IOLA units. During this phase, the unit was implemented with approximately 175 students at four institutions. Finally, as of this chapter's writing, the team is finalizing the unit's instructional materials for publication on the project website[2] by the end of 2022. Once published, it will be accessible to over 700 registered users.

16.2.1 Design Phase

The goal of the Design Phase is to create a first draft of the unit tasks and associated sequencing. This relies on all three RME heuristics: didactical phenomenology, guided reinvention, and emergent modeling. In practical use, these heuristics are

[2] https://iola.math.vt.edu

intertwined and mutually informative. For instance, as designers, we first tasked ourselves with identifying an experientially real context that “begged to be organized” (Larsen, 2018, p. 27) using the determinant concept; within that, we considered the influence of both guided reinvention and didactical phenomenology.

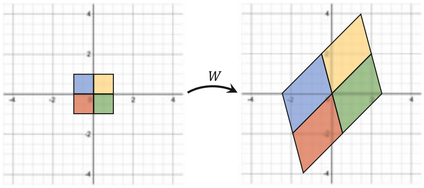
In the determinants unit, the design work was most foundationally influenced by didactical phenomenology. We began with the goal of creating a task setting in which students mathematize phenomena in a way that creates the need to develop the concept of determinants. Gravemeijer and Terwel (2000) give the example of length: for students to construct length as a mathematical object, they should be confronted with situations where phenomena have to be organized by length. We also considered the emergent modeling heuristic; in particular, we sought to create a task sequence that could leverage students’ intuitive knowledge towards the development of more formal ways of reasoning about the concept of determinant. In addition, we leveraged our practice- and research-based knowledge of student reasoning in linear algebra and of student participation in disciplinary practices such as symbolizing or theoremizing (Rasmussen et al., 2015; Zandieh et al., 2017).

Although the historical origination of the determinant concept is rooted in characterizing the solvability of systems of equations, in both Japan and Europe in 1600–1700’s (Andrews-Larson, 2015), we chose the geometric interpretation of linear transformations in R^2 and R^3 as our contextually relevant starting point. In particular, we chose *measure of distortion of space* as an experientially real setting that could “give rise” to the notion of determinants for the students as they engaged in the task. This leverages a graphical interpretation of matrices as the carrier of information about linear transformations, which productively builds off of the existing IOLA “Italicizing N” unit (Andrews-Larson et al., 2017). In addition, this setting provides a rich set of ways for interpreting the meaning of a zero determinant (e.g., in relation to the invertibility of a matrix transformation) and reasoning about important determinant properties related to composition and inverses of linear transformations and their matrix representations. Thus, the driving phenomenon of the determinants unit is to draw on students’ knowledge of matrices as linear transformations to build a conceptualization of matrix determinant as a measure of (signed) multiplicative change in area or volume.

The unit has four tasks. The main purpose of Task 1 is to generate a need for students to suggest *change in area* as a way to quantify the distortion that objects experience under various 2×2 single and composite matrix transformations. Rather than tell students explicitly from the start to look for change in area, the task is set up to bring students into a problem situation where the notion of change of area arises from a need to answer that problem, and it arises from the students’ mathematics; this is in line with the goals of didactical phenomenology. This sets the foundation for the guided reinvention of the 2×2 determinant formula (Task 2) and the introduction of the term “determinant,” which is in line with the intent of guided reinvention. Task 3 gives students a new transformation graphically, and they are to determine the matrix that “undoes‘‘ that transformation as well as the determinant of the “undoing” matrix. The main learning goals for Task 3 (see Fig. 16.3) are that students will (a) coordinate their newly developing understanding of determinants as

The Window Task

Suppose you know there is a linear transformation U: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ that graphically distorts the 4-paned "window" as shown in the diagram, but you don't yet know the transformation or its matrix representation $W = \begin{bmatrix} & \end{bmatrix}$.



1. Without knowing the exact transformation or matrix, based on our previous work, what is the measure of this transformation's change in area for objects in $\mathbb{R}^2$? In other words: without knowing W, what is $det(W)$? Why? Show your work.

2. You now want to know about the transformation that "undoes" the effect of matrix transformation W. What is the change in area measure (i.e., the determinant) of that matrix, and why?

3. If you haven't done so already, determine the entries of the matrix W and the "undoing" matrix from #2. Then find the determinants of these matrices and compare them with your measures of the change in area from #1 and #2 above. Describe at least one thing you notice and why you think that might be sensible.

Fig. 16.3 A condensed version of Task 3 from the determinants unit

change in area with their previous experience with invertible linear transformations to reinvent $\det(A^{-1}) = {}^1/_{\det(A)}$ for an invertible 2×2 matrix A, and (b) conceptualize the determinant of the matrix of an "undoing" transformation as the multiplicative reciprocal of the determinant of the original transformation.

In Task 4, students explore the geometric interpretation of matrix transformations and their determinants via GeoGebra applets for 2×2 and 3×3 matrices. Each applet consists of a matrix, sliders to control each matrix entry, a realtime calculation of the determinant, and a realtime dynamic parallelogram or parallelepiped showing the image of the unit square or cube under the matrix transformation. With these components, students are able to create matrices within the parameter constraints and visualize the effect that these changes have on the preimage. The goal is for students to create their own observations and conjectures about determinant (e.g., under what conditions $\det(A) = 0$ seems to occur) without specific prompting by the teacher so that, when discussed, it is the students' mathematics that is leveraged and explored. However, the implementation notes also have prompts prepared to direct students towards important observations and conjectures if students don't generate those on their own. By the end of the unit, the remaining learning goals include the determinant properties: $\det(AB) = \det(A)\det(B)$, $\det(kA) = k^n\det(A)$, and $\det(A) = 0$ *iff* columns of A are linearly dependent *iff* A is not invertible. Taken together, in addition to its reliance on didactical phenomenology and guided reinvention, the determinants unit aligns with emergent modeling by fostering that students first develop models-of their mathematical activity about change in area induced by matrix transformations. These later become models-for more sophisticated mathematical reasoning about the determinant in general.

16.2.2 PTE Phase

During the PTE phase, our team uses an adaptation of the paired teaching experiment methodology (Steffe & Thompson, 2000) to investigate 1–2 pairs of students' interactions with these early iterations of the tasks within a controlled interview setting. Though consistent with Steffe and Thompson (2000), our methodology departs in ways that are consistent with other RME-based curriculum design projects at the university level (e.g., Lockwood, 2019). Our primary goal during the interviews is to teach students specific content by engaging them in theoretically grounded tasks with a teacher-researcher guiding the trajectory of the participants' activity. Further, we are interested in participants' observable mathematical reasoning during interview sessions and rely on these data to inform future instructional approaches and implementations. Steffe and Thompson (2000) also focus on modeling students' cognition through frequent iterations of hypothesis testing; however, our team is focused foremost on iteratively refining the tasks themselves and identifying the changes that might better support students' mathematical activity while engaging in the tasks. To this end, we focus on participants' mathematical activity to the degree that it informs (1) iterative revisions to the tasks themselves and (2) our expectations of student solutions in later implementations of the materials, specifically during CTE and OWG phases. This informs the development of instructor materials that highlight possible student responses so that future IOLA instructors can anticipate how to handle a variety of responses during small group work or whole-class discussions.

In the determinants unit PTE, our data highlighted the importance of students' prior knowledge about linear transformations and their standard matrix representations. The students interviewed in the PTE were enrolled in an introductory linear algebra course that was completely asynchronous and online due to COVID-19. This course was not an IOLA course; rather, it consisted of watching video lectures recorded by the course instructor while completing a note-taking worksheet. At the time of the interview (towards the end of the course), the students had learned determinants with respect to matrix invertibility with a focus on computation and properties; they had not yet learned about linear transformations. Therefore, the interviewees did not have a robust geometric interpretation for linear transformations. As such, significant time during the PTE was spent helping them develop an adequate sense of linear transformations for them to engage in the task sequence. To gain additional perspectives on the task sequence, a slightly revised version was piloted with the project team after the PTE and prior to the CTE, which again led to minor revisions. One insight from this experience relevant to Task 1 was the variety of quantities individuals may attend to when working to quantify distortion, a finding that was echoed in the subsequent CTE. A second insight was the accessibility of Task 3 for the conceptualization of the determinant of a matrix A's inverse as the multiplicative inverse of the determinant of A.

16.2.3 CTE Phase

During the CTE phase, the research team implements the unit in a Classroom Teaching Experiment (Cobb, 2000). According to Cobb, a CTE has two main aspects: the first “is concerned with instructional development and planning and is guided by an evolving instructional theory,” and the second “involves the ongoing analysis of classroom activities and events” (p. 314). In the CTE phase in the Design Research Spiral, we begin with the task sequence, refined during the PTE, as well as the knowledge of potential student thinking that surfaced during the PTE. The teacher-researcher for the CTE responds to student thinking during the task implementation by making revisions to the task sequence as needed. Ad hoc preliminary analyses inform further revisions to the task sequence and instructor implementation notes prior to its enactment in the OWG phase.

In the context of the determinants unit, one of the research team members served as the teacher-researcher for the CTE. This CTE served as an effective catalyst for both slight and more substantive revisions to the task sequence and instructor implementation notes. For instance, the CTE implementation of Task 1 further illuminated the vast range of strategies for quantifying a distortion that students can develop given time to do so, and their activity did not always lead toward developing understandings directly related to the determinant. In other words, although an open implementation of Task 1 has the potential to provide opportunities for students to explore systematizing mathematical measures and definitions, this is broader than the intended goal of measuring the multiplicative change in area caused by matrix transformations. Thus, the project team concluded that it was important to communicate clearly in the implementation notes that instructors would find it advantageous to limit the amount of time spent considering possible ways of measuring distortion of space, and to shift toward a focus on change in area relatively early in subsequent iterations; an alternative task statement that suggested “change in size” rather than “distortion” was added to the unit, as an option that could be used to further help guide students towards change in area. An additional insight from the CTE was that the algebraic formulation of the 2×2 determinant may serve as an important linchpin for students in reasoning graphically with the dynamic geometry applets in Task 4 to make discoveries and conjectures about determinant properties. This confirmed that reinventing the 2×2 determinant formula early in the task sequence is appropriate so students can leverage this in their subsequent mathematical work.

16.2.4 OWG Phase

During the OWG phase, the research team works with participating instructors who are experienced IOLA implementers to test the unit in their own intact classrooms. To support the participating instructors’ understanding of the materials as designed, the research team works through the tasks as a group with the participating instructors. This process also allows the research team to discuss implementation strategies for keeping students at the center of the mathematical development of those ideas (e.g., eliciting, responding to, and building on student contributions). To support this goal, OWG meetings adopt a modified lesson study approach so as to hear the experiences of instructors as they implement tasks and to provide opportunities for reflection and collaborative conversations among implementing instructors (Fortune et al., 2020).

The goals of the OWG for the research team are to obtain feedback on task design, formatting, and sequencing – as well as information about the ways in which instructors' institutional and individual resources, orientations, and goals influenced their implementation. Further, the research team learns about how participating instructors interpret the tasks and their intended learning goals, how the tasks were implemented, and how they played out in various classrooms with various student populations. These insights inform the development and refinement of instructor notes and the tasks themselves.

The remainder of this section gives substantial detail regarding the OWG for the determinants unit. We highlight this phase of the design research spiral to illustrate its relation to IOI and instructional change literature.

Context of Participating Instructors: Teaching During A Pandemic Data collection took place in the North American Spring 2021 semester (January-May), when many instructors around the globe were teaching in online and hybrid formats due to the COVID-19 pandemic. By design, half of the OWG was research team members who were currently teaching linear algebra and half was participating instructors who were new to the determinants task sequence. Two of the implementing instructors were teaching fully online (one with relative autonomy over course content and two classes of approximately 40 students each, one with coordinated large lectures of several hundred students – which made use of zoom breakout groups – and which also had additional activities during smaller recitations). The other two were teaching in a hybrid format in which some students attended in person and others simultaneously attended online (both of whom had section sizes between 20–40 students and autonomy over course content). Both participating instructors who were new to the task sequence used their own course notes in lieu of a course text; both research team members used extant textbooks to support their implementation of IOLA materials (Lay et al., 2016; Poole, 2015). All instructors had previously taught linear algebra and had experience in curriculum design; thus, the OWG participant group reflected a set of pedagogically sophisticated mathematicians and mathematics educators. The online working group met with research team members about once per week during the semester of implementation.

Illustrations of Connections to Instructional Change Literature We highlight contributions of OWG participating instructors who were not members of the research team that offered insights into connections between RME, IOI, and instructional change literature. The OWG phase of the research project was particularly important toward informing the development of instructor support materials, including needed supports and flexibility. At this stage, teachers' resources, orientations, and goals (which include consideration of their institutional context) are taken explicitly into account. In particular, we highlight examples of how instructors' current understandings and valuation of content and its mathematical development shapes their initial interpretations of and reactions to materials, as well as ways in which their current practices and institutional context shape their implementation of the materials and choices about what needs to be made explicit in instructor support materials

Exemplar 1: Differing Orientations Toward Mathematical Learning Foci and Takeaways Prior to sharing details about the determinants unit with the participating instructors, we asked them to share what they saw as the most important ideas and non-negotiables in their teaching of determinants. The two participating instructors had different orientations regarding the organization and focus of the content related to determinants. Specifically, Dr. T, whose background was influenced by mathematical physics, articulated a vision that was highly aligned with our approach – emphasizing the geometric interpretation at the core of what he wanted students to take away.

> Dr. T: For me, determinants as a change in volume, is a non-negotiable. I don't actually care about determinant formulas, although students are accountable for a 2×2 and a 3×3 formula. And I would like them to know that the seemingly natural extrapolation of those formulas does not work on a 4×4.

When asked about seemingly natural extrapolation, he said "products of the diagonals minus products of the off-diagonals." Dr. T elaborated that there is "Absolutely zero mention of cofactors" in his course, but later added that he also wanted students to understand the relationship between invertibility and the determinant being zero. Dr. F then weighed in, emphasizing the role of the determinant in standard methods for finding eigenvalues and eigenvectors (later commenting that this was inspired by his own research and learning, which was more heavily focused on differential equations than linear algebra).

> Dr. F: So for the exact opposite side, for me it is about solvability conditions. Because to me, the goal is to say why is it that we do the determinant A-λI equals zero? Why that determinant? Why that equals zero? So that we have infinitely many solutions. To me, that's the goal. I have really bad geometric intuition about the volume transformation. Period. So I tend to downplay that.

To us, this exchange was illustrative of the way in which two instructors whose instructional orientations were both deeply committed to implementation of active learning can carry vastly different orientations toward the mathematical development of a particular topic. Further, these orientations are rooted in instructors' own expertise (a resource), which shapes their views of the ways in which particular mathematical ideas are most useful. Dr. F stated that determinants had value primarily algebraic in nature, with a focus on the information they provide about when there is a unique solution to a matrix equation. In contrast, for Dr. T, the geometric view was of primary importance. Interestingly, after working through the IOLA determinant tasks together, Dr. F re-articulated his view of the value of the geometric approach, highlighting how geometric understandings of the determinant could be helpful for understanding eigenvectors and eigenvalues in terms of a diagonalization. This suggests that, while he still viewed the primary use of determinants in relation to eigenvectors and eigenvalues, he was now able to articulate a way of making that connection geometrically rather than just algebraically.

Exemplar 2: Institutional Context and Instructional Approaches We also learned that the instructors had different approaches to supporting their students' learning, particularly in relation to the commitments of RME. Both identified the value of student exploration that preceded formal presentation of information as important for the development of creativity and modeling capabilities. Only Dr. F, however, who had small classes and complete instructor autonomy, conducted his classes in alignment with these goals. Dr. T had large coordinated sections with smaller lab sections, and thus he felt he needed to adhere to a set pacing schedule. As noted in Apkarian et al. (2021), this is often viewed in tension with efforts to be responsive to students' reasoning – both in terms of ability to meaningfully elicit and respond to student thinking, and in terms of adhering to a pacing schedule due to time constraints. Although Dr. T stated that he viewed student exploration coming before formal treatment (e.g., reading the course text) as better for students' learning, he found it more efficient to require students to read prior to class – while acknowledging that this approach "destroys creativity." He elaborated:

> Personally, I think it's better to explore and then read. But I think it's more straightforward to ask the students to read before coming to class. I find holding them accountable for reading before class, is um, something that's easy to justify. And I tell my students one hour of studying before class is worth two hours of studying after class. I've got a convincing justification for them. I personally think that they would learn more if they did the two hours of studying after class after the one hour of exploring, but I don't know how to exactly pull that off.

For the purpose of the study, Dr. T reserved a few lab sessions in which students would be asked *not* to read ahead of time so as to learn what ideas they might bring forward when given the tasks and asked to reason about them without reading about the content ahead of time.

In our view, this exchange highlights the way in which one's institutional context can shape one's perception of what is instructionally possible. Time and flexibility emerge as two resources that are intertwined with one's institutional context, and Dr. T identified tensions between goals of learning and efficiency (an effect of perceived lack of the resource of time). In his instruction, he deemed efficiency as easier to "justify" to students and thus made the instructional choice reflective of that orientation. However, this choice is in conflict with the goals and orientations espoused by RME. Tensions such as those described by Dr. T are present throughout the research literature and thus likely influence how curricular resources are taken up at scale. This underscores the importance of both flexibility in the form of emergent rather than prescribed outcomes (Henderson et al., 2012), as well as the importance of ongoing feedback and support to help instructors know when their pedagogical choices may be at odds with the intended design of curricular materials (e.g., as when to formalize certain concepts, which relates to the fourth component of IOI: formalizing language and notation).

16.2.5 Web Phase

During the Web phase, the research team creates the finalized version of the tasks and instructor support materials and adds them to the IOLA website. These materials allow instructors to build a course for their needs from the units and material available. The initial structure for the website content was based on the work of Sean Larsen (e.g., Lockwood et al., 2013) with the *Inquiry Oriented Abstract Algebra* materials. For each task, three main components comprise the instructor support materials:

- *Learning Goals and Rationale*: Addresses how the task contributes to meeting instructional goals and what kinds of thinking are meant to be evoked, leveraged, or challenged;
- *Student Thinking*: Elaborates, through photo and video examples of student work, ways in which students might think about or approach the task, answers/strategies they will likely develop; and
- *Implementation*: Includes suggestions for implementing the task, what discussion topics might be most productive, and what types of student ideas teachers should anticipate.

The instructor support materials also contain a lesson overview, editable task sheets, implementation video clips, and homework suggestions. The website has a "Goals for a Typical Day" page, which highlights various classroom interactions that help foster a productive class environment. It describes various interpersonal structures (small group work, partner talk, whole class discussion, and telling) that comprise an inquiry-oriented classroom, and it provides detailed suggestions of how to foster productive whole class discussions.

16.3 Discussion

In this chapter, our core goal was to connect literature on RME, IOI, and Instructional Change in a way that is both empirically and theoretically based – via the Design Research Spiral, as exemplified via the new IOLA unit on the concept of determinant. In the IOLA project we draw on previously successful design-based research models for instructional material development grounded in the RME design and IOI pedagogical traditions. By introducing the OWG Phase and extending the revision process to incorporate multiple instructors at multiple sites, the design cycle is extended in ways that facilitate materials refinement based on feedback from experienced instructors in varying instructional contexts. We conclude the chapter by considering the practical and theoretical work of aligning instructional design theories with research on instructional change. That is, we consider how these bodies of work can be mutually informative and explore possible points of compatibility or tension, with the goal of connecting research and practice.

As a general instructional theory, RME does not explicitly take up issues of instructional change at scale. Rather, the focus is the development of task sequences that support guided reinvention of mathematical ideas on the part of students. Gravemeijer (2020b) does elaborate on the role of instructors in this process, and on assumption that students work collaboratively with peers as they develop and refine these ideas. IOI aligns with and is rooted in RME by design (Kuster et al., 2018) and provides further definition of what a classroom that implements RME-inspired instruction looks like in practice.

When considering the relation between our design research spiral and these bodies of literature, we note that the earliest stages (Design and PTE) are most strongly informed by RME, with IOI becoming a stronger focus in middle stages (PTE, CTE, and OWG), and Instructional Change literature taking on the strongest focus at the latest stages (OWG and Web). In some ways, this may suggest that it is challenging to put RME literature in direct dialogue with literature on instructional change. However, we include two key examples from our data that stimulate a direct dialogue. First, instructors' orientations toward content may differ from the didactical phenomenological choices made by the design team. This emerged from discussions in the OWG focused on instructors' goals as they related to those of the instructional materials. Second, we found that instructors' resources and goals shape their instruction in ways that may be aligned with or in tension with the intent of RME. For instance, requesting that students read ahead of time is certainly not a rare instructional choice, but it does not place students at the center of the reinvention process. Furthermore, in high-enrollment courses, it may be unrealistic for an instructor to generate and build on student reasoning in a manageable way. Thus Henderson et al.'s (2012) distinction between emergent and prescribed outcomes becomes helpful. It is productive to distinguish features of a research-based curriculum that can be flexible in implementation so that outcomes can be viewed in an emergent way rather than a prescriptive one, yet also acknowledge possible tension between intent of curricular materials and the goals of an instructor.

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Chapter 17
Profession-Specific Curriculum Design in Mathematics Teacher Education: Connecting Disciplinary Practice to the Learning of Group Theory

Lena Wessel and Timo Leuders

Abstract In this chapter, we report on design decisions for teaching group theory in a profession-specific advanced mathematics class for prospective secondary teachers. We relate our teaching design to previous suggestions on Abstract Algebra teaching and suggest an overarching framework of theorizing for locating already established and still needed theoretical contributions (kinds of theory elements) for tertiary curriculum designs. We draw on Mathematical Knowledge for Teaching and connections of secondary mathematics and advanced mathematics teaching according to Wasserman as categorial theory elements. With those, we take a closer look at what we already know for answering questions concerning profession-specific Abstract Algebra learning, before we illustrate the core design decisions realized in our teaching design, and summarize challenges and conditions of success that led to further changes in the teaching design.

Keywords Content knowledge in mathematics teacher education · Teaching group theory for future teachers · Curriculum design in tertiary mathematics

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17.1 Profession-Specific Teaching Designs: Introducing Theory Elements for Reflecting on Design and Content Decisions

For many years, teacher educators involved in curriculum development for prospective mathematics teachers at university are aware of and work on eliminating or at least mitigating the double discontinuity first described by Klein (1908). Changes and innovations have been tested and implemented in order to diminish the second discontinuity between school and university mathematics for future teachers, so that students no longer perceive their studies as too detached from their profession. That this is a serious problem can be seen from the fact that mathematics courses often get poor students' evaluation especially with respect to the necessity and relevance for their later teaching at school (Ticknor, 2012; Gray, 2021; for the German system Mischau & Blunck, 2006). For curriculum designers, this perceived discrepancy between tertiary mathematics and school practice raises important questions: Are there ways to structure and design tertiary mathematics courses so that they become more meaningful teacher preparation experiences? Can mathematics courses influence prospective teachers' pedagogical practice? How can mathematics courses be designed so that they are perceived more relevant to future mathematics teachers?

There is rising attention to the complex design questions, which are often subsumed under the notion of profession-specific courses, meaning in the context of this paper mathematics courses for prospective teachers. Methodologically, they require design research studies for suggesting design principles for profession-specific mathematics teaching designs at university level. With a prominent focus on primary or secondary school teaching and learning, Design Research has been established as a research methodology that systematically combines two aims: (1) improving subject-matter classroom teaching by designing teaching-learning arrangements for a certain topic and (2) generating theoretical contributions by empirical research in order to understand the initiated teaching-learning processes for a certain topic (Cobb et al., 2003; Barab & Squire, 2004; Gravemeijer & Cobb, 2006). In this research paradigm, mathematics education lecturers and researchers also develop design innovations for various topics in the tertiary mathematics curriculum pursuing the dual aims of *improving* instructional designs and *theorizing* (e.g., for the case of Abstract Algebra teaching, Larsen, 2013; Larsen et al., 2013; Leuders, 2016a).

However, for the above case of profession-specific mathematics courses, we still lack systematic reviews of what the label *profession-specific* refers to in the various studies dealing with university mathematics classes being particularly tailored for prospective teachers, and how the learning processes of prospective teachers are affected by such changes. One simple definition could be that next to pure mathematics learning goals, additional goals are pursued, such as positively influencing prospective teachers' orientations towards teaching (e.g., on how mathematics should be taught). For structuring such additional goals, Wasserman (2018a, p. 7) puts the focus on connecting the particular advanced mathematics to secondary mathematics and opens up a continuum of connections: The connections range from "content connections" to "disciplinary practice connections", "classroom teaching connections", and "modeled instruction connections" (to be further elaborated in Sect. 17.1.3). For enhancing such connections in Abstract Algebra university teaching, a growing body of research has been published rather recently (e.g., Ticknor, 2012; Wasserman, 2018a; Gray, 2021). Cummings et al. (2018) claim that next to these theoretical contributions on *what* prospective teachers can learn in a profession-specific Abstract Algebra class, we still need "both theory and data on *how* prospective teachers can and do learn from an Abstract Algebra course" (Cummings et al., 2018, p. 327). Further, we need theory that explains the process quality of prospective teachers' learning by reconstructing challenges and conditions of success in initiated learning processes.

For keeping track of design decisions as well as illustrating research gaps concerning the what and the how in outcomes of theorizing tertiary curriculum design innovations, we suggest to draw on a framework elaborated by Prediger (2019). It captures what theorizing means in Design Research and suggests a language for theory elements according to their function and structure. The theory elements are structured according to the intertwining levels of practical and theoretical contributions which Design Research studies aim at: (1) the *design* of the teaching-learning arrangements for providing answers on *how-questions* "How to design a teaching-learning arrangement so that the initiated learning processes reach an intended aim?", and (2) the *structure* of the learning content for providing answers on *what-questions* (Gravemeijer & Cobb, 2006; Prediger & Zwetzschler, 2013; Bakker, 2018).

In these dimensions of what- and how-questions, Prediger (2019) provides a methodological foundation for categorial, normative, descriptive, explanatory, and predictive theory elements as outcomes of theorizing in Didactical Design Research, summarized in Table 17.1.

All five kinds of theory elements (categorical, normative, predictive (humble or refined), descriptive, and explanatory) are consequently intertwined in Design Research studies, while design principles in their complete structures provide the theoretical background for the functioning of the design: "design principles is probably the most prevalent term used to characterize the kind of prescriptive theoretical understanding developed through educational design research … [as they] integrate descriptive, explanatory, and predictive understanding to guide the design of interventions" (McKenney & Reeves, 2012, p. 35). When it comes to Design Research on profession-specific teaching designs, structuring the learning content is highly complex due to the different facets of teacher knowledge. We believe the framework of theory elements to be valuable for reflecting and classifying one's design decisions: The theory elements help illustrating where insights and recommendations for designing are already at hand as well as identifying aspects which are not yet researched sufficiently. This also accompanies our chapter when we work on the guiding question of "How to adapt an Abstract Algebra curriculum focusing on learning group theory so that profession-specificity is enhanced by

Table 17.1 Theory elements on teaching designs (how) and content structure (what) (Prediger, 2019, p. 14)

	How-questions for theory elements on the design of teaching-learning arrangement	*What-questions* for theory elements on structuring the content
Categorial theory elements	Categories for design principles, process qualities, characteristics of design elements	Categories for distinguishing and relating aspects of the learning content
Normative theory elements	Which process quality should be reached in order to achieve a later learning goal (and why)? (process qualities)	What should students learn (and why)? (unpacked learning content goals)
Humble predictive theory elements	Which design principles should be applied for which aim?	In which (still vague) learning trajectory can the learning content be structured?
Descriptive theory elements	Which situational effects can the design principles and design elements unfold in the teaching-learning pathways? And how does that relate to the intended effects?	What learning pathways do students usually take along the intended learning trajectory? And how does that relate to the intended learning trajectory?
Explanatory theory elements	Which background do the (non-)effects of design principles and design elements have? Under which conditions of success do they have the intended effects?	What can explain the students' typical perspectives, learning pathways and obstacles? (e.g., which aspects are crucial for learning the next one?) What can explain the differences between the intended learning trajectory and the individual learning pathways?
Refined predictive theory elements	Elaborated design principles: Which design characteristics and design elements can be applied for which intended aim and which explanatory element justifies the expectation of these effects and which conditions of success must be considered?	What relations between aspects of the learning contents must be considered? In which refined learning trajectory (or learning landscape) can the relevant aspects of the learning content be structured in order to increase access for all students?

content-, disciplinary practice-, content teaching- or modeled instruction connections?"

17.1.1 Facets of Teacher Knowledge as Categorial and Normative Theory Elements

In this section, we give a short overview of categorial and normative theory elements that became relevant for distinguishing and relating aspects of the learning content in mathematics teacher professionalization contexts. With Shulman's (1986) distinctions of content knowledge from pedagogical content knowledge (PCK) as differentiations of facets of teachers' professional knowledge, and Bromme's framework (2001) for facets of teacher knowledge, important foundations have been provided for theorizing in the field of tertiary curriculum design in mathematics teacher education programs. Today, their conceptualizations serve as theoretical frameworks in numerous studies which aim at more detailed understandings of teachers' professional knowledge, its development, and its relation to teachers' practices (overview with a focus on PCK see Depaepe et al., 2013). For a further specification of mathematics teacher knowledge, parallel lines of frameworks evolved over the last decades. Whereas Shulman outlined professional knowledge for all teachers, Ball and colleagues (e.g., Ball et al., 2008) refined the notion of facets of teacher knowledge specifically for mathematics teachers. Their development of the Mathematical Knowledge for Teaching (MKT) framework has been widely adopted in mathematics education. In addition to being explicit about teacher knowledge related to the practices of teaching (i.e., a practice-based approach to teacher knowledge), the primary contribution of MKT was providing three sub-domains of *content knowledge* in the sub-domains of "common content knowledge (CCK)", "specialized content knowledge (SCK)", and "horizon content knowledge (HCK)"; and *pedagogical content knowledge* in the sub-domains of "knowledge of content and students (KCS)", "knowledge of content and teaching (KCT)", and "knowledge of content and curriculum (KCC)".

While pedagogical content knowledge became quite popular in mathematics education research for addressing and working on the second discontinuity, also the content knowledge facet of "horizon content knowledge" is gaining more theorizing attention recently. In Ball and colleagues' framework it is defined as the "awareness of how mathematical topics are related over the span of the curriculum" (Ball et al., 2008, p. 403). This knowledge is particularly important when we think of the content connections between school algebra and more advanced Abstract Algebra. Wasserman (2018b) picked up the notion of "horizon content knowledge" in the construct of *knowledge of nonlocal mathematics for teaching*. In this framework, mathematical ideas in the local neighborhood of the mathematics being taught at school and the nonlocal mathematical neighborhood are distinguished (ibid.). While the four domains SCK, CCK, KCT and KCS primarily address local mathematical ideas, Wasserman conceptualizes "horizon content knowledge" and "knowledge of content and curriculum" as part of nonlocal mathematics. He also claims that for nonlocal mathematical knowledge to have an impact on the teaching of school mathematics, it seems necessary, but not sufficient, to establish connections between the nonlocal and local content, and specific connections have to "reshape one's understanding of the local neighborhood" (Wasserman, 2018b, p. 7) in order to influence teaching practices. As an example, learning that (Z, +) is a group and that the steps to solve linear equations like $x + 4 = 10$ rely on the group axioms may change one's perception about solving these equations (Wasserman, 2018b).

In the above table of theory elements (Table 17.1), we can locate the different constructs of mathematics teachers' professional knowledge as categorial and normative theory elements for structuring the learning content of an advanced mathematics class such as Abstract Algebra and for discussing design principles that have proven fruitful in previous research. In the next section, we start with the latter and summarize predictive theory elements in the how-dimension and give an overview of elaborated design principles for Abstract Algebra teaching designs.

17.1.2 Learning Abstract Algebra: What We Learn from Previous Research for Answering How-Questions

Students' difficulties in Abstract Algebra courses have been well documented (e.g., Asiala et al., 1997; overview in Weber & Larsen, 2008). Clinical interview studies show how students struggle not only with proving theorems in Abstract Algebra, but also with the level of abstraction and complexity of the fundamental concepts (Hazzan, 1999) as well as the algebraic structuralism of Abstract Algebra in general (Hausberger, 2015). Drawing on the notion of algebraic structuralism (referring to "structure" used as a meta-concept that is typically used but not mathematically defined in algebra classes), according to Hausberger (ibid.) it seems not surprising that students are struggling when they are supposed to implicitly learn by themselves and from examples what is meant by a structure. In consequence, Hausberger develops activities for students to reflect explicitly on structuralist thinking (e.g., Hausberger, 2017).

For the specific case of group theory in Abstract Algebra, empirical studies have repeatedly shown that "undergraduates tend to avoid using their conceptual knowledge of the relevant mathematical objects by relying on well-known procedures" (Hazzan, 2001). This can be traced back to a lack of sustainable mental models (Hazzan, 1999), an observation that also Weber and Larsen share: While students are often able to reproduce formal definitions or procedures, they "have no informal ways of thinking about groups other than by reciting the group axioms" (2008, p. 142). Studies across different sub-topics of a typical Abstract Algebra curriculum came to similar conclusions that students "have no intuitive descriptions, e.g. of what it meant for two groups to be isomorphic" (Weber & Larsen, 2008). As a consequence of observed shortcomings in students' conceptual understanding, extensive Design Research was carried out in the field of teaching and learning Abstract Algebra (including a phase of scaling up with the result of the *Teaching Abstract Algebra for Understanding* (TAAFU) Curriculum, overview in Larsen et al., 2013 as well as in-depth epistemological and cognitive investigation of selected concepts such as the homomorphism concept, see e.g. Hausberger, 2017). The TAAFU research aimed at developing a local instruction theory (Gravemeijer, 1998) that takes empirically identified typical students' difficulties and challenges into account. Several years of research resulted in a local instruction theory which follows the principle of *guided reinvention* of crucial concepts of group theory in an inquiry-based teaching design (for further elaboration on *Realistic Mathematics Education principles* (RME) see Freudenthal, 1991; Gravemeijer, 1998). The following design principles of the TAAFU Curriculum have been implemented in three instructional sequences (one for each of the main course concepts: group, isomorphism, and quotient group) (Larsen, 2009; Larsen et al., 2013):

Guided Reinvention and Emergent Modelling The backbone of the teaching designs are instructional sequences which feature a reinvention phase and a deductive phase. During the reinvention phase, the students work on a sequence of tasks designed to develop and formalize a concept by drawing on their prior knowledge and informal strategies. The instructional sequences start with a problem situation in which students encounter and develop an understanding of a certain concept, such as group, as 'models of' in line with the design principle of emergent modelling (Gravemeijer, 1999; Leuders, 2015). The product of the reinvention phase is a formal definition and a collection of conjectures concerning the concept the students have been working on. The reinvention phase is followed by a deductive phase in which students work on proving theorems – which are often based on conjectures arising during the reinvention phase – using the conventions, formal definitions and previously proved results. Guided reinvention in the sense of *RME* has proven fruitful for developing conceptual understanding which is assumed to be more sustainable (for the case of algebra in comparison to knowledge acquired in a rather typical deductive algebra class), because "[k]nowledge and ability when acquired by one's own activity stick better and are more readily available than when imposed by others" (Freudenthal, 1991, p. 47). Having completed both, reinvention and deductive phases, students have acquired the certain concept (e.g., group) as 'models for' referring to the notion of emergent modelling that a concept which initially emerged as a 'model of' students' activity in problem situations. Cuoco and McCallum (2018) support this design principle in their conclusions on working on the second discontinuity, stating that abstractions should be motivated with concrete examples whenever possible and call for "experience before formality".

Proofs and Refutations Heuristics Following suggestions by Larsen and Zandieh (2008) for processes of guided reinvention in undergraduate mathematics education, the TAAFU designs rely on this heuristic as design principle for the transitions between each reinvention and each deductive phase. This design principle captures students' learning processes in which conjectures are revised by analyzing proofs especially in light of comparing and contrasting examples and counterexamples (Rittle-Johnson & Star, 2011).

With these design principles, the research team provided important theoretical contributions in the dimension of the above how-questions: If we want students to be actively engaged in developing sustainable conceptual understanding, we need to offer learning opportunities that allow for guided reinvention processes in which students not too quickly move from conceptual understanding to formalized symbolic representations and procedures.

17.1.3 *Learning Abstract Algebra: What We Learn from Previous Research for Answering What-Questions*

Abstract Algebra is defined as “the study and generalization of algebraic structures” (Wasserman, 2016, p. 30). For prospective mathematics teachers in a secondary teacher education program in Germany, the course curriculum typically comprises binary operations, identities, inverses, commutativity, groups and subgroups, isomorphisms/homomorphisms, and rings (while the curriculum for prospective teachers often differs from a curriculum of mathematics major programs). Next to symmetry and quotient groups, permutations (as re-arrangements of elements of a set) serve as a typical example of exploring the group concept (Alcock, 2021, pp. 136 ff.). Within such a curriculum that is typically structured according to the textbook or lecturer’s scripts, the different subtopics have various potentials for enhancing profession specificity for prospective teachers (with different emphasis for primary or secondary/upper secondary teachers).

Already Felix Klein illustrated how Abstract Algebra is relevant for school algebra, and younger research particularly investigates the connections between both (Suominen, 2015; Gray, 2021; Leuders, 2016a). In these investigations, it is often the content connection perspective that is taken (Wasserman, 2018a). Content connections refer to a meaningful relationship between a concept discussed in the advanced course (e.g., a group in Abstract Algebra) and some secondary content (e.g., invertible functions). Unfortunately, as Cuoco (2001) points out, “Abstract Algebra is seen as a completely different subject from school algebra” (p. 169) despite the clear connections to secondary school mathematics. This is one reason why explicitly discussing such mathematical content connections is a way that instructors of advanced mathematics courses choose in order to make the advanced content more relevant to secondary teachers. For identifying fruitful content connections, Abstract Algebra curriculum designers can rely on a study by Wasserman (2016) who reconstructed how ideas in Abstract Algebra can be beneficial for algebra (and early algebra) teaching. The content of school algebra only implicitly draws on the structures of groups, fields, and rings (largely fields and rings). This is why the study aimed at identifying common content areas that provide a synthesis of the types of school mathematics potentially informed by knowledge of Abstract Algebra. As a result, the four content areas *arithmetic properties, inverses, structure of sets*, and *solving equations* have been shown as most relevant. Further support for instructors searching for fruitful connections in an Abstract Algebra curriculum to secondary algebra is provided by Suominen’s (2015) comprehensive connections list between concepts found in Abstract Algebra and secondary school mathematics. For the case of German course material, Leuders (2016b) proposes a curriculum and textbook for explicating the multiple connections between elementary algebra and Abstract Algebra. More specifically, Leuders (2016a) analyzes how and why aspects of Galois Theory are relevant for secondary teachers (with a focus on upper secondary mathematics teaching) by unfolding the content connections to solving equations.

Next to *content connections*, Wasserman (2018a) introduces the *disciplinary practice connection* as a point of connection to secondary teaching. Here, the disciplinary practice that one engages in while studying advanced mathematics is also engaged in while studying secondary mathematics. That is, the processes that one engages in while "doing" advanced mathematics are related to some of the important mathematical practices that have been identified by mathematicians and mathematics educators as process standards or mathematical practice standards (such as problem solving, modeling, proving etc.). According to Wasserman, these kinds of connections serve a dual purpose: Mathematically, they intend prospective teachers to become better "doers" of mathematics so that they have a better grasp on mathematics itself (the epistemological nature of mathematics, mathematical norms and sensibilities, etc.). With a more pedagogical purpose, disciplinary practice connections intend that secondary teacher's pedagogical choices in their future teaching will engage their own students in these forms of thinking and doing. The already mentioned Abstract Algebra curriculum (Leuders, 2016b) for prospective teachers implements such disciplinary practice connections with an emphasis on inquiry-based learning. Moreover, the suggested units are meant to initiate processes of horizontal and vertical mathematization (Treffers, 1987; Gravemeijer, 1999; Leuders, 2015). When we see mathematizing as the organization of a kind of reality problem with mathematical means, Treffers (1987) differentiates horizontal from vertical mathematization. While "[t]he attempt to schematize the problem mathematically is indicated by the term 'horizontal' mathematization (...), activities that follow and that are related to the mathematical process, the solution of the problem, the generalization of the solution and the further formalization, can be described as 'vertical' mathematization (Treffers, 1987, p. 71). For vertical mathematization we draw on models, schemes, symbols and diagrams (ibid.). By implementing the principle of horizontal and vertical mathematization in Leuders' (2016b) curriculum, the students get the chance of perceiving modeled instruction connections. *Modeled instruction connections* capture connections to secondary teaching that are realized by intentionally modeling particular kinds of instruction in mathematics so that teachers learn mathematics in particular ways that can shape their own approaches to teaching (Wasserman, 2018a). The intended implication for secondary teachers is about their pedagogy: Aside from the mathematical content, the implicit intent behind modeling instruction in mathematics courses for secondary teachers is that particular kinds of pedagogical choices or pedagogical models would be incorporated into their teaching.

As a fourth type of connection, Wasserman (ibid.) defines the *classroom teaching connection* as some connection regarding the content of advanced mathematics, but as applicable to a specific secondary teaching situation. That is, the advanced mathematics is serving as a means to motivate particular and specific kinds of pedagogical actions in the classroom. The teaching situations from mathematics classrooms serve as a means to connect to advanced mathematics topics. The primary implication is about professionalizing teacher's pedagogical reactions to or in a specific teaching situation (e.g., designing problems for teaching, responding to students, sequencing activities, etc.). Substantial work by Cuoco and Rotman

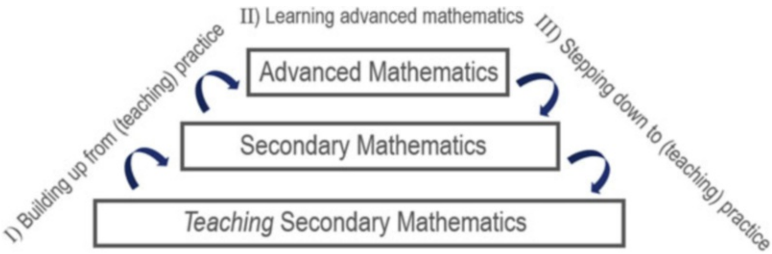


Fig. 17.1 Instruction model (Wasserman et al., 2019, p. 386)

(2013), Cuoco and McCallum (2018) in the thematic field of Abstract Algebra combines content and classroom teaching connections while Prediger (2013) puts a stronger emphasis on the classroom teaching connections for addressing the second discontinuity in upper secondary teacher education. A variety of teaching innovations structured according to the intended aims and kinds of connections are also summarized in Wasserman (2018b).

In sum, the four types of connections serve as categorial theory elements, and as such provide a language for distinguishing and relating aspects of learning contents and learning goals in profession-specific teaching settings. On the categorial and normative level, we find a growing body of theoretical contributions for guiding instructors' decisions who want to include specific learning goals addressing connections to secondary algebra teaching. When we come back to the different kinds of theory elements, the predictive theory element is of particular importance for backing the structure of learning trajectories. So far, only little is known about students' learning pathways along profession-specific learning trajectordes or insights into learning processes which *identify* or *explain* students' typical perspectives, learning pathways and obstacles.

Reacting to this empirical research gap, Wasserman et al. (2019) developed an instructional model according to which connections between the advanced mathematics learning content (conceptualized as knowledge from the nonlocal mathematics) can be connected to secondary mathematics and its teaching. The model covers the three phases of *building-up from (teaching) practice*, *learning advanced mathematics* and *stepping-down to (teaching) practice* (see Fig. 17.1).

For example, students are asked to evaluate the quality of a teachers' statement about the power rule statement in the first phase (*building-up*). In the next phase (*learning*), different proofs of the power rule statement and other derivative rules typical for real analysis are covered. Finally, the students are supposed to apply their newly acquired nonlocal knowledge to reevaluate the teachers' statement about the power rule statement (*stepping-down*) (Wasserman et al., 2019). Empirical research indicates that learning trajectories structured according to this sequencing principle have the effect that students' perceived relevance of the course for their prospective teaching increases (McGuffey et al., 2019).

17.2 Design Principles and Design Elements for Enhancing Profession-Specificity in an Abstract Algebra Class for Prospective Teachers

Since Abstract Algebra is a university mathematics field for which previous research provides a fruitful theoretical base, we started modifying our university mathematics teaching for prospective primary and secondary teachers in an Abstract Algebra class. We draw on insights from Didactical Design Research cycles, conducted by both authors as teacher educators and course designers starting in the academic year 2018/19 at the University of Education Freiburg and being continued since 2020 at Paderborn University. Table 17.2 gives an overview of the implemented design

Table 17.2 Overview of design experiment cycles

Design experiment cycle 1 (2017/18)	Design experiment cycle 2 (2018/19)	Design experiment cycle 3 (2020/21)
Aim and research setting		
Testing inquiry learning tasks and Cinderella/GeoGebra Applets for guided reinvention phase	Implementation of	Implementation of
Whole class setting (n = 23)	Reflection tasks on disciplinary practice and modeled instruction connections	Content connections and classroom teaching connections by realizing the Wasserman et al. (2019) sequencing principle
	Structural scaffolding “inquiry staircase”	Relating registers and representations for realizing content connections
	Whole class (n = 49) and laboratory setting (4 groups of 4 students: inquiry unit on permutations)	Whole class (n = 84) and laboratory setting (n = 8 in 4×2 pairs of students)
Collected data		
Weekly uploaded journals with students reporting on discovered phenomena, conjectures, challenges and open questions	Weekly uploaded journals with students reporting on discovered phenomena, conjectures, challenges and open questions	Weekly uploaded journals with students reporting on discovered phenomena, conjectures, challenges and open questions
Standardized evaluation questionnaire with open and closed items	Standardized evaluation questionnaire with open and closed items	Standardized evaluation questionnaire with open and closed items
	Video data: inquiry learning process on permutation problem	Video data: complete learning process of building up, learning and stepping down in learning trajectory on associativity/commutativity and inverses

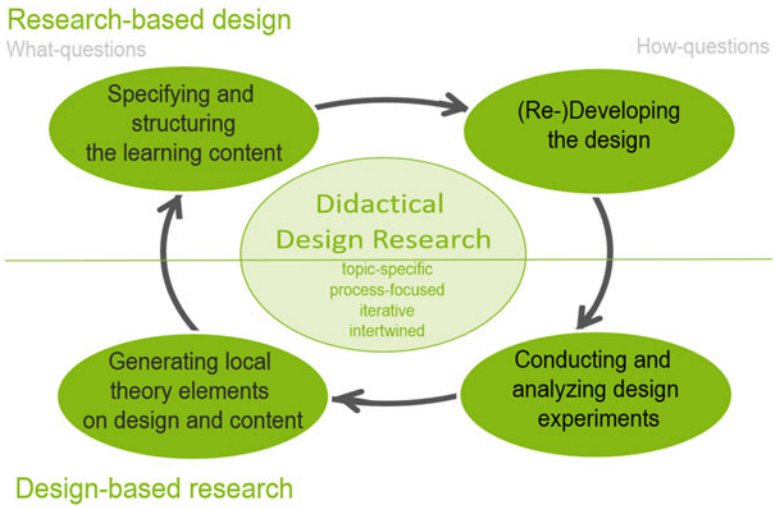


Fig. 17.2 Process model with working areas for topic-specific Didactical Design Research (Prediger, 2019)

decisions and collected data in each cycle. Methodologically we rely on the iterative Design Research cycle (see Fig. 17.2) which considers the what- and how-questions explicitly.

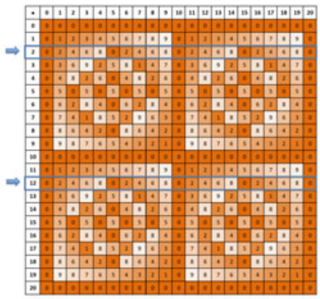
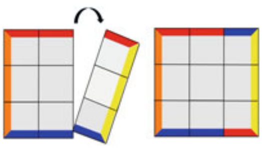
The following sections show how profession-specificity is interpreted and realized in the teaching design. Furthermore, we explain in what way design decisions for how-questions ought to be modified so that the intended effects of the teaching design can be achieved. In this way, we unfold theory elements for working on how-questions in tertiary didactical Design Research, particularly with a focus on predictive theory elements ("Which design characteristics and design elements can be applied for which intended aim?").

17.2.1 First Design Experiment Cycle

For implementing our teaching design, we analyzed the tasks of the (online available) TAAFU teaching material with respect to their potential of being combined with the inquiry learning tasks of the Abstract Algebra curriculum by Leuders (2016b). As a result, we orchestrated *four parts* of guided reinvention in different mathematical situations and contexts: the first three parts (they comprise a geometrical, arithmetical, and combinatorial approach, see Table 17.3) each cover 3–4 weeks of the course.

The first three parts also have in common that they all aim at developing conceptual understanding of the group concept for mathematization and interpretation for working on the given problem. Having developed a sustainable

Table 17.3 Overview of guided reinvention phases and concepts in focus of the course curriculum

Three parts of guided reinvention in the contexts of	Horizontal mathematization (group as 'model of')	Vertical mathematization (group as 'model for')
Geometric operations	Relations between movements (invariance, inverse operation)	Lifting strategies and thinking to support development of formal concepts:
Symmetry groups 	Structural properties: associativity, invariance	Work with axiomatic definition of a group
	Develop informal strategies (e.g., for working with Cayley tables)	Systematize informal strategies by agreeing on conventions and rules
	Subgroup and order	Formalize and proof theorems on general phenomena (e.g., Lagrange's theorem)
Arithmetic operations	Relations between actions (changing order of operation)	
Quotient groups 	Structural properties: commutativity, inverse operation	
	Cyclic groups	
Combinatorial operations	Relations between actions of rearrangements	
Permutation groups 	Structural properties: inverse operation, parity	
Fourth part:		
Guided reinvention of the concept of isomorphism and isomorphic groups 		

understanding of groups, the fourth part of the course covers 2–3 weeks in which students reinvent the concept of isomorphism and isomorphic groups including arguing for or against isomorphism in different ways. Along these four parts, further neighboring concepts (see Table 17.3), accompanying procedures and conventions, such as working with Cayley tables and the use of formal notations, are developed.

Each week of the course follows the structure of *(1) inquiry phase* in which the students work together with a partner on a given problem, note their observations and conjectures in their journals which are uploaded for the course instructor; *(2) phase of systematization* in a lecture which reacts adaptively to students' work, and extends their ideas; *(3) application phase* in a tutorial setting in which students apply or prove concepts and theorems.

In the first cycle of working with the teaching material, a focus was put on each week's inquiry phase in which Cinderella and GeoGebra Applets served as core design elements for initiating students' reinvention processes (Cinderella and GeoGebra Applets provided by Leuders, 2016b, freely available from the author). For reflecting and structuring the students' learning processes, the students have been asked to collect their thoughts in accompanying learning journals (Artigue & Blomhøj, 2013). The journals consist of reflections and notes from all three phases of inquiry, systematization, and application (conjectures, open questions, insights, critical moments in the sense of aha moments, mistakes or misunderstanding, conventions, thoughts of transfer, deeper or more tangible understandings). Journal entries on the inquiry phase have been uploaded weekly so that the course instructor could follow students' thinking and adaptively designed the lecture aimed at systematization of discovered phenomena. On the basis of these uploads, the functioning of the inquiry tasks has been evaluated, and tasks could be modified for the next Design Experiment cycle.

The extensive phase of working with problems of inquiry aimed at disciplinary practice connections: We intended prospective teachers to become better "doers" of mathematics and experience the creative mathematical practices of inquiry and problem solving (stating conjectures, generating examples, seeing and generalizing connections, etc.). We also hope that the prospective teacher's pedagogical choices in their future teaching will engage their own students in these forms of thinking and doing, so far, the methodological setting does not allow any insights into these potential effects. In line with Hußmann and Selter (2013), we conject that learning trajectories need not only these phases of experiencing inquiry-based learning, but also moments in which prospective students can try out inquiry-based learning in teaching experiments, e.g., with small groups of secondary students. Throughout the whole course, content connections to school algebra stayed rather implicit.

As the core insights of this cycle with respect to necessary adaptations of the teaching design, we observed that next to explicit statements in uploaded student work, also the course evaluation showed that the students encountered many difficulties in the phase of inquiry: Students have been insecure about what to write down in their journals; they often stated that they have been challenged by the openness of the inquiry-learning tasks and that they were not used to work on such open problems in their mathematics studies. As a positive effect, they highlighted their general motivation and that they often enjoyed doing the mathematics in this class. In consequence, we sticked to the general structure of the course (inquiry, systematization, and application), but identified the need for additional scaffolding the first phase of inquiry as a condition of successful mathematical learning processes.

17.2.2 *Second Design Experiment Cycle: Scaffolding Guided Reinvention and Noticing Connections*

As a reaction to the challenges that students faced when working on the open problems of inquiry, the course material was extended and included a didactical design element combined with making expectations more explicit to students. In their mathematics education courses, prospective teachers deal with the didactical principle of inquiry-based learning. As a means for speaking about the typical processes that mathematics teaching designers hope to initiate in inquiry learning, one can use the model of "inquiry staircase" (Rösike, 2022, drawing on Schelldorfer, 2007). In this model, the back and forth between capturing the problem, trying out particular calculations, comparing examples and finding patterns, making conjectures, or giving reasons is captured (see Fig. 17.3). The staircase model gives the students a language for speaking and writing about their mental processes of inquiry which we asked them to do in their journals. Using the staircase model and filling the steps with concrete examples from the first inquiry process (symmetries of

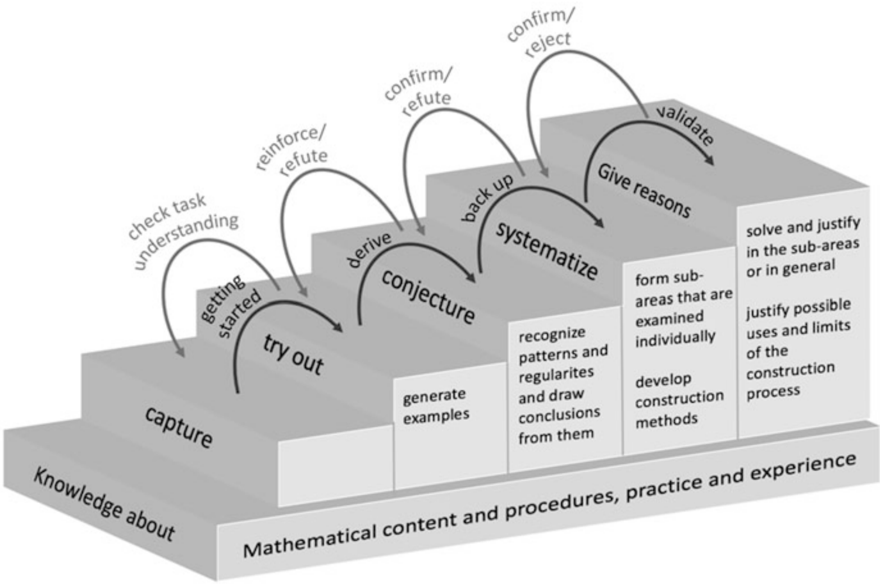


Fig. 17.3 Inquiry staircase as categorial structural scaffolding (translated from Rösike, 2022)

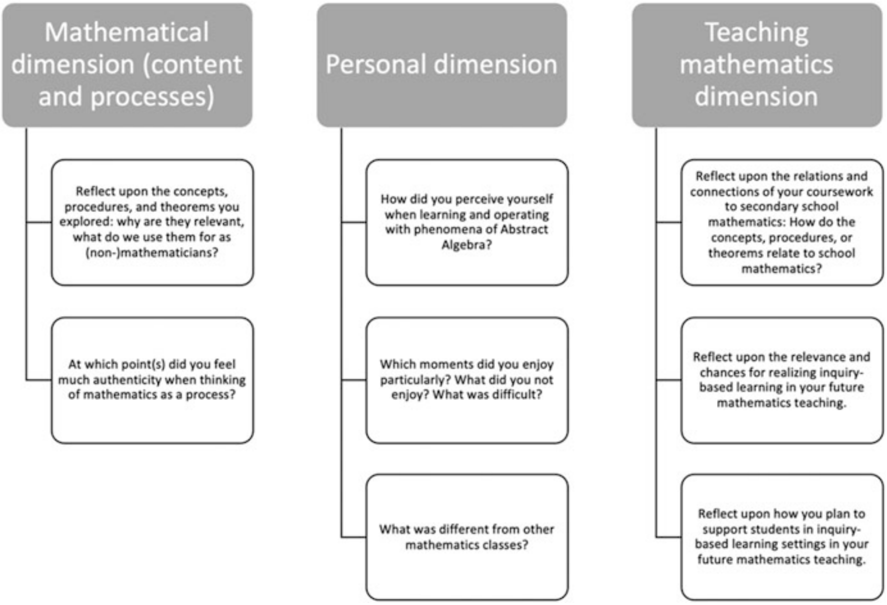


Fig. 17.4 Reflection tasks as design element

geometrical figures, reinvention of the identity and inverses) intended to make the students reflect on the typical phases of inquiry and reasoning (perception, trying out, conjecture, systematize, give reasons as well as the notion of moving back and forth between them) and to compare these to their own processes of inquiry. The staircase model was introduced in a short teaching-related article and from then on should serve as a self-reflection tool for one's own learning and as a connecting element in the sense of disciplinary practice connection (see Sect. 17.1.3).

As a second major need for revision, we identified means to make the disciplinary practice and modeled instruction connections more explicit, because our students not rarely perceived how the course relates to and obviously has connections to school mathematics. After having completed the first three processes of guided reinvention and deductive reasoning (this equals the first 3 weeks of the semester which focused on geometric problem contexts), the prospective teachers first encountered the *reflection tasks* (Fig. 17.4) we asked them to work on three times throughout the semester or after the semester. These reflection tasks and questions refer to a *content dimension* by asking to reflect upon the crucial concepts, procedures, or theorems they have explored so far and why they seem relevant; to a *personal dimension* by asking to reflect upon one's own learning practice (including emotions and challenges) and to a *classroom teaching mathematics dimension* by asking to reflect upon secondary school relations content- and process-wise. The prospective teachers chose which phase and context of guided reinvention (geometric, arithmetic, combinatorial) they explicitly relate to which reflection dimension (mathematical, personal, teaching mathematics).

Analyzing the prospective teachers' texts shows how they perceived and interpreted connections to school algebra, while at the same time the dual purpose described by Wasserman (2018a) (becoming better "doers" of mathematics, changing teacher's pedagogical orientations towards inquiry-based teaching approaches) as the excerpts written by Jonathan and Stephanie (both prospective secondary teachers in their bachelor program at University of Education, Freiburg) exemplify:

Jonathan: "For me, it was very helpful to work on the problems intensively by myself. I had the feeling being much closer to the mathematics. It was fun making conjecture, proving them, and reject or modify them. This way, I came up with new conjectures. For me it felt good, because consistently inquiring new problems brought new insights into mathematics every time."

Stephanie: "Especially for my future teaching, I experienced valuable moments of understanding: For students, it makes so much more sense inventing the concepts and discovering how they relate to another concept or how a procedure works. EVERY student is able to discover something."

17.3 Outlook on the Third Cycle and Discussion

In spite of the positive effects which we experienced with the connections we implemented in the Abstract Algebra curriculum for prospective teachers especially with a focus on disciplinary practice connections, the potential of content connections and the sequencing principle suggested by Wasserman et al. (2019) have not been exploited satisfactorily. This is why the third cycle of design experiments focuses on these aspects: All course materials have been analyzed for their potential of implementing the sequencing principle. As a result, the course units on arithmetic properties and inverses have been re-designed and now include material for *building up from teaching practice* and *stepping down to teaching practice* (next to the learning phase for which only small changes were needed). The three phases of the sequencing principle have been intertwined with the phases of inquiry, systematization, and application, which worked out more successfully for the pairs "systematization/learning" and "stepping down to teaching practice/application" in comparison to the pair "inquiry/building up from teaching practice". The laboratory design experiments have been transcribed so that the analysis can now focus on reconstructing the initiated learning processes. With this third cycle and the clear focus on leaning processes, we aim at the research gap of descriptive, explanatory, and predictive theory elements that wish to address profession-specific or "connected" learning processes (in the Wasserman sense of connections between secondary and tertiary algebra).

With this chapter, our main theoretical contribution to practice-oriented research in tertiary teaching designs refers to the notion of profession-specificity and how this might be operationalized by drawing on categorial theory elements from teacher professional knowledge frameworks. We suggest theory elements as fruitful categories for thinking of and designing integrated teaching designs that aim at learning goals from a more pedagogical (PCK) dimension in addition to pure content learning goals which can be combined by different kinds of connections.

In the Design Research paradigm of tertiary mathematics education, developing teaching designs for learning Abstract Algebra already has a long tradition so that we could draw on insightful suggestions for structuring learning contents (see Sects. 17.1.2 and 17.1.3). Given the specific situation of German mathematics teacher programs for lower secondary schools, we identified design elements as modifications so that the inquiry-based design functions with respect to the double aim of high-quality algebra learning while making connections to mathematical practice and classroom teaching. These design elements (inquiry staircase and reflection tasks) would probably also function in tertiary mathematics classes with other mathematical foci to be learned in inquiry-based settings.

The presented suggestion of a profession-specific Abstract Algebra teaching design incorporates profession-specificity-enhancing design elements. We see potential of those design elements in making advanced mathematics classes more meaningful for students with respect to the courses' relevance for future teaching. However, the questions of interrelations in the *learning processes* can only be answered as a next step with analyzing transcripts of the initiated learning.

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Chapter 18
Drivers and Strategies That Lead to Sustainable Change in the Teaching and Learning of Calculus Within a Networked Improvement Community

Wendy M. Smith, Matthew Voigt, Antonio Estevan Martinez, Chris Rasmussen, Rachel Funk, David C. Webb, and April Ström

Abstract Actively engaging students in learning mathematics is crucial to student success and equitable teaching and learning. Yet, this practice requires instructors to shift teaching strategies, which is not easily accomplished, particularly by themselves. In this chapter, we report on a longitudinal study of mathematics departments in the process of shifting department norms and practices in support of active learning and inclusive teaching. The research-informed change efforts have drawn on theories of institutional change and Networked Improvement Communities (Bryk

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AS, Gomez L, Grunow A, LeMahieu P. Learning to improve: how America's schools can get better at getting better. Harvard Education Publishing, 2015). Data include interviews with tertiary mathematics instructors, course coordinators, department chairs, deans, other campus administrators, and students, as well as document analyses. Analyses of the data from three institutions, through the lens of networked improvement communities, reveal that some of the most important drivers of change related to institutional change to enact active learning are shared tools and resources, professional development, policies and structures, and connections across a network to other mathematics departments engaged in similar efforts.

Keywords Mathematics department change · Driver diagrams · Leveraging institutional change · Active learning mathematics · Equitable tertiary mathematics outcomes · Networked improvement communities for institutional change

18.1 Introduction

Grounded in research on institutional change (Kezar, 2013), the Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL) project has studied 26 tertiary mathematics departments in the United States that have initiated and sustained departmental change initiatives designed to improve calculus programs. Our findings point to a systemic set of influential drivers that formal and informal departmental leaders—acting collaboratively as change agents—proposed, motivated, and enacted in undergraduate calculus. The focus of all the mathematics departments was on transforming high-enrollment introductory courses: Precalculus through Calculus.[1] In this chapter, we selected three cases that highlight how common drivers were used to articulate a vision for active learning, incentivize collaboration, enhance professional development, and disseminate instructional resources. All three departments demonstrated evidence of greater use of active learning and improved student success (Miller et al., 2020; Oliver & Olkin, 2020; Vandenbussche et al., 2020); these departments were also selected as cases due to differences in state-level educational policies and types of institutions (e.g., relative emphasis on research and teaching). These three cases provide examples of how similar drivers for change were applied using locally adapted strategies. As a note, *change drivers* are broad categories to guide attempts to change, whereas *change strategies* are particular to local contexts and comprise the planned activities that were enacted. As partners in the SEMINAL project, their activities and findings contributed to a growing Networked Improvement Community designed to share resources and lessons learned among the SEMINAL participants.

[1] In the U.S., Precalculus focuses on functions, modeling, trigonometry, and other requisite topics for Calculus 1 (differential calculus through the Fundamental Theorem of Calculus) and Calculus 2 (integral calculus which includes sequences and series).

Actively engaging students in learning mathematics is crucial to equitable teaching and learning (Theobald et al., 2020); we follow Laursen and Rasmussen's (2019) definition of inquiry-based mathematics education to encompass our approach to active learning and equity:

1. Students engage deeply with coherent and meaningful mathematical tasks.
2. Students collaboratively process mathematical ideas.
3. Instructors inquire into student thinking.
4. Instructors foster equity in their design and facilitation choices.

Actively engaging students often requires instructors to shift teaching strategies, which is not easily accomplished in isolation. Curricular materials designed with "group-worthy problems" (Boaler, 2006, p. 366) and instructors trained in orchestrating inclusive mathematical class discussions are important components of active learning practices that engage students. As there are increasing calls to improve equitable student outcomes and broaden participation in the mathematical sciences, it is important to approach the issue of improvement systemically. Thus, in line with Reinholz et al.'s (2020) call for more research on institutional change, this chapter focuses on research about improving teaching and learning in the calculus sequence, with the tertiary-level mathematics department as the unit of change. Although the focus of this chapter is around active learning and equity in mathematics, the principles of how a department might seek to change its culture of teaching and learning are broadly applicable across tertiary education. This research project's focus is on the department as the unit of change, and thus does not delve into the finer-grained details of classroom activities that engage students.

One of the reasons we engage in this work is the need for research on programmatic change. Curricular and instructional innovation is an important starting point for implementing active learning practices and increasing student engagement. However, to ensure the work of individual or small groups of instructors is sustained, such work needs to find its roots in the context of a mathematics department that, at the very least, recognizes and modestly supports active learning. Research into how mathematics departments change is limited; while there has been research on programmatic change in STEM departments more broadly, mathematics departments and courses differ from other STEM disciplines in ways that significantly impact the change process (Reinholz et al., 2020). For example, calculus is not taught in isolation of other undergraduate mathematics courses; more typically, it is one course in a sequence that begins with precalculus and culminates with single or multivariable integral calculus. To spread, support, and sustain active learning, programmatic change is necessary. The better we can understand how mathematics departments and particular drivers for change can be used to support active learning, the more likely we will be able to prepare and support the "missing millions" (p. 17) for STEM majors and career pathways (National Science Board, 2020).

The drivers that mathematics departments leverage have the potential to affect individual teaching practices as well as the culture of the department. One type of driver is to directly address a barrier. For example, if student evaluations are viewed as a barrier because they privilege a teacher-centered mode of teaching, then making changes to student evaluations could turn the barrier into a driver. Since student evaluations feed into policy decisions about promotion and tenure, revising these evaluations might encourage faculty to adopt active learning pedagogies (Apkarian et al., 2021; Dennin et al., 2017). However, directly addressing a barrier may not lead to changes in teaching practices. For example, instructors often cite time constraints, both in terms of the time needed to prepare lessons and the time needed to cover material, as a reason they do not make greater use of active learning pedagogies (Henderson & Dancy 2007; Brownell & Tanner, 2012; Shadle et al., 2017). Barriers such as class size, student preparation, and classroom infrastructure are not readily changed, and, even if they were, it is not clear that the removal of these barriers would result in significant changes to teaching practices.

Rather than removing a barrier directly, another approach is to focus on modifying the conditions and culture in which faculty work. Such drivers focus less on individual instructors and more on the department. An important insight from literature on institutional change (Henderson et al., 2011; Kezar, 2013; Laursen, 2019) is that widespread uptake of active learning pedagogies requires a shift in the unit of change from individual instructors to the department. Recent work has begun to identify key drivers for change—drivers that take a more holistic view of the change process. In an empirical study of 169 science, engineering, and mathematics faculty at the same four-year institution, Shadle et al. (2017) reported on 15 faculty-identified drivers, with four being the most important drivers for change: expand on current practice, encourage collaboration and shared objectives, improve teaching and assessment, and align with existing resources. In a review of important change drivers across different academic disciplines, Laursen (2019) identified professional development, communities of practice, resource collections and digital libraries, local data on student outcomes, and collaboration with other departments or disciplines.

The SEMINAL project's overarching research question is: What conditions, strategies, and actions at the departmental and classroom levels contribute to the initiation, implementation, and institutional sustainability of active learning in the undergraduate calculus sequence across varied institutions? In this chapter, we focus on the sub-question: What strategies do mathematics departments involved in a networked improvement community use to accomplish desired transformations in introductory tertiary mathematics courses across varied institutions? We address this question through a cross-case analysis of mathematics departments at three universities.

18.2 Theoretical Background

Initiating and sustaining departmental change is facilitated by attending to and articulating the context of the local change project, the current state, a desired state, and the hypothesized drivers that will enable progress toward the intended outcomes. The articulation of these elements is referred to as a *theory of change*

(Anderson, 2005; Reinholz & Andrews, 2020), and includes the local, context-specific set of strategies and assumptions that guide a specific change project. Such local theories are often captured pictorially in a *driver-strategy diagram* to show the drivers and strategies in relation to the intended aims or improvement targets. Figures 18.1, 18.2, and 18.3 show the driver-strategy diagrams for each of the three sites. Informed by a review of the change literature and our ongoing work with each site, we were able to identify the following four major drivers that were leveraged by all three sites: shared tools and resources, professional development, policy and structures, and networking. The specific strategies are the local, department-initiated change drivers that derive additional power when they are informed by a more global *change theory*. Change theories are overarching frameworks that provide a generalizable explanatory model.

We draw on Networked Improvement Communities (NICs) as the overarching change theory behind SEMINAL (Bryk et al., 2015; Martin et al., 2020). NICs are built on several principles, including collaboration across institutions (departments) and a focus on systemic improvement. Bryk et al. (2015) describe the key theory of NICs as including: focus on specific aims (improvement targets); deep understanding of the problem and system that led to the current problem; disciplined application of improvement science via cycles of transformation and collecting and using local data; and coordination to accelerate the development, testing, refinement, and scale up of improvement strategies in multiple contexts. Within the broader theory of NICs, a networked community can also serve as the driver for change; the SEMINAL project included an overarching community (NIC) and multiple interconnected local communities (NIC), all working toward the common goal of improving student engagement in mathematics courses.

Some of the departments involved with SEMINAL combined NICs with Communities of Practice (CoPs; Wenger, 2000). CoPs involve a community focused on practice in a particular domain—in this case, instructors seeking to improve teaching and learning in introductory mathematics courses. CoPs focus on the members of the community acting collectively and collaboratively, whereas NICs focus on cycles of improvement making incremental progress toward improvement targets. NICs also include the dimension of sharing across communities to accelerate change efforts; CoPs may or may not choose to share their lessons learned outside their local community. Combining CoPs and NICs brings more emphasis to developing a community of instructors within the broader NIC framework.

18.3 Methods

SEMINAL research is built around comparative case studies (Yin, 2012); for this chapter, we focus on three sites of incentivized case studies, with data collected 2018–2021. Mathematics departments submitted proposals to join the SEMINAL project in 2017–2018 and were selected through a competitive review process. The three sites were selected from a collection of nine longitudinal case studies and thus allow for robust conclusions based on the impact of participating in the SEMINAL NIC. More information about SEMINAL methods can be found in Smith et al. (2021); more information about the cases discussed in this chapter can be found in Oliver and Olkin (2020), Vandenbussche et al. (2020), and Miller et al. (2020). The three cases were chosen as representative of some of the diversity of the SEMINAL sites, overall. Each site had an articulated theory of change and actively participated in the SEMINAL NIC to make progress toward their own transformation aims. The SEMINAL project holds monthly NICcasts—interactive webinars—to engage local SEMINAL members in ongoing conversations and sharing information.

California State University East Bay (CSUEB) is a public Hispanic-serving institution with an enrollment of 16,000 students. CSUEB offers a master's degree in mathematics and lower-division mathematics courses are taught by part-time, career-line, and tenure-track faculty in small sections of 20–40 students. Kennesaw State University (KSU) is a large public university with over 40,000 students; 51% of the KSU student body is classified as white, with another quarter Black and 12% Hispanic, thus KSU shares many characteristics with minority-serving institutions. KSU is a doctoral degree granting institution with high research expectations. Lower-division mathematics courses are taught in small to medium sections (approximately 35–60 students) by some tenured/tenure-track faculty, and many full- or part-time contract instructors. The Ohio State University (OSU) is a large, public university with over 60,000 students. The student body is predominantly white (66%). OSU is a research-intensive university and its mathematics department is ranked third in terms of National Science Foundation funding for Mathematical Sciences. Lower-division mathematics courses at OSU are taught primarily in large sections of 300–500 students by tenured/tenure-track faculty, with recitations (i.e., subsections of the course in which students review material and work on problems) of 25–30 students taught by graduate student instructors.

This chapter focuses on data collected from Fall 2018 to Fall 2020, and includes:

- interview and observation data from at least two site visits per institution, including researcher notes from the site visits and periodic progress meetings with each site;
- artifacts such as reports of those site visits, which underwent member-checking and revisions to summarize the findings from each site visit; and
- secondary data sources such as locally generated reports and manuscripts by members of the targeted institutions, including their proposals to join SEMINAL, annual progress reports submitted to the SEMINAL team, documentation of local efforts, and manuscripts written by local team members (Miller et al., 2020; Oliver & Olkin, 2020; Vandenbussche et al., 2020).

Although the SEMINAL project also collected additional data from students and instructors, those data are not included in the analysis for this chapter.

To create the transformation stories for each department, we conducted an extensive document analysis, starting with the institutions' proposals to join the SEMINAL project that outlined their goals, intended change drivers, and strategies. We analyzed the reports the SEMINAL research team wrote summarizing each site visit, the institutions' annual reports, and published research of their progress and findings (Miller et al., 2020; Oliver & Olkin, 2020; Vandenbussche et al., 2020). Our analysis focused initially on the major change drivers identified by the SEMINAL research team, based on the SEMINAL change theory, as informed by Shadle et al.'s (2017) change drivers and barriers. After analyzing each case separately, we then looked for commonalities across cases and distilled the major change drivers to those related to shared tools and resources, professional development, policies and structures, and networking. As part of our axial coding process, we depicted the drivers and strategies visually via driver-strategy diagrams for each of the cases. The specific strategies were determined by each mathematics department and were informed by: norms, needs and resources for each department, information shared within the SEMINAL NIC, and decisions made by project leaders, These themes are supported by our theory of change and are instantiated differently based on local contexts. Throughout this chapter, quotations come from the various documents analyzed, including direct quotations from participant interviews, and are representative of the findings overall.

18.4 Findings and Results

We present a summary of the changes (including theories of change) for each university mathematics department in this section. For each case, the findings are clustered around the major drivers for change: shared tools and resources, professional development, policies and structures, and networking.

18.4.1 California State University East Bay (CSUEB)

The CSUEB mathematics department entered into the SEMINAL project with a targeted focus of improving its calculus sequence, particularly to help support its underrepresented and racially minoritized (URM) students. A major driving factor for this change was a noticeable achievement gap between URM students and their non-underrepresented peers with average course grade (on a 0–4 scale) gaps of 0.46, 0.33, and 0.28 in Precalculus, Calculus 1 and Calculus 2, respectively (2017–2018 academic year). Drawing on the four major drivers as described in the previous section, we present the specific strategies that CSUEB enacted in order to successfully implement, sustain, and ultimately teach others about active learning strategies in introductory mathematics classrooms. Figure 18.1 illustrates connections between drivers and specific strategies.

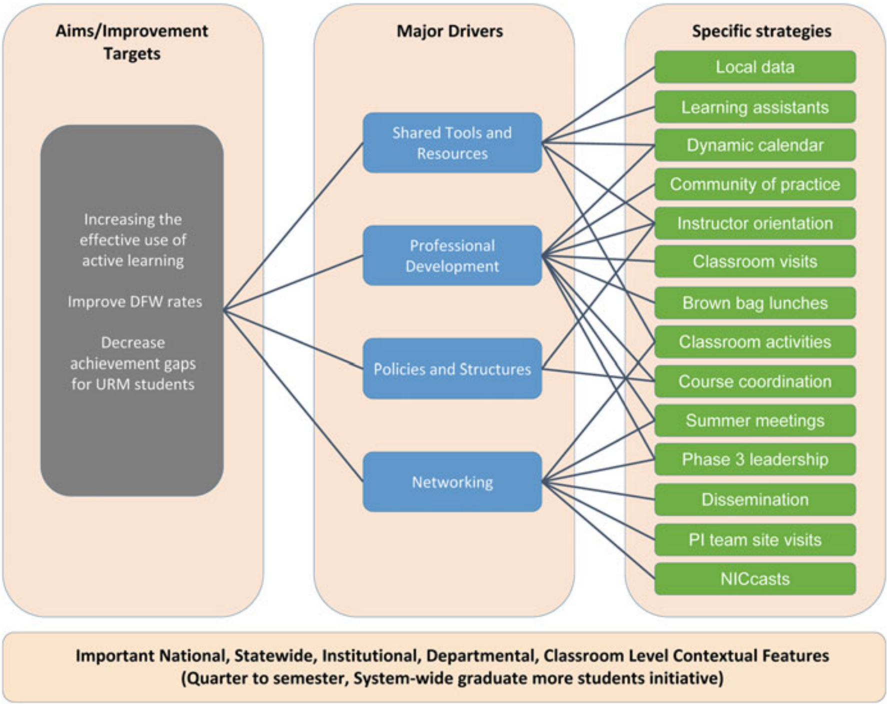

Fig. 18.1 CSUEB's driver-strategy diagram
Note: "DFW" stands for grades of D, F, or withdraw, so improved DFW rates means fewer people are failing/needing to repeat a course

18.4.1.1 Shared Tools and Resources

Five specific strategies were central to CSUEB's efforts. At the department level, one important strategy was the use of local data related to passing rates and persistence in the calculus sequence. Results from this data prompted action to change the delivery of the calculus curriculum. One of these changes was the implementation of a dynamic calendar, an online resource shared with all instructors of a particular course that contains the pacing information for the delivery of the material; this resource also contains myriad active learning activities to be used in class (Oliver & Olkin, 2020). New instructors attend an orientation in which they discuss a suggested weekly class schedule, active learning activities, and historical grade distributions for the course. This orientation is led by the coordinator for the course, who oversees the logistical management of the multi-section course and is responsible for guiding and supporting instructors in their professional growth.

18.4.1.2 Professional Development

Effective coordination not only consists of proper logistical management but also includes an explicit focus on community building and professional development (Rasmussen & Ellis, 2015). CSUEB coordinators are responsible for running the pre-semester new instructor orientation and supporting the instructors by establishing a community of practice (Wenger, 2000), which involves all of the instructors for one particular course. The value of the CoP can be seen in one representative quotation from an instructor reflecting on their experience being a part of the CoP and teaching five courses in one semester:

> Because of this community, it had taken that stress off of my shoulders. So, personally I really appreciated feeling like I had some guidance and somewhere to go to. Not only people to talk to, but just an information source to rely on.

At these CoP meetings, instructors talk about their teaching, what is working well, and what is not working well. They have the opportunity to ask for help related to classroom management, and they also schedule classroom visits. CSUEB's mathematics department has been able to establish a culture where classroom visits by colleagues are highly encouraged and widely accepted as a normative practice. Other opportunities for instructors to discuss effective active learning activities or strategies exist outside of the CoP meetings, such as monthly lunches.

18.4.1.3 Policies and Structures

New-instructor orientation and course coordination are two permanent structures that are crucial to building an environment where teaching is valued, and instructors are supported in their professional growth. These structures foster a culture within the department where faculty view them as normal operating procedure rather than an imposed policy that restricts instructor growth and autonomy.

At the base of the driver diagram are two important institutional factors: the California State University graduation initiative to graduate more students by 2025[2] and the change in 2018 from 10-week academic terms to 15-week terms. The 2025 initiative was a statewide focus for the entire CSU system, but was an important motivator for the changes that needed to be done to support students at CSUEB, particularly URM students. The move to the 15-week term was an opportunity to examine the curriculum and led to the implementation of course coordination and other important changes, described previously, to the CSUEB calculus sequence.

[2] https://www2.calstate.edu/csu-system/why-the-csu-matters/graduation-initiative-2025

18.4.1.4 Networking

Networking significantly contributed to making and sustaining meaningful change related to using active learning material throughout the calculus sequence. CSUEB leveraged six strategies related to networking—which ultimately led to the emergence of CSUEB as a leader within the SEMINAL project—namely: utilizing the support and feedback from the leadership team during the initial site visits in 2018 to make changes to the system; disseminating in-class activities and attending webcasts so the SEMINAL NIC could learn from CSUEB's organizational structure, classroom activities, general approach to CoPs and coordination; and participating in summer meetings with other SEMINAL participants. As of the 2018–2019 academic year, the achievement gaps between the URM students and their non-underrepresented peers had entirely vanished (Oliver & Olkin, 2020).

18.4.2 Kennesaw State University (KSU)

KSU joined the SEMINAL project with aims of increasing student success rates throughout the Precalculus through Calculus sequence and eliminating disparities in success rates of students from different demographic groups and different sections of the same course. Between Fall 2014 and Spring 2016, success rates for Precalculus, Calculus 1, and Calculus 2 were approximately 66%. Further, success rates for URM students were generally lower (5–10%) than their peers. In this section we describe the specific strategies KSU used to address its goals (see Fig. 18.2).

18.4.2.1 Shared Tools and Resources

One major incentive for joining the SEMINAL project was that KSU's team would be able to leverage the support of the SEMINAL network and funding to develop a more robust coordination system. Starting from the current common textbook and recommended syllabus, the KSU team developed an online repository of common learning objectives and topics, sample syllabi, literature about the benefits of active learning, active learning tasks, and clicker questions for Precalculus through Calculus. Some of these active learning tasks were taken from the Boulder/Omaha Active Learning Alliance group (Hodge et al., 2020a, b), while others were developed by instructors. New coordinators of the courses have taken over refining and developing new materials.

The KSU team views common assessments as an important component of a coordination system that supports student learning. While progress in this area has been slow due to logistical complications, the pandemic, and some faculty pushback, the department did pilot a common assessment in College Algebra, with an eye toward expanding this to all coordinated courses.

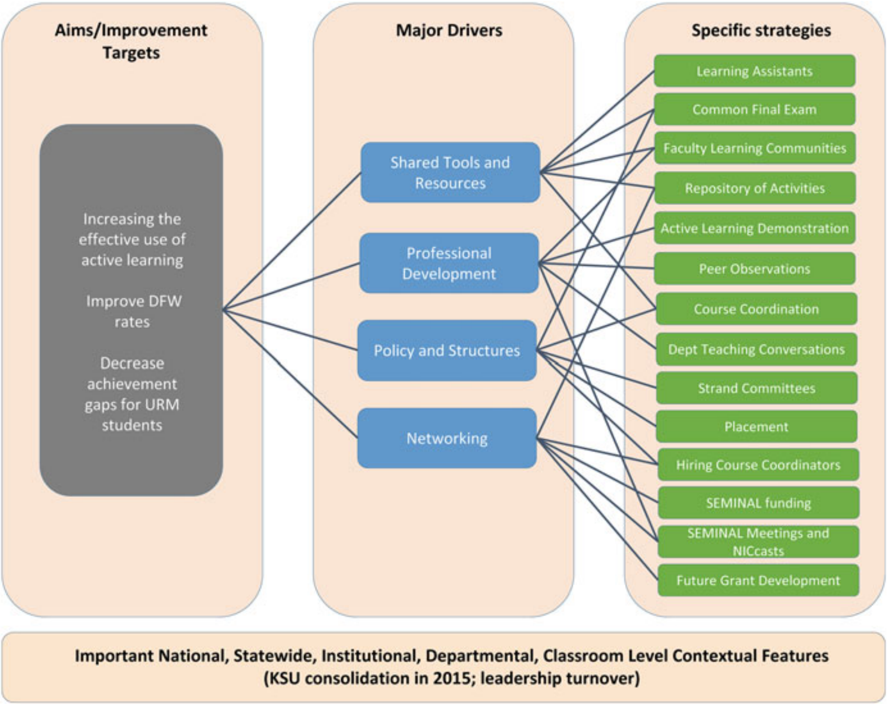

Fig. 18.2 KSU's driver-strategy diagram

18.4.2.2 Professional Development

Much of the success of the change efforts at KSU can be attributed to leveraging existing departmental and college structures. Pre-dating SEMINAL, KSU devoted significant resources to structures called *faculty learning communities* that act as a form of long-term professional development for faculty. These communities involve groups of five to seven faculty members (often from different disciplines) who meet throughout a semester or year to work on improving some aspect of teaching. These structures have been used to develop active learning instructional materials for faculty in Precalculus and Calculus and were explicitly leveraged by the local team to encourage more faculty to participate. In Fall 2018 the department also started monthly seminar-style "teaching conversations" for instructors to discuss teaching and normalize the practice of having conversations about improving teaching. In addition to college and department-wide professional development, the KSU team facilitated one-time professional development opportunities so that instructors can envision how an active learning classroom might look. They also worked to conduct peer observations in Precalculus through Calculus 2 classrooms, in cooperation with the university Center for Excellence in Teaching and Learning, again leveraging an institutional structure.

18.4.2.3 Policies and Structures

Three structures have had a significant impact in change efforts at KSU: placement policies, strand committees, and hiring practices. Before joining SEMINAL, the department worked extensively to reform mathematics placement procedures for incoming students. Although this strategy predated SEMINAL, it directly supports their project goals. As one KSU team leader stated, "We just started a placement procedure which is going to have a huge effect on success rates, so it's going to be impossible to untangle that [from the impacts of SEMINAL]." This strategy emphasizes factors we see as key to the successful efforts at KSU to improve Precalculus through Calculus 2 courses: change efforts were successful, in part, because people at multiple levels (e.g., department, college, university) prioritized improving these courses, and local leaders effectively tied their goals to overall university goals.

At the start of joining SEMINAL, the KSU mathematics department was still in the early stages of developing a new culture. Following the university's consolidation and move toward higher research intensity, faculty members needed to reposition themselves within the new combined university system and department, leading to some uncertainty among faculty members about the role of teaching in this new environment. Local change agents recognized the importance of attending to this context when building a vision for their course improvement efforts, being particularly cognizant of the need to give instructors a sense of ownership in the change process. A department structure established in 2016 known as *strand committees* has been used to guide and garner support for change efforts. Strand committees are run on a volunteer basis (any department member is welcome to join). These committees oversee particular groups of related courses (e.g., Calculus 1 and 2 make up one strand) and make recommendations for course policy changes to the department.

One of the most significant successes from the work done at KSU is the establishment of semi-permanent course coordinators for College Algebra, Precalculus, and Calculus 1 and 2. A strong commitment to active learning and to supporting others' use of active learning were explicit criteria for these positions. In addition, hiring practices for instructor and faculty positions have been evolving over time to include conversations about active learning, expressing the department's desire to hire instructors who are committed to using active learning.

18.4.2.4 Networking

SEMINAL provided funds for faculty to participate in faculty learning communities and the development of instructional materials. Beyond funding support, KSU leaders expressed that they value the feedback and ideas generated when meeting with other institutions from the SEMINAL network. For example, they reached out to CSUEB for suggestions about coordination after seeing CSUEB's dynamic calendar. Another leveraged membership in the Mathematics Teacher Education Partnerships' Active Learning Mathematics group (Smith et al., 2020) to access co-created active learning materials for Precalculus and Calculus. In addition to the SEMINAL funds, KSU leaders believe that participating in the SEMINAL project lent credence to their efforts, generating an excitement about change efforts that directly impacted the college's decision to create these positions. One leader commented that:

> I think all of the coordination was born out of the SEMINAL effort. It's possible that the department would have come around to it another way. But the coordination that we have done over the past few years on the grant, and then I think the hiring of the new coordinators moving forward, is all a direct effect of the grant.

The KSU team leaders also learned about an equitable practices observation protocol (EQUIP; Reinholz & Shah, 2018) through the SEMINAL network and brought in a facilitator to learn more about how to implement it. Team leaders also used the network to define roles and responsibilities they wanted for newly hired course coordinators:

> In the summer, there was a discussion about data and collecting it and using it. We ended up using that to help us to shape what sort of responsibilities we might want our new coordinators to have...so that was really helpful where I think we're having conversations now just about what's feasible and how to make it so that the data is something that we can get on a regular basis.

Finally, being a part of SEMINAL has opened the door to other grant opportunities for KSU. KSU has partnered with other SEMINAL universities to submit additional grant proposals to study professional development and equitable teaching practices as well as CoPs.

18.4.3 The Ohio State University (OSU)

A team of lecturers, faculty, and administrators at OSU engaged in a decade of experimentation with various instructional approaches in Precalculus through Calculus sections designated for instructional innovation. Supported by internal campus grants, they investigated how technology could be used to: "flip" courses using online videos of lectures that students could watch before class; support formative assessment in large lectures of up to 500 students using polling software; and create interactive worksheets using Ximera (The Ohio State University, 2021) to support individualized feedback to students. Its SEMINAL proposal synthesized lessons learned from this experimentation and focused on expanding the orientation for new instructors and incentivizing and scaling up instructor use of formative assessment tools. The sections that follow describe how OSU used locally viable strategies to address the four major drivers (see Fig. 18.3).

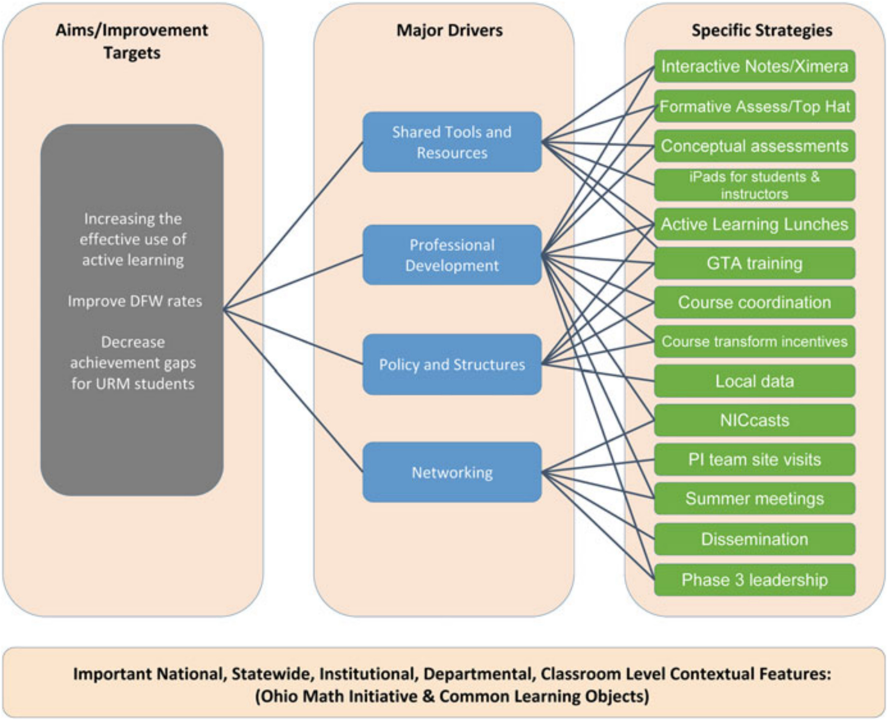

Fig. 18.3 OSU's driver-strategy diagram
Notes: "GTA" refers to graduate teaching assistants. Phase 3 leadership refers to the SEMINAL strategy of positioning members from Phase 2 (including faculty at The Ohio State) in leadership roles with the next phase of the SEMINAL research

18.4.3.1 Shared Tools and Resources

Ximera is a program that can convert LaTeX documents into interactive PDF files, allowing students to digitally complete lecture notes and mathematics tasks, and add in their own responses to prompts as they observe and participate in the lecture. OSU used Ximera to design an interactive textbook and related instructional resources that could be distributed to all calculus sections. Student homework activities were also adapted with Ximera so that instructors could receive real-time information regarding problematic tasks and common errors. This resource strategy allowed for rapid distribution of materials and informed adaptation of lessons, tasks, and activities. Piloting and revision of the Ximera Calculus I textbook to use as a department-wide text was a major undertaking when OSU initiated its work with SEMINAL.

Incentivizing instructor use of Top Hat (live polling software) through the distribution of iPad Pros was another resource strategy that the OSU team used to secure buy-in from reluctant instructors of large lectures. OSU provides all incoming students with iPads. To leverage this institutional policy, the team used Top Hat to design polls to promote discussion in large lectures. They further assumed iPad Pros could be used to incentivize instructor use of Top Hat and found great success with this approach, with the demand being greater than the supply. A co-leader noted that this approach should be expanded even further, "I would give all incoming graduate students iPads, just like we do the undergrads, so that they can use them to teach and to learn."

18.4.3.2 Professional Development

Commitment to collaboration during the experimentation phase demonstrated tangible benefits for professional learning and community, which led the OSU team to expand its Math Education Forum to an "Active Learning Lunch," whose acronym of ALL denotes inclusivity.

Coordination was also used as a strategy to provide ongoing professional learning and build community among course lecturers and graduate student instructors who lead recitations. Coordinators serve in their roles for many years, which allows them to develop institutional knowledge and learn how to negotiate policies and structures. Their professional and institutional knowledge was recently leveraged as they took on significant roles as instructional designers, to abstract the best ideas and lessons learned from prior experimentation to realign Ximera textbooks, homework assignments, Top Hat questions, and conceptual assessments.

> Having a text written from the perspective of the course coordinators had an unexpected benefit for the large, coordinated classes.... [T]he coordinator [laid] out, in their own voice, their vision for the material. This led to a level of course alignment that was not possible previously (Miller et al., 2020, p. 10).

Miller et al. (2020) also noted that this instructional design work had an "impact in Calculus 2 where success rates (percent of students who earned an A, B or C) increased 10-15 percentage points" (p. 15).

18.4.3.3 Policies and Structures

Structures for professional development also include four weeks of graduate student instructor training in the summer and a one-day instructor orientation. Both of these activities have been sustained for many years and serve important roles in communicating the goals and expectations of the department and course coordinators, as well as building and sustaining a sense of community.

The role of the course coordinator is an established departmental policy in which some degree of authority is given to the coordinator to design and select the content and resources used in lectures and recitations. The coordinators' provision of syllabi, schedules and resources to instructors incentivized their use of related resources (e.g., interactive textbook, polls). The only remaining variable to advance active learning is pedagogy. However, in interviews we found that pedagogical approaches can be encouraged but not dictated. Pedagogy was left to the discretion of instructors since instructional leaders argued that actively learning is more effective when it is aligned with related attitudes and beliefs (Miller et al., 2020). A related influential policy that motivated statewide discussion of course goals was the Ohio Math Initiative, which aligned content goals and transfer policies between tertiary institutions.

18.4.3.4 Networking

The OSU SEMINAL team has been actively involved in and contributes to the SEMINAL network. Some have adopted leadership roles in ongoing SEMINAL research efforts; some facilitated NICcasts; and all have participated in the SEMINAL NICcasts and summer workshops. The OSU team articulated that one of the reasons it submitted a SEMINAL proposal was to join a professional network that it recognized as beneficial to achieving its goals. When asked to recently reflect on the benefits, one of the team members noted, “There’s a lot of emotional support and creative support [so] that you’re not doing this alone. I think that makes a big difference for me.”

Other team members noted how their involvement in summer meetings and NICcasts informed the redesign and expansion of the summer instructor orientation to support active learning in lectures and recitations and incorporate more equitable and inclusive practices.

18.5 Reflections and Synthesis

18.5.1 Shared Tools and Resources

Across the three departments, course coordination was a common strategy for sharing tools and resources. Each department focused its attention on coordinating course-level materials by leveraging technology to facilitate the sharing of resources. CSUEB created a dynamic calendar that not only provided active learning activities but also served as a resource for pacing instruction and facilitated the onboarding of new instructors. KSU’s strategy of coordination included an online repository of both course resources (e.g., sample syllabi, research literature on active learning), but also curricular resources (e.g., active learning tasks, clicker questions, common final exam) to which instructors could use and contribute as a way of enhancing the repository. Finally, OSU leveraged technology through the use of Ximera to house interactive instructional materials for students to access through their institution-issued iPads. The Ximera platform organized guided notes so that students could access these notes on their iPad and fill in blanks while following along during class, and it also served as the platform for homework, which provided instructors with real-time feedback on student performance. Although each department’s strategy for sharing tools and resources for increased course coordination differed, they all had two main similarities: use of technology and a shared goal of providing resources for instructors in support of their course as well as materials for student utilization.

Additionally, departments found ways to use other tools to support their efforts. CSUEB used local data of DFW rates and persistence as a critical driver for change to provide both momentum for change and affirmation of their efforts. Its building of a CoP is also a central strategy. At KSU, faculty who engaged in a learning community could request an undergraduate learning assistant to support in-class implementation of active learning. OSU initiated Active Learning Lunches and provided their faculty with an iPad to help incentivize their use of Top Hat to support instruction and formative assessments in large lecture courses. The strategies implemented by these institutions demonstrate creative ways to share tools and resources with instructors, which is a necessary and vital approach for increasing course coordination in support of active learning.

18.5.2 Professional Development

In comparing the three departments' approaches to professional development, it is clear that they all leveraged the idea of building community among all who were teaching Precalculus to Calculus courses, including lecturers and graduate student instructors. Activities such as summer meetings, informal lunches, and seminar-style teaching conversations, provided the means for instructors to engage in conversations about implementing active learning and the challenges and opportunities nested within this targeted change. Thinking about professional development as a means for coordination provides synergy within the community and allows for instructional growth. At CSUEB, instructors of the CoP engaged in peer observations of one another which allowed for a culture of open-door teaching to flourish and become normative. KSU and OSU also incorporated peer observations: KSU leveraged its teaching and learning center to implement observations, and OSU used peer-observation to support instructor use of iPads and polling software.

Through each institution's local professional development activities, opportunities arose for improving the active learning activities shared among the instructors, as well as through SEMINAL NICcasts and research activities where they could showcase active learning instruction and learn from others on how they were implementing active learning.

18.5.3 Policies and Structures

The policies and structures in mathematics departments need to be negotiated and adapted to endorse and sustain course transformation. A common strategy in each case was using course coordination to reduce inequities in student opportunities for learning and increase the likelihood of successful completion of the calculus sequence. Coordination also actuates instructor engagement in ways that benefit the department: collaboration, communication, course planning meetings, professional development, and distribution of resources. Coordination also has the benefit of reducing the collective workload and improving the alignment among goals, curriculum, and assessment. Course coordinators are imbued with discretionary authority, and the manner in which they choose to make decisions (e.g., collaborative, authoritative) allows them to develop leadership skills that can stabilize and sustain how courses are taught and assessed.

External policies were influential in all three cases and could be summarized as state-level policies that required either curriculum alignment (CSUEB and OSU) or structural changes (KSU). As public universities, all three institutions are subject to various degrees of state administrative and fiscal authority, and they are expected to abide by legislation mandates and initiatives. Private institutions may have different policy constraints.

18.5.4 Networking

Literally and figuratively, an institutional network is at the heart of SEMINAL. From conceptualization through enactment over five years, SEMINAL has engaged leaders in department and university administration by partnering with the Association of Public and Land-grant Universities and mathematics department chairs. SEMINAL also included numerous researchers and educators with expertise and knowledge in local and systemic reform efforts in mathematics education. SEMINAL was designed to operate as a network that collaborated in problem solving, shared local findings and resources, and tested conjectures across varied contexts working to infuse active learning in undergraduate calculus.

The interviews and reports from these cases all point to how networking was valued and achieved through:

- in-person site visits, to bring greater visibility of the project to other faculty and administrators;
- summer meetings to build community and share resources and tools; and
- monthly NICcasts to address contemporary challenges and issues articulated by the network members.

Within the SEMINAL network, partnerships and teams emerged to support grant writing, think through the implementation of innovative ideas, and advance more equitable and inclusive practices. Participation in a national network offered local teams political cover to make progress with proposed innovations; regular meetings and progress reports motivated teams to share tools, resources, and lessons learned.

18.6 Implications and Limitations

Focusing on the department as the unit of change has been crucial to the success of SEMINAL's efforts to impact positive cultural changes to infuse active learning strategies in Precalculus and Calculus courses. The focus on mathematics provided data for this research analysis, but similar NIC-based change efforts could be transferred to other tertiary departments with high-enrollment courses. At the same time, the specific change drivers and strategies appropriate for a given context will vary depending on the discipline and existing climate, culture, and structures. For example, unlike in other STEM disciplines, lower-division mathematics courses (e.g., Precalculus, Calculus) are typically taught by assisting teaching staff (e.g., graduate students, postdoctoral researchers, adjunct faculty). Thus, although different STEM departments might share a driver of instructor professional development, the enactment might look quite different across mathematics courses taught by graduate students, science labs led by graduate students, or computer science courses led by tenured faculty. It is worth noting that each mathematics department, to participate in SEMINAL, developed a proposal in which they completed a self-assessment of the local context, and articulated a change strategy that identified resources, structures and practices that were needed to expand the use of active learning in Precalculus through Calculus 2 courses.

Changing instructional practices is enacted individually by each instructor in each classroom, and support for such practices and efforts to institutionalize educational innovations require the ongoing support of a departmental unit. Although change efforts do not need all members to actively support the educational innovations, successful transformations require a critical mass of actively involved individuals who have sufficient authority to seek resources and make changes—and who can collectively provide institutional memory for sustaining changes. The active involvement of both formal and informal leaders has emerged as one of the most important change drivers for departments seeking to improve the teaching and learning of tertiary mathematics.

The establishment of such a critical mass of change agents does not happen by chance, but requires a specific focus on the vision, goals, and establishment of a community of instructors enacting change. The development of a CoP was fostered through a shared vision of active learning and the structure of course coordination to bring together the necessary people and resources. We see great promise in the use of driver-strategy diagrams both for researchers to understand institutional change and for change agents seeking to build a common vision and goals for institutional change.

Research on institutional change is a fairly recent and emerging field in university mathematics education; although, such research is long-established in the business world. The SEMINAL project was positioned to contribute to the field to understand how to support mathematics departments in enacting and sustaining cultural changes to infuse active student engagement in Precalculus and Calculus courses. Although research on active learning is promising for improved equitable student outcomes

(Freeman et al., 2014; Stains et al., 2018; Theobald et al., 2020), having students work in groups is not automatically more equitable and can actually extend or exacerbate implicit biases (Johnson et al., 2020). The change efforts at these three sites are in a United States context, focused specifically on racial inequity in achievement for URM students; however, equitable learning is a global phenomenon and change efforts must attend to the identities and communities of students being served (or underserved) in local contexts (e.g., women, indigenous students, ethnic or religious identity).

Changing departmental cultures does not occur in a vacuum: such transformation efforts need to include a wide array of people, from students and instructors to department and university administrators. Although every mathematics department has unique dimensions to its local context, working collaboratively in a NIC is a productive way to accelerate local transformation efforts. As such, the credibility of the larger NIC can help provide more credence and enthusiasm for the localized change strategies. The SEMINAL NIC model is structured so that departments can learn from one another and benefit from learning about how other departments overcame barriers and leveraged effective change strategies.

All research is necessarily bounded; one decision SEMINAL made was to focus on institutions with graduate programs in mathematics, in order to include a focus on graduate student instructors. Future research (some of which is ongoing through related projects) should investigate similar questions in a wider variety of institutional contexts, including community colleges (Ström et al., 2020). Future research and transformational efforts are also needed to scale up the work of SEMINAL to an even broader community. Finally, and perhaps most importantly, more research is needed to understand the necessary conditions for improving equitable student outcomes.

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Part IV
Research on University Students' Mathematical Inquiry

Chapter 19
Real or Fake Inquiries? Study and Research Paths in Statistics and Engineering Education

Marianna Bosch, Ignasi Florensa, Kristina Markulin, and Noemí Ruiz-Munzon

Abstract Study and research paths (SRPs) are a teaching format proposed by the Anthropological Theory of the Didactic based on the inquiry about an open question. The question that is initially proposed is a critical component of an SRP. It can sometimes lead to "fake inquiries" when the instructional purpose is not to answer the question but to meet some specific curricular content during the inquiry process. This paper presents different choices of generating questions for SRPs implemented in various university degrees. We focus on how these choices condition the development of the SRP, especially in what concerns the situation in which the generating question appears and its answer is received.

Keywords Inquiry-based learning · Anthropological theory of the didactic · University mathematics education · Statistics · Elasticity · Study and research paths

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19.1 Introduction

Inquiry-based mathematics education (IBME) has become a central goal in education in many countries during the last two decades, and higher education is not an exception. Governments and international organisations such as the OECD have promoted its expansion. Adopting competency-based curricula within the European Higher Education Area was considered a significant opportunity to foster the transition (Van der Wende, 2000). Artigue and Blomhøj (2013) analyse how this transition process has been framed and modelled by six research frameworks that share some general principles but also produce a diversity of methodologies and research tools. This paper focuses on those provided by the anthropological theory of the didactic (ATD) in the case of university education and illustrates some connections found with the theory of didactical situations (TDS).

The ATD (Chevallard, 2019) opposes the pedagogical *paradigm of visiting works* that proposes a curriculum based on works to study, to the pedagogical *paradigm of questioning the world* where the curriculum should ideally consist of questions to investigate. In this new paradigm, the works to be studied "are no longer given by an omniscient instance but are determined by the very logic of the inquiry" (op. cit., p. 100). Therefore, the new paradigm includes the visit of works, even if it gives it a new function: works are not an end in itself; they are valued by their capacity to elaborate answers to the questions approached.

The ATD proposes *study and research paths* (SRPs) as a model to describe inquiry processes and design inquiry-based instructional formats (Chevallard, 2015). An SRP is initiated by a generating question addressed by a community of study constituted by a group of students and some guides of the study – the teachers. The selection of this initial question is part of the curriculum problem: the social project that defines what is to be studied in a given school institution.

At the moment, we can consider that university institutions are still in the transition between the old and the new paradigm, proposing competence-based curricula but still mainly formulating them in terms of works of knowledge to study – or to visit. The implementation of SRPs under these conditions requires compromise solutions. Our proposal relies on the identification of didactic phenomena (Chevallard, 1985) that are linked not only to the prevailing curriculum structure but also to the conceptions of knowledge underlying it. Florensa et al. (2018a) analyse the tension between the two components of the transition between paradigms: proposing SRPs based on the inquiry of questions and using them to overcome didactic constraints produced by the paradigm of visiting works. Bosch (2018) illustrates the different tools proposed by the ATD to question and model the knowledge to be taught and the evolution of these tools to adapt to the emergence of the new pedagogical paradigm. In this line of research, the ATD research group in Spain has already designed, implemented and analysed diverse SRPs in different higher education degrees and institutions (see Table 19.1) (Florensa et al., 2019).

Our research focuses on the SRPs' *ecology*, that is, the conditions needed and the constraints hindering their running as normalised instructional activities (Barquero

Table 19.1 SRPs implemented in higher education by the ATD research team in Spain

Subject	Level/Degree	Generating question	Period	References
Mathematics	1st year. Degree in Chemical Engineering	Population dynamics	2005–2009	Barquero (2009) and Barquero et al. (2011, 2013)
Mathematics	1st year. Degree in Business Administration	Sales fore-casts Bike share system	2006–2014	Serrano (2013) Serrano et al. (2010)
Mathematics	1st year. Degree in Business Administration	Facebook users forecast	2015–2017	Barquero et al. (2018)
Statistics	2nd year. Degree in Business Administration	Consumer behaviour	2019–2021	Markulin et al. (2021a, 2022)
Strength of materials	2nd year Mechanical engineering	Slatted bed design	2015–2020	Bartolomé et al. (2018)
Elasticity	3rd year Mechanical engineering	Bike/device/car part design	2015–2020	Florensa et al. (2016) Florensa et al. (2018b)

et al., 2013). The implementation of SRPs reveals diverse ecological challenges related to their fitting as normalised activities in different types of higher education institutions. One of the issues pinpointed in all the previously cited studies is the existing tension between the curriculum (the formulation of the "knowledge to be taught") and the SRP implementation, constraining the selection of generating questions to the (partial) coverage of the course contents. This tension generates difficulties in fitting SRPs into course plans, which explains that some of the SRPs have been implemented as elective courses to minimise their effect on curricular aspects of mandatory courses. The force of the university pedagogical tradition also manifests in the strong dependency observed between the teachers running the SRPs and its sustainability. For instance, Barquero et al. (2011, 2013) show that when the implementation of an SRP stops being under the didacticians' responsibility, it runs the risk to be reduced into a traditional problem-based or tutorial session, where the teacher ends up solving most of the questions raised in front of the students. Since curricula are usually formulated in terms of knowledge organisations and not of questions to address, teachers experience inquiry processes as a slowdown regarding the expected progress of the didactic time.

All these observations show a constant tension between "taking the generating questions seriously", that is, letting the generating question nourish the inquiry process, and the teacher's problem of linking the SRP to specific curriculum content. It may lead to the development of SRPs where the generating question is conceived as a mere excuse to visit specific knowledge organisations instead of being the driving force of the inquiry dynamics. One can think that the adoption of competency-based curricula would facilitate teachers' detachment from content-based curricula. However, the prevailing epistemology in higher education institutions with content-oriented programs seems to endure.

We present in this work two of the SRPs that have been implemented by a team of teachers and didacticians in two higher education institutions. Specifically, we will describe an SRP in Elasticity in a Mechanical Engineering degree and an SRP in Statistics in a Business Administration Degree. We want to analyse to what extent SRPs detach from classical content-oriented curriculum using the concepts of *adidacticity* and *situation* developed by Brousseau (1997).

19.2 Theoretical Framework, Research Questions and Empirical Methodology

A line of research we are following these past years consists of studying the conditions and constraints affecting the transition from the paradigm of visiting works to the paradigm of questioning the world. In this line of research, the description, design, implementation, and analysis of SRPs become critical, and the ATD has proposed diverse theoretical developments to sustain this work. We use various analysis tools to analyse SRPs' development. The first one is the *Herbartian schema* we are presenting here in its short form:

$$[S(X; Y; Q) \rightarrow M] \hookrightarrow A^{\heartsuit}.$$

In the schema, $S(X; Y; Q)$ is a didactic system where a group X of students with the help of a group of teachers (or study guides) Y addresses a question Q to provide an answer $A^{\heartsuit}$. The study of Q generates an *inquiry* process involving a *didactic milieu* M including questions Q_i derived from the initial one, "ready-made" answers $A_j^{\diamond}$ one can find in the literature or by consulting works and experts, together with empirical data D_k and other material and knowledge works W_l. The expanded version of the Herbartian schema is then:

$$[S(X; Y; Q) \rightarrow \{Q_i; A_j^{\diamond}; D_k; W_l\}] \hookrightarrow A^{\heartsuit}.$$

If the Herbartian schema points at the elements of the inquiry, its dynamics is described in terms of *dialectics*. We are only considering two of them in this paper. The first one is the *question-answer dialectic*, which will provide a first description of the inquiry structure. Approaching a question Q_i can be done by exploiting the inquiry milieu M, but also by searching available answers $A_j^{\diamond}$ and studying them to integrate them into the milieu (through a deconstruction-reconstruction process). This study generates a provisional answer $A_i^{\heartsuit}$ to Q_i, but also new questions about the validity and limitations of $A_i^{\heartsuit}$ (and $A_j^{\diamond}$), its adequacy to Q_i, the adaptations required, etc. (Bosch & Winsløw, 2015). The question-answer dialectic shows the progress of the inquiry, how it moves forward. Following Barquero and Bosch (2015), we attribute it to the *chronogenesis* of the SRP. The second dialectic we are considering here is the *media-milieu dialectic*. *Media* refers to any system emitting intentional messages, like books, articles, the internet, but also experts. The search for preestablished answers A_j° is done by exploring the media resources. In a sense, the media's answers have to be integrated into the milieu – turning into "sure" knowledge – and the elements of the milieu have to be worked out to make them produce new messages. The evolution of the milieu during the inquiry process by including new objects and partial answers constitutes the inquiry *mesogenesis* (relating "meso" to "milieu"). Let us notice that the notions of *chronogenesis*, *mesogenesis* and *topogenesis* come from the first developments of the theory of didactic transposition (Chevallard, 1985, pp. 71–79; see also Barquero & Bosch, 2015). Even if they correspond to the evolution of didactic processes within the paradigm of visiting works, they can be easily extended to the broader paradigm of questioning the world. In this context, the topogenesis corresponds to the share of responsibilities between teachers and students (the "topos" corresponding to the place or position occupied by X and Y). It corresponds to the evolution of the "didactic contract" from the Theory of Didactic Situations (TDS, Brousseau, 1997) we are using here in its original meaning.

An essential element of the SRPs dynamics is to be "question-driven". By this, we mean that the study community (X, Y) approaches Q with the aim of elaborating an answer $A^\heartsuit$, which remains the final goal of the inquiry. This statement might seem obvious, and it is so when we consider an inquiry process from the perspective of the paradigm of questioning the world. However, in the paradigm of visiting works, many times teachers present questions Q because they want X encounter some specific knowledge organisations and learn them (incorporate them in the milieu). In this case, the inquiry appears conditioned to some previously established answers A_j° that X and Y are supposed to find or use in the elaboration of $A^\heartsuit$. We talk in this case of *finalised* SRPs (Chevallard, 2011).

In the SRPs experimented by our research team, the tension between the paradigm of visiting works and the one of questioning the world is always present. Teachers Y try to design and manage the inquiry process in a "question-driven way", which requires important changes in the didactic contract that are not always easy for the students – and neither sometimes for the teachers. For instance, students expect the teachers to assume many responsibilities in the SRP management, like planning the work to do, proposing the questions to address and the media to consult, validating the answers they propose, etc. In a way, in the traditional didactic contract teachers are expected to lay out the paths students will then follow, as if the "question-driven" inquiry was only driven by the teacher, not by the students.

To analyse the changes of responsibilities and better grasp the constraints originated in the prevailing paradigm of visiting works, our analyses have led us to use the distinction between *didactic* and *adidactic situations*, two key notions of the TDS. We consider, in a first step, the contrast between didactic and adidactic, introducing a sort of continuity between them by talking about the *adidacticity* of a teaching situation. In the TDS, the adidacticity of a teaching situation is established when the activity – or game – proposed to the students can be carried out without the teacher's intervention and, therefore, without knowing the teacher's intentions about the knowledge to be taught. Students know that the teacher has designed the game or activity for a didactic purpose: this is the difference between an adidactic and a *non-didactic* situation. However, students engage in the activity to win the game, not to fulfil the teacher's didactic intentions. In this case, the situation milieu needs to be rich enough to provide feedback to the students and help them evolve in the strategies used:

- In *didactical Situations*, the teacher maintains direct responsibility for all stages of the lesson. She tells the students her intentions, what they will have to do, and what the results should be. She intervenes freely to keep the class traveling on the desired route. [. . .]
- In *a-didactical Situations* it is the students who have the initiative and the responsibility for what comes of the Situation. The teacher thus delegates part of the care for justifying, channeling and correcting the students' decisions to a *milieu* (a problem statement, a physical set-up, a game, an experiment). (Brousseau et al., 2014, p. 147)

However, in our use of the notion of didactic situation, we do not include its functioning as a *fundamental situation*, that is, as a reconstruction – and epistemological model – of the mathematical knowledge to be taught: "Each item of knowledge can be characterised by a (or some) adidactical situation(s) which preserve (s) meaning; we shall call this a fundamental situation" (Brousseau (1997, p. 30). What we will adopt from the TDS notion of situation – which is at the core of Brousseau's epistemological proposal – is the consideration that any question or problem is never raised in a vacuum but always appears to X (and Y) under specific circumstances or conditions, and with some particular available resources (and other unavailable ones), a certain *milieu*.

Therefore, in our analyses of the experienced SRPs, we will talk in general about *adidacticity* to refer to the very moments when students make decisions primarily considering the generating question Q and the final answer $A^{\heartsuit}$, without prioritising the instructional process that envelops the inquiry. This does not mean that students take Q – and $A^{\heartsuit}$ – seriously because they *like* Q. For instance, they can be interested in Q because they will be assessed by the "quality" of $A^{\heartsuit}$ as an answer to Q. But it means that the aim of the inquiry remains Q and not any preestablished answer A° or the particular intermediate works A_i° they will have to find and made available. Looking for these adidactic moments can help measure to what extent an SRP is overcoming the constraints of the paradigm of visiting works. Adidactic moments can be good indicators that students are going beyond the traditional didactic contract and are assuming their new inquiry responsibilities, thus producing an evolution of the topogenesis.

From the previous considerations, we raise the following research questions:

> To what extent can SRPs generate *adidacticity* (where X and Y prioritise the generating question as the main goal of the inquiry) in higher education institutions? Under what conditions? What is the role of the *situation* and *contract* within which the generating question is raised?

Our study is based on the exposition and comparison between two case studies considering an SRP on Elasticity implemented in an Engineering degree, and an SRP on Statistics from a Business Administration degree. Both SRPs have been designed, experienced and analysed using the same methodology, derived from the didactic engineering methodology as described by Barquero and Bosch (2015). The design of the SRP considers a didactic phenomenon to address – usually derived from the visiting works paradigm – and proposes an a priori analysis based on a description of the generating power of the initial question Q in terms of expected potential derived questions and partial answers. This a priori analysis is then completed with a proposal of schedule and didactic devices for the organisation of the inquiry work: presentation of Q, grouping of students in teams, use of a regular logbook per each team of students, sharing and validation of intermediate results, presentation of the final answer, continuous and final assessment, etc. The a posteriori analysis relies on empirical data produced by these devices (students' logbooks and productions, teacher preparation notes and classroom presentations, assessment) completed by a common survey administrated to the students and a semi-structured interview to a small sample of them. In each SRP in Elasticity here considered, two teachers (a lecturer doing research in didactics and a lecturer doing research in another area) run the SRP with two different groups of students. In the second SRP in Statistics, a researcher in didactics acted as a teacher assistant in both groups, also adopting the role of an external observer. The developments presented in the next sections mainly rely on the observations made by the teachers and observers during the implementation of the SRPs. These observations regarded the students' productions and the interactions in class. The final students' surveys and interviews were then used to validate and complete them. Details about the methodology and role of collected evidence can be found in (Florensa et al., 2018a, b) and (Markulin et al., 2021a, 2022).

19.3 An SRP in Elasticity

The first SRP we consider here corresponds to an Elasticity course in the third year of a Mechanical Engineering Degree in the EUSS School of Engineering in Barcelona (Spain). Florensa et al. (2016) describe the first two versions of the SRP and the didactic phenomena identified by researchers. Since 2015–2016, the SRP has been experienced each academic year (last implementation on year 2019–2020).

The didactic phenomena that motivated the SRP design concern the institution's epistemological conception of Elasticity and its influence on three interrelated facts: the monumentalisation of the discipline, the high level of algorithmisation of the type of tasks promoted during the course, and the detachment between lab and theory sessions. These phenomena are linked to the paradigm of visiting works. The activity "labelled" as Elasticity in the institution is mainly related to the General Elasticity Model presentation and its use to solve paper and pen problems (Florensa et al., 2016). Because of the model's mathematical nature, the problems which are solvable using paper and pencil are far from real engineering problems: geometries and loads are extremely simplified and, consequently, far from real-world situations and disconnected from workplace activity. The proposed SRP intended to overcome these phenomena by offering an alternative raison d'être of Elasticity: enabling engineers to design and optimise parts of any geometry and under any load working in an elastic regime.

According to this change in the conception of the discipline, the SRP was proposed as a project lasting for the last 6 weeks of the semester, after a traditional organisation (theory and paper and pencil problems) for the nine first weeks. The lecturers presented the generating question for the first SRP implementation as follows: "A bike company asks you to design a bike part using one of three given materials. You will have to provide an answer in a final report. The report will include the dimensions and shape of the proposed design, the considered loads and their justification, the safety factor, the deformations, as well as the project time planning and cost". During the project, students submitted weekly reports accounting for the progress of the SRP in terms of lists of questions and answers.

The SRP incorporation in the Elasticity course was not problematic in terms of curricular restrictions, for two main reasons. On the one hand, the SRP took place during the last 8 weeks of the semester, leaving the first part of the course with the traditional lecture and tutorials structure. This organisation made the transition between the two parts of the course easier for the lecturers. On the other hand, the students' weekly and final reports facilitated the assessment according to the curriculum, which is stated in terms of learning goals (and not contents), as is the general situation in many Spanish universities.

Florensa et al. (2018b) describe the effects of the two first implementations on the content at stake in relation with the didactic phenomena addressed. However, they also note that students found difficulties in adapting to the evolution of the didactic contract between the two parts of the course, especially when validating answers:

> Lecturers have to refrain from validating students' answers and students have to assume a lot of new responsibilities, like raising questions the teachers are not answering, searching for new information to address the questions raised—and validating them—, sharing answers with their classmates without the teacher's interference, deciding when and how to test their results in the lab. (Florensa et al., 2018a, b, p. 9)

The validation was problematic because the bike company was fictive, and the productions were only validated through comparisons with other existing bike parts or by lecturers' comments on the weekly and final reports. This point relates to the adidacticity involved in the SRP: the generating question was intended to generate an adidactic situation where the use of the elasticity model and simulation software would enable students to obtain a final answer. However, the validation capacity of the milieu was not sufficient. This introduced students failing to assume full responsibility for their productions, and relying too much on the lecturers' validation. The absence of a strong validating milieu thus led to a loss of adidacticity: the validation of the answers was done "outside" the SRP (by the lecturers).

In order to overcome these issues, after the two first editions, researchers, together with lecturers, changed the generating question and intended to facilitate the validation by creating a stronger milieu. The new generating question took advantage of the creation of a Formula Student[1] team at the engineering school. The question was stated as follows: "The Formula Student team asks you to (re)design and optimise a part of the racing car. You will have to provide an answer in a final report. The report will include the dimensions and shape of the proposed design, the considered loads and their justification, the adequation to the formula student rules, the factor of safety, the deformations, as well as the time planning of the project and the cost of the project".

The new question modified the way students addressed the questions, as the answers were not to be presented to a fictive company, but to a real formula student team composed of peers, some even participating as students in the SRP. The validation process changed substantially: lecturers' validation became secondary, as the students were now attending to competition rules and the team members' feedback. Besides, part of the lab used for the subject was also used as the Formula Student car garage. Elasticity students and Formula Student's team members thus shared the same space, which facilitated their interactions. The new generating question emphasised another fact that already existed in the first edition. Many students addressed relevant topics in mechanical engineering that are usually addressed in other courses, such as manufacturing processes, surface treatments, and fatigue-related aspects. In the first SRP, the teacher could tell students they would see these topics later. In the second one, these topics had to be addressed during the SRP.

19.4 An SRP in Statistics

The second SRP was implemented in a Statistics course on the second year of a Business Administration Degree at IQS School of Management in Barcelona. Markulin et al. (2021a, 2022) present the first edition of the SRP together with its analysis. We are considering two implementations of the SRP, a pilot one in the year 2019–2020 and the second one in 2020–2021. These two courses followed the same organisation as the Elasticity course: a first part with traditional lecture/tutorial sessions and a second part wholly devoted to the SRP.

The hypotheses underlying the SRP give priority to data analysis processes in statistics using appropriate software, including gathering data and cleaning datasets. The technical and theoretical elements of the course include descriptive analysis, probability distributions and an introduction to inference and hypothesis tests. They were introduced as the answer to some concrete questions embedded in case studies, with data to summarise and results to generalise. The reason for designing and

[1] Formula Student is a student engineering competition held annually in diverse countries worldwide. Student teams from around the world design, build, test, and race a small-scale formula style racing car.

implementing the SRP in the course was twofold. On the one hand, as a teaching process, it aims at proposing a real case where students have to gather data and perform a complete statistical analysis on a general issue. On the other hand, from a research perspective, it aims at providing some evidence for the study of the ecology and management of an inquiry-based work in a university setting, considering that the ecological dimension is often taken for granted in the literature on Statistics project-based learning (Markulin et al., 2021b).

The first edition of the SRP was proposed as a project in a second part of the course, lasting for the last 3 weeks of the semester with six 2-h sessions. The first part of the course was organised in bi-weekly case studies that were a mixture of theory sessions and students' analysis of open questions to be partially answered: providing a descriptive analysis, using models of distributions, checking the descriptive analysis with hypothesis tests, etc. All the cases preceding the project, and the project itself, were worked on in teams of four or five students. The generating question for the SRP was: "How is the behaviour and what are the motives of vegan and vegetarian people for choosing their diet (motivations, values, purchasing behaviour, etc.)?" The Marketing department of the IQS School of Management proposed a survey elaborated from a research perspective. Students had to gather data by administrating the survey to relatives and friends of different ages. All answers were put together, and students had to elaborate an analysis of the gathered dataset. The students' expected final production was a poster describing the sample obtained by disseminating the survey and a summary of the findings, depending on the research focus that teams chose to develop: vegans' personal values, habits of purchasing, motivations for the diet and social engagement etc. Students elaborated one intermediate report presenting the progress on the SRP in terms of a "done and to be elaborated" summary during the project. Lecturers proposed a structure of the final report, containing a sample description and validation, some questions raised about the chosen focus and the statistical analyses needed to answer them. When the results produced were not sufficiently clear, hypothesis tests could provide additional tools to sharpen the conclusions. Students presented the final report in a poster session, and the project assessment counted 30% of the final grade.

The incorporation of the SRP was not problematic in terms of curricular restrictions for the same motives as the before mentioned SRP in Elasticity. Moreover, given the students' profile, the practical focus of the subject and the Marketing department's collaboration appeared as positive conditions. Regarding the teamwork as well as the theoretical and technological basis provided during the sessions preceding the project, the pilot SRP ran as expected. Students obtained data from their friends and relatives mainly, which of course, introduced strong biases, but they related their findings to the type of sample considered. Nevertheless, Markulin et al. (2021b) concluded that the generating question about vegans' behaviour was not well posed because it was too closely dependent on the survey structure, which was decided by others and for a research purpose (outside of the SRP). This result shows the importance of the situation in which the generating question is raised, that is, the target or receiver of $A^{\heartsuit}$, the conditions of this reception and its purpose (what will be done with it).

The final poster presentations was carried out with a combination of "internal" and "external" evaluators, internal being lecturers of the Statistics course giving feedback on the intermediate report and external being experts from different departments within the institution where the SRP was implemented. Among external evaluators, one was familiar with the research issue about vegan and vegetarian diets, being involved in creating the proposal and the survey. The external evaluator's implication, who actually opened and proposed the topic of the project, seemed as an appropriate solution for strengthening the adidacticity of the project work with the chance of the final posters being possibly assessed by the "client". However, due to the project proposal's lack of clarity, the process and the goal were often blurred.

The second edition of the SRP was proposed within the constraints of doing the course partially or even completely online due to the COVID-19 pandemic. With that in mind, the proposal implied a real consultancy study with a practical purpose, rather than as a research project. The client was a society that was funding a cooperative supermarket, FoodCoop BCN, the first of its kind in Barcelona, wishing to analyse their target members and customers. Two members of the cooperative supermarket introduced the project to the students in a real-time online session at the beginning of the semester and could answer their doubts and clarify any initial misunderstandings of their business idea. According to their demands, the generating question of this second SRP was stated by the lecturers in collaboration with the students as: "Who are the FoodCoop BCN target members? Which is the best district in Barcelona to place the supermarket?"

The new SRP assumed a change of the course organisation. Instead of concentrating everything at the end, the SRP was distributed among the course at the semester's beginning, middle, and end. Two reasons justified this choice. First, the fact that students needed time to collect the answers to the survey. Second, the last 3 weeks of the semesters were programmed entirely for the project work to avoid overloading students with different cases once they have already collected the answers. Also, to make students working on the project in the middle of the semester while waiting for the collected data, lecturers proposed an intermediate task to describe Barcelona's districts' sociodemographic profile based on data available on the web, which could help prepare the sample validation. In total, the SRP was done for 4–5 weeks, implied three intermediate reports, each having aimed at different aspects of the project process and ended up with a presentation session. The results were validated like in the first edition, combining the internal evaluators for the intermediate reports and the external ones for the final presentation session (which was held online instead of as a poster presentation).

The first intermediate report about Barcelona districts had unexpectedly to be done entirely online, just when all the classes were transferred to online modality exclusively. Such report relied on the work with Excel carried out during the previous semester's course of Computer Science and Systems, a course that had also suffered a change to online modality. The conditions resulted in a too simplified study for the intermediate report and a certain discouragement of some teams of students. The other two intermediate reports focused on the survey data analysis. These reports were to be elaborated describing the personality and the intentions of the potential cooperative's members. Despite some teams' interesting efforts in the proposed directions, a considerable number focused their analysis more on describing rather obvious characteristics of general Barcelona residents than providing valuable consultancy information for the client. But at the time of writing this paper, a rigorous a posteriori analysis of the second edition of the SRP has still not been carried out.

An essential strength of an SRP, in theory, is the richness of the question that generates the whole path. The second Statistics project precisely relied on a question that emerged from a real business project (a cooperative supermarket) and required a practical solution. However, to avoid complexity in the design of the survey, lecturers requested the help of some marketing department experts, which, unfortunately, led (again!) to prioritising a research approach to cooperative customers' behaviour to the consultancy work that was the idea of the proposed SRP. In the end, we did not observe a noticeable evolution from the first to the second SRP, despite the changes in the project format, type of generating question and involvement of a real client as receptor and validation agent. We attribute it to the continued strong guidance from the lecturers in the kind of decisions made when running the inquiry: a sudden proposal to search for secondary source data about Barcelona districts; imposition of an external survey prepared by an expert team; lack of intermediate interactions with the client (before the final one). Despite initially proposing a real consultancy question by the affected cooperative organisers, lecturers did not succeed in engaging students in the inquiry as professionals, including the joint enterprise of making a rich enough milieu available. The passage to the online teaching modality represented a critical difference as well, making comparison with the first version difficult. In a way, COVID-19 amplified weaknesses of the implemented SRP that might may had remained implicit in normal class conditions.

19.5 Conclusions and New Open Questions

This paper is based on the analysis of two experienced SRPs, one for the teaching of Elasticity to Engineering students, another one for teaching Statistics in a degree of Business Administration. In both SRPs, the generating questions have evolved from the first to the second edition, resulting in different conditions in how the dialectics of questions-answers and media-milieu have taken place. The results obtained shed new light on the ecology of SRPs and the limitations of the instructional devices used to enact the dialectics. Considering them under the light of the notions of adidacticity and situation from the TDS helps us formulate three remarks that open new questions about the relationships between the TDS and the ATD to be contemplated in further investigations.

19.5.1 The Choice of the Generating Question and the Curriculum Constraint

The choice of the generating question *Q* of an SRP – or of any other kind of inquiry-based or project-based instructional proposal –appears as an open problem in educational research. Where does it come from? Who selects it? Under what criteria? It also appears as an important open issue when designing an SRP for a given instructional purpose. Once we have decided to implement an SRP in subjects like Elasticity or Statistics, what is a “good” generating question? How to choose it? Are some questions more productive than others? Is it possible to measure their “generating power”? This questioning is crucial at different levels: for the educational project, for the facility to implement and manage the SRP, for how students will receive and accept it, etc. However, we consider that this problem is usually raised from the perspective of the visiting works. In a way, when one asks “how to choose *Q*?” the implicit concern is “to teach a given knowledge organisation predetermined by the curriculum”. In the paradigm of questioning the world, the question “what *Q* to address?” corresponds to the curriculum problem, that is, to the problem of defining the instructional project for a given set of students *X*. In the paradigm of visiting works, the curriculum is defined in terms of subjects and knowledge organisations students are required to visit and learn. In the paradigm of questioning the world, it is defined in terms of questions to address. Therefore, the choice of the generating question *Q* is primarily part of the curriculum definition and only secondarily of the instructional process’s organisation. Even if it currently corresponds to a crucial question in our research, it partially appears because we are still conceiving instructional processes according to the old paradigm of visiting works or, at least, in the transition between paradigms.

19.5.2 *Taking* **Q** *Seriously and Creating* **Adidacticity** *During the Inquiry Process*

In our research about SRPs, we are starting to consider a broader notion of *adidacticity* in the moments where the rules of the activities followed by (*X*, *Y*) do not seem to include their didactic component. This happens when the generating question of the inquiry becomes the primary goal of the study community, and the instructional project of learning some predetermined subjects and topics (the *didactic* project) remains in the background. We often refer to these moments by saying that “the generating question is taken seriously” and, thus, it is not seen as a “fake question” only addressed for another aim, the one of teaching and learning some “content”. These adidactic moments are often associated with the existence of external validation instances (external to (*X*, *Y*)) incorporated in the inquiry process through the media-milieu dialectics. In these cases, teachers act as partial validators of some of the intermediate answers or tools obtained; they act as helpers to prepare a final answer they are not validating. They intervene now and then to make the inquiry progress but do not assume the final product's responsibility. Both X and Y are responsible for a final answer $A^{♥}$ assessed by an external instance – the *receivers* of $A^{♥}$. However, the existence of external validators is not enough.

19.5.3 Changing the Generating Question or Changing the Situation *in Which It Arises?*

As we have seen in the second edition of the Elasticity SRP and the Student Formula project, it is not a change of generating question Q that modifies the didactic contract, but a modification of the conditions or *situations* in which this question arises, which determine what is at stake for the students. Between the first and second SRP, there is no real change in the question raised: students have to find material for a given part of a mechanical machine in both cases. However, in the bike case, it appears first as a fake demand, since only the teacher is receiving the report about the material needed to build it. On the contrary, with the Student Formula team, it becomes vital to get a part as light as possible and do it correctly for the sake of the students' team race.

We can find here one of the main raisons d'être of the notion of *situation* in the TDS, which includes the question raised, the conditions under which it arises – the initial *milieu* – and the conditions for its reception and validation. In a way, a question also includes a contract about the type of answer $A^{♥}$ expected, the persons who are receiving and validating it, and what they are doing with it – its destiny. The conditions to manage an SRP in a question-driven way, and the students' engagement in the process, depend on the initial contract established when proposing the question to the students and maintaining this contract all along with the inquiry. Our second case study about the SRP in Statistics for students in Business Administration shows the importance of the initial formulation of the generating question Q and the fragility of the conditions provided by the contract established with the external validators about the kind of acceptable answers $A^{♥}$. It is an illustrative example of "real" generating questions – existing outside the classroom and the instructional project – that did not appear sufficiently real to the students. The students' role in the contract passed with the clients appears as a critical condition for developing the SRP and its dialectic, both for teachers and students. New dialectics, like the one of the dissemination and diffusion (Chevallard with Bosch, 2020, pp. xxi), need to be incorporated into the analysis and the design of SRPs, to improve their ecology and ensure their sustainability under less favourable conditions.

19.5.4 The Inclusion of TDS Notions into ATD Analyses

Let us add a final comment about the inclusion of notions taken from the TDS into the analysis of didactic processes that have been designed and implemented within the principles of the ATD. In 2007, Guy Brousseau developed some connections and distinctions between key notions of TDS and what could be considered as their analogues in ATD. His methodological proposal consists in considering these notions "as points of contact, [. . .] to move from one to the other, rather than as borders" (Brousseau, 2007, p. 24, our translation). And he concludes:

> The ATD and the TDS complement each other. But in my opinion, it would be absurd to simply juxtapose them. In many issues they are intertwined, they must be considered together. What problems do they pose for each other? What answers do they offer each other? What advances do they promise together? (Brousseau, 2007, p. 22, our translation)

In recent developments of the ATD, Chevallard (2021) introduces a relative perspective by considering the instance (person or institution) for whom a given situation appears as didactic or non-didactic, distinguishing the cases of isodidactic and antididactic situations and talking more generally about *possibly didactic situations*.

These two networking strategies worked at the theoretical level of research praxeologies (Artigue & Bosch, 2014). The strategy we adopt in this paper is bottom-up, starting from the needs raised by the research techniques when analysing and designing SRPs. In this situation, combining notions from the ATD and TDS is something natural to us. In a way, ATD is a development of TDS, and they share many basic theoretical assumptions. Our approach corresponds to a pragmatic research perspective, resorting to the notions and methodological proposals that seem most appropriate for the problems we address – a pragmatic attitude much in line with the paradigm of questioning the world.

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Chapter 20
Fostering Inquiry and Creativity in Abstract Algebra: The Theory of Banquets and Its Reflexive Stance on the Structuralist Methodology

T. Hausberger

Abstract This chapter centres around the "theory of banquets", an invented structure outside the standard Abstract Algebra syllabus. This theory has been elaborated to facilitate students' access to structuralist thinking at large through the use of the meta-lever. Students are guided in an investigation of the meaning of a "structure" as they engage in crucial steps of the structuralist method. The study has been carried out with the methodology of didactic engineering: the activity has been designed, implemented and analyzed using Brousseau's Theory of Didactic Situations and an epistemological and semio-cognitive framework, the "objects-structures dialectic". The purpose of this chapter is both to introduce the non-francophone community to this research, published in Recherches en Didactique des Mathématiques, and to connect and contrast it with selected other studies. The results of a classroom experiment with third year Bachelor students are presented and discussed synthetically. New data are also presented in the form of lab sessions, with more advanced students, in order to emphasize the inquiry and creativity that the theory of banquets is able to generate. Theoretical ideas from Fischbein and Tall are thus combined with the objects-structures dialectic in order to account for creativity processes.

Keywords Abstract algebra teaching and learning · Didactics of mathematical structuralism · Theory of banquets · Didactic engineering · Inquiry-based learning · Creativity in university mathematics education

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20.1 Introduction

Abstract Algebra designates in this chapter an upper division undergraduate course typically required for mathematics majors and centered on the structures of groups, rings, and fields. It is a subject generally recognized as challenging by students whose difficulties, in particular regarding abstraction processes and the acquisition of “structural sense”, are well documented in university mathematics education research. As far as teaching is concerned, it seems that instructors rarely adopt new pedagogical approaches (Fukawa-Connelly et al., 2016): lecture is predominant and tutorials are often dedicated to working out standard examples and basic techniques that leave little room to students’ creativity. A few design-based instructional approaches have been experimented, but these focus primarily on the teaching and learning of Group Theory so that research in this area remains limited. Altogether, Abstract Algebra “offers many challenges to researchers in order to develop inquiry-based approaches that may promote adequate conceptualization and understanding” (Hausberger, 2018b).

In fact, the structuralist point of view inherited from German algebraists and systematized in the mid twentieth century by the Bourbaki group shaped mathematics as a research field to give it its contemporary face (Corry, 1996). As explained, this contrasts with the reality of mathematics classrooms where this powerful and insightful vision struggles to be transposed. What kind of classroom activities may be envisaged to foster inquiry and creativity in Abstract Algebra learning, in line with genuine mathematical research practices?

This chapter centres around the “theory of banquets” (an invented structure outside the standard Abstract Algebra syllabus). This theory was elaborated to facilitate students’ access to structuralist thinking at large through the use of the meta-lever (Hausberger, 2021). The study has been carried out with the methodology of didactic engineering (Artigue, 2014): the activity has been designed, implemented and analyzed using the Theory of Didactic Situations (TDS; Brousseau, 1997) and an epistemological and semio-cognitive framework, the “objects-structures dialectic” (Hausberger, 2017b). The latter has been drawn upon in reference to the interplay between semantic work on mathematical objects and syntactic work on the axiomatic structures that unify these objects. This dynamic is as a fundamental epistemological (and didactical) dialectic that characterizes structuralist thinking.

The results of a classroom experiment with third year Bachelor students are presented and discussed, more synthetically than was done in the author’s RDM paper (2021), written in French. The purpose of this chapter is both to introduce the non-francophone community to this research and to connect and contrast it with selected other studies. In relation to the full paper, less emphasis is given to the debate on the existence of a fundamental situation (in the sense of Brousseau, 1997) for structural concepts, and the point of view of the teacher (how the situation may be managed) is not provided. However, two supplementary lab sessions with more advanced students are discussed, in order to emphasize the inquiry and creativity that the theory of banquets is able to generate. Finally, an early draft of this research was first presented in 2016 at the KHDM conference in Hannover (Hausberger, 2017a). It is a pleasure to take the opportunity of this book in honor of Reinhard Hochmuth to disseminate the advances of this work to the international community of research on university mathematics education.

The chapter is structured as follows: in Sect. 20.2, main theoretical ideas underlying innovative aspects of the theory of banquets are presented and compared with other instructional approaches. In the next section, mathematical aspects of the didactic engineering are described and a priori analyses of main tasks are provided as a reference. In Sect. 20.4, the data from the classroom experimentation and lab sessions are analysed and the learning affordances of the theory of banquets are discussed with respect to inquiry and creativity. A conclusive section summarizes striking elements of this practice-oriented study, and discusses further avenues of research.

20.2 Inquiry and Creativity in Abstract Algebra Teaching and Learning

The goal of this section is twofold: firstly, to situate the theory of banquets among different approaches that draw emphasis on either inquiry, creativity or both; secondly, to develop a few of the main underlying theoretical ideas before the more detailed presentation of the design in the next section.

20.2.1 Inquiry

Good examples of innovation can be found in the Teaching Abstract Algebra for Understanding project (Larsen, 2013), whose ambition was the creation of a research-based inquiry-oriented curriculum for Abstract Algebra. The design followed the Realistic Mathematics Education framework (RME), in the tradition of Freudenthal, and was centered on Group Theory (GT). Under the guidance of the teacher, students investigated the set of symmetries preserving geometric figures, developed a calculus for computing their combinations, and axiomatized the set of rules that governed the algebraic computations. The instructional device culminated with the “reinvention” of the definition of a group and similar processes were used to handle quotient groups and the group isomorphism concept.

Within the Anthropological Theory of the Didactic (ATD), Bosch et al. (2018) also launched a similar program. The global vision is a shift of paradigm, from “monumentalism” (the critical view that contents are rarely questioned and problematized in the current curricula) to a new epistemological and pedagogical paradigm called emblematically “questioning the world” (Chevallard, 2015). With a focus on GT, they looked for problems external to GT that could lead to the reproduction of a substantial part of GT as a means to ascribe some rational to it. They concluded that a counting problem on symmetries of a square might be a suitable candidate for a reconstruction of elementary GT. The main difference with Larsen lies at the implementation level of the inquiry process: ATD proposes a general instructional device called Study and Research Paths (SRP; Chevallard, 2015) and endowed with several theoretical tools (in the form of dialectics) to organize and measure the development of the inquiry process. But the envisaged SRP has not been carried out by the authors and therefore remains hypothetical.

Moreover, Bosch et al. wondered if the counting problem was substantial enough to motivate the study of the isomorphism theorems. Such a question points to the structuralist approach conducted by the author: if groups encode symmetries, a substantial part of the rationale of GT relies in its relationship with the structuralist methodology in general. In other words, some attention must also be paid to the meta-concept of structure itself and to the methodological dimensions of mathematical structuralism, that could also be questioned.

In these lines of thought, the notion of structuralist praxeology was introduced (Hausberger, 2018a) together with the experimentation of a SRP on Ring Theory that used a transcription of an online forum as a crucial component of the milieu. A general interpretation of "questioning the world" in Abstract Algebra was proposed, based on the idea that formalization was both a mathematization of the world (the extra-mathematical reality) and, at a higher level of abstraction, a conceptual rewriting of previous (pre-structuralist) mathematics in terms of structures, usual mathematical objects being taken as the (intra-mathematical) reality. In this context, questioning the world amounts to questioning mathematical objects in such a way that a fruitful dialectic between objects and structures may be developed. Such a vision meets the point of view of RME and in particular its notions of horizontal and vertical mathematization. We will see below that abstraction processes in Abstract Algebra may be distinguished from these two notions.

Before getting into these details, let us introduce a third theoretical framework that inspired the theory of banquets as a second type of inquiry-oriented innovation for the teaching and learning of mathematical structuralism. The banquets have been designed as a problem that may be regarded as a partly a-didactical situation in the sense of TDS. The inquiry thus takes the form of the epistemic actions of the learners who play against an antagonist milieu. Moreover, the meta-lever is used, that is "the use, in teaching, of information or knowledge about mathematics. [. . .]. This information can lead students to reflect, consciously or otherwise, both on their own learning activity in mathematics and the very nature of mathematics" (Dorier et al., 2000, p. 151). Concretely, a meta-discourse is explicitly introduced in the milieu; for example, the worksheet begins in these terms:

> A structuralist theory is an abstract theory: it therefore deals with objects whose nature is not specified. They are then noted by symbols: x, y, z or α, β, γ, etc. In the theory of banquets, there is only one type of objects [. . .] Since the nature of the objects is not specified, it is the *relations* between the objects that are the focus of the theory [. . .]. (Hausberger, 2021, Appendix 2)

The meta-discourse also aims at fostering a level of meta-cognition, along the lines of Piaget's reflective abstraction (Piaget & Beth, 1961), which is viewed as an essential part of the inquiry process. Indeed, the main questions that are, explicitly or implicitly raised by the theory of banquets, are the following: What is a banquet? What does it mean to classify banquets? What do we mean by "structure" in mathematical structuralism? Answers may be found by reflecting on the classification of groups in order to classify banquets using similar structuralist means.

20.2.2 Creativity

Discourses on creativity draw back to Poincare's essay *L'invention mathématique* (1952) on the topic of mathematical discovery, creativity and invention, and to Hadamard's lectures on the psychology of mathematical invention (1945) which emphasized four stages (initiation, incubation, illumination and verification) in the journey to invention in the mathematical field. Those writings influenced the work of Fischbein (1994) on the interaction between the formal and intuitive components of mathematical activity. One of the main point made by Fischbein is that "a world of intuitive models act tacitly and impose their own constraints" (loc. cit. p. 236), even at formal stages of intellectual development. For instance, the abstract notion of set comes with the idea of a collection of objects, with all its connotations. Intuition is often accompanied by figural representations; these lead to idealized mental entities that interplay with axiomatic or deductive systems. Fischbein met here the views of Freudenthal (1983) who advocated in his *didactical phenomenology of mathematical structures* that mathematical concepts should be taught together with their underlying mental images. Fischbein therefore investigated cases of fertile symbiosis – or cases of conflict – between figural/intuitive and conceptual properties of mathematical objects in the elaboration of a mathematical proof, for instance in geometry. As an extension of this work, Kidron (2011) studied means to help students be aware of their tacit models and achieve a complete synthesis between formal and intuitive representations in the sense that the mental structure was flexible and avoided conflicts. A situation in analysis was designed and experimented with students.

To the author's knowledge, there are very few studies on the role of intuition and mental models in Abstract Algebra. In a pilot study, Stewart and Schmidt (2017) used Tall's three worlds (embodied/symbolic/formal; 2013) framework to compare a mathematician and one of his students' mathematical experience as the class was about to prove the Fundamental Theorem of Galois Theory. According to Tall, "natural proof builds on concept imagery involving embodiment and symbolism, which may build on embodiment, symbolism, or a blend of the two" (p. 286). Mathematical activity and access to formal knowledge therefore requires one to navigate between the three worlds, including the embodied world based on gestures and perception of patterns. This raises a didactical issue, since "there are significant problematic changes in meaning that must be addressed to move to another plane of mathematical thinking" (p. 414). In the case of Galois Theory, the student struggled to revisit the concrete examples in the light of the abstract concepts of the theory, and therefore had a quite different mathematical experience than the professional mathematician. According to Stewart and Schmidt, the difficulty lies in "blending conceptual embodiment and operational symbolism" (p. S47) as a path to formal mathematics.

The theory of banquets is an educational device that aims to facilitate the access to structuralist thinking. As evidenced by Fischbein and Tall, such an access involves the development of mental models and the integration of intuition and logic to build flexible and coherent schemes. The name of the theory (banquets) was chosen in order to facilitate such an integration, as the mental image of guests sitting around tables should be evoked. An important step before designing tasks consists in clarifying the cognitive and epistemological dimensions of structuralist thinking from a theoretical point of view. The objects-structures dialectic aims at providing such a framework.

20.2.3 The Objects-Structures Dialectic

According to Cavaillès (1994), two movements of abstraction are at work in structuralist thinking, *idealisation* and *thematisation,* which apply transversally to each other (one is perceived as vertical, the other horizontal). They follow one another dynamically to express a dialectic between form and content, which Cavaillès calls the "dialectic of concepts". Roughly speaking, idealization consists of extracting a form, which is then thematized into a higher-level object theory. Precise definitions are given in (Hausberger, 2017b). In fact, idealization may be linked to the horizontal mathematization of RME and thematization to vertical mathematization. However, idealization is not centered on real-life phenomena (but on the epistemic action of identifying invariant properties attached to a plurality of heterogeneous situations), and thematization is a particular vertical mathematization, specific to the structuralist project.

Moreover, in the case of Abstract Algebra (unlike elementary school algebra), two levels of organizing principles of phenomena need to be distinguished: on the one hand, the level of the given structure (of group, ring, etc.), which appears as the organizing principle of phenomena involving objects of a lower level; on the other hand, the meta-concept of structure itself, which is playing an architectural role in the elaboration of mathematical theories, in relation to the structuralist methodology. Indeed, similar questions and tools govern the application of the abstract unifying and generalizing point of view of structures and characterize the process of thematization. For instance: which identity principle to adopt (which are the natural morphisms between objects of a given type of structure)? How to classify objects up to isomorphism? Which structuralist theorems govern the decomposition of objects into simpler ones? We recover here key questions that will be used to design the theory of banquets and its reflexive stance (the inquiry dimension).

Let us now develop the cognitive dimension that relates to intuition, creativity and mental models. As mental representations cannot be accessed, semiotic considerations on external representations must also be considered. A first didactical idea is to use contributions of model theory, which offers a fertile point of view to bridge intuition and logic through the distinction between syntax and semantics and the articulation between these two aspects. First of all, a definition by axioms is, from a logical point of view, an open sentence. The models (the instances that satisfy these statements, in other words the objects in the sense of the objects-structures dialectic) constitute the *semantic* content of the structure, in relation to the system of axioms that defines it syntactically. Referring to Fischbein, models may include mental models built from perceptual intuition or embodiment in the sense of Tall. This will be the case with the banquet structure, whence its very name. By contrast, syntactical work with the axioms is carried out in the symbolic world; articulation between syntax and semantics thus amounts to what Fischbein called a fertile symbiosis. This leads us to distinguish between a syntactic point of view on idealization, which consists in abstracting the particular nature of objects and isolating the formal properties of relations (the "logic" of relations), and a semantic point of view which emphasizes the *isomorphism classes of models*. The latter mediate the concrete semantic domain of objects and the abstract syntactic domain of the structure, but the price to pay is the transition from elements to classes. From this point of view, the task of classifying models (up to isomorphism) appears fundamental for the conceptualization of an abstract structure. The conceptual aspects include concept formation but also the structural horizon of the structuralist theorem of decomposition of objects into simpler ones: in the case of banquets, the decomposition of a banquet in a disjoint union of tables. This is a clear illustration of the role that conceptual embodiment, in the sense of Stewart and Schmidt (2017), may play on the journey to the so-called formal world (in other words, structuralist thinking).

The second idea is to use Duval's theory (2006) to handle representations and work in the symbolic world, in other word the manipulation of signs. According to Duval, the mental model is an internal representation, which serves to objectify the banquet structure; whereas the observables are the external representations produced by learners (*semiosis*), *in* particular during the conceptualization process (*noesis*). As a means to investigate creativity in students' work, we will pay particular attention to these representations, as well as to the semiotic manipulations (treatments and conversions in the sense of Duval), which are used to determine classes of banquets.

20.3 The Theory of Banquets: A Didactic Engineering

The theory of banquets was designed according to the methodology of didactic engineering (Artigue, 2014). As stated in the introduction, our main focus in this chapter is on learners' activity that will be analysed in the light of the theoretical elements, centered on inquiry and creativity, that were just presented. The choice of values of didactic variables and the orchestration between didactical and a-didactical dimensions of the situation in the sense of Brousseau (thus the role of the teacher) are discussed in the RDM paper (Hausberger, 2021). We will restrict our account to a brief presentation of mathematical aspects of the theory of banquets and provide an a priori analysis of the tasks that relate to the data discussed in the next section.

20.3.1 Mathematical Presentation of the Theory of Banquets

A banquet is a set E endowed with a binary relation R which satisfies the following axioms: (i) *No element of E satisfies xRx;* (ii) *If xRy and xRz then* $y = z$; (iii) *If yRx and zRx then* $y = z$; (iv) *For all x, there exists at least one y such that xRy.*

In part I of the worksheet (which has been distributed in one go and may be processed linearly), students are asked the following questions:

1 a. Coherence: is it a valid (non-contradictory) mathematical theory? In other words, does there exist a model?
1 b. Independence: is any axiom a logical consequence of others or are all axioms mutually independent?
2 a. Classify all banquets of order $n \leq 3$
2 b. Classify banquets of order 4
2 c. What can you say about **Z/4Z** *endowed with* $xRy \Leftrightarrow y = x+1$*?*
2 d. How to characterize abstractly the preceding banquet (that is, how to characterize its abstract banquet structure among all classes of banquets, in fact how to characterize its class)?

The abstract/concrete relationship is reversed in part II, which begins with the empirical definition of a *table* of cardinal number n to mean a configuration of n people sitting around a round table. The following questions are raised:

1 a. What relationship between people could be used to abstractly define a table?
1 b. State a system of axioms abstractly defining a table.
2 a. Propose a definition of sub-banquet *and* irreducible *banquet. Let* $b = (E,R)$ *be a finite banquet and* $x \in E$*. Define and characterize the sub-banquet* <x> *generated by x.*
2 b. What is the link between tables and irreducible banquets?
2 c. Define the operation of union *of banquets. State and prove the structure theorem of finite banquets.*
2 d. Apply the theorem to banquets of cardinal number 4.

The banquet structure possesses a large variety of models since the system of axioms may be interpreted in quite different worlds, beginning with the empirical interpretation of guests sitting around tables (a component of Tall's embodied world): *xRy* if x is sitting on the left (or right) of y. Other domains of interpretation include Set Theory (the binary relation is represented by its graph), Functions ($xRy \Leftrightarrow y = f(x)$) defines a function f according to axioms (ii) and (iv); the other two axioms mean that it is injective without fixed points), Permutation Groups (*f* is a bijection when *E* is finite, in other words a permutation without fixed points) or even Matrix Theory (the relation is seen as a function $E^2 \rightarrow \{0,1\}$ and represented by the corresponding matrix; the axioms express rules on the number of 1 in each row and column) and Graph Theory (xRy if and only if the vertices x and y are connected by an edge oriented from x to y).

The structure theorem of banquets (decomposition in a disjoint union of tables) thus corresponds to the well-known theorem of canonical cycle-decomposition of a permutation, but the analogy remains hidden since the binary relation of banquets is different from binary operations that define groups. A complete rewriting in terms of permutations is not expected from students. These remarks explain why the theory of banquets is mathematically rich but may not be found in any textbook (it is less general than permutation groups). Moreover, it is a simpler theory (in the sense of mathematical technicality) than Group Theory and it carries the underlying intuition and mental image of guests sitting around tables (a wedding banquet).

In the language of TDS, the theory of banquets decomposes into 4 main (sub-) situations:

- the logical analysis of the system of axioms (I 1),
- the classification of banquets of small cardinal numbers (I 2);
- the axiomatic definition of tables (II 1);
- theoretical elaboration and the structure theorem (II 2).

We will now apply the theoretical framework (the objects-structures dialectic, Sect. 20.2) to the a priori analysis of the tasks dedicated to the classification of banquets of small cardinal numbers (I 2). The main prerequisite is a course in elementary Group Theory.

20.3.2 A Priori Analysis of the Classification Tasks

The methods may be divided into two categories: on the one hand a syntactic-dominant approach, which is similar to the reasoning used in the case of the classification of groups of small orders, and on the other hand a semantic-dominant approach, which uses generic models borrowed from matrix or graph theory. It will be necessary, however, in each case to articulate syntax and semantics at a given point in the reasoning.

In the syntactic-dominant approach, let us take the case of three elements x, y, z. Up to permutation of elements, we can assume $x\,R\,y$ (under (i) and (iv)); necessarily, (yRx or $y\mathrm{R}z$) and (zRx or zRy), again under (i) and (iv). Of the four cases, only yRz and zRx is possible, by virtue of axioms (ii) and (iii). The reasoning is similar with four elements, but it requires repeating several times the “up to permutation” argument. This leads to two classes: xRy, yRx, zRt, tRz and xRy, yRz, zRt, tRx. One may expect students to stop at this stage, while it remains to justify that these two classes are distinct (and nonempty, by providing a model). The first point requires

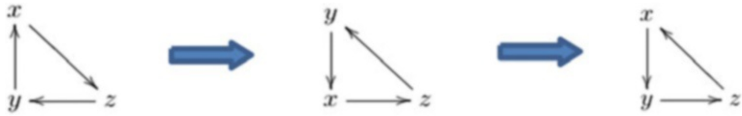

Fig. 20.1 Identification of isomorphic models through treatments within the semiotic register of graphs and pattern recognition

the notion of isomorphism, in fact the knowledge of properties invariant under isomorphism, which allow to distinguish the two classes. In the case of groups of order 4, well-known to students, the presence or absence of an element of order 4 is usually invoked. Working out the analogy with banquets consists in identifying a pattern of cyclicity: reasoning about the order of an element amounts, in our context, to reasoning about the cardinal of the "chain" generated by an element (by iteration of the relation), which is a closed loop in the case of finite cardinality. Cyclic groups, including those formed by the roots of unity, also rely on this mental image of the circle. While it is unlikely that students will engage in such formalization, except for those who are particularly comfortable with formalism, it is likely that the cyclic pattern will be recognized and emphasized, the more so as it is suggested by the mental image of banquets from the embodied world. The aim of questions (c) and (d) is to lead students to make this mental image explicit and formalize a notion of cyclic banquet.

In the semantic-dominant approach, matrix or graph theory is used to produce generic models that may represent all possible cases. It is therefore a question of differentiating classes. Graphs allow to quickly deal with the case of 3 elements by replacing analytic reasoning on axioms with a succession of actions, as in a Lego game: there are only two possibilities of endowing three letters x, y, z with arrows such that the resulting directed graph fulfills the axioms (interpreted in graph theory). To convince oneself that the direction of rotation is not important, a treatment (in the sense of Duval) within the graph symbolic semiotic register may be applied: the first step consists in re-establishing the counter-clockwise direction, which does not change the directed graph; the same cyclic pattern is then recognized up to permutation of x and y (step 2). Without formalizing a notion of isomorphism, the principle of abstraction, in its naive sense of abstracting elements, thus allows to figure out that both models lie in the same class, in the etymology of isomorphism (having the same shape) (Fig. 20.1).

The situation is more complex in the case of 4 elements, as the number of configurations is higher. Nevertheless, knowledge from graph theory (treatments to remove the crossings of arrows) makes it easy to come to either the case of the cyclic graph or the case of the graph with two connected components of two elements related by a double arrow. The visual process of pattern recognition allows to conclude, by forgetting the labeled vertices of the graph.

The formal definition of isomorphism requires to have integrated the syntactic point of view of bijection that preserves relations. In GT, isomorphisms preserve operations, which is conceptually different, but the syntactic proximity of x*y and

xRy should allow students to easily find the condition $\forall(x, y) \in E^2$, $xRy \Rightarrow \varphi(x)\ R'\ \varphi(y)$ defining a morphism $\varphi: (E,R) \to (E',R')$ of banquets. The actual construction of the isomorphism, for example between the 2 previous banquets of cardinality 3, may be carried out by comparing xRy, yRz, zRx and $x'Rz'$, $z'Ry'$, $y'Rx'$: if φ maps x to x', it will also map y to y' and z to z'. In fact, in writing relations in such sequences, we have implicitly identified the cyclic pattern. The latter can be made explicit by conversion (in the sense of Duval) to a register that underlines the pattern graphically (such as graphs or empirical banquets from the embodied world).

20.4 Learning Affordances of the Theory of Banquets

A classroom experiment with third year Bachelor students took place in 2014 (4 sessions of 1h30 each for the full worksheet). To encourage the meta-cognitive dimension, students worked in small groups of 3–4. Each group had to return its research notebook to the teacher after each session and phases of devolution and institutionalization (in the sense of Brousseau, 1997) took place when appropriate according to the scenario (Hausberger, 2021). The experiment was later supplemented by two lab sessions (outside the classroom), with two pairs of more advanced students called Alice/Bob and Chris/Debby in this chapter. Alice had completed a PhD in mathematical physics and occupied several post-doctoral positions before passing exams to become an upper-high school/upper-secondary teacher. Bob was about to begin a PhD in differential geometry and Chris/Debby were more standard Master students. The data that will be discussed in this section comprise excerpts of students' notebooks and excerpts of transcripts of dialogues among pairs of advanced students.

20.4.1 What Is a Banquet? Students' Creative Processes in Making Sense of a Formal System of Axioms

Unsurprisingly, Alice readily connected the axioms to the mental model of wedding banquets and blended symbolic manipulation of axioms with intuitive reasoning in the embodied world. Alice is indeed close to being a professional mathematician and her work is a wonderful illustration of Tall's claims on symbolism and embodiment.

Alice: Classical, we specify the structure through relations, okay.
Bob: Antisymmetry [about axiom (i)].
Alice: It's not quite like that, it's non-reflexivity; there's one guy on the right and one on the left, that's the idea, [laughter]; there's nobody sitting alone at a table.
Bob: The elements are people? And in relation if together at the table?

Alice: Yes, that's it. The relation is to sit on the right (or left). However, you can have at most one guy on the right and at most one on the left, there is at least one guy on the right. Yes… there is theory and models. To show that it's not contradictory, you can show that there exists a model. I suggest we take one guy. No, one guy doesn't work, 2 guys sitting next to each other. So you take E = {*x*,*y*}. You can also put {0,1}.

Bob: {1,2}?

Alice: Let's take E = {*a*,*b*} and for the relations the couples (*a*,*b*) and (*b*,*a*). So it is indeed a model.

Bob: [after reflection] ok

Alice: Yes, a set with 2 elements, they are sitting opposite each other… obviously, there is at most one on the right and one on the left, they are in relation with the one opposite…

Bob: So this is existence. And consistence?

More surprisingly, Chris and Debby did not relate the axioms to wedding banquets (before part II of the worksheet). Nevertheless, they spontaneously introduced semiotic representations from graph theory through cognitive processes that also relate to embodiment: "Globally, we have a point *x* which maps to *y* and *z*, we necessarily have equality" (discussion of axiom (ii)). The movement of the pencil, from *x* to *y*, thus the gesture, led them to represent the relation in the form of an oriented arrow. They then borrowed from permutations the notation (*x y z*), more condensed, to designate the resulting directed graph in the case of 3 elements (without linking banquets neither to graph theory nor GT).

In the classroom, nearly every group of students began by representing banquets from the real world in a more or less idealized manner (top of Fig. 20.2). In order to solve the assigned tasks, generic models with more affordances towards mathematical treatments had to be produced, therefore the teacher had to introduce the repertoire of either graphs or matrices. Examples of such representations are provided at the bottom of Fig. 20.2, which also includes in the middle a purely symbolic representation in Set Theory.

20.4.2 What Does It Mean to Classify Banquets? Students' Creative Processes in Developing a Structuralist Point of View

Let us now analyze how the representations from the embodied and symbolic worlds may be used to potentially achieve the journey to the formal world and develop a structuralist sense. We will begin with the expert practice of Alice who is playing the role of teacher towards Bob:

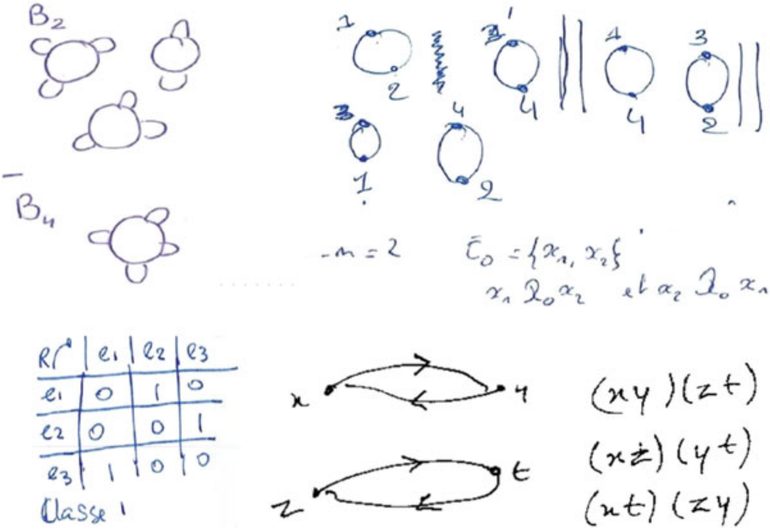

Fig. 20.2 Semiotic representations produced by students throughout part I

Bob: Cardinal number 3…

Alice: *The circular thing, people a,b,c around the table*. (*a*,*b*), (*b*,*c*), (*c*,*a*). It remains to be seen that this is the only one. (*a*,*b*) by numbering, it is still valid.

Bob: (*a*,*c*), (*c*,*b*), (*b*,*a*)?

Alice: It's the same model, up to isomorphism.

Bob: That's true.

Alice: (*b*,*a*)… there's going to be a problem, because *c* is going to be sent on what? If *c* is sent on *a* or *b*, as *a* and *b* are already reached, we will deny (ii).

Bob: If we had (*a*,*b*) and (*b*,*a*) we wouldn't know what to do with *c*…

Alice: Yes, that's it. *Because his two potential right-wing neighbors already have one neighbor*

Bob: So it's necessarily (*b*,*c*) and we complete.

Alice: Perhaps cardinal number 4 will be more interesting. Shall we say {*a*,*b*,*c*,*d*}?

Bob: Yes.

Alice: So there is the circular model… are you following me?

Bob: Always… but, in this case, there can be several *if you put them a,b,c,d around a table…*

Alice: Yes, but you'll be able to find a bijection, which amounts to a renumbering. If you want, the natural morphisms in there will be… is there a way to send E on E′ by a bijection that sends *R* on *R*′? So if you have a circular model, you're going to be able to send it on a circular model by a permutation.

Bob: Uh, yes…

Alice: So we always have (a,b); we always have (b,c)... ah, can b send itself to a? That would make a first case separation.

Bob: *It would make a two-table banquet, so to speak.*

Alice: Yes, this is a possibility. You can have (a,b), (b,a), (c,d), (d,c). In fact, we're going back to the previous banquets. We have the circular banquet $R_{C,4}$, and we have, one could say, finally a direct sum in fact. It is a direct sum of banquets: $R_4 = R_2 \bigoplus R_2$. Are there others? I don't think so.

Bob: Are there other direct sums possible? No, because there is no one-person banquet.

Alice: In theory, you can have irreducible models, which do not break down into direct sums, and which are a priori different from the circular model. But here, if we have (a,b) and if we put (b,a), then the rest is specified; so we will try to put (b,c). If we put (c,d) we fall back to the circular banquet; (c,a) we're screwed. So this is the only possibility, I don't know if you follow me...

Bob: OK, so we have our two models.

Striking features of this dialogue include a fertile symbiosis between figural/intuitive and conceptual properties of banquets that result in operative symbolism (salient sentences are underlined in italic). Another feature is the conceptual perspective of Group Theory (and Abstract Algebra in general): the direct sum of banquets has not been defined yet (the operation of union of banquets is the focus of part II.2.c), therefore Alice's reasoning cannot be understood but as analogical thinking with, for instance, the decomposition of the Klein group V_4 in a direct product of two cyclic groups of order 2. Students anticipate part II and also introduce a notion of irreducibility (which was not forecasted in the a priori analysis centered on undergraduate students).

Let us now describe how Chris and Debby proceeded within the semiotic register of graphs:

Chris: There would be 9 of them.

Debby: Nevertheless, we only considered objects that we know. But since the beginning, we have been talking about a structure.

Chris: But wait, the elements can always be numbered. What could go wrong?

Debby: Our own consistency.

Chris: But here, we thought about relationships, we didn't think about the objects themselves, we didn't take a particular relation.

Debby: Never mind.

The conclusion they drew is mathematically inaccurate. Nevertheless, the reflexivity shown by the students is remarkable: they emphasized that they abstracted both the nature of elements and the semantics of the relations. However, the algebraic symbolism (the letter) gives the illusion that the process of abstraction is complete. This is not the case, since labels of the graph vertices should also be removed. Although they were visualizing the pattern (left part of Fig. 20.3), students did not develop the intuition that the list of 9 models consisted in 2 classes, and as a

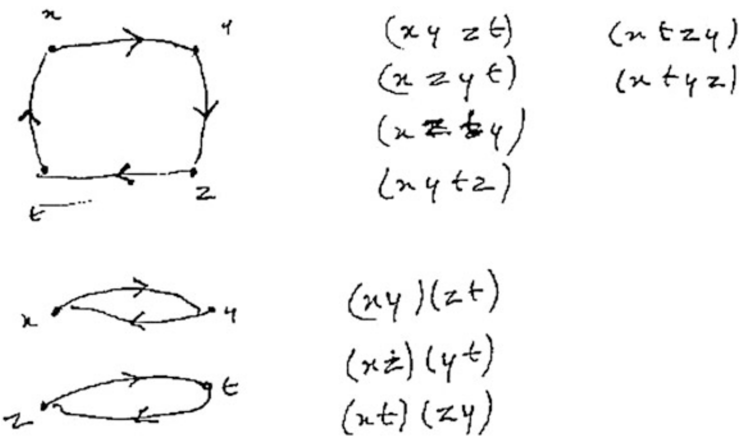

Fig. 20.3 Chris/Debby's classification of banquets of cardinality 4

consequence they did not formalize a notion of isomorphism. The intervention of the instructor (I) was required to achieve this, which proved to be a long journey:

Debby: So there would be 2 classes up to isomorphism, this kind of object and this kind.

Chris: There, **Z**/4**Z** and there **Z**/2**Z** × **Z**/2**Z**, in fact.

I: Are you thinking about the classification of groups?

Debby: Necessarily, we think about the classifications we know.

I: So there are 2 types of objects and here you have listed them all on *x*, *y*, *z*, *t* [...] You have listed all the possible oriented graphs on *x*, *y*, *z*, *t* that fulfills the axioms. [...] And why do you say there are two classes?

Chris: Two classes? We have put all the permutations behind, anyway.

I: And why would (*x y z t*) and (*x y t z*) be the same?

Chris: No, not the same, of the same type.

I: What does it mean to be of the same type?

Chris: I am thinking of permutations. One will loop faster than the other. I am clearly thinking about the order behind it.

Debby: A bijection. One can pass from one element of this class to another by a bijection, but not between the 2 classes.

I: Isn't it always possible to find a bijection between two sets of same cardinal number?

Chris: Yes it is!

Debby: Ah yes, but will it respect the structure? [...]

Again, the journey involved Tall's three worlds (a loop is part of the embodied world), but the process of accommodation of knowledge on groups to achieve a structuralist classification of banquets proved to be difficult. The situation is different from the case of Alice who developed a complete mental structure (or scheme) regarding structuralist decompositions that could easily integrate the case of banquets with the support of its mental model. Outside help was therefore needed, but the last question raised by Debby showed that the inquiry process tackled crucial issues in the development of structuralist sense.

Let us finally point out the creative processes generated during the classroom experiment, in particular the role played by the mental model. Unlike Chris/Debby who did not mention wedding banquets at all (maybe, due to a didactical contract that separated real world phenomena from genuine mathematical objects), there were groups who engaged in classifying empirical banquets without questioning abstraction processes (Fig. 20.4). The mental model is here an obstacle towards bridging the symbolic world and students did not manipulate the formal axioms at all.

To the other end, there were groups who were proficient in syntactical manipulation of axioms but did not produce any synthesis of intuitive and symbolic representations in their semiosis. In between, a few groups made a fertile use of the mental model in identifying the class of the banquet ($\mathbf{Z}/4\mathbf{Z}$, R) (Fig. 20.5). Students referred to the "table of 4", which was idealized in their drawing and superposed with another representation in the semiotic register of graphs. In the terms of our theoretical framework, this is a convincing example of conceptual embodiment. Nevertheless, the notion of isomorphism was not formalized by any group without the intervention of the teacher who had to renegotiate the didactical contract (by emphasizing the need of formal definitions) and clear a path to the formal world as in the Chris/Debby case.

Most of the students perceived analogies with GT, on an intuitive basis (informally) and for various obvious reasons: the notation $\mathbf{Z}/4\mathbf{Z}$ in the worksheet (whereas it is essential to distinguish the additive group $\mathbf{Z}/4\mathbf{Z}$ from the banquet ($\mathbf{Z}/4\mathbf{Z}$, R)), the type of task (classifying objects) and the similarities in the results obtained (which is not a coincidence, given the link with permutation groups). However, those who made the analogy most explicit did not manage to expand the mental structure

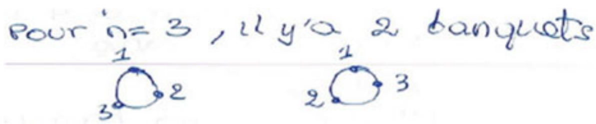

Fig. 20.4 The mental model as an obstacle: empirical classification of banquets

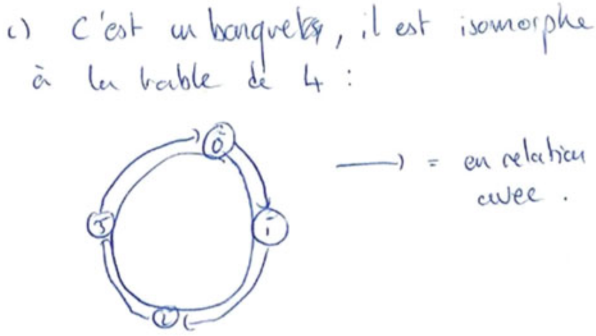

Fig. 20.5 Identification of the isomorphism class by conversion to a graphical register related to embodiment and pattern recognition

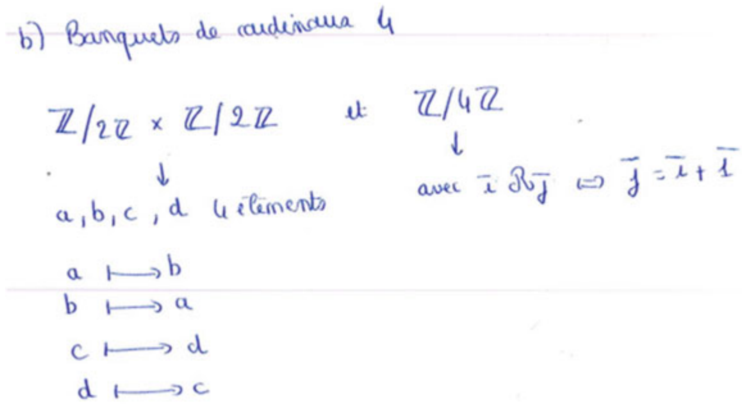

Fig. 20.6 Students' difficulties in expanding the mental structure from groups to banquets

(or schema) from groups to banquets as Alice did. For instance, a group of students used the symbolic representation $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ to designate the 2-table banquet (Fig. 20.6) without noticing conflicting aspects with the Cartesian product in Set Theory. Indeed, the disjoint union of tables adds up cardinal numbers whereas the Cartesian product is multiplicative. The same conclusion as Kidron (2011, p. 125) draws, applies here: "it highlights the need for mathematical educators to help students be aware of their tacit models, and to complete the synthesis between the formal and the intuitive into one mental structure".

20.5 Conclusion and Perspectives

The theory of banquets is the fruit of a practice-oriented study that sheds light on mathematical creativity in Abstract Algebra, in the sense of a fertile interplay between intuition and formalism, as it is experienced by professional mathematicians (represented by Alice), and by graduate or undergraduate students. The goal is also to induce a meta-cognitive shift (a dimension of inquiry with a focus on the meta-concept of structure) in order to facilitate the access to structuralist thinking. The mathematical experience varied according to students' personal level of advancement in the mathematical journey through Tall's 3 worlds. However, the 3 dimensions (embodied, symbolic and formal) were almost always present, whatever the level. Most students could take advantage of the cognitive affordances offered by the mental model of the embodied world, to some extent, but only advanced students were able to achieve a complete synthesis between the formal and the intuitive in the development of a structuralist classification of banquets. Unsurprisingly, the intervention of the teacher was needed to point out conflicting aspects between the formal and the intuitive or stimulate the inquiry process on the meaning of a structuralist classification for students to move forward.

The main novelty of the theory of banquets in comparison with previous didactical designs in Abstract Algebra presented in Sect. 20.2 is its ambition to tackle the issue of students' access to structuralist thinking at large. GT thus serves as prerequisite, and meta-cognition is crucial in the inquiry process developed by students. If research results may be expressed as above using theoretical constructs from Fischbein and Tall, the framework of the objects-structures dialectic allowed a finer-grained analysis of the interplay between the formal, symbolic and embodied components of mathematical activity than previous accounts (e.g. Stewart & Schmidt, 2017).

The study can still be deepened in different directions, depending on the theoretical framework that is used to complement the analyses in the spirit of networking. Studying structuralist praxeologies (Hausberger, 2018a) in the ATD framework can shed more light on what it means and requires, at the level of praxis and logos, to successfully work out the analogy between groups and banquets in the classification tasks. The inquiry process can also be modeled as a SRP in order to get a finer control on its vitality, economy and ecology. Inside TDS, the different levels of the milieu (its structuration) may be analyzed in order to have a clearer picture of the relationship of the epistemic subject with the milieu during the different phases of the situation, particularly those of meta-cognition, and finally tune didactic variables with higher granularity. Cognitive and semiotic aspects of the analysis may also be deepened, for instance by using semiotic frameworks that pay more attention to embodiment, or by attempting to study the structuralist schemes (or schemas) involved in the theory of banquets using cognitive frameworks.

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Chapter 21
Following in Cauchy's Footsteps: Student Inquiry in Real Analysis

Sean Larsen, Tenchita Alzaga Elizondo, and David Brown

Abstract Proof-based mathematics courses are often taught in a lecture-based format that deprives students of the opportunity to engage in authentic mathematical activity. Students are presented with mysterious answers to questions they have never even been asked to consider. Inquiry-oriented instruction (IOI) provides an alternative approach in which teachers engage students in deep mathematical sense-making as they inquire into both the mathematics and one another's thinking. As part of a larger project, we have designed an inquiry-oriented instructional sequence that engages students in the reinvention of several real analysis concepts. In this chapter, we explore the mathematical activity of both the students and instructors in one course using this instructional sequence. Our results suggest that the principles of inquiry-oriented instruction provide important support for instructors who are motivated to provide students with an opportunity to engage in legitimate mathematical activity while also meeting expectations for content coverage.

Keywords Inquiry oriented · Real analysis · Proof · Realistic mathematics education · Mathematical practices · Convergence of sequences

21.1 Introduction

Advanced, proof-based mathematics courses are overwhelmingly taught through lecture using a "definition-theorem-proof" format (Weber, 2004). However, research (Lew et al., 2016) has demonstrated that even good lectures do not result in students understanding the ideas the teacher intends to convey. More importantly, this traditional approach denies students the opportunity to engage in authentic and creative mathematical activity even though research (e.g., Lampert, 1990; Larsen & Zandieh, 2008) has demonstrated that students are quite capable of doing so. Inquiry-oriented instruction (IOI) provides an alternative approach in which

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teachers engage students in deep mathematical sense-making as they inquire into both the mathematics and one another's thinking (Laursen & Rasmussen, 2019). While the students engage in mathematical inquiry, the teacher inquires into this activity both to assess the students' learning and with the aim of leveraging the students' ideas to drive the mathematical agenda (Laursen & Rasmussen, 2019). In this chapter we explore student inquiry in the context of real analysis. The participating students were enrolled in a university course meant to support them in transitioning to advanced proof-based mathematics. Our purpose in this chapter is to illustrate the principles of inquiry-oriented instruction (Kuster et al., 2018) and exemplify the kind of student inquiry that this approach can support.

Real analysis is a topic most mathematics students encounter at some point in their university studies. It also has an interesting history in that it was created as a result of digging deeply for the rigorous foundations of the well-established computational calculus. So, it is a particularly good context for illustrating the way that definitions and axioms emerge near the end (and not the beginning) of the development of an area of mathematics. However, the students in a traditional introductory analysis course do not experience the course in a way that reflects its historical development. Many real analysis courses begin with the unmotivated introduction of field and order axioms before a similarly unmotivated introduction to the least upper bound property. These foundations along with some basic definitions are then used to prove theorems like the Archimedean property and the density of the rational numbers in the real number line. These early concepts that student meet in their first real analysis course are not only disconnected from the computational calculus the students are familiar with, but they also provide mysterious answers to questions the students have never even been asked to consider.

Our research team conducted a series of design studies resulting in the development of an instructional sequence that captures the spirit of the historical inquiry that led to the development of modern real analysis. This design process is described elsewhere (Larsen et al., 2021; Strand et al., 2021; Strand, 2016; Vroom, 2020). Here we examine the mathematical activity in a course that utilized this instructional sequence in order to provide an illustration of the principles of IOI in action. Specifically, we frame this illustration in light of the four principles described by Kuster et al. (2018):

1. Generating student ways of reasoning
2. Building on student contributions
3. Developing a shared understanding
4. Connecting to standard mathematical language and notation

We will focus on one portion of the instructional sequence and use it to demonstrate how the instructor moves, student inquiry, and instructional tasks worked together to realize these principles of IOI. *Generating student ways of reasoning*, refers to the process of eliciting student activity and thinking that anticipate the formal mathematics that is the goal of instruction. The principle of *building on student* *contributions* emphasizes the idea that the students' mathematical activity drives the agenda with the formal mathematics being developed as students further mathematize their own mathematical activity. It is crucial that this principle is enacted in concert with the principle of *developing a shared understanding* so that all of the students have the opportunity to experience the mathematics as the result of their own inquiry. Finally, the principle of *connecting to standard mathematical language and notation* acknowledges the instructor's role as a broker between the classroom community and the greater mathematics community (Rasmussen et al., 2009). In this role, the instructor both supports the classroom community in becoming legitimate participants in the mathematics community and supports them in accessing the more efficient and powerful language and notation of that community.

21.2 Context and Brief Description of the Instructional Sequence

21.2.1 Intermediate Value Theorem as Starting Point

The inquiry we will be describing takes place in the context of an instructional sequence that was designed with the support of the theory of Realistic Mathematics Education (RME), which holds that students should learn mathematics by mathematizing both reality and their own mathematical activity (Gravemeijer & Doorman, 1999). Of particular importance was the RME heuristic of didactical phenomenology (Gravemeijer & Terwel, 2000; Larsen, 2018) which posits that if the designer wants to support students in reinventing a mathematical idea, they should strive to identify a phenomenon that can be productively organized by that idea. In particular, we wanted students to reinvent the idea of a least upper bound, so we developed a task context in which the concept of a least upper bound could emerge as a commonsense solution to a problem. Note that this design heuristic directly reflects the IOI principle of *generating student ways of thinking* in that the idea is to provide a task situation that is likely to elicit informal ideas that anticipate the formal mathematics that is the goal of instruction. The starting point for the instructional sequence is the Intermediate Value Theorem (IVT). The task of proving this theorem by drawing on approximation techniques, motivated by Cauchy's historical proof (Grabiner, 2012), is a context in which several important ideas like sequence, limit, monotonicity, and least upper bound emerge as useful tools. In the following subsections we will (1) describe the course and participating students from which we draw our illustration, and (2) briefly describe the historical proof of the IVT that inspired the instructional sequence.

21.2.2 Context: The Course and the Participating Students

Our instructional approach was developed by conducting a series of design experiments over the course of several years. The classroom data shared in this chapter come from a university transition-to-proof course that was conducted near the end of the design process when the design was relatively stable. The course was taken by 14 students all of whom had taken at least a year of computational calculus. Only one had previously taken a real analysis course. Six of the students were women and eight were men where two of the men were Hispanic. The course was co-taught by the first two authors (one a professor and one a graduate student) with the assistance of the third author (also a graduate student). Due to the COVID-19 pandemic, the course was taught online in a remote synchronous format. Mathematical inquiry was conducted collaboratively, with students working individually, in pairs, in small groups, or as an entire class using various technological tools including Zoom, an online whiteboard app, Google Docs, and a wiki-text (Katz & Thoren, 2014). The course consisted of 19 sessions, each approximately 110 min in duration. We screen recorded all class and office hour zoom sessions. We also collected all of the students in-class work (Google Docs etc.) and homework in digital form. Additional data included screen recordings of instructor debriefing and planning sessions, lesson plans in Google Docs form, and the Wiki-Text in which we created "permanent" public records of the collective mathematical progress of the class.

21.2.3 Starting Point and Cauchy's Proof of IVT

The primary instructional sequence begins with a question that is designed to elicit a justification that relies on some version of the Intermediate Value Theorem. We ask the students, "Does every 5th degree polynomial have a real root?" The students typically argue that a 5th degree polynomial must have a real root because it approaches ∞ in one direction and $-\infty$ in the other, and because polynomials are continuous it must cross the x-axis at some point. Subsequent discussions result in the formulation of a conjecture that is a special case of the IVT: Any continuous function with a sign change will have at least one root.

The focus of the students' inquiry for the next several weeks is the development of a proof of this conjecture. Our approach is motivated by Cauchy's original proof of this theorem (Grabiner, 2012). Cauchy supposed that $f(a)$ and $f(b)$ have opposite signs where $a < b$. He then described a process in which the interval $[a, b]$ is partitioned into m equal sized intervals. One of these sub-intervals must have endpoints that have opposite signed output values. The partitioning is repeated indefinitely, generating two sequences a_n and b_n such that for every n: (1) $a_n < b_n$, (2) $f(a_n)$ and $f(b_n)$ have opposite signs, and (3) $b_n - a_n = (b_{n-1} - a_{n-1})/m$. Cauchy then claimed that these two sequences must converge to the same limit because they will differ from each other "by as little as desired" (p. 168). He then concludes that

$f(a_n)$ and $f(b_n)$ must both converge to the output of this limit value since f is continuous, and because these sequences always have opposite signs this common limit must be zero. It is the task of rigorously proving Cauchy's claim that a_n and b_n both converge (and have the same limit) that motivates the development of several important concepts of real analysis including the idea of a least upper bound. The illustration we present here is situated within this portion of the instructional sequence. Specifically, we pick up the story at the point where the students had just completed creating approximation methods in their small groups and we conclude our description at the point where the students are generating informal arguments for why an increasing sequence that is bounded above must converge to its least upper bound.

21.2.4 Data Analysis

Our goal for this chapter is to illustrate Kuster et al.'s (2018) four IOI principles in action. We began by reviewing all data related to the core part of the instructional sequence (beginning with the development of approximation methods and ending with a proof of the monotone convergence theorem). Then, convinced that each of the principles were represented by several instances, we selected a subsequence that was short enough to allow us to present a detailed discussion while also representing a coherent mathematical story. We then conducted an in-depth analysis of the data related to these tasks. We looked both for evidence of intent (in the task statements and instructor moves) and of student activity that aligned with each principle. For example, in the case of *developing a shared understanding*, we looked for task statements that required students to explore and make sense of an idea that had been introduced by one or more students. We also looked for instances in which students exhibited an effort to understand such an idea (e.g., by asking a question) or did mathematical work with the idea (e.g., approximating a root using a method they did not develop). Finally, we constructed chronological narratives for each of the instructional tasks, highlighting activity related to the IOI principles. We chose to construct these narratives chronologically in order to illustrate (and better understand) how the IOI principles interact over time to support students' progressive mathematical activity.

21.3 Classroom Inquiry: From the Bisection Method to Least Upper Bounds

The first major task of the instructional sequence is to develop an approximation method for finding the root of a continuous function that has a sign change. The students worked individually, then in pairs, and finally in small groups to develop approximation methods. Each small group created a viable method. Two groups created different versions of what we referred to as "decimal expansion" methods and one group created what we referred to as the "bisection" method. We have consistently seen both of these methods in our research, and our usual approach has been to eventually focus on the bisection method because it provides a bit better support for subsequent proving activity. However, it has been our habit to have all of the students develop some understanding of both methods before selecting one to focus on for the rest of the instructional sequence.

21.3.1 Developing a Shared Understanding of the Two Approximation Methods

As we observed above, the IOI principle of *building on student contributions* must be realized in tandem with the principle of *developing a shared understanding*. There are two powerful reasons for this. First, as Kuster et al. (2018) argue, developing a shared understanding helps the instructor to provide an equitable and inclusive learning environment for the students. Second, as a practical matter, students need to have ownership of a given idea in order to further develop it mathematically, so developing a shared understanding facilitates the process of building on student contributions. This is particularly true of the bisection method (Fig. 21.1) in our instructional sequence because it is leveraged as a starting point for the development of a number of important concepts.

In order to develop a shared understanding of the approximation methods, we asked each student individually to "sketch a generic continuous function and illustrate the first 3 steps and the n-th step". They were asked to do this using both a bisection and a decimal expansion method. We then selected a few students to

To keep track of which iteration we are on, let n be the step number. For the n-th step do the following:

Average a_n and b_n and call it $c_n = \frac{a_n+b_n}{2}$.

Find $f(c_n)$.

If $f(c_n) > 0$,

c_n becomes b_{n+1}

If $f(c_n) = 0$,

you have the root exactly.

Otherwise,

c_n becomes a_{n+1}.

Add one to n.

Repeat until $f(c_n) = 0$ or a_n and b_n are both the same to at least the desired number of digits. Your approximation c_n is then correct to that many digits.

Fig. 21.1 Description of the bisection method as recorded in the course Wiki-Text

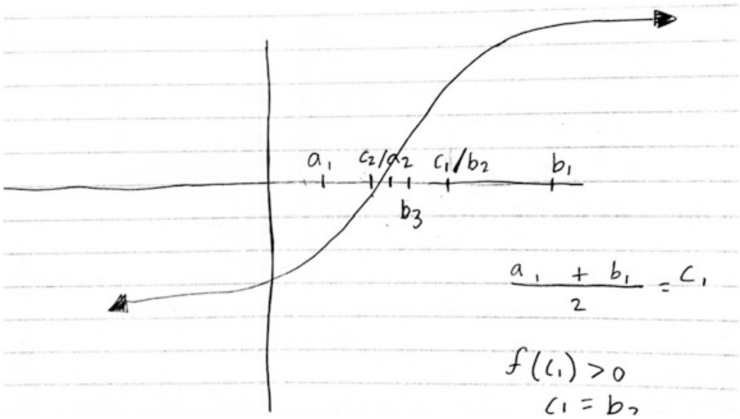

Fig. 21.2 A student's illustration of the bisection method

share their drawing with the whole class. One of the shared drawings for the bisection method is shown in Fig. 21.2.

This individual task and the subsequent discussion of the drawings provided each student a chance to develop their own understanding of the bisection method and provided the classroom community the opportunity to develop a shared image of how the method worked. Subsequent tasks provided further opportunities for this shared understanding to develop as it was leveraged to develop more advanced and formal mathematics.

21.3.2 Connecting the Approximation Method to Formal Mathematical Language and Notation

Besides developing a shared understanding of the approximation method, the task of illustrating each method had the additional goal of beginning the process of mathematizing the bisection method using the sequence concept. The drawings (as seen in Fig. 21.2) naturally used notation consistent with sequence notation. The instructors leveraged this aspect of these student artifacts to connect the students' approximation methods to the sequence concept along with its formal definition and conventional notation.

First, Sean (Instructor) references the pictures and makes the observation that the bisection method seems to generate some infinite lists of numbers. He then asks the students, "Does anyone know what the word is for a list of numbers that goes on forever? It's a thing you studied in calculus." The students in our class were quick to make the connection to sequences. We then informed the students that formally, a sequence is defined as a function. In an effort to support all of the students connecting the formal definition of sequence to their bisection approximation method, we asked them to think about what the domain and codomain of a sequence would be when conceptualized as a function:

Tenchita (Instructor): Formally, we like to define a sequence as a function. So, my question for you is, does that make any sense to define a sequence as some sort of function? [. . .] if we look back at our definition of a function, right, it went from, it had some domain, codomain, right? And can we connect that at all to our sequences?

Maya: So, our domain is the steps like the natural numbers as we're stepping forward. And we're mapping that to each individual value in this sequence.

Tenchita (Instructor): Yeah, exactly [. . .] like Maya was saying, the n is kind of our input value, right? It's the thing that we're saying, assign one to a_n. Right? So, we're like plugging in, if you're thinking about a function, the natural number, and your output will be what [. . .] the n th term of that sequence is.

In connecting the students' bisection approximation method to the formal concept of sequence we provided them with language and notation that they could use to further mathematize their informal mathematical activity. We immediately leveraged and reinforced this by asking the students to identify and name (using sequence notation) several specific sequences that would be generated by iterating the bisection method. The resulting list of sequences included the left and right endpoint sequences (denoted as a_n and b_n), the midpoint sequence (denoted as c_n), the corresponding output sequences (denoted as $f(a_n)$, $f(b_n)$, and $f(c_n)$), and the sequence of interval lengths (denoted by $b_n - a_n$).

21.3.3 Eliciting Student Reasoning: Conjectures About Sequences Generated by the Bisection Method

While principles of *eliciting student reasoning* and *building on student contributions* are conceptually distinct, in practice the same instructional task will often reflect both principles. This is a natural consequence of the RME idea of progressive mathematization in which students' mathematical activity in response to a given task will subsequently function as a new starting point for further mathematizing. This is the case with a key task of our real analysis instructional sequence in which we ask students to make conjectures about the various sequences they identified as being generated by the bisection method. This task builds on the students' previous contribution (specifically their approximation method). However, the task also is meant to elicit new reasoning about how these various sequences behave – reasoning that is subsequently built on to develop several key real analysis concepts. In Fig. 21.3 we see a list of conjectures as recorded in the course Wiki Text.

Notice that several of these conjectures assume the existence of a root. One of the first things we do to build on these student contributions is to remind them that our overall objective is to (like Cauchy) leverage our approximation method to prove the existence of a root. We then ask them to generate some new conjectures (inspired by Conjecture 6) that do not assume the existence of a root. In response to this task, the students in the course generated some additional conjectures including one that simply asserted that a_n will always converge.

Let's Recall our Conjectures From Last Time edit

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous function that has a sign change.
Suppose you apply the bisection method with a_n being the sequence of left endpoints, b_n being the sequence of right endpoints, and c_n being the sequence of midpoints.

Then...

1. Sequence $f(c_n)$ converges to zero
2. Sequence $f(a_n)$ converges to zero
3. Sequence $f(b_n)$ converges to zero
4. Sequence $b_n - a_n$ converges to zero
5. If $b_n - a_n = 0$ then we found the root
6. a_n will converge to the root
7. b_n will converge to the root
8. c_n will converge to the root
9. $f(a_n) > 0$ for all n
10. $f(b_n) < 0$ for all n
11. Sequence of left endpoints, a_n, will always increase (never decrease)
12. Sequence of right endpoints, b_n, will always decrease (never increase)

Fig. 21.3 Screenshot of the wiki-text page documenting conjectures

21.3.4 Building on Students' Ways of Reasoning: General Conjectures About Sequence Convergence

The development of a proof of the convergence of the left endpoints is an important step in constructing a proof of the IVT that is based on the bisection approximation method. Further, the development of such a proof supports students in reinventing a general theorem (e.g., Monotone Convergence Theorem) that motivates the development of the least upper bound concept and the completeness axiom. Thus, we were interested in *building on students' ways of reasoning* about the left endpoint sequence to formulate conjectures at a higher level of generality. Specifically, we asked students to develop conjectures of the form "If __________, then the sequence x_n converges" where the hypotheses were to be selected from among the properties the sequence of left endpoints (a_n) was known to possess.

We approached this task in two phases to support the students in drawing on their knowledge of the context and in making the shift to more general activity. First, we sought to elicit student ways of reasoning by asking the students to list reasons why they think the sequence of left endpoints converged to a real number. They worked in three small groups, recording ideas in a Google Doc. A subset of the (unedited aside from math typesetting) ideas that two small groups recorded appears in Table 21.1.

Table 21.1 Nine student ideas about why the left endpoint sequence converges

	Student ideas about why left endpoint sequence converges
1	Reminds me of Zeno's paradox goes half way then half way are you ever going to get where you are running? As limit goes to infinity it will converge to b_n
2	We are adding the geometric series (at most) to the first term which converges. With the max distance it can travel being $b_1 - a_1$
3	Because is stuck between two values and is only increasing in value so it must converge on some value
4	b_n is never increasing and a_n is never decreasing and they won't pass each other so they have to converse on something
5	b_n and a_n are getting closer together so they must converge
6	a_n are increasing
7	Because a_n keeps same value or increase it will approach the point that is the root
8	What we know is that our a_n 's and b_n 's take on the form of c_n. If we apply the bisection method as $\lim\limits_{n\to\infty}$ our midpoints (c_n) will converge to a number. Since c_n must converge to a number, our a_n sequence must also converge to a number
9	Since $f(a_n)$ is always positive then a_n converges to a number

Table 21.2 A collection of the unique generalized conjectures

	General conjectures generated
1	Let x_n be a sequence of real numbers. If limit of $x_{n+1} - x_n$ is 0, then x_n converges
2	Let x_n be a sequence of real numbers. If the sequence is increasing and bounded above, then x_n converges
3	Let x_n be a sequence of real numbers. If there exists another sequence y_n such that the distance between them ($y_n - x_n$) converges to zero. Then x_n (and y_n) must converge
4	Let x_n be a sequence of real numbers. If x_n increases, then x_n converges
5	Let x_n be a sequence of real numbers. If x_n increases and there exists another sequence y_n such y_n decreases. Then if x_n remains less than y_n, then x_n converges
6	If x_n is equal to another convergent sequence sometimes then it converges
7	If x_n gets closer to another sequence y_n then it converges
8	If x_n is increasing then converges
10	If x_n is equal to a geometric sequence then it converges
11	If x_n is adding a convergent series then it converges

After this initial brainstorming task, we initiated a shift toward more general activity by asking the students to build on their ideas to create *general* conjectures of the form "If ______________ then x_n converges". Notice that the task statement subtly reinforced the idea that we are looking for a general conjecture by using the notation x_n instead of a_n which directly references the left endpoint sequence. Table 21.2 contains a complete (removing duplicates) list of the general conjectures developed by the groups.

Again, we will see that this task not only served to build on the students' previous reasoning but also to elicit ways of reasoning that would subsequently be built on in order to generate some important concepts and theorems.

21.3.5 Building on Students' Ways of Reasoning: Investigating the False General Conjectures

To further *build on the students' contributions* in a way that would further the mathematical agenda of the course, we chose to follow-up on the first four conjectures in Table 21.2. Investigating these four conjectures provided the class with the opportunity the engage with the three most common ideas we observed in the small group discussions (so we see the principle *of developing a shared understanding* in action as well).

The first idea was to focus on the monotonicity of the left endpoint sequence (Conjecture 2 and Conjecture 4). This was the idea we intended to ultimately pursue. The second idea (Conjecture 1) was inspired by the fact that the first differences of the left endpoint sequence decrease to zero. This idea does not quite work (the partial sums of the harmonic series provide a counterexample). The third idea (Conjecture 3) was inspired by the fact that the two endpoint sequences are approaching one another. While Conjecture 3 is untrue as stated, the idea can be refined to generate either a monotone convergence theorem or something like the Nested Interval Theorem.

We initiated the process with a brief whole class discussion to establish that the fourth conjecture was not true because any unbounded increasing sequence would provide a counterexample. Then in small groups, the students were asked to consider whether the first and third conjectures were true and to generate counterexamples if they were not. Here, we briefly share some of the discourse about Conjecture 3 from one of the small group discussions.

As soon as her entire group was in the Zoom breakout room, Tenchita (Instructor) asked, "so, what do we think?". One of the students, Emily, replied immediately saying, "Well, the $y_n - x_n$ [Conjecture 3] implies one is positive and one is negative. And it's implying that the difference between the two is zero. I think it's true, I can't come up with a counterexample." This response reflects the cognitive difficulty in navigating the two levels of generality at play. In the context of the bisection method, the two sequences that satisfy the hypothesis of this conjecture are a_n and b_n which have the further property that one generates only negative function values and the other generates only positive function values. It is this property (ultimately not relevant to the conjecture) that Emily is referencing when she says one is positive and one is negative. In addition to possessing this distracting property, these two specific sequences are also bounded and non-oscillating so they do not exhibit the kinds of behaviors that a counterexample would need.

After a brief discussion to help Emily navigate this shift to more general activity by discussing what the conjecture does *and does not* say one can assume about the sequences x_n and y_n, another student, Eduardo, suggested a clever trick for generating a counterexample. He asked, "Why don't you let both sequences be the same?" Then a third student, Dylan, elaborated saying, "if you just say the n th term is n. Those are both, that is a sequence that just grows forever, and definitely does not converge. But if you did do this conjecture and say, you know $y_n - x_n$, then you get zero every time." Tenchita (Instructor) followed this up by helping the students generate more complex examples (by adding $1/n$ to different kinds of non-convergent sequences).

Conjecture 1 was more challenging to address using the students' own ideas because the most accessible examples involve infinite series. Specifically, the most obvious (based on students' prior experiences) counterexample is the sequence of partial sums of the harmonic series. In this course, we were content to *present* this counterexample to the students. While we cannot guarantee all of the students fully understood this counterexample, one student was able to observe that it could be contrasted with a_n which can also be seen as a sequence of partial sums, but one that is clearly bounded.

21.3.6 Building on Students' Ways of Reasoning: Investigating the True General Conjecture

After dealing with the false conjectures, we turned the students' attention to the increasing and bounded conjecture [Conjecture 2]. The goal was to build on this student contribution to eventually develop the concept of least upper bound and a proof that an increasing bounded sequence would converge to its least upper bound. This required us to elicit students' ways of reasoning about why such a sequence would converge *and* what its limit would be.

First, we asked them *why* a sequence must converge if it is increasing and bounded above. After giving them a couple of minutes to think about this, we then asked the students a different, more specific, question: "If x_n is increasing and bounded above, what does it converge to?" They were given a few minutes to discuss this at the end of a class session, and then asked to respond to it as part of a homework assignment. Notice that by asking the students to generate responses individually, we were setting the groundwork for establishing a shared understanding of the ideas we would subsequently pursue, but ensuring that each student had an opportunity to develop their own ideas first.

Ten of the fourteen students responded in their homework by saying that the sequence would converge to "the" upper bound. This student contribution provided an opportunity for us to elicit the new idea of a least upper bound. We started the next class period with a poll in which we asked the students about the sequence $n/n + 1$. The students were asked to choose between:

(a) a_n is bounded above and 1 is THE upper bound
(b) a_n is bounded by ANY number greater than or equal to 1, and
(c) a_n is not bounded since n goes to infinity.

One student selected (c), while seven selected (a) and six selected (b). We followed up the poll with a whole class discussion. Sean (Instructor) asked if someone who selected (a) could say why. A student, Leo, responded saying, "I put option one, because you can never actually get to one. So, it can't actually be greater than one, right? Because ... if the denominator is always $n + 1$ you'll never actually reach." Notice that this response explains why one is an upper bound, but it does not explain why the numbers larger than one are not.

Sean (Instructor) responded by asking if a different student could engage with Leo's idea by explaining, "why isn't two an upper bound?". Emily responded saying, "It can be and that's why there's an option two 'cause it could be anything. You just know that it doesn't pass one. One is just the *smallest possible bound*."

These two student contributions provided us with the opportunity to accomplish two goals. First, they provided a chance to establish a shared understanding of the idea that a bounded sequence will have many different upper bounds. Second it provided a chance to elicit the idea that one of the upper bounds might be special in the sense of being smaller than all of the others. We capitalized on this opportunity first by calling attention to the distinction between the two students' contributions:

> Sean (Instructor): So, you want, we want to kind of distinguish between those two. And so, I can tell you what the official math answer to this question is, is option two, because by upper bound, we don't require it to be the best one. And so, any number bigger than one- one or bigger, would be considered an upper bound. There's a whole extra special concept about a smallest upper bound, which I would not be surprised if we spent a bunch of time talking about that later today.

We then opted to attempt to build on Emily's contribution by revisiting the question of the limit of an increasing bounded sequence. We solicited ideas and one of the students, Will, said that it would be the lowest of the upper bounds. All of the students indicated agreement in a whole class poll intended to support the development of a shared understanding. At this point, another student, Amelia (the one student who had taken a real analysis class before), remembered that the standard term was *least* upper bound. As instructors, we then *acted as brokers* between the math community and the classroom community to confirm that "least upper bound" was the normative name for this concept.

To further formalize the concept of least upper bound and to develop a shared understanding of the idea, we then worked with the students to write a definition. Again, acting as brokers, we provided the students with a definition stem that involved normative notation for least upper bound (i.e., Let x_n be a sequence. α is the *least upper bound* of x_n if...). In a brief whole class discussion, we solicited initial student ideas and then Tenchita (Instructor) proposed constructing a two-part definition where the first part asserts that α is an upper bound and the second asserts that it is the smallest upper bound. Several students suggested formalizations of the first part, including one who used the Zoom chat feature to propose, "for all n, α is bigger than x_n" which is a standard formulation aside from the inequality apparently being strict.

In an attempt to formalize the second part of the definition a student, Eduardo, made a viable suggestion asking, "Can you make it into a set of alphas and just saying that ... the first, the alpha naught is the one we're looking for? Alpha naught being the lowest one." Aside from the implication that the set of upper bounds can be listed in order, this suggestion does capture the required condition and could be built on to finish formalizing the definition of least upper bound. However, instead of creating a formalization, we opted to proceed with a definition that stated the second condition simply as "α is smaller than all the other upper bounds". Because the next phase of the instructional sequence involved leveraging this definition to prove that an increasing bounded sequence converges, we expected that if there was a need to further formalize this definition it would emerge at that time and could be addressed then.

21.3.7 Generating Student Ways of Reasoning: Brainstorming Why *the Least Upper Bound Will Be the Limit*

Or next aim was to build on the students' contributions (the increasing and bounded conjecture and the idea of a smallest upper bound) to prove a theorem that would guarantee the convergence of the left endpoint sequence. We began by generating student ideas about why an increasing and bounded sequence must converge to its least upper bound. Students were asked to develop and share this reasoning first in small groups and then in a whole class discussion. As the discussion below illustrates, this activity can generate useful informal ideas that can be built on to generate a formal proof. The following is a portion of the whole class discussion.

Leo: . . .if the sequence passes the bounds, then it's not truly bounded above for the definition that we've come up with at least. Tell me where I'm wrong.

Sean (Instructor): Well, the good news is we don't have to have a proof at this moment now, because we actually can't prove it yet until we do another thing. So, this is just us coming up with our initial ideas of why we think it's true. So,

Tenchita (Instructor): So one thing you did say is that if it goes past it, that means that the sequence isn't bounded above, does that necessarily mean it's not bounded above? Or what does that mean about the least upper bound?

Joseph: It means that *that* wasn't a bound.

[Having established that the sequence cannot go beyond the least upper bound, we then built on this reasoning by asking if there is anything else needed to ensure the sequence will converge to the least upper bound.]

Tenchita (Instructor): Is that the only thing we need? For it to not go past?

Emily: It can't stop before.

Tenchita (Instructor): Por que? why?

Emily: Because if stops before it. . .there could be still something between that point where it stopped and where you selected that bound to be.

Sean (Instructor): And that bound is supposedly, is the smallest upper bound.

Emily: Exactly. So how can it be the smallest if it's stopping here [gesturing to suggest a sequence stopping short of the supposed smallest upper bound]? There's all this space!

Notice that through this discussion, the instructors (1) encouraged students to generate and share their reasoning, (2) built on that reasoning by using it to direct the conversation, and (3) connected their reasoning to the formal concepts they had discussed earlier (i.e., least upper bound). While students still have significant work to do to create a formal proof of the MCT, we see this discussion as critical step in building on students' ideas so that they are connected to more formal ways of reasoning that the students can use to create the proof. In subsequent lessons, the classroom community developed a definition of limit for sequences and eventually used it to capitalize on these informal arguments to produce a formal proof of the increasing and bounded conjecture on the way to proving the IVT.

21.4 Conclusion

This chapter shows how taking an IOI approach to real analysis can introduce students to the foundational topics in a way that is driven by the students' activity. In particular, our analysis illustrates the way that the four principles of inquiry-oriented instruction support a process of progressive mathematizing. For example, the students' approximation methods were *connected to standard mathematical language and notation* by identifying how the artifacts (list of numbers) produced when carrying out the methods connect to the formal idea of sequences. We then *generated student ways of reasoning* about these sequences by asking the students to develop arguments for why the sequence of left endpoints converges. The subsequent task *built on these student contributions* by having students transform their ideas into conjectures about general sequences; ultimately leading to a conjecture resembling the Monotone Convergence Theorem. Throughout this process, we engaged students in critically thinking about one another's ideas to *develop a shared understanding* of the concepts. As a result, the classroom community was able to reinvent significant mathematical ideas (e.g., MCT) while engaging in important advanced mathematics practices including conjecturing, defining, generalizing, and justifying. The students were also able to make substantial progress toward developing a proof of the Intermediate Value Theorem inspired by Cauchy's historical proof.

It is not a novel concept to build on student thinking when teaching real analysis concepts. For instance, Cory and Garofalo (2011) designed an instructional intervention in which students used a computer program that supported them in making sense of epsilon-*N* relationship of sequence convergence. This intervention leverages a tool developed by Roh (2010) that aims to confront and build on common student conceptions of sequence convergence. However, we argue that the IOI approach represented by the four IOI principles (and illustrated by our analysis) goes further by positioning the students as responsible for the mathematical activity in a way that promotes their ownership of the mathematics being developed. As Gravemeijer and Doorman (1999) suggest, an RME approach to instruction (in our case in real analysis) goes beyond bridging "the gap between their [students'] informal knowledge and the formal mathematics", to "transcend this dichotomy by aiming at a process in which the formal mathematics emerges from the mathematical activity of the students." (pp. 115–116).

In summary, the four principles are at the core of an RME approach to real analysis and when enacted together combine to produce an instructional approach that supports the classroom community in engaging in collective authentic mathematical activity (an important goal in its own right) that results in the production of the advanced mathematics targeted by the curriculum. It is a considerable challenge to create an educational experience that provides all of the students with an opportunity to engage in legitimate mathematical activity while also meeting expectations for content coverage. The four principles of IOI provide important support for instructors who are motivated to take on this challenge.

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Chapter 22
Examining the Role of Generic Skills in Inquiry-Based Mathematics Education – The Case of Extreme Apprenticeship

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Abstract Generic skills are considered important in learning and for employment, but these skills are rarely explicitly taught or even mentioned as learning objectives. In this chapter, we study generic skills in Inquiry-Based Mathematics Education (IBME). We investigate how the presence of generic skills has evolved during the development of an inquiry-based teaching model in university mathematics, called Extreme Apprenticeship. We use a historical approach to study the course descriptions and learning objectives of 64 different implementations of courses taught with the Extreme Apprenticeship model since its conception. The data reveals that the variety of generic skills mentioned as learning objectives has gradually become more enriched and they have assumed a position equal to content skills. We describe how the nature of Extreme Apprenticeship has supported these changes. Based on our analysis, we infer that some of the driving forces behind the changes are professional development of teachers, development of teaching and assessment methods, and close collaboration among teachers involved with Extreme Apprenticeship. Also, the specific nature of mathematics, such as the emphasis on proving, has shaped the way generic skills are incorporated in the learning objectives.

Keywords Generic skills · Inquiry-based mathematics education · Extreme apprenticeship · Tertiary mathematics education · Historical method in education research

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22.1 Introduction

University students are expected to develop not only disciplinary knowledge and skills, but also several generic skills, such as analytical, communication, teamwork and problem-solving skills (e.g. Chan et al., 2017; van Dierendonck & van der Gaast, 2013). Generic skills are important in any discipline and needed in studies as well as in working life (Strijbos et al., 2015; Tuononen et al., 2019). However, there is evidence that university students develop more content knowledge than generic skills (Stiwne & Jungert, 2010; Monteiro et al., 2016; Tynjälä et al., 2006), and generic skills are rarely explicitly taught or even mentioned as learning objectives (De La Harpe et al., 2000). There is often a lack of consistency between beliefs of importance of generic skills and the degree to which they exist in teaching (Chan et al., 2017; Jones, 2009). This may be due to teachers' focusing on teaching of content knowledge and not realising their role in developing students' generic skills (Barrie, 2006; De la Harpe et al., 2000).

The development of generic skills can be promoted by different pedagogical practices. It is important that generic skills are mentioned in learning objectives and taken into account in assessment (Crebert et al., 2004; Hyytinen et al., 2019; King et al., 2017). Many studies have shown that collaborative learning environments promote the development of generic skills (Ballantine & Larres, 2007; Choi & Rhee, 2014; Knipprath, 2017; Virtanen & Tynjälä, 2019). In mathematics, one framework of providing such learning environments is Inquiry-Based Mathematics Education (IBME; Laursen & Rasmussen, 2019; Artigue & Blomhøj, 2013). Indeed, Laursen and Rasmussen (2019) have suggested that IBME offers rich opportunities to teach generic skills alongside mathematical content, and they advocate for more research in this direction.

To facilitate the adoption of generic skills in university curricula, more research is needed on how generic skills can be implemented in the teaching of different disciplines, as well as what affects teachers' decisions to include or omit generic skills in their teaching. In this chapter, we take a historical approach (Berg, 1998; Sáez-Rosenkranz, 2016) to study one case of incorporating generic skills in university mathematics education through IBME, the Extreme Apprenticeship method (XA; Rämö et al., 2020). The study is based on an analysis of course syllabi and learning objectives written by the teachers. Learning objectives offer a fruitful point of view to generic skills research, as curriculum has a central role in the learning of generic skills (Leung et al., 2014). Our intention is to describe the gradual change in the presence of generic skills in the course descriptions, infer potential reasons behind teachers' choices to include these generic skills, and analyse how the generic skills are affected by the inquiry-based nature of the teaching. The results can be used to guide university teachers – especially mathematics teachers – developing new teaching methods and desiring to include generic skills in their courses.

22.2 Generic Skills and Their Role in Mathematics Curricula

Many countries and universities have implemented competence-based education, and competences and generic skills have been included in curricula as learning objectives (Clarke, 2017). Therefore, in addition to disciplinary knowledge and skills, learning of generic skills, such as analytical, communication, teamwork and problem-solving skills, is a key aim of university education. The term generic refers to the idea that these skills are common to all disciplines, although there is variation in skills that are emphasised in different disciplines (Barrie, 2006). However, it can be argued that although different disciplines emphasise different competences, there are still more similarities than differences between disciplines (Krause, 2014). Several other terms are used to refer to generic skills, such as key skills, transferable skills, employability skills, core skills, academic competences and generic attributes (Barrie, 2006; Tuononen, 2019). A coherent definition of generic skills does not exist, but there are various kinds of lists of generic skills (Barrie, 2006). These lists vary from simple technical skills to complex intellectual abilities and ethical values (Barrie, 2006), and there are discipline-specific lists as well as lists created at different universities and in different countries (Badcock et al., 2010; Jones, 2009). For example, Virtanen and Tynjälä (2019) have explored generic skills such as creativity and innovation, critical thinking, decision making, learning skills and self-assessment skills.

Attempts have been made to describe which generic skills are important in mathematics. In Australia, the generic skills students are expected to acquire in a mathematics programme include communication of mathematical knowledge to experts and non-experts, ethical conduct of mathematics, quantitative problem-solving, writing skills and teamwork skills (FYiMaths, 2013, as cited in King et al., 2017). The Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM) recommends that every college-level mathematics course should support students in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind (Barker et al., 2004). In Hong Kong secondary education, critical thinking, creativity, collaboration and communication skills, information technology skills, numeracy skills, problem solving skills, self-management skills and study skills are the generic skills that are expected to be developed (Leung et al., 2014).

Mathematics students, graduates and employers think that generic skills are important but find that they do not develop well enough by studying mathematics (King et al., 2017; Inglis et al., 2012; Rayner & Papakonstantinou, 2015). Literature suggests several means to teaching generic skills, such as various teaching methods, using different kinds of tasks, combining theory and practice, and collaborative learning as well as working alone (Virtanen & Tynjälä, 2019). Generics skills should be integrated in domain specific courses, and not taught in separate courses (Hyytinen et al., 2019; Virtanen & Tynjälä, 2019). Since generic skills are lacking from university mathematics courses, it would be valuable to find ways to implement these suggestions in university mathematics education.

22.3 Extreme Apprenticeship, a Form of Inquiry-Based Mathematics Education

Inquiry-Based Mathematics Education (IBME) encompasses two strands of active learning and teaching (Laursen & Rasmussen, 2019): Inquiry-Based Learning (IBL; e.g., Laursen et al., 2014; Yoshinobu & Jones, 2011) and Inquiry-Oriented Instruction (IOI; e.g., Kuster et al., 2018; Rasmussen & Kwon, 2007; Wawro et al., 2012). IBME is built upon four central ideas: students engage deeply with meaningful mathematics, students collaborate to make sense of mathematical ideas, teachers inquire into the thinking of students, and teachers foster equity in their classrooms.

The form of IBME discussed in this chapter is a teaching model called Extreme Apprenticeship (XA). It was originally created in the University of Helsinki in 2011 for teaching computer science students (Vihavainen et al., 2011), but was soon developed to suit also mathematics teaching (Hautala et al., 2012; Rämö et al., 2019). The theoretical background of XA is in situated view on learning (Lave & Wenger, 1991) and Cognitive Apprenticeship (Collins et al., 1991). In the XA model, students take part in activities that resemble those carried out by professionals, which supports them in becoming experts in their field (Rämö et al., 2020). In the case of mathematics, expert skills that have been emphasised are, for example, reading and writing mathematical texts and participating in mathematical discussions. Students start studying a new topic by solving introductory problems together with each other and with the guidance of the teaching team. They receive feedback on their work from the teaching team. After that, the topics are discussed in the lectures in order to build a bigger picture and deepen understanding. After the lectures, students are given more challenging tasks, and at the same time they start studying new topics via new introductory tasks. The teaching team consists of the responsible teacher and tutors who are undergraduate or graduate students. The XA model has been used in both small and large scale, the student cohorts varying from 10 to 500.

In the XA model, the central themes of IBME are addressed in several ways (Rämö et al., 2020). The task sequences given to the students are designed carefully so that they support deep understanding. An open learning space has been created, in which students can discuss and collaborate with each other and course tutors. The tutors take part in pedagogical development that teaches them how to inquire into the students' learning. Finally, sets of explicitly shared norms support equal participation, as does the open, accessible learning space. In order to find a meaningful way of assessment supporting active and inquiry-oriented learning, self-assessment has been included in the teaching of many of the XA courses in the form of the DISA model (Digital Self-Assessment; Häsä et al., 2019, 2021). The students perform self-assessment exercises in which they evaluate their own skills against the learning objectives and receive feedback on their assessment.

22.4 Research Problem and Hypotheses

This study takes a close look at how the presence of generic skills has evolved in the course descriptions during the development of a particular IBME type teaching model. The purpose is to describe the process of incorporating generic skills to the courses, as well as to bring to light the most likely factors behind the process. The focus is on the point of view of teachers who are developing the model: what generic skills are they including, in what form and in what order? What influences the teachers' decisions at each point of the development?

To offer answers to these questions, we examine the learning objectives as well as other course descriptions written and published each year since the beginning of the XA model. We study the materials in their context, taking note of the timing of each change and relating them with major steps in the development of the model. The written learning objectives form a good source of information for the XA model, as they have been paid special attention to by the teachers and are described in detail in the courses under study. In the context of XA, the learning objectives are created by the teachers and can be considered to reflect the intentions of the teachers developing the model, as opposed to external requirements dictated by the department or programme of study.

Based on earlier literature on generic skills in university education as well as the nature of IBME and the XA teaching model, we expect to find support for the following ideas. Firstly, since IBME style learning environments are theorised to support the teaching of generic skills, mentions of these skills should arise during the development of XA. In particular, we expect to see skills such as reading mathematical texts and participating in mathematical discussions, as these are emphasised in the philosophy of XA. Secondly, as the teachers are involved in the development of a new teaching model, they are actively revising their course descriptions, learning objectives and teaching methods of their courses. Therefore, we expect to see improvements in how the learning objectives are formulated and how objectives and methods are aligned in the courses. Thirdly, as generic skills are integrated into mathematics courses, we expect them to reflect certain characteristics of mathematics, such as a strong focus on problem-solving and a particular mode of communication, mathematical proof.

22.5 Method

The historical method involves a systematic process examining past events or phenomena in order to create an account of what happened in the past (Berg, 1998). The events are usually accessed via written documents, called sources. The aim is not merely to collect information and describe the events, but to search for causes and offer explanations to them. According to Sáez-Rosenkranz (2016), the historical method can help to characterise education systems or study the development of education cultures, and it can be applied when the aim is to explain the causes of observed phenomena in their social context.

The research process in this study is based on the processes of historical research outlined by Berg (1998) and Sáez-Rosenkranz (2016):

1. Specification of the research problem
2. Building a theoretical framework and developing hypotheses
3. Collection and critical observation of sources
4. Organisation and analysis of sources
5. Building a narrative synthesis.

First, the research problem was chosen and narrowed down in light of scientific interest as well as available source material. Then, literature on generic skills and the XA model was studied, and tentative hypotheses discussed. Source materials were collected from websites and dated. As some sources consisted of wiki pages and other changeable documents, edition histories were inspected to confirm dating and authenticity of documents. Missing sources (e.g. removed web pages) were noted.

The analysis phase began by preparatory inspection of sources in three parts:

- The course websites were examined for mentions of generic skills, learning objectives and teaching and assessment methods, and the temporal development of these was recorded on a timeline, noting any major and minor changes in content and wording.
- Similarly, the learning objectives mentioned in the course websites (usually in the form of matrices) were examined, recording any major and minor changes in content and wording.
- Learning objectives matrices of all XA courses from two time periods, the beginning of their use and the most recent study year, were analysed using content analysis guided by the theoretical framework. The generic skills were coded based on the previous studies (Leung et al., 2014; Virtanen & Tynjälä, 2019). After finding the range of generic skills, the overall change of generic skills during the years was analysed.

Following the preparatory inspection, the authors met to discuss their findings. At this point, four subthemes were selected for further focusing: (1) which generic skills are present in the learning objectives and how does this change in time (2) how the generic skills are presented (how prominent they are, how they are worded etc.) (3) how the generic skills objectives are aligned with other parts of teaching and assessment, and (4) how the generic skills objectives of different courses are related. Then the sources and prepared material were consulted again in the light of these subthemes, and narratives were composed to describe and explain the developments under each subtheme. The four narratives were discussed among the authors and developed iteratively. Finally, conclusive remarks were based on the resulting narratives, as well as the theoretical framework, to offer a response to the research problem.

The first and second authors of this manuscript have used XA in their teaching and taken part in developing the teaching model. Their expertise was used when inferring potential reasons behind teachers' choices. They could also act as additional sources for some events. The third author is a researcher of generic skills and transition to working life. She guided the content analysis of learning objectives and acted as an external corroborator to reduce bias.

22.6 Context and Sources

The focus university of this study is a Finnish research-intensive university. Finnish university studies typically last five years with three years of Bachelor's studies and two years of Master's studies. Students declare a specific major when they enter the university, and their studies focus heavily on their chosen discipline from the beginning. Typical majors for students who take mathematics undergraduate courses are mathematics (including teacher education), computer science, economics and statistics.

The source data for the study consists of course websites and what is linked to those pages, in particular the written learning objectives. In the focus department of this study, there has been very little programme level control or planning of the learning objectives of individual courses. Since 2017, the study programmes have set learning objectives for each course, but objectives have been concise and written at a general level. In particular, there has been no programme level requirement of including generic skills among the learning objectives. All this means that individual teachers of the courses have been able to write more detailed learning objectives and use them in their teaching. The learning objectives we use as data in this study are the learning objectives written by the teachers of XA courses. The sources of our study can be considered primary sources, as they were produced by the teachers themselves at the time of teaching.

Data were gathered from courses that were taught with the XA model since its adoption to teaching mathematics in 2011. They are described in Table 22.1. Apart from Algebra II, all the courses are undergraduate courses. The table contains abbreviations that are used for the courses in the subsequent sections.

In total, 64 different implementations of these courses were included in the study. Nine course pages no longer existed, and these implementations were excluded from the analysis (1 instance of IUM, 3 instances of LM1 and 3 instances of LM2). All these courses were summer courses offered by the Open University, and they had visiting teachers who could not be reached at the time of this research. The courses included in the study were taught in Finnish, and quotes presented in the following section are translated from Finnish. The only exception is the course A2 which was taught in English in 2017–2019.

Table 22.1 Description of courses included in the study

Course name	Abbre-viation	Level	Credits (ECTS)	Topics	Implementations included in the study
Introduction to university mathematics	IUM	Bachelor (BSc)	5	Sets, functions, proving	Autumn 2011–2019 Spring 2013–2019 Summer 2017–2018
Linear algebra and matrices I	LM1	BSc	5	Vectors of R^n, matrices, linear independence, basis, eigenvalues	Autumn 2011–2019 Summer 2013, 2016–2019
Linear algebra and matrices II	LM2	BSc	5	Axiomatic definition of a vector space, linear mappings	Autumn 2011–2018 Spring 2020 Summer 2013, 2016–2019
Algebra I (split into AS1 and AS2 in 2015)	A1	BSc	10	The first half of the course corresponds to AS1, and the second half to AS2	Spring 2011–2014
Algebraic structures I	AS1	BSc	5	Groups, rings	Spring 2015–2020
Algebraic structures II	AS2	BSc	5	Quotient groups, homomorphisms	Spring 2015–2019
Algebra II	A2	Master (MSc)	10	Group theory, commutative algebra, field extensions	Spring 2016–2019

22.7 Results

The results are organised under four subthemes: which generic skills are mentioned among the learning objectives, how the generic skills are presented, alignment of the generic skills objectives with teaching methods, and how the generic skills objectives of different courses are related.

22.7.1 Generic Skills as Learning Objectives

Generic skills that have been mentioned in the XA course descriptions over the years have varied. The following categories were identified among the learning objectives of XA courses: problem-solving, oral and written communication, group work, reading, innovativeness and creativity, information technology skills, giving and receiving feedback, and developing a mathematician’s identity. Table 22.2 illustrates

Table 22.2 Categories of generic skills that appear in course syllabi in 2013 and 2019. In 2013, three categories were identified among generic skills, whereas in 2019, nine categories were identified

Generic skills in 2013	Generic skills in 2019
Reading skills Written communication Oral communication	Reading skills Written communication Oral communication Information technology skills Collaboration skills Problem-solving skills Creativity Giving and receiving feedback Developing a mathematician's identity

the change in diversity that has happened over the years in generic skills, listing the categories of generic skills in 2013 and in 2019. It can be seen that the variety of generic skills has risen from three to nine categories. This implies that over the years, teachers have become more proficient in recognising and verbalising generic skills.

When the XA model was implemented for the first time in 2011, learning objectives concerning generics skills did not exist. In autumn 2013, the learning objectives for the courses IUM and LM1 were described in detail as learning objectives matrices, including also generic skills. As learning the kind of skills experts use is in the core of the XA method (Vihavainen et al., 2011), the first learning objectives emphasised skills that were considered important to expert mathematicians. They concerned reading skills (e.g. "Reads course material"), oral communication (e.g. "Is able to form precise questions for one's mathematical problems"), and written communication (e.g. "Writes solutions whose language and logical structure are so clear that another person can make sense of them"). Reading mathematical proofs is mentioned separately ("Reads proofs and is able to follow their logical structure"). Oral and written communication have had a prominent role among the generic skills throughout the years. Of all the skills, they have been mentioned most often. Coincidentally, these skills have been recognised as important in mathematics curricula also in other sources (FYiMaths, 2013, as cited in King et al., 2017; Leung et al., 2014). After the emergence of first learning objectives, matrices were created also for other courses, and the existing ones evolved slightly. In the autumn 2014, information technology skills were added to the courses LM1 and LM2. These learning objectives concerned mathematical programming (e.g. "Is able to make small alterations to the code in order to obtain desired results", "Is able to search for new commands").

A big step in the evolution of generic skills happened in spring 2016 when the XA model was implemented in a Master's level course A2. Because the course had a smaller number of students than undergraduate courses, group work could be easily incorporated into the teaching of the course and, for the first time, collaborative skills were mentioned (e.g. "Does one's own share in group work", "Takes the other members of the group into consideration"). The appearance of collaborative skills in other XA courses was not until the course AR1 in spring 2020. All this implies that the teachers found it difficult to include group work into teaching. Indeed, there is evidence from previous studies that group work and collaboration skills do not develop enough during studies (King et al., 2017; Tuononen et al., 2019), and should be taken into account more often in curricula and assessment (Challis et al., 2009; Inglis et al., 2012; Leung et al., 2014).

Another new category that emerged in the course A2 in 2016 was problem solving skills. These skills included, for example, linking concepts and coming up with new ideas in solving challenging tasks. They were added because the teacher of the course felt that it was difficult to describe advanced content skills in mathematics as all of them seemed to boil down to problem solving. Problem-solving skills are typical learning objectives in mathematics (Barker et al., 2004; Challis et al., 2009; Leung et al., 2014), so it is somewhat surprising that they did not occur among learning objectives prior to this. Maybe this was due to the fact that so much of mathematics is related to problem solving that it was difficult to "see the forest for the trees", that is, to distinguish it from the content categories. The generic skills in the course A2 included also creativity ("I can come up with proofs that require linking different concepts and creative thinking").

The next major evolution happened in summer 2017 when the learning objectives of the course LM1 were expanded significantly. For example, oral communication skills now included new skills such as "I have mathematical discussions in which I express my own thoughts and listen to other people's ideas" and "I am able to maintain a mathematical discussion that benefits both parties". Written communication was broadened to include skills related to writing proofs, such as "I define the variables I use in proofs". Also, a completely new category of skills, giving and receiving feedback, appeared when peer feedback was added among the teaching methods of the course. These skills concerned reacting to written feedback received from teachers and peers (e.g. "I do not take personally the feedback I have received but understand that it was given so that I could learn more", "I am able to operate in situations in which I receive contradictory feedback from different sources") and giving feedback to other students ("I give constructive peer feedback that aims to make the other students' work better").

"Developing a mathematician's identity" was a new category of skills that appeared in the course IUM in autumn 2018. These skills concerned the epistemology of mathematics (e.g. "I can explain the meaning of definitions, theorems and proofs in mathematical communication"), cultural symbolism (e.g. "I aim to adopt phrases used in mathematics") and motivation (e.g. "I am persistent when facing difficult problems"). Developing an identity is not something that is typically listed as a learning objective. It bears resemblance to "acquiring mathematical habits of mind" which the Mathematical Association of America recommends to be taught in mathematics courses (Barker et al., 2004).

The ability to give and receive feedback as well as developing a mathematician's identity relate closely to the development of expertise and self-regulative knowledge and skills (Tynjälä et al., 2016). It seems that workplace relevance and students' employability were at this point considered more than before, which implies that a new step in the evolution of generic skills was taken.

22.7.2 Communicating the Generic Skills

During the development of the XA model, the way generic skills are formulated in the course websites and in the learning objectives has changed. In the beginning of the model in 2011, the focus of course websites was on practical instructions and course content, and learning objectives with complete descriptive sentences were virtually absent. This was the tradition in mathematics course pages: precedence was given to practical information, such as lecture times and how to complete the course. Course content was given as a list of topics in the lower part of the page, as it was not thought to be referred to as often as the practical information.

Generic skills make a first appearance in the course page of LM1 in autumn 2012 as an add-on sentence after a list of mathematical topics: “In addition, the student will develop their skill in reading mathematical texts and practices to produce clear and well-structured solutions.” Soon, however, learning objectives were brought to the courses in the form of matrices, divided into “foundation skills” and “course topics”. These corresponded roughly to generic skills and content skills. Matrices themselves were given in separate pages, which enabled them to be moved to the top of the course page, and “foundation skills” appeared above “course topics”. This seems to convey a message that the “foundation skills” were considered something that underlay the learning in the course.

Teachers’ personal development shows in the way they gradually break away from tradition. First, the teachers realised the importance of including the learning objectives, and soon moved them to the top of the course page. However, the teachers struggled with the proper formulation of generic skills in the learning objectives and instead gave them in the form of practical instructions directly applicable to studying, such as “Familiarises oneself with the topic of the following lecture with the help of the course material, in order to get full benefit from the lecture.” It seems that the teachers of the XA model considered it important to tell students that learning in these courses consists of more than the mathematical content and procedures, but they did not yet know how to express these as objectives instead of instructions. On the other hand, this operationalised formulation may have been more approachable to students who were also not yet familiar with learning objectives.

The introduction of self-assessment was a major catalyst in the way learning objectives were expressed in XA courses. For successful self-assessment, it was essential that learning objectives were formulated in such a way that it would be easy for the students to relate their skills to the requirements of the course (Andrade & Du, 2007). Earlier, the objectives had been written in third person, but now they changed to first person, likely to assist with reflection. Also, the requirement levels of the objectives had been labelled as “approaching required skills” and “achieving required skills”, which had emphasised the learning path leading to course completion; now they were changed to grade levels “1–2”, “3–4” and “5”, to assist with self-grading. This in turn prompted the need for more careful analysis of different levels of skills and rewording of the objectives. This improved the quality of the learning objectives matrices, but on the other hand, the earlier idea of a learning path that takes place before the passing level was lost.

As the learning objectives matrices were being completely redesigned, it gave the opportunity for the teachers to pour in everything they had learned during the years since the first matrices were written. They had also gained confidence and were ready to embrace bold ideas and stride even further from tradition. The generic skills and course-specific skills were combined in the same matrix. Study skills and expert skills were properly combined: for example, the formulation “Participates in discussion in lectures and workshops” was replaced with “I participate in mathematical discussions, in which I express my own ideas and listen to others’ ideas”. Here, the objective is written as a generic skill that is useful also beyond university studies, and the expanded wording points out what is important in these discussions. However, the full generality is still curbed by the use of the word “mathematical”; this word could have easily been removed without real difference to the meaning.

An issue specific to mathematics concerns mathematical proving. Proving is an activity almost unique to mathematics. In terms of generic skills, it could be said to incorporate at least three skills: problem-solving, written communication and creativity (e.g. CadwalladerOlsker, 2011). In the learning objectives matrices, proving first appears as its own category in generic skills, including reading and constructing proofs. In 2017, this category disappears, and skills related to proving appear under a new heading “Reading and writing mathematics”. It seems that at this point, teachers have wanted to put special emphasis on the communication aspect of proving. However, with this choice, other aspects, such as problem solving and creativity may have become undervalued.

Throughout the development of the XA model, the evolution of generic skills in the course websites and objectives reflects the development of pedagogical competence and confidence of the teachers, as they gradually break away from tradition. The largest visible changes happened in two stages: the introduction of learning objectives matrices in courses and redesigning the matrices at the introduction of self-assessment. However, the teachers’ skills have clearly been developing in the meantime, as when the visible change happens, the result is much more sophisticated and developed than before.

22.7.3 Interplay of Objectives, Methods and Assessment

The concept of alignment in course design refers to choosing the teaching methods so that they support the learning objectives and basing the assessment of learning outcomes on the same objectives (Biggs & Tang, 2007). Analysis of the source material reveals that regarding generic skills, the XA courses have always struggled with alignment. Most strikingly, the courses kept the traditional end exam assessment for many years, although learning was supported by many new types of formative assessment during the course.

In some cases, the objectives have been tied strongly to a particular method, instead of choosing methods to reach the objectives. This can become an issue when objectives are shared between different courses or course instances. For example, learning to read and understand proofs is part of the general goals in the XA framework. However, it is not dictated how this goal is supported, apart from that it should happen primarily through students' activity. In autumn 2015, a method called "self-explanation training" (Hodds et al., 2014) was adopted as an activity in the courses IUM and LM1. Later, to emphasise this to the students, a new skill was added to the learning objectives matrices: "I know how to use self-explanation to read and understand proofs". This then raised the question whether future instalments of the same course – or indeed other XA courses – would need to include the self-explanation training activity.

There have been even more substantial examples of changes in methods leading to changes in objectives. In spring 2016, the course A2 introduced small group work in the classroom. In accordance, group working skills were added as a separate topic to the learning objectives matrix. Similarly, in summer 2017, peer feedback exercises were added to the course LM2, and accordingly, a new category of "Giving and receiving feedback" was added to the learning objectives matrix. Later, it was noticed that the course workload was becoming too heavy because of too many different kinds of activities, but – according to the teacher – it was difficult to give up the peer feedback exercise as it was now considered as part of the learning objectives.

Concerning assessment, the XA framework does not take a position on how to conduct it. For many years, XA courses were assessed in the traditional way of final exams, often coupled with mid-term exams and extra points awarded by coursework. However, generic skills were mostly not assessed. This changed when summative self-assessment was adopted in the course A2 in spring 2016. The self-assessment was performed in private assessment discussions with the teacher at the end of the course; the students were allowed to choose a final grade and offer justifications based on the learning objectives. As the teacher had followed the activities in class closely, this enabled a reliable assessment of practically all generic skills. The new assessment spurred developments also in other courses: self-assessment was added in different amounts, exams were changed to project work, and so on.

It can be seen that throughout the development of XA that there has been an increasing amount of striving for better alignment. Regarding generic skills, there have been difficulties, as the assessment methods that teachers were used to did not support direct assessment of generic skills. This was particularly problematic, as researchers have noted (e.g. Kember, 2009) that in order to support the learning of generic skills, teaching and assessment methods should be varied and aimed at the desired capabilities. However, as the XA model evolved and teachers gained experience, the alignment improved. This is again in line with previous research that stresses that teachers need to have pedagogical competencies to integrate generic skills and to use various teaching and assessment methods (Hyytinen et al., 2019; Jones, 2009).

22.7.4 Programme-Level Development

Analysis of the source material indicates that learning objectives regarding generic skills vary less from course to course than those regarding content skills: content skills are unique to each course, but some courses share the learning objectives for generic skills. For example, the first learning objectives matrix was created for the course LM1, and the courses IUM and LM2 used this same matrix. Even though the learning matrices for the courses LM1 and IUM were initially identical, they evolved individually over the years, resulting in two different matrices. Over the years, the courses have influenced each other in many ways. Individual learning objectives may have been copied from one course to another, or the objectives of one course have served as prerequisites for another course. For example, taking into account the feelings of another person in a conversation was added to the learning objectives matrix of the course A2 in spring 2018. In autumn 2018, this was mentioned also among the learning objectives of the course LM1. Another example is the course AS1 which has the course LM1 as a prerequisite. Almost all prerequisite skills for the course AS1 are learning objectives from the course LM1.

Because learning generic skills requires time and practice (Bunney et al., 2015), it is good that the same generic skills are mentioned as learning objectives in different courses. On the other hand, the fact that so many courses in our study share the same generic skills in their learning objectives can also be seen as a downside, as students are expected to develop diverse generic skills. In addition, one would assume that students' generic skills develop during each course and therefore the subsequent courses should have more advanced learning objectives.

Collaboration between teachers has had a central role in the evolution of generic skills in the XA courses. For example, in 2013, when the first learning objective matrices were created for the courses LM1 and IUM, they were written in collaboration with the teachers of the two courses who were close colleagues. Also, the significant rewriting of learning objectives in 2017 was done in cooperation with teachers from different courses. From previous studies, it is known that development and integration of generic skills into disciplinary courses requires collaboration between teachers, and that teachers understand individual and collective responsibility for teaching generic skills (Hyytinen et al., 2019).

In many cases, ideas have spread from one course to another when the same person has taught those courses. For example, the courses LM1, AS1 and A2 have all been taught at some point by the same person who has used the learning objectives of one course in writing the learning objectives of another course. Those learning objectives have then been passed on to the new teacher. This way, knowledge about learning objectives has been transferred to new people. However, in some cases, the new teachers have only published the mathematical learning objectives but not the ones concerning generic skills (LM1, summer 2013; LM2, summer and autumn 2017). It may be that the teacher did not find the generic skills relevant or thought that it would be too radical to publish them.

In the focus department of this study, individual teachers have had a central role in shaping how generic skills are taken into account in the teaching of mathematics courses. They have developed teaching methods and learning objectives and have spread information among undergraduate courses. While it has been important that the initiative to develop generic skills has come from the teachers, also programme level planning would have been needed. To ensure that the core generic skills are taught during the study programme, and different courses emphasise different generic skills, programme level development and collaboration are essential (Bunney et al., 2015).

22.8 Concluding Remarks

In this chapter, we studied the visible presence of generic skills during the development of an IBME-type teaching model, Extreme Apprenticeship (XA). Our analysis revealed that during the development of XA, the variety of generic skills mentioned among the learning objectives has become much wider, and generic skills have gradually assumed a position equal to content skills. Moreover, the way generic skills are described has become more sophisticated. Assuming that the written learning objectives reflect the teachers' intentions and actions in their teaching, we can infer that the role of generic skills has increased and diversified in XA courses. In this way, our study lends support to the claim that introducing IBME style teaching has potential to teach generic skills in university mathematics.

The development of the XA model and of the learning objectives concerning generic skills have been intertwined. New additions to the teaching model, such as self-assessment, peer feedback and group work, have spurred new learning objectives and vice versa. The introduction of self-assessment seems to have been a particularly significant reform that has changed drastically the way generic skills are presented and assessed. These findings indicate that within IBME, the active role of students as well as varied tasks and assessment methods may support the learning of generic skills. Also, our results suggest that hands-on teaching experience of the developers and programme-wide collaboration by the teachers may foster effective incorporation of generic skills to the curriculum.

This study used the historical method to describe past events and search explanations for teachers' choices. A lot of care was taken to include all relevant sources and to confirm the dating and authenticity of the documents. However, it is possible that some details may have been overlooked. Another potential error is biased interpretation. To mitigate this, the authors first collected and processed all material systematically. Then they compared their findings and interpretations. As two of the authors were developers of the XA model, and the third was a non-mathematician and a researcher of generic skills, the team had both insight that helped with interpretation, and an external eye to reduce biased conclusions.

Generic skills are important for the students' future, and time and effort should be dedicated to them in university teaching. It can be argued that integrating generic skills into domain courses is an effective way of including them in the curriculum. It is therefore important that more studies are dedicated to investigating the effectiveness of such integration attempts. We also invite researchers to report on students' perceptions, which are lacking from our study. Furthermore, to complete the picture, research is needed to assess to what degree these endeavours are successful in actually teaching generic skills to students.

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Chapter 23
On the Levels and Types of Students' Inquiry: The Case of Calculus

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Abstract The learning of mathematics organized around carefully designed questions is vital for students' quantitative literacy development at both elementary and advanced stages. Deliberating on how the inquiry-based instruction could be supported, we further theorize about the idea that inquiry comes in many levels and through different types of activities. We use the Herbartian schema and related constructs of the Anthropological Theory of the Didactic in order to embrace the continuum of levels of inquiry, labeled as confirmation, structured, guided, and open. We review calculus textbooks' descriptions of these inquiries and their roles. We also suggest additional activities of the types that complement those found in the textbooks. The new types include tasks for comparison and recognition of objects, evaluation of the validity of statements, modification of questions, and evaluation of reasoning - the actions common in mathematical research. We conclude by commenting on possible relations between the activity's inquiry level, type, and its learning potential (e.g. adidactic, linkage, deepening, and research).

Keywords Calculus textbooks · Herbartian schema · Levels of inquiry · Milieu construction · Praxeology · Types of activities

23.1 Introduction

Enhancement of quantitative literacy, that is, the ability to employ quantitative arguments in various contexts, is one of the major goals of mathematics education. However, it is very much possible that university students, whose program requires a significant mathematics component, may not be successful in responding to relatively elementary questions involving numbers and figures. Agustin et al. (2012)

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conducted a diagnostic test focusing on practical applications and interpretations that entail

> sophisticated reasoning with elementary mathematics rather than elementary reasoning with sophisticated mathematics (Steen 2004, p. 9).

Agustin et al. (2012) found that students had difficulties with geometric applications, numerical and algebraic relations, and drawing logical conclusions from numerical information. About a half of Calculus 1 students failed to solve problems like the following one.

Sample Diagnostic Question
A box is sitting on a floor in a room. It has the base 3×6 sq ft and height 2 ft. What are the (integer) dimensions of the smallest rectangular tarp that will cover the box to the floor?

Hughes-Hallett (2003) proposed,

> mathematics courses that concentrate on teaching algorithms but not on varied applications in context, are unlikely to develop quantitative literacy (p. 93).

Indeed, it is important for learners to go beyond the formal use of rules. Mathematics courses based on memorization, passive learning and reproduction of given ideas and formulas do not prepare students for simple, practical applications of mathematics, let alone for dealing with abstract ideas and formal reasoning that is central to university mathematics (Nardi 2008).

Research on undergraduate teaching indicates the advantages of instructional practices that actively engage students in learning mathematics (Freeman et al. 2014; Rasmussen & Kwon 2007; Kogan & Laursen 2014). The results of recent surveys of mathematics departments in the United States that offer a graduate degree in mathematics suggested that one of the main characteristics of successful calculus programs was using active learning strategies, such as students working on problems in class, peer-to-peer and whole-class discussion, either as the primary instructional approach or in conjunction with lecture (Rasmussen et al. 2019). In addition, the study highlighted the importance of including challenging units

> offered with high expectations for students, including engaging, conceptually oriented content beyond an emphasis on procedures and skills related to calculus (ibid, p. 100).

However, it was also concluded that many mathematics departments across the United States are struggling with implementation of these recommendations even if they recognize their value. In particular, according to universities' self-reports only

> 22% of Precalculus courses used some or mainly active learning, and this shrank to 20% of Calculus 1 courses and 14% of Calculus 2 courses (ibid, p.103).

Could the quality of assigned tasks be a part of the problem? Lithner (2004) discusses that many textbook exercises can be done by copying solved examples without any conceptual grasp of mathematical properties and ideas. He reports that at least 70% of the 598 exercises that he had reviewed, fall in this category. Similarly, Tallman et al (2016) found that 85% of 3735 randomly selected calculus exam questions could be done by memorization of facts and procedures, without understanding of an idea or why a procedure is valid.

This chapter partly addresses the issue of implementing active and meaningful learning in teaching calculus and looks at selected learning resources from the perspective of their potential support and promotion of students' quantitative reasoning. For that, we first revisit the notion of inquiry as a nucleus of active learning. We will theorize that inquiry comes in many levels, forms and different types of activities. We will look at various tasks presented in the textbooks and offer inquiry tasks of some additional types.

23.2 Theoretical Background and Framework

The inquiry-based approach in teaching mathematics is grounded in the epistemological view that knowledge emerges from such processes as thinking, experimentation, and reflection. Knowledge cannot be simply imposed on learners because its acquisition

> depends upon the activity, which the mind itself undergoes in responding to what is presented from without (Dewey 1902/1956/1990, p. 209).

This is consistent with the constructivist view on cognitive development of mathematics knowledge by active mind (Piaget 1950; Bruner 1968) when participants are engaged in problem solving, specialization, generalization, conjecturing, and proving (Pólya 1945; Mason et al. 2010; Schoenfeld 1985).

In order to prevent students' knowledge from becoming 'inert' (Collins 1988) mathematical thinking is stimulated through various challenges and opportunities for reflection on and improvement of mathematical understanding (von Glasersfeld 1987). These and other characteristics of the inquiry-based learning (IBL) of mathematics have been identified and studied from several theoretical perspectives (Artigue & Blomhøj 2013).

In particular, the Anthropological Theory of the Didactic (ATD) (Chevallard 1999; Chevallard 2019) offers several constructs useful for our further deliberation on the nature of inquiry. In the ATD, inquiry is defined as

> the action taken to provide an answer A to a question Q (Chevallard & Bosch 2019, p. xxv).

In the course of an inquiry, a group X of (one or more) learners and a group Y of (zero or more) teachers form a didactic system $S(X, Y, Q)$ around the question Q. The learners are confronted with information received from various sources such as experts, books, the internet, and peers. These sources of information, each of which may have its own agenda, are regarded as *media*. The process of inquiry involves critical revision of the statements received from media concerning possible answers to the question Q, while looking for evidence and proofs of such statements.

In this process, an adidactic *milieu* M is formed by the didactic system: $S(X, Y, Q) \rightarrow M$. From the dialectics of media and milieu, from the pool of statements viewed as conjectures and their proofs, refutations or alterations, an answer A is produced. In short, an inquiry is represented by the Herbartian schema

$$H[S(X, Y, Q) \rightarrow M] \rightarrow A. \tag{23.1}$$

Milieu is the central part of an inquiry. It consists of answers $A' = \{A'_1, A'_2, \ldots\}$ available in media, questions $Q' = \{Q'_1, Q'_2, \ldots\}$ derived from study of Q and A', and works W that help to make sense of A' and build up A. Milieu also includes data D that is used in support of conclusions drawn within the didactic system $S(X, Y, Q)$. Thus,

$$M = \{A', Q', W, D\}. \tag{23.2}$$

In the ATD, work W is viewed as any intentional product of human activity. Questions and answers are considered as a type of work. Answers are regarded in ATD as praxeologies. A praxeology consists of four elements: a type of tasks T, a technique τ required to accomplish the task, a technology θ that explains or justifies the technique and a theory Θ that provides the foundation of the above items, with all the notation used and assumptions made. The components constitute the praxis block $\Pi = [T, \tau]$ and the logos block $\Lambda = [\theta, \Theta]$ of a praxeology. Therefore, a *complete* answer to any concrete question includes a practical and theoretical parts

$$A \ni (\Pi, \Lambda). \tag{23.3}$$

The task type is specified by a verb and the object (e.g. ‘to solve a quadratic equation’). Tasks are united in the same type because of the same method (technique) applicable to them. As noted by Polya & Szego (1978/98),

> An idea that can be used only once is a trick. If one can use it more than once it becomes a method (p. viii).

The need to understand and explain a technique calls for a technology and a theory.

Praxeologies emerge in human activity in response to humans’ practical needs. However, they undergo a transposition when appearing in the teaching context. They often become detached from the original context, disintegrated and reorganized for the purpose of better exposition. This leads to the teaching paradigm of ‘Visiting Works’ when students are let to appreciate certain techniques, technologies and even theories without much familiarity with the origin of the considered tasks. Within this paradigm it is the teacher or the book author who inquires about certain questions in order to present a result of their inquiry to the learners.

> Students are not expected to take part in the inquiry proper: they receive a ready-made answer and have to accept, understand and adopt it as the class’s answer. [In this scenario] a lecture [...] addresses ‘topics’, deals with ‘subjects’, and but rarely struggles to explicitly answer explicit questions (Chevallard 2019, pp. 101–102).

The ATD advocates for an alternative didactic organization that puts questions in the centre of learning. 'Questioning the World' paradigm implies the presence of active learners searching for truth and evidence among the answers and theories available to them (including the results of their own reasoning). However, as we saw in the Introduction, the implementation of this teaching paradigm presents difficulty to many instructors. Indeed,

> Most students, regardless of age, need extensive practice to develop their inquiry abilities and understandings to a point where they can conduct their own investigation from start to finish (Banchi & Bell 2008, p. 26).

Banchi & Bell (2008) proposed an approach addressing this issue in the context of elementary science education. Viewing inquiry as a less formal type of research activity, they proposed four consecutive levels of inquiry that students can experience. These levels are explained below. The research questions of this chapter are:

1. What new details that are useful in the context of calculus arise from an ATD-based review of the Banchi & Bell approach?
2. How do calculus textbooks describe intended levels and roles of inquiry?
3. What are examples and purposes of additional problem types supporting IBL of calculus?

23.3 The Levels of Inquiry in Calculus Textbooks

The level of inquiry depends on the amount of guidance and information given to students by a book author or instructor, facing the question:

> Where do I draw the line between content I must impart to my students versus the content they can produce independently? (Ernst n.d.).

In the context of science education (Banchi & Bell 2008) divided the information into 3 categories: (i) guiding question, (ii) procedure, and (iii) expected results. We use the Herbartian schema (23.1) to further theorize about the classification proposed in Banchi & Bell (2008).

The presence of a *guiding question* is explicitly shown in the Herbartian schema as a part of the didactic system $S(X, Y, Q)$. Answering a question could be reduced to a sequence of related tasks. *Procedure* then is identified with techniques τ corresponding to these task types T and supported by theory Θ. As for *expected results*, in mathematics we should distinguish an effort that produces a short answer a (e.g. a number or formula) from an inquiry producing a complete answer (23.3), namely:

$$A = \{a, \Pi, \Lambda\}. \tag{23.4}$$

According to the authors' descriptions there are different textbook organizations. In a traditional calculus textbook

> each chapter is divided into sections and at the end of almost every section, variety of problems is given. [...] Each topic is followed by examples, simple and complex alike, solved in detail and graphs are presented whenever they are needed. In addition, we provide answers to selected problems (Friedman & Kandel, 2011, p. viii).

The key difference of the IBL resources is that questions are given after only some minimal information is provided.

> In this book the students will encounter a question or problem and then they will be given the opportunity to answers the question or to solve the problem before moving on. The organization of the material here maximizes the reader's opportunity to participate in the creative process (Falbo, 2010, p. xiii).

As Polya & Szego (1978/98) explained, the most important goal is to

> stimulate the reader to independent work and to suggest to him useful line of thought. [...] The impartial and factual knowledge is for us a secondary consideration. Above all we aim to promote in the reader a correct attitude, a certain discipline of thought (pp. vi–vii).

Next, we review several traditional and IBL calculus textbooks used at the undergraduate university level. While we realize that our selection of textbooks could be complemented with more choices, we believe that it already allows us to identify references to different levels and roles of inquiry.

23.3.1 The Structured and Guided Inquiries

According to Banchi & Bell (2008), in the case of *structured* inquiry the question and procedure are both provided to students and their goal is to find the expected result. In the case of *guided* inquiry only the question is provided, and it is anticipated that students find their own procedures and results. Using (23.1), we can schematically represent both cases as[1]

$$H[S(X, Y, Q_y^{\diamondsuit}) \to M] \to A_x^{\heartsuit}, \tag{23.5}$$

where $Q_y^{\diamondsuit}$ is the question posed to students by teacher $y \in Y$ and $A_x^{\heartsuit}$ is a complete answer (23.4) produced by student $x \in X$. The difference shows up in the milieu M in (23.5). In the case of structured inquiry the praxeology is provided $(\Pi_y^{\diamondsuit}, \Lambda_y^{\diamondsuit}) \in M$, while in the case of guided inquiry it needs to be constructed by the student: $(\Pi_x^{\heartsuit}, \Lambda_x^{\heartsuit}) \in M$.

[1] Following the ATD tradition, we mark by the rhombus ($Q^{\diamondsuit}, A^{\diamondsuit}, \Pi^{\diamondsuit}$, etc.) the pieces of information given to learners and by the heart ($Q^{\heartsuit}, A^{\heartsuit}, \Lambda^{\heartsuit}$, etc.) - the elements produced by them. The subscript y symbolizes the teacher $y \in Y$ who supplies the information and the subscript x symbolizes the learner $x \in X$ who produces the items while possibly working with other students and teachers.

Exercises in the end of a section in a traditional textbook are typical examples of these inquiries. Authors may deliberately vary the level of difficulty with the pedagogical goal to include both kinds of inquiry, as illustrated below.

> (a) The first exercises are routine, modeled almost exactly on the examples; these are intended to give students confidence. (b) Next come exercises that are still based directly on the examples and text but which may have variations of wording or which combine different ideas; these are intended to train students to think for themselves. (c) The last exercises in each set are difficult. [Doing them] requires insight into what calculus is really about (Marsden & Weinstein, 1985, p. vii).

Group (a) in this quote gives an example of structured inquiry. IBL resources may introduce concepts by breaking them in a sequence of short tasks. The following structured inquiry aims at making students invent the definition of the derivative.

An Example of Structured Inquiry
Consider the function $f(x) = x^3 + 3x^2 + 2$.

(a) Calculate the average rate of change $\frac{\Delta f}{\Delta x}$ of f from $x = 0$ to $x = 3$.
(b) We choose a sequence of points that gets incrementally close to $x = 3$.
  Let $x_1 = 2$. Then $f(x_1) = \_\_\_\_$, $\frac{\Delta f}{\Delta x} = \_\_$.
  Let x_2 to be closer to $x = 3$ than x_1. Then $\frac{\Delta f}{\Delta x} = \_\_$.
  Let x_3 to be closer to $x = 3$ than x_2. Then $\frac{\Delta f}{\Delta x} = \_\_$.
(c) What is happening to $\frac{\Delta f}{\Delta x}$ in part (b) above?
(d) This suggests to us that the instantaneous rate of change at $x = 3$ is __.

This assignment tells students exactly what to do. Because of that Greene & von Renesse (2017) are concerned that students

> will successfully complete each task, but probably not understand the bigger ideas and connections (p. 657).

As well, speaking of the dominance of structured activities (e.g. routine exercises) in a traditional textbook, Lithner (2003, 2004) describes students' approach to solve problems by identifying surface similarities and mimicking sample solutions. We represent such undesirable scenarios by the schema leading to an answer $a_x^{\heartsuit}$ with no or little conceptual understanding:

$$H[S(X, Y, Q_y^{\diamond}) \to M] \to a_x^{\heartsuit}. \tag{23.6}$$

Lithner (2004) suggests, possession of plausible reasoning based on intrinsic mathematical properties is an essential conditions for obtaining complete answers $A_x^{\heartsuit}$ (23.4).

One possible scenario of the structured inquiry is worth noting. Students may discover an alternative way of doing the given task while performing a method suggested by a textbook or a teacher. Kondratieva (2019) discusses a case when a student accidentally, by the virtue of making a mistake, found an alternative technique $\tau_x^{\heartsuit} \neq \tau_y^{\diamondsuit}$ for finding the osculating plane to a 3D curve at a point. The analysis of the mistake led the student (prompted by his instructor) to a theoretical justification $\theta_x^{\heartsuit}$ of the new technique. This outcome is more desirable compare to (23.6):

$$H[S(X, Y, Q_y^{\diamondsuit}) \to M] \to \{a_x^{\heartsuit}, \tau_x^{\heartsuit}, \theta_x^{\heartsuit}\}.$$

A general recommendation is to move from too much structured inquiry towards a guided one that engages learners e.g. by means of a more authentic question, and forces them to construct or invent rather than copy or follow an idea of a solution.

In the next task students supposedly will perform calculations similar to the ones explicitly suggested in the structured inquiry example. However, now the calculations will follow from the logic supported by the physical model. Thus, the numerical answer will be associated with the concept of instantaneous speed.

An Example of Guided Inquiry

Consider a situation where cameras are located along a roadway and the following information about a passing car is captured:

Time (mins past start camera)	0.5	1	1.5	2	2.5	3	3.5
Distance (miles from start camera)	0.06	0.1	0.16	0.28	0.47	0.79	1.33

How fast is this car traveling at the exact instant 2.7 min past the start camera?

(a) Provide a strategy for obtaining your most accurate estimate to answer the above question.
(b) Find a function that models the distance data provided above. Using this model, calculate another estimate to answer the question (Greene & von Renesse 2017, p. 658).

In the guided case, milieu still could contain some relevant ready-made works and answers chosen by students from a book, lecture or the internet. However, learners take more initiative for creation and verification their own approach. Due to this increased complexity, Greene & von Renesse (2017) warn that,

> you may find students become frustrated and give up on the question before reaching the level of inquiry that would lead to deep understanding (p. 657).

Therefore, textbooks offer a combination of different levels of inquiries, sometimes accompanied by hints or explanations. Exercises could be embedded in the text prompting learners to pause and think about emerging questions as they read a section. Some IBL explorations can be no more than a long series of interconnected exercises (Schumacher 2007, p. xv). As well, the authors value learning projects.[2]

> One way of involving students and making them active learners is to have them work (perhaps in groups) on extended projects that give a feeling of substantial accomplishment when completed (Stewart 2008, p. xiii).

Here is a description of a discovery project found in a traditional textbook.

> By observing the patterns that occur in the integrals of several members of the family, you will first guess, and then prove, a general formula for the integral of any member of the family (Stewart 2008, p. 494.).

The families include $\int \frac{1}{(x+a)(x+b)}\ dx$, $\int \sin(ax)\cos(bx)\ dx$, $\int x^n e^x\ dx$, $\int x^n \ln x\ dx$, and the project comprises the following tasks: (a) evaluate particular (given) cases using CAS; (b) guess general formula based on these cases; (c) predict other values based on your general formula and verify with CAS; (d) identify the restrictions on the parameters; (e) prove your formula using basic techniques of integration.

Another example of guided inquiry, a sequence of tasks that provides a link between calculus and analysis by making use of several techniques learned in the former to match a theoretical construct of the latter, is discussed in Kondratieva & Winsløw (2018).

It is important that authors offer general guidance on how to approach an inquiry and challenge (Adams & Essex 2010, p. xiv), elaborating on Polya's principles of problem solving Pólya 1945, and exemplifying them in the calculus context (Stewart 2008, pp. 76–78, 241, 322). Pólya & Szegő (1978/1998) advise to build up, connect and apply ideas as learners work through sets of problems given in the book.

> Many problems, which would be intractable even for an advanced student if set in isolation, are here surrounded by preparatory and explanatory problems presented in such a context that with some perseverance and a little inventiveness it should be possible to master them (Pólya & Szegő 1978/1998, p. xi).

We will now consider another two levels of inquiry.

23.3.2 *The Confirmation and Open Inquiries: Two Extremes*

At the *confirmation* level of inquiry

> students are provided with the question and procedure (method), and the results are known (Banchi & Bell 2008, p. 26).

Using (23.1), we can schematically represent this case as follows

$$H[S(X, Y, Q_y^{\diamond}) \to M] \to A_x^{\heartsuit} = A_y^{\diamond}. \tag{23.7}$$

[2]For ATD-based analysis of project-based learning see Markulin et al. (2020).

In (23.7), student x is provided by teacher y with both the question $Q_y^\diamond$ and components of milieu M: the short answer $a_y^\diamond$ explained in the context of given methodology[3] and supported by numerical or pictorial data $D^\diamond$. In the process of internal confirmation $A_x^\heartsuit = A_y^\diamond$ of the complete answer (23.4) offered in an example illustrating a general method, the student is supposed to get a sense of how to approach questions of the given type. As it is evident from the introduction to a traditional textbook,

> there are a lot of examples with complete solutions to help you with the exercises (Marsden & Weinstein 1985, p. xi).

Further, an IBL textbook explains the aim of questions with solutions and answers.

> The solutions [. . .] are stated in a step-by-step manner that lets the student uncover the solution one line at time to check his or her work. [. . .] Students may also use solutions as hints. [. . .] Cover all but the first line, and go back and try to work the problem on your own (Falbo 2010, p. xiv).

Reading provided solutions after one's own attempts, successful or not, to solve the problems should result in deeper thinking processes.

> If repeated efforts [to answer a question] have been unsuccessful, the reader can afterwords analyze the solution [...] with more incisive attention, bring out the actual principle, which is the salient point, assimilate it, and commit it to his memory as a permanent acquisition (Pólya & Szegő 1978/1998, p. xi).

Thus, in view of the authors, a confirmation inquiry is a useful learning tool because it requires careful analysis and validation of given information, understanding of the presented ideas and remembering them for future use. Despite the fact that all the information is available to students, their critical stand is signified by the ability to interpret and explain and then to accept or reject provided items, and as a result, by the appearance of their own questions $Q_x^\heartsuit$ in the milieu M.

The emergence of questions is a crucial characteristic of any level of inquiry. They may have various degrees of depth and sophistication, pivoting students' way towards a complete answer. Some of these new questions may give rise to an independent inquiry. In this case the inquiry is said to be *open*. At this highest level students

> have the purest opportunities to act like scientists, deriving questions, designing and carrying out investigations, and communicating their results (Banchi & Bell 2008, p. 27).

When students use actual questions $Q_x^\heartsuit$ that they have about calculus (Perrin & Quinn 2008), they construct a milieu to develop and test own conjectures:

$$H[S(X, Y, Q_x^\heartsuit) \to M] \to A_x^\heartsuit. \tag{23.8}$$

[3] Students may be given either only the praxis block $\Pi_y^\diamond$ requiring own explanations $\theta_x^\heartsuit$ or both praxis and logos blocks $(\Pi_y^\diamond, \Lambda_y^\diamond)$ requiring a validation.

This is how an author of IBL resources describes intended students' behaviour:

An Example of Open Inquiry

> [...] students investigate the area and perimeter of Koch's snowflake. They naturally develop an example of a series and ask just the right question: Is this infinite or not? How do we know when a (geometric) series is convergent or not? Looking at many examples on wolfram alpha [a CAS] can then lead to a conjecture. Now the students are ready to think about how to prove their conjecture. Notice that any language needed (series, geometric series, convergence, divergence, sum notation, limit, . . .) can be provided during the exploration, when the students need and want the vocabulary to express their thinking (von Renesse 2014).

Learning to conduct an open inquiry involves mastery of the lower-level inquiries and simultaneous experience of several levels of inquiry within the same activity (Banchi & Bell 2008, p. 28).

In a sense, a goal of any textbook is to teach students how to conduct open inquiries, that is to model how to ask right questions and the ways to approach their answers. Learning to inquire is a gradual process. Pólya & Szegő (1978/1998) begin with questions that do not require any special mathematical knowledge. However, some of their problems from the first (1924) edition of the book have given rise to extensive research (ibid, p. v).

Describing IBL, textbook authors emphasize supporting role of the instructor encouraging students' initiative and engagement in the learning process:

> a teacher must be willing to become an 'interested bystander', while the student is the one who presents solutions on the board (Falbo 2010, p. xi).

Students' collaboration in small groups should be promoted, while the instructor

> supports the students' exploration by helping them make conjectures and see important relations (Schumacher 2007, p. xvi).

Instructors also should help students develop meta-skills such as being confident, curious, motivated, and persistent (Greene & von Renesse 2017). In any case, IBL instructors should be interested in and promote student thinking (Rasmussen et al. 2017).

Next section models more types of tasks that one performs during mathematics inquiries.

23.4 Additional Types of Activities That Promote Students' Inquiry

Calculus textbooks include a wealth of material for instructors to select from, adjust and compose the tasks to be offered in their courses, based on students' needs (Adams & Essex 2010, p. xv). Meantime, Swan (2005) provides examples at the secondary school level and advocates for usefulness of activities of the following five types: classification of objects, recognition of objects in various forms, evaluation of statements, modification of problems, and analysis of reasoning. We follow the original intention of each of the types and illustrate how an instructor can compose new "inquiry-training" questions from typical assignments found in textbooks.

23.4.1 Classifying Mathematical Objects

In this type of activities students compare objects from a given set using either own or given classifications.

> They learn to discriminate carefully and recognize the properties of objects. They also develop mathematical language and definitions (Swan 2005, p. 16).

Textbooks often ask to evaluate individual series. In the next example the quest is open for discussing and using other criteria for series comparison.

Activity 1
Examine the following three series and identify, in turn, why each one might be considered the *odd one out*.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n, \qquad \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n, \qquad \sum_{n=0}^{\infty}(-1)^n\left(\frac{\pi}{2}\right)^{2n+1}\frac{1}{(2n+1)!}.$$

The first two series are geometric, while the third is not. The second and the third series are alternating, while the first is not. The first and the last series converge to 1, while the second converges to $-1/3$. If unable to evaluate all series students could estimate that the second sum is negative, while the first and the last are positive.

In the next example students will see how organization of information may lead to new conjectures and address misconceptions.

Activity 2
Classify the p-sequences $\{a_n = \frac{1}{n^p}\}_{n=1}^{\infty}$, parameterized by real numbers p in the two-way grid by providing relevant values of p in each cell

General term $a_n = \frac{1}{n^p}$, $n \geq 1$	$\sum_{n=1}^{\infty} a_n$ converges	$\sum_{n=1}^{\infty} a_n$ diverges
$\lim_{n\to\infty} a_n = 0$		
$\lim_{n\to\infty} a_n \neq 0$		

A possible (incomplete) answer could consist of selected examples.

General term $a_n = \frac{1}{n^p}$, $n \geq 1$	$\sum_{n=1}^{\infty} a_n$ converges	$\sum_{n=1}^{\infty} a_n$ diverges
$\lim_{n\to\infty} a_n = 0$	$p = 2$	$p = 1$
$\lim_{n\to\infty} a_n \neq 0$	none	$p = -1$

The ultimate goal of the activity is the complete classification that summarizes properties of p series and sequences.

General term $a_n = \frac{1}{n^p}$, $n \geq 1$	$\sum_{n=1}^{\infty} a_n$ converges	$\sum_{n=1}^{\infty} a_n$ diverges
Here parameter p is any real number.		
$\lim_{n\to\infty} a_n = 0$	$p > 1$	$0 < p \leq 1$
$\lim_{n\to\infty} a_n \neq 0$	none	$p \leq 0$

The second row of the table suggests that for sequences that converge to 0, related series may or may not converge. However, in the third row of the table, one cell contains 'none'. This illustrates the Test for Divergence. Students often misapply it by assuming that if $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges, while the test really claims that $\lim_{n\to\infty} a_n \neq 0$ implies the divergence of the series.

23.4.2 *Interpreting Multiple Representations*

In this type of activities students collect cards depicting the same mathematical object or idea in different ways, by making sense of information encoded by diagrams, formulas, word descriptions, etc.

> They draw links between different representations and develop new mental images for concepts (Swan 2005, p. 16).

Activity 3

Select the cards showing several representations of the same object.

1. The region enclosed by a parabola $y = x^2$ and the line $y = x$

2. The region within the circle $x^2 + y^2 = 1$ and above the x-axis

3. $\begin{cases} -1 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1 - x^2} \end{cases}$

4. $\begin{cases} 0 \leq x \leq 1 \\ x^2 \leq y \leq x \end{cases}$

5. $\begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \pi \end{cases}$

6. $\begin{cases} 0 \leq y \leq 1 \\ y \leq x \leq \sqrt{y} \end{cases}$

Students should recognize that cards 1, 4, 6 describe one object while cards 2, 3, 5 describe another one. They could also add graphical representation of the regions and other possible descriptions. Students' difficulties with the change of order of integration and the use of polar coordinates in double integrals could be reduced if they have understood multiple representations of the regions of integration. Similar activity with 3D domains is helpful for performing triple integration.

23.4.3 Evaluating Mathematical Statements

In this type of activities students need to determine under what conditions (if any) a given statement is true. By doing that, they learn

> to develop rigorous mathematical arguments and justifications, and to devise examples and counterexamples to defend their reasoning (Swan 2005, p. 16).

Statements could highlight common misconceptions and errors.

Activity 4
Decide whether the statement is *always*, *sometimes* or *never* true, and give explanations for your decision: "When one evaluates an iterated integral over a rectangular region of functions in the form $f(x, y) = g(x)h(y)$, the order of integration does not affect the efficiency of calculations."

In general, $\iint\limits_{[a,\,b]\times[c,\,d]} g(x)h(y)\;dA = \int_a^b g(x)\;dx \int_c^d h(y)\;dy = \int_c^d h(y)\;dy \int_a^b g(x)\;dx$.
As each integral (dx and dy) is done separately, the order of integration should not affect the efficiency of calculations. However, if one of the integrals is zero and the other integral is hard to evaluate, this makes a difference. Once counterexamples are identified, students can adjust the statement to make it always true.

23.4.4 Creating Problems

In this type of activities students devise or modify problems for others to solve.

> This offers them the opportunity to be creative and *own* problems. While others attempt to solve them, they take on the role of teacher and explainer. The *doing* and *undoing* processes of mathematics are vividly exemplified (Swan 2005, p. 16).

For instance, problems could reflect on the connection between the chain rule for differentiation of composite functions and the substitution rule in integration, as illustrated in the next *doing-undoing* example

$$\frac{d}{dx}\sin(u(x)) = \cos(u(x))\cdot u'(x) \rightarrow \int \cos(u(x))\cdot u'(x)dx$$
$$= \int \cos u du = \sin(u(x)) + C$$

Besides the inverse problems, students create variations of a given problem. They need to pay attention to various aspects of the problem that affect its difficulty.

Activity 5
Modify the following question: "Starting from point A, you make one step in any direction. If each of your next steps is a half of the previous one, what is the farthest distance that you can move away from A? "

Possible modifications include:

- How far away from A can you move in 10 steps?
- What is the least number of steps needed to move 1.9 metres away from A?
- What if each of your next steps is a 3/4 of the previous one?
- What is the region that you can eventually step in while moving in random directions away from A?
- What is the infinite sum of all natural powers of $\frac{1}{2}$?

Students' understanding of the initial question in ordinary words may support 'mathematization' of further versions of the question. By inventing some alternative questions students may also develop a solution strategy for the original problem. However, some variations may be unnatural, clumsy, undefined, or unsolvable. To ask reasonable questions is a skill that comes through experience and guidance.

23.4.5 Analysing Reasoning and Solutions

In this type of activities students focus on the solution process itself. They are asked to (a) compare given alternative pathways through a problem, (b) logically organize given fragments of a solution, and (c) verify a proposed solution.

By discussing given strategies to solve a problem, students concentrate on such characteristics as generality, applicability and efficiency of different methods.

Activity 6

Compare the following answers to the question "How to find the coefficient of x^{11} in the Taylor expansion $\sum_{n=1}^{\infty} a_n x^n$ of the function $f(x) = e^{3x^4}$?"

A:. Use the known expansion $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$ and set $y = 3x^4$.

B:. Use the formula $a_n = \frac{f^{(n)}(0)}{n!}$ for $n = 11$.

C:. Notice that the function f is even, so only even powers will appear in the Taylor expansion.

Strategy C is the most efficient due to specific property of the function. At the same time, the most general strategy B is the least efficient in this case.

Comparison of several strategies of solving the same task could be useful for students' development of mathematical connections. Kondratieva & Bergsten (2021) discuss a task of drawing a parabola and its tangent line by three different methods. In that study students compared a calculus-based and two geometrical constructions shown to them using dynamic geometry software.

The next activity deals with chains of reasoning consisting of several subtasks. Each card contains a fragment of the process of evaluating a series by the telescoping method. Cards related to two series are mixed together. When students sort these cards producing two complete solutions, they focus on logic rather than on computations.

Activity 7

Sort the cards into two logically ordered solutions and report the answers.

1. $x^2 + 3x + 2 = (x+1)(x+2)$

2. $9x^2 + 3x - 2 = (3x-1)(3x+2)$

3. $\frac{1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$

4. $A(x+2) + B(x+1) = 1$

5. $A(3x+2) + B(3x-1) = 1$

6. $\frac{1}{9x^2+3x-2} = \frac{A}{3x-1} + \frac{B}{3x+2}$

7. $\begin{cases} A + B = 0 \\ 2A + B = 1 \end{cases}$

8. $\sum_{n=1}^{N} \frac{1}{9n^2+3n-2} = \frac{1}{6} - \frac{1}{15} + \frac{1}{15} - \frac{1}{24} + \cdots - \frac{1}{9N+6}$

9. $\begin{cases} A + B = 0 \\ 2A - B = 1 \end{cases}$

10. $\sum_{n=1}^{N} \frac{1}{n^2+3n+2} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{N+1}$

11. $\lim_{N\to\infty} \left(\frac{1}{6} - \frac{1}{9N+6}\right) = \frac{1}{6}$

12. $\lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+1}\right) = \frac{1}{2}$

13. $A = 1/3, B = -1/3$

14. $A = 1, B = -1$

15. Answer: $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} =$.

16. Answer: $\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2} =$.

The sequence of card 1, 3, 4, 7, 14, 10, 12, 15 leads to the answer 1/2 in card 15. The sequence 2, 6, 5, 9, 13, 11, 16 leads to the answer 1/6 in card 16.

Verification of given solutions is a tricky business, because errors could be of different nature: computational or logical, including hidden assumptions, incorrect inference, and redundancy. Ideally, students always have to verify the details of solutions and proofs given in a book or lecture, even if they 'make sense' overall.

Activity 8
Consider the problem and examine the solution for possible pitfalls.

Problem: Find whether the series $\sum_{n=0}^{\infty} \frac{n^3}{\sqrt{1+n^4}}$ converges or diverges.

Solution: Consider the function $f(x) = \frac{x^3}{\sqrt{1+x^4}}$, so that $a_n = f(n)$. The integral $\int_0^{\infty} \frac{x^3}{\sqrt{1+x^4}} dx$ could be evaluated by the substitution $u = 1 + x^4$, $du = 4x^3 dx$. The resulting improper integral diverges: $\int_0^{\infty} \frac{1}{4\sqrt{u}} du = \frac{1}{2} \lim_{a \to \infty} \sqrt{a} = \infty$. By the Integral Test, we conclude that the series diverges as well.

Application of the Integral test requires the function f to be continuous, positive and decreasing on some infinite interval $[x_0, \infty)$. While the first two conditions are met, the last condition is not. So, technically the Integral Test in not applicable. At the same time, the answer provided in the solution is correct: the series is divergent. By discussing this situation, students review the role of the conditions of the Integral Test in their explanations of why this test works. They should also conclude whether the idea implemented in the solution is incorrect, or it only leads to redundant calculations.

23.4.6 Different Types of Activities and Milieu Construction

The five types of activities support different ways of thinking (Swan 2005) and contribute in the development of inquiry ability in different ways.

Classification of objects is critical for distinguishing intrinsic mathematical properties from surface similarities, the skill emphasised by Lithner (2004) in the context of structured inquiry. It is helpful for identifying the type of task and related methods. If criteria for comparison are not specified (Activity 1), learners face a guided inquiry (23.5). The activity of sorting a collection of objects unites individual problems in a small project, providing an opportunity for generalization and making new conjectures (Activity 2), thus leading to open inquiries (23.8). Learners develop mathematical definitions and language useful for working with the milieu ingredients (23.2).

By interpreting multiple representations of the same object, learners focus on the meaning related to the technical skills required for producing each particular representation. Careful consideration and explanation of matching choices (Activity 3) strengthen connections between praxis Π and logos Λ, leading to more complete praxeologies populating learner's milieu (23.2).

By evaluating and properly quantifying statements (Activity 4), students develop their ability to convince and prove. Since milieu includes possible answers obtained from media, the ability to be attentive to details of statements is indispensable for developing one's own answer based on other works. Typical misconceptions may be addressed through this type of activities.

By modifying problems, students reflect on the praxeologies that they possess. They play with a given problem's conditions by possibly altering types, methods and justifications. This also fosters students' ability to pose questions related to the generating question (Activity 5) and thus, to shape the milieu development. Making new problems is a sign of students' progress towards open inquiry.

Analysis of reasoning aims to produce a shift from focusing on a short answer (23.6) to making a complete answer (23.4) that includes theoretical justification. Learners become aware that the same short answer could be produced in several ways and they compare different praxeologies related to the same question (Activity 6). A short answer while formally correct, could follow from a flawed solution. A proper construction of milieu (23.2) depends on the learner's ability to spot mistakes (Activity 8). Also, it relies on producing extended chains of smaller connected tasks (Activity 7). Verification of reasoning within confirmation and structured activities is a necessary step towards performing independent thinking at the guided and open levels.

Most of the above activities are in the 'Visiting Works' spirit as they facilitate the study of bodies of knowledge. Yet, they foster milieu construction in (23.1), the skill critical for conducting research within the 'Questioning the World' paradigm.

23.5 Concluding Discussion

The IBL of mathematics stimulates the development of quantitative literacy because

> students are learning how to learn. [...] They come to realize that the learning process involves properly identifying questions/problems and then applying knowledge and problem-solving skills to collect, analyze and synthesize information and arrive at viable conclusions and solutions (Blessinger & Carfora 2015, p. 11).

In this chapter, we use the ATD theoretical frame to elaborate on the four-level continuum of inquiry: confirmation, structured, guided and open. Besides clear theoretical distinction between the levels in terms of the origin of question, procedure and result, some new details become apparent by looking at the classification through lenses of Herbartian schema (23.1). The goal of the formalization is to scrutinize the structure of each case in terms of its building blocks. Such a look reveals that at any level of inquiry the result of students' work ought to include both practical and theoretical components: the ultimate goal of their study is formation of a complete (23.4) rather than a short answer (23.6). As prominent numerical analyst R. W. Hamming once noted, the purpose of computing is insight, not numbers.

Therefore, the procedure is viewed as more than just a method. A theory that explains practical steps should be acquired by the learner and become a part of the result. Theories may differ in their depth: calculus praxeologies could be rather pragmatic and less formal than those found in real analysis (Job & Schneider, 2014). Finally, the questions raised in the course of students' work define the construction of milieu from which an answer and possibly new inquiries emerge. The quality of questions affects the quality of reasoning. A guiding question perceived as unimportant may lead to superficial reasoning, while engaging questions stimulate learners' persistence and sense making.

In the open inquiry, much like in the scientific research, learners generate own questions from their environment; then they search for answers, methods and justifications. At the other extreme, everything: the question, the procedure and the answer—are supplied in a textbook. A crucial shift in students' learning attitude occurs when they start to conceive the latter case as a confirmation inquiry. Instead of blindly accepting or memorizing the given information, students begin to question it and look for supporting evidence. The confirmation inquiry is a concluding part of *any* inquiry. Students need to know when a problem at hand is solved correctly and when an argument is valid. Working with confirmation inquiry tasks, they find examples and models of such instances.

Structured inquiry activities require students to solve given tasks by given procedures. Each specimen of tasks brings its own mathematical nuance in the way the procedure is employed. Focusing on those nuances helps students develop the infrastructure used at the higher levels of inquiry.

The guided inquiry presents carefully crafted questions that engage students in problem-solving activities. In practice, it may be hard to draw a line between the structured and guided levels of inquiry. In both cases students construct a milieu to find their own answer to a given question and they consult other works. However, in the guided case students take more initiative in their search and invention of ideas, compared to the structured case where the approach is suggested to them.

Traditional and IBL textbooks employ different teaching philosophies. The former gear towards confirmation and structured inquires while the latter—towards guided and open levels. Nevertheless, in both traditional and IBL calculus textbooks, the authors build on the dialectic of study and research, providing careful explanations and the room for reader's action, reflection and initiative. They explicitly talk about learning situations which could serve as inquires at any of the four levels. They aim to bolster students' confidence, feeling of accomplishment, and to encourage independent work. The authors provide advice on students' learning behaviour and the role of instructor.

Open or guided inquiry activities have research potential. Besides that, an activity may be characterized by its potential for students to (i) engage and work autonomously (adidactic potential); (ii) build upon existing knowledge (linkage potential); and (iii) elaborate and construct inferences (deepening potential) (Gravesen et al., 2017). Confirmation and structured levels of inquiry strengthen activities with adidactic and linkage potentials. Students work with given information by making connections with knowledge they already possess.

Perhaps, in order to enhance the deepening potential of an activity, one needs to emphasize a greater variety of tasks. Students should learn how to compare mathematical objects and classify them based on similarities and distinctions (Activity 1 and 2); recognize different representations of the same object (Activity 3); evaluate statements paying attention to special cases and conditions (Activity 4); modify given problems (Activity 5); analyze given reasoning (Activities 6,7,8). These processes are useful for constructing a milieu from different inputs and appear in mathematical research. Therefore, reliance on these types of questions is instrumental for developing students' knowledge in progress and their abilities to conduct more open inquiries. The mathematical ideas behind such questions, the level of inquiry, and learning potentials could be foreseen by or evident to the designer. However, as it is reminded in Gravesen et al. (2017), the realization of theoretical characteristics of an activity strongly depends on the instructional conditions of its implementation.

Our discussion about the range of inquiry levels and types of questions aims to assist calculus instructors in the process of selecting learning resources and developing their teaching and assessment strategies to support IBL. Since students exhibit various reading and working habits (Oktariani et al. 2020), including short-cut strategies with suspension of sense-making (Lithner 2004), they will benefit from explicit indication of learning goals and expectations related to each type of assigned tasks.

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Chapter 24
From "Presenting Inquiry Results" to "Mathematizing at the Board as Part of Inquiry": A Commognitive Look at Familiar Student Practice

Igor' Kontorovich, Rox-Anne L'Italien-Bruneau, and Sina Greenwood

Abstract The mathematics education literature occasionally suggests inviting students to the classroom board to share results of their inquiry. In this conceptual chapter, we discuss how this practice can be investigated. Linking across different bodies of literature, we illuminate the special status of boards in mathematics and its teaching, and elaborate on the affordances of a board as a physical place for mathematizing. Building on the commognitive framework, we conceptualize the practice in terms of situations where students engage in a public communicational activity and generate narratives about the mathematical objects of their inquiry. We refer to these situations as "mathematizing at the board" and argue for special opportunities that they provide for students' learning. To present the conceptualization in action, we use two proofs that students generated at the board as part of our ongoing project on topology teaching and learning. We use this data to illustrate the analytical potential of the commognitive construct of routines to capture nuanced differences in students' mathematizing at the board.

Keywords Proving at the classroom board · Commognitive framework · Presentation of inquiry results · Topology teaching and learning

24.1 Introduction

As a pedagogical concept, *inquiry* has travelled a long way from the work of John Dewey to the mainstream of mathematics education research (for a comprehensive review see Artigue & Blomhøj, 2013). The Online Etymology Dictionary (n.d.) suggests that the word "inquiry" originated from "enquiry" in Old French, where, in

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the early fifteenth century, it stood for "a judicial examination of facts to determine truth". In the contemporary educational literature, inquiry is used to refer to a family of distinctive practices of learning and instruction, when some scholars maintain that inquiry can underlie mathematics education in a broader sense (e.g., Artigue & Blomhøj, 2013; Laursen & Rasmussen, 2019). Encyclopedia of Mathematics Education defines *inquiry-based mathematics education* (IBME) as

> [. . .] a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions [. . .], interpret and evaluate their solutions, and communicate and discuss their solutions effectively (Dorier & Maaß, 2020, p. 384).

With this chapter we hope to make a conceptual contribution to the IBME literature (cf. Gilson & Goldberg, 2015). Our focus is on what the IBME literature captures in terms of "students presenting and sharing their work" (Laursen & Rasmussen, 2019, p. 134), "communicating the results" (Maaß & Artigue, 2013, p. 781), and "the process of presenting outcomes of an inquiry phase or of the whole inquiry cycle to others (peers, teachers) and collecting feedback from them" (Pedaste et al., 2015, p. 54). Some researchers suggest that this practice can unfold on a classroom board (e.g., Ernst et al., 2017; Laursen et al., 2014; Yoshinobu & Jones, 2011), but only a hand full of studies refer to the use of boards in inquiry classrooms (e.g., Johnson, 2013; Wagner et al., 2007). The issue pertains not only to the literature being silent about how students mobilize the board to communicate their mathematical inquiries but also to the need for the research machinery to investigate this practice in depth. Hence, we are concerned with the question: "*How can this student practice be investigated?*"

The openness of the question is intentional as it is aimed to ignite a conversation in the IBME community about a practice, which some position at the core of inquiry (Hayward et al., 2016). We use this chapter to share our perspective, which grows from our theoretical stance, personal experience, and an ongoing project on learning and teaching in a topology course (Kontorovich, 2021b; Kontorovich & Greenwood, 2022).

In a digital age, spotlighting students' engagement with an old-school artefact requires a justification. Accordingly, in Sect. 24.2, we deliberate on the special place of boards in mathematics and its teaching, through linking works across different bodies of literature and with an eye to inquiry. Afterwards, we turn to the commognitive framework and capitalize on its comprehensive toolkit to conceptualize the focal practice, to consider its potential in terms of students' learning, and to offer possible strands for its analysis (Sects. 24.3 and 24.4). In Sect. 24.5, we illustrate these ideas with data from our project.

24.2 Boards, Inquiry, and Mathematics

A lingering marker of a classroom that remains is an imposing vertical board hanging on the wall. Barany (2020) trace the blackboards' entrance to mathematical higher education back to elite engineering training in Napoleonic France. Other sources attribute the invention of a board to Reverend Samuel Reed Hall (1795–1877), a minister and an education innovator, who is renowned for establishing the first school in the US for preparing teachers, writing the first American book on teaching, and organizing earliest education associations (e.g., Dobbs, 2001). Notably, some resources suggest that Hall used the first board to explain arithmetic. His first board was a large sheet of dark paper that could be written on and erased easily (the invention of the eraser has been also attributed to Hall). Boards were initially met with ridicule, but Hall persisted with their usage (Currier, 1878).

Inviting students to the board to prove mathematical statements has been a famed instructional feature of the so-called "Moore method". In the mathematics community, Robert Lee Moore (1882–1974) has been recognized for his contribution to the education of many mathematicians. Devlin (1999) goes as far as naming Dr. Moore as "the greatest math teacher ever". F. Burton Jones (1977) describes Dr. Moore's method of teaching as follows,

> After stating the axioms and giving motivating examples to illustrate their meaning he [Moore] would then state some definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened, omitted, or partially omitted. [...] When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present the proof. Then the other students, especially those who hadn't been able to discover a proof, would make sure that the proof presented was correct and convincing (pp. 274–275).

The reiterated rigidity of the Moore's method (e.g., Coppin et al., 2009) may overshadow several points of connection between this pedagogy and aspects often associated if not with IBME then at least with inquiry-based approaches (e.g., Hayward et al., 2016; Mesa et al., 2020; Rasmussen & Kwon, 2007). Jones (2017), a former student of Moore, argues that "the most important aspect of Dr Moore's teaching philosophy [...was] to develop the skill of thinking" (p. 301). S. L. Jones juxtaposes *learning to think* and *learning knowledge*, associating the former with generating original proofs and being critical about them. This echoes Dewey's (1938) notion of *reflective inquiry* as a controlled and reflective process, in which students develop general habits of mind for learning (see Artigue & Blomhøj, 2013 and Laursen & Rasmussen, 2019 for the impact of John Dewey's works on IBME).

The instructional design at the core of Moore's pedagogy, i.e., assigning mathematical statements for students to prove, providing them with space and time to do so, and inviting them to the board to share their proofs, resonates to us with an inquiry cycle (Pedaste et al., 2015). Such a cycle can enable "students to learn new mathematics through engagement in genuine argumentation" and empower "learners to see themselves as capable of reinventing mathematics and to see mathematics itself as a human activity" (Rasmussen & Kwon, 2007, p. 191). Furthermore, in spite of its general emphasis on individual learning, Moore's format of students working at the board has a distinctive social and even collaborative character. S. L. Jones (2017) stresses that Moore would remain silent throughout a student's proof, initially leaving it up to the class to spot issues. This classroom dynamics opens the space for communication where students explain their work to others, listen and attempt to make sense of classmates' arguments. Mesa et al. (2020) note that communication of this sort is characteristic to inquiry classrooms.

Laursen et al. (2014) comment that inquiry approaches in college mathematics in the US have grown from the Moore method. In this way, the suggestions of some researchers to invite students to the board as part of their inquiry (e.g., Ernst et al., 2017; Johnson, 2013; Laursen et al., 2014; Wagner et al., 2007; Yoshinobu & Jones, 2011) may be seen as a pedagogical heritage of Dr. Moore. Yet, additional reasons for making these suggestions can be offered.

Greiffenhagen (2014) notes the "iconic status" and "omnipresence" of blackboards in mathematics, referring to them as "indispensable" for mathematicians (e.g., see Wynne, 2021 for a remarkable gallery of mathematicians' chalkboards and reflective essays). Greiffenhagen notes that in popular culture, mathematicians are often depicted as standing in front of a board, when many mathematicians use similar photographs of themselves on their webpages. Furthermore, there are accounts attesting to the role of boards in mathematicians' work. For instance, Lightman (2019) recalls how in the early 1970s, a renowned physicist, Richard Feynman, developed equations describing spontaneous emission from black-holes on the blackboard in Lightman's office. This was a contravention to a then accepted tenet, assuming that black-holes were "completely black" and emitted no energy on their own. The ideas emerged as part of a casual conversation, and no one in the room bothered to copy down the equations. When Lightman returned to his office the next day, the blackboard had been wiped by the building cleaners. And a year later Stephen Hawking became famous for his work on black-hole emission. This story speaks to the value, spontaneity, materiality, and temporarily of mathematics that one can generate on a board.

Paul Halmos refers to a blackboard when describing his famous realization about epsilons in complex function theory. In his words,

> Then one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilons. I understood what limits were, and all of the stuff that people had been drilling into me became clear (Albers, 1991, p. 8).

In his automathography, Halmos (1985) describes additional instances where he arrived at significant mathematical insights when standing next to the board and communicating with someone. This expands the conception of a board from a material surface for capturing mathematics to a distinctive physical space where one thinks, acts, and communicates mathematically. In accord with sociological research (e.g., Artemeva & Fox, 2011; Barany, 2020; Greiffenhagen, 2014), we suggest that people do mathematics *at* the board rather than only *on* it.

The last set of reasons for engaging university students with a classroom board stems from sociological research. This research illuminates boards, and especially the black ones, as having "particular affordances that lend themselves to the presentation of mathematics" (Greiffenhagen, 2014, p. 523). These affordances include the close relations between mathematical thought and writing (e.g., Rotman, 1993), the easiness with which words, symbols, and diagrams can be combined at the board, and the possibilities that boards provide to mathematicians to "think with eyes and hands" (Latour, 1986). These possibilities are relevant to those at the board and those observing this process. For instance, in their investigation of a diverse cohort of 50 mathematicians, Artemeva and Fox (2011) found that all of their participants accounted for a classroom board as a means of providing students with an experience of mathematical processes. In the words of Halmos (1970), "the blackboard [...] provides the opportunity to make something grow and come alive in a way that is not possible with the printed page" (p. 149). Furthermore, many mathematicians in Artemeva and Fox appreciated the way writing on the board slows down their writing and supports their students' focus and understanding. These affordances seem especially relevant to inquiry classrooms, where students are expected to act in ways that are similar to how mathematicians work, reason, and engage with each other.

24.3 Mathematizing at the Board from the Commognitive Standpoint

We propose that the commognitive framework is a viable candidate for investigating the practice in the focus of this chapter. Our choice in commognition grows from the increased attention that this framework has received in the last decade in the community of university mathematics education (e.g., Nardi et al., 2014; Winsløw et al., 2018). Furthermore, given the communicational nature of the focal practice (see Sect. 24.2), it seems only reasonable to consider it through the lens of a theory that is all about communication. We start this section by reviewing the framework in brief and proceed to offering a commognitive perspective on the practice. The latter enables us to consider the potential contribution of this practice to students' learning of mathematics.

24.3.1 Commognition in a Nutshell

Commognition is a socio-cultural, discursive, and participationist framework of learning (e.g., Sfard et al., 2001). From the commognitive standpoint, in a university mathematics course, a teacher and students participate in a mathematical *discourse* (e.g., topology). Their participation manifests in their usage of characteristic *key-words* (e.g., “topology”, “topological space”), *visual mediators* (e.g., “(X, τ)”, a diagram), *routines* (e.g., proving, topologizing a set), and *narratives* that are generally endorsed by mathematicians (e.g., “A set X together with all its open subsets constitute a topological space”).

In a classroom setting, the four abovementioned characteristics rarely feature in their pure form as they are embedded in different types of communication. The first type is *mathematizing*, and it occurs when students narrate about mathematical objects (e.g., “So, let x not equal to y and suppose they’re in X”). Another frequently occurring talk is that of *subjectifying*, that is, communicating about mathematizers themselves (e.g., “I wanted to…”). One kind of subjectifying through which students navigate their mathematical moves is *meta-mathematizing* (Chan & Sfard, 2020). It revolves around what mathematizers did so far, what they are currently doing, and what they are planning to do next (e.g., “Suppose for a contradiction”).

Commognition situates students’ participation in a mathematical discourse on a continuum, the two edges of which can be described as *ritualistic* and *outcome-oriented*.[1] Rituals refer to situations where one’s performance is driven towards carrying out a mathematical procedure. The word “ritual” highlights the anthropological and sociological roots of the term. There, rituals are often viewed as culturally laden ways of acting that are valued for reinforcing traditions and promoting social bonds (e.g., Bell, 2009). Similarly, Lavie et al. (2019) maintain that rituals are appreciated for their performance and often executed for social reasons. On the other end of the continuum, one’s actions are outcome-oriented when they are targeted at growing mathematical narratives and developing routines that are new to the performer.

Distinguishing between ritualistic and outcome-oriented participation entails a methodological challenge. Nachlieli and Tabach (2019) suggest addressing it through attending to circumstances under which a performer implements a routine. Drawing on Sfard (2008), the researchers focus on the circumstances in which a routine was evoked (initiation) and considered complete (closure).[2] In this way, rituals appear as socially encouraged processes, in which a performer adheres to a particular procedure that has been formerly demonstrated. In outcome-oriented cases, a performer is agentive and chooses among several options to pursue the assigned task.

While recognizing the necessity of rituals for mathematics learning, commognition is unequivocal in its wish for students to participate in a mathematical discourse in outcome-oriented ways. Accordingly, contemporary commognitive research focuses on teaching and instructional designs that afford such participation. Nachlieli and Tabach (2019) discuss a special kind of tasks “that could not be

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[1]The original commognitive term for the latter edge of this continuum is “explorative”. We use “outcome-oriented” with the same meaning to avoid possible confusions with the notion of inquiry.

[2]Lavie et al. (2019) offer fine-grained characteristics for diagnosing one’s routine in terms of a ritual and outcome-oriented performance. A “looser” approach to this issue is sufficient for our purposes.

successfully solved by performing a ritual. Rather, a successful completion of the task can only be achieved by participating exploratively" (p. 257), i.e. in an outcome-oriented manner. Next, we argue that the practice in our focus can emerge in such a task.

24.3.2 Mathematizing at the Board

Now that theoretical and conceptual blocks have been laid out, we are ready to offer a commognitive perspective on the focal practice. We refer to it as *mathematizing at the board* and associate it with classroom situations, in which a student (or a group of students) is invited to engage in a public discursive activity and generate a mathematical story (or a narrative) about the objects of their inquiry. In a university setting, this story can be expected to unfold as a deductive sequence of endorsed narratives that are substantiated in accord with a conventional mathematical discourse and norms that have been established in a particular classroom. Once generated, such a story can become a focus for a classroom discussion in terms of its holistic consistency, comprehensiveness, and cohesiveness of its constituting narratives.

We argue that the above perspective is epistemologically different from the ones featuring in the IBME literature in three ways. *First*, growing from a participationist standpoint, we reject what appears as an implicit acquisitionist assumption (cf. Sfard, 1998 for a acquisitionist perspective on learning and Heyd-Metzuyanim & Shabtay, 2019 for an acquisition pedagogical discourse), according to which a student "reflects", "represents", or "communicates" mathematics that they constructed in the previous phases of their inquiry. Instead, we suggest that when mathematizing at the board, students participate in a mathematical discourse in a manner that is distinct from the ones that took place in preceding inquiry steps (for instance, when students initially constructed mathematical stories about the objects of their inquiry) (for a detailed examples see Kontorovich & Greenwood, 2022). *Second*, our perspective underscores the mathematizing component of students' communication, turning mathematical objects into the main characters of their stories. This is different from subjectification-dominated instances, in which students focus on what *they* did with these objects in the preceding steps. *Third*, by construing a classroom board as a physical space for communication, we attend to students' stories as they unfold in various media, including the components that are captured on the board (e.g., text, diagrams, annotation), oral verbatim (e.g., utterances, intonation), gestures, movement, face expressions, gazes, and so on.

Subject to an appropriate instructional design, mathematizing at the board can require students to participate in a mathematical discourse in an outcome-oriented manner (Nachlieli & Tabach, 2019). Similarly to how previously studied procedures are not sufficient to complete a meaningful inquiry, it is hard to imagine a "recipe book" that can prepare students for mathematizing at the board. Specifically, we refer to myriad micro-routines that a student needs to initiate and lead to a closure, such as generate a mathematical text, provide a verbal commentary, use notes, and engage with the rest of the class. Thus, when considered in fine grain, the scope for individual agency and decision making seems too large to “solve the task” of mathematizing at the board ritualistically. Furthermore, Laursen and Rasmussen (2019) argue that “as students discuss, elaborate and critique ideas together, they deepen their understanding and build communication skills, collaborative skills, and appreciation for diverse paths for solution” (pp. 138–139). Accordingly, when accepting the invitation to the board, students implicitly agree not only generate a story that is expected to meet high mathematical standards but to do so in the presence of a particular audience, usually a teacher-mathematician and classmates. Then, if appropriate inquiry norms have been established, the focal task can be inseparable from the need to elaborate, revisit, and address potential critique of the generated story (cf. F. B. Jones, 1977; S. L. Jones, 2017). Successful completion of these processes is unlikely without the storyteller participating in a mathematical discourse in an outcome-oriented manner.

24.4 From a Broad Practice to More Focused Routines

In the previous section, we conceptualized mathematizing at the board in relation to students’ stories that unfold in multiple media. An empirical study summons an operationalization of this conceptualization that would afford exploring this practice through the lens of more focused routines that students implement. The few mathematics education studies that we found casually mention that students put some mathematics on the board, without delving into this process. Thus, we turn to the neighboring bodies of research to extract three intertwining aspects that can inform researchers’ selection of routines to notice and analyze: chalk talk, accounting for the audience, and the distinction between oral and written mathematics.

24.4.1 Chalk Talk

Artemeva and Fox (2011) associate chalk talk with classroom situations where teachers speak aloud “while writing on the board, drawing, diagramming, moving, gesturing, and so on” (p. 355). The researchers consider chalk talk as a pedagogical genre within a discipline “which is realized in the social practices and discursive accounts of key stakeholders” (ibid, p. 346). The complexity of chalk talk stems from myriad communicational and physical routines that a teacher undertakes either in parallel or in a close proximity to each other. These include writing a mathematical text, articulating and providing a metacommentary on it, moving in space, pointing, referring to notes or problem sets, raising rhetorical questions, pausing for reflection, turning to students, and asking questions.

There are reasons to expect students’ mathematizing at the board to bear resemblance to the chalk talk of their teachers. Indeed, when this practice is planned (rather than spontaneous), students can make preparatory notes, which seems not very different from how university instructors get ready for their lectures. Furthermore, Artemeva and Fox (2011) argue for the pervasiveness of chalk talk in many mathematics classrooms as their findings emerged from university teachers at different stages of their career, teaching experiences, countries, educational, cultural and linguistic backgrounds. Bazerman (2010) notes that "knowledge is produced, stored and accessed in specific genres associated with different activity systems" (p. 445). Then, when put in an essentially pedagogical situation, it seems reasonable for students to draw on chalk talk as a system of activities (or routines) of their teachers. Given all the research on the dominance of lectures in university mathematics, this might be the most familiar system to the students.

24.4.2 Audience

Given the public nature of mathematizing at the board, it seems reasonable to attend to how students account for their audience. In the context of argumentation, Perelman and Olbrechts-Tyteca (1969) distinguish between the universal and particular audience. The universal audience consists of all the people that an arguer considers competent, when a particular audience stands for a concrete group of people that the arguer addresses explicitly.

In a classroom setting, it may be tempting to assume that peers and a teacher-mathematician constitute the particular audience for students' public mathematizing. Yet, empirical research shows that even mature students can impose on themselves different and rather idiosyncratic tasks in what may appear as a standard situation (e.g., Kontorovich, 2021a; Krupnik et al., 2018). Then, special analytical tools are needed to delineate the audience and the way it is considered in one's mathematizing.

One potentially useful tool of this sort could be attending to students' use of personal pronouns. Mathematics education has been focusing on teachers' use of pronouns. Pimm (1987) suggests that the ambiguity of "we" allows the teacher to appeal to different referents: the teacher and the mathematics community, the teacher and the students, the teacher only, the students only, or a combination of these. In Rowland (1992), pronouns are interrogated from the perspectives of social positioning, interpersonal power, and generalization. Herbel-Eisenmann and Wagner (2010) attend to personal pronouns as markers of interpersonal positioning of authority (e.g., "I want you to"). Nachlieli and Tabach (2015) show that through personified utterances a teacher can make mathematical objects more accessible to students (e.g., "when *we* move on the graph"). In this way, scrutinizing students' oral and written narratives for pronouns may turn insightful for learning about the audience the students have in mind.

24.4.3 What Is Said and What Is Written

Much has been said on the characteristics of academic mathematical texts (e.g., for a comprehensive review see Morgan, 1998). These have been often described as being dense with terminology and symbols, modest in their use of "grammatical" words, impersonal and authoritative formulations, adhering to nominators rather than verbs, underscoring deductive reasoning, and so on. Many of these descriptors are not unique to mathematics and can also be found in scientific texts. Some of these descriptors are explicitly recommended in purposeful resources for mathematics writing (e.g., Knuth et al., 1989), which endows them with a status of disciplinary norms. Thus, it may be interesting to see whether these descriptors apply to texts that students put on the board, especially in advanced mathematics courses.

Lew et al. (2016) and Fukawa-Connelly et al. (2017) found that in their instruction of advanced courses, university teachers do not only provide definitions, formulate theorems, and prove them, but also address more informal aspects, such as modelling mathematical behaviors, giving examples and methods that can be useful for a range of problems. Notably, while formal mathematics unfolded in writing, more often than not, the informal elements were stated orally. This research justifies paying attention to routines that students employ on the board compared to those implemented orally. The routines through which the students coordinate their work in the two media are also noteworthy.

24.5 Illustrations

To put the conceptualizations presented in Sects. 24.3.2 and 24.4 in-action, we turn to our ongoing project. The project is contextualized in a topology course offered by the mathematics department in a large research-intense university in New Zealand. This course brings together mathematics majors at the advanced stages of their studies and graduate mathematics students. The course has an interactive and collaborative character, providing its students with multiple opportunities for active participation in a topological discourse (Kontorovich, 2021b).

A key activity of the course was inspired by the "Moore method". In this activity, the students are allocated topological statements to prove and provided with space and time to work on them. More complicated statements are usually allocated in advance, giving students at least a week before proving them at the whiteboard for the whole class. Each student is in charge of a different statement and its proving gets scheduled for a particular lesson. In other cases, students work on the statement during the lesson, which is followed by a volunteer proving at the board. The typical course size is about 10 students, and each of them comes to the board several times throughout a semester.

Once a statement is proved, it is absorbed into the classroom discourse. This legitimizes drawing on this statement in the future as well as a teacher-mathematician’s request to re-prove it as part of a test or an exam. This also applies to the statements that students prove. Moreover, if a proof at the board is problematic, the teacher-mathematician does not “fix” it in a traditional sense. Instead, she facilitates a discussion encouraging the whole class to scrutinize the proof, discuss its strengths, and offer suggestions for improvement. This instructional design creates a system of dependencies between the student who mathematizes at the board and the rest of the class.

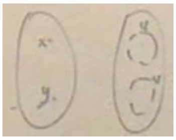
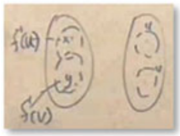
In our project, we video-record the course lessons and collect students’ written work. We believe that the project’s data on students’ proving at the board is suitable to illustrate the preceding conceptualization of the focal practice. Next, we present two proofs generated by Jonah and Virginia (pseudonyms) and proceed to analyze them in terms of three routines: coordination between oral and written narratives, accounting for the class audience, and meta-mathematizing. We choose the specific proofs and routines for the readers to appreciate the richness that can emerge when mathematizing at the board is analyzed in high resolution.

24.5.1 Jonah’s Proof

Jonah’s proof took place in the lesson on Hausdorff spaces. At the beginning of the lesson, the teacher-mathematician defined Hausdorff spaces as those where every two elements can be separated by open sets (i.e. for each $x \neq y$ in X there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$). After discussing this definition and specific examples, the students were divided in pairs and asked to prove propositions, including the following: “Given $f: X \rightarrow Y$ is a continuous function and Y is Hausdorff. If f is one-to-one then X is Hausdorff”. After group work was completed, Jonah volunteered to prove this statement at the whiteboard. He approached the board holding the classroom notes with definitions and propositions but the notebook with his previous work on the focal statement remained on the desk.

Table 24.1 presents the transcript of the proof that Jonah generated at the whiteboard. The table’s columns demarcate between the oral and written components of his story and the rows distinguish between his utterances. Specifically, in some cases, Jonah spoke as he wrote (more or less), when in other instances there was an evident time gap between his articulated and written narratives. We use “. . .” to point at cases where Jonah did not continue his oral sentences, “$^{\downarrow}$” to mark him glimpsing at his notebook, and square brackets for our commentary. Throughout the proof Jonah stood facing the board and with his back to the class. Figure 24.1 presents a snapshot of Jonah’s board on the completion of his proof.

Table 24.1 Transcript of Jonah's proof at the board

Move	Time-stamp	What Jonah said at the board	What Jonah put on the board
1	19:38–19:56	So, let x not equal to y and suppose they're in X	Let $x \neq y \in X$.
2	19:57–20:12	Then f of... f of x is not equal to y because f is one to one	Then $f(x) \neq f(y)$ because f is 1–1
3	20:12–20:45	And... mmmh... Y is Hausdorff so there exist U V open in Y such that	Y is Hausdorff, so there exist U, V open in Y such that
4	20:45–21:00		[a] $f(x) \in U, f(y) \in V$, [b] $U \cap V = \emptyset$.
5	21:00–21:14	And... mhh... suppose [glimpses at the statement to be proved and at what he wrote on the whiteboard; looks hesitant]	
6	20:14–21:29		[sketches on the left down part of the board] 
7	21:30–21:34	[giggles and smiles like of an embarrassment. Then, goes back to his seat and returns with the notes that he made when working on the statement earlier.]	
8	21:35–21:58	[↓] oh yeah! Because f is continuous so... the pre-image of U and the pre-image of V [points at the symbols of $f^{-1}(U)$ and $f^{-1}(V)$ that he just wrote] are open in X	f is continuous, so $f^{-1}(U)$, $f^{-1}(V)$ are open in X.
9	21:59–22:08	[↓] suppose for a contradiction [says in a fading voice]	Suppose for a contradiction
10	22:08–22:23	[as he sketches the dotted circle around y] so this is the pre-image of V	[continues the sketch from 6] 
11	22:24–23:11		, $\exists z \in f^{-1}(U) \cap f^{-1}(V)$ Then $f(z) \in f(f^{-1}(U)) = U$ and $f(z) \in f^{-1}(f(V)) = V$[a] *□

[a] We assume that Jonah meant to write $f(z) \in f(f^{-1}(V)) = V$

24.5.1.1 Coordinating Between Written and Oral Narratives

The story that Jonah generated at the board can be divided into two parts: [1–10] where he speaks as he works on the whiteboard (i.e. "chalk talk" in terms of Artemeva & Fox, 2011), and [11] where he completes the proof in silence. In the former part, probably the most evident routine can be called duplicating since his oral utterances mirror the sentences that he put on the whiteboard (e.g., [1–3] and

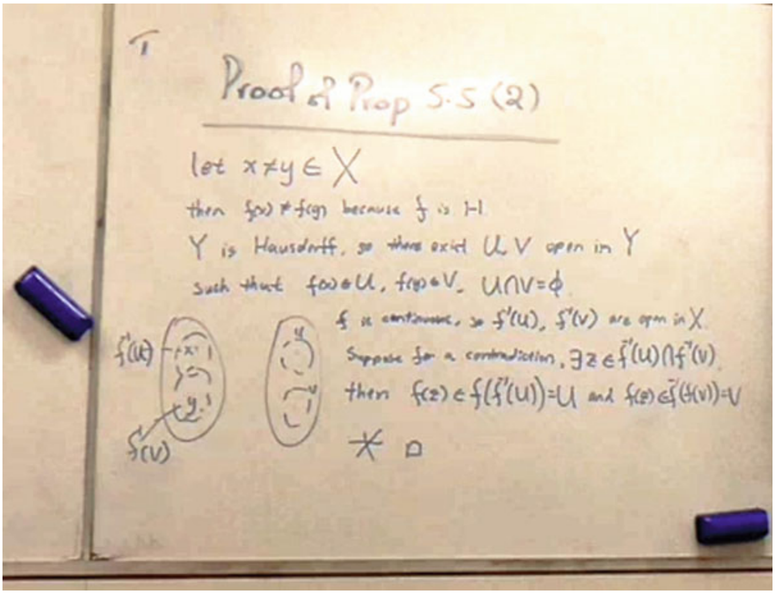


Fig. 24.1 Snapshot of Jonah’s board

[8–9]). The duplication is perfect in the case of written words (i.e., every written word is articulated), when symbols are replaced with the names of the corresponding objects (e.g., “$f^{-1}(U)$” is verbalized as “a pre-image of *U*” and not as “*f* to negative one of *U*”).

24.5.1.2 Accounting for the Audience

Jonah’s narratives both oral and written, appear purely objectified and do not leave much room for human agency. Indeed, even in [5] and [9], the people whom he expected to “suppose” remained unnamed, which is typical for academic mathematical texts (cf. Morgan, 1998). This allows wondering whether Jonah was the particular audience of his own proof. Considering his mathematizing as a proof for himself explains why he practically ignored his classmates and the teacher throughout the process, which was evident in Jonah standing with a back to the class, often blocking the text that he put on the board with his body. This also explains him gradually “turning off” the oral component of his proof – when one is communicating with themselves, the talk is loud even when all others hear is silence.

It may seem peculiar when someone mathematizes publicly but expresses almost no markers of acknowledgment of the people who witness their craft. However, let us recall that Jonah was proving the same statement that the whole class worked on beforehand. Moreover, the teacher-mathematician stopped the group work by asking whether students were finished and “ready to present”. Accordingly, Jonah might have assumed that similarly to him, his peers had completed their proofs already, and then have little interest in his work. This interpretation also relates to Jonah's usage of his notes.

Jonah came to the board with the classroom notes, leaving the record of his previous work on the focal statement behind. He half-heartedly grabbed this record in [7], when it seemed like he was incapable to pursue his mathematical story. He glanced at his notebook only twice in [8–9] and completed the rest of the proof without resorting to it again. This pattern allows proposing that Jonah was driven to prove the statement at the whiteboard "on his own" as much as possible, through minimizing his reliance on his previous work.

24.5.1.3 Meta-Mathematizing

The point at which Jonah resorted to his auxiliary notebook does not appear accidental to us. Indeed, in [1–6] and [8] he unpacked the notions of "Y is Hausdorff", "f is one-to-one", and "f is continuous" into detailed narratives about open sets and points, and into a diagram. However, the central problem of the statement (cf. Selden & Selden, 1995) was in showing that the statement's hypothesis necessitate that "X is Hausdorff", which required a meta-mathematizing move. This move occurred in [9], where Jonah set up the course of his proof towards a contradiction. And once he assumed the existence of z in the intersection of the pre-images of U and V – first in the diagram and then in text – he reached this contradiction rather effortlessly.

Jonah's casual "Suppose for a contradiction" in [9] remained implicit regarding the specific narrative that he presumed to be false. Indeed, until now, each of his written narratives was deductively inferred either from the statement hypothesis (e.g., [2–3], [8]) or from the narrative that was generated just beforehand (e.g., [3–4]). However, made in the sixth line on the whiteboard, the supposition contradicted to [4b] – the narrative about the sets U and V being disjoint. Similarly, Jonah's last sentence [11] can be continued with "then $f(z) \in U \cap V$, a contradiction since U and V are chosen so that $U \cap V = \varnothing$ from [4b]". Perhaps, these connections and inferences were obvious to Jonah; yet, for us as external observers his meta-mathematizing left room for further specification.

24.5.2 *Virginia's Proof*

The second proof comes from the lesson on compactness. In the classroom, a space was defined as compact if every collection of open sets covering it has a finite subcover. Before that lesson Virginia was assigned the statement positing that if X is compact and $\mathcal{C}$ is a collection of its closed subsets with the finite intersection property (*FIP* hereafter) then $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. To recall, a collection of subsets is said to have FIP if every finite sub-collection has a non-empty intersection. Table 24.2 contains a transcript of Virginia's proof at the board. In addition to the notation explained in Sect. 24.5.1, we use "↗" for instances where Virginia turned her gaze, head or body from the board towards the class, and "↙" when she turned back to the whiteboard. Figure 24.2 presents a snapshot of the whiteboard when Virginia completed her proof.

24.5.2.1 Coordinating Between Written and Oral Narratives

Virginia's proof at the board can be decomposed into pairs of oral and written narratives. Only in [1] and [6] both narratives were generated more or less at the same time. In the remaining cases, there was an apparent time gap between what Virginia said and what she wrote (e.g., for first-written-then-said see [3–4] and [18–19], for first-said-then-written see [7–8] and [12–13]). Then, the two components rarely duplicated each other in full, when the oral utterances seem to elaborate on their written congeners. For instance, in [1–4], Virginia orally combined two written sentences into a single narrative; in [2–3] and [19–20] she used the logical connectors "then", "because", "so", and "cause" that spelled out the relations between her written statements.

Another notable routine pertains to Virginia replacing conventional names of the mathematical objects that she symbolized in writing with deictic "this", "that", "all these" (e.g., [6–7], [9], [23]). While research occasionally interprets students' use of deictic words as evidence of them lacking an appropriate terminology (e.g., Nardi et al., 2014), this does not seem to be the case of Virginia. For instance, in [4], she verbally acknowledges $\{C_i\}_{i \in I}$ as a collection of closed subsets; in [6], she articulates their intersection; and in [7] she stresses its complement. Accordingly, we suggest that by using deictic words in combination with conventional names of mathematical objects, Virginia verbally shifted attention from the former to the latter in preparation for manipulating with the corresponding objects in her next step.

24.5.2.2 Accounting for the Audience

While in Jonah's proof his accounting for the audience is hard to find, in Virginia's proof it is difficult to miss. Indeed, the transcript presents instances of her humanizing the written sentences by using "we" (e.g., [4]), turning and gazing at her classmates (e.g., [9]), and breaking the lines of symbolically compound statements with expanded oral utterances (e.g., [7–11]). These patterns are consistent with the context of Virginia's proof: she was allocated the statement in advance, which endowed her story at the board with value for the whole class. This interpretation is also in tune with her not only proving in the sense of generating a deductive sequence but also elaborating on the intermediate narratives (cf. proofs that prove and proofs that explain in Hanna, 1990).

Table 24.2 Transcript of Virginia's proof at the board

Move	Time-stamp	What Virginia said at the board	What Virginia put on the board
1	9:38–9:43	Suppose *X* is compact	Suppose *X* is compact
2	9:44–9:49	And emh… [↓]	
3	9:50–10:28		Suppose $\{C_i\}_{i \in I}$ [↓] is [↓] an arbitrary [↓] collection of closed subsets with the finite intersection property
4	10:29–10:37	[gazes at what she wrote in 1–3 and says while pointing with a marker] OK, so it says suppose that *X* is compact and suppose we have a collection of closed subsets with the finite intersection property	
5	10:38–10:47	[↓] and then I [↗↙] wanted to prove it by contradiction so if we …	
6	10:48–10:57	Suppose that the intersection [↓] of <u>this</u> [intonates when writing $\bigcap_{i\in I} C_i$] is the empty set [↗↙]	Suppose $\bigcap_{i\in I} C_i = \varnothing$.
7	10:58–11:05	Emh… then *X* is a complement of <u>this</u> [points at $\bigcap_{i\in I} C_i$ in 6]	$X=$
8	11:05–11:11		[continues the previous line] $\left(\bigcap_{i\in I} C_i\right)^c$
9	11:11–11:12	Because <u>that is an empty set</u> [points at $\varnothing$ in 6 and ↗ when saying "because"]	
10	11:13–11:17	[↙]and then… emh… which is equal to [↓]	
11	11:18–11:23		[continues the previous line] $=\bigcup_{i\in I} C_i^c$
12	11:24–11:30	And.. Emh… and these all these si-ai-es are closed [points at $\{C_i\}_{i \in I}$ from 3] so, emh… /si: Iz/ are open [points at C_i^c in 11]	
13	11:30–11:40		C_i close $\Rightarrow C_i^c$ open
14	11:30–11:48	So then… emh… [↓] the set of all the si-ai complements is an open cover *X*	
15	11:48–12:05		So $\left\{C_i^C\right\}_{i\in I}$ is open cover for *X*
16	12:06–12:10	Emh… emh… then… because *X* is compact… emh…	
17	12:11–12:15		By compactness [↓]
18	12:16–12:20	Emh… there is	

(continued)

Table 24.2 (continued)

Move	Time-stamp	What Virginia said at the board	What Virginia put on the board
19	12:21–12:37		There is [↓] a finite subset $J \subseteq I$ [↓] such that [↓] $X = \bigcup_{i \in J} C_i^c$ [↓]
20	12:38–12:54	[↗] cause we got... cause it's... cause X is compact then there is a finite subsets of the... such that X is the union of finite subsets [points at $\bigcup_{i \in J} C_i^c$ from 19] and...[↓][↙]	
21	12:54–12:58	Oh and Soo... and this is the same as...	
22	12:58–13:05		[continues the previous line] $= \left(\bigcap_{i \in J} C_i\right)^c$
23	13:05–13:08	This [points at the brackets in 22] so the inverse is the empty set	
24	13:09–13:11		[continues the previous line] $\bigcap_{i \in J} C_i = \varnothing$
25	13:11–13:22	Which is a contradiction cause we said that... because it [↗↙] got the finite intersection property.	

24.5.2.3 Meta-Mathematizing

Virginia articulated the meta-mathematical move of contradiction in [5] but did not capture it in writing. The written component of her proof ended in [24] with "$\bigcap_{i \in J} C_i = \varnothing$" without elaborating on what enables this narrative to conclude the proof. The oral congener of this narrative in [25] explains that "it got the finite intersection property", when "it" stands for "$\{C_i\}_{i \in I}$" whose FIP was noted in [3]. And since $\{C_i\}_{i \in J}$ is a finite sub-collection of $\{C_i\}_{i \in I}$, the contradiction follows. Accordingly, we suggest that the key meta-mathematical move in Virginia's proof could be elaborated further.

24.6 Summary

In this chapter, we drew attention of the IBME community to the core inquiry practice of students' communicating the results of their inquiry; a practice, that according to some can unfold on a classroom board but is often overlooked in empirical research. Then, we focused on how the board version of this practice can be investigated.

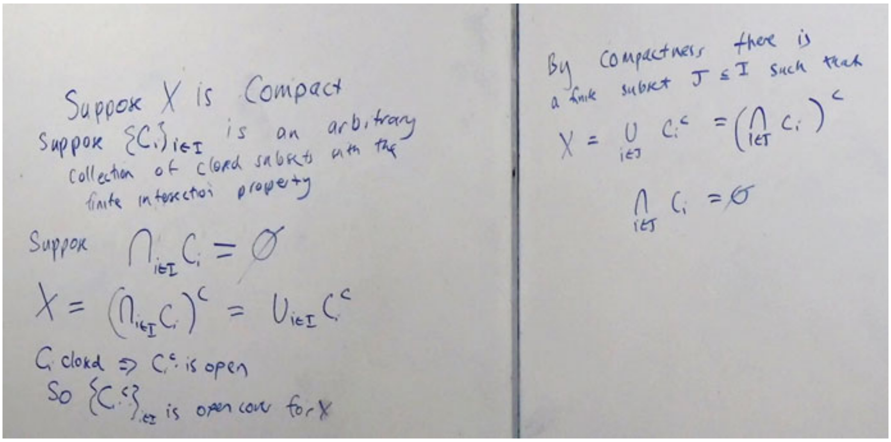


Fig. 24.2 Snapshot of Virginia's board

In attempt to address this question, we linked across different bodies of literature to illuminate the special status of boards in mathematics and its teaching, and the affordances of a board as a physical place for mathematizing. These links contribute to the IBME literature by putting forward arguments for engaging students with a classroom board as part of the focal practice. We believe that drawing on the literature that has been concerned with mathematicians and how they work, makes these arguments especially relevant to IBME. These arguments may also be convincing for practitioners who orchestrate inquiry in their classrooms and wish to maximize its affordances.

Then we grounded the focal practice in the commognitive framework, giving rise to the notion of mathematizing at the board. Capitalizing on the framework that has not been used yet in the context of inquiry, we illuminated this practice in a light that is different from what is accepted in the IBME literature: not as a "mere" reflection on or representation of mathematical work that was completed beforehand but as mathematizing in its own right. Specifically, we associated this practice with students generating stories whose main actors are mathematical objects; stories that in the university setting can emerge as deductive chains of well-substantiated narratives and that the classroom can scrutinize for their holistic consistency, comprehensiveness, and cohesiveness. This conceptualization enabled us to put forward an argument for the outcome-oriented potential of mathematizing at the board.

Next, we proposed that students' mathematizing at the board can be analyzed in fine grain with the commognitive construct of routines. Drawing on the sociological research and studies with university mathematics teachers, we proposed attending to how students coordinate between what they say and what they put on the board, their accounting for the audience, and their meta-mathematizing. To illustrate these proposals, we presented two proofs from our project in a topology course. These instances demonstrated a range of communicational and physical routines that students can implement at the board even within a proof that is only a few minutes long. The two protagonists of the illustrations showcased how differently mathematizing at the board can play out.

By grounding a single inquiry practice in commognition, we offer a "proof of concept" or of the viability to bring this theoretical framework to IBME research. The emerging insights may be a marker of the broader potential of commognition to promote the IBME agenda. Indeed, Artigue and Blomhøj (2013) maintain that "[f]or research it is relevant to pinpoint the mechanisms responsible for the learning outcome of IBME" (p. 798) and that "different theoretical frameworks can support the conceptualization of IBME [. . .], and its implementation in practice" (p. 809). The acknowledged conceptual and analytical powers of commognition, and our personal experience with this framework make us optimistic about its promise to contribute to these endeavors.

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Chapter 25
Preservice Secondary School Teachers Revisiting Real Numbers: A Striking Instance of Klein's Second Discontinuity

Berta Barquero and Carl Winsløw

Abstract The real numbers form the basis for most of secondary school mathematics, from analytic geometry to calculus and normal distributions. However, both in secondary school and in undergraduate mathematics, it is customary to stay with an informal description of what the real numbers are, while the university courses on analysis certainly stipulate that they form a complete metric space, and prove many advanced theorems based on that. Future secondary school teachers typically see little relation between the advanced theory they learnt from undergraduate mathematics courses and what they are going to teach in secondary school. The general "gap" is often called "Klein's double discontinuity". This chapter focuses on the particular case of real numbers. More concretely, we focus on identifying certain challenges related to the second discontinuity and the efforts and effects of trying to bridge it through a "capstone course" at the University of Copenhagen. We analyse students' work with assignments designed to enable them to draw on university mathematics to solve problems involving real numbers and software commonly used in secondary school. This analysis is complemented by a survey and some interviews to get more insight into students' perceptions about the relevance of the real number theme for high school teaching.

Keywords Mathematics teacher education · Real numbers · Klein's second discontinuity · Computer algebra systems · Anthropological theory of the didactic

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25.1 Introduction

In Denmark, Spain, and many other countries across the world, university based initial education of secondary school mathematics teachers consists of a more or less extensive study of academic mathematics, complemented by some courses that are specially dedicated to preparing them for teaching.

Felix Klein, in his seminal book on "Elementary mathematics from a higher viewpoint" from 1908 (Klein, 2016), warmly supports the idea that future teachers should be well acquainted with contemporary mathematics, but at the same time laments the distance which appears between academic mathematics courses and the task of teaching mathematics in secondary school. He introduces the so-called "double discontinuity" encountered by secondary mathematics teachers. Here we focus on the second discontinuity which concerns the transition from university studies to secondary school teaching. Klein's book was based on the lectures he gave for several years at Erlangen University, in view of bridging this second discontinuity by revisiting, with the students, the school mathematics topics (from numbers over algebra to geometry and basic analysis), while drawing on the more advanced knowledge they had acquired at university.

This idea has been adopted in different ways for teacher education, from the *Stoffdidaktik* ("content didactic") courses taught in many German universities to the "capstone" and "methods" courses offered in North American teacher education (Buchholtz et al., 2013; Ferrini-Mundy & Findell, 2001; Sultan & Artzt, 2018). Such courses are often welcomed by students who envisage a teaching career, but they are also frequently criticised for being too far from what students perceive as relevant for their profession, or from the students' university background in mathematics, or both. In fact, substantial uses of university mathematics courses in secondary school mathematics teaching are not so easy to identify or establish, as evidenced for example by the recent study of Yan et al. (2022).

In this chapter, we focus on a particular piece of mathematics: the real number system. Real numbers appear everywhere in the advanced study of functions and geometry, both at university and in secondary school. The (up to isomorphism, unique) complete ordered field of real numbers certainly forms the lifeblood of Mathematical Analysis, both as this area is taught at universities and in contemporary mathematical research. It is usually less clear what role the more subtle features of real numbers play (or may play) at the secondary level (González-Martín et al., 2013; Winsløw, 2015; Licera et al., 2019). Certainly, secondary school pupils encounter numbers that are not rational, and functions and function properties which rely, theoretically, on the real numbers being complete. Analytic geometry, as taught from lower secondary school onwards, certainly operates with points and lines that cannot be restricted to, for instance, objects with only rational coordinates. However, the traditional curriculum at secondary level is often informal, if not entirely mute, when it comes to the raison d'être, definition and properties of real numbers. Their consequences for other (more visible) mathematical themes, such as functions, derivatives or analytic geometry, are often taken for granted or postulated as facts of nature, while they formally rely on subtle arguments involving assets of the real number field such as completeness. In fact, these subtleties are also often given a relatively superficial treatment in undergraduate courses, where they typically appear as preliminaries to real analysis. The question arises about what secondary school teachers need to know about such aspects of real numbers and what could or should be done to fill these needs.

We start by drawing on a reformulation of Klein's second discontinuity (Winsløw & Grønbæk, 2014) in terms of the Anthropological Theory of the Didactic (ATD) (Chevallard, 1999). Next, we introduce the main tools from ATD that allow us to problematise this second discontinuity, in relation to the institutional positions that individuals can occupy, and use this to develop Klein's questioning of the standard view, that there is a simply continuity between the mathematical praxeologies encountered in the two institutions. More concretely, this chapter investigates how these matters appear in a capstone course taught at the University of Copenhagen. Most students in this course are doing a minor in mathematics and have only done parts of the undergraduate programme in pure mathematics, with the explicit goal of teaching mathematics at secondary level along with their major subject. We analyse one of the tasks proposed to question elementary contents about real numbers in this particular university context, and students' individual and collective work with the tasks and the difficulties they encounter – both at the technical level of mathematical work, and at the level of perceived relevance of the subject for high school teaching. The point of this case study is to identify certain challenges relating to the second discontinuity and efforts to bridge it in the specific case of the real number system.

25.2 Formulating Klein's Double Discontinuity Within the ATD

The Anthropological Theory of the Didactic (ATD) models practices and knowledge as residing in institutions, using the notion of *praxeology* (Chevallard, 1999). We outline the main tools used here. A praxeology $O = (P, L)$ consists of a *praxis* P (types of tasks, techniques to solve them) together with *logos* L (discourse about the praxis, and theory that explains, relates, and justifies the practical discourse in more general ways). Institutions are considered as configurations of *positions* that individual persons may occupy within the institution—such as being mathematics teacher in Spanish upper secondary school. These positions are more or less characterised by how they are supposed to relate to the praxeologies carried out in the institution. For a position p and a praxeology O living in the institution I, we denote that relationship by $R_I(p, O)$. It is important here that praxeologies exist in more than one institution (e.g., fraction arithmetic appears in both primary and secondary school) but still with subtle differences (such as preferred tasks, techniques, terminology, notations).

In this paper, we consider praxeologies related to the real numbers. As an example of a common type of task, we could name producing a decimal representation of a given fraction of integers. Techniques may be both manual, as a division algorithm and rules for rounding off, and instrumented, such as buttons to press and settings to set on a calculator. In different schools, the discourse on these practices may differ in subtle ways, and the theory supporting that discourse may be more or less formal. In school institutions, syllabi and textbooks (cf. González-Martín et al., 2013) may help to get some impression on the relationship that students are supposed to develop to this praxeology. It is usually much less explicit what relationships are expected from teachers.

As further explained by Winsløw and Grønbæk (2014), Klein's double discontinuity can be interpreted as the transitions.

$$R_S(p, O) \rightarrow R_U(\sigma, \omega) \rightarrow R_S(t, O) \tag{25.1}$$

where $S =$ school, $p =$ pupil in S, $t =$ teacher in S, $U =$ university, $\sigma =$ student in U, and finally O and ω are mathematical praxeologies in S and U respectively, which involve the same mathematical objects or tasks – such as representing a fraction of integers as a decimal. Typically, student teachers occupy position σ after position p, and before t.

As for the second transition in (Eq. 25.1), the question is how to develop $R_U(\sigma, \omega)$ in order to strengthen the continuity and improve the contribution of $R_U(\sigma, \omega)$ to a future $R_S(t, O)$. In a capstone course, praxeologies that are somehow between ω and O may be addressed. For instance, one could aim for the relation of t to an informal or missing mathematical theory within O to be strengthened by mobilising some theoretical elements from a relevant ω. To be concrete, to explain and justify the first step in

$$\frac{2}{3} = 0.\bar{6} \approx 0.67$$

the calculation

$$\frac{2}{3} = \frac{6}{10} \cdot \frac{1}{1 - \frac{1}{10}} = 6 \cdot \sum_{n=1}^{\infty} \frac{1}{10^n}$$

connects the meaning of $0.\bar{6}$ with a convergent infinite series encountered in basic Calculus. While the technique in O to establish the identity is likely to be either long division or calculator use, it could be important for $R_S(t, O)$ to include the meaning of periodic infinite decimals as *limits* of sequences of rational numbers, and more generally to connect limits to the right-hand side of the identity

$$0.c_1c_2c_3\ldots = \frac{c_1}{10} + \frac{c_2}{100} + \frac{c_3}{1000} + \ldots$$

so that it is in fact a definition of the left-hand side as a number, whose "existence" can be inferred from the *logos* of some (not too fancy) university praxeology ω. We can denote such a new relationship to O – to be obtained within U – as $R_U(\sigma, O \cup \omega)$. In fact, the goal of Klein's book, and the typical capstone course, is to establish links for students between mathematics learned at university and the mathematics taught and learnt in schools. A "capstone course" for future teachers thus, in general, consists of adding an intermediate link to the transition:

$$R_S(p, O) \rightarrow R_U(\sigma, \omega) \xrightarrow{capstone} R_U(\sigma, O \cup \omega) \rightarrow R_S(t, O) \tag{25.2}$$

with further specifications of the goal being the concrete combinations $O \cup \omega$ of school and university mathematics, and the new relation to be defined in terms of the concrete praxeologies involved. Looking again at the example pursued above, one could, for instance, aim for the students to know and use the relation between (certain) infinite series and (decimal representations of) real numbers, both in praxis and logos.

The research questions of our study can now be formulated as follows:

*RQ*1*:* How can the "capstone" passage in (Eq. 25.2) be organised in the case of O and ω being praxeologies concerning the real number system, its uses and its properties?

*RQ*2: What are the effects of such organisations, both in terms of students actually developing links between O and ω, and in terms of students' perceptions of those links as being relevant to teaching?

It is of course only possible to give partial answers to these questions: examples of organisations of the capstone passage, and examples of the links and perceptions such a concrete organisation produces in a given context. The main purpose of the chapter is to show, by just some aspects of one case, what ideas and challenges can be involved when setting up such an organisation, and especially how to use the theoretical framework proposed above.

In the next section, we use the theoretical tools established above to review previous work related to the above questions, which also includes proposals that fall under the scope of RQ1.

25.3 Real Numbers in Capstone Mathematics for Future High School Teachers

A common situation in many universities is still the classical situation (Eq. 25.1), which corresponds to a kind of empty answer to RQ1. In this case, one may still investigate RQ2, which then asks what links between O and ω that students may develop by themselves, for instance if their contacts with university theory and practice on real numbers lead to any progress in their command of corresponding school praxeologies. There is no shortage of studies on university mathematics students' and secondary school teachers' relationships to more or less basic aspects of the real numbers, for instance, the meaning of irrational numbers (Fischbein et al., 1995; Sirotic & Zazkis, 2007), infinite decimals and square roots (Bronner, 1997), completeness (Bergé, 2010), or identities of type $0.\bar{9} = 1$ (Ely, 2010). In most of these studies, we do not know exactly how $R_U(\sigma, \omega)$ was formed or if the subjects have been exposed to any systematic effort to build $R_U(\sigma, O \cup \omega)$, but such studies often find that a considerable part of the informants have not advanced beyond $R_S(p, O)$, at least when it comes to the specific points tested.

As regards $R_S(p, O)$, this is typically limited to *meeting* and occasionally *using* a few instances of irrational numbers (especially π and a few square roots). Even if their irrationality is discussed, it may remain rather marginal to the practical situations in which students use decimal representations of such numbers (González-Martín et al., 2013). The increasing use of calculators may cause all practical use of numbers to involve only finite decimals. If the set of these is denoted $\mathbb{D}$, one could say that $\mathbb{R} \approx \mathbb{D}$ for many students at this level, and even the official requirements for $R_S(p, O)$ may not challenge that. Note that student work with fractions and algebraic expressions involving quotients is usually still required, even when concrete numbers appear in the end to be from $\mathbb{D}$.

Given this state of affairs, one may in fact consider that $R_U(\sigma, \omega)$ is more or less irrelevant to $R_S(t, O)$. Moreira and David (2008, p. 37) went a step further and argued that "the academic mathematics approach to number systems may conflict with the kind of mathematical knowledge teachers need in practice", essentially because the formal approach neglects the meaning of different models of the real number system – which could indeed be important at various stages of $R_S(p, O)$ and hence for the teacher. Refusing the idea that altogether $R_U(\sigma, \omega)$ could contribute to $R_S(t, O)$ may not rule out the existence of "capstone" or supplementary university courses for students aiming to become teachers. When these are disconnected from the previous experience, one would have a variant of (Eq. 25.1) that could be represented as:

$$R_S(p, O) \to R_U(\sigma, \omega) \oplus R_U(\sigma, O') \to R_S(t, O) \tag{25.3}$$

where $R_U(\sigma, \omega)$ and $R_U(\sigma, O')$ are more or less independent or perhaps even in conflict; here the notation O' reflects that the version of O' presented in the institution U could be somewhat different from O. A common strategy for building $R_U(\sigma, O')$ would then be to work with situations from (or very close to) school mathematics that are, on the other hand, very challenging, with much more autonomous study and research required from students than is usual in academic courses. An ambitious design in this direction, given to future secondary school teachers and focused specifically on the real number system, was designed, and experimented by Licera (2017). In her proposal, the author works on the construction of the real numbers based on the measurement of quantities that differ from both the university and the secondary school praxeologies. While being closer to S in terms of the type of activities carried out during the construction, the kind of O' here considered aims to stablish a new is the relationship between real numbers and the measurement of quantities.

From a position closer to that of Klein (2016), it may be contended that "a construction of the real numbers, a proof that they satisfy the (...) field axioms, and a proof that they satisfy the Completeness Axiom, are necessary for teachers", since these "need to know how to prove what is unstated in high school in order to avoid false simplifications" (CBMS, 2010, p. 60). Kramer (2014) agrees with this position and proposes that $R_U(\sigma, \omega)$ needs to include not only the full formal construction of $\mathbb{R}$ following Cauchy, but also formal versions of more school relevant models ($\mathbb{R}$ as points on a line, $\mathbb{R}$ as "all" infinite decimals), and complete proofs of the isomorphism between these models and Cauchy's construction. A full exposition is given by Kramer and von Pippich (2013, pp. 119–153). It is clear how this links to and extends $R_U(\sigma, \omega)$, but the link to $R_S(t, O)$ is less obvious. One could easily end up with a variant of (Eq. 25.1), with new potential links that often remain invisible to individuals in the transition.

A midway position seems to be needed to realise (Eq. 25.2), by identifying and organising situations for students (σ) to with substantial and challenging links between university mathematics (ω) and the treatment and use of real numbers at school (O). Sultan and Artzt (2018, pp. 285–357), in a capstone textbook for secondary teachers, can be said to propose such a midway. Drawing on students' knowledge from basic analysis and naïve set theory, several "school level themes" are taken from first intuition to precise results. These themes include properties of the number line (density properties etc.), basic arithmetic rules, construction of exponential and logarithmic functions, decimal expansions of rational and general real numbers, and the existence of many (uncountably many) non-algebraic numbers as evidence that $\mathbb{R}$ hugely exceeds $\mathbb{D}$ and $\mathbb{Q}$. The "answers" to RQ1, whose effects we study to answer RQ2, is largely based on this text, naturally with additional didactic choices and emphases that we now proceed to describe.

25.4 Context of the Capstone Course UvMat and Methodology for the Case Study

The capstone course called UvMat (an abbreviation for the Danish equivalent of *Mathematics in a Teaching Context*) is held yearly in the University of Copenhagen. Most of the (typically, 25–30) students in this course are finishing a 2-year minor in mathematics to become high school teachers in this subject along with their major (at Master level). Some students take the mathematics minor much after their major, and then often have substantial teaching experience in the other subject. Due to the shortage of qualified mathematics teachers, there are even temporary positions to teach mathematics with less than full qualifications, meaning that a larger number of students usually have some experience with high school mathematics teaching.

The overall goal of the course is to enable the students to work with upper secondary school mathematics by relating it to relevant parts of their academic bachelor courses. It runs over a quarter (7 weeks), during which students devote half of their time to it. The course combines weekly lectures with exercise sessions, students' individual and collective work on weekly group assignments, for which they also receive supervision and written feedback. The final exam is written and individual.

The mathematical topics focused on come from combinatorics, arithmetic, algebra, and statistics. One theme, covered over 2 weeks, concerns real numbers and functions defined on them (such as exponential and logarithmic functions). In particular, the existence and (quasi-)uniqueness of decimal representations of real numbers is worked on rigorously based on completeness and series known from first-year courses.

Winsløw and Grønbæk (2014) describe some of the challenges faced by the theme on real numbers. The Danish high school curriculum does not treat the properties of real numbers (and its main subsets) in any detail and thus develops Calculus in an intuitive way, with massive use of instrumented techniques. The first-year courses in Analysis swiftly pass through topological "preliminaries" like the completeness of $\mathbb{R}$ and classical theorems related to continuity, compactness and so on, to focus on a rigorous treatment of differential and integral calculus of functions defined on $\mathbb{R}^n$, along with more advanced topics. Students generally find these courses difficult and disconnected from what they learnt in high school (cf. also Gravesen et al., 2017).

In the real number topic, which run over 2 weeks (weeks 3 and 4), UvMat mainly seeks to deepen the students' knowledge about $\mathbb{R}$ and connect with questions related to basic functions met in high school to build a new relationship of type $R_U(\sigma, O \cup \omega)$. We consider here some elements of the weekly assignment for week 4. This assignment included some tasks close to what could be encountered in high school (analysing graphs of rational functions) but with specific questions to make students draw on more advanced knowledge, including material from the book on the decimal representation of real numbers (details below). As in many other course activities, the purpose is to place students in an intermediate position between U and S, to make them realise that problematic phenomena of mathematical praxeologies in S can be theoretically explained while drawing on praxeologies encountered in U. The study of these problematic phenomena also implies moments of *questioning* the school mathematical knowledge, in the sense that students should realize how its theoretical elements are sometimes insufficient to account for results of common practices, such as graphing done with computer software (cf. below).

Our aim here is to analyse some salient details and effects of this effort, both in terms of concrete designs (RQ1), the links students actually develop between O and ω, and students' perceptions of those links as being relevant to teaching (RQ2). On the one hand, we analyse students' written work on the weekly assignment from course week 4. We have selected the 14 groups' answers to the last task (quoted later) over six in total that composed the weekly assignment. Groups' answers are analysed based on an *a priori* analysis done by the researchers, before its implementation, in terms of subtasks likely to be considered by the students when addressing a particular task or question asked to be solved. We also present the analysis of students' responses to a survey, distributed at the end of the course, and complemented by in-depth interviews with four of the students. The interviews were carried out in English by the first author, video recorded and transcribed, and then analysed by the authors jointly.

The first five tasks in the assignment explore $\mathbb{D}_n = 10^{-n}\mathbb{Z}$ $(n \in \mathbb{N})$ as subsets of $\mathbb{R}$. In particular, students construct a sequence (a_n) such that $a_n \in \mathbb{D}_n$ and $a_n \to \sqrt{3}$. The last task [*T*] is:

Using Maple, investigate what the graphs of the functions given by

$$f(x) = \frac{x^2 - 3}{x - a_{10}} \quad \text{for } x \neq a_{10}$$

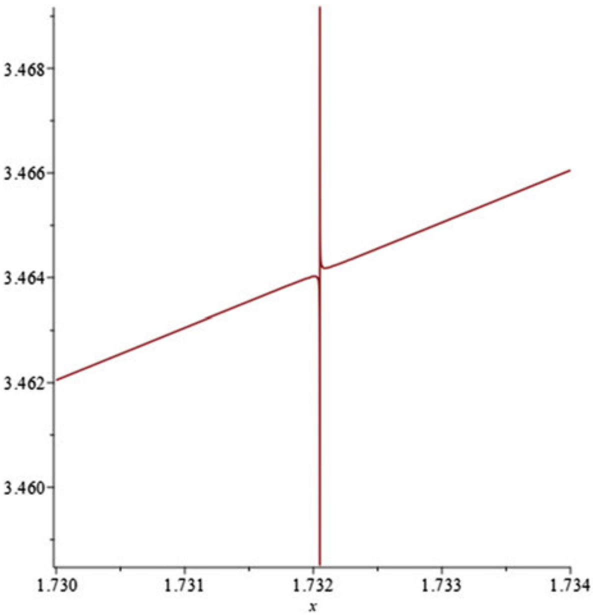
$$g(x) = \frac{x^2 - 3}{x - \sqrt{3}} \; for \; x \neq \sqrt{3}.$$

look like. Comment.

We note here that, depending on the interval of plotting, Maple may show the function graphs as

1. identical straight lines, or
2. different straight lines, or
3. very similar *X*-shaped figures (see Fig. 25.1; in fact, due to rounding error, even the graph of *g* may get a kind of "vertical asymptote" in Maple, when plotted on a very tiny interval around $\sqrt{3}$), or

Fig. 25.1 *X*-shaped graph of *g*, produced by student group 5



4. g as a line and f as anything between a strange X and a nice smooth curve with a vertical asymptote.

The standard plot shows (1), but the students are supposed to doubt the identity of the graphs, as they know $a_{10} \neq \sqrt{3}$. This is likely to push them towards experimenting with the plot settings, in particular, by modifying the plot domain.

Our purpose is to analyse the students' praxeologies, as reflected in their written work. This analysis is based on an initial analysis of the particular mathematical task T. This first step consisted of dividing T into possible subtasks, as described below, and to anticipate the kind of techniques and justifications associated to these tasks, and their roots in O and ω. Moreover, as the task required the use of Maple, we also indicate the expected interaction of instrumented and non-instrumented techniques.

The first description of subtasks concern what can be done to investigate f and g separately:

T_1: Plot the graph of the function on one or more intervals (instrumented O-techniques)
T_2: Identify visual properties of the graphs, such as asymptotes (O-techniques)
T_3: Look for algebraic developments of the function expression (O-techniques)
T_4: Investigate the convergence and boundedness of the function at a_{10} and/or $\sqrt{3}$ (potentially, instrumented O-techniques and ω-techniques)

Some of these subtasks are most likely to be carried out once certain differences appear from carrying out T_2 in both cases. Then, the students may engage in further subtasks related to the "Comment" part of the question (understood as, explain the differences and similarities of the graphs):

T_5: Numerical or graphical comparisons of values of the functions (instrumented O-techniques)
T_6: Discuss if the functions are identical, anywhere, somewhere? (O-techniques, including instrumented ones to plot the functions together, and ω-techniques, such as polynomial division, which are covered previously in UvMat, or other courses)
T_7: Compare convergence properties of the functions at a_{10} and/or $\sqrt{3}$ (potentially, instrumented O-techniques, and ω-techniques)
T_8: Discuss the nature of the singularities (removable/essential, ω-techniques)
T_9: If defining f_n in analogy with f (using a_n in the place of a_{10}) does $f_n \rightarrow g$ in some sense, and what does the answer mean? (ω-techniques)
T_{10}: Investigate if the properties of certain number sets, such as completeness, countability or denseness, help to interpret the differences between the graphs, and how they apparently depend on the interval used for plotting (ω-techniques)

This *a priori* analysis was then used to analyse students' weekly reports by identifying the particular subtasks, techniques, and elements of the technology and theory that appear in the students' responses. The analysis was carried out by the two authors independently and then compared, with a high degree of convergence.

Concerning the students' responses to the survey and interviews, our purpose was to get more insight into their perceptions about the relevance of the course, and in particular the real number theme, for high school teaching. The survey and interview asked about: details about the experience in teaching mathematics at secondary school (Part A); the challenges and achievements of the course, with respect to the theme on real numbers (Part B); initial expectations from students (Part C); and the role of this course in relation to the transition between university and school mathematics (Part D). We present the quantitative results from the survey, complemented by some of the students' responses from interviews, to identify the achievement and the challenges relating to the second discontinuity and efforts to bridge it in the specific case of the real number system. Again, the analysis was carried out jointly by the authors.

25.5 Student Work on the Task *T*

We present the results from the analysis of 14 written reports on the weekly assignment introduced above, delivered by students in groups of 1–3 students as the theme on real numbers was finishing. From their answers to task T (introduced in the previous section), we explain the traits of corresponding student praxeologies.

As could be expected, all the students carry out T_1 about plotting the graph of each function, but with significant variation, including one or more plots for each function, the two functions plotted together, and so on. Two groups conclude the graphs are "more or less the same". The first states as the conclusion (with some informal form of solution to T_9 as justification):

> [Group 12] We first note that $a_n \to \sqrt{3}$ as $n \to \infty$ for the concrete a_n we found in the previous question. Thus, $f(x)$ must be an approximation of $g(x)$ which becomes more and more precise, as n grows. (...) We see that the graphs coincide and thus $f(x)$ must be an approximation of $g(x)$.

Group 5 also had this conclusion and provided X-shaped plots at very small intervals (the one for g is shown in Fig. 25.1), and states that the graphs are almost identical, just with divergence at slightly different spots.

The other groups get somewhat correct solutions to T_6 when discussing where and how function could be considered identical. Eight groups show only the graphs that are clearly different, while six also include or mention the plots which appear identical; four of these then argue that this is somehow "wrong". The 12 groups who provide plots showing the difference (on one or more smaller intervals) provide various explanations, based on these graphs and sometimes other ideas:

> [Group 9] We define the two functions and plot them in the same diagram. In the first plot we see clearly that $f(x)$ cannot be defined in 1.7320508075, while it seems that Maple has no problem defining $g(x)$ in $\sqrt{3}$. [...] It is eye-catching that Maple seems to define $\sqrt{3}$ in a smaller interval than 1.7320508075, although both numbers should "fill" the same space on the number line.

In the last phrase, one may see an informal attempt at T_{10}, when interpreting the differences between the graphs according to the properties of numbers, but the main source is still the graphs, and Maple is blamed for the difference between them. Another attempt to consider T_{10} is equally strange:

> [Group 10] This illustrates very well the completeness of $\mathbb{R}$, as $\sqrt{3}$ is irrational with infinitely many decimals, and we can therefore get arbitrarily close to the actual value without being equal to it, while this is not the case for a_n since this has finitely many decimals.

Other groups provide comments related to limits, divergence, and asymptotes, also based mainly on the graphs. Four groups solve T_2 by pointing out the asymptote for g, and six groups provide some form of limit interpretation (T_4 and T_7, in two cases also involving T_3, otherwise referring just to the graphs). One group provides a characterisation of the singularities (T_8) while two groups are content with showing by a plot that the graphs are different (T_5). In fact, they appear as slightly different line segments on some small interval.

The overall dominance of informal descriptions of various graphs is not so surprising, given the wording of T. However, four of the groups begin their discussion by carrying out the algebraic developments of g (T_3), using O-techniques (product of sum and difference). They all note that although Maple draws the graph as a continuous line, $g(x)$ is only defined for $x \neq \sqrt{3}$. None consider this similar subtask for f, so this is only used to explain what the graph of g looks like:

> [Group 4] The function $g(x)$ therefore becomes a straight line, which is not defined in the point $x = \sqrt{3}$, because the denominator (of the original rational function) cannot be zero. We see that the Maple plot of $g(x)$ is linear for all x, but $g(x)$ is not defined at $x = \sqrt{3}$.

Three of the groups note that the necessary degree of zooming is related to the accuracy of Maple in plotting graphs, for instance:

> [Group 13] [...] if $f(x)$ is plotted in Maple with a very small interval, an asymptote is seen. Since Maple selects a number of points in the graph, this is not seen when we consider a larger interval for x.

Although, these remarks are not backed by numerical calculations.

Instrumented techniques were thus mainly related to plotting on various intervals but were not used in students' algebraic analysis of the functions, or for the calculation of limits of functions. None of the students made numerical investigations using tables of function values, as they had done in previous tasks in the same weekly report. The limited use of Maple is eye-catching in the cases where the students attempted more than informal and visual arguments.

It is also remarkable that only a few students attempted to relate the task to previous questions in the assignment or to other elements of ω, and that the few who did mostly fail. There is thus some evidence that students σ spontaneously rely mainly on a past relation $R_S(p, O)$ or even $R_S(t, O)$, if they have teaching experience already. It is as if when students judge T as a possible task belonging to O, they call for the expected praxeological elements in this institution.

This does not mean that they cannot learn from the written report and oral feedback provided for their work but reveal about the likely conflict that can be felt by σ when choosing a position closer to $R_U(\sigma, \omega)$ or to $R_U(\sigma, O)$ and reacting in consequence. It also shows that the assignment might have to be modified to enable a different relationship of the type $R_U(\sigma, O \cup \omega)$ to bridge the gap through the construction of the necessary praxeological infrastructure to analyse T from the midway position between U and S. Still, it is undeniable that not one of the 14 papers provide a solution that reflects a fully satisfactory relationship of that type, or for that matter clarity and correctness of exposition reflecting an adequate relation of type $R_S(t, O)$.

25.6 Students' Perceptions

In this section we present the main outcomes from a survey that was distributed to all 25 students, and from in-depth follow-up interviews with four respondents. The survey was answered by 11 students. Three of the four interviewees and 7 of the survey respondents had some experience with teaching mathematics in high school. In the sequel, *Interviewees A*, *B* and *C* designate the ones who already had taught in secondary school, with 1, 4 and 20 years of experience respectively, while *Interviewee D* had no teaching experience.

In the survey, students were asked about the challenges and achievements of the course, in particular, about the theme of real numbers and about what they consider important for secondary school teachers to learn about this topic. Table 25.1 summarises the results.

A basic observation is that while students think their theoretical knowledge about real numbers increased, they are less certain about the practical implications for teaching.

The interviews provide a more detailed idea about students' viewpoint. For instance, one student suspects the new knowledge could mainly serve at the advanced high school level (level A):

Table 25.1 Results of the degree of agreement on the assertions concerning the achievements of the course about the topic of real numbers (1 = totally disagree, 5 = totally agree)

	Mean	*Median*
The topic of real numbers is important at secondary school education	3.3	3
This topic provided me more knowledge about ...		
... how to conceptualise and use real numbers in secondary school	3.3	4
... how to manipulate and operate with real numbers in secondary school	3.9	4
... the rationale and usefulness of real numbers at secondary school	3.7	4
... how the topic of real numbers is related to other topics (measure, round off, errors, approximation, limits, etc.)	3.8	4
... questions about teaching real numbers in secondary school that I have never reflected on before	3.7	4

> I think it's important for me to know, but I don't think, I haven't been teaching that in secondary school. I've only been teaching at the level B. I think it might be more important if I, one day, teach the level A, this level, but right now I don't think this is something that I need to teach them, but I like to know myself. [. . .] I like that we are focusing on the topics that we have to teach at the secondary school. I like that we will have another background view of the mathematics theory. I don't think that I've learned all the things here at the courses, but it's another way to look at it in this course. I like that. It is making me think of how can I teach maybe in another way, what can I, what should I focus on? *Interviewee A*

Another interviewee is more critical of the topic and think that the course should focus on other ones:

> I'm not sure that all the work we've done with building up the real number system chapter is that relevant compared to other areas [. . .]. There are other topics that are part of the curriculum, and that are part of the textbook as well, which we could have studied as part of this course: trigonometry, geometry, for instance, or the topic we are starting today about functions and regressions. This topic is probably well justified. But the real number system [. . .], I suppose it does give us more familiarity with the sort of the different, number systems and especially the abstraction from the rational numbers to the real numbers. *Interviewee C*

However, other students so consider the real number theme as an important, but difficult, part of the foundations for high school mathematics:

> I think it's really important that you have control over the basics. [. . .] Definitely [real numbers] has been the most difficult topic. If I want to take some of this (referring to the tasks proposed in the course about real numbers), I have to use like many hours. *Interviewee B*
>
> It makes us possible to explain things in different ways in primary school, and all the way up. [...] It has helped me a lot to be able to explain the same things in another way that I have experienced. *Interviewee A*

The following part of the survey and the interviews focused on the expectations student had prior to taking the course (Table 25.2). The dominant expectations were to "extend my knowledge about some topics of Mathematics relevant for Secondary school" and "to get ideas and methods for designing activities". The highest rated item shows the importance given by the students of extending their knowledge to

Table 25.2 Results from the students' expectations before starting the course (1 = totally disagree, 5 = totally agree)

I expected to. . .	Mean	Median
. . . extend my knowledge about some topics of mathematics relevant for secondary school	4.5	5
. . . get ideas and methods to design activities in secondary school	4.4	5
. . . get more knowledge about how to adapt university mathematics to secondary school mathematics	3.3	3
. . . be trained about some teaching methods to teach mathematics at secondary school	3.8	4

something "new" to be created in U which is relevant at S. Indeed, the course material on decimal representations of real numbers does involve mathematical results which are not covered in the students' other courses although it is built on key ideas from them (such as geometric series and completeness). In other words, $R_U(\sigma, O \cup \omega)$ involves combinations of O and ω which are neither present in $R_U(\sigma, \omega)$ or in $R_S(s, O)$.

All the interviewees express that they expected more proximity with secondary school. For instance, Interviewee A and C would have preferred the course to include more about teaching techniques, as opposed "academic mathematics".

> My expectations before the course were to learn more about how to design or construct tasks of progression of tasks to give a picture to students, [...] What we're doing now is more talking about the theory behind what we are going to teach in secondary school. And yeah, I think I need more practical. [...] I'm not sure. No, I don't know why but I think I need something else. Yeah. But that might be the Didactics. [...] *Interviewee A*

Although they emphasise their need for practical knowledge, when they expand on what this knowledge might be, they also include some generic aspects, such as "motivation of students", "dealing with diversity of levels in the class", "IT in education":

> I still think that I am needing more practical things. [...] All of them teaching how to give to students' feedback, to motivate, construct the task for them, how to design a lesson to motivate again, all about motivation. *Interviewee A*

The last part of the interviews and surveys were about the discontinuity between university and high school mathematics. In both, students were asked to place UvMat on a scale between university mathematics and high school mathematics (shown at the extremes of a line segment). None of the four interviewees hesitated much when placing this course in this line segment. What surprised us the most was how *Interviewees A*, *B* and *C*, with experience in high school, quickly place this course closer to university mathematics:

> Until now, closer to university mathematics. We have used some things that we have never learned in secondary school. But what we will do in a few minutes [referring to the next topic of the course] is more secondary school mathematics [...] Today is regression and statistics. But what we have learnt about real numbers and exponential growth; these are university maths [...]. *Interviewee A*

> This would be university mathematics and high school mathematics. So that, there are plenty of steps, on the span of the bridge so we have here, the course UvMat, which is a sort of "building the bridge" or "attempting to build a bridge over here". So that, it insists that the bridge can be built. [...] So that, we may trust on this belief and see how far, how far you can build the bridge from the secondary school reality towards university mathematics. [...] It makes sense to me also that you need experience from, from both worlds to, you know, to actually build the bridge. *Interviewee C*

By contrast, *Interviewee D*, who has no teaching experience, placed the course closer to high school:

> I would say it is closer to this one [pointing at secondary school mathematics]. [. . .] I would say we have looked at the mathematics closer to here, but we are proving things like we are in university [. . .] but I don't find it that abstract [compared to other mathematics courses]. *Interviewee D*

Furthermore, *Respondent 4* in the survey compared the difficulty and formalism of this course in comparison to others:

> The subjects are the same as the ones found in the gymnasium, but the approach is more advanced. Still, it is nowhere nearly as advanced as regular courses–it felt like a holiday from these. *Survey respondent 4*

In fact, assuming the course offers some combination $O \cup \omega$ of praxeologies from the two institutions, the unexpected O parts are clearly more visible when viewed from the (exclusive) position of a university student σ, while the ω-parts may appear more visible when one can also assume the position t, based on relations of type $R_S(t, O)$. Considering the solutions of the task T that were analysed in Sect. 25.5, we may also observe a tendency of course participants to assume the position s (school student) as they perceive the task to pertain to a high school praxeology, and to then draw mainly on past relations of type $R_S(p, O)$ (and perhaps $R_S(t, O)$ as well).

25.7 Discussion and Conclusions

One major challenge hidden in RQ1 is that the real numbers are assumed, in many institutions, as a transparent mathematical object which requires no separate work. When Cartesian coordinate systems are introduced, we may first work only with integer coordinates, but sooner or later we learn that all points have coordinates and that all points on the "number line" correspond indeed to a number. The real numbers actually named in school are either rational, roots or a few other exceptions. Even in high school this does not change, but the introduction of Calculus (beginning with limits) makes certain properties of real numbers appear, albeit exclusively through properties held by *functions* defined on $\mathbb{R}$, and usually treated as "evident". Even at university courses in Calculus, this is often the case. By contrast, textbooks for rigorous analysis courses – which are taught in many European universities from the first or second year – usually begin with some "preliminaries" concerning the topology of $\mathbb{R}$, often passed over relatively quickly. Students may have been shown, within the same lecture, "facts" like the uncountability of $\mathbb{R}$ and the density of $\mathbb{Q}$ in $\mathbb{R}$. And such quickly passed (and maybe quickly forgotten) facts are not easily mobilised by students, for instance, to explain how Maple plots graphs and why that may sometimes lead to surprising or bewildering results (like in the example). It would then appear even less likely that they use such knowledge to develop a teacher relation to these kinds of phenomena.

In this paper, we have examined one strategy (materialised in a course at a specific university) for helping students invest some of university mathematical knowledge in high school mathematics and its teaching. The approach to the theory of real numbers is much less formal than in other university courses on mathematics, but also much more formal than the approach to real numbers and functions in high school. The assignments include mathematical problems that are close (or identical) to what students and teachers work with in high school, but with demands and contexts that allow mobilisation of more advanced knowledge. In the example studied, the need for more subtle aspects of real numbers and functions comes from the bewildering output given by Maple. We notice here that, both in high school and at university, Maple is mostly used to carry out simple standard tasks, not to carry out more subtle investigations like the one proposed in T.

Both the analysis of student work, and of their own conceptions of the wider experience, illustrate that drawing on advanced knowledge is neither easy nor automatic for students. They are familiar with expectations in the two institutions, secondary school and university, as seen apart, but the requirement to analyse a problem from one institution (S) with the tools offered by the other (U) is very unfamiliar to them. Our analysis shows the general tendency that when they are confronted with a task, students seek to identify it as belonging to *either* O *or* ω, in order to be able to draw on previously established relations $R_S(p, O)$ or $R_U(\sigma, \omega)$, rather than establishing a new relationship of the type $R_U(\sigma, O \cup \omega)$. Overcoming that tendency requires the construction of new didactic contracts (in the sense of Brousseau, 1997, chap. 5) and a new didactical infrastructure (in the sense of Chevallard, 2009) involving task design, among other elements.

To establish relationships of type $R_U(\sigma, O \cup \omega)$, simply siding with either O or ω will evidently not do. Establishing even fragile and very local links requires careful design and redesign of situations and assignments. The example studied here was far from perfect or successful. And the case study here only sheds some light on the effects of such an effort to construct links between the two compartments of students' mathematical background and how students react and reflect on it. Another, and no doubt just as difficult and important question, is to investigate the last step in (Eq. 25.2), namely the role that such links can play in their practice as teachers.

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Part V
Research on Mathematics for Non-specialists

Chapter 26
Challenges for Research on Tertiary Mathematics Education for Non-specialists: Where Are We and Where Are We to Go?

Avenilde Romo-Vázquez and Michèle Artigue

Abstract The challenges raised by the teaching of mathematics for non-specialists in tertiary mathematics education were pointed out by mathematicians long ago. Since the beginning of the twentieth century, innovations and reflections have developed around the world, but research in mathematics education in this area is still underdeveloped. Where are we and where are we to go? In this chapter, we address these two issues with a specific focus on engineering education. Returning to the history of the field, we discuss these challenges, how they have been perceived and dealt with, and how they are regularly renewed by scientific and technological advances, or societal evolution and emerging concerns. Then, using examples selected from recent research and development work and through the institutional and epistemological perspective offered by the anthropological theory of the didactic, we illustrate the progression of theoretical approaches and knowledge over the past two decades, and discuss the perspectives for future research that this analysis opens.

Keywords Tertiary mathematics education · Mathematics for non-specialists · Epistemological and didactical perspectives · Engineering education

26.1 Introduction

Research concerning mathematics education for non-specialists at the tertiary level deals with the large numbers of students who do not intend to pursue academic careers as researchers or teachers in this discipline, and for whom the tertiary

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institutions that admit them consider their mathematics background insufficient. A wide variety of courses and professions are involved that receive students with different backgrounds since, with few exceptions, educational systems differentiate teaching in high school.

For a long time now, tertiary institutions have been aware of the challenges that mathematical education raises. At the beginning of the twentieth century when the CIEM –today known as the ICMI, International Commission for Mathematical Instruction– was created, one of the first reports it produced was devoted to the mathematics that engineers in training would need (d'Ocagne, 1914). When ICMI Studies were launched, under the Presidency of Jean-Pierre Kahane, a pure mathematician who specialised in Fourier analysis and was regularly in charge of mathematics service courses, he immediately proposed to devote one study to the topic of *Mathematics as a Service Subject*, a theme he considered of utmost importance (Howson et al., 1988). A more recent ICMI Study launched and conducted jointly by the ICMI and ICIAM (International Council for Industrial and Applied Mathematics) focused on the educational interfaces between mathematics and industry (Damlamian et al., 2013). Throughout the twentieth century, innovations and reflections have developed worldwide, as shown by these studies and many others, and have analyzed scientific, technological, professional, and pedagogical developments. Nevertheless, as several surveys have pointed out (Artigue et al., 2007; Winsløw et al., 2018), until recently research in mathematics education in this area has been scarce, even at the university level where most students study mathematics only as service-courses. This situation, however, is changing as is reflected in the new entry on *Service-Courses in University Mathematics Education* in the *Encyclopaedia of Mathematics Education* (Hochmuth, 2020). The challenges raised by the teaching of mathematics to non-specialists are increasingly being addressed by research in mathematics education. With the support of networks like INDRUM (International Network for Didactic Research in University Mathematics), researchers are now better connected and organized to capitalize on the knowledge produced, and to address the crucial divide between research and practice (González-Martín et al., 2021). So, the question is, where are we today and where do we go from here?

In this contribution, we address these two issues, especially in the context of engineering education. For this purpose, we returned to historical debates and reflections regarding mathematics education for non-specialists at the tertiary level since we postulate that historical knowledge contributes to the understanding of current states of affairs. After elucidating this background and the encyclopaedia entry mentioned above, we present our vision of the main challenges in this area. Most of them are not new, but even the old ones are renewed by scientific and technological advances and societal changes. Using a selection of recent studies on engineering education and adopting the institutional and epistemological perspective offered by the anthropological theory of the didactic, we show the progression of conceptual tools and empirical knowledge as they relate to some of these challenges in the twenty-first century. We conclude with suggested perspectives for future research.

26.2 A Historical Perspective

The historical perspective presented in this section is adapted from the study proposed by the first author on the interesting case of the *École Polytechnique* in France and developed in her doctoral thesis (Romo-Vázquez, 2009), as well as on the three CIEM/ICMI studies mentioned in the Introduction. The resulting vision is, inevitably, partial, but we consider it sufficient to support the reflection.

26.2.1 A First Historical Lens: The École Polytechnique

This school was created in 1794 in the context of the French Revolution. Drawing on Belhoste (1994), Romo-Vázquez showed that tensions already existed at that time, and that three models of education would follow one upon another in less than a century, products of choices that combined academic, professional and social considerations. These models were driven, respectively, by the ideas of Monge, Laplace, and Le Verrier. The first model, described as Encyclopaedist, was influenced by the ideas of Monge, the founder of descriptive geometry. It was divided into theoretical and practical teaching. Mathematics and physics formed the basis of theoretical teaching, while geometry occupied a predominant place in the latter, legitimized by its important role in applications in the period. The courses on theoretical sciences and their applications were closely connected. Taught by members of the Academy of Sciences, they preceded practical sessions held in laboratories and taught by more advanced students. The creation of Schools of Application in 1795, like the *École des Mines* or the *École des Ponts et Chaussées* where polytechnicians specialised after graduating from the *École Polytechnique*, destabilized this first model as those schools claimed responsibility for teaching applications. In that same year, the creation of an exit examination based essentially on mathematics that determined who would be admitted to those schools, meant that this discipline became a key selection tool. Under the influence of Lagrange and Laplace, analysis, which provides general methods for applications, became the dominant field. Analytical methods penetrated into courses on mechanics, physics, machine theory, geodesy, and probability. However, Cauchy's course on analysis became highly theoretical, so links with applications weakened. The application schools denounced this evolution, arguing that mathematics was for engineers becoming something like Latin; that is, a selection tool quickly forgotten as soon as students left school. At the same time, the booming technical and industrial revolution imposed its needs. This led, in 1829, to the creation of the *École Centrale des Arts et Manufactures*, based on a new model that taught a new approach to industrial science, one devoid of theoretical references that sought to resolve the tension between pure abstraction and applications. The *École Polytechnique* later underwent reforms in a mission entrusted to the astronomer Le Verrier in 1850. At that point, the criterion of usefulness for applications became the basis for organizing teaching and analysis as a domain was no longer taught.

This narrative describes an episode of French history but as Schubring (2007, 2019) sustains it has a more general value. Tensions between mathematics and its applications were also observed in the polytechnical schools created in Europe and beyond in the nineteenth century despite evident structural differences with respect to the original *École Polytechnique*. Clearly, for over two centuries debates have raged over just what kinds of mathematics should be taught to future engineers, and the correct relationship between mathematics and applications and between theory and practice. Different answers have led to the elaboration of different models. This history also shows that, from the beginning of the nineteenth century, and driven essentially by the industrial and technical revolution, a new scientific field emerged, that of the engineering sciences, and with it new models for the place of mathematics in the training of engineers.

26.2.2 A Second Historical Lens: CIEM/ICMI Studies

These debates would continue throughout the twentieth century, parallel to the development of engineering sciences. In this section, we focus on three studies that allow us to assess how this situation evolved over a period of more than a century. The first two have been analysed in Romo-Vázquez (2009); here, we continue to build on her analysis.

In 1914, the CIEM met to deal with two issues, one of which was "the place and role of mathematics in higher technical education". A report on the education of engineers in ICME member countries was presented. It showed that, in general, differential and integral calculus constituted the basis of mathematics education for engineers. Mathematics was perceived as a broad, autonomous discipline, one that was expected to provide a solid base of knowledge for studying other disciplines, such as mechanics and physics –the classical disciplines of application– as well as the engineering sciences. The dominant model was, therefore, of a rather Laplacian type. However, the engineering sciences had come to occupy an important place in most curricula, leading to a reduction in the number of hours allocated to mathematics. Some people even argued that the necessary grounding in mathematics should be acquired beforehand. At that meeting, d'Ocagne, a mathematician, engineer, and professor at the *École Polytechnique* and the *École des Ponts et Chaussées*, gave a lecture entitled, "The role of mathematics in the engineering sciences". Drawing on numerous examples, from undersea telegraphy to the propagation of liquid waves in elastic pipes (the Kelvin effect), he sought to show that, while engineers may be under the impression that they use only basic mathematics in their daily work routines, solving engineering problems often required much more advanced mathematics. He also illustrated the role that the engineering sciences play as an interface between mathematics –theoretical and abstract– and the everyday practices of the engineering sciences. He defended the view that engineers require a solid mathematics education in order to understand and use, in a non-blinded way, the knowledge produced by these sciences in their practice. This claim seems to have been rather widely shared at the time. Debates centred on the issues of how to provide such a mathematics education and organize its links to the engineering disciplines and the questions that emerge in practice.

The second ICMI Study devoted to this theme is the third in the series, but by the time it was issued, circumstances had changed. That study made it very clear that scientific and technical developments had created new educational needs. Moreover, it also showed the emergence of a movement from the traditional theory-applications vision towards modelling perspectives. Pollak's contribution illustrates this evolution quite well. He explained, for example, how the development of the telecommunications sector led *Bell Laboratories* (where he had worked for 35 years) to set up, as early as the 1940s, training courses to complement university engineering courses: "Linear algebra, complex variables, Fourier series and Fourier and Laplace transforms, probability theory, statistics, semi-conductor physics, and a number of other topics which at that time were not part of the regular university education of electrical and mechanical engineers" (Pollak, 1988, p. 30).

The courses that *Bell* organised at that time followed the classic theory-application pattern, but retrospectively, and in light of his experience, Pollak came to question that approach and emphasize the diversity of mathematical forms of thought: not only those at stake in the classically recognized fields of analysis, algebra, and geometry, but also statistical, probabilistic, and algorithmic forms, and others that underlie optimization and operational research activities. He also underscored the need for employees not only to understand the mathematics they used, but also to be aware of the fact that those various forms of thinking can be applied to the real world and provide valuable insights there by preparing students to deal with open-ended situations, and to think about how they might use mathematics to solve them. According to Pollak, modelling courses were particularly well-suited to meet those needs. At the time, as we mentioned at the outset, mathematics had connections with a growing number of fields and service courses in addition to engineering subjects. This increasing diversity is not, however, reflected in the study volume.

The panorama in the ICMI-ICIAM study launched in 2008 was much broader (Damlamian et al., 2013). Its Discussion Document states that the study is based on a wide definition of industry, interpreted by the OECD as "any activity of economic or social value, including the service industry, regardless of whether it is in the public or private sector" (p. 4). The goal of studying the interfaces between mathematics education and industry was to find "a balance between the perceived needs of industry for relevant mathematics education and the needs of learners for lifelong and broad education in a globalized environment" (p. 5). This study shows a clear evolution from the earlier ones. The vision of modelling was strongly present in both the contributions of mathematicians working in industry or universities, and those of mathematics educators. Industrial mathematicians emphasized the interdisciplinary nature of their projects and the importance of developing good communications and collaboration skills. The study also stressed the important changes induced by technological evolution. Questions linked to the invisibility of mathematics despite its growing role, and the relationship that needed to be established with the digital tools that encapsulate mathematical techniques and lead to explorations of complex processes through model simulation were omnipresent in the document. Several contributions also emphasised the importance of more advanced mathematics in certain fields, such as finance, and the attractiveness of these new fields that tended to upset traditional balances.

Many achievements are presented in the study based on work in both communities: the ICIAM and the ICMI, despite their different experiences, goals, approaches, practices, time scales, language, and overall diversity. One main challenge that was of interest in the study was, undoubtedly, to foster communication, to cross borders. As the study's conclusions highlight, for example, industrial practices tended to induce forms of training in industrial mathematics that were in line with the idea of communities of practice promoted in mathematics education research (p. 449):

> Modelling weeks in general education as well as internships and industrial workshops seem to blur the traditional separation of those who know already and those who have to learn. Instead, people from university and industry –in a joint, social effort– hope to approach problems unsolved to date with the help of mathematics.

These practices also tended to promote forms of assessment that were less individual than those generally used in mathematics education, because in these contexts cooperation is a key condition for success. In both fields, education and industry, certain question arise: which software black boxes should be opened, to what extent (grey or white), and how, though the constraints and challenges are not the same (p. 450):

> ... hiding a process in a black box can be a marketing strategy for industry to secure superiority over competitors in the market [...] in education, black boxes are a challenge for the learning process —even if some learning processes definitively rely on black boxes remaining black.

Differences were also stressed regarding modelling practices (pp. 450–451):

> For industry it is the gateway into the use of mathematics... the extra-mathematical part of the famous modelling cycle is the reason for using mathematics at all. Generally speaking, it is only if mathematics offers additional insights into an extra-mathematical question that industry will make use of mathematics. For education, modelling with the help of mathematics is often an important aim in itself for classroom activities and curricula and it is definitively an important competence to be acquired.

This study presents many interesting and innovative realizations concerning students at all levels, up to the most advanced, and teachers. However, as far as research is concerned, its contribution remains limited. For its authors, "research on the use of mathematics in industry (taken in the wide sense indicated at the beginning of the paper) has just started in various institutions around the world and in several academic disciplines (such as Didactics of Mathematics, Applied Mathematics, or Sociology of Work)" (p. 451).

What exactly is the state of this research today, specifically in the didactics of mathematics mentioned as one of the contributing fields? How does this relate to the challenges that this brief historical perspective has highlighted? And which challenges does it identify? We address these questions in the following sections, first considering the new entry in the *Encyclopaedia of Mathematics Education* mentioned above.

26.3 Mathematics Education for Non-specialists Through the Lens of the *Encyclopaedia of Mathematics Education*

Hochmuth (2020) acknowledges that most research in this area is recent (20 of the 30 references are less than 10 years old) and highlights a number of challenges that researchers have identified. His discussion centres on the persistent challenge of the relationship between mathematics service courses and those in the major disciplines for non-specialists, the difficulties created by the organization of such courses and the limited knowledge that we still have of mathematical practices outside of mathematics itself: “Whereas topics like differentiation, integration, or stochastic distributions can easily be identified and considered in curricula, the discipline-related adequate mathematical practices and conceptions are often not sufficiently determined or are not even known and require further research.” (p. 771).

Hochmuth links this problem to the persistence of applicationist visions of the relationship between mathematics and other disciplines, and to the institutional disconnection that this vision engenders. The studies cited show that this leads to numerous mismatches between the uses of mathematics in service courses and in basic or advanced major subject courses, mismatches at the level of symbolism, practices, modes of reasoning, and validation. These mismatches are sources of major difficulties for students, who tend to develop an incoherent mixture of concept definitions and concept images. The case of the concept of derivative is cited as an example with reference to research concerning students in mechanical engineering, economics, and biomedical science. As that research shows, this situation affects more globally the students’ recognition of the relevance of what is taught in mathematics for solving the problems encountered in their major subject courses. Other difficulties are mentioned more briefly, notably the fact that these students, at least in the early years of their university education, often have little confidence in their mathematical abilities. A final point is that developing this confidence is not something that pedagogical forms –which are essentially transmissive– or evaluation instruments foster.

In addition to research analysing current practices and their effects, Hochmuth discusses studies designed to make mathematics service courses more relevant to students and better connected to the teaching of their major disciplines. Collaboration between mathematics lecturers and professors from other disciplines plays a crucial role in this process. However, Hochmuth points out that:

> [. . .] there is still a lack of systematic research on detailed studies about the use of mathematics in other disciplines, on possibilities, demands, and limits of the use of technology and software tools, in particular in the implementation of complex life-like examples from other disciplines, as well as on dealing with an increasing heterogeneity of cognitive and affective-emotional prerequisites of students in large service-courses (p. 773).

Returning to the initial question of the challenges posed to didactic research by the mathematical education of non-specialists at the tertiary level, this entry clearly identifies an essential challenge: the one posed by the insufficient connection between mathematics teaching and the teaching of the students' major disciplines. Meeting and eventually overcoming this requires additional knowledge, first of all about how mathematics 'lives' in different disciplines. This demands an understanding of the corresponding practices and their rationale. It is also necessary to enhance our understanding of the systems of conditions and constraints that seem to lock service courses into this state of disconnection; for example, the organization of service courses that bring together large cohorts of students with a variety of professional goals and backgrounds, and courses that must cover a large amount of content in a short time to respond to such diverse needs. These conditions tend to legitimize an applicationist vision of mathematics as a discipline that provides general concepts and methods expressed in a purely mathematical discourse. Beyond mere understanding, this field needs research that explores the construction of possible connections, and the possibility of crossing borders, developing alternatives, and studying the conditions for their ecological viability.

But there are also other challenges. As shown in Sect. 26.2, from the outset engineering disciplines have played the role of mediators between, on the one hand, the scientific disciplines to which mathematics belongs and, on the other, practice. This level of practice cannot be overlooked. The research initiated by researchers like Noss et al. (2000) clearly shows that workplace mathematics have characteristics that distant them from those of university courses. Once again, our knowledge of these aspects is still fragmentary. In the following section, based on a selection of studies, we present in greater detail some conceptual and empirical advances that have been achieved in confronting these challenges.

26.4 Mathematical Training for Non-specialists from an Institutional Perspective

In a concerted effort to recognize where we are concerning the challenges mentioned above and the direction in which we need to head, this section revisits the research analysed in Hochmuth (2020) in light of a perspective nourished by the Anthropological Theory of the Didactic (ATD) regarding the circulation of praxeologies among institutions. The discussion focuses on institutional epistemologies and potential tensions and relations among them.

26.4.1 Selected Theoretical Elements of the ATD

The ATD, proposed by Chevallard (1999, 2019), is an epistemological model that allows studying human activity in its institutional dimension. A praxeology $[T, \tau, \theta, \Theta]$ is a minimal unit of analysis of human activity. Its four components are task type (T), technique (τ), technology (θ); and theory (Θ). 'Task' refers to what is to be done; 'technique' to how it is to be done; 'technology' to a discourse that produces, justifies, and explains the 'technique'; and 'theory' to that which produces, justifies, and explains the 'technology'. The first two elements correspond to the *praxis* block [T, τ], the latter two to the *logos* block [θ, Θ]. In any institution there exist different levels of praxeologies or praxeological organizations, called pinpoint, local, regional, and global. The pinpoint level corresponds to a praxeology unit with only one technique for performing one type of task and one *logos* block. Linking all pinpoint praxeologies with the same *logos* block gives rise to the local level. Associating local praxeologies with the same theory corresponds to the regional level, while global praxeologies, or domains, group together specific regional praxeological organizations. Discipline is the top level and combines all domains.

An institution is a stable social organization that makes human activities possible by providing its subjects with conditions and resources. Subjects can occupy different positions in one institution and belong to different institutions at the same time. Romo-Vázquez (2009) proposed a classification of institutions according to their relation to knowledge as a predominant vocation: producing, teaching, and using.[1] Producing, or Research, institutions, *Ir*, include disciplines that generate knowledge (e.g., mathematics, electronics); Teaching institutions, *It*, are responsible for transmitting knowledge (e.g., school mathematics, school electronics); and Using institutions, *Iu*, are involved in utilizing or applying knowledge (e.g., practical training, manufacturing). Each institution has a specific epistemology. When praxeologies pass from one institution to another they undergo transpositive processes (Chevallard, 1999). Disciplinary mathematical praxeologies thus undergo didactical transposition to become school mathematical praxeologies (Chevallard, 1991). In the other direction, through some transposition user mathematical praxeologies become disciplinary mathematical praxeologies. An exciting example is Heaviside's work (Lützen, 1979), another is *Bell Laboratories* (mentioned in Sect. 26.2.2), where mathematical praxeologies such as Laplace Transformation were taught long before they were introduced into university mathematics courses (Pollak, 1988).

Based on the above, we consider that the challenges of mathematics education for non-specialists can be viewed advantageously through relationships among research, teaching, and using institutions. This leads to two key questions: what relations and tensions among these institutions have been identified, and what new connections can we establish among them? To address these issues, we organized the studies analysed by Hochmuth (2020) into three categories: those developed in workplaces, those focused on analysing math courses and courses in major disciplines, and those focused on the design of didactic proposals. This classification helps us reinterpret their results in light of our institutional and epistemological perspective, identify advances and limitations, and build a coherent overview of where we are and where we are heading.

[1] A praxeology can be created, taught, and used in any of these institutions. Distinguishing them makes it possible to analyze the circulation of mathematical praxeologies.

26.4.2 Mathematical Praxeologies in Workplaces

Interest in analysing the role of mathematical praxeologies in workplaces has increased in the last two decades (e.g., Alpers, 2011; Bissell & Dillon, 2000, 2012; Frejd & Bergsten, 2016; Gainsburg, 2006, 2007; Hall et al. 2007; Kent & Noss, 2002). Pioneering research in engineering workplaces developed by Kent and Noss (2002) identified, for example, a division of mathematical work in the development of civil engineering projects associated with two types of engineers: designer-specialists and analysts. Designer-specialists tend to be more experienced:

> [. . .] so, as an engineer grows up, he may no longer be using the mathematics that they started out using, they are still using the understanding that they derived earlier in their experience, and some of this is difficult to describe as to the sort of knowledge it is (Kent & Noss, 2002, p. 3).

Analysts (5% of employees), in contrast, are usually younger engineers who handle the mathematical-analytical problems that other engineers cannot solve. A large part of their work consists in interpreting the "codes of practice", which are similar to the "methodological guides" that Vergnaud (1996) identified in his study of an aeronautical company. These documents contain practical, experimental, and analytical knowledge that have been verified by communities of practice. They coincide with Bissell and Dillon's (2000) affirmation that "models have to be mediated and negotiated within a community of practice to make any sense" (p. 6). We thus identify a specific epistemology that underpins the relationships among different types of knowledge and is sometimes profoundly embedded in the technology. According to Kent et al. (2007), this concerns Techno-mathematical Literacies (TmL). As these authors explain (p. 66), the prefix "techno" emphasizes "the mediation of mathematical knowledge by technology", while the plural form of "literacies" points to "the breadth of knowledge required in the context of contemporary work." Van der Wal et al. (2017) characterize, among other aspects of the TmL, technical software skills related to encapsulated mathematical knowledge at three levels of transparency, called white, grey, and black boxes (a distinction also mentioned in Sect. 26.2.2). Velten (2009) sustains that these levels characterize the use of mathematical models, while Frejd and Bergsten (2016) relate them to three types of activities that professional mathematical model constructors perform: data-generated modelling, theory-generated modelling, and model-generated modelling. The first is characterized by "the work of gathering, interpreting, synthesizing, and transforming data as the main underlying base for identifying variables, relationships, and constraints about a phenomenon used in the model development process" (p. 20). This type of activity leads to the production of grey and black boxes for social or biological sectors, among others. The second entails constructing theory. It is related to grand projects and is developed by a work team using computer support to produce white boxes closely related to mechanical systems. Finally, model-generated modelling produces an empirical confrontation of already-constructed (existing) mathematical models –that is, grey or white boxes– that may be related to other modelling activities. The schemes of data-generated, theory-generated, and

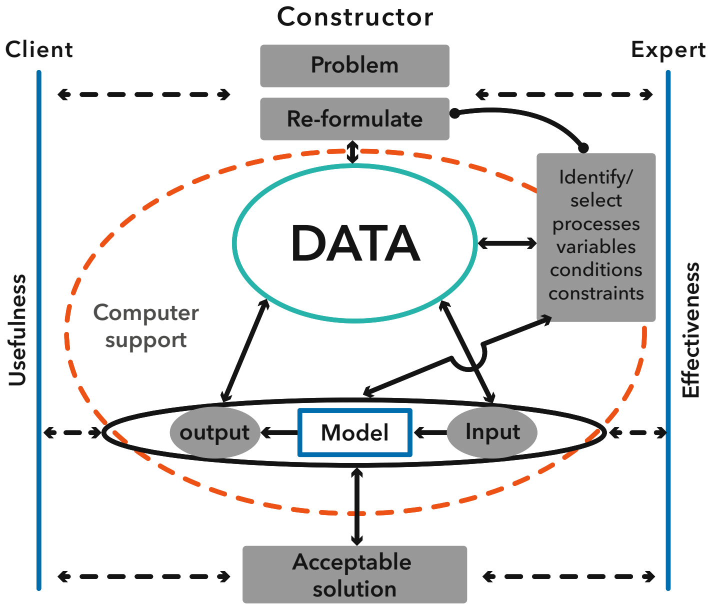

Fig. 26.1 Data-generated modelling activity. (Similar to Frejd & Bergsten, 2016, p. 21)

model-generated modelling activities are shown in Figs. 26.1, 26.2, and 26.3, respectively.

The effectiveness of this model is determined through communication with other "experts".

These approaches led us to identify various workplace praxeologies of engineers and professional mathematical model constructors. Mobilizing them requires conceptualizing structures, systems, materials, behaviours, data, and information obtained through mathematical methods and models. Moreover, having experience in performing a wide variety of tasks allows those specialists to make decisions to control their practice, a phenomenon that Gainsburg (2007) identified as "engineering judgment". Likewise, we identify specific workplace conditions: collaborative work, hard use of technology, engineering roles, and mathematical work division. Characterizing workplace praxeologies and conditions is the first step in the process of transposing them into the training of non-specialists. It was in this vein that Frejd and Bergsten (2016, p. 31) developed their research and stated: "Here, our characterization of the constructors' modelling work may contribute to the didactic transposition process by being a source of information about central components and processes used by the professional model constructors." It is also crucial to develop the theoretical tools to more finely analyse the transpositions of mathematical praxeologies and the workplace needs that guide and motivate them.[2] Indeed,

[2]See the Chap. 30 by Castela & Romo-Vázquez.

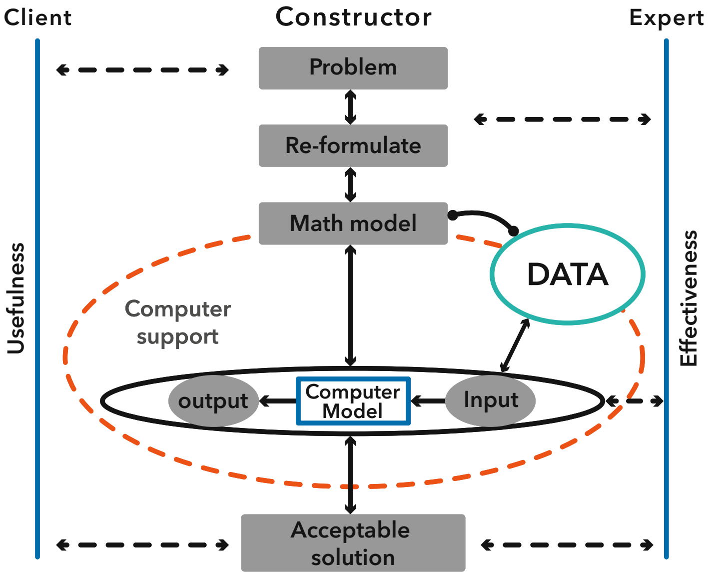

Fig. 26.2 Theory-generated modelling activity. (Similar to Frejd & Bergsten, 2016, p. 25)

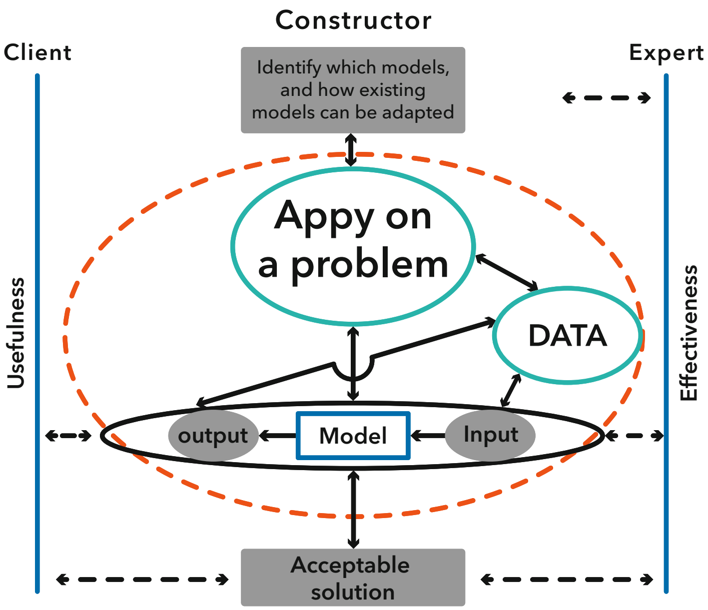

Fig. 26.3 Theory-generated modelling. (Similar to Frejd & Bergsten, 2016, p. 27)

these and other workplace analyses provide, increasingly, an epistemological reference that can support graded didactic proposals: starting from the analysis of situations that have been solved, and running through proposing improvements, identifying errors, and exposing practical knowledge. This offers a way to confront the challenges mentioned in Sect. 26.3 that need to be explored.

26.4.3 Mathematics and Major Discipline Courses

Major disciplines and mathematics courses have been analysed in the context of engineering programs (Allaire & Willcox, 2004; Castela & Romo, 2011; Flegg et al., 2011; González-Martín & Hernandes-Gomes, 2018; Guedet & Quéré, 2018; Hochmuth et al., 2014). Relations between them are established mainly through common mathematics praxeologies. Laplace transformation, notions of trigonometry, matrix models, and functions, among other elements, are all taught in math courses (e.g., Fourier analysis, linear algebra, differential and integral calculus), while their transposed forms are included in courses in other disciplines (e.g., control theory, materials strength, circuits, etc.). Faulkner et al. (2020) analysed the relations between calculus courses in mathematics and engineering courses on circuits and statics. Specifically, they considered homework problems and, from the perspective of mathematics-in-use, identified the types of problems, skills –"procedural sequences of steps used to solve a particular type of problem" (p. 5)– and concepts involved. Their results demonstrate that concepts from calculus were applied in 20% of the problems in the course on circuits and 8% of those in the one on statics. They further identified the existence of mismatches between how the same concepts (e.g., derivative, limit, continuity) were taught in calculus *versus* engineering courses. For instance, continuity, a crucial notion in calculus, is taught in mathematics courses as a property to be checked, but in engineering courses it is applied as a "guarantee desirable of physical properties" (p. 12). Overcoming such mismatches requires a deeper understanding of transpositions, their origins, and the development of the disciplines of circuits and statics up to the present, which would constitute a solid base for relating these courses. For instance, referring to electrical engineering, Gueudet & Quéré (2018) acknowledge that the transposed trigonometric forms presented in courses reveal connections among concepts, functions, vectors, semiotic registers, graphical curves, and arrows.[3] Specifically, in the study of alternating sinusoidal regimes, the trigonometric functions considered as signals – $s(t) = A\sqrt{2}sin(\omega t + \phi)$ – are studied and represented geometrically by vectors, in this case, a Fresnel vector (phasor). What is the origin of these connections and what motivated the transpositions from mathematics to electrical discipline? Bissell (2012, p. 73) stated: "The phasor approach allowed powerful geometrical representations (based on Argand diagrams) to be used for a wide variety of applications:

[3] See Chap. 27 by González-Martín et al.

from electrical power transmission to electronic circuits and electromagnetic wave transmission." However, the usefulness of mathematics is not the only principle governing such transpositions. In the case of engineering and based on the work of various historians (e.g., Dalmedico, 1996; Kline, 1994, 2000), Bissell and Dillon (2003) recognizes two broad paths in the recent development of electronic engineering. The first refers to processes of mathematization and "scientification" adopted and accentuated around World War II, the second to the tradition of producing knowledge through practice. Bissell goes on to elucidate that the evolution of the techniques of practice differ in each domain, but observes that this occurs "most of the time with the objective of avoiding complicated mathematics!" (p. 6). Likewise, Bissell acknowledges that the need to develop telegraphy and new telephony motivated the existence of meta-languages (e.g., circuit diagrams) that correspond to other transposed forms of mathematics. Indeed, meta-languages can be the base for computational engineering tools, where some mathematical models work as black boxes. These three criteria –giving scientific status to the discipline, avoiding complex mathematics, and solving new problems– can be analysed from a didactic perspective and used to propose new relationships between electrical engineering and mathematics courses. We see this as a new direction to address the challenges mentioned in Sect. 26.3.

Another way consists in analysing contemporary engineering research institutions. To illustrate this way, we consider biomedical engineering and the blind source separation method (BSS), jointly analysed by researchers in biomedical engineering and mathematics education (Vázquez et al., 2016). The BSS is used to separate signal mixtures. It is a case of inverse modelling because the sources and the mixed model are both unknown. In the context of biomedical signal-processing, the BSS is applied to separate cerebral and extra-cerebral sources (s) under the assumption that an electroencephalogram (EEG) is a linear instantaneous mixing; that is, source signals reach the sensors simultaneously (Sanei & Chambers, 2007). More specifically, EEG recordings are the result of a combination of diverse sources, s (cerebral and extra-cerebral), through a mixing matrix A. Signals are captured by electrodes placed on the scalp, denominated 'observations', x, as shown in Fig. 26.4. The mixing model is given by $x = As$.

The BSS praxeology is as follows: task type: separate the sources in cerebral and extra-cerebral origin; technique: using the BSS algorithms, the inverse B of the mixing matrix A can be obtained and the original sources can be estimated as $y = Bx$, where y represents the estimates of s. Technology: consists of estimating unknown P signals s (sources) based only on the knowledge of Q mixes of the signals x (observations). The term 'blind' means that both the sources, s, and the mixing matrix, A, are unknown. The spatial model of the mix at instant k is defined as follows for the ideal case, without noise: $x(k) = As(k)$, where:

- $x(k) = [x_1(k), \ldots, x_Q(k)]^T$ is the vector of the observed signals (channels),
- $s(k) = [s_1(k), \ldots, s_P(k)]^T$ is the vector of the sources of origin (unknown), and
- $A(Q \times P)$ is the mixing matrix (unknown).

The theory is signal-processing.

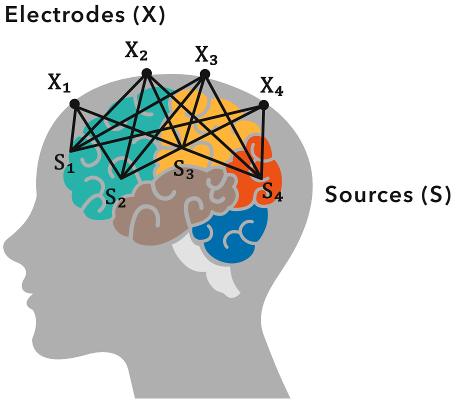

Fig. 26.4 Mix of sources recorded by the electrodes. (Similar to figure in Romo-Vázquez, 2010, p. 35)

Several BSS algorithms have been proposed to obtain the separation matrix, *B*, using either high-order statistics (HOS), thus explicitly addressing the 'independence' (Independent Component Analysis, ICA), or second-order statistics (SOS) on time-delayed or windowed signals, as well as combined (ICA + SOS). Vázquez et al. (2012) presents a broad examination of these BSS algorithms and an analysis in simulated signals is made. Those authors propose simulated sources, *s*, and known mixing matrices, *A*, to compare *s* and its estimated sources *y*. Once the BBS algorithms were tested, the methodology was applied to real EEG signals. The BSS praxeology is fascinating, and we argue that it can be an epistemological referent for designing interdisciplinary teaching proposals (see Sect. 26.4.4). Indeed, similar contexts, such as radiography and tomography, have been integrated into a linear algebra course in the IMAGENMath[4] project. Access to the materials (instructor's notes, students' notes, solutions to exercises) is available on request, but once obtained cannot be shared. As a result, it is not possible to determine the use and impact of this digital resource in engineering education, as proposed by Pepin et al. (2021). Nevertheless, it is an inspiring project for linking mathematics and engineering.

Research in engineering and mathematics courses has shown tensions (Hochmuth, 2020) as an effect of transpositions of mathematical praxeologies that have led to misunderstandings, incoherent mixtures of concept definitions and concept images, changes in notations and meanings, and replacing mathematical concepts with tables and formulas, among others. We believe that a promising direction for future research consists in analysing the disciplines, including their historical evolution, or their contemporary development. The purpose here is two-fold: to understand the factors that motivated the transpositions of mathematics, and to obtain an epistemological referent to create didactic proposals. From ATD theory, the epistemological referent could be structured as a nesting of praxeologies of increasing level: pinpoint, local, regional, and global. We also consider it essential

[4] Available at www.imagemath.org

to analyse the potential of existing didactic devices that promote relationships between mathematics and engineering courses (e.g., Schmidt & Winsløw, 2021).

26.4.4 Didactic Proposals for Mathematical Training for Non-specialists

Promising didactic proposals for the mathematical training of non-specialists have been generated from the perspectives of mathematics modelling (Kaiser, 2020) using inter-, trans-, and co-disciplinary approaches. According to Takeuchi et al. (2020), "interdisciplinarity refers to the amalgamation of two or more disciplines, whereas transdisciplinarity goes beyond the amalgamation; the relationship among disciplines is not additive, rather reflexive and emergent" (p. 11). In this regard, Chevallard (2013) affirms that co-disciplinarity makes it possible to bring together tools from different fields, as illustrated in the 'Questioning the World' paradigm, which is characterized by studying open, researchable questions, *Q,* that lead to the development of Study and Research Paths (SRP). Determining inter-, trans-, and co-disciplinary relations in a didactic proposal is, however, closely associated with the research methodology and design adopted. The emblematic research methodology called didactic engineering (Artigue, 2020), for example, has been adapted for designing SRP. Preliminary analyses –the first phase of this methodology– focus on the epistemological and didactic dimensions of non-mathematical knowledge that can come from engineering, economy, or the Arts, etc. In the case of engineering, Bartolomé et al. (2019) analysed a Strength of Materials course (SM) to determine a question, *Q*, that would satisfy three conditions: "being project-oriented and include some engineering context; involving a real object, which could be taken to the lab for mechanical tests; and potentially inducing the study of most of the important chapters of the SM course" (p. 336). Within the same SRP perspective, Galindo (2019) analysed the Quantitative Structure-Activity Relationships (QSAR) methodology that is utilized to predict activities or properties as a function of their chemical structure, using mathematical models (see Fig. 26.5). This analysis is based on documents both scientific (e.g., Balaban et al., 1992; Golbraikh et al., 2012) and scientific-didactic, including one entitled "Molecular Descriptors for Chemoinformatics" (Todeschini & Consonni, 2009). The *Preface* to that work states:

> Indeed, this new edition has been conceived not only for experts and professional researchers but also for PhD students and young researchers who wish to enter the field of molecular descriptors and related areas, giving special attention to a didactical use of the book and suggesting some possible routes for didactical purpose (Mannhold et al., 2009, p. XI).

The analysis of these documents and the proposal of a general *Q* (see Fig. 26.6) as the basis for the design of various SRP was carried out in conjunction with an expert in QSAR.

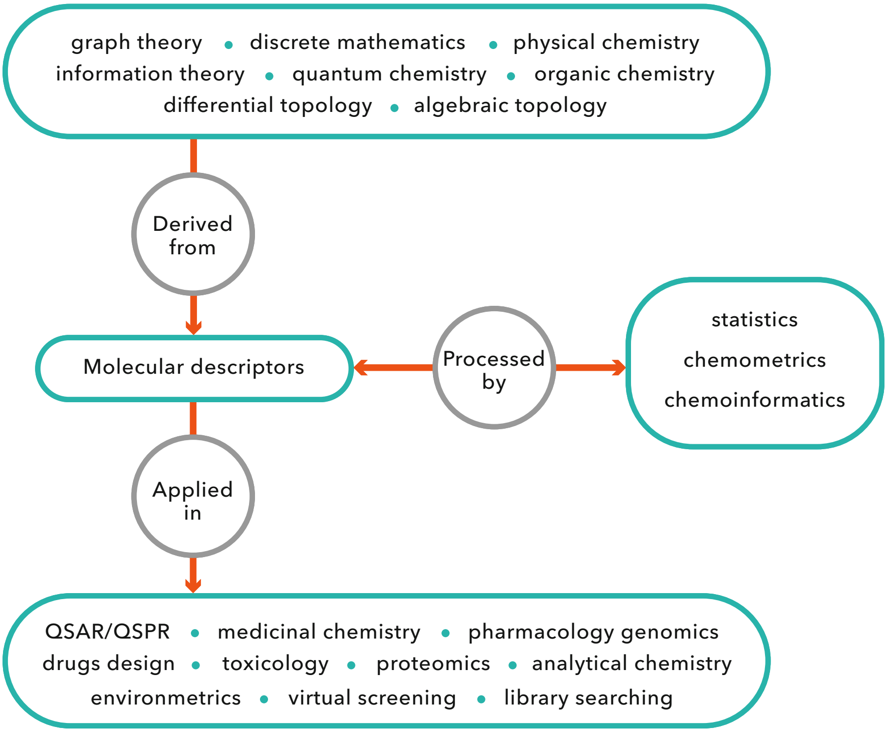

Fig. 26.5 QSAR modelling. (Similar to Todeschini & Consonni, 2009, p. XVI)

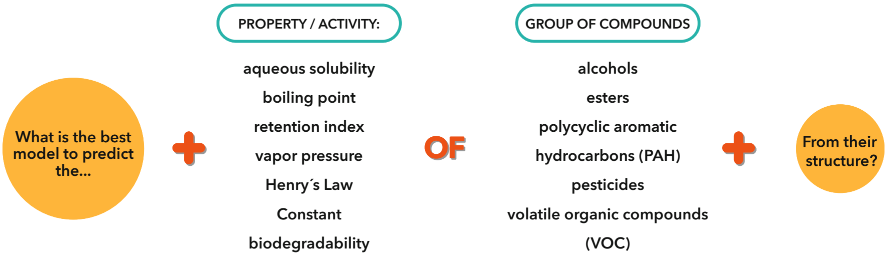

Fig. 26.6 *Q* to chemical engineering training (translation by the authors)

In addition to transposing tasks from the different engineering institutions – courses, research, practical training, and workplace – it is also possible to transpose the conditions under which those tasks are developed. Siero et al. (2017), for example, proposed a project to develop a tactile sensorial therapeutic ramp in three parallel courses, one mathematics service course, and two majors courses. The professors involved, one mathematician and two engineers, followed their syllabuses and the project ran in parallel throughout the semester. Thus, students from different majors and semesters worked in teams (3–5 students) simulating some workplace conditions: mathematical work division with a mix of novices and experts.

Implementing inter- or co-disciplinary activities in the first university semesters often becomes quite complex because students have such limited backgrounds. What is required is a balance between non-mathematical tasks (or those of mathematical modelling) and the mathematics that must be taught according to established study plans. Along these lines, Vázquez (2017) proposed a didactic activity for a first-year university linear algebra course based on the method BSS (see Sect. 26.4.3) used to separating mixtures of signals in various areas of engineering (Common & Jutten, 2010). The *Q* proposed was: how to separate a mixture of voices (audio signals) based on their register? The complexity of this question was attenuated by considering the linear mixture at one sole point in time and replacing real voices with pure tones, with modelling based on the wave function $y(t) = a{\cdot}sin(2\pi\omega t)$. The mixture A of n sources (s) registered by n observations (x) is associated with the matrix model $As = x$, seen as the transformation T: $R^n \rightarrow R^n$. This permits the emergence of a technique for calculating T^{-1}: $R^n \rightarrow R^n$ such that $T(x) = s$. The didactic design based on a dialogue between the ATD and APOS (Actions, Processes, Objects, Schemes) theories allowed the matrix transformation and its inverse –key elements of the school BSS praxeology– to be constructed as objects of linear algebra instead of being approached from a utilitarian perspective (Kaiser, 2020).

This discussion demonstrates that inter- and co-disciplinary approaches offer the possibility to frame teaching proposals that link disciplines and their teachings. Didactic engineering is postulated as a solid research methodology that can sustain these didactic designs, SRP, and even proposals developed outside ATD as mini-(Alpers, 2011) and larger-scale projects. However, a question emerges concerning the preliminary analyses: what level of analysis is necessary to access the epistemology of other disciplines and their teachings? From the institutional perspective, the epistemological referent could correspond to local, regional, and global praxeological levels. One criterion for determining the optimal level is the type of teaching proposal intended, as mentioned at the end of Sect. 26.4.2, and the kind of relationship between institutions pursued; that is, inter- or co-disciplinary. According to our analysis, the participation of experts from other disciplines, and mathematicians or math professors, in developing inter- or co-disciplinarity is fundamental for generating a minimal expression of the mixed or interdisciplinary epistemology. Generally-speaking, the didactic proposals analysed permit the construction of disciplinary knowledge and the identification and use of resources (e.g., research articles, software, Internet, experts), while also promoting diverse forms of teamwork (students from one or diverse groups). We see promise in these inter- and co-disciplinary approaches to address the heterogeneity of students in mathematics service courses.

26.5 Conclusion

This historical review of mathematical education for non-specialists shows that the constant evolution of disciplines, the profession, and social demands have increased the complexity of achieving educational models characterized by a balance between theory and practice. Research to date has focused mainly on engineering careers, which limits its scope. Very often, it has focused, as well, on classic mathematics service courses, especially differential and integral calculus. As the contributions of professionals to the latest ICMI study show, there is a need to broaden the spectrum to include more advanced mathematics. It is also necessary to enhance considerations of the growing role that probabilistic models are taking in all fields of activity, and question the evolution induced by big data and artificial intelligence whose techniques break with the usual forms of mathematisation, as Jablonka (2017) points out. We must also rethink, in light of technological developments, the ongoing issue of the relationship that is to be established with the digital tools in which mathematical techniques are encapsulated and, in another direction, the role that mathematics plays in software development (Beynon, 2012; Dahl, 2017). Our analysis reveals that recently developed research has focused on workplaces, mathematics courses, and courses in other disciplines, as well as on the design of teaching proposals that have generated contributions at the global and local levels. On the one hand, a theoretical framework for curricular innovation in engineering mathematics has been generated (Alpers et al., 2013), and didactic perspectives on mathematical modelling (Kaiser, 2020) and the inter-, trans-, and co-disciplinary approaches have been further developed. On the other, we have attained a better understanding of the logic that determines the use of mathematics in workplaces, the types of problems faced there, the different mathematical modelling activities that coexist, the relationships among modelers, clients, and experts from different areas, and the crucial role of the computer tools and levels of transparency associated with encapsulated mathematics (black box, grey box, white box). All these elements seem very distant from university mathematics, but investigations focusing on courses in other disciplines show possible points of convergence with mathematics courses and the possibility of establishing a university interdisciplinary epistemology (the mixed epistemology) likely to support new didactic proposals closer to the type of tasks performed in workplaces. Although research along this line needs to be encouraged, we consider that identifying praxeologies related to a type of modelling may offer a way to go beyond applicationist visions. For example, inverse modelling can be approached through the BSS (Vázquez et al., 2020) since it is used in several areas of engineering and biology. Likewise, didactic proposals characterized by studying questions or the development of mini-projects or larger ones based on didactic engineering show a way to establish more local relationships between teaching institutions and those that use mathematics. However, the complexity entailed in entering into the logics of other disciplines makes generating a multiplicity of proposals difficult, and their diffusion also seems to be limited to the sphere of the research community in mathematics education. These didactic proposals still constitute today specific cases that impact only a small number of students. Their permanent integration into mathematics training will require changes in existing educational models and in university institutions to foster greater collaboration between math departments and those of other disciplines, as well as with industries. In other words, it is necessary to create and ensure solid relations among the institutions involved in producing, teaching, and using mathematics. To these challenges we must add those mentioned in Sect. 26.3 regarding the need to take into account recent scientific developments, such as the increasing influence on practices of accessing and treating big data.

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Chapter 27
Mathematics in the Training of Engineers: Contributions of the Anthropological Theory of the Didactic

Alejandro S. González-Martín, Berta Barquero, and Ghislaine Gueudet

Abstract In recent years, there has been a considerable increase in the number of studies examining the role of mathematics courses in engineering students' education. Researchers have identified some important differences between the way mathematics is taught to engineering students and how engineers actually use mathematics in the workplace. In this chapter, we present tools provided by the anthropological theory of the didactic (ATD), which offers a useful framework for investigating issues related to the role and use of mathematics in engineering courses, as well as for designing innovations in mathematics and engineering courses. Using examples, we demonstrate how praxeological analyses can uncover differences in the way mathematical tools are used in mathematics courses and engineering courses. We also provide examples of implementation of study and research paths (SRPs) aimed at reducing the gap between educational and professional practices with respect to mathematics for engineers.

Keywords Anthropological theory of the didactic (ATD) · Mathematics in engineering · Mathematics in the workplace · Study and research path (SRP) · Inquiry-based processes · Modelling

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27.1 Introduction

Around the world, engineers are in demand to help solve many of the challenges related to sustainability, development, and infrastructure. For this reason, concerns are being raised in several countries where the number of engineering graduates is in decline.[1] Some reports indicate that, in some cases, up to half of students do not finish their engineering degree within the expected time frame and a significant number (in some cases, up to one-third) of students enrolling in an engineering degree drop out after failing a mathematics course (Faulkner et al., 2019; Faulkner et al., 2020). This has prompted some engineering departments to take over the teaching of mathematics courses, with economic repercussions for mathematics departments (Faulkner et al., 2019). Furthermore, recent technological advances have had an impact on engineers' professional practices and needs, making it reasonable to question the relevance and role of current mathematical training for engineers; this leads to a need for research to better understand these phenomena. However, much of university mathematics education (UME) research implicitly focuses on the practices of mathematicians[2] (Artigue, 2016), ignoring the specific needs of professionals such as engineers.

To tackle these issues, this chapter provides an overview of research examining how engineering students actually use mathematics, along with a few research-based interventions. In accordance with this book's main goal, which aims to present innovations in the context of theoretically grounded practice, we have chosen to focus on contributions from the anthropological theory of the didactic (ATD). ATD can help guide effective change by providing tools for analysing current practices in engineering courses, identifying gaps between mathematical knowledge taught in mathematics courses and mathematical knowledge deployed in professional courses and in the workplace. Furthermore, ATD can also help identify innovative ways of bridging these gaps, in particular by introducing real-world examples that give meaning to the knowledge being taught. This approach, through the innovative proposal of study and research paths (SRPs), aims to introduce students to questions encountered in today's engineering workplace, lending prominence to mathematical modelling. According to Pepin et al. (2021), mathematical modelling has played a central role when it comes to innovation in education for engineers, as it more closely resembles actual engineering practices (e.g., designing and testing models). As a tool, modelling is now used in a variety of ways beyond the more traditional modelling cycle description (e.g., Blum & Leiß, 2007); for instance, as part of a mathematical competencies framework (Niss & Højgaard, 2019), to measure students' modelling abilities, when completing tasks in engineering courses as a

[1] For instance, in the United States, the President's Council of Advisors on Science and Technology reported in 2012 that approximately one million more STEM graduates would be needed in order to meet the demands of the U.S. workplace (PCAST, 2012).

[2] That is, the use of mathematics and the rigour and structure of content in structure of content in mathematics research papers.

complement to other problem-solving techniques (Kortemeyer & Biehler, 2017), and, more recently, in questioning the role of mathematics and of modelling in engineering education, as part of more institutional perspectives.

In the case of ATD, mathematical modelling has been linked to the notion of mathematical activity since the framework was first developed (Chevallard, 1989), with the assumption that doing mathematics mostly consists in producing, transforming, interpreting, and developing mathematical models (García et al., 2006). In the context of engineering education, modelling becomes central to addressing real-life problems in the engineering workplace, exposing students to professional practices and deepening their understanding of mathematics' usefulness and its applications in various engineering subdisciplines. That said, this chapter neither focuses on nor examines different approaches that conceptualise modelling through other epistemological models (see an overview in Barquero et al., 2019). We also note that our focus is on the training of engineers and not on their workplace, which is covered in the Chap. 30 by Castela and Romo Vázquez.

This chapter is organised as follows: The next section (Sect. 27.2) provides a review of literature on issues concerning mathematics in engineering training,[3] which leads to an explanation in Sect. 27.3 of tools from ATD that we use to present and discuss the specific examples given in this chapter. The following two sections then illustrate how these tools are used: Section 27.4 presents the results of the analysis of engineering courses, revealing the latter's use of mathematical tools and how this use differs from that normally seen in a mathematics course. Section 27.5 presents specific examples of innovations in engineering courses, discussing how epistemological tools can help inspire a different way of organising content, and how some of these interventions are being deployed in the institutions where they were first implemented. We conclude the chapter with some observations based on the examples of research discussed.

27.2 Problems with Mathematics Courses for Engineers

How should mathematics be taught to engineers? The question is not an easy one, and its complexity has long been acknowledged (e.g., Howson et al., 1988). In addition to this, most research studies on UME still consider the practice of mathematicians to be an implicit reference (Artigue, 2016), which is not helpful in addressing this question. This situation is changing, however. Recent years have seen a considerable increase in the number of studies examining the role of mathematics courses in engineering students' education (González-Martín et al., 2021). Pepin et al. (2021) observe that, since 2003, the mathematical needs of future

[3] We keep this section brief and note that a detailed discussion of challenges for research in mathematics for non-specialists, with examples specific to engineering, is the focus of the Chap. 26 by Romo Vázquez and Artigue.

engineers have been increasingly addressed in terms of "mathematical competence," following Niss' (2003) definition of this term,[4] both in research literature and in curricular propositions. For instance, the European Society for Engineering Education (SEFI) proposes to construct curricula for future engineers based on competences such as "thinking mathematically," "reasoning," "representing," and "communicating" (Alpers, 2013).

One element that the literature has identified as the origin of many difficulties for engineering students is related to the "classic" structure present in many engineering programmes around the world, where mathematics courses are presented mostly in the first years, separate from engineering courses (e.g., Artigue et al., 2007; Engelbrecht et al., 2017; González-Martín et al., 2021). This structure is influenced by the original model of the *École Polytechnique*, founded in France in 1794 (Belhoste, 1994), and it seems to hinder students' ability to make links between concepts (Christensen, 2008). Under this approach, mathematics courses are taught in a purely "mathematical" way, often focusing more on technical skills and "on mathematical concepts and understanding rather than applications" (Loch & Lamborn, 2016, p. 30), with significant differences between the mathematical terminology and notation adopted by engineering and mathematics teachers (Flegg et al., 2011). Prerequisite mathematics courses have high failure and dropout rates (Faulkner et al., 2020); the lack of connection between mathematics and engineering courses may lead students to see mathematics content as irrelevant, reducing their interest and motivation. Research on the background and practices of teachers in engineering programmes also shows that teachers with a mathematical background may focus more on rigour (e.g., González-Martín & Hernandes-Gomes, 2020), and that teachers with a science or mathematics background may tend to see academic excellence in mathematics courses as a kind of prerequisite for engineering, reinforcing approaches to teaching that emphasise formalism and downplay practical applications (Nathan et al., 2010). In particular, these practices do not seem to develop the mathematical skills that subsequent engineering courses require (Faulkner et al., 2019).

In questioning the approaches used in mathematics courses, some researchers argue that these courses have two major problems: (1) they often fail to develop competences expected of engineers, such as the ability to transfer knowledge to other applications (e.g., Harris et al., 2015) and modelling (Faulkner et al., 2019); and (2) only a small portion of their content is actually applied in engineering courses (Faulkner et al., 2020). Regarding (1), Faulkner et al. (2019) provided data from 24 teachers of engineering courses that list Calculus I, Calculus II, Calculus III, Linear Algebra, or Differential Equations as prerequisites or corequisites. These teachers reported that their students had not developed modelling skills in their previous mathematics courses. Quéré (2019) also demonstrated that a significant

[4]Niss (2003) defined mathematical competence as "the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role." (pp. 120–121)

proportion of engineers in the workplace report that their training did not teach them modelling skills, and that their university mathematical training was not well adapted to their current professional needs. Regarding (2), Faulkner (2018) and Faulkner et al. (2020) analysed a first-year engineering statics course, showing that only seven out of the 84 homework exercises (≈8%) required some explicit knowledge of calculus; their analysis of a circuits course also revealed that only 14 out of the 70 assigned homework problems (20%) required an understanding of calculus concepts or skills in order to be solved. Research indicates that, in many cases, engineers do not recognise the mathematics they use (Artigue et al., 2007; Gainsburg, 2006; Kent & Noss, 2003), since it is imbedded in their practice in such a way that "only the vestigial traces of the college mathematics taught [...] remains in the mathematics that they actually use in activity" (Noss, 2002, p. 54). Therefore, it is reasonable to assume that this phenomenon also occurs in engineering courses.[5]

The literature on these issues tends to conclude that traditional content and teaching methods are not meeting current professional needs (e.g., van der Wal et al., 2017) and do not allow students to adequately develop mathematical skills for the workplace (Sevimli, 2016). Although the number of studies on engineering students' difficulties is increasing, more research is needed that provides explicit analyses and models of engineers' practices (in university courses and in the workplace). This could provide a better understanding of students' mathematical needs and offer insights into how professional engineers actually deploy mathematical knowledge. ATD is one approach that has helped provide insight into these questions. In the next section, we present some of its key aspects.

27.3 Some Key Notions from ATD[6]

From the perspective of ATD, the concept of knowledge is extremely broad. It embraces theoretical constructions but also includes the practical dimension of knowledge – the *know-how* that underpins all kinds of human activities – and both components (the practical and theoretical components) can be jointly analysed. The notion of *praxeology* is viewed as the basic unit used to analyse human action in general (Chevallard, 1992) and, in particular, mathematical knowledge in different institutions. A praxeology is understood as an entity formed by a combination of *praxis* – the know-how or ways of doing – and *logos* – an organised rationale about the praxis. Praxeologies consist of a type of task, a set of techniques, a rationale about the technique (called technology), and a theory, and this quartet provides a

[5] In this section, we focus mainly on research concerning mathematics courses for engineers. Castela and Romo Vázquez provide more information on the engineering workplace in their chapter.

[6] In this section, we provide a brief synthesis of notions that are used in the rest of the chapter. For more details, the reader is invited to read the Chap. 19 by Bosch et al.

unified vision of different activities. Praxeologies are shaped by the institutions where they take place. While tasks may be similar, the praxeologies can differ depending on the context (e.g., a mathematics course for future engineers, an engineering course, or in the workplace). In the Chap. 30 by Castela and Romo Vázquez, they use ATD for an epistemological analysis of the production of praxeologies in different institutions (in particular, mathematics courses and industrial settings), and to discuss the transposition of knowledge from one institution to another. In this chapter, we analyse outcomes of transposition processes by examining praxeologies involving mathematics in engineering courses (Sect. 27.4). Drawing on ATD, we then propose possible interventions to reduce the gaps this transposition creates (Sect. 27.5).

As Bosch (2018) explains, the dissemination of praxeologies takes place through what we call *didactic systems*. A didactic system is a set of three elements S (X; Y; $\wp$) formed by a person or a group of persons Y (the teachers) who do something to help another group of persons X (the students) to learn a given body of knowledge or praxeology $\wp$. We can imagine a particular didactic system in a university class, where X is a group of first-year students, Y is the course teacher, and $\wp$ is the praxeology to be taught (for instance, diagonalisation of matrices or methods for solving differential equations); in this praxeology, the teacher might plan particular tasks and techniques to teach to her students, as well as rationales justifying this particular praxis. The evolution of the didactic system S (X; Y; $\wp$) undoubtedly depends on what X and Y can do, but also on how the praxeology $\wp$ is being transposed. It is also important to analyse the conditions and constraints under which didactic systems interact and evolve (called the *ecology*, Barquero et al., 2013; Chevallard, 2002).

Chevallard (2015) describes several important constraints related to what has been called the *paradigm of visiting works*. In this paradigm, didactic systems S (X; Y; $\wp$) are determined by the selection of a set of praxeologies $\wp$ that students are asked to "visit" under a teacher's guidance. This situation leads to what the author calls the "monumentalisation" of curriculum, whereby each selected mathematical work appears as "a monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raisons d'être*, now or in the past" (p. 175). This stands in contrast to the *paradigm of questioning the world*, which offers a quite different perspective. In this paradigm, didactic systems S (X; Y; Q) are not formed around a given praxeology to be studied, but rather around a question Q, to which X, under the guidance of Y, might provide a final answer A.

One recent important contribution of ATD involves the notion of *study and research path* (SRP). During the past decade, a considerable amount of research has been developed by designing and implementing various formats of SRPs in different universities (see Barquero et al., 2022). The goal of implementing an SRP is twofold. On the one hand, SRPs can be understood as a didactic device that promotes a shift from the paradigm of visiting works to the new paradigm of questioning the world (Bosch, 2018; Chevallard, 2015). On the other hand, SRPs can also be seen as a research tool, one which can be used to identify and study didactic phenomena, i.e., regular facts that are observable in teaching and learning processes and are specific to the content involved.

SRPs have adopted different instructional formats depending on the educational level, the conditions assumed, and the constraints and didactic phenomena considered. However, some commonalities may be observed. The starting point of an SRP is a generating question Q_0 posed by the teacher and addressed to the community of study: X and Y. The main goal is for X and Y to collaborate on elaborating a final answer A to Q_0. Students are usually divided into small teams X_i, and different responsibilities are assigned to each team, according to the derived questions Q_i generated by Q_0. The community of study's collective work and the knowledge involved can be described as a concatenation of derived questions and their associated answers that will lead to the elaboration of A. The inquiry process combines moments of "study" of available information with moments of "research" (in the sense of *inquiry*) into and creation of new questions and answers.

In the next section, we illustrate the uses of the ATD to uncover specific uses of mathematics in engineering courses, allowing for a better understanding of the gaps and disconnections mentioned in the previous section.

27.4 Practices in Engineering Courses

This section illustrates the potential of ATD for analysing how mathematics is used in engineering courses, with two examples. It also examines how research in this field has led to developments in ATD constructs.

In the first example, Castela and Romo Vázquez (2011) conducted a study of reference materials in order to analyse and compare the use of the Laplace transform in a mathematics course and in two control theory courses,[7] which are considered to be different institutions. They analysed the use of Laplace transforms in each of these institutions in terms of their relation to knowledge (production, teaching, and using). Given a function $f(x)$ of real variable, its Laplace transform is defined as: $L[f(t)] = F(p) = \int_0^\infty e^{-pt}.f(t)dt$. Laplace transforms are necessary in professional practice as follows:

> The problem at stake is the automatic regulation of systems: if a quantity is to be kept constant, an electronic gauge measures its value; when variation is recorded, an appropriate regulation process is triggered to go back to the desired value. The less time needed to get the quantity back to this value, the more efficient the control system. The evolutions of the different systems involved are described by differential equations, turned to algebraic ones by the Laplace transform and easily solved, with a rational function $F(p)$ as a solution. To inverse the Laplace transform, the online textbook recommends using a table of Laplace transforms. (Castela, 2017, p. 422)

[7] In detail, the courses were: (1) a mathematics course on holomorphic functions in an engineering school; (2) an online course in automatics for electrical engineering and industrial computing; (3) a course in an engineering science and technology programme.

First, Castela and Romo Vázquez (2011) show that, in the control theory courses, results and properties concerning Laplace transforms are presented as formulae, while their validity or the conditions that make these properties true are not addressed; rather, students are provided with justifications based on professional practices:

> [. . .] we assume the following hypothesis: the system that will generate the function *f*(*t*) as a response to a starting excitation must be initially **at rest**, meaning that *f*(*t*) must be **constant** before a signal of command is applied. Therefore, in the previous expressions: *f*'(0) = 0 and *f*'(0) = 0 and, in general, all initial values of successive derivatives of *f*(*t*) are zero. (Castela & Romo Vázquez, 2011, p. 105, our translation, bold characters in the original)

Their analyses of the two engineering courses where Laplace transforms are introduced lead the authors to characterise different levels of validation[8] of mathematical properties and results, whereby the mathematics research institution is ignored, evoked, or invoked as an epistemological guarantee, or convoked (detailing the mathematical proof) (p. 113). There are also differences at the level of the techniques; for instance, in mathematics courses, rational functions are decomposed using the general technique which presents each new denominator as one of the factors of the original denominator:

$$\frac{2x+1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3},$$

This differs from the technique used in certain areas of engineering; in automatics, the technique writes the expression in the denominator as: $k.(1 + T_1 x)\,(1 + T_2 x)$, etc. This is due to the fact that if the Laplace transform is written in the format $F(p) = \frac{1}{1+1.5p}$, then $f(t) = k.(1 - e^{-t/1.5})$, with 1.5 being the time constant of this function; its higher value determines the reactivity, and hence the quality, of the initial system of differential equations that was to be solved and that is transformed into an algebraic system of equations by the Laplace transform.

Their work leads to an enlarged version of the praxeology model, rendering more explicit the fact that different sources of knowledge related to a mathematical technique may exist. While a mathematical institution may produce results that validate a given technique, the user institution produces and validates its own norms in order to foster an effective use of techniques. Figure 27.1 shows an initial version[9] of this expanded model. Here, the two θ indicate the coexistence of a theoretical (mathematical) and a practical (or empirical) rationale for techniques, I_u

$$\left[T, \tau, \begin{array}{c} \theta^{th}\,\Theta \\ \theta^{p} \end{array} \right] \begin{array}{l} \leftarrow P(M) \\ \leftarrow \quad I_u \end{array}$$

Fig. 27.1 Expanded praxeological model proposed by Castela and Romo Vázquez (2011, p. 126)

[8] See more details in the first section of the Chap. 30 by Castela and Romo Vázquez.

[9] This model has evolved further. See Castela (2017).

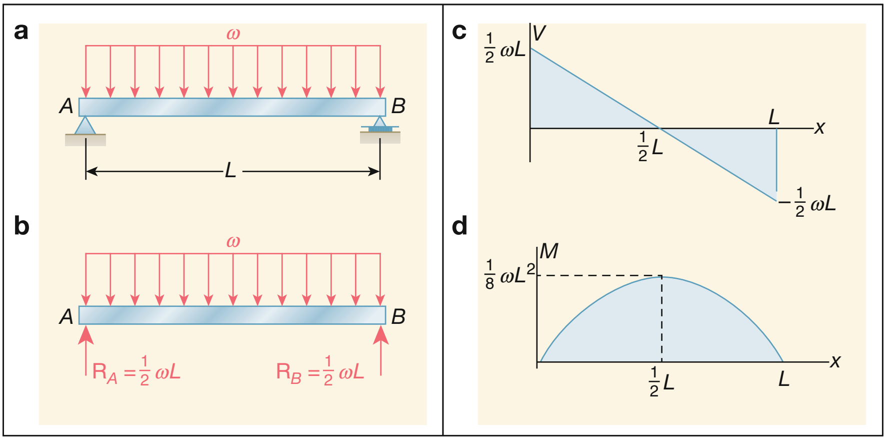

Fig. 27.2 Solved example with a uniformly distributed load (Beer et al., 2012, p. 362). The top right graph represents the shear force (antiderivative of a constant), and the bottom right graph represents the bending moment (the antiderivative of the shear force). The maximum value of the bending moment indicates where there is more tension in the beam (in this case, at its mid-point)

denotes the institution to which mathematical tools have been imported, and $P(M)$ the institution where mathematical knowledge and techniques are produced.

This model is considered in our second example, found in the work of González-Martín and Hernandes-Gomes (2017) and González-Martín (2021). Their work analyses the use of integrals in reference textbooks and teaching practices in two engineering courses: in strength of materials, to define bending moments; and in electricity and magnetism, to define electric potential. Their analysis reveals that, although integrals are used to define these notions and to introduce related techniques (integrals are present in the logos block), as the above model illustrates, the techniques' explanations use elements derived from professional practices, with certain properties of integrals and derivatives employed only implicitly. An entanglement of elements from mathematics and engineering is evident both in the technique and in the technology, with the mathematical rationale undergoing an adaptation (for example, limits are not explicitly used).

For instance, in the task concerning the sketching of bending moment diagrams (see a simple example in Fig. 27.2, and a full analysis of a complex task here), the final product is the graph of an antiderivative, but the technique is based on arithmetic procedures and not on calculus techniques. As noted by other researchers on workplace practices (e.g., Noss, 2002), only vestigial traces of calculus are visible in the technique and in the final result, confirming that these phenomena also occur in engineering courses. Moreover, as Faulkner et al. (2020) indicate in their analysis of engineering course materials, the study of González-Martín (2021) argues that if the same task were to be solved using integrals explicitly, the technique would become longer, whereas the simple techniques applied (which are supported implicitly by integrals) produce a simpler solution, acceptable for the field of engineering.

This fact is confirmed by the teachers of both courses. Their teaching reproduces the praxeologies present in the reference textbooks, and they state that these praxeologies are coherent with engineering practices.[10] They also both confirm that in their student assessments, they focus primarily on the mastery of techniques proper to engineering, and that students' ability to explicitly use integrals has a low impact on their mark.

The two examples discussed in this section illustrate how ATD allows for the detailed analysis of practices in various fields of engineering, examining the mathematical content that is used, how it is used, the rationales that underlie this use, and the type of tasks that call for this use. The tools provided by ATD facilitate analysis of the tasks that need to be solved, pinpointing the implicit and explicit mathematical content at stake. In this sense, it gives researchers a better understanding of the kind of mathematics that engineering students require and how they use it, which is connected to the notion of mathematical competence proper to these fields. The praxeological model also helps identify differences in the actual use of mathematics and the classic presentation (including techniques and rationales) in mathematics courses discussed in the second section of this chapter. In the next section, we illustrate how ATD has enabled the development of teaching interventions.

27.5 SRPs in Engineering Programs

In this section we present and discuss three selected examples of SRPs, illustrating how ATD can inform practice-oriented research in the context of engineering programs. These SRPs were designed and implemented to address the issue of the lack of connection between mathematics and engineering courses or workplace practice. In Sect. 27.5.1, we present the case of an SRP concerning statistics (Quéré, 2019). Through this example, we illustrate how ATD provides epistemological tools for designing and managing such an intervention, and, in particular, the importance of both the generating question and the use of the question-answer map (Florensa et al., 2019) to anticipate various possible paths chosen by students. In Sect. 27.5.2, we present two cases of SRPs in engineering courses, concerning elasticity (Florensa et al., 2016, 2018) and strength of materials (Bartolomé et al., 2019). These examples form the basis for a discussion on various modalities of an SRP's integration into regular courses in an engineering programme, examining the conditions that facilitate its implementation and make it sustainable, as well as possible constraints. For this purpose, we have selected two SRPs with contrasting modalities of integration: presented at the end of a traditional course, or presented to students at the beginning of the course and later used to organise subsequent course content based on the questions raised during the SRP.

[10] In standard practices, specific values are handled and calculations can be obtained using tables. Only in the cases of specific designs do these calculations need to be made, usually with the help of a computer to calculate the integrals.

In Sect. 27.5.1, we examine an SRP in a mathematics course (statistics, in this case), while in Sect. 27.5.2 we look at two SRPs in engineering courses. However, we consider this difference to be superficial, focusing our discussion on the issues mentioned above.

27.5.1 Epistemological Tools for Designing and Managing an SRP: An Example in Statistics

Barquero et al. (2013) demonstrated the potential of SRPs for changing the teaching paradigm within engineering programmes. They analysed the implementation (over five academic years) of SRPs on population dynamics for first-year chemical engineering students. Since this early study, a systematic method for designing and managing SRPs has been progressively developed (see Florensa et al., 2019, who analysed and compared five different SRPs). In exploring the outcomes of SRPs and in reviewing the latest developments in their design and management, we draw on work by Quéré (2019), who designed and implemented an SRP in statistics in the context of a chemical engineering school in France, with third-year students.

Quéré worked with a colleague, Roger, a chemistry teacher, who also teaches statistics to third-year students. Quéré outlined the main principles of an SRP for Roger, providing him with a few research articles on the topic. This collaboration between a researcher in mathematics education (Quéré) and a specialist in chemistry (Roger) played an essential role in the design and management of the SRP. Roger proposed the generating question for this SRP, which was:

> Q_0: "In the pharmaceutical industry, how can you check that the product (drug) corresponds to the dosage indications on the package?"

The generating question is the starting point of any SRP. It has to be "a 'lively' question of genuine interest to the community of study" (Barquero et al., 2013, p. 327); this is crucial for the change of paradigm foregrounded by ATD. As a chemistry specialist with knowledge of statistics, Roger knew that checking to make sure a drug actually meets the intended dosage is a professional problem that chemical engineers encounter in their workplace. As evidenced in the Chap. 30 by Castela and Romo Vázquez, controlling the variability of a production process is a central type of task in the industry institution. This question provides connections between the statistics course and the engineering workplace.

One potential difficulty when implementing an SRP is tied to the fact that students propose and investigate their own sub-questions, derived from the generating question. This can give rise to unexpected issues, which can be challenging for teachers. An essential tool for overcoming this challenge (and to help manage an SRP more generally), is the question-answer map (Florensa et al., 2019). The a priori question-answer map for this SRP designed by Quéré and discussed with Roger is represented in Fig. 27.3. This map presents the general themes of the students' possible sub-questions. For example, "if the drug is very expensive, what kind of statistical

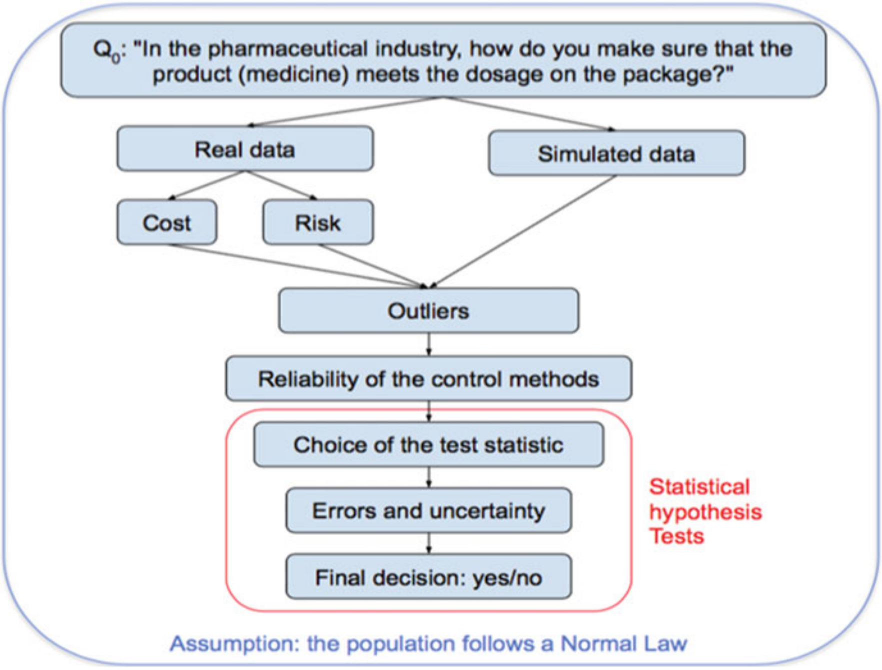

Fig. 27.3 Initial question-answer map for the SRP (Quéré, 2019)

tests should we use, given that we should test as small a sample as possible?" is a question concerning real data and cost.

The SRP was implemented in the context of a project-based course, whose outcome was to be a website. The students investigated the theme (provided by the generating question) and presented their results on a website to be consulted by future third-year students. The students had previously taken a course with Roger on probabilities and statistics,[11] which included statistical tests (in particular Student's t-tests and z-tests). Fourteen students took part in the SRP, over five sessions, with each session lasting between one and two hours.

The generating question was presented during the first session. The students were eager to work on a professional question, and collectively they decided to investigate it in order to pinpoint sub-questions in advance of the next session. During this second session, the sub-questions were presented and discussed. Finally, six sub-questions were selected, and the students were divided into seven pairs. One pair was in charge of coordinating the work while the other pairs were assigned to study one sub-question each. The third and fourth sessions were dedicated to presenting and discussing the results obtained by the various pairs of students

[11] Generally, students are not usually exposed to an SRP's content in a previous course. In Sect. 27.5.2 we look at SRPs whose content is presented during the same course.

(based on their work outside the sessions). The teachers guided the discussions to ensure the application of content previously studied in the statistics course. During the fourth session, the students planned the design of the website which would house the results of their work; this helped coordinate the work of the various groups and contributed to their understanding of the study's overall structure. The students' assessments considered mathematical criteria (correctness and relevance of the methods proposed), communication criteria (clarity of the website and of the presentation), while also taking into account each individual student's involvement.

Quéré observed the students' activity during the SRP and analysed the logbooks of each student pair. Due to space constraints, rather than present the full results of the students' activity, we have selected a few examples illustrating various important aspects. After searching for documentation concerning pharmaceutical industry processes, the students proposed to develop three different methods, distinguishing between a common drug, a rare drug, and an intermediate drug. The students used methods and concepts learnt during their statistics course. For example, they used a Student's t-test (or a z-test, depending on the kind of drug) to compare the expected mean (of the active ingredient's proportion) with the actual mean. They had learnt these tests in Roger's course; however, the work they performed during the SRP contributed to their appropriation of the course praxeologies. In the SRP, they were responsible for choosing the most relevant test and for choosing the acceptable level of risk. In the exercises associated with the traditional course, the test and the risk are provided; students only need to apply the test and to decide, according to its output, if they accept or reject the hypothesis, based on the given risk (usually 5%). Moreover, the students also discovered new mathematical content during the SRP. For example, the tests mentioned above can only be applied to "normal samples" (those which can be considered as following a Normal law), and in Roger's course the students had worked only on normal samples. In the SRP, they had to investigate the normality of the sample. During their investigations, they discovered tests that allowed them to accept or reject a sample's hypothesis of normality.

The students' final website described the three methods they designed. Each part of these methods presented what can be interpreted in ATD as a complete praxeology: the type of task, the technique to be used, and technological-theoretical elements (e.g., explanations about the choice of a given risk).

Quéré's selection of statistics as the theme for this SRP was inspired by a previous survey of working engineers (Quéré, 2017, 2019), in which more than half the participants reported using statistics in their professional activity. The generating question was one that chemical engineers working in the pharmaceutical industry are apt to encounter in their professional practice. During the SRP, the students employed statistical software commonly used in the industry. Several other aspects of the SRP led Quéré (2019) to assert that it can help bridge the gap between the engineering programmes and the workplace: (1) because the end result was a website, throughout the process the students were aware of the need to communicate their results clearly; (2) because the teams working on the sub-questions had to be effectively coordinated, an important aspect of the workplace.

Through questionnaires completed by the students and an interview with Roger, Quéré (2019) observed an overall positive reaction to this SRP. The students recommended implementing similar SRPs during their fourth and fifth years of study as well. They noted that while it should not replace a traditional course, it had helped them deepen their knowledge of previously studied content. Roger noted that starting with a generating question and asking students to then determine sub-questions to investigate was a successful process, one he had not experienced before, even in project-based courses.

27.5.2 *Teaching Formats of SRPs and Their Ecology: Two Examples of Engineering Courses*

In this section, we focus on two particular cases of SRP that have been implemented in two engineering courses: elasticity and strength of materials. We have selected the cases of Florensa et al. (2016, 2018) and Bartolomé et al. (2019) for two reasons. On the one hand, both cases examine SRPs that were implemented in the same engineering school, EUSS-School of Engineering (Universitat Autònoma de Barcelona) in Barcelona. In 2014, this engineering school introduced innovative teaching practices: first, by having the teacher-researchers guide the SRPs' implementation and, soon after, by sharing and transferring this responsibility to non-researcher teachers. On the other hand, the changes to the way that the SRPs have been integrated into the compulsory courses are noteworthy. We refer to two main modalities of integration: an SRP at the end of a course and a course organised around the question/s introduced by an SRP (see Barquero et al., 2020, 2022 for more details about the different teaching formats of SRPs).

In the first case, Florensa et al. (2016, 2018) present an SRP that was implemented at the end of an elasticity course (6 ECTS,[12] first semester, third year) in a mechanical engineering undergraduate programme. The course ran over 16 weeks. The first 9 weeks followed a more traditional structure, including lectures and lab sessions in which students were given hands-on experience related to the course content. This left the last 7 weeks for the SRP's implementation. This modality of integrating the SRP at the end resulted in major changes to the course's traditional structure. A careful analysis of the existing syllabus was crucial in deciding what to include in the first part of the course. The researchers (one of whom taught one of the courses), started by analysing the kind of tasks proposed and the prevailing pedagogical approach adopted in previous academic years. This analysis revealed two important issues: rather than actual workplace-based scenarios, the students were mostly presented with ideal situations in the form of (easy) analytically solvable problems/exercises; furthermore, several classes were devoted to lab sessions whose main goal was to familiarise students with certain computation programmes and

[12] 6 ECTS (European Credit Transfer System) represents 60 hours of teaching and 150 hours of work for students.

methods, far removed from actual professional practices. Based on these findings, the two teachers reorganised the first part of the course to provide students with the practical, technical, and theoretical knowledge they believed that students would need for the SRP.

The SRP implementation was initiated by a generating question addressing the design of a bike part (such as a brake lever, a gear, or bike lock key), and the choice of material (with unknown mechanical properties). Students were expected to produce work that was more similar to workplace tasks and that included the finite-elements method as part of the primary knowledge mobilised. This SRP was presented at the beginning of the semester, and students were informed that the first part of the course would cover material in preparation for the proposed project. Students worked in small groups of three to four. Each group was tasked with addressing Q_0 with a particular bike part and chosen material. Once the SRP was initiated, the generating question became a central focus; the goal was to provide a suitable solution for the bike part problem, while the knowledge required to solve the problem was incorporated into the content of the engineering activity. The SRP was assessed by means of weekly reports describing the inquiry activity and a final report that included the design and justification of the part. First implemented during the academic year 2014–2015, this process has been repeated in six additional academic years. This modality of the SRP's integration seems to offer benefits, as it represents a balance between the study of works in the first part of the course, and the inquiry into questions derived from the generating question in the second part.

In the second case, Bartolomé et al. (2019) piloted another modality of an SRP's integration – in which the SRP serves to guide the course – resulting in a more drastic reorganisation of the course content and structure. This SRP was implemented for the first time in 2016–2017 with second-year mechanical engineering students in a strength-of-materials course (6 ECTS, second semester). Similar to the previous case, the initial epistemological analysis of the course revealed that the course content and exercises did not accurately reflect actual problems engineers encounter in the workplace. The generating question used in the first implementation of this SRP concerned the design of a slatted bed. It was presented in the following context (Bartolomé et al., 2019, p. 337):

> You are working as an engineer in a company manufacturing slatted-beds. Your company supplies beds to an American client (a chain of motels). Recently, you have been commissioned to provide them with single slatted-beds, capable of supporting the weight of a 120 kg person. You have some slatted-beds in stock, which you could supply immediately if meeting the clients weight specs. . .
>
> – Is the slatted-bed able to resist?
> – What is the maximum load it can resist?
> – If necessary, redesign the slatted-bed to meet the clients' specs.

The SRP was implemented throughout the entire semester, which ran over 17 weeks with two two-hour sessions per week. Each class was divided into four parts.[13] In the

[13] In Bartolomé et al. (2019) more details can be found about the question-answer map (Figure 1 on page 334) and the organisation of work (Figure 2, p. 336).

first part, the whole class discussed the project's status and identified the derived questions that each group of students considered relevant for their work and for the class. The questions were assessed, and the ones deemed most relevant were then assigned among the student groups. The students then spent time working on these questions for the remainder of the session. Finally, each team presented their answers to the class. As in Florensa et al. (2019), the use of question-answer maps played a crucial role for the researchers and the teachers with respect to the description and institutionalisation of knowledge emerging from their work. The teachers used these maps in class to open and close the sessions and to trace the path followed, together with students. This tool was also useful for students, who used these maps to describe their work in terms of the questions addressed, the answers provided and the new questions raised moving forward. Student assessment was based on the students' work as observed by the teachers (question-answer maps generated – with points for clarity, quality of questions tackled, and sources of information – the presentation of work to the rest of the class, and teamwork). The students' final output was also taken into account.

27.6 Conclusions

In this chapter, we focused on the potential of ATD to offer a theoretically grounded approach that can inform practice-oriented research. We organised our discussion in terms of: (1) the tools provided by ATD for analysing existing practices and the outcomes of such analyses; and (2) work on the instructional design and the analysis of its implementation in the context of engineering education.

Regarding (1), as we illustrated in Sect. 27.4, ATD offers tools for analysing engineering practices (both in the workplace and in educational settings). These tools can be used to reflect on course content and teaching methods in mathematics courses within engineering programmes, and how these courses prepare students to take on professional tasks. Considering the difficulties that engineering students experience with mathematics courses within the classic structure of engineering programmes (as discussed in Sect. 27.2), and considering the current needs of professional engineers, research is urgently needed to clarify the differences in practices between educational and professional settings and to find solutions for bridging these gaps. As demonstrated, ATD can help reveal precisely how mathematical content is taught and used in a given institution. It is therefore possible to compare practices in different institutions where the same content is taught. This is especially relevant in the case of engineers (or other professionals), who encounter mathematics in mathematics courses, in engineering courses, and ultimately, in the workplace. However, it is important to note that ATD-based analyses do not lead to the conclusion that SRPs should replace existing practices. Rather, less drastic changes to the curriculum are suggested, by indicating that a praxeology, or a part of a praxeology, is missing and could be productively added. For instance, in the engineering context, praxeologies about communicating mathematics (e.g., Quéré,

2019) or technological rationales concerning the mathematical content in strength of materials courses (e.g., González-Martín, 2021) may be developed. Moreover, as we discuss below, SRPs should not necessarily replace the usual teaching practices and can instead be combined with them in various ways. Finally, it is worth highlighting the fact that empirical research has led to theoretical developments within ATD. In particular, the key notion of praxeology has evolved and been adapted to service courses, revealing that engineering practices present a mix of rationales originating from mathematics and from professional practices (e.g., Castela, 2017). In this sense, Peters et al. (2017) have also proposed an extended ATD model, where they consider that even in the techniques, it is possible to find a mix of steps drawn from mathematics and from typical professional practices.

Regarding (2), in Sect. 27.5 we focused on the construct of the SRPs and on the analysis of the conditions put into place in implementing SRP-based interventions. In these interventions, the traditional course structure and the knowledge to be taught are questioned and reorganised in response to the need to provide a collective answer to the generating question(s) presented in the SRP. SRPs have the advantage of narrowing the gap between course content and real-world practices, since modelling is a key element. This is an important contribution, since recent research indicates that engineering teachers complain of students completing mathematics courses without acquiring modelling skills (e.g., Faulkner et al., 2019), while professional engineers have also reported that their mathematics courses did not adequately prepare them for modelling activities (e.g., Quéré, 2019). Therefore, considering our comments on (1), we can see that the tools ATD provides help achieve two different, although complementary, goals: pinpointing possible challenges for students, as well as proposing interventions that are ecologically more viable and epistemologically closer to real-world engineering activity. In considering this epistemological dimension, SRPs go beyond the classic project-based courses and help change the paradigm of visiting works. Moreover, as shown in this chapter, there are various ways to implement SRPs and researchers have provided analyses on the main conditions that facilitate and hinder their implementation (Barquero et al., 2020, 2022). An analysis of the various SRPs implemented in university programmes yields the following key points:

- The need for epistemological tools for questioning and rethinking the knowledge at stake, as well as the need to place mathematical modelling at the core of the activity;
- The importance of developing tools (such as question-answer maps) that help researchers and teachers (non-researchers in mathematics education) to participate equally in this epistemological questioning;
- The various formats for integrating SRPs into mathematics and engineering courses, as well as the advantages and inconveniences of these formats;
- The identification of the conditions that facilitate and constrain the dissemination of SRPs in engineering education.

Since, at present, SRPs are being implemented regularly in various engineering training programmes (Florensa et al., 2019), future research should be able to identify the common characteristics of their successful implementation, as well as the differences that are proper to the institutions in which they are being implemented. This will provide more information concerning SRPs' ecological dimensions. We anticipate that in the years to come, ATD will provide valuable insights for practice-oriented research on the training of engineers; furthermore, this practice-oriented research will continue feeding theoretical developments.

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Chapter 28
Modifying Exercises in Mathematics Service Courses for Engineers Based on Subject-Specific Analyses of Engineering Mathematical Practices

Jana Peters

Abstract This contribution presents the idea of modifying exercises from a mathematics service course on the basis of analyses (in the sense of the Anthropological Theory of the Didactic) of mathematical practices from electrical engineering. The core of this small-scale approach is to use the respective specific conceptualisation of mathematical knowledge in electrical engineering and in mathematics service courses for teaching design. In earlier work, this specifically conceptualised mathematical knowledge could be methodologically grasped with two different institutional mathematical discourses. The example shows how an existing exercise of a mathematics service course can be modified to support connections to mathematical practices from the engineering mathematics discourse. This illustrates exemplarily the importance of recognising the subject specificity of institutional mathematical practices in electrical engineering.

Keywords ATD · Mathematical practices of engineers · Modifying exercises · Connecting engineering and mathematics · Mathematical discourses

28.1 Introduction

Mathematics has at least two locations in engineering study programs: Firstly, in mathematical service courses for engineers, usually students from different engineering study programs learn basic mathematical practices, often taught by mathematicians. Secondly, mathematical practices are also taught in specific engineering courses (e.g. Signal Theory), usually by lecturers of the engineering faculty. Research on university mathematics education shows, that both, mathematics in service courses *for* engineers, and mathematics *in* engineering courses (see also

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Pepin et al., 2021), are different and often not connected (Hochmuth, 2020; Winsløw et al., 2018, section 2.5). For example, Gueudet and Quere (2018) show differences between mathematics in service courses for engineers and mathematics in engineering courses in terms of connections within the subject matter. With a focus on trigonometry, they show that while in engineering courses multiple connections are made between contexts, representations and concepts, hardly any of those connections are found in the mathematics courses studied. Schmidt and Winsløw (2018) show that both types of courses are separated on an institutional level. In this regard, they note that "the selection of mathematical contents to be taught may be based on needs and priorities from the engineering disciplines, while the actual teaching [in mathematics service courses] is carried out according to generic standards and methods for teaching mathematics." (p. 165). In a study on the views of engineers and mathematicians on the concept of continuity, Alpers (2018) shows that this separation is also reflected in the different views of mathematicians teaching service courses and lecturers of the engineering faculty.

Most attempts to establish and support connections in mathematics service courses to mathematics in engineering courses[1] are based on the introduction of application examples from the engineering sciences in mathematical service courses (e.g. Härterich et al., 2012; Schmidt & Winsløw, 2018) or on more innovative course structures for mathematical service courses, such as project work (e.g. Alpers, 2002) and study and research paths (e.g. Barquero et al., 2008).

Both approaches can be problematic: The teaching and learning of mathematics, like any other subject, is situated in societal and institutional conditions that constitute the possibilities and restrictions of action. From the standpoint of the Anthropological Theory of the Didactic (ATD), Barquero et al. (2013) study institutional constraints and limitations within the educational system that hinder the large-scale dissemination of modelling activities.[2] In addition to a survey of literature, which shows that difficulties and barriers in this respect are a general problem, they systematically identify problems at different levels in a detailed study of one of their own projects. They categorise the constraints under the headings of monumentalism, individualism, and protectionism (Barquero et al., 2013, p. 322 ff). Furthermore, consolidated course structures (time tables, distribution of working hours and teaching, weekly assignments) are not always changeable and teachers are usually not empowered to change the traditional organisation of the mathematical content.

[1] Other approaches that attempt to establish connections to mathematics within engineering courses are not considered here.

[2] Within the ATD mathematical modelling is understood as a specific mathematical activity, more precisely as "processes of reconstruction and connection of praxeologies of increasing complexity . . . that should emerge from the questioning of the rationale of the praxeologies that are to be reconstructed and connected." (García et al., 2006, p. 243). This includes both intra-mathematical modelling as well as processes starting from extra-mathematical questions.

In addition to these large-scale approaches, the inclusion of isolated application examples from engineering in mathematics service courses represents a small-scale approach that enables changes in teaching under existing conditions. But this can be problematic in an epistemological sense:[3] Using isolated application examples could promote the image of engineering science as application of mathematics in specific extra-mathematical contexts (Barquero et al., 2011). This can promote the view of engineering per se as extra-mathematical and thereby establish a distinction between a mathematical and an engineering world[4] that somewhat contradicts the attempt to connect: Application examples can provide connections between mathematics and the engineering context. But those connections must explicitly take the specificity of mathematics *in* engineering into account. Otherwise they presuppose an understanding of the relationship of mathematics and engineering as per se disconnected. However, in everyday teaching, i.e. outside of larger teaching development projects, recourse to isolated application examples without considering the different conceptualisation of mathematics *in* engineering may appear to be the only option, especially in view of institutional constraints.

In this contribution I present a third approach that focusses on establishing and promoting connections within mathematical practices. The idea of modifying exercises from a mathematics service course according to reconstructed aspects from engineering mathematics practices is a small-scale approach that is based on the research perspective and results from an ongoing research project with Reinhard Hochmuth (Hochmuth et al., 2014; Hochmuth & Peters, 2020, 2021b; Hochmuth & Schreiber, 2015; Peters & Hochmuth, 2021). One aspect that distinguishes this approach from the approach of using application examples (the other small-scale approach) is that identified important aspects from engineering mathematical practices are brought into the mathematics service course without also introducing the engineering context.

In the following, I will build on our theoretical conceptions and the results of our research (Sect. 28.2); show by means of a detailed example (Sect. 28.3) how an existing exercise in a mathematical service course can be modified in such a way that the mathematical discourse related to service courses could be internally expanded with regard to the engineering mathematical discourse; and finally discuss (Sect. 28.4) the connections to aspects of mathematics *in* engineering, that could possibly be established and supported by this small-scale approach as well as further considerations.

[3] See our considerations in (Hochmuth & Peters, 2021a). I would like to note that such epistemological considerations are also relevant for large-scale projects. In addition, epistemological aspects are also part of institutional and societal conditions. No teaching development approach is free from possibly coming into conflict with existing conditions.

[4] Barquero et al. (2011) refer to this phenomenon as "applicationism". That the separation of the non-mathematical engineering context and the mathematical world is not adequate is also observed by Biehler et al. (2015) in the context of modelling cycles.

28.2 Theoretical Perspective and Previous Research

The idea of modifying existing exercises in mathematics service courses is based on the one hand on the general research perspective of ATD on mathematical practices and on the other hand on concrete study results on mathematical practices in Signal Theory by Hochmuth and Peters (2021b) and Peters and Hochmuth (2021). Therefore, I will first give a brief overview of ATD concepts relevant here. I will then summarise the research context and findings relevant to the exercise modification from our studies that follows in Sect. 28.3. This then not only provides the background for exercise modification but also shows how through the process of analysing materials, connections can be reconstructed that each in itself hold potential for change.

28.2.1 Concepts of the Anthropological Theory of the Didactic

ATD is a research programme to study human practices from an institutional perspective.[5] The concept of institution in ATD is based on the work by Douglas (1986). She elaborates the idea that all knowledge is dependent on (social) institutions and, conversely, that all institutions are based on shared knowledge (p. 45). Castela (2015) defines an institution *I* as "a stable social organisation that offers a framework in which some different groups of people carry out different groups of activities. These activities are subjected to a set of constraints, – rules, norms, rituals – which specifies the institutional expectations towards the individuals intending to act within the institution *I*." (p. 7). Any form of knowledge, and thus also actions in relation to this knowledge, is thus located in institutions and subject to institutional conditions.

Praxeology is the concept for the detailed subject-specific specification of institutional knowledge. In ATD praxeology is used to describe knowledge in terms of two inseparable, interconnected blocks: The praxis block consists of types of tasks (T) and relevant techniques (τ) used to solve them. The logos block consists of a two-level reasoning discourse. At the first level, technology (θ) describes, justifies and explains the techniques and at the second level, theory (Θ) organises, supports and explains the technique. A praxeology is usually represented in short as the 4 T-model $[T, \tau, \theta, \Theta]$. An important aspect of technology, i.e. part of the logos block, is the raison d'être of a body of knowledge. This is the reason why it exists in an institution, what it is good for, and why it is studied. When considering a particular topic in different institutions, different praxeologies emerge: different types of tasks

[5]Fundamental elaborations on ATD can be found, for example, in Bosch and Gascón (2014) and Chevallard (1992, 2019); in addition, Bosch et al. (2011) and Bosch et al. (2019) provide insight into typical studies in this research programme.

are relevant, different solution techniques are adequate, different raisons d'être exists and different reasoning discourses are acceptable and constitutive. This is referred to as the institutional dependence of knowledge.

While praxeologies allow mathematical knowledge to be grasped rather statically in its institutional conception, the concept of (didactic) transposition offers the possibility to investigate and describe dynamic aspects of the production, development, change and dissemination of knowledge between institutions (e.g. Bosch & Gascón, 2014). The basic model of the didactic transposition process is based on a distinction between three relevant institutions: First, scholarly knowledge is produced by experts in universities or research institutes. The knowledge to be taught is determined by official curricula. Finally, this becomes the taught knowledge that is taught in courses. The transition from scholarly knowledge to knowledge to be taught is also referred to as external didactic transposition, the transition to taught knowledge as internal didactic transposition. Schmidt and Winsløw (2018) refer to these concepts and show in particular that the specific institutional conditions of engineering thus enter into the external didactic transposition, but not into the internal one. They call this "the parallel model for didactic transposition in engineering education" (p. 165).

28.2.2 Mathematical Practices in Signal Theory

Schmidt and Winsløw (2018) focus on mathematical knowledge for engineering students that is provided through mathematics service courses. In our own studies, though, we point out that mathematical practices especially in higher-level engineering courses, such as Signal Theory, are rather a mixture of practices of mathematics from service courses, mathematics as developed and used in basic electrical engineering courses, and specific signal theory content (Hochmuth & Peters, 2020, 2021b; Peters & Hochmuth, 2021). The various combinations of dark- and light grey techniques and technologies in Fig. 28.3 are an example of such a mixture. Moreover, our analyses show that mathematical practices in these courses cannot be understood solely as the application of mathematical concepts taught in mathematics service courses.

To grasp this mixture of mathematical practices in Signal Theory, we introduced an extended praxeological 4 T-model and two corresponding mathematical discourses[6] (Peters & Hochmuth, 2021). The starting point for the idea of exercise modification presented in Sect. 28.3 are then analyses of an exercise with a lecturer's sample solution in the context of amplitude modulation and associated student solutions (Hochmuth & Peters, 2021b). The exercise and the lecturer's sample solution are presented in the Appendix.

[6]Our understanding of discourse, i.e. its meaning and analytical status, is clarified and linked to Weber's (1904) concept of ideal types in (Hochmuth & Peters, 2021b).

The amplitude modulation context will provide us with interesting insights into the role of complex numbers in electrical engineering, which will eventually be used in the exercise modification. To this end, I will first introduce the context and contrast the role of complex numbers in electrical engineering with the role of complex numbers in the mathematics service course.[7] Those different roles are grasp within our work as different mathematical discourses on complex numbers. The analysis of the roles of complex numbers in electrical engineering and in mathematics service courses is based on standard literature, lecture notes, and students' notes for consolidated standard courses which are held at the University of Kassel. Both, the mathematics service course and the introductory course on electrical engineering are courses that students attend before attending the course on signal theory. The described mathematics service course is also the setting for the exercise modification in Sect. 28.3. Secondly I will summarise the results of the analysis of the lecturer's sample solution and address some of the results of the analyses of the student solutions.

28.2.2.1 Amplitude Modulation and the Role of Complex Numbers in Electrical Engineering and in Mathematics Service Courses

Amplitude modulation (AM) is a central topic in signal theory. With amplitude modulation, several message signals (e.g. for different radio stations) with different carrier frequencies can be transmitted (e.g. via antenna) and received without crosstalk between signals at the receiver (e.g. radio set) depending on the chosen carrier frequency. The principle of amplitude modulation is illustrated in Fig. 28.1: The amplitude of a high-frequency carrier signal $\cos(2\pi f_0 t)$ (left) is varied in relation to that of the low-frequency message signal $s(t) = \cos(\Omega t)$ (centre). The AM signal can then be represented as $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$ (right).

In Fig. 28.1 amplitude modulation is visualised using waveforms of the corresponding signals. In the exercise we have analysed, an AM signal is to be represented as a rotating phasor in the complex plane. This change to phasor-representation (see Fig. 28.6 in the appendix) makes it possible to study properties of AM that are not apparent in the waveform representation. The connection between waveform, phasor-representation and algebraic description with complex numbers of a periodic signal is also a basic topic of introductory courses on electrical engineering. Albach (2011), a standard textbook for introductory courses on electrical engineering, first introduces phasors with the purpose to graphically describe time-dependent sinusoidal functions. The relationship between phasor- and waveform-representation is shown in Fig. 28.2, left side: The phasor with length $\widehat{\mathrm{i}}$,

[7] The subject-specific context is also relevant in other publications within the research project on mathematical practices in Signal Theory. These or similar presentations of amplitude modulation and connections to complex numbers can therefore also be found in (Hochmuth & Peters, 2021a, b; Peters & Hochmuth, 2021).

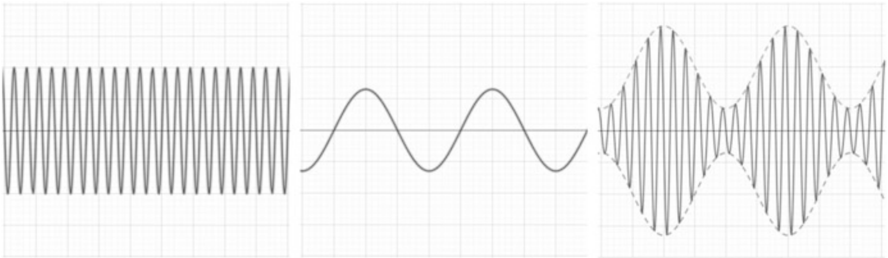

Fig. 28.1 Visualisation of amplitude modulation: high-frequency carrier signal (left), low-frequency message signal (centre), and AM signal (right), created with GeoGebra

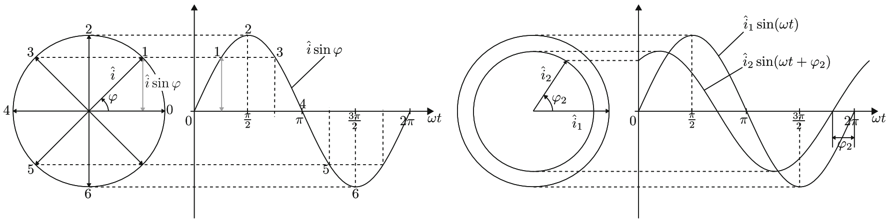

Fig. 28.2 Relationship between phasor and time-dependent functions. (Redrawn similar to Albach, 2011, p. 32)

rotates constantly counterclockwise with angular velocity ω. The projection onto the vertical axis provides the waveform of $\widehat{i}\sin\varphi$. At the right side of Fig. 28.2 two sinusoidal currents with different amplitudes $\widehat{i}_1$ and $\widehat{i}_2$ and phase difference φ_2 are shown.

As both phasors (Fig. 28.2, right side) rotate with the same angular velocity ω, the rotation of the phasors can be neglected. When analysing electrical components, the amplitude ratio of the input signal to the output signal and the phase shift between input and output caused by the components are of primary interest. Therefore, phasors are important graphic tools for interpretation and analysis of electrical engineering processes.[8] Current- and voltage ratios in electrical networks can be displayed and analysed graphically in static phasor diagrams (or Argand diagrams). For the purpose of an algebraic description of phasors, the plane in which phasors are drawn, is considered as the complex plane. The phasor can now be understood as a complex quantity that symbolically represents the time-dependent periodic signal. The compatibility of the geometric rules for manipulating phasors and the calculation rules of complex numbers is justified via physical relations (e.g. Kirchhoff's laws).

[8] In this contribution I focus on the role of complex numbers in the electrical engineering conceptualisation of phasor. The introduction of complex numbers in electrical engineering had a much bigger significance (e.g. Bissell, 2004; Bissell & Dillon, 2000, 2012).

Furthermore, for a sinusoidal quantity the following holds: $A\cos(\omega t+\varphi)=\Re(Ae^{j(\omega t+\varphi)})=\Re(Ae^{j\omega t}e^{j\varphi})$, where A is the amplitude and j denotes the complex unit in electrical engineering. The algebraic representation of the rotating phasor is $Ae^{j\omega t}e^{j\varphi}$. In circuits where all quantities change with the same angular velocity, the time dependent factor ejωt can be factored out, i.e. the rotation of the phasor can be neglected. Here the Euler representation is important for separating rotational and constant components.

In the case of amplitude modulation, rotational aspects can no longer be neglected because the carrier signal and the message signal have different angular velocities. The algebraic representation of the phasor for amplitude modulation is given in line (3) of the sample solution in the appendix. Here, also the Euler representation is important for separating the different frequencies of carrier- and message signal.

In the mathematics service course, complex numbers are considered in the first part of the course in the context of Linear Algebra (Strampp, 2012). Their introduction is motivated by the solvability of the equation $x^2 + 1 = 0$. For this purpose, real numbers are extended by a number i with the property $i^2 = -1$. This approach is typical for the whole chapter: the rational is aimed at an elaboration of the solvability of equations. Calculation rules for complex numbers are derived without introducing and proving formal concepts, but by stating that all rules which are relevant for calculating with real numbers should continue to be applicable (p. 59). Also, it is pointed out that various terms are an extension of already known concepts from real numbers. For example, the complex exponential function $\mathrm{e}^{\phi \mathrm{i}}$, which is introduced to serve as a pointwise convenient abbreviation for $\cos(\phi) + \sin(\phi)i$ (p. 74). Although the chapter is clearly designed to develop a practical approach to the concepts and rules of calculation, it is subject to an orientation towards the inner-mathematical, generalisation-oriented formal rational of academic mathematics. In addition to the algebraic view on complex numbers, the chapter also contains a geometric view: An analogy to vectors is established, but the vector concept is also distinguished from complex numbers: "We speak of phasors[9] [Zeiger] and not of vectors, since complex numbers, unlike vectors, can also be multiplied. This multiplication extends the multiplication of real numbers." (p. 60, translated by author). Phasors provide an illustrative justification for the formal conceptualisations of complex numbers. As Felix Klein (1967) notes, this is a view of complex numbers that was already held by Gauss. Klein states that Gauss "justifies the legitimacy of operating with complex numbers by the fact that one can give them and the operations with them that illustrative geometric interpretation…" (p. 64, translated by author). This meaning

[9]We translated the German term Zeiger with the term phasor, which already refers to electrical engineering concepts. But electrical engineering aspects play no role in the course and Strampp (2012) does not refer to them either. In German the term Zeiger is used both in electrical engineering and in mathematics service courses, but with different meanings (reference to electrical engineering concepts vs. geometrical object with no further references). By using the term Zeiger Strampp (2012) can thus establish a connection to the electrical engineering courses without dropping the inner mathematical conception of complex numbers. This aspect of using the same term, that has different meanings in different course-contexts is in jeopardy of being lost through translation.

of complex numbers in the mathematics service course differs from the meaning of complex numbers in electrical engineering (see also the work of Steinmetz (1893) who first introduced complex numbers to electrical engineering). Furthermore, the compatibility of the rules for graphically manipulating phasors and the calculation rules of complex numbers are justified via physical laws.

28.2.2.2 ATD Analyses of the Lecturer's Sample Solution and Student Solutions

From the institutional standpoint of the ATD, courses of a study program can be understood as institutions. In the following we will differentiate two institutions: an institution HM associated with the mathematics service course and an institution ET associated with electrical engineering.[10] According to the institutional dependence of knowledge, the different institutions give rise to different conceptualisations (praxeologies) of complex numbers. The two different characterisations of complex numbers in the previous section can be understood as descriptions of institutional aspects that shape the logos blocks of the respective institutional praxeologies and thus, due to the dialectic of praxis and logos, also the practical part, i.e. they can each be understood as part of two associated institutional mathematical discourses: one associated with the institution HM, i.e. the HM-discourse, and one associated with the institution ET, i.e. the ET-discourse. An important difference between the two institutional discourses is the difference between the respective raisons d'être: In electrical engineering the raison d'être for complex numbers is to describe periodic signals, together with strong connections to phasors and waveforms. In the mathematics service course the raison d'être for complex numbers is to serve for generalising concepts from real numbers, to solve equations, and as formal objects of calculation. There is also a connection to phasors but the phasor concept is different and usually serves to visualise properties of complex numbers.[11]

In our research on mathematical practices within a signal theory course, we used the notion of institutional discourses to capture the mixture of mathematical practices that occurred in a praxeological analysis of the lecturer sample solution of an exercise in the context of amplitude modulation. We identified the two mathematical discourses and associated praxeological elements to the HM-discourse (τ_{HM} and θ_{HM}) or the ET-discourse (τ_{ET} and θ_{ET}) depending on the respective institutional orientation within the solution steps.

[10]The acronyms HM and ET were introduced by Peters and Hochmuth (2021) to denote the two relevant contexts of "Höhere Mathematik" (HM, mathematics service course) and "Elektrotechnik" (ET, electrical engineering) and the associated discourses. HM and ET are the standard German acronyms for these contexts.

[11]Nevertheless, this visualisation aspect is important because it contributes to the logos block, i.e. the reasoning discourse, e.g. with regard to abstract calculation rules.

In addition, our approach refers to the work of Artaud (2020) that allows to connect the two mathematical discourses with two different didactic transposition processes: Artaud has considered two different ways how mathematical knowledge arise in fields such as electrical engineering: (1) Either the mathematical knowledge required in electrical engineering institutions is already elaborated and developed in other institutions, for example academic mathematical research institutes. This knowledge then enters the electrical engineering institution via didactic transposition processes, so to speak externally, and serves the mathematical education of future electrical engineers. Here, one can localise the HM-discourse and the idea of mathematics *for* engineering. Through the didactic transposition process, however, the academic mathematical knowledge is changed and adapted especially to the needs of electrical engineering institutions for the education of future engineers, but maintains the orientation towards academic mathematics. Schmidt and Winsløw (2018) also note this and it is to this aspect that their parallel model for didactic transposition refers. (2) Or, the relevant mathematical knowledge has been developed in the course of a historical process by actors specialising in electrical engineering. In this case, the mathematical knowledge entered the electrical engineering institution a long time ago via an institutional transposition process to be put to use. Bissell's (2004) investigation of the introduction of complex quantities in electrical engineering, driven by Steinmetz (1893) among others, that allows to manipulate graphical and pictorial representations instead of complicated mathematical expressions and also led to systems thinking and black box analysis (p. 309), give a glimpse on such an institutional transposition process. In the course of time this knowledge was used in electrical engineering and was didactically transformed in order to be taught. This didactic transposition process is endogenous. Here the ET-discourse and the idea of mathematics *in* engineering can be situated.

A graphical representation of our analysis result of the lecturer's sample solution is shown in Fig. 28.3, see also the detailed analysis in (Hochmuth & Peters, 2021b).

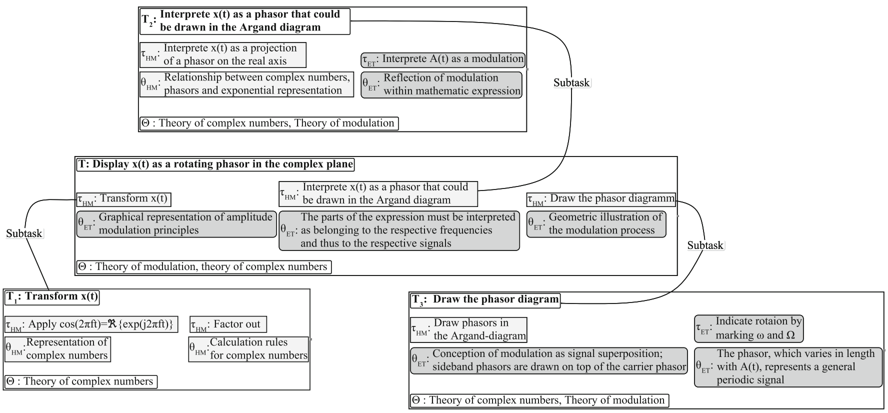

Fig. 28.3 Graphical representation of the ATD Analysis (Hochmuth & Peters, 2021b)

The exercise is solved in three steps (see Appendix): Transforming mathematical expressions, interpreting the mathematical expression to draw a diagram, and drawing the phasor diagram. The main part of the exercise, to display $x(t)$ as a rotating phasor in the complex plane, is a task (T) in the sense of the ATD. We then assigned techniques and technologies (τ_{HM}, θ_{HM} in light grey, τ_{ET}, θ_{ET} in dark grey) to each of the three solution steps. This is shown in the bold framed rectangle in Fig. 28.3: Without focussing on the detailed analysis one can see, that for each solution step HM-techniques are accompanied with ET-technologies. We characterised this as "an embedding of HM-techniques in the ET-discourse" (Hochmuth & Peters, 2021b). We further refined the analysis in a second analysis step, in which the three techniques assigned to the three solution steps are considered as subtasks T_1 to T_3, see the corresponding light framed rectangles in Fig. 28.3: In this step we were able to further enlight the nature of the respective embeddings.

Although we will not go into the details of the analysis here, it is clear from Fig. 28.3, that the two mathematical discourses, the ET-discourse (dark grey) and the HM-discourse (light grey), interact in various ways. The view that mathematics is simply applied in electrical engineering is not adequate, practices in Signal Theory contain aspects of both mathematical discourses. Dealing adequately with both discourses is therefore a requirement for engineering students.

In Hochmuth and Peters (2021b) we used this result as a reference for analyses of students solutions. In the following I will show clips from two student solutions[12] in which the adequate switching between the discourses was not present and the correct phasor diagram was not produced (Fig. 28.4). Two decisive steps in the course of the sample solution are firstly the calculation step from line (2), in which $x(t)$ could be interpreted as a real part of three rotating phasors drawn in the origin, to line (3), in which $x(t)$ can be interpreted as a rotating carrier phasor with time-dependent amplitude $A(t)$. And secondly the change of representation from the algebraic expression in line (3) to the phasor diagram.

The student solution on the left side of Fig. 28.4 does not contain the step from line (2) to line (3) from the sample solution. Instead this student reproduces the HM-discourse by drawing a diagram similar to diagrams from the mathematics service course where the Argand diagram and the unit circle are used to illustrate properties of complex numbers. This student solution also contains further information on properties of complex numbers like the complex conjugate $e^{-j2\pi ft}$ that are not relevant for the solution of the exercise. The three terms from line (2) are drawn as three separate phasors. Important aspects of amplitude modulation and references to the ET-discourse are not present.

The student solution on the right side of Fig. 28.4 mainly contains ET-discourse aspects but significantly deviates from the sample solution. References are made to previous topics in the lecture (Fourier transform and low pass filter), but these are not

[12] In order to protect the privacy of the students, the student solutions are translated from German and rewritten by the author without correction marks. In the detailed analyses in (Hochmuth & Peters, 2021b) those two student solutions are labelled I2 and I3.

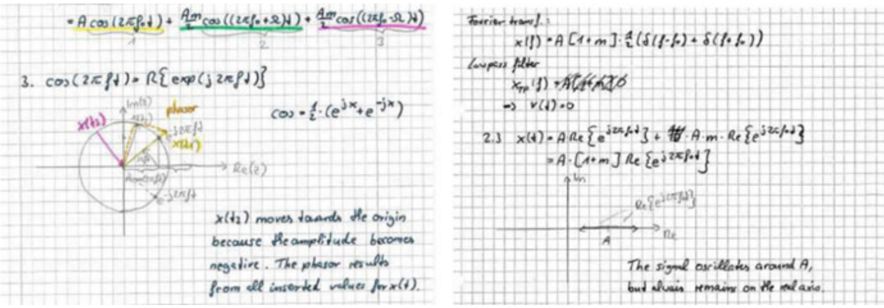

Fig. 28.4 Left: student solution solely within the HM-discourse. Right: student solution mainly within the ET-discourse (Hochmuth & Peters, 2021b)

goal-oriented and appropriate. Although this solution maintains an orientation towards the ET-discourse and some rotational aspects are present, the connection between the mathematical concepts and their graphic representation in terms of modulation principles is missing.

28.3 From Analyses of Engineering Mathematical Practices to Modifying Exercises in Mathematics Service Courses

The summary of the analyses in the preceding section showed that adequately working with two different mathematical discourses is a requirement for engineering students. Furthermore, some difficulties of students, that were not able to flexibly switch between the two discourses are shown. Subject-specific aspects, that also seem to be at the core of those difficulties are: dynamic aspects cannot be neglected; more than one rotating phasor is relevant and phasors have to be drawn in a specific way; a complex algebraic expression must be represented graphically; time-dependent exponential function.

Those analyses results now shall serve to inform an exercise modification in the mathematics service course. The mathematics service course under consideration is a two-semester consolidated course that is regularly held at the university of Kassel. Material from this course consists of student lecture notes, standard literature (Strampp, 2012), and exercise sheets with sample solutions from 2013. It consists of two lectures, one exercise session and one special exercise session where selected, important topics are presented, per week. Students are expected to individually work on weekly exercise sheets, that are eventually handed in and graded. Application examples are not present in the material. With focus on the chapter of complex numbers we characterised the mathematical discourse, i.e. the HM-discourse, as orientated towards the inner-mathematical, generalisation-oriented formal rational of academic mathematics. The raison d'être for complex numbers is that they allow for generalisation, they are useful to solve equations, and they are formal objects of calculation. We also noted that in the chapter of complex numbers no connections to phasor representations other than for illustrative reasons are made. Furthermore, the Euler equation, with which the internal relationship of the exponential function, sine and cosine could be recognised, only serves a useful shortcut to simplify calculations. Important connections between complex numbers and trigonometric functions, that go beyond this convenient calculation tool, are not present.

The basic idea behind the proposal for exercise modification is now to modify an existing exercise from the mathematics service course such that the HM-discourse on complex numbers could be enriched or expanded towards the ET-discourse in order to establish connections to mathematical practices that are relevant in electrical engineering. However, neither the course organisation nor the general orientation towards academic mathematics is to be changed.

To demonstrate this idea with an example, it is first noted that the exercises in the chapter of complex numbers are mainly standard exercises: change between Euler- and Cartesian representation, training of basic calculations, and determining the roots of polynomials. The following exercise is an exception.[13] It is the only exercise in which a time-dependent exponential function occurs:

Which curves are described in the complex plane by

$$ae^{-ti} + be^{ti},\ a, b \in \mathbb{R} \text{ constant},\ t \in \mathbb{R}?$$

This exercise was marked "too difficult" in the student's notes.

The sample solution from the student's notes is:

$$\begin{aligned}
ae^{-ti} + be^{ti} &= a(\cos(-t) + \sin(-t)i) + b(\cos(t) + \sin(t)i)\\
&= a(\cos(t) - \sin(t)i) + b(\cos(t) + \sin(t)i)\\
&= (a+b)\cos(t) + (-a+b)\sin(t)i\\
&= x + yi\\
&= \left(\frac{x}{a+b}\right)^2 + \left(\frac{y}{-a+b}\right)^2 = 1
\end{aligned}$$

This then is recognised as the equation of the ellipse, that was introduced in the preceding special exercise session. The cases $a + b = 0$ and $-\mathrm{a} + b = 0$ are treated separately, in which the ellipse becomes a straight line: for $a + b = 0$ we get $2b \sin(t)i$, a straight line on the imaginary axis between $-2bi$ and $2bi$. For $-a + b = 0$ we get $2a \cos(t)$, a straight line on the real axis between $-2a$ and $2a$.

The question of the exercise already points to the raison d'être of the ET-discourse (complex numbers are useful to describe periodic signals). But in the sample solution, only elementary relations such as Euler's equation and Pythagorean identity are used to give the expression a form that could be recognised from

[13] Exercise and sample solution are translated from German by the author.

previous lectures. Why $ae^{-ti} + be^{ti}$ describes an ellipse, or why the special cases generate straight lines is not explained, periodic or rotational aspects are not present.

This changes when software like GeoGebra (Hohenwarter et al., 2018) is used. With digital tools like GeoGebra, dynamic aspects can be visualised and explored and an otherwise too difficult exercise becomes accessible. Since students may be inexperienced in using GeoGebra, I will present the modified exercise with a step-by-step construction of the ellipse with phasors below:[14]

Which curves are described in the complex plane by

$$C(t) = ae^{-ti} + be^{ti}, a, b \in \mathbb{R} \text{ constant}, t \in \mathbb{R}?$$

- Plot the corresponding locus curve in GeoGebra by following the steps below:
  - Write C = a e^(−i t) + b e^(i t) in the Input and confirm.
  - Create sliders for a, b, and t.
  - Write Locus in the Input and chose
    Locus (<Point Creating Locus Line>, <Slider>).
  - Replace <Point Creating Locus Line> with C and <Slider> with t.

  Try different values for a and b and describe the curves. Use the slider for t to explore how the point C moves on the curve. What happens for $a = 0$, $b = 0$, $a = b$ and $a = -b$?
- In the next step we construct phasors for ae^{-ti} and be^{ti}:
  - Write P= a e^(−i t) in the Input and confirm.
  - Write u = Vector in the Input, chose Vector (<Point>) and replace <Point> with P.
  - Write v = Vector in the Input, chose Vector (<Start Point>, <End Point>). and replace <Start Point> with P and <End Point> with C.

  We have now represented the point C as the sum of the two phasors. Again, try different values for a and b and explore how the rotating phasors construct the curve. What is the consequence of the different signs in the exponents?

By using software like GeoGebra, dynamic aspects can be visualised (see also Fig. 28.5). In addition, connections can be made to the phasor representation, which now goes beyond only serving to visualise properties. In this exercise, it can be explored how the combination of two rotating phasors describes a closed curve and how the algebraic representation of a complex number is connected to its phasor-representation. The cases $a = 0$ or $b = 0$ result in circles with one phasor

[14] For more experienced students, this exercise could be formulated in less detail:

(a) Create the locus curve in GeoGebra including sliders for a, b and t. How does the locus curve change depending on values for a and b?
(b) Represent the components of the equation ae^{-ti} and be^{ti} as phasors respectively (use the GeoGebra Vector function) and represent $C(t)$ as the result of adding the two phasors so that point C moves on the locus as you vary t. What is the consequence of the different signs in the exponents?

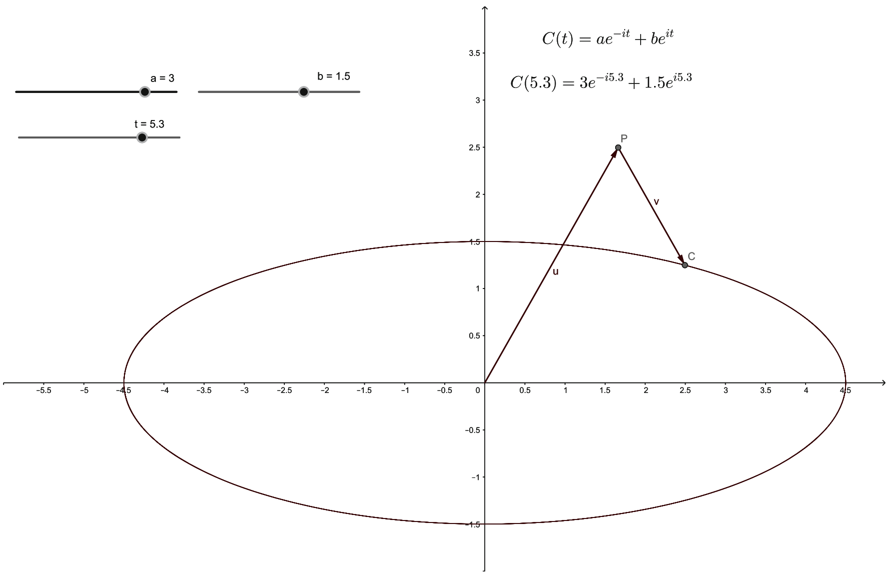

Fig. 28.5 Ellipse with corresponding phasors, created with GeoGebra

each. This is familiar from the lecture, as properties of complex numbers, in particular the introduction of Euler's formula, are illustrated with the unit circle. So, connections to previous aspects of the lecture are established. Furthermore, the raison d'être for complex numbers in the HM-discourse can be extended by the aspect that complex numbers are suitable to describe periodic functions or closed curves.

This modified exercise also fits in the course structure, e.g. this exercise can be part of a weekly exercise sheet. The exercise does not violate the orientation towards the rational of academic mathematics of the HM-discourse. It is also possible to embed the task in a mathematical-historical context: The method of constructing the ellipse by two rotating phasors is very similar to historical conic section drawers (e.g. Van Maanen, 1992).

28.4 Discussion

The aim of this contribution is to show, and illustrate with an example, how subject-specific analyses of mathematical practices from signal theory can serve to modify exercises from mathematics service courses, even within restricted institutional conditions. The focus is to support connections between mathematical practices from service courses and from electrical engineering within the mathematics. The approach presented here thus represents, in addition to large-scale development projects and the inclusion of application examples, a further possibility for changes in teaching.

The ATD concept of the institutional dependence of knowledge is at the core of this approach: The same mathematic topic, e.g. complex numbers, is conceptualised differently in different institutions, the subject-specific rationales and meanings, the raisons d'être, overall, the mathematical discourses are different. This is associated with a specific research stance: Within this approach, mathematical practices of engineers are acknowledged as institutional mathematical practices in their own right. This stance is not compatible with the introduction of application examples in the sense of applicationism, i.e. without taking the engineering specific conceptualisations of mathematical knowledge into account. From this stance, it is possible to reconstruct engineering-specific mathematical discourse aspects like, besides other, the engineering-specific raison d'être of a mathematical concept. These aspects are mathematical aspects, and not aspects from an extra-mathematical engineering context, that have been endogenously developed and modified over time within engineering institutions (cf. Artaud, 2020). Therefore, they can be included in the mathematical discourse of mathematics service courses, that are often oriented towards academic mathematics. At this point changes are necessary: The ET-context should be removed but the discourse aspect kept. The analysis of the AM exercise shows that the orientation towards academic mathematics is important in engineering courses such as Signal Theory. This is also one of the reasons for maintaining this orientation for the HM-discourse in this approach for exercise modification. In the example, the raison d'être for complex numbers in the ET-discourse was, among other things, describing periodic signals. In the HM-discourse, this can be changed to: describing periodic functions or curves. If the students encounter complex numbers in engineering the mathematical discourse on complex numbers (i.e. the raison d'être) in the ET-discourse, to describe periodic signals, is not entirely different from the extended mathematical HM-discourse. Therefore this approach can contribute to reduce the metaphorical distance between the mathematical discourse of the mathematics service course and the engineering mathematical discourse inner-mathematically. This concerns both the internal didactic transposition (cf. Schmidt & Winsløw, 2018) and the establishment of connections within the subject matter (cf. Gueudet & Quere, 2018), e.g. connections of HM-techniques with ET-technologies. Many of the differences in the views of mathematicians and engineers addressed by Alpers (2018) can also be understood as aspects of a respective institutional mathematical discourse. From the perspective presented here, however, it is not enough for a mathematician to read engineering books and talk to engineers, for example. To really take the engineering view seriously, it is necessary to take it seriously in its own institutional conception. Under this precondition, however, discussions with engineers, textbooks by engineers, but also historical and philosophical studies are useful in order to characterise mathematical discourses specific to engineering.

Of course, this small-scale approach presented here is not free from problems and of coming into conflict with societal and institutional conditions either. I have shown how, in the process of analysing institutional mathematical practices, potential for change in teaching can be identified within existing conditions. This is double-edged, as it can also support the position of not needing to change social and institutional conditions, and thus act as a counter-argument for approaches that aim precisely to such changes. On the other hand, while acknowledging this criticism, it can be stated that the ATD-specific research stance is also relevant for lecturers and entails that retreating to the position that changes in teaching are entirely possible without conflict with and change in social and institutional conditions is short-sighted. The approach presented here presupposes lecturers to question their own institutional standpoint, their own mathematical discourse. But this stance does not solve contradictions and possible conflicts. There is no clear solution for societal and institutional conflicts. Societal struggles cannot be solved on the basis of ATD[15] analyses. However, such an analysis, as presented here, offers a differentiated view of what is possible at the exercise level and what is not.

ATD focuses specifically on institutional and subject-specific conditions. In order to take social struggles and contradictions into account, a research perspective that addresses a more general level is necessary. A promising approach in this direction is the subject-scientific approach from the field of critical psychology (e.g. Holzkamp, 1985; Schraube & Osterkamp, 2013). Various studies have already shown that this approach is compatible with ATD (Hochmuth, 2018; Hochmuth & Schreiber, 2015; Ruge et al., 2019).

Consideration of the relationship between lecturers of mathematics service courses and teaching approaches developed in research brings the focus to the sustainability of teaching development research and therefore also to professional development. Ruge and Peters (2021) develop an understanding of professional growth based on the subject-scientific approach which, besides other things, adopts a view of professional development that goes beyond deriving practical and applicable tools from research. In this sense, the approach to exercise modification presented here also does not provide a directly applicable tool, but shows how there is potential for teaching development within the process of analysing respective institutional mathematical discourses and reflecting the institutional situatedness of mathematical practices.

Appendix: Exercise with Lecturer Sample Solution

The exercise under consideration is structured in three items:

1. A message signal $s(t) = \cos(\Omega t)$ has to be amplitude modulated. The result is $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$
2. The result of item 1. Has to be written as the sum of three harmonics. The result is $x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$
3. The result of item 2. Has then to be displayed graphically in the complex plane as a rotating phasor with varying amplitude.

[15] Or any other theoretical approach.

The ATD analysis focusses item 3. of the exercise. The exact problem definition of item 3 is (my translation):

> Graphically display $x(t)$ in the complex plane as a rotating phasor with varying amplitude using the relationship $\cos(2\pi ft)\Re\{\exp(j2\pi ft)\}$ and the result under item 2.

Lecturer Sample Solution
One first writes

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \tag{1}$$

$$= A\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\{\exp(j(2\pi f_0 t + \Omega t))\} + \frac{Am}{2}\{\exp(j(2\pi f_0 t - \Omega t))\} \tag{2}$$

$$= \{\exp(j2\pi f_0 t)\underbrace{[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)]}_{A(t)}\} \tag{3}$$

and interprets the expression in the square bracket as a real-valued time-dependent amplitude $A(t)$, which modulates the carrier phasor $\exp(j2\pi f_0 t)$ rotating at frequency f_0 in Fig. 28.6.

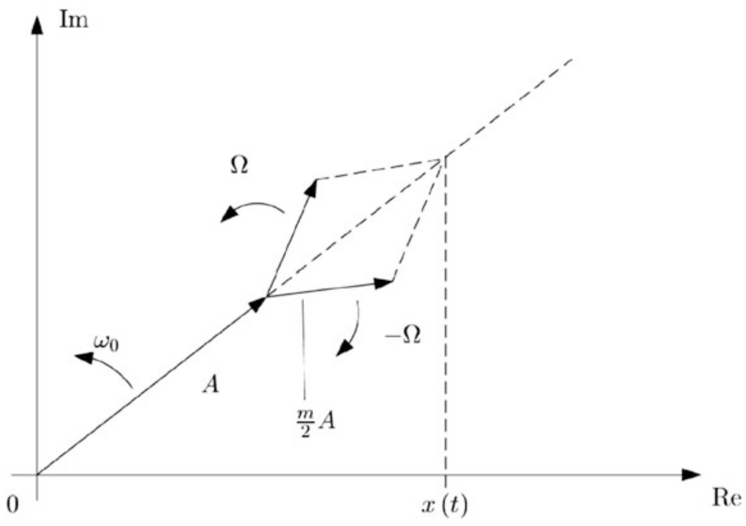

Fig. 28.6 Representation of $x(t) = A[1 + m\cos(\Omega t)]\cos(2\pi f_0 t)$ as the real part of a rotating phasor $A(t)\exp(j2\pi f_0 t)$ with $\omega_0 = 2\pi f_0$

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Chapter 29
Learning Mathematics in a Context of Electrical Engineering

Frode Rønning

Abstract This paper reports from an early phase of a project where first-year students on a programme in electronic systems learn mathematics in close contact with their engineering specialisation. Using concepts from the Anthropological Theory of the Didactic (ATD), the connection between mathematics and electrical engineering will be analysed based on concrete examples. On the basis of interviews with teachers in both fields, challenges and opportunities with teaching mathematics in an engineering context are described. The analysis reveals a complex interplay between mathematics and engineering, and the teachers emphasise division of labour as a crucial issue.

Keywords Mathematics for engineers · Electrical engineering · ATD · Praxeology

29.1 Introduction

Mathematics has always been regarded as an important subject for engineering students, and many different approaches to the teaching of mathematics for engineers can be identified. The traditional approach is to teach mathematics as part of a package of general courses, often over the first two years, assuming that this will provide the students with the necessary background to make use of the mathematics in engineering courses later (Winkelman, 2009). A critique towards this approach is that it may lead to mathematics being taught with a focus only on mathematical concepts and understanding and not on applications (Loch & Lamborn, 2016). Another critique, of a more general nature, can be connected to the challenges of transferring knowledge from one context to another (e.g. Evans, 2000). Acknowledging that knowledge is context dependent, one might argue that mathematics for engineering should be learnt within the engineering context where it is going to be

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used. And indeed, at many universities mathematics is taught in courses specially designed for particular engineering programmes (Alpers, 2008; Enelund et al., 2011; Klingbeil & Bourne, 2014). This model gives good opportunities for including programme specific problems in the mathematics teaching, and it is assumed that this will increase the perceived relevance of mathematics. However, this solution also raises some issues. Providing specialised mathematics courses for each study programme will be expensive if the university offers a large number of study programmes, and it may cause complications for students who wish to switch from one study programme to another. Another argument used in favour of general mathematics courses is that one of the strengths of mathematics is exactly the fact that it *is* general and that one of the competencies that students should acquire by studying mathematics is to adapt to new and unknown situations. There are, however, strong arguments for creating better connections between mathematics and the engineering subjects since many students find it challenging to apply mathematics they are supposed to have learned when they need it later in the engineering courses (Carvalho & Oliveira, 2018; Harris et al., 2015). There seems to be no obvious solution to these issues and therefore it is of interest to try out different models and study these models in practice.

In this paper I will report from an early phase of a project at the Norwegian University of Science and Technology, NTNU, where the aim is to redesign mathematics courses for engineering programmes. The project is given the acronym MARTA, and its full title would translate to English as *Mathematics as a Thinking Tool*. MARTA is so far restricted to one study programme, Electronic Systems Design and Innovation (ELSYS), but will later also include other engineering programmes. MARTA is part of a process aiming at redesigning all the technology programmes at NTNU, a process referred to as Technology Studies for the Future (Fremtidens teknologistudier, 2022). This paper is based on experiences from the first semester of the project MARTA, where a basic course in mathematics is taught in close connection with the course Electronic System Design and Analysis (ESDA) to first-year students. Using the Anthropological Theory of the Didactics (ATD), (e.g., Bosch & Gascón, 2014; Chevallard, 2006), I study the discourses that develop to see how the praxeologies in mathematics and engineering influence and interact with each other. I will inquire into the challenges and opportunities that arise at the interface between mathematics and electronics, as seen from the viewpoint of the teachers in the two subjects.

29.2 Background and Context of the Study

Engineering education has from early on experienced a tension between theory and practice, between academic and professional aims. Edström (2018) describes engineering education in the United States before 1920 as highly practical. After that time a change took place, influenced by European-educated engineers with a more mathematically oriented background. Edström writes that the development was slow, with some exceptions in "newer fields, such as chemical and electrical engineering, which grew from science disciplines" (2018, p. 40). The development got a boost after the Second World War. This is reflected in a report from a committee appointed to review the state of the education at Massachusetts Institute of Technology. In this report there are several warnings against a development of engineering education towards becoming too far separated from practice and also a critique against routine learning:

> [M]any students seem to be able to graduate from the Institute on the basis of routine learning, and . . . though fully equipped with knowledge of standard procedures . . ., they lack the critical judgement, the creative imagination, the competence in handling unique situations (Lewis, 1949, pp. 28–29).

Further, it is emphasised in the report that it is important to "explore vigorously every means for confronting the student with basic data in genuine problem situations", and a belief is expressed that it is possible to find problems that "are simple enough to be used in the early years and complex enough to be challenging" and that "abstract concepts are best taught through their applications" (Lewis, 1949, p. 29). Edström (2018) remarks that many of the issues in the Lewis report are still valid today. The more specific question of what kind of mathematics should be taught to engineers also has a long history (Alpers, 2020, p. 5). First, this question addressed only the actual content of mathematics for engineers but later also issues about the connection between mathematics and engineering and who should be teaching mathematics to engineers were included (Ahmad et al., 2001; Bajpaj, 1985; Cardella, 2008).

Several recent studies show that the tension between usefulness and scholarliness, and the challenges with applying theory to practical engineering problems, still persist (Carvalho & Oliveira, 2018; Harris et al., 2015; Loch & Lamborn, 2016). The Conceive, Design, Implement, Operate (CDIO) Initiative, launched in 2000, addresses this issue. It is described as "an innovative educational framework for producing the next generation of engineers" (www.cdio.org). Further details are given below.

The CDIO approach has three overall goals: To educate students who are able to

1. Master a deeper working knowledge of technical fundamentals
2. Lead in the creation and operation of new products, processes, and systems
3. Understand the importance and strategic impact of research and technical development on society (Crawley et al., 2014, p. 13)

Crawley et al. (2014) emphasise that it is not memorisation of facts and definitions, nor the simple application of a principle that is important, but *conceptual understanding*, seen as ideas that have lasting value. In addition, the CDIO approach values *contextual learning*. This means, among other things, that new concepts should be presented in situations familiar to students and in situations they recognise as important to their current and future lives (Crawley et al., 2014, pp. 32–33). The CDIO approach involves combining ideas of learning in context and maintaining deep, or conceptual, understanding (Marton & Säljö, 1976). These ideas are in line with those presented by Scanlan in 1985 in a talk about mathematics in engineering education. In his talk Scanlan concluded by stating that mathematics should be an essential part of the students' formation and "not a set of 'tools' to be acquired before proceeding to the 'important' part of the course" (Scanlan, 1985, p. 449).

The project MARTA that I am reporting from, has as its main aim to create a closer connection between mathematics and engineering programmes, while maintaining conceptual understanding in both fields. An overarching goal for the project is to develop mathematics as a 'tool for thinking'. The programme Electronic Systems Design and Innovation (ELSYS) has been chosen as a pilot for MARTA. Other programmes will follow. ELSYS is one of 17 five-year Master of Technology programmes at NTNU, admitting approximately 1700 new students in total each year, approximately 100 in ELSYS. All these programmes traditionally contain four mathematics courses distributed over the first three semesters, with almost identical content for all programmes. MARTA represents a break with the traditional model. In MARTA, the idea is to make adaptations by shifting the emphasis on various topics as well as changing the sequencing of the topics, in order to better suit the needs of the engineering programmes. It is expected that this approach will make the students better see the relevance of mathematics for their engineering specialisation. The approach is in line with the idea of contextual learning from CDIO.

This paper is based on experiences from the first semester of the five-year programme, which is also the first semester of the project. Based on these experiences, my aim is to get a better understanding of the interplay between mathematics and topics from electrical engineering, which may be of value when developing the project further.

29.3 Theory and Methodology

Concepts from ATD will be used in the analysis. A central notion in ATD is the notion of *praxeology*, "the basic unit into which one can analyse human action at large" (Chevallard, 2006, p. 23). A praxeology is composed of two blocks, the praxis block, P, and the logos block L. P is seen as consisting of two parts, *types of tasks* (T) and a set of *techniques* (τ) to carry out the tasks. L also consists of two parts, a *technology* (θ), or justification for the techniques used to carry out the tasks, and the *theory* (Θ), which provides the basis and support for the technological discourse (Bosch & Gascón, 2014, p. 68). I will write $P = [T, \tau]$, $L = [\theta, \Theta]$, and $\boldsymbol{P} = [P/L] = [T, \tau, \theta, \Theta]$ for the whole praxeology. This is often referred to as the 4 T-model

A social situation is called a didactic situation

> whenever one of its actors (Y) does something to help a person (x) or a group of persons (X) learn something (indicated by a heart ♥). A *didactic system* $S(X; Y; ♥)$ is then formed. The thing that is to be learned is called a *didactic stake* ♥ and is made up of questions or praxeological components (Bosch & Gascón, 2014, p. 71).

In my case X can be seen as made up of students at the ELSYS programme. Y is made up of two components, Y_M and Y_E, where Y_M consists of teachers and learning resources involved in the teaching and learning of mathematics to X, and Y_E consists of the corresponding components in the Electronic System Design and Analysis (ESDA) course.

The driving force in a praxeology is the desire for X to find answers (A) to questions (Q). The questions depend on the praxeology they emerge within. In the process of finding the answers, a didactical milieu, M, is developed, consisting of material and immaterial tools that X gathers, with the help of Y, in the process of inquiring into the question Q. This situation is represented with *the reduced Herbartian schema* $S(X; Y; Q) \hookrightarrow A$ (Chevallard, 2020). The milieu is seen as consisting of several components: existing answers (A_i) offered by other persons or institutions, works (W_j) of different kinds that can be accessed, and new questions (Q_k) that may arise during the work: $M = \{A_1, A_2, \ldots, A_m, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p\}$ (Chevallard, 2020, p. 44).

Since there are separate courses in mathematics and electronic systems, there will also be separate didactic stakes, $\heartsuit_M \neq \heartsuit_E$. Hence, there are two didactic systems, $S(X; Y_M; \heartsuit_M)$ and $S(X; Y_E; \heartsuit_E)$, and two praxeologies, one for mathematics, $\boldsymbol{P}_M = [T_M, \tau_M, \theta_M, \Theta_M]$, and another for electronic systems, $\boldsymbol{P}_E = [T_E, \tau_E, \theta_E, \Theta_E]$. Learning in context should have as a consequence that the didactic stakes in the two praxeologies should overlap ($\heartsuit_M \cap \heartsuit_E \neq \emptyset$), and therefore I find it of interest to study the interplay between $\boldsymbol{P}_M$ and $\boldsymbol{P}_E$, within the didactic system $S(X; Y_E; \heartsuit_E)$. I focus on the system $S(X; Y_E; \heartsuit_E)$ since I consider $\boldsymbol{P}_E$ to be the central praxeology in ELSYS, with $\boldsymbol{P}_M$ playing a role as a "supporting praxeology" for $\boldsymbol{P}_E$. On the basis of selected questions from $\boldsymbol{P}_E$, I will identify elements of the milieu used to answer these questions. In particular, I will be looking for similarities and differences regarding technologies (θ) and techniques (τ) that are applied to solve a given task, coming from $\boldsymbol{P}_E$. The aim of this investigation is to answer the following question: In which ways can techniques and technologies from mathematics and electronic systems in combination contribute to finding answers to questions arising in $S(X; Y_E; \heartsuit_E)$?

29.4 Previous Relevant Research

One issue regarding mathematics in engineering education is to find the right balance between theory and practice. Flegg et al. (2012, p. 718) argue that "[w]ithout the explicit connection between theory and practice, the mathematical content of engineering programs may not be seen by students as relevant". They also claim that in cases where mathematics departments teach the mathematical content to the engineering students, the engineering departments may have little idea of what mathematical content the students are exposed to. Loch and Lamborn (2016) observed that first-year mathematics is often seen as irrelevant and distracting by engineering students, who are more interested in applied engineering subjects. This lack of relevance was attributed partly to mathematics being taught in a 'mathematical' way, "with a focus on mathematical concepts and understanding rather than applications" (Loch & Lamborn, 2016, p. 30). Loch and Lamborn report from a project where higher year engineering students were asked to create multimedia artefacts meant to show the relevance of mathematics. The project resulted in two animated videos showing how mathematics was used to plan and construct a building and a car. In interviews with first-year students after they had seen the videos, some students said that the videos did demonstrate the relevance of mathematics, and that "there is probably a reason we're being taught what we're being taught" (Loch & Lamborn, 2016, p. 38). However, students also reported that they found the videos overwhelming because of the amount of mathematics that was shown. Regarding the purpose of mathematics for engineers, Cardella (2008) claims that mathematics should be more than learning some specific topics. It is about learning a way of working and thinking that is of value for the work as an engineer. Faulkner et al. (2019) use the term "mathematical maturity" to cover what many teachers in engineering subjects hope that students learn from their mathematics coursework.

Booth (2004) discusses various approaches to learning mathematics by presenting a table of different strategies, with corresponding intentions and goals. These approaches constitute a hierarchy where the lowest level is made up of the strategy "Just learning" with the intention "To learn the content" and the goal "To know the content for use when needed". The highest level is made up of the strategy "Studied reflection" with the intention "To be able to take different perspectives on problems" and "To relate content to the world outside of mathematics". The goal is here formulated as "To be able to use mathematics to solve problems" and "To understand how mathematics applies to other situations" (Booth, 2004, p. 15). Scanlan, a professor of electrical engineering, warned against seeing mathematics for engineers just as a set of tools, but rather as an essential part of the students' formation (Scanlan, 1985, p. 449). It could be argued that in order to be able to use mathematics in a meaningful way, e.g. in engineering, it is necessary to learn mathematics to the level of *studied reflection* (Booth, 2004). This could also be related to *mathematical maturity* (Faulkner et al., 2019).

Booth also argues that mathematics should not be taught by engineers but that "mathematicians and engineers could unite some of their courses so that the students experienced a team of teachers leading their learning of mathematics in the world of engineering they intend to enter" (Booth, 2004, p. 21). This is in line with the ideas of contextual teaching from CDIO (Crawley et al., 2014), and also with the ideas behind MARTA.

Gueudet and Quéré (2018) report that a gap can be observed between mathematics taught in mathematics courses and the way mathematics is used to solve problems in engineering courses. An important explanation that they give for this gap is that the mathematics courses do not make enough connections. As examples of relevant connections, the authors list links between mathematics and the real world, between different mathematical contents and between different representations (Gueudet & Quéré, 2018). Connections are also seen as important by Wolf and Biehler (2016) who present 10 examples of what they denote as *authentic problems* in mathematics for mechanical engineering. To secure connection, one of the basic principles that is presented, is that the problem should be authentic in the sense that it should not just be a dressed-up mathematical problem with unrealistic numbers (Wolf & Biehler, 2016).

Authentic problems are also discussed by Schmidt and Winsløw (2021), using the theory of didactic transposition (part of ATD). They create a model for what they call *Authentic Problems from Engineering*, defined as "a problem which comes from current research and innovation in some specific institution of scholarly engineering" (Schmidt & Winsløw, 2021, p. 266). In their paper, they present a model for task design, where tasks in the mathematics course are created, based on the problems from engineering. As an example, they present an assignment based on the problem to compute and control the magnetic field induced by a so-called Halbach magnet (Schmidt & Winsløw, 2021, p. 272).

Recently several researchers have shown how ATD can be a useful tool for investigating mathematics for engineering students, (e.g. González-Martín, 2021; González-Martín & Hernandes-Gomes, 2017, 2018, 2019; Peters et al., 2017). The main focus of González-Martín and Hernandes-Gomes is to compare presentations in Calculus textbooks with presentations in textbooks for professional engineering courses, to identify connections between the fields. Most of the examples presented by these authors are from mechanical engineering, but also a course in electricity and magnetism is studied (González-Martín, 2021). The results, in particular in mechanical engineering, indicate a lack of connection between the praxeologies. A similar analysis on the topic of Fourier series in mathematics and signal theory has been made by Rønning (2021). Also here, there are differences but it seems that signal theory makes more explicit use of results from mathematics than what may seem to be the case in mechanical engineering.

Summing up, it seems that there are two main challenges that are reported on. One is that students do not see the relevance of mathematics for their engineering profession. This in turn may reduce the motivation for mathematics, and perhaps also for the study as a whole, and may lead to drop-out (Faulkner et al., 2019). The second challenge is the lack of connection between mathematics and engineering subjects (e.g., Flegg et al., 2012; Gueudet & Quéré, 2018; Loch & Lamborn, 2016). Recently, some approaches to create connections have been presented (e.g., Schmidt & Winsløw, 2021; Wolf & Biehler, 2016). There seems to be agreement that it is important to develop problem solving abilities. This can be expressed as making mathematics a tool for thinking. And for this to happen, deep knowledge is required (e.g., Booth, 2004; Cardella, 2008; Crawley et al., 2014; Scanlan, 1985), as well as good problems.

29.5 Analysis of Data

The question raised in this paper is the following: In which ways can techniques and technologies from $\boldsymbol{P}_M$ and $\boldsymbol{P}_E$ contribute to finding answers to questions arising in the didactic system $S(X;\ Y_E;\ \heartsuit_E)$? As data for the study, I used teaching material (problem sheets, lecture notes, textbooks, video lectures) from the ESDA course. With the video lectures (Lundheim, 2019) as the main source, supported by a textbook that was recommended for the students (Nilsson & Riedel, 2011), I performed an open coding of utterances as representing a *technique* or a *technology*. In each case, I also coded according to whether I saw the utterance as arising from $\boldsymbol{P}_M$ or from $\boldsymbol{P}_E$. To further strengthen my analysis, I conducted a joint interview with both the mathematics and the ESDA teacher after the end of the semester. The purpose of the interview was to get further insight into issues arising from studying the teaching material, as well as getting insight into the teachers' experiences from the first semester of the project. The interview was audio recorded and partly transcribed. From the teaching material I selected as my main example a situation with modelling an electric circuit (see Fig. 29.1). This example provides the main question for the reduced Herbartian schema $S(X; Y; Q) \hookrightarrow A$ (see Sect. 29.5.1). In the interview, I inquired into the techniques and technologies behind the main example and I asked both teachers to formulate their ideas about learning and teaching in context, and to explicate their view on how the two subjects could mutually support each other. I intend to show some possibilities for making connections between mathematics and electrical engineering, and to show the interplay between the praxeologies $\boldsymbol{P}_M$ and $\boldsymbol{P}_E$ in making this connection. The analysis will show that knowledge from both praxeologies is needed to solve the given problem.

29.5.1 Example: An Electric Circuit

The electric circuit I will use as an example is illustrated in Fig. 29.1. This, and similar circuits, are used frequently in the early phase of the ESDA course and can therefore be seen as an important basic example for the students at ELSYS. The circuit consists of two resistors, with resistance R_1 and R_2, a capacitor with capacitance C and an inductor with inductance L. The problem is to determine the voltages v_1 and v_2 at the points A and B shown in Fig. 29.1, given the input voltage $v(t)$.

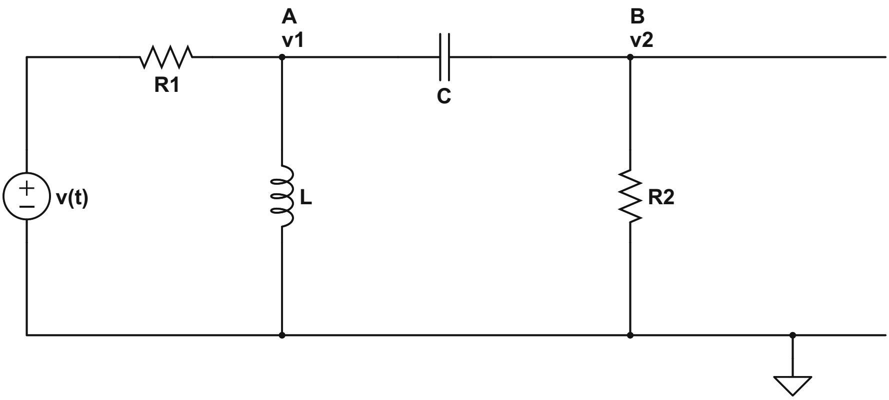

Fig. 29.1 A circuit with a given input voltage and two unknown voltages

This is a problem from $\boldsymbol{P}_E$ where the question Q is to find the voltages v_1 and v_2. The expression for these voltages will be the answer A. I will discuss the reduced Herbartian schema $S(X;\ Y;\ Q) \hookrightarrow A$ for this problem by identifying elements of the didactical milieu coming both from $\boldsymbol{P}_E$ and from $\boldsymbol{P}_M$. My data for this discussion come from video lectures by Lundheim (2019) and a textbook on electric circuits (Nilsson & Riedel, 2011). Both these resources are central in the ESDA course, and hence in the didactic system $S(X;\ Y_E;\ \heartsuit_E)$.

In the video lectures, two Equations, (29.1) and (29.2), are presented, based on the currents at the points A and B, with voltages v_1 and v_2 respectively.

$$\frac{v_1 - v}{R_1} + \frac{1}{L}\int v_1(t)dt + C\frac{d}{dt}(v_1 - v_2) = 0 \tag{29.1}$$

$$C\frac{d}{dt}(v_2 - v_1) + \frac{v_2}{R_2} = 0 \tag{29.2}$$

Equation (29.1) models the current out of the node at the point A (v_1). The left-hand side of (29.1) contains three terms, one for the resistor, one for the inductor and one for the capacitor. I will look at how each of these terms are justified in Lundheim's (2019) presentation. For each justification, I will indicate, either by θ_E or by θ_M, which praxeology I interpret the justification to be based on. The first term is justified by saying that "this is just regular circuit analysis", i.e. Ohm's law is used: The current through the resistor is proportional to the voltage over the resistor (θ_E). The two other terms are more interesting. For the second term, it is said that "the current through an inductor is proportional to the integral of the voltage over the inductor" (θ_E), and for the third term that "the current through a capacitor is proportional to the derivative of the voltage over the capacitor" (θ_E). Furthermore, the principle used is what is known as Kirchhoff's law of currents, stating that the sum of the currents out of the node at v_1 is zero (θ_E). Equation (29.2) is obtained in a similar way by analysing the current going out of the node at the point B (v_2). Now Lundheim observes that Eq. (29.2) is a first order differential equation whereas Eq. (29.1) contains terms including both an integral and a derivative (an integro-differential equation). To transform this to a "pure differential equation" he takes the derivative with respect to time on both sides of (29.1) to obtain (τ_M)

$$\frac{d}{dt}\frac{1}{R_1}(v_1 - v) + \frac{1}{L}v_1(t) + C\frac{d^2}{dt^2}(v_1 - v_2) = 0 \tag{29.1$'$}$$

Lundheim now observes that a system of differential Eqs. (29.1′) and (29.2), has been obtained and that in principle this system can be solved (within $\boldsymbol{P}_M$). He says that he finds this to be complicated, and therefore he will look for an alternative way to find the answer A to the question Q. This "alternative way" is based on the assumption that the input signal (v) is sinusoidal. This is a reasonable assumption in $\boldsymbol{P}_E$, but in $\boldsymbol{P}_M$ it would probably be seen as a (very) special case.

The following reasoning is presented. For a given trigonometric signal $x(t) = A\cos(\omega t + \varphi)$, define its complex form $X(t) = Ae^{j(\omega t + \varphi)} = Ae^{j\varphi}e^{j\omega t}$. Then $x(t) =$ Re $X(t)$. The complex number $Ae^{j\varphi}$ is called the *phasor* or the *complex amplitude* of the signal[1]. An important point made is that $\frac{d}{dt}X(t) = j\omega X(t)$ and $\int X(t)dt = \frac{1}{j\omega}X(t)$. Although this technique is purely mathematical, it would rarely be seen as a technique for differentiating and integrating in $\boldsymbol{P}_M$, since it would apply only to a very limited choice of functions. These functions, however, play a very important role in $\boldsymbol{P}_E$ and therefore it makes sense to introduce this technique.

Applying this technique to the system of Eqs. (29.1) and (29.2) and replacing the voltages v with their complex form V, the following system of algebraic equations is obtained.

$$\frac{V_1 - V}{R_1} + \frac{1}{Lj\omega}V_1 + Cj\omega(V_1 - V_2) = 0 \tag{29.3}$$

$$Cj\omega(V_2 - V_1) + \frac{V_2}{R_2} = 0 \tag{29.4}$$

Solving this system for V_1 and V_2 the unknown voltages v_1 and v_2 are obtained by taking the real part. The given task T belongs to $\boldsymbol{P}_E$ but the techniques and technologies belong to $\boldsymbol{P}_M$ (properties of complex numbers). However, the techniques, although purely mathematical, would not have been given such a prominent role in a mathematical praxeology. This shows that the choice of technique may depend on the praxeology: A technique (τ_M) from $\boldsymbol{P}_M$ is considered more important because it is used in $\boldsymbol{P}_E$ compared to if it had been used in $\boldsymbol{P}_M$.

I now return to the modelling process resulting in the system of the integro-differential equation Eq. (29.1) and the differential equation Eq. (29.2), to take a closer look at the justifications for setting up these equations. Of particular interest are the terms $\frac{1}{L}\int v_1(t)dt$ and $C\frac{d}{dt}(v_1 - v_2)$ in Eq. (29.1). For the term with the integral (the inductor), the principle used is that the current through an inductor is proportional to the integral of the voltage. For the term with the derivative (the capacitor), it is claimed that the current through a capacitor is proportional to the derivative of the voltage. These are technologies (θ_E) from $\boldsymbol{P}_E$ leading to the application of techniques (τ_M) from $\boldsymbol{P}_M$.

Concerning the capacitor, Nilsson and Riedel (2011) write:

> [A]pplying a voltage to the terminals of the capacitor … can displace a charge within the dielectric. As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the **displacement current**. At the terminals, the displacement current is indistinguishable from the conduction current. The current is proportional to the rate at which the voltage across the capacitor varies with time (p. 204).

This technology (θ_E) gives the relation $I = C\frac{dv}{dt}$. For the inductor, Nilsson and Riedel just state that the following relation holds, $v = L\frac{dI}{dt}$ (Eq. 6.1, p. 198). Then they state:

[1] j is used for the imaginary unit, in accordance with the tradition in $\boldsymbol{P}_E$.

Table 29.1 The didactical milieu for the electric circuit in Fig. 29.1

	$\boldsymbol{P}_M$	$\boldsymbol{P}_E$
Main question, Q		Determine the voltages v_1 and v_2 given an input voltage v.
Sub-questions, Q_j	How to solve a system of differential equations How to solve a system of algebraic equations	Model the current flow at the nodes v_1 and v_2
Works, W_k	Transforming Eq. (29.1) to Eq. (29.1′) Solving a system of algebraic equations Properties of complex numbers	Kirchoff's current law Behaviour of current over resistors, capacitors and inductors Properties of the phasor
Partial answers, A_i	The Eqs. (29.1′) and (29.2) The Eqs. (29.3) and (29.4) Solution of the system of Eqs. (29.3) and (29.4)	
Main answer, A		The values v_1 and v_2

"Note from Eq. 6.1 that the voltage across the terminals of an inductor is proportional to the time rate of change of the current in the inductor" (p. 198). Hence, they give a mathematical interpretation of a relation between electrotechnical quantities, without justifying why this particular relation, $v = L\frac{dI}{dt}$, holds. Accepting this, again by mathematical techniques (τ_M), one gets $I = \frac{1}{L}\int v(t)dt$.

For the circuit in Fig. 29.1 I expressed the question Q as determining the voltages v_1 and v_2 at the points A and B, given an input voltage v. The answer A in $S(X; Y; Q) \hookrightarrow A$ contains the values of the unknown voltages. In search of this answer a didactical milieu M was generated, $M = \{A_1, A_2, \ldots, A_m, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p\}$, consisting of partial answers, A_i, works (results), W_j, and new questions Q_k (sub-questions), used to find the answer A to the original question Q. Some of these components are formulated within $\boldsymbol{P}_M$ and some within $\boldsymbol{P}_E$. Table 29.1 shows the didactical milieu associated with the electric circuit in Fig. 29.1.

Table 29.1 shows the interplay between the praxeologies $\boldsymbol{P}_M$ and $\boldsymbol{P}_E$ for the given problem. Although both Q and A belong to $\boldsymbol{P}_E$ the didactical milieu also includes questions and answers of a purely mathematical character, and the process of finding the values v_1 and v_2 draws on works from $\boldsymbol{P}_M$. However, works from $\boldsymbol{P}_M$ are not sufficient. In order to model the current flow, justifications from $\boldsymbol{P}_E$ are needed to formulate the system of Eqs. (29.1) and (29.2). I find the behaviour of current over capacitors and inductors to be of particular interest. Why can this be modelled with derivatives and integrals as shown in Eqs. (29.1) and (29.2)? I will return to this question in Sect. 29.5.2. I have previously pointed out that a key word pertaining to the challenge of teaching mathematics for engineers is *connections* (e.g. Gueudet & Quéré, 2018). The analysis resulting in Table 29.1 shows how the didactical milieu involved in solving the problem with the circuit consists of elements from both praxeologies $\boldsymbol{P}_M$ and $\boldsymbol{P}_E$ and that both praxeologies are essential in the path leading to the solution of the problem. In the next section I will look into how the project MARTA creates opportunities for connections, as well as going deeper into some of the justifications given in the analysis of the circuit.

29.5.2 Opportunities for Connections

The example described in Sect. 29.5.1 comes from $\boldsymbol{P_E}$, but the analysis shows that elements from both $\boldsymbol{P_E}$ and $\boldsymbol{P_M}$ are used to solve the problem (Table 29.1). Therefore, it is necessary that the students have some knowledge from mathematics in order to make sense of what is going on. This is in itself nothing special, so to see what extra can be gained by teaching mathematics and electronics in close connection, I interviewed the teachers Marc, who was teaching the mathematics course, and Eric, who was teaching the ESDA course to the same students in their first semester. When asked about the main differences in the current approach compared to a traditional approach, Marc emphasises that in addition to changing the sequencing of topics, he tries to include circuits into mathematics as often as possible. He continues: “But I don’t know the electronics and it is difficult to find circuits that give good mathematical problems. Then I have to ask Eric or look in a textbook”. Here Eric comments that a crucial point is *division of labour*. “I think that mathematics must live on its own premises, and that the learning goals in mathematics must be mathematical. We cannot make plans that presuppose that the mathematicians know a lot of electronics. The most important is continuous communication.”

Marc gives an example of a circuit which is modelled by a non-linear differential equation. The mathematical purpose of this example was to motivate the introduction of numerical methods for solving differential equations, and Marc felt that the students thought it was fun. The problem was given as solving the differential equation using Euler’s explicit and implicit methods, and it was just claimed that the differential equation would model the given circuit. The purpose of this example was purely mathematical, namely to introduce Euler’s methods. This could have been done without connection to the electrical circuit, but the circuit worked as a link between the praxeologies, perhaps contributing to the students seeing increased relevance.

Below is a dialogue following another of Marc’s examples.

Eric	The mathematicians have a habit of setting all values of the components equal to one, because then it gets much tidier. With this, the physics disappear.
Marc	I defend this based on the principle of division of labour. The mathematical principles are easier to comprehend if you leave out the physical constants.
Eric	Then you are left with the structure of the problem. I think this is the kind of division of labour we should have. We can “dress the problem up” later.
Marc	I think it is a good pedagogical trick to clean away the mess when you learn something for the first time.

This dialogue shows a fundamental difference between the praxeologies. In engineering one is concerned with units and with physical constants that are important for understanding the physical principles. In mathematics, however, one is more concerned with the structure, and this structure may come better to the fore if for example (non-zero) constants are set equal to one. Based on the principle of division of labour, both teachers find that this difference is not problematic, but on the contrary, that it can be an asset.

One issue that Eric finds particularly important is that the close connection to mathematics gives the possibility to justify principles from electrical engineering better. As an example, he mentions *the principle of superposition*. This is explained in the following way by Nilsson and Riedel:

> A linear system obeys the principle of **superposition,** which states that whenever a system is excited, or driven, by more than one independent source of energy, the total response is the sum of the individual responses (Nilsson & Riedel, 2011, p. 144).

The strategy chosen in the book by Nilsson and Riedel is to deactivate all sources of energy but one, and study the system that is then created (τ_E). Solving for the currents in each of the circuits with just one source of energy, it is claimed, with reference to the principle of superposition (θ_E), that the complete solution is obtained by adding the currents. Eric finds this argument unsatisfactory, and he is happy that he can use mathematical arguments to justify the principle. Eric says: “I was always told that, ‘this is how it *is*’. Now we can argue that this is actually how it *has* to be”. The mathematical justification of the superposition principle is based on linear algebra. Each of the circuits with only one source of energy can be modelled with a system of linear equations $Ax_i = b_i$, $i = 1, \ldots, n,$ where n is the number of energy sources. The complete circuit can then be modelled by $Ax = b,$ where $b = b_1 + \ldots + b_n$. Since A is a linear operator, the complete solution is given by $x = x_1 + \ldots + x_n$ (θ_M). This is an example that a technology from mathematics is used to justify a technique in electrical engineering.

In Sect. 29.5.1 it would appear both from Lundheim (2019) and from Nilsson and Riedel (2011) that the justification for the modelling of the circuit shown in Fig. 29.1 was somewhat unsatisfactory. I therefore asked Eric in the interview how he would justify the modelling of Eq. (29.1). Regarding the capacitor, Eric says:

> Current is the derivative of charge with respect to time. How much charge passes through a crosscut per unit of time. The number C indicates how much charge a capacitor can hold. Q = CV, so I = dQ/dt = C dV/dt.

In the justification he bases his argument on the definition of current, as the rate of change of charge (Q) with respect to time (θ_E). And the charge that a capacitor can hold is proportional to the voltage, where the proportionality constant C is a characteristic of the capacitor. This is in line with the argument given in Nilsson and Riedel (2011) that “[t]he current is proportional to the rate at which the voltage across the capacitor varies with time” (p. 204).

I observed in Sect. 29.5.1 that the argument in Nilsson and Riedel (2011) for the behaviour of the inductor was rather vague. Below is the explanation provided by Eric in the interview.

> For the inductor it is more tricky. You cannot use the concept of charge. You need flux, which is physically much heavier. So here I often use some analogies, e.g. analogy with mass. Imagine you will push a car. It is heavy in the beginning but as the car starts to roll, you need less and less force and finally the car rolls by itself. Mass as inertia. An inductor functions as inertia for the current. In the beginning high voltage is needed to get the current going, but as the current starts to flow, the voltage goes down. So an inductor exerts inertia towards changes in current. If you want a quick change in the current you need high voltage. When the current evens out, the voltage goes down. When the current is zero, the inductor works as a short circuit.

The justification he gives is in the form of an analogy, thinking of the inductor as an element that resists change, like mass at rest. The crucial formulation here is "[i]f you want a quick change in the current you need high voltage". This means that to get a large value of $\frac{dI}{dt}$, v needs to be large, motivating the relation $v = L\frac{dI}{dt}$. This example shows that Eric draws on yet another praxeology for his justification, by comparing with pushing a car. This he does because the justification within $\boldsymbol{P}_E$ (using flux) would not be accessible for the students at this point. Then, using a mathematical technique (τ_M), the relation can be written as $I = \frac{1}{L}\int v(t)dt$, as in Eq. (29.1).

Although recognising the value of the interplay between the two praxeologies, the teachers argue that they also, to some extent, should be kept apart. This is expressed using the expression *division of labour*. It is the role of $\boldsymbol{P}_M$ to work with the *structure* of a problem, and the role of $\boldsymbol{P}_E$ to see the problem, and its solution, in an engineering context.

29.6 Discussion

In the literature, there are some particular challenges that are frequently mentioned: lack of relevance of mathematics for engineers, lack of connections between mathematics and engineering, and challenges with applying mathematics to engineering problems (e.g., Carvalho & Oliviera, 2018; Flegg et al., 2012; Gueudet & Quéré, 2018; Harris et al., 2015; Loch & Lamborn, 2016). There is also criticism against mathematics being taught too "mathematically" (Loch & Lamborn, 2016). However, there is evidence to support that there is a need for a deep knowledge of mathematics, to avoid mathematics becoming just a set of tools (e.g., Booth, 2004; Cardella, 2008; Crawley et al., 2014; Scanlan, 1985).

An intention with the project MARTA is to teach mathematics and engineering in close connection, with much of the mathematics contextualised through problems and examples from engineering, in line with the ideas of the CDIO approach (Crawley et al., 2014). An overarching goal is to develop mathematics as a way of thinking (Cardella, 2008; Faulkner et al., 2019) and obtaining deep learning, both in mathematics and in the engineering subject (Crawley et al., 2014; Marton & Säljö, 1976; Scanlan, 1985).

With the above principles as a background, I performed a praxeological analysis of an example from $\boldsymbol{P}_E$ in order to investigate how techniques and technologies from

$\boldsymbol{P}_E$ and $\boldsymbol{P}_M$ in combination contribute to finding answers to questions arising in the didactic system $S(X; Y_E; ♥_E)$. The analysis shows that applications of mathematics in electrical engineering involve a complex interplay between the praxeologies to establish a functional didactical milieu. Techniques and technologies from two praxeologies are intertwined and although both the problem and the answer lie within $\boldsymbol{P}_E$, it is necessary to use elements from $\boldsymbol{P}_M$ to get to the answer. This interplay between the praxeologies I see as evidence that deep knowledge in both fields is necessary. Not only techniques, but also technologies (justifications) from $\boldsymbol{P}_M$ are necessary, so using mathematics just as "a set of 'tools'" (Scanlan, 1985, p. 449) will not suffice. A certain degree of "mathematical maturity" (Faulkner et al., 2019) is needed to master the interplay between the praxeologies.

I also identified some issues that are seen as important from the viewpoint both of the mathematics teacher and of the electronic systems teacher. Their main message is that of division of labour. They recognise that they enter the work with the students with different competencies. They work closely together but the mathematics teacher says that "I don't know the electronics" and he admits that he finds it difficult to find examples from electrical engineering that give good mathematical problems. The electronic systems teacher says that "I think that mathematics must live on its own premises", and he recognises the value of mathematics for example to see the structure behind a method. It will be of great interest in the further work with the project to get information from the students, both in surveys reflecting their perceptions of the collaboration between the fields, and in direct observation of students working on problems. Another issue is to see the effect of including other study programmes into the project. It will not be sustainable to have specially designed mathematics courses for each study programme, so an important line of inquiry will be to study the interplay between the praxeology $\boldsymbol{P}_M$ and a given praxeology $\boldsymbol{P}_Z$, where Z represents an engineering field, for various choices of Z.

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Chapter 30
Towards an Institutional Epistemology

Corine Castela and Avenilde Romo-Vázquez

Abstract In higher education, the difficulties of implementing teaching sequences in which several academic and engineering disciplines, or even professional worlds, coexist have been widely documented. We hypothesize that these difficulties stem, especially, from a series of conditions and constraints that determine the knowledge life in these different universes. In this chapter, we propose using tools from the Anthropological Theory of Didactics (ATD) to analyze these epistemologies and illustrate their application with examples from land surveying, industrial, and computer science contexts.

Keywords Anthropological theory of the didactic · Engineering education · Interdisciplinary mathematics · Inter-institutional transposition · Institutions' epistemic activities · Industrial epistemology

30.1 Introduction

Several didactic approaches in higher education relate mathematics to other disciplines: the competency-based approach (Niss & Højgaard 2019), mathematical modelling perspectives (Kaiser, 2020), project-based learning (Kolmos, 2009), challenge-based learning (Gallagher & Savage, 2020), interdisciplinary education (Roth, 2020), and transdisciplinary mathematics education (Jao & Radakovic, 2018; Klein, 2013). But integrating these approaches in the medium and long term poses a major challenge that requires considering several epistemological and didactical aspects, including the nature of professional knowledge (González-Martín et al.,

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2021), the role of technologies that encapsulate the mathematics used in workplaces (Kent et al., 2007), the variety of student profiles in engineering, economics, and management, among other fields (Winsløw et al., 2018), assessment beyond content (Borrego & Cutler, 2010), institutional teaching conditions, and the tensions that professors face (Hernandez-Martinez, et al., 2021). On the epistemological level, it is important to recognize that insufficient knowledge of professional contexts may cause curricular conflicts –on a small or large scale— between disciplinary theoretical knowledge and other kinds of knowledge (Young & Muller, 2016), in our case, professional knowledge. This necessitates making the different epistemologies underlying these distinct kinds of knowledge visible to build a solid reference for curricular and didactic innovation. Specifically, in this chapter, Sect. 30.2 proposes using tools from the Anthropological Theory of Didactics (ATD) to analyze the different epistemologies. Sections 30.3, 30.4 and 30.5 illustrate their functionality with examples from surveying, industry, and computer science.

30.2 Theoretical Framework

A socio-cultural conception of humans underpins ATD, one centered on institutions as an absolute precondition for humanity's development and social activities. An institution is a stable social organization that provides a framework in which different groups carry out different groups of activities. Institutions foster collective processes to confront and solve human problems. They favor disseminating innovations and provide the necessary resources (material and cultural) for activities to proceed. Conversely, institutions also constrain the different types of activities that it expects people to carry out in the social environment they build. The fact that an institution *I* enables and imposes on its subjects, −the people who held the different positions in *I*- specific ways of doing and thinking is presented by Chevallard (2003, p. 82) as a feature of institution.

The ATD thus considers that human activities are framed institutionally. Some institutional constraints are universal, others derive from the resources, norms, and values of the institution *I*, which are partly determined by a network of institutions that influence *I*. Consequently, the praxeologies[1] related to these activities are guided by institutional influences. This explains why Chevallard (2006, p. 23) considers praxeologies as social idiosyncrasies, we would rather adopt a more explicit formulation: institutional idiosyncrasies.

What derives from these epistemological hypotheses is that when praxeologies produced in one institution move to another, such boundary-crossing processes are likely to generate transformations, called transposition effects, due to changes in the conditions and constraints (Chevallard, 1999, p. 231). In Sect. 30.3 we present an

[1]For a presentation of the concept of praxeology, see the 3rd section of the chapter by González-Martín, Barquero and Gueudet.

example of such a scenario, involving mathematics and land surveying. This phenomenon of institutional transposition is a generalization of the didactic transposition outlined in the second edition of *La transposition didactique* (Chevallard, 1991, p. 214).

Multiple institutions converge and, presumably, collaborate in training engineers and technicians: academic sciences, engineering sciences and professions, at least through the didactic corresponding institutions; for example, mathematics, physics, mechanics, and the machining industry in mechanical engineering curriculum. We argue that to analyze the complexity of such educational environment, epistemological studies must recognize the central role of institutions in issues involving the production and circulation of knowledge. We propose calling this an institutional epistemology.

The fundamental unit that ATD gives to these epistemologies can be schematized as $[T, \tau, \theta, \Theta] \leftarrow I$, where T is a task type, τ a technique used to perform some tasks from T, θ a rational discourse, called a technology, about the *praxis* $[T, \tau]$,and Θ a second level of discourse, called a theory. This scheme, present in Castela (2016, 2017), is not limited to indexing the praxeology by I, but also introduces as study topics, the processes, represented by the arrow, by which I produces, legitimizes and institutionalizes the praxeology. This means that some *I's* subjects perform specific task types whose objects are the praxeologies used at I. These task types develop the institution's knowledge capital, or its institutional *épistémê*. Therefore, we have chosen to use the adjective 'epistemic' to qualify these task types and related praxeologies.

What epistemic task types are involved? We must recognize, first, that, under the above assumptions, an institution's subject must be sufficiently lucid that they do not pretend to know everything about the epistemic activity of another institution, including the task types involved. While some task types can be assumed as universal, others are specific and, at best, shared by certain categories of institutions.

To propose an initial list of epistemic task types, we hypothesize that the technology of a technique accounts, though perhaps succinctly, for the epistemic activities to which it has been subjected. Based on an analysis of three university courses on the Laplace transformation for future engineers or senior technicians, Castela and Romo (2011) introduce six categories, expressed as verbs, that fits well with the idea of task type: *Describing* the technique; *Validating* it (i.e. proving that this technique actually does produce what is expected); *Explaining* why it is efficient (concerning causes); *Motivating* its different stages (regarding objectives); *Facilitating* its use; and, finally, *Appraising* it (with respect to the field of efficiency and to the comfort of use relative to other techniques available). We illustrate here the final four categories using technological elements of the technique derived from the integration by parts theorem for calculating an integral $\int_a^b u(t)dt$.

Two differentiable functions f and g are chosen, with continuous derivatives so the product $f.g' = u$.

Facilitating: an invariant presentation of the calculations, such as $\begin{matrix} f(x) & f'(x) \\ g'(x) & g(x) \end{matrix}$ can help remember the formula and prevent confusions.

Motivating the choice of functions*:* f and g are chosen such that deriving f eliminates the problem of calculating an antiderivative of $u = f.\ g'$.

Explaining efficiency: this technique may be effective because, in some cases, the derivation changes the nature of the function, e.g., logarithm.

Appraising: this technique seems to be particularly suitable when integrating a product of functions but can also be tried if no product is visible, by taking $g' = 1$ (see $u = ln$).

Not all integrals can be calculated with this technique. Other techniques exist, even for products (see $u = f'.f^n$).

Section 30.5 presents an analysis of a technology that will exemplify some of these categories (for additional examples from different scientific or professional contexts, see Castela, 2017; Castela & Elguero, 2013; Covián, 2013; Solares-Pineda et al., 2016).

Following the formulation in Chevallard's definition of praxeology,[2] Castela and Romo (2011) considered these epistemic task types to be *functions* of the technology, but now we ask if the associated *praxis* can be strictly discursive in nature? The answer is yes, for Describing, Motivating and Explaining. Otherwise, although in mathematics a *logos* (a demonstration) suffices to Validate a technique, this is not true for most *praxis*, especially in the experimental sciences. This also holds for Appraising. The answer is probably negative for Facilitating if this task type is conceived as the ergonomic dimension of a process Designing-Appraising-Improving a technique. In summary, for most institutions, and for most epistemic task types involving a *P praxis*, the technology of *P* at least states the conclusions of the epistemic study, which are institutionally certified in this way. It can also include the discursive part of the epistemic work realized and specify explicitly the knowledge used throughout this work.

Finally, we must mention certain task types not yet mentioned. These aim to *Define the institutional specifications of a given praxis* by, first, characterizing the task type: what exactly is to be done? and under what conditions? If there is no description of what is expected, it is impossible to validate a technique. The next step is to describe the requirements related to the technique, what Castela (2020) calls technical standards. Briefly, these standards may address the technique validity, with criteria related not only to its effectiveness (completing the task), but also to its efficiency (reliability, generality, etc.), usability, safety; its rational intelligibility; its adequacy for dissemination and learning. Appraising the degree to which the technique complies with institutional standards is for us an essential aspect of the praxeology legitimation process in an institution. Section 30.4 presents examples of technical standards for measurement *praxis* and production processes in relation to the associated praxeologies of Appraising, Validating, and Improving, to evidence their close relation in the industrial world.

[2] « une deuxième fonction de la technologie est d'*expliquer*, de *rendre intelligible*, d'*éclairer* la technique. [. . .] Enfin une dernière fonction correspond à un emploi plus actuel du terme de technologie: la *production* de techniques. » (1999, pp. 226–227).

30.3 From Mathematics to Land Surveying, an Example of Transpositive Effects

In this section, we illustrate the phenomenon of the institutional transposition of a mathematical praxeology. This approach allows us to highlight another source of difficulties that may arise in the training of engineers and technicians; namely, that it is often considered essential to teach these students more sophisticated mathematics than the ones that are employed in normal professional practices because they may be useful for possible career advance, but also for adapting to future changes in professional practice, or, in the short term, for dealing with certain critical situations. This means that it is difficult to design interdisciplinary sequences based on authentic professional situations that give mathematics teachers opportunities to make students work with the knowledge to teach.

Our example comes from Covián's PhD dissertation (2013) on the contribution of mathematics to topographic surveying and plotting activities in historical, professional, and academic contexts. We borrow the following from a professional context in which an expert is asked to describe the technique he would use professionally to compute the area of a polygonal terrain (see Figs. 30.1 and 30.3). The technique he advocates and considers the one most often used today, is based on coordinate geometry in orthonormal bases, with the following formula for the lot in question:

$$\begin{aligned} 2S = {} & x_0(y_1 - y_5) + x_1(y_2 - y_0) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_5 - y_3) \\ & + x_5(y_0 - y_4) \end{aligned}$$

This formula assumes that the coordinates of the polygon's vertices are determined in a given orthonormal system. The practice in topography is to choose a West-East

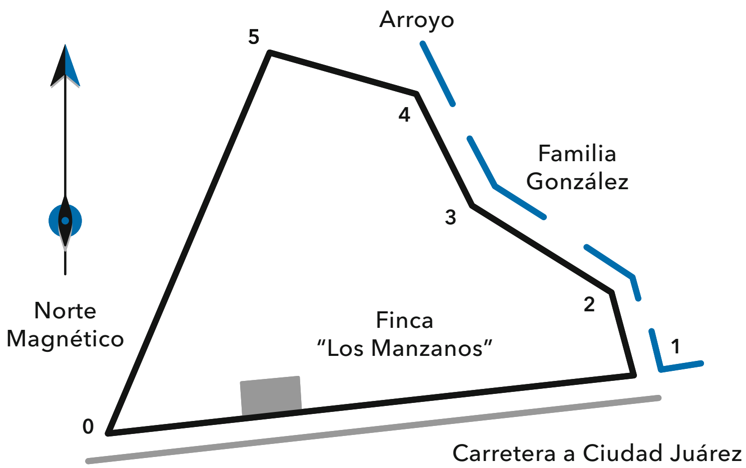


Fig. 30.1 Polygonal terrain. (Redrawn similar to Covián, 2013, p. 142)

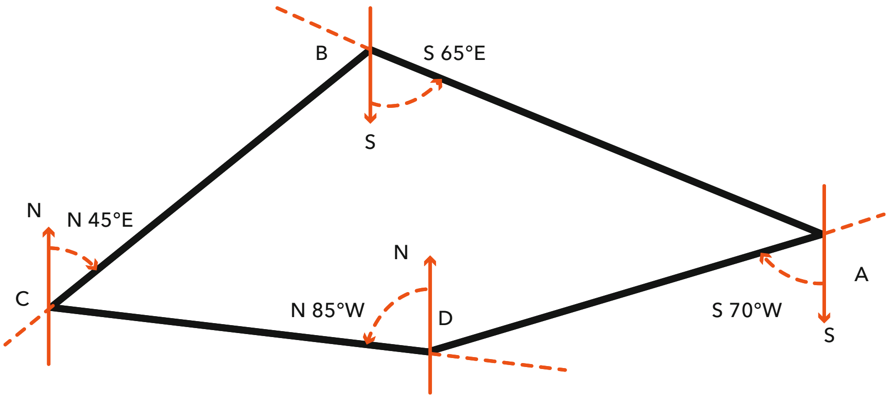


Fig. 30.2 Examples of bearings. (Redrawn similar to Natural Resources Conservation Service-US Department of Agriculture, 2008, p. I–24)

oriented axis as the *x*-axis, a South-North one as the *y*-axis, and the origin in a way which ensures that the coordinates are positive.[3]

What steps does the expert's technique follow to compute these coordinates?

Measurement phase: produce a geometric model of the terrain (Fig. 30.3) using the 'closed traverse' technique

- Measure the length of the sides in meters.
- Measure the bearing of side 0–1. A *bearing* (*rumbo magnético*) is an angle of 0–90° measured from the north or south pole, whichever is closer, and from east or west (see Fig. 30.2). The conventional notation in Mexico and the US is N45°E.
- Measure the angles of the polygon by placing the instrument at each vertex, traveling in a counterclockwise direction. The measurement of the angles is made in a clockwise direction. The internal angles of the polygon are therefore measured (Fig. 30.3).

Calculation phase

- Check the measurements of the angles.

Measuring errors in a closed traverse can be quantified by the sum of the interior angles of the polygon so formed. The sum should be $(n - 2)180°$, where n is the number of sides in the traverse. Empirical usage in surveying holds that the total angle should not vary from the correct value by more than $\sqrt{n}$ times the precision of

[3]For Anglo-Saxon references, see for example Chapter 1, Part 650- Engineering Field Handbook edited by the Natural Resources Conservation Service of the US Department of Agriculture (2008) https://directives.sc.egov.usda.gov/OpenNonWebContent.aspx?content=25276.wba

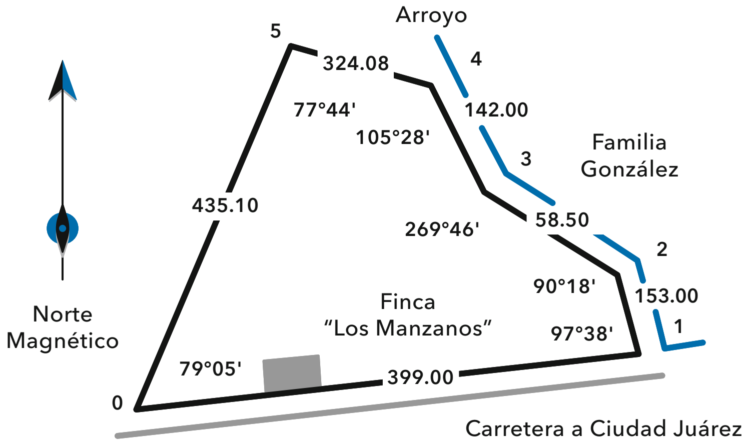


Fig. 30.3 Geometric model of the polygonal terrain. (Redrawn similar to Covián, 2013, p. 149)

Table 30.1 Corrected measures (Covián, 2013, p. 150)

Lados	Ángulos internos observados	Ángulos internos corregidos	Rumbos magnéticos calculados	Distancias en metros
0–1	79°05′	79°05′10″	*NE* 82°00′00″	399.0
1–2	97°38′	97°38′10″	*NW* 0°21′50″	153.00
2–3	90°18′	90°18′10″	*SW* 89°56′20″	58.50
3–4	269°46′	269°46′10″	*NW* 0°17′30″	142.00
4–5	105°28′	105°28′10″	*NW* 74°49′20″	324.08
5–0	77°47′	77°47′10″	*SW* 2°54′50″	435.10
Sumas	719°59′	720°		1511.68

the instrument (1′ in the present context). If the difference is reasonable, it is evenly distributed among the angles; if not, the measurements are repeated (Table 30.1).

- Successively calculate the bearings of the sides based on the measures of the internal angles and the bearing of the side 0–1.

- Compute the side projections on the axes (Table 30.2).

The multiplication by the *cosine* of the bearing determines the North (+) and South (−) projections (on the *y*-axis), while the *sine* gives the East (+) and West (−) ones. See Fig. 30.4 for an example with data retrieved from a Civil Engineering Analysis project at University of Memphis, available online.[4]

- Calculate the coordinates of the vertices using the corrected projections.

[4] Unknown author and date. http://www.ce.memphis.edu/1112/notes/project_3/traverse/Surveying_traverse.pdf

Table 30.2 Computing the projections (Covián, 2013, p. 157)

Lados	Rumbos Magnéticos calculados	Distancias En metros	Cálculo de Proyecciones			
			Norte (+)	Sur (−)	Este (+)	Oeste (−)
0–1	*NE* 82°00′00 ″	399.0	55.53		395.12	
1–2	*NW* 0°21′50 ″	153.00	152.99			0.97
2–3	*SW* 89°56′20 ″	58.50		0.06		58.49
3–4	*NW* 0°17′30 ″	142.00	141.99			0.72
4–5	*NW* 74°49′20 ″	324.08				312.77
5–0	*SW* 2°54′50 ″	435.10				22.12
Sumas		1511.68	435.36	434.59	395.12	395.07
			Error en las *y*: $E_y = 0.77$		Error en las *x*: $E_x = 0.05$	

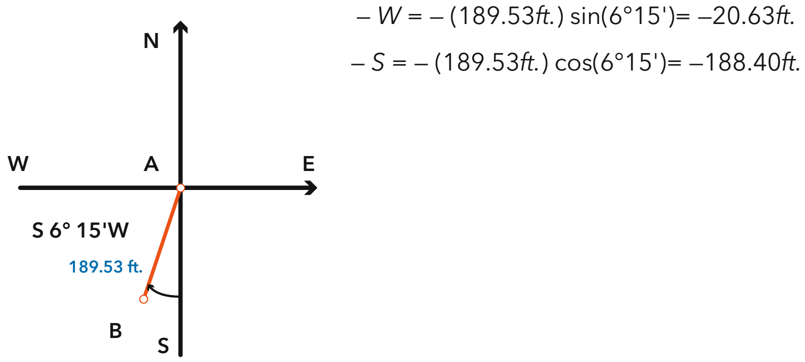


Fig. 30.4 Side projection. (Redrawn similar to Civil Engineering Analysis-university of Memphis, p. 3)

In this surveying work, a mathematician will recognize a task type that can be described succinctly as calculating the Cartesian coordinates of a vector in an orthonormal coordinate system when its polar coordinates are known. To this end, they will refer to the functions of a real variable *sine* and *cosine* defined through the trigonometric circle. The supposedly known angle in a classic situation corresponds to the angle of the *x*-axis with the vector considered, with no differentiation of the quadrants. This is measured in radians. The measure is generally chosen between $[-\pi, \pi]$. The coordinates are $x = rcos\alpha$, $y = rsin\alpha$, where r stands for the vector norm. This technique is taught from high school onwards in countries like France and Brazil (see Fig. 30.5).

This is not the praxeology the surveyor uses, for his is based mathematically on the trigonometry of the right-angle triangle, through the key-notion of ‘bearing’. Contrary to the mathematical usage just outlined, the bearing is an angle measured from the *y*-axis, that is the North-South direction whose predominance reflects the

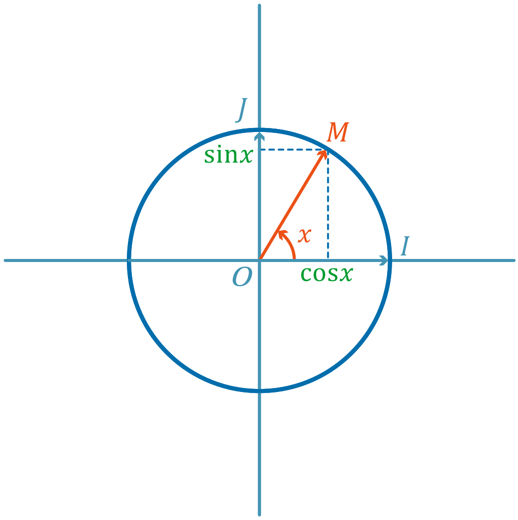


Fig. 30.5 Circular definition of sine and cosine. (Redrawn similar to Loeng, 2019, p. 296)

influence of astronomy and navigation on the development of topography. Another specificity is how the technique considers the quadrants in order to obtain the algebraic value of the projections by translating the symbols N, S, E, and W present in the bearing notation in + for N and E, − for S and W. Note that these algebraic values are avoided in Table 30.2, where only positive values appear in the four columns differentiated by a reference to a cardinal point (in association with a sign). Another semiotic difference with mathematics involves the representation of the length measures which appear in decimal form always with two figures after the decimal point, even if they are 0. This corresponds to the notion of significant figures that links numerical writing to the precision of the measurements.

In summary, we offer the following comments: The source of the surveyor's praxeology can be understood as a mathematical praxeology developed in the trigonometry of the right-angled triangle, but it has undergone transposition processes to adapt to the profession. Therefore, if we take the type of surveying tasks (Calculating the area of a polygonal terrain) analyzed above as the basis for an interdisciplinary sequence, the mathematics teachers will be disappointed if they wish to have the students work with circular trigonometry, unless they impose a technique that does not consider the real conditions of surveying work. This will likely entail agreeing to work with knowledge that is not included in the course syllabus and whose importance in surveying is probably unknown to them: triangle trigonometry, the sum of polygon angles. Yet, due to the transposition effects that we could denominate adaptations of the mathematical knowledge in the Double Approach framework, students may encounter difficulties that could surprise both the mathematics and the topography teacher; for instance, when they determine the bearing from the internal angle (sexagesimal calculations such as $180^\circ - 179^\circ 38' 10''$) or the projections (using the formulas $x = rcosa$, $y = rsina$ when the bearing is not the polar angle). This illustrates the importance of working conjointly.

30.4 Aspects of an Industrial Epistemology

This section discusses some results of an inquiry into metrological practices in industrial contexts that we were encouraged to pursue by Aldape-Carillo's Master thesis (2016). In order to design Study and Research Activities (or Paths[5]) for high school, Aldape-Carillo looked for potential questions in the automotive industry, a major job provider for young Mexicans with a Bachelor's degree. The thesis presents the epistemological investigation conducted, based on the field study of a car door production plant (visits to production lines, interviews with production and quality control engineers), and an analysis of a metrology handbook recommended by the general manager (Chrysler Group LLC et al., 2010, MSA-2010 in the following).

MSA-2010 was developed by a Measurement Systems Analysis Work Group, sanctioned by three automotive companies (Chrysler, Ford, General Motors) under the auspices of the Automotive Industry Action Group (AIAG). During our research for this chapter, we came across a similar document: The *Statistical Quality Control Handbook* published in 1956 by Western Electric (WE-1985in the following) for internal use. This manual was provided by the company's personnel department. It was republished ten times until 1985.[6] According to the Editorial Board Preface:

> This book is not a treatise on statistical quality control. [. . .] Its main purpose is to describe procedures that, if followed, will tend to preserve the essential aspects of the quality control at Western Electric. It can be considered as a kind of a collection of techniques and methods that have proven to be most useful for the success of these programs. Much of the material is based on training courses that have been given over the past six or seven years to engineers, managers, and executives at all levels of management (p. 5).

Note that the AT & T group, a Western Electric subsidiary, has been a pioneer (since 1924) in developing statistical quality control through a specific department in its research center. Therefore, we hypothesize that most praxeologies presented in the manual are transposed versions of praxeologies developed at that research institution, enriched by their use in production contexts, a process represented in the following figure inspired by (Castela, 2017, p. 422):

Here, *Ir* stands for a research institution that produced the praxeology and *Iu* for an institution that uses it. The asterisks represent the transposition changes in the components of the original praxeology, which are examined by I_r^*, a noosferian[7] institution created by *Ir* and *Iu*. The latter adds its own technologico-theoretical contribution $\theta_u - \Theta_u$.

These two manuals reveal the extent to which a company, or branch of industry, operates as an institution by disseminating to their subjects praxeological resources

[5]For a presentation of this pedagogy, see the chapter by González-Martín, Barquero and Gueudet.

[6]The Western Electric Company closed in 1985. The book is now almost impossible to find, but a French translation is available online.

[7]See Chevallard (1991, p. 214).

that constitute a reference which constrains the practices. In addition to organizing the related training courses, publishing these handbooks contributes to institutionalizing certain praxeologies.

MSA-2010 focuses on the analysis of measuring systems, while WE-1985 deals with the control of production processes. The questions we pose concerning these manuals are: why are these issues so important in industrial contexts? And, why do all the praxeologies presented contain a statistical component?

30.4.1 General Conditions and Constraints

In a specific industry, a core activity, broken down into a set of task types, is the production of a certain product (e.g., semiconductors, screws). This product must comply with the specifications issued by R & D teams and clients. Two aspects are decisive for an industrial epistemology:

On a manufacturing line, the quality characteristics of a product vary from one realization to another.[8]

Five basic components of the process are possible sources of variability: machinery, labor, method, environment, and input material. Some causes can be identified and the company can seek to correct them, others are inherent to the production process. The latter are called the random variability component. This explains why variability cannot be eliminated, but only reduced to a minimum. Due to this uncertainty regarding manufactured objects, specifications are always accompanied by tolerance limits.

Variability control cannot be achieved once and for all; over time, new process disturbances will appear.

We can now begin to perceive the epistemological distance that lies between the scientific world and the world of industrial production. Although both the experimental and engineering sciences confront the problem of uncertainty because measurement systems are not free of variability, they work with models that simplify reality, allowing them to assume that “all things are otherwise equal”. But this assumption is not applicable in industrial contexts. When a technique produced by an R & D laboratory, and validated in the model, is imported into a factory for large-scale, long-term use, the ‘things’ assumed to be constant begin to vary, while others that the model ignored (e.g., workers’ skills, vibrations in the workshop) turn out to influence the production. As a result, the technique implemented becomes an object of epistemic activities specific to industrial institutions. Due to the causes of the random variability component and the impossibility of inspecting every part produced, techniques based on statistical theory are widely used in a procedure called Statistical Process Control (SPC). SPC is a set of praxeologies that permits, first,

[8] We rely here on Bettayeb (2012), especially p. 36.

appraising the extent to which a production process actually does what is expected of it by producing items that conform to specifications and, second, adjusting the process when this does not occur. SPC is defined by the AFNOR (the French Branch of ISO) as

> . . . a set of actions to appraise, regulate and maintain a production process in a state where it manufactures all its products in accordance with the specifications adopted and, above all, with characteristics that are stable over time. SPC is one of the dynamic elements of the quality system and, as such, contributes to the continuous improvement of production. [. . .] a monitoring system allows a quick and efficient reaction to any drift, thus avoiding the mass production of non-compliant products (quoted by Bettayeb, 2012, p. 35, our translation).

Here, we highlight the importance of three task types: Appraising, Improving, and Monitoring the process. Due to the variability, the specifications applied do not take the form of measures to be exactly achieved. Rather, they define tolerance intervals. Production is described through intervals that must be estimated and compared to the specifications. It is important to add that, as is expected in descriptive statistics, the answer given is not fixed but at a certain statistical threshold. On this basis, the process will be validated or not. In the latter case, an investigation into the causes will be undertaken to improve the process by acquiring better knowledge of the variables that influence the effective implementation of the process in the specific context of the workshop. This is an extension of the designing activity conducted by R & D which generates an evolution of the process (see Fig. 30.6). This point is central in WE-1985, which stresses the importance of monitoring all processes (see 30.4.3).

30.4.2 Measurement System Analysis

Appraising a production process requires using measurement systems (MS) that are by no means free of variability.[9] Indeed, the true value of the quantity being measured is inaccessible. "Uncertainty is the value assigned to a measurement result that describes, within a defined level of confidence, the range expected to contain the *true* measurement result" (MSA-2010, p. 63). This condition means that MS can only be used on production sites if they are subject to regular assessments of their qualities within specific context of use. The aim is to appraise their adequacy with what is expected of them. But what are the qualities that can be expected from an MS (MSA-2010 Ch. I. A, pp. 6–7)?

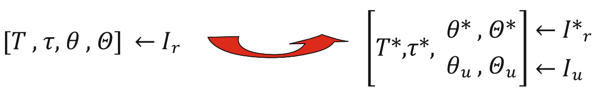


Fig. 30.6 From *Ir* to *Iu*, the transposition model

[9] For a complete presentation of this problematic, see (Joint Committee for Guides in Metrology, 2008).

- **Sensitivity:** responsiveness of the MS to changes in a measured feature. Quantified by the smallest input that results in a detectable output signal.
- **Accuracy**: "closeness" to the true value, or to an accepted reference value.

The accuracy of an MS is appraised according to three components.[10]

Bias: difference between the observed average of measurements and the reference value.

It quantifies a defect in the accuracy of an MS. If it is significant in size relative to the required measurement accuracy, a correction can be applied to compensate for the effect (see Sect. 30.3, angle measurement checking) (Fig. 30.7a).

Linearity studies quantify changes in bias over the normal operating range (Fig. 30.7b).

Stability studies quantify changes in bias over time (Fig. 30.7c).

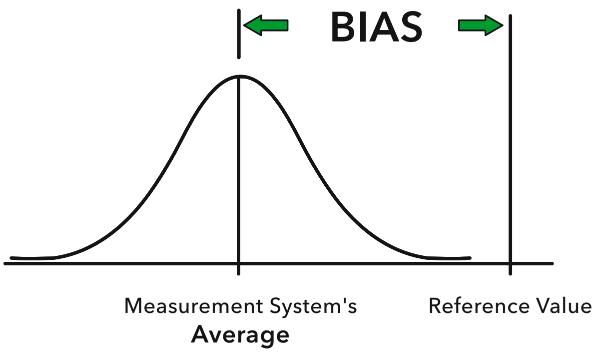


Fig. 30.7a Bias. (Redrawn similar to MSA-2010, p. 6)

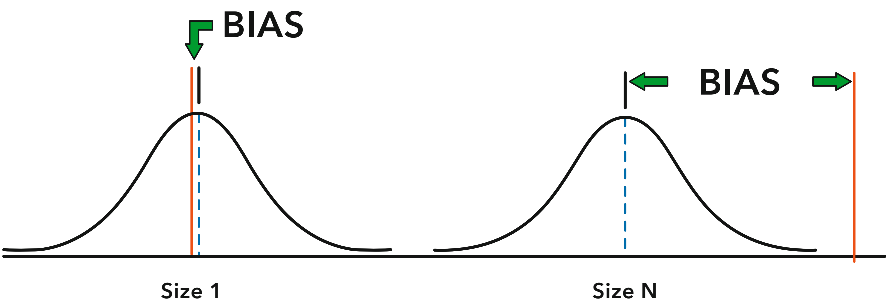
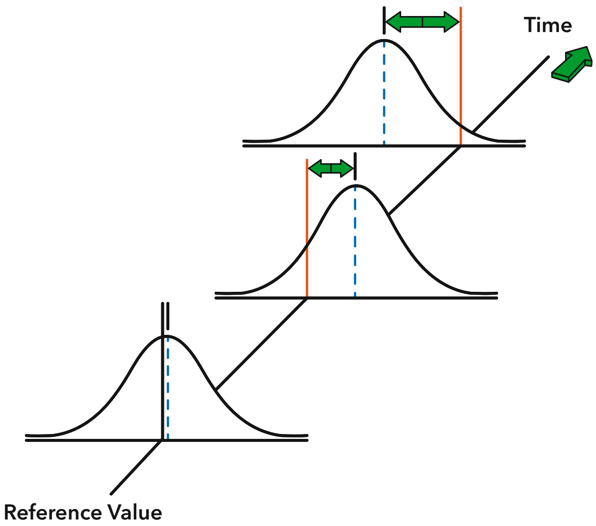


Fig. 30.7b Linearity. (Redrawn similar to MSA-2010, p. 6)

[10] Associating a specific level of confidence requires certain assumptions regarding the probability distribution, characterized by the measurement result and its standard deviation. Normal distribution is often applied as a principle assumption for MS, this explains the shape of the curves in the figures.

Fig. 30.7c Stability. (Redrawn similar to MSA-2010, p. 6)



- **Precision:** "Closeness" of repeated readings to each other.

The precision of an MS is appraised according to two components.

Repeatability studies quantify the variation in measurements obtained with one gage[11] when used several times by an appraiser who measures an identical characteristic on the same part over a short time period.

Commonly referred to as *E.V.* – Equipment Variation (Fig. 30.8a).

Reproducibility studies quantify the measurement average variation of measurements made by different appraisers using the same gage when measuring a characteristic of one part.

Commonly referred to as *A.V.* – Appraiser Variation (Fig. 30.8b).

The extent to which an MS possesses these qualities is the subject of various praxeologies. One example is presented below. The results provide the basis for the institutional validation of the system.

Example of praxeology: A Gage Repeatability and Reproducibility (GRR) study

A GRR study helps investigate:

- Repeatability: how much of the variability in the MS is caused by the gage?
- Reproducibility: how much of the variability is due to differences among operators?
- Whether the MS variability is small compared to that of the production process.
- Whether the MS can differentiate among distinct parts.

Various techniques are available for conducting studies of this kind (see MSA-2010 for a detailed presentation). We focus on the Average and Range technique used by the engineers that Aldape-Carillo (2016) interviewed in the automobile factory.

[11] Measurement device

Fig. 30.8a Repeatability. (Redrawn similar to MSA-2010, p. 7)

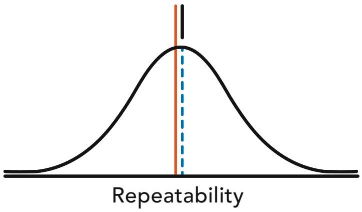
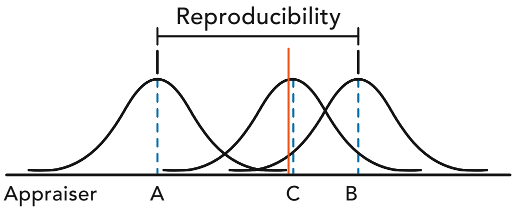


Fig. 30.8b Repeatability. (Redrawn similar to MSA-2010, p. 7)



Description of the technique

- Obtain a selection of n parts ($n \geq 10$) that represent the expected range of the production variation of a given part.[12] Three operators (A, B, C) measure the n parts, three times per part, in random order.
- Calculate the average and range (largest reading - smallest reading) of the three readings for each part and each appraiser.
- Calculate the average ($\overline{R_A}$, $\overline{R_B}$, $\overline{R_C}$) of the ten ranges and the average ($\overline{X_A}$, $\overline{X_B}$, $\overline{X_C}$) of the ten averages for each appraiser.
- Calculate the average of the 9 readings for each part and the average $\overline{\overline{X}}$ of these 10 results, which is also the average of $\overline{X_A}$, $\overline{X_B}$, $\overline{X_C}$. Then determine the range of the part averages (Rp), i.e., the largest part average – the smallest one.
- Calculate the average $\overline{\overline{R}}$ of $\overline{R_A}$, $\overline{R_B}$, $\overline{R_C}$ and the range of the averages $\overline{X_A}$, $\overline{X_B}$, $\overline{X_C}$ $\left(\overline{X_{DIFF}}\right)$

Once collected (see Table 30.3), the data are subjected to graphical and numerical analysis. We focus on the second dimension.

$\overline{\overline{R}}$ quantifies the average variation of three measures regardless of the part and the appraiser, $\left(\overline{X_{DIFF}}\right)$ quantifies the variation among appraisers. These figures are used to estimate, respectively, the MS repeatability variation EV and reproducibility variation AV using formulas not shown here due to space limitations. We simply state that they entail constants derived from the probability distribution of the statistic parameters.[13]

[12] This choice supposes that the enterprise's metrology laboratory has a good knowledge of the range of variation of a given product.

[13] See (Clément, 2017), p. 15.

Table 30.3 Example of a completed GRR data collection sheet (MSA-2010, p. 118)

	PART										
Appraiser/Trial nb	1	2	3	4	5	6	7	8	9	10	Average
A 1	0.29	−0.56	1.34	0.47	−0.80	0.02	0.59	−0.31	2.26	−1.36	0.194
2	0.41	−0.68	1.17	0.50	−0.92	−0.11	0.75	−0.20	1.99	−1.25	0.166
3	0.64	−0.58	1.27	0.64	−0.84	−0.21	0.66	−0.17	2.01	−1.31	0.211
Average	0.447	−0.607	1.260	0.537	−0.853	−0.100	0.667	−0.227	2.087	−1.307	$\overline{X}_a$=0.1903
Range	0.35	0.12	0.17	0.17	0.12	0.23	0.16	0.14	0.27	0.11	$\overline{R}_a$=0.184
B 1	0.08	−0.47	1.19	0.01	−0.56	−0.20	0.47	−0.63	1.80	−1.68	0.001
2	0.25	−1.22	0.94	1.03	−1.20	0.22	0.55	0.08	2.12	−1.62	0.115
3	0.07	−0.68	1.34	0.20	−1.28	0.06	0.83	−0.34	2.19	−1.50	0.089
Average	0.133	−0.790	1.157	0.413	−1.013	0.027	0.617	−0.297	2.037	−1.600	$\overline{X}_b$=0.0683
Range	0.18	0.75	0.40	1.02	0.72	0.42	0.36	0.71	0.39	0.18	$\overline{R}_b$=0.513
C 1	0.04	−1.38	0.88	0.14	−1.46	−0.29	0.02	−0.46	1.77	−1.49	−0.223
2	−0.11	−1.13	1.09	0.20	−1.07	−0.67	0.01	−0.56	1.45	−1.77	−0.256
3	−0.15	−0.96	0.67	0.11	−1.45	−0.49	0.21	−0.49	1.87	−2.16	−0.284
Average	−0.073	−1.157	0.880	0.150	−1.327	−0.483	0.080	−0.503	1.697	−1.807	$\overline{X}_c$= − 0.2543
Range	0.19	0.42	0.42	0.09	0.39	0.38	0.20	0.10	0.42	0.67	$\overline{R}_c$=0.328
Part average	0.169	−0.851	1.099	0.367	−1.064	−0.186	0.454	−0.342	1.940	−1.571	$\overline{\overline{X}}$= 0.0014 R_P=3.511
$([\overline{R}_a = 0{,}184] + [\overline{R}_b = 0.513] + [\overline{R}_c = 0.328])/[nb\ of\ appraisers = 3] = 0.3417$											$\overline{\overline{R}}$=0.3417
$[Max\,\overline{X} = 0.1903] - [Min\,\overline{X} = -0.2543] = \overline{X}_{DIFF} = 0.4446$											

The MS variation for repeatability and reproducibility (*GRR*) is calculated by the following formula:

$$GRR = \sqrt{(EV)^2 + (AV)2}$$

The part variation PV (part-to-part variation without measurement variation) is estimated from the range of part averages R_p. Total variation *TV* is then calculated by:

$$TV = \sqrt{(GRR)^2 + (PV)2}$$

The following standard[14] can be used to determine whether the MS is acceptable:

- GRR under 10% of *TV*: acceptable MS
- GRR over 30% of *TV*: MS needs improvement.
- Between 10% and 30%, decision will depend on use.

In this example, GRR = 26,68% of TV, so the MS will likely not be validated without improvement.

In the praxeology shown, variation is quantified on the basis of range. The Analysis of Variance method, in contrast, uses variance and standard deviation, it is preferred because of its flexibility, as long as the user has access to appropriate software and a solid grounding in statistics.

30.4.3 Process Capability Analysis

We are now interested in validating a new manufacturing process, based on the appraisal of a capability index that is considered to quantify the extent to which the process meets certain specifications. Of course, the MS must be previously appraised under what may be considered as reference conditions. For the process stability being checked, the statistical parameters of the process are estimated through a Repeatability and Reproducibility study with Analysis of the Variance. The capability of the process is quantified by the following index where σ is an estimate of the standard deviation (see Fig. 30.9 for an illustration):

$$Cpk = \frac{distance\ from\ mean\ to\ the\ nearest\ specification\ limit}{3\sigma}$$

The process is appraised as being "capable" if *Cpk* is greater than 1.33.

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[14] Automotive Industry Action Group (AIAG) sustains these positions. They appear in MSA-2010: Table II-D 1.

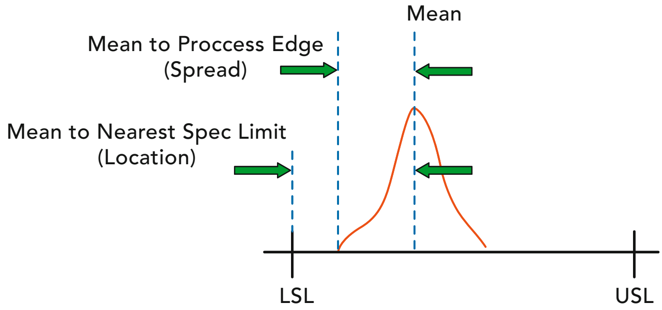

Fig. 30.9 Interpretation of the index Cpk. (Redrawn similar to 1.Factory: https://www.1factory.com/quality-academy/guide-to-process-capability-analysis-cp-cpk-pp-ppk.html)

Example of praxeology: Improving the process through graphic analysis

As mentioned above, this index is relevant for stable processes, but when a new process is being implemented, it may show variations deemed abnormal because they do not show a random distribution. Highlighting such anomalies to identify and correct the causes, if possible, at the moment they exert a negative effect on production, is what (WE-1985) calls a "Process Capability Study", an approach based, essentially, on producing Control Charts (introduced by W.A. Shewhart, a statistician working in the R & D branch at AT & T, see Clément, 2017, p. 10). For each one of at least 20 samples of the same number n of parts with probable homogeneity, i.e., produced over a short period, the average and range of the n measures are calculated and plotted to obtain Average & Range charts, which (WE-1985) posits as the most sensitive indices. The grand average $\overline{\overline{X}}$ and control limits (UCL and LCL for Upper and Lower Control Limit) determined using the average range[15] are also plotted. See Fig. 30.10 for an example (5 units samples every hour for 20 h) (Fig. 30.10).

These maps are examined for anomalies. What is a normal profile? First, its points fluctuate randomly, so there is no recognizable order. Moreover, since the values of distributions tend to cluster near the mean, most points on the control graph will naturally fall close to the $\overline{\overline{X}}$ line and balance on either side. Finally, since most distributions have "tails" up to $\pm 3\sigma$, it is natural that a point on the graph sometimes approaches a limit of 3σ. In this case, the process is considered stable. The profile is considered abnormal if one of these three characteristics is missing. Western Electric has laid down rules for distinguishing among different profiles. We illustrate some of them in Fig. 30.11a–c.

The first meaning of an abnormal profile is that important causes with the capacity to exert a great influence on the process are present in a form that can be studied (WE-1985, p. 42). Abnormal variations produce significant profiles on a control chart that, in turn, allows detection and studies of the cause-and-effect relations

[15] See (Clément, 2017), p. 10.

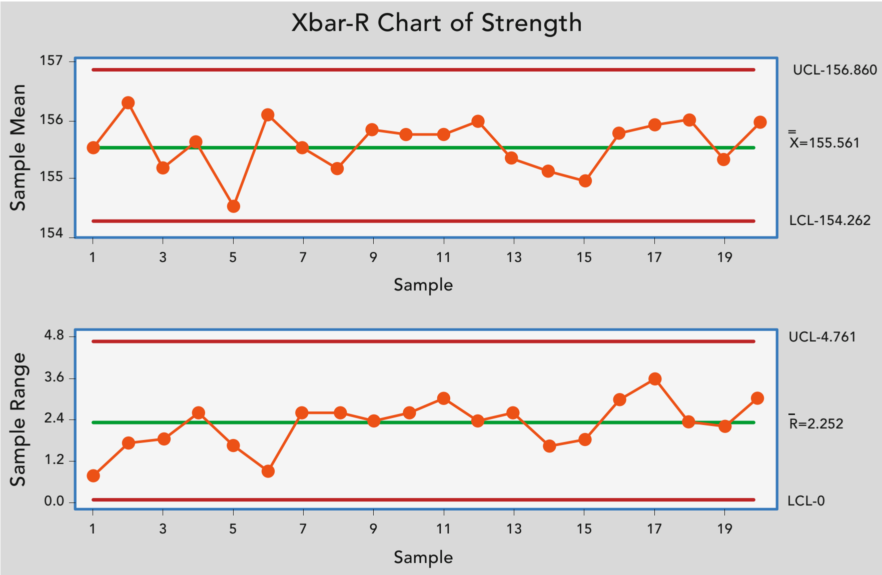

Fig. 30.10 Average & Range charts. (Redrawn similar to Minitab 18 Support: https://support.minitab.com/en-us/minitab/18/help-and-how-to/quality-and-process-improvement/control-charts/how-to/variables-charts-for-subgroups/xbar-r-chart/before-you-start/overview/)

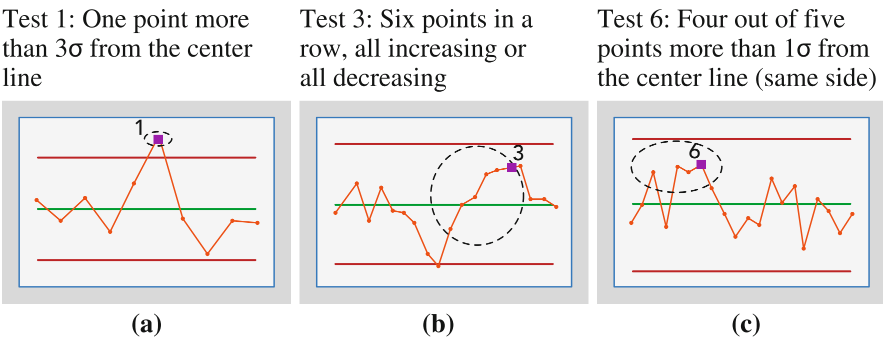

Fig. 30.11 (**a**–**c**) Abnormal profiles. (Redrawn similar to Minitab 19 Support)
[a]The rules give rise to 8 tests in the software Minitab. https://support.minitab.com/en-us/minitab/19/help-and-how-to/quality-and-process-improvement/control-charts/how-to/variables-charts-for-subgroups/xbar-r-chart/interpret-the-results/all-statistics-and-graphs/#control-limits

involved. Based on their knowledge of the process, and of the kinds of causes associated with abnormal profile types, the engineers and technicians who conduct the study can make hypotheses about the possible causes of the anomalies. After correcting one of these causes, a new sampling will generate new maps, possibly easier to interpret because fewer causes are superimposed. WE-1985 (p. 48) presents an example: the study included 5 samplings, each followed by a development applied to the production process, until an acceptable capability index resulted. (WE-1985) advocates that such studies, in lighter forms, should be performed regularly in workshops for adequate process monitoring.

In summary, we have shown that measurement systems and production processes are appraised through analyses of certain qualities for which the user institutions define standards to be met. If necessary, cycles of appraisal and improvement will be conducted recursively until validation is possible. Concerning the praxeologies of appraising, improving, and validating implemented in this field, we argue that the techniques are not entirely discursive since they require sampling. Moreover, and despite the absence of a specific study, we assume that the *logos* has a strong statistical component and contains empirical knowledge on the manufacturing reality (θu in Fig. 30.6). Finally, it is important to recognize that the list of expected qualities presented here is not exhaustive because, for example, it leaves out the dimensions of profitability and duration.

30.5 Making the PageRank Algorithm Intelligible

Section 30.4 focused on analyzing the *praxis* component of the praxeologies that correspond to the appraising, improving, and validating task types, in an effort to show that they are not entirely discursive. This section, therefore, focuses on the task types that Sect. 30.2 presented as discursive: describing and above all motivating and explaining, two aspects of making intelligible. Note that these task types contribute to the transmission of the technique, they fulfill didactical functions.

Our example is based on the work Patricio-Martínez (2016) has devoted, in his Master thesis, to the PageRank algorithm, in order to design mathematical modeling activities that make it possible to link a modeling activity, derived from professional practice, to mathematics teaching for computer systems engineering students.

What is the task type that the PageRank algorithm proposes to solve? A search engine determines all the sites where the object of the search appears, but because it must be able to choose among the multitude of sites found, it has to rank them. PageRank was developed “to improve the quality of web search engines” (Brin & Page, 1998, p. 108) and help users find answers in the first 10 sites proposed. This entails working on the ranking criteria.

The main difficulty of Patricio-Martínez’s project lies in the secrecy imperatives that govern R & D institutions in cutting-edge competitive sectors. In their article published in a research review, Brin and Page (1998), PageRank’s developers, revealed very little about the algorithm. Therefore, Patricio-Martínez had to rely on documents written by mathematicians for clearly didactic purposes, that show the mathematics hidden behind PageRank (D’Andrea, 2020[16]; Fernández, 2004). These

[16]Patricio-Martinez has worked with a 2012 version of d’Andrea’s paper, which is no longer available. We refer to a 2020 version, similar to the previous one, edited in a review. English translations of the citations are ours.

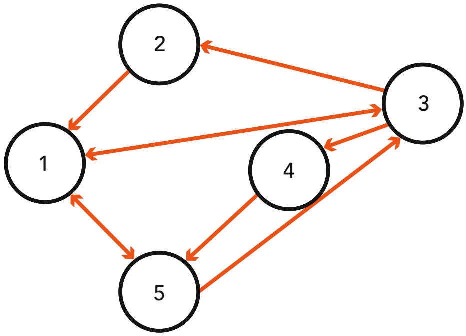

Fig. 30.12 The internet graph. (Redrawn similar to D'Andrea, 2020, p. 27)

texts propose a reconstruction of the steps that led to the development of the algorithm. In this didactic *logos* on the Pagerank technique, the task types Describe, Appraise and Motivate occupy a central place, with the results of the evaluation providing the rationale for successive adaptations of the model. To illustrate this, we rely on D'Andrea (2020).

The technique may be described through a list of subtask types (see Chaachoua et al., 2019):

T1: Representing the Web structure mathematically.

In PageRank, the web is a directed graph in which the websites are the nodes and the links between them are the edges (see Fig. 30.12).

T2: Defining a process to assign an importance score (or PageRank) to the sites.

The key issue here is to define a criterion of importance. The appraising-improving process gives rise to four successive proposals described below.

One potential criterion would be to count, equally, the citations that a website receives from other websites. D'Andrea appraises this criterion as one that can be manipulated easily because "one could quickly 'inflate' the importance of a given website by simply creating several websites that have links to the same website" (*ibidem*, p. 28). This negative appreciation motivates the introduction of a second criterion, this one based on PageRank's postulate that "the importance x_i of page P_i is directly proportional to the sum of the importance of the websites linking to it." (*ibidem*, p. 28). This is illustrated by a new graph (see Fig. 30.13) where "C" is more important than "F" though both count one citation since "E" is less important than "B".

D'Andrea proposes moving to a matrix representation, *motivated* by the inefficiency and huge computational cost of a graph representation for the number of websites involved. He thus introduces the so-called incidence matrix of a graph, a classical tool of graph theory. M_I is defined as the square matrix of size equal to the number of nodes in the graph, so that $m_{i,j} = 1$ if there is a link from the page P_i to the page P_j and 0 if not.

This matrix may be used to represent the system of linear equations that derives from the PageRank postulate: "If M_I is the incidence matrix of the Web graph and

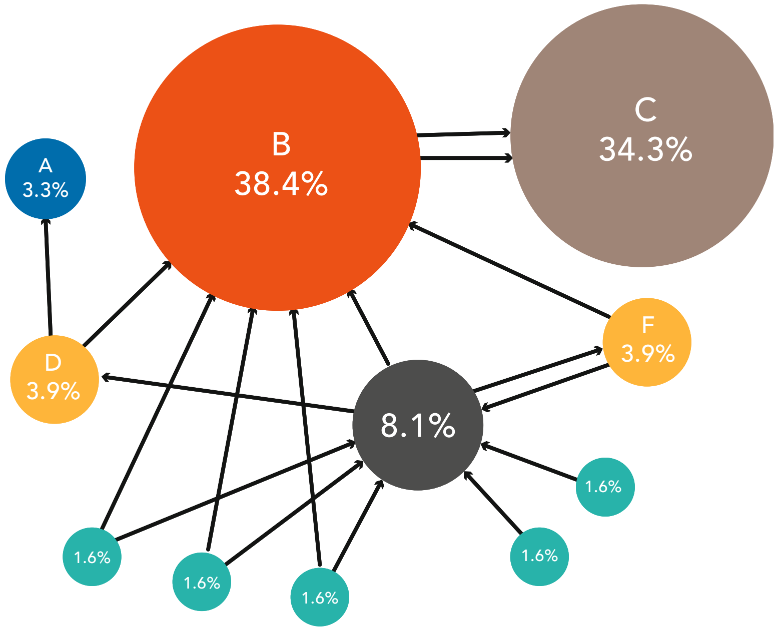

Fig. 30.13 Importance graph. (Redrawn similar to D'Andrea, 2020, p. 28)

$x = (x_1, \ldots, x_n)$ the importance vector in $(R^+)^n$, then $M_I^t x^t = \lambda x^t$, where $\lambda \in R^{+*}$ is the proportionality constant." (*ibidem,* p. 29). At this point, D'Andrea introduces mathematical technological elements: definition of eigenvector and eigenvalue. He then states the following property: the importance vector is an eigenvector of M_I that corresponds to a strictly positive eigenvalue (Fig. 30.14).

After experimenting with this technique for the example shown above using a software to determine the five eigenvalues and corresponding eigenvectors, D'Andrea inquires as to the unicity of the eigenvalues related to a single eigenvector:

> One might assume that what happens in this example is a general fact, that of any square matrix with zeros and ones there will be a single positive eigenvalue, and associated with it a single positive eigenvector which will be the solution to our problem (*ibidem*, p. 30).

We may assume that unicity allows PageRank to avoid the need to choose among several solutions. Regarding the eigenvector, the answer lies in the following remark by Brin and Page (1998, p. 110): "PageRanks forms a probability distribution over web pages, so the sum of all web pages' PageRanks will be one".

A third proposition is introduced, motivated by the following negative appreciation of the previous proposition: "if a page has only one link, this link is worth the same as any other link from another page that produces a million links" (D'Andrea, 2020, p. 31). In the incidence matrix, the $m_{i,j}$ term is divided by the number of links leaving from *Pi.* The new matrix denoted $M_{I,\ E}$ is a right stochastic one in which the sum of each line terms is 1.

$$\begin{cases} 0\cdot x_1 + 1\cdot x_2 + 1\cdot x_3 + 1\cdot x_4 + 1\cdot x_5 = \lambda x_1 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 1\cdot x_4 + 1\cdot x_5 = \lambda x_2 \\ 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 1\cdot x_4 + 1\cdot x_5 = \lambda x_3 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 = \lambda x_4 \\ 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + 1\cdot x_4 + 0\cdot x_5 = \lambda x_5, \end{cases} \qquad \mathbf{M}_0^t\mathbf{x}^t = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^t \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \lambda \cdot \mathbf{x}^t.$$

Fig. 30.14 Linear Equations and matrix representation associated with Fig. 30.12. (Redrawn similar to d'Andrea, 2020, p. 30)

$$\mathbb{M}_{I,E}^{\varepsilon} = (1-\varepsilon).\,\mathbb{M}_{I,E} + \frac{\varepsilon}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}$$

Fig. 30.15 The perturbed incidence matrix. (d'Andrea, 2020, p. 34)

This brings us back to the issue of unicity, for the third proposition must now be appraised on its ability to always provide matrices with at least one simple, strictly positive eigenvalue (1), with one associated eigenvector with strictly positive coordinates (2). If there are several such values, a criterion must be proposed to select one. One theorem from Perron-Frobenius's theory provides a selection technique (rank the values in descending order of their modulus) and sufficient conditions for successful use of the matrix: one that has only strictly positive entries has a unique eigenvalue strictly larger in modulus than the other ones (dominant eigenvalue) and satisfying (1) and (2) conditions. But the incidence matrices do not meet these conditions because some entries $m_{i,j}$ are equal to 0.

This motivates the proposition to slightly perturb the mathematical model, a rather common procedure in computational mathematics and numerical linear algebra. Thus, a convenient matrix is added to the incidence matrix to obtain a matrix with only strictly positive entries. The perturbed matrix is defined as follows, with ε being a very small strictly positive real number (Fig. 30.15).

The fact that such perturbation provides a correct technique for the Ranking task type is explained as follows:

> The underlying principle of this idea is that the "importance" function is continuous, and if I can calculate it "close" to the situation where I am, it is already enough for what I want, which is to order the importance and not actually calculate them (*ibidem*, p. 34).

This, the final evolution of the technique presented by D'Andrea, returns us to the last subtask type.

T3: Calculating the site's importance

D'Andrea notes that this calculation is extremely large, so that techniques from a linear algebra course cannot be used. He explains that PageRank uses the 'power method', an algorithm which, in the case of a diagonalizable matrix with a dominant eigenvalue λ, produces a number and a non-zero vector, which approximate λ and its eigenvector. The elements of mathematical *logos* provided allow us to understand the need to work with a dominant eigenvalue. However, one question remains: is it possible to choose the perturbed matrix so that it is always diagonalizable?

What should be highlighted in this example of didactic transposition related to PageRank?

It is not possible to assess the distance of this reconstruction from PageRank because of secrecy constraints. Moreover, the reconstruction is not unique: "Several options exist for modeling the behavior of a random Web surfer after landing on a dangling node,[17] and Google does not reveal which option it employs." (Wills, 2006, p. 6). This is not important because the rationale of the transposition project is to teach certain mathematical knowledge. However, the logos is not limited to theoretical mathematics. Rather, if one pays close attention to the motivation behind the successive changes to the model, it is possible to observe appraisal steps that are largely based on practical knowledge related to the Web and computational practices. Finally, we should mention a technique not found in D'Andrea (2020): the use of analogies to make the technique intelligible. For instance, Fernández (2004) refers to the way in which play-offs are determined in the USA to show the interest in "normalizing" the importance of the sites.

30.6 Conclusion

This chapter draws on the hypothesis that every institution develops the following epistemic activities: designing-transforming, legitimating (Castela, 2020), and disseminating praxeologies. Sections 30.4 and 30.5 illustrate seven epistemic task types relating to the *praxes* used in an institution: Describing, Motivating, Explaining, Appraising, Improving, Validating, and Monitoring. Two dialectics are especially highlighted: Appraising-Improving in designing-transforming activities and Appraising-Motivating in didactical ones. The praxeologies so derived take specific forms that reflect institutional conditions and constraints: nature of the institutional objectives (Sect. 30.4: industrial production, Sect. 30.5: teaching mathematics), product specifications, role of measurement, conditions variability, secrecy constraint. The techniques applied may involve mathematical knowledge as evidenced by the didactic resources provided by the industrial companies, this suggests possible developments for mathematics education in higher education. These possibilities, unfortunately, can hardly be described as straightforward, due to the phenomenon of transposition that goes hand-in-hand with the circulation of praxeologies (Sect. 30.3).

[17] Both Brin and Page (1998) and Fernández (2004) mention this analogy to a "random surfer" who clicks on a page and then randomly links to another one.

Each institution has its own epistemic regime. We assume that this hypothesis explains the difficulties encountered in efforts to design didactic activities for higher education based on inter- and transdisciplinary approaches (Klein, 2013; Jao & Radakovic, 2018; Roth, 2020; Takeuchi et al., 2020) and mathematical modeling perspectives (Kaiser, 2020). We defend the idea that mathematics teachers should collaborate with representatives of different institutions of production and use of the knowledge involved (Schmidt & Winsløw, 2021; Siero et al., 2017), but this is no easy task due to institutional constraints. Hence, we advocate for further research on the mathematics used in workplaces, such as Frejd and Bergsten (2016), Gainsburg (2007), and of course, the three theses by Mexican students on which this chapter is based. We sustain that this approach will provide mathematics teachers with a better knowledge of institutions using mathematics in non-academic ways such as other sciences and professions. In this way, they might design, even (almost) alone, sequences based on concrete, genuine problems that occur in those institutions. Clearly, this a complex task, but one that is unavoidable in the training of professionals in light of the demands of the twenty-first century.

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Chapter 31
Concept Images of Signals and Systems: Bringing Together Mathematics and Engineering

Margret A. Hjalmarson, Jill K. Nelson, John R. Buck, and Kathleen E. Wage

Abstract This paper will examine students' reasoning with conceptual problems in signals and systems, a subfield of electrical engineering. In particular, we study how students consider multiple representations that model real phenomena. Also, signals and systems problems require students to think about multiple representations of the same signal, namely interpreting amplitude variation as a function of time and energy variation as a function of frequency. We aim to understand how students connect representations across different domains for the same signal. We adapt a framework to describe representational contexts and process-objects to signal and systems concepts. While some of these mathematical applications are unique to electrical engineering, this provides one example of exploring students' concept images that include applications of mathematics.

Keywords Electrical engineering · University education · Engineering education · Student learning · Mathematics learning

31.1 Background

Mathematics is "the queen of the sciences" according to a quote widely attributed to Carl Friedrich Gauss. This is sometimes interpreted as mathematics being necessary for all the sciences. However, the sciences and, in our case, engineering have their

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own influence and interpretation of mathematics. The relevant purposes and structures of mathematics may be different for other disciplines. The engineering student is learning mathematics as a necessary tool for engineering - not for the sake of mathematics itself. In addition, advanced engineering coursework includes mathematical content taught from the perspective of its role in engineering. We use the framework of concept image (Tall & Vinner, 1981) to adapt Zandieh's framework for describing derivatives to signals and systems concepts. Zandieh's framework provides the opportunity to describe the different aspects of signals and systems in terms of different representations or contexts (e.g., graphical, symbolic). The framework for derivatives is useful for adapting to signals and systems because they have a clear physical form that is characterized using symbolic and graphical contexts.

Signals and systems problems require students to consider multiple representations of the same signal, namely interpreting amplitude variation as a function of time and energy variation as a function of frequency. We aim to characterize how students employ representations across different domains for the same signal and how they also make connections to applications they will need as engineers. Overall, our chapter explores the following. How do students interpret, describe and reason with graphical representations of signals and systems problems as part of a concept image of signals and systems?

31.2 Literature Review

Our study explores students' conceptual understanding of signals and systems. More specifically, the students need to interpret graphical representations that describe different aspects of a signal. The connection between graphical representations and engineering is compelling as an example of the intersection between the disciplines because signals and systems (and electrical engineering more generally) is a particularly mathematics-intensive part of engineering.

31.2.1 Engineering and Mathematics

Faulkner et al. (2019) studied the views of engineering faculty about their definitions of mathematical maturity for students. They found three themes that were important practices of mathematically mature students: (1) "uses and interprets mathematical models", (2) "chooses and manipulates symbolic and graphical representations" and (3) "Computational tools reshape 'what needs to be known' to be mathematically mature" (p. 111). For our purposes, we are most interested in the first two. The representation of a signal could be described as a model consisting of both a symbolic and a graphical representation. This model also represents a real phenomenon and attempts to describe a real situation. However, students need understanding of the related symbolic and graphical representations in order to understand the phenomenon. The engineering faculty perspective on the role of mathematics in their study affirms both the disciplinary needs and the necessity of mathematical reasoning as part of engineering.

From the student perspective, Harris et al. (2015) describe how students understand the role of mathematics in engineering programs. Some students were surprised at the extent of the mathematics required both in terms of mathematics courses and how much mathematics is in engineering courses. At the same time as mathematics was seen as essential, the students were also seeking to better understand the connections and applications of mathematics to engineering and felt it would support their learning to have those connections made explicit in courses (e.g., via engineering examples in mathematics courses). Similarly, Flegg et al. (2012) found variation in how students identified the relevance of mathematics to engineering either in terms of engineering coursework or later professional practice.

There has not been extensive investigation into students' conceptions of signals and systems. However, other studies of students' conceptions of signals examine the different praexologies of mathematics and engineering (Hochmuth & Peters, 2021; Rønning, 2021). For instance, textbooks for mathematics and engineering describe or foreground different concepts (Hochmuth & Peters, 2021). Coppens et al. (2017) explored students' understanding of frequency and phase shifts, and they point to students being able to define phase shifts but not being able to consider the physical context. A common theme in all of the above studies is how students need to access different contexts for considering signals and systems concepts including physical applications, mathematical representations, and engineering representations.

31.2.2 Use of Representations as Contexts

The use, creation and interpretation of multiple representations for phenomena is a common practice of engineering and mathematics. Different frameworks for working across different representations include the Lesh translation model which includes symbolic, graphical, real-world phenomenon, written descriptions, and images (Cramer, 2003). In this model, the focus is on how the same phenomenon can be represented in different ways that might highlight some aspects of the phenomenon and obscure other aspects. Mathematical representations in this sense are a key component of modeling, which is often a goal in engineering work (Magana et al., 2020; Magana & de Jong, 2018). Zandieh characterizes these different representations as "contexts" (2000) including symbolic, physical, verbal, and graphical. These contexts are then part of students' concept image of a topic.

Other studies have examined how students use representations in engineering tasks. For example, Johnson-Glauch and Herman investigated representations in statics and digital logic (2019). In a second study, they investigated students' conceptual knowledge and the connections to domain-based representations (Johnson-Glauch et al., 2020). A salient recommendation from their study is "Our findings suggest that we should be careful about documenting only students' misconceptions without considering the factors that may prompt them and the useful knowledge that students may be employing." (p. 460, 2020) Their study explored which aspects of a representation students found "task-relevant" as part of understanding how students' interpret and use such representations in problem-solving. Their study also explored the role of disciplinary notational conventions in obscuring or highlighting task-relevant features of representations. In signals and systems, this could include how students interpret both the symbolic representations and the graphical representations. Part of their findings include updating the Lesh translation model to include salient representations for engineering.

31.3 Signals and Systems Courses in Engineering

Signals and Systems, also sometimes called Linear Systems, are electrical engineering classes focusing on the mathematical representations of physical signals and the manipulations of these signals by linear and time-invariant systems. Within the engineering curriculum, these classes bridge the theory students learn in differential equations with the implementation learned in circuits. These classes present common techniques used to embed, transmit, and extract the information in signals. In physical terms, these signals include, for example, electromagnetic and acoustic waves, temperatures, tides and sea level, and populations of healthy or infected individuals. These techniques for manipulating information are critical infrastructure for much of the modern information economy, as well as foundational technologies like AM and FM radio, radar, sonar, and GPS. In mathematical terms, signals are represented by functions where the independent variable most commonly represents time. Systems are operators on these functions. Most, but not all, of the systems studied, belong to the class of linear time-invariant systems. These systems are the subset of linear operators represented by a Toeplitz matrix, possibly of infinite dimension. For continuous-time signals and systems, the domain of the signal function is the set of real numbers.

Frequency domain representations are a central theme in signals and systems classes. Students learn dual representations both for representing signals and the behavior of systems. Central to these representations is the role of complex exponentials as eigenfunctions of linear time-invariant systems. Students gain practice with properties of frequency representations commonly used in filters and modulation systems. For example, the convolution operator in one domain corresponds to Hadamard multiplication in the dual domain. In engineering systems, such filters can be designed to keep signals at some frequencies while removing others (e.g., an ideal lowpass filter). This property underlies the filters selecting a single television channel or mobile phone user from among many simultaneous transmissions.

As engineering classes, signals and systems classes explore practical implementations as well as the ideal mathematical systems. These classes encourage students to develop rich concept maps providing them with the tools to reason with dual representations such as time and frequency. When successful, students can use the intuition developed in these classes to reason about the behavior of signals and systems without explicit mathematical equations or formulas for the systems. This framework empowers students to make sound strategic choices about which domain simplifies the analysis of a given system or signal: "When the going gets tough, the smart switch domains." In other words, a problem often lends itself to the time domain or the frequency domain; knowing which domain allows for an elegant solution is critical to mastering signals and systems material. This conceptual and graphical reasoning is frequently an important basis for innovation in signal processing.

A first course in continuous-time signals and systems requires differential equations as a prerequisite. Students are also expected to have a firm grasp of complex numbers and to have experience with phasors and with representing complex numbers in the complex plane. The topics in the signals and systems courses considered in this study are designed to help students master the analysis of signals and systems via differential equations, the Laplace transform, and the Fourier transform. In a typical schedule, the course would begin with system properties and convolution, all treated in the time domain. Fourier series would follow to introduce the frequency domain and filtering. Fourier series would then be extended to the Fourier transform, and properties of the Fourier transform (notably the convolution property) would be studied. Interpretation of magnitude and phase of the Fourier transform, in particular what magnitude and phase of the frequency response tell us about a system, would also be studied. The Laplace transform, its application for computing system outputs, and its relationship to differential equations, may be introduced before, during, or after the study of Fourier transforms depending upon the textbook used and instructor preferences.

As mentioned above, continuous-time signals and systems can be viewed as an application of differential equations characterize signals and systems. Analog circuits are the most common physical realization of signals and systems in the electrical engineering curriculum. How signals and systems and circuits courses are sequenced in the curriculum varies across institutions, but students are often taking an introductory circuits course concurrently with or immediately before/after a signals and systems course. As such, students are building connections among the mathematical foundations of differential equations; application of differential equation modeling to continuous-time signals and systems; and design, implementation, and analysis of physical signals and systems through analog circuits. In addition, they are learning new domains (Fourier and Laplace) and the use of those domains in signal/system analysis. Anecdotally, students are often most comfortable working in the time domain, as it has been the predominant domain in their math courses and in mathematics applications throughout their education. Hence, helping students achieve fluency working in multiple domains and selecting the best domain for a particular task is an important objective of a signals and systems course. The extent to which students are explicitly challenged and encouraged to draw connections across domains and representations (equations, graphs) depends upon the instructor; helping students learn to build these connections was a primary goal for the instructors teaching the signals and systems courses considered in this study.

31.4 Interviews and Analysis

This study considers interviews conducted as part of an effort to better understand students' thinking when solving problems in the Signals and Systems Concept Inventory (SSCI) (Wage et al., 2005). The 47 participants had recently taken a signals and systems course, typically in the second or third year of their undergraduate coursework in electrical engineering. The students interviewed were from two different institutions involved in the study. The interviews were conducted over two consecutive years, so the students were not necessarily from the same section of the course even if they were enrolled at the same institution. The interviewers were faculty in electrical engineering; they were the designers of the SSCI and are three of the co-authors of this paper. The interviewers did not interview students from their own classes, but given the technical nature of the problems, we found it important for interviewers to have the relevant expertise to be able to prompt students for more information or respond to students' questions.

The interview protocol included two parts: a set of signals and systems problems and a set of open-ended questions about students' perceptions of the content, the tasks, and the course. The interview protocol was initially designed as a cognitive interview using tasks from the Signals and Systems Concept Inventory (Buck et al., 2007; Wage et al., 2005, 2006) in order to understand students' interpretations of the tasks. We describe the problems as conceptual because they required students to interpret graphs and symbolic representations without the necessary numerical values to perform calculations or write equations. We explore later how students interpreted the lack of ability to write equations or perform calculations, but this structure forced them to reason with the graphs and explain what they noticed about them that illustrated their conceptual understanding rather than use of procedures or algorithms. This is similar to how Engelbrecht et al. characterize conceptual vs. procedural tasks (2012).

Other analyses of this set of interview transcripts have focused on the concepts that students found difficult or important (Nelson et al., 2010) and what can be learned from the interviews to help instructors consider how to teach signals and systems courses. Using the transcripts of the interviews, we looked for instances where students specifically discussed graphing and how they were making sense of the content. Our focus in this analysis arises from things we noticed in the process of prior analyses that focused on the correctness of students' reasoning and their approaches to reasoning through the tasks. The two lead authors conducted an initial, open coding of the interviews looking first for evidence of students' use and interpretations of graphing and their perceptions of the mathematics in the course. The first author focused on the open-ended questions and the second author focused on the signals and systems tasks. They discussed the themes that were emerging and brought together their perspectives as a mathematics educator and an electrical engineer. The third and fourth authors are electrical engineering faculty who designed the SSCI and participated in earlier analyses of the transcripts.

Excerpts were identified from a set of the tasks themselves and the general questions at the end of the interview using Dedoose. For their perceptions of mathematics, graphing and their relationship to engineering, students described this most often when explaining the most important and most difficult concepts in the course. Given this is not something we explicitly asked students about but was emergent, we do not make claims about how many students might hold these epistemic beliefs about mathematics and engineering knowledge. Rather, we point to themes that emerged across multiple students or comments that seemed to point in directions for further investigations. Zandieh's (2000) framework was introduced later as a framework for organizing the findings in terms of the major concepts of signals and systems and the associated contexts or representations.

All 47 interviews had at least one excerpt we initially coded as "graphing" to indicate they were discussing an aspect of graphing, and there were 215 excerpts coded as "graphing". Following the initial coding, we refined the coding scheme and coded excerpts for three types of graphing comments: connect graphs across domains ($n = 73$), connect math to visual ($n = 28$), connect visual to phenomenon ($n = 88$). Only one interview did not have at least one of these codes assigned to an excerpt. All other transcripts had an excerpt with at least one of these sub-codes. In a different coding from the graphing focus, we also used the code "real world applications" for any excerpts ($n = 39$) where students described connections (or lack of connections) to real world problems or real engineering work. The graphing excerpts and the real-world applications excerpts provided a sense of their concept images and point to a few intriguing aspects of their thinking about the concepts as well as comments that were less frequent but still illuminating.

31.5 Signals and Systems Concept Image and Conceptual Problems

31.5.1 Concept Image for Signals and Systems

For our study, we suggest an interpretation of the components of signals using a framework adapted from Zandieh's characterization of the derivative (2000). In her paper, she explains two dimensions of the concept image: process-objects and contexts. She includes three process-objects within the layers of her derivative framework: ratios, limits, and functions. We examine two layers in our study for signals and systems: frequency and filtering. Zandieh defines process-objects as structures that can be interpreted both as actions or dynamic processes and as objects. This sense of dynamic action (or process) paired with an object is useful for signal processing tasks because each component can be considered to be dynamic and as an object. Similar to considering the limits of ratios as part of the concept image of derivatives, filtering is an operation on frequency of a signal. The filter is similar to a function in that it operates on input signals which then have output

Table 31.1 Process-objects and Contexts for Signals and Systems

Process-object	Graphical	Physical/Applications	Symbolic	Verbal
Frequency	Time-domain	Energy	Equations	"Squished", "slower", "more oscillations", "not as stretched out"
	Frequency domain (see question 7 for graphs)	Energy	Equations	"Not as active"
Filtering		Amplifying or attenuating frequencies (e.g., low-pass filters), noise	Equations	"Passed through", "cut off", "magnify the output"

signals, but the filter can also be described as an object. Each of these layers can be represented within different contexts (e.g., graphical, symbolic, physical). Other authors have expanded Zandieh's characterization of the contexts for derivative (Roundy et al., 2015).

We have adopted Zandieh's (2000) terminology of "contexts" since it is a broader term to describe ways of thinking about a concept than representations, and "contexts" can include spaces that do not necessarily have the structure of a representational system such as physical applications (p. 105). Filtering tasks include two graphical contexts. The first is a time-domain plot that shows the amplitude of the signal as the dependent variable and time as the independent variable. The second graphical context is a frequency-domain plot which shows the signal in terms of energy variation across different frequencies. Table 31.1 describes these two dimensions of a concept image for signals and systems and includes some examples of the students' terminology (in quotes from the interview analysis described later). Zandieh's framework refers to the physical context to include velocity as part of derivative. In Table 31.1, we have modified this to be "physical/applications". As we will discuss later, the applications of the concepts are important for helping engineering students make meaning of the concepts. These applications are physical contexts for considering signals.

Students who participated in the interviews were asked to think aloud as they solved several conceptual signals and systems problems, most of which were drawn from the Signals and Systems Concept Inventory (SSCI). For this paper, we analyzed students' think-aloud solutions to three of these problems, all of which tie to the concepts of frequency and/or filtering and all of which include graphical contexts in both the time and frequency domains. While students typically bring some intuition about the concept of frequency, the Fourier transform is often introduced for the first time in a signals and systems course. Representation and analysis of signals and systems in the frequency domain, as well as connecting time- and frequency-domain representations, is key content in a signals and systems course. Figure 31.1 shows Problem A, one of the problems students were asked to solve in

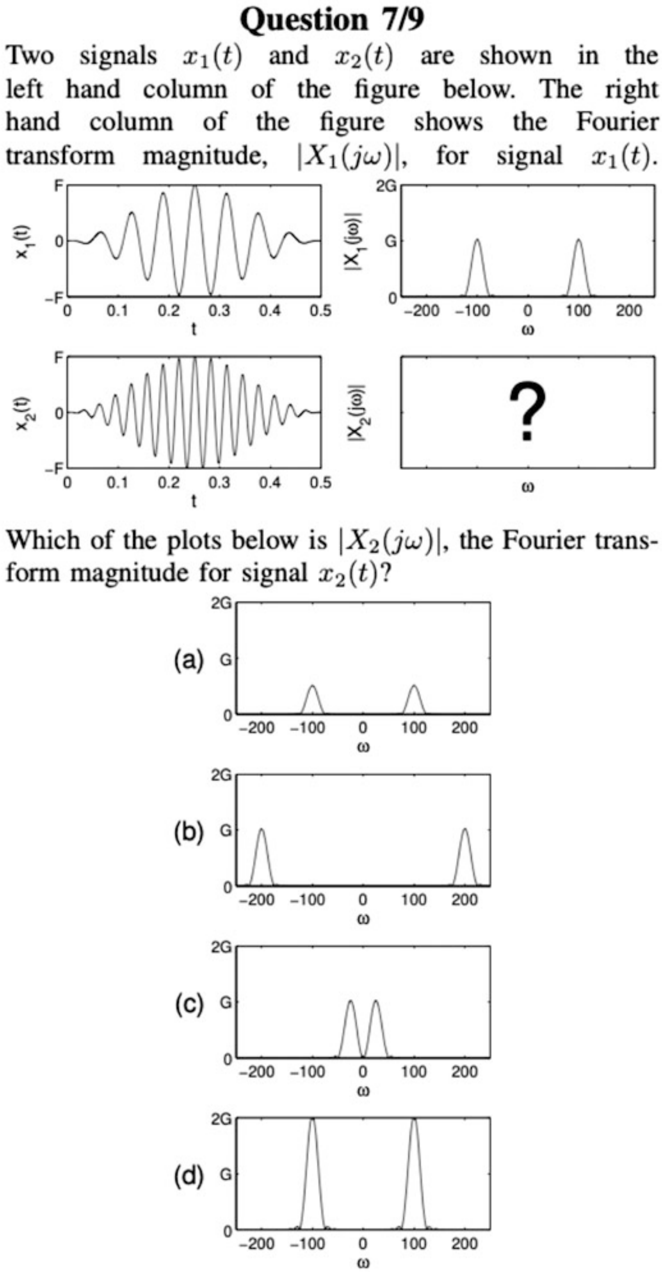
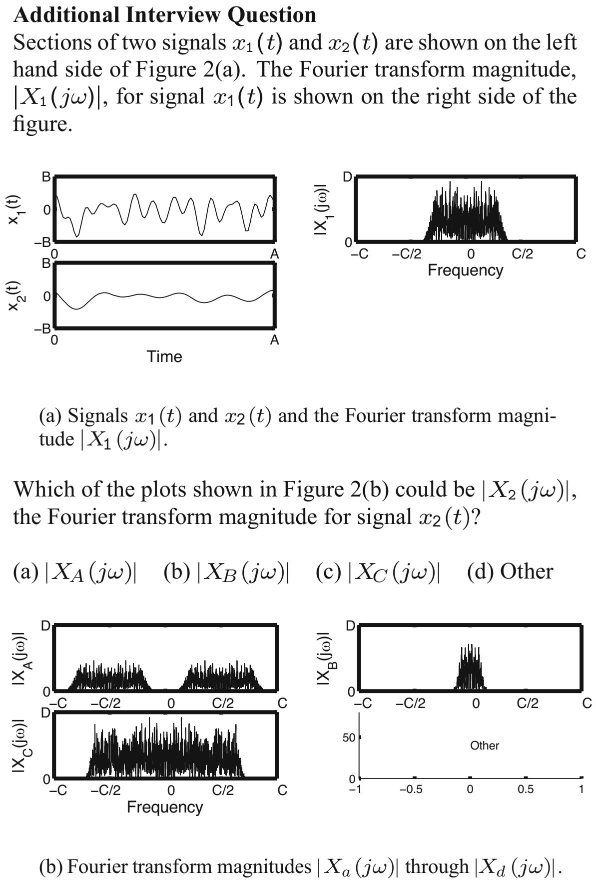


Fig. 31.1 Problem A (Question 7/9 on the SSCI)

Fig. 31.2 Problem B


Additional Interview Question

Sections of two signals $x_1(t)$ and $x_2(t)$ are shown on the left hand side of Figure 2(a). The Fourier transform magnitude, $|X_1(j\omega)|$, for signal $x_1(t)$ is shown on the right side of the figure.

(a) Signals $x_1(t)$ and $x_2(t)$ and the Fourier transform magnitude $|X_1(j\omega)|$.

Which of the plots shown in Figure 2(b) could be $|X_2(j\omega)|$, the Fourier transform magnitude for signal $x_2(t)$?

(a) $|X_A(j\omega)|$ (b) $|X_B(j\omega)|$ (c) $|X_C(j\omega)|$ (d) Other

(b) Fourier transform magnitudes $|X_a(j\omega)|$ through $|X_d(j\omega)|$.

the interviews. This problem demonstrates a task that requires connecting time- and frequency-domain representations of signals.

Specifically, Problem A deals with relating time and frequency-domain representations of a narrow-band signal (specifically a windowed sinusoid). Students are given a time-domain graph of a low-frequency sinusoid and a graph of its Fourier transform magnitude. They are then given a time-domain graph of a higher-frequency sinusoid and asked to identify the correct frequency-domain graph from a set of four possible answers.

Problem B (Fig. 31.2) is similar in structure to Problem A but rather than considering windowed sinusoids, the problem considers realizations of random noise with different bandwidths (i.e., low-pass filtered white noise). Students are given the time and frequency-domain graphs of one random noise signal. They are also given the time-domain graph of a second random noise signal with narrower bandwidth than the first. They are asked to identify the correct frequency-domain graph of the second signal from a set of four possible answers, one of which is “none of the above.”

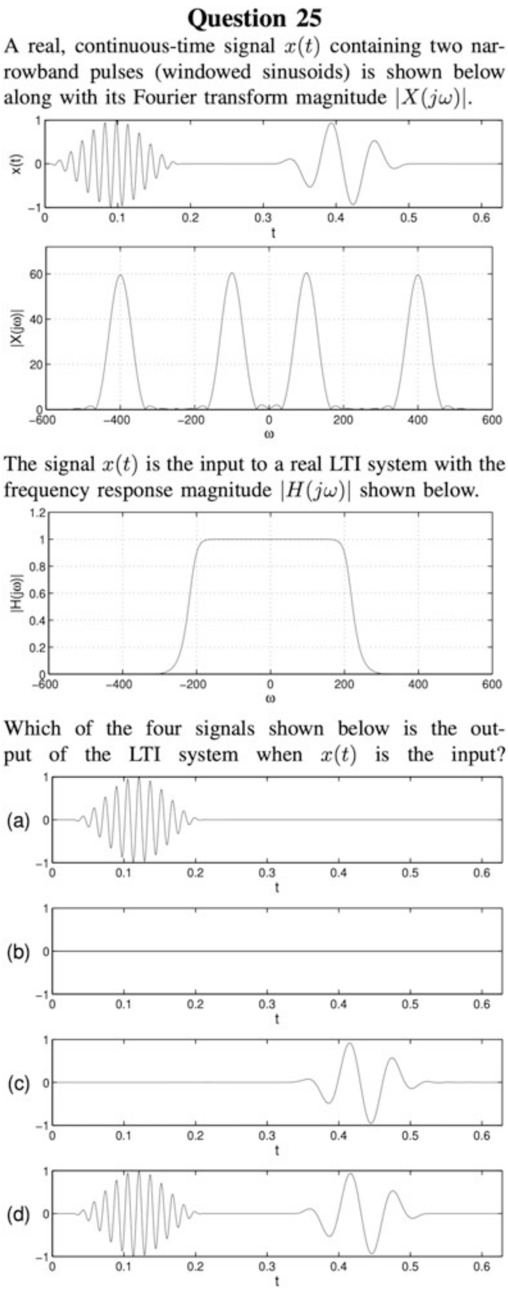
Problem C considers time and frequency representations of signals and also integrates a filtering operation (see Fig. 31.3). Students are given the time- and frequency-domain graphs of a signal that includes two windowed sinusoid pulses, one at four times the frequency of the other. Students are also given the frequency-domain graph (Fourier transform magnitude) of a low-pass filter. They are asked to identify the correct time-domain graph of the filter output when the given signal is the input to the filter.

31.6 Analysis of Concept Inventory Questions

31.6.1 Students' Descriptions of Frequency

In all three problems analyzed, students needed to understand how to interpret frequency from a time-domain plot. Based on the interview data, students were comfortable with verbal descriptions of the relative frequency of two windowed sinusoids. When asked to explain how they knew which signal had a higher frequency, some described the rate of oscillation using terminology from the field, e.g. "The frequency of $x_2(t)$ is greater than that of $x_1(t)$. There's more oscillations in the period of time" or "There are more periods within the same timeframe." In other cases, students described frequency in a more colloquial way (e.g., "$x_2(t)$ has a higher frequency than $x_1(t)$ cause it's more squished") but still suggesting an image of frequency and how it appears in time-domain representations. Problem B required students to consider the frequency of time-domain signals in a more challenging task. In this case, the random signals contain energy across a range of frequencies. Several students recognized that these signals did not have a closed-form equation representation and instead were more of a "real world" example. Still, many students were able to extend their conceptual understanding of frequency to compare the frequency content of the two signals, using descriptions such as "In $x_1(t)$, there's a lot more going on. ... So, this one ($x_2(t)$) has less going on in the given time. So, it seems like it should have a smaller frequency."

Moving beyond interpreting frequency from time-domain graphs, in all three problems students needed to connect time-domain graphs to frequency-domain graphs. For Problems A and C, this involves understanding that the frequency-domain representation of a windowed sinusoid shows the signal's energy centered around the frequency of the sinusoid (i.e., a magnitude peak at that frequency). Hence, for a higher-frequency windowed sinusoid, the peak in the frequency-domain representation will occur at a larger value on the horizontal axis, farther from the origin. The majority of students were able to make this connection and to use it to select the correct answer to Problem A. For many students, however, it was not clear from their think-aloud responses that they understood that the frequency-domain representation was showing the amount of energy at each frequency. Instead, they may have been relying on a memorized "rule" that a higher frequency corresponds to the peak moving out in the frequency-domain representation. One student expressed


Question 25

A real, continuous-time signal $x(t)$ containing two narrowband pulses (windowed sinusoids) is shown below along with its Fourier transform magnitude $|X(j\omega)|$.

The signal $x(t)$ is the input to a real LTI system with the frequency response magnitude $|H(j\omega)|$ shown below.

Which of the four signals shown below is the output of the LTI system when $x(t)$ is the input?

Fig. 31.3 Problem C. Note: Question previously appeared in Wage et al. (2006). Used with permission

this thinking explicitly: "The trick that I learned from one of my classmates was if you see double the cycles, it's just shifted off that much in the Fourier transform . . . almost like a match game kind of thing." It is worth noting that this student explicitly counted the number of cycles in each windowed sinusoid to determine the frequencies rather than leveraging the relative frequency difference through visual inspection.

In Problem B, students needed to move beyond knowing how the frequency-domain representation changes when the frequency of a windowed sinusoid increases. To identify the correct solution, they needed to understand how a random baseband signal with a narrower range of frequencies would be represented in the frequency domain. Working with this signal required them to extend beyond a single-frequency pulse and beyond signals with clear equation-based representations. To attack this problem, students had to first recognize that the second signal was lacking the higher frequency components present in the first. One student described the second signal as "not as active as the first one." Students who were able to identify the correct answer expressed that the reduced frequency content would correspond to larger magnitudes closer to the origin in the frequency domain. Many applied the duality principle, even though they didn't name it, "The period of $x_2(t)$ seems more stretched out. It seems like just slower and therefore the frequency would be more compact in the Fourier transform." Others relied on an understanding of Fourier series (and by extension the Fourier transform) and perceived the random signals as combinations of sinusoids, e.g. "This has a bunch of signals all added together. I would look at it as being a bunch of things all added together." Another student explained their reasoning as "If you look at $x_1(t)$ ... when you do a Fourier series representation, it's gonna be a sum of a whole bunch of cosines and likewise with $x_2(t)$, and since you have a sum of a whole bunch of cosines you have several omega values. That's why you have all these multiple spikes in the spectrum representation." Several students did not have a strong enough understanding of how the reduced-bandwidth signal would be represented in the frequency domain, instead expecting a lower magnitude in the frequency-domain representation or a frequency-domain representation with less rapid oscillation, similar to the difference between the two time-domain representations.

Students' responses to all three problems indicated that the time domain was the preferred "native" domain for interpreting signals. Even when they could easily translate from time to frequency, they studied the time-domain signal first. One student described their thought process this way: "I'm going to look for similarities between $x_1(t)$ and $x_2(t)$. If I can define $x_2(t)$ in terms of $x_1(t)$ or somewhat close to that, then I could define $X(j\omega)$ of $x_2(t)$ in terms of $X(j\omega)$ of $x_1(t)$." Students' think-aloud responses to Problem C provided insight about how their preferred representations differed for signals vs. systems. Students were given both time- and frequency-domain graphs of the input to the system. The system response is shown in the frequency domain, but the possible system outputs are shown in the time-domain. Problem C revealed that the frequency-domain representation provided more intuition for systems. Students' responses suggested that they were most comfortable thinking of systems as ideal filters (or approximations thereof) with frequency-domain representations. When considering the signals in all three problems, most students began by studying the time-domain graph and then translated that information to the frequency domain. In contrast, students gleaned information about the system response directly from the frequency-domain plot provided; none of the participants attempted to determine a time-domain representation for the system or identify the correct system output using time-domain approaches. Arguably, filters and filtering are most intuitive in the frequency domain, as filters are conceptualized in terms of how they act on signal content at different frequencies. We speculate that students' comfort with frequency-domain representations of systems may be related to a simplified view of systems as idealized filters that pass (retain) some frequencies while eliminating others. This is supported by previous analysis in which we studied common misconceptions tied to students' responses to SSCI problems (Wage et al., 2006).

Across the problems analyzed and across the students interviewed, there were several instances in which students grasped for numbers and/or equations rather than relying on conceptual understanding. For example, some students computed the frequencies of the signals in Problems A and C (by counting the number of peaks in a certain time period) rather than using the information readily available in the frequency-domain plots to determine approximate frequencies or recognizing that the problem required only a qualitative characterization of the signals' frequencies. The random signal in Problem B did not allow students to compute/estimate a frequency, and some expressed discomfort with this. In describing their lack of confidence in their answer to Problem B, one student said "I don't have the actual values for $x_1(t)$and $x_2(t)$. I need to actually be able to do the math. . . . This seems that it was just data gathered somewhere, so we don't actually have an equation for it. . . . If it had an equation then I'd be able to manipulate that."

31.6.2 Connections Between Graphical and Symbolic Contexts

Eighteen interviews included some comments about how they were thinking about connecting the math and the visual (28 excerpts across these interviews). Within student responses either to the signals and systems problems or to the more general questions about their understanding and perceptions of the content, we noticed some comments within the code initially described as "connecting the math to the visual" of students distinguishing "doing the math" from either interpreting the graphs as visual representations or from thinking about the concepts they were using. In terms of the concept image in Table 31.1, they make a distinction between the symbolic context, graphical context, and the applications. Some of them describe this explicitly as a translation process between the visual representation and the "math". Based on their choice of wording in the tasks, we interpret their reference to "the math" as being writing equations or working with symbolic representations. The following exchange is an example where a student refers to "doing the math".

Interviewer: Any inkling as to why Fourier series was so hard?

Student: It has more to do with the math behind it. I'm hoping that when I graduate, I'm a senior this year, and hoping when I graduate that

> some of this stuff can be done in computers and I don't have to worry about, doing that stuff by hand. It was the math. It's the understanding of... I have... I have to tell you, I've heard this feedback from a lot of the students from the year before, which is when I took it, we had a hard time really translating the visual to the math. There were some people who the hardest time was translating the visuals that were given to how the math came out.

In other comments, there is also a disconnect for students between the graph and "the math" such that they don't know how to or don't feel confident about what to do with the graph alone. For example, "I wouldn't have much trouble figuring it out but if - you know - I just see an image and like I don't know how to do it in math." An intriguing phenomenon is how some of them sought an equation or symbolic representation in order to confirm their interpretation of the graph. For instance, "in order for me to be very confident, I would actually need to get an equation for it" in response to how confident they were about an answer. But, also recognized when the graph might represent real phenomenon or real data sets that might not have a clear equation. For instance, "...this seems that it was just data gathered somewhere so we don't actually have an equation for it... so if I could... if it had an equation then I'd be able to manipulate that or at least I had the raw data and I was able to manipulate that..." This student seems to be seeking some numbers rather than the graph as a source of certainty or as confirmation of their interpretation.

In a related set of comments, some students also identified that "the math" was harder than the graph or that the graph helped them understand "the math". For instance, "Really in calculus we used to integrate one term or just a straight double integral and this required actually conceptualizing the shifting. As soon as [you] put it visually that you were flipping and shifting then the integral made sense". They recognize that there should be a connection between the two representations - the symbolic and the graphical - but may find it either challenging or illuminating to identify that connection. In this way, their concept image includes both graphical and symbolic contexts. However, students may have difficulty finding the relationships between those. Or, the relationships may be complicated because it may be easier to create a graphical representation for the real signal than to create a symbolic representation of the signal.

31.6.3 Applications and the Concept Image

We identified excerpts in 29 interviews that discussed connections to applications of the content or real-world engineering. Not all students mentioned connections to either real world applications or other engineering content, but there was a consistent emphasis on making connections among the responses to describing the most important concept or the most difficult concepts. For real world connections, understanding the applications of the technique and when it would be useful was an indicator of importance. Some students pointed to specific applications (e.g., transmitters, cell phones, music technology), and others referenced this in a more general way. The other indicator of importance was connections to other engineering content usually noted by referencing other engineering courses where the concept re-occurred in some fashion. Students' concept image included connections to other topics or content in engineering.

Conceptual understanding was described as finding meaning, making connections, or identifying applications. A few students recognized the need for conceptual understanding as part of real-world engineering in the workplace. This is consistent with Harris et al.'s (2015) study where engineering students described the value of mathematics as an applied tool for the workplace. One student had a clear statement about these connections: "I actually think it helped me a lot having tests like the one we just had, where it was very conceptual, it made me understand the material better as opposed to going and listening to a professor lecture and say and do all this math. I mean don't get me wrong we have to do the math to actually understand that stuff too, but it's very good to understand real world applications and do these kind of conceptual problems because that's what you do in the real world." In Table 31.1, we combine physical contexts and applications because for the students these contexts are linked.

One student was particularly detailed in making the connection between applications in the workplace and the importance of conceptual knowledge. The student explains in the excerpt below how the conceptual knowledge of mathematics is needed as a check on a simulation (i.e., a computer-generated output). As with other students, "doing the math" appears to imply doing the calculations or working with symbols. Conceptual understanding for the student is for making sense of the quantitative information.

> I am more technical minded. I used to work in the field. I feel it's better if you can put more conceptual questions in the class. Even in the class, I think we had one part out of the five or six questions in a quiz or exams, but I think it's very important because doing the math is a matter of practice, and this is the way I put it, but understanding is the hard part. This course is more of understanding, because in the field we may not do a lot of math except for quick checks. We do a quick check for a system, but when you do real design, accurate design, we need to put it through a simulator, but at the end of the day I need to know conceptually, does that make sense, plus of course, I need to do a quick math just to make sure that I'm having a good result or it's just the simulator is not doing its job. We cannot trust that all the time.

This student may be exceptional in recognizing these connections, but we found it important to note as an example of how students might understand the relationship between mathematics and engineering knowledge and practice. In terms of the students' concept image, the student seems to consider the "math" (or the symbolic context) in an intuitive or informal way as a check for the interpretation of the physical or applied context for the problem. One question this raises relative to concept images is how students move between formal mathematics and more informal checks or intuitions that may be essential to their future work as engineers. In particular, the graphical context may be easier or more accessible for real engineering tasks than symbolic contexts that do not have simple (relatively speaking) representations.

31.7 Discussion

In this paper, we have explored components of students' concept images for signals and systems. The tasks presented to the students focused on the graphical contexts for presenting the signals and systems. Overall, the students made some connections to the physical contexts and had a foundational understanding of frequency. What becomes more challenging for them is reconciling the two graphical contexts of time-domain and frequency-domain. A basic question for us is whether this is simply because students have more experience with functions generally where time is the independent variable. Other authors point to students' challenges with signals that include more than one frequency (Coppens et al., 2017) which suggests students may have a less formulated concept image overall for thinking about frequency as a variable within a system.

Zandieh's framework for process-objects was originally developed for calculus tasks, but we found it useful for organizing the different aspects of the signals and systems concepts we explored here, and we suggest it may be helpful as an organizer for other applied mathematics topics. In the case of signals and systems, students need to learn to use all of the representations available (e.g., symbolic, graphical) and connect these representations to the real phenomena that they are used to describe and interpret. Zandieh's framework helps to make explicit that the phenomena are dynamic and that there are multiple representations of a process that students need to understand. The concept image includes both signals and systems that act upon those signals (e.g., filters). As seen in the figures for the tasks, the graphs are complex and the symbolic representations are similarly complex (or more complex). In the interviews, students' concept-images then include pieces of the different components (e.g., graphical, symbolic, verbal), but they are still developing their understanding and learning to connect these pieces together.

Practical questions continue to exist between engineering and mathematics departments about how and when mathematics content should be taught for engineering students (Engelbrecht et al., 2012; Faulkner et al., 2019), including how mathematics content should be distributed among departments. Students may experience mathematics as shaped by different departments, but there are some common recommendations and implications. First, engineering students recognize engineering as an applied discipline and are concerned with real-world applications of mathematics (Harris et al., 2015). This shapes their use of it as a resource and a tool. It also shapes their perspective on the relative importance of different aspects of the concept image (e.g., symbolic, graphical). Students reported relying on symbolic representations as part of mathematical certainty when faced with tasks that were largely graphical. Students tend to equate "doing the math" with symbolic contexts. Admittedly, we have inferred this meaning from their statements in context but they present "doing the math" in relationship to either graphical representations or conceptual knowledge. However, it raises the question of how math plays into their conception of engineering knowledge. Which parts of other concepts in mathematically-heavy engineering courses are mathematics or engineering from students' perspectives?

As in other studies, students in interviews were seeking understanding of real-world applications to find meaning for the mathematics and as part of coming to understand what was happening in the system represented. At the same time, there are examples of where knowledge they developed in mathematics classes is not necessarily translating to the engineering context. For instance, they are sometimes challenged by graphs with frequency as the independent variable and magnitude as the dependent variable. We connect this to what other researchers have called "task-relevant features" (Johnson-Glauch et al., 2020) so that it may be harder for students to interpret frequency and magnitude as relevant to the problem. These graphical contexts highlight different aspects of the signals and systems represented while obscuring other aspects but need to be used in concert to reason through the task.

31.8 Limitations

All students interviewed had taken electrical engineering coursework, so our analysis is situated in that sub-discipline of engineering. However, other studies report similar experiences in statics tasks about students translating between representations and considering different features of the representations (Johnson-Glauch & Herman, 2019; Johnson-Glauch et al., 2020). Also, we are interviewing engineering students who are not yet professional engineers. They are upper-level students in their third or fourth year of the program. They are making the transition from mathematics courses taken in the early years of their program to upper-level engineering courses that are building on that mathematical experience so are building more specialized content knowledge. Some students also report experience in the field via internships or other jobs. This is a relatively small sample of students at two institutions so may not be representative of the experience at other institutions. Even with those limitations, there are some important areas to explore based on students' responses.

31.9 Conclusion

Across the analysis of the students' thinking in the tasks and the open-ended questions about their perceptions of the content, we found examples of how they incorporated different pieces of their concept image for signals and systems. They may start from their foundations in other mathematics topics in the graphical contexts that are most familiar (e.g., preferring time-domain to frequency-domain graphs). They are also still learning how to translate and choose among graphical contexts and symbolic contexts for interpreting the signals and systems phenomenon. As students in an engineering course with significant levels of mathematics content, they are also seeking to understand or recognizing the ways in which the mathematics is applied and looking for connections to their future work as engineers.

Some students also recognize the need for conceptual understanding that goes beyond procedures and calculations, but that helps them when interpreting graphical or physical contexts that may not have clear symbolic representations.

What is the relationship of mathematics and engineering? While this is not a new question, it is still an open question. Other studies suggest a greater emphasis and focus on engineering tasks when teaching mathematics courses for engineering students. As a logistical issue, the two disciplines have some divergence in terms of notation (e.g., *i* and *j* both used for complex numbers depending on the discipline) that can create difficulties for students when moving between mathematics courses and engineering courses while trying to make sense of what is relevant for each (Rønning, 2021). These differences may seem simple to engineers or mathematicians but may not be simple for students who are still building mental frameworks of important concepts. In addition, is the mathematics coursework attending to some of the subtleties engineering students experience when "doing the math"? If students perceive "doing the math" as limited to working with symbols, then is graphical interpretation of a real phenomenon mathematical for them? However, the concept image we suggest includes aspects that might be referred to as either mathematics or engineering. Students were conscious of the different contexts as representations or ways of thinking about the concepts. As students are developing a concept image, both disciplines are at play and may be indistinguishable as the mathematics is applied in engineering. For describing concept images in other topics, we may explore how these contexts and process-objects come together to inform students' understanding and use of what they're learning.

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Chapter 32
Analyzing the Interface Between Mathematics and Engineering in Basic Engineering Courses

Jörg Kortemeyer and Rolf Biehler

Abstract This chapter studies the mathematical skills required in technical subjects of bachelor programs in engineering. The interface between mathematical subjects and introductory engineering courses is conceptualized to study this topic. We developed a transferable framework for the analyses of exercises, related problem-solving strategies, and typical sources of errors. In exercises from an engineering exam after the second semester, the transitions between engineering contexts and the use of mathematical methods, whose use and shape can stem from courses on mathematics for engineering or from their use in lectures on engineering themselves, are characterized, also concerning the tensions they create. Finally, we present the results of applying this framework to an exercise used in an exam, which was given after the first year studies in electrical engineering.

Keywords Normative solutions of engineering exercises · Expert interviews for competence expectations · Mathematics in engineering education · Ordinary differential equations in engineering sciences · Solution processes of experts · Basic electrical engineering courses

32.1 Introduction

Engineering education for students usually consists of two strands: courses in the engineering sciences and courses in mathematics. This separation is based on the conviction that offering a course that integrates the necessary mathematics into engineering science is not a satisfactory solution. However, the separation requires students to make a high transfer effort to use the mathematical resources they have

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learned in the mathematics course when solving problems in the engineering sciences. Other mathematical resources stemming from school mathematics education or learned in an integrated way in the engineering course may be relevant in the solution process. There is some evidence that this integration does not always work successfully. Innovative approaches suggest that the mathematics courses should be explicitly and better related to engineering as a field of application. Vice versa, it may be productive for mathematical resources to be taught when needed in engineering courses. Two examples of such innovations are the following: The study by Wolf (2017), who developed mathematical modeling tasks from the context of mechanical engineering to be included in a mathematics course for engineering students and the study of Hennig and Mertsching (2013), who developed situated mathematical resources in a course "Foundations of Electrical Engineering".

Within mathematics education research, including applications in a course is – as a rule – conceptualized as adding mathematical modeling tasks to the course. The conceptualizations of mathematical modeling include a distinction between mathematics and "the rest of the world", implying that after a phase where the mathematical model is set up, the mathematical problem is solved and then reinterpreted in the context at a later stage. However, when we look into exercises in standard books on engineering or physics, the tasks to be solved do not appear as "mathematical modeling" tasks. There is no clear separation between mathematics and, say, electrical engineering (EE). The basic concepts are physical magnitudes that are related by equations. Therefore, it is unclear when and how students must enter into decontextualized mathematics in the problem-solving process. In this process, they can apply the decontextualized methods they have learned in the mathematics course. The situation is even more complicated as the implicit mathematical practice in engineering contexts is different from the practice in mathematics courses. Well-known examples are the use of differentials as "infinitely small" quantities in most courses and the use of vectors as directed quantities – very different from how differential equations and vectors (as elements of vector spaces) are treated in a mathematics course.

Our study aims at a deeper understanding of this situation, i.e., how mathematics and what kind of mathematics is used and needed when students are asked to solve problems in their engineering course.

As a case study, we took four tasks from a final course examination on Fundamentals of Electrical Engineering (FoEE) at the end of the first study year. We aimed at combining several perspectives. First, we analyzed the tasks from our own (external) focus regarding the kind of mathematics needed. Second, we asked EE experts to explain what competencies and skills they require from their students when solving the tasks. Based on these analyses, we looked into students' written solutions in an authentic final exam of a FoEE-course, and we designed an observational study watching students (not part of the course but with similar prerequisites) working on the same problems. We used our theoretical analyses as an a priori analysis and analytical background to better understand students' cognitive processes and difficulties in the written solutions.

This chapter presents the results of the analyses on these issues on one of these exercises dealing with oscillating current.

32.2 Theoretical Backgrounds

This section presents the theoretical tools used to develop the newly constructed methodology for our investigations. We build on conceptualizations of mathematical modeling processes and mathematical problem solving and related heuristics. In addition, we add perspectives from the didactic of physics, namely the analysis of the relation between mathematics and physics by Bing (2008) and Tuminaro and Redish (2007). Moreover, we refer to the reconceptualization of the mathematical modeling process from the perspective of physics education (Uhden et al., 2012).

32.2.1 Different Mathematical Practices and Disparities in Mathematics and Engineering Courses

Engineering students often learn mathematics in two parallel contexts: through the "Math for Engineering Students" (MfES) course and the usage of mathematics in courses on engineering sciences, such as the "Fundamentals of Electrical Engineering", FoEE, which we will analyze in this paper. The students are expected to use their mathematical competencies and skills to analyze and solve exercises in EE. This situation entails several challenges: There are asynchronicities between lectures on MfES and FoEE so that mathematical methods required in FoEE-courses may be taught only later in MfES-courses. The MfES-course has the deductive conceptual structure of typical lectures in mathematics, while the FoEE-course also has a specific order of its topics according to traditions in structuring courses according to electromagnetic topics and theories. Moreover, the mathematics in MfES and FoEE can be characterized as different mathematical practices (Alpers, 2017) or different praxeologies in the sense of the Anthropological Theory of Didactics (ATD, see, e.g., Winsløw et al., 2014). A praxeology consists of four related components: theory and technology (the theoretical block with justifications for the practical block), techniques and tasks (the practical block). This approach allows us to characterize the different usages of "the same" mathematical methods such as differential equations in mathematics (MfES) and FoEE. Rønning (2021) applies ATD in mathematics in/for engineering, and the Chap. 30 of Castela and Romo and Chap. 27 of Gonzales-Martin et al. in this book provide further uses. The different practices may compete as resources in students' minds when an EE task requires, for instance, solving ordinary differential equations (ODE), and resources from MfES or FoEE for solving ODEs can be applied. Moreover, a recent study by Hochmuth and Schreiber (2016) addresses this challenge using ATD to determine what content, concepts, and heuristic strategies are relevant for successfully completing advanced engineering courses, such as “Signals and Systems”, where praxeologies from MfES are also competing.

Redish (2005, p. 2) elaborates on how mathematics is used differently in physics, for example, regarding the use of constants and variables: “We mix things of physics and things of math when we interpret equations.” Symbols in physics are not chosen arbitrarily but are associated with particular mental ideas about physical quantities or measurement results. This assignment of physical meaning to mathematical expressions is both powerful and valuable, allowing one to work with complex mathematical expressions without going into mathematical depth. The critique of the separation of mathematics and the rest of the world in the conceptualisations of the modeling process is also elaborated by Schürmann (2018a, b). Fettweis (1996) gives three explanations for the specific practices of using mathematics in engineering that differ from mathematics itself: (1) solution methods are based mainly on existing mathematical theories, (2) solution correctness can be substantiated by physical principles, and (3) the universality of physically motivated principles with which applicability can be justified, which goes beyond what has been shown in the mathematics courses, as long as there are no mathematical contradictions.

Alpers (2017) does a document analysis of typical engineering and mathematics textbooks to conceptualize differences in two domains, the use of vectors and the use of differentials. In school mathematics, vectors are usually understood as a class of arrows of equal length, parallel and in the same direction, which can be transferred into each other by shifting. Force vectors in statics are bound vectors that can be moved along a so-called line of action without losing their static effect. This affects the vector operations because if two vectors that are not on the same line of action are added, they are treated as free vectors, and the line of action of the resulting free vector must be determined. Other differences arise in connection with the terms component, coordinate and magnitude of a vector. For this purpose, different notions are used in the MfES and FoEE-courses, leading to inconsistencies. Alpers (2017, p. 138) describes a further difference in the use of differentials, which tend to be regarded as “infinitely small quantities” in the engineering sciences. In engineering, a part that is infinitely small in at least one dimension is selected and multiplied by an infinitely small length (dl), area (dA), volume (dV) or mass (dm), which again results in an infinitely small quantity. These infinitesimally small quantities are “added up” by integration, resulting in a (finite) property of the whole object.

32.2.2 Conceptions of Mathematical Modeling

As we mentioned in the introduction, the modeling cycle’s conceptualization may not be adequate for FoEE. A prominent view of the modeling cycle is provided by Blum and Leiss (2007) to describe idealized modeling processes of real-world problems that can be solved using mathematics. In broad outline, it divides the modeling process into two distinct parts, the so-called “rest of the world” and "mathematics". The solving of a mathematical modeling exercise is divided into seven steps: (1) understanding of the exercise and the underlying situation as well as the construction of the so-called "situation model", (2) simplifying and structuring of the situation and construction of the so-called "real model", (3) translating into a mathematical problem (entering the "world of mathematics"), (4) carrying out mathematical work, (5) interpreting the result in the real world, (6) validating and (7) presenting of the results. The cycle consists of two parts, the "rest of the world" with steps (1), (2), (6), (7), and the step within mathematics (4). The changes between the two worlds happen in steps (3) and (5). This modeling cycle description is considered as an idealization, probably only applicable in school contexts. Nevertheless, this approach is helpful for us as a tool to show essential features of our "modeling example", which differ even on an idealized level.

Uhden et al. (2012) suggest a "physical-mathematical model" which distinguishes different degrees of mathematization. The starting point is as well the "world", which, in a physical problem, has to be structured and simplified to enter the physical-mathematical model, which is left for validation in the final step. The physical-mathematical model names three facets of mathematization: (a) mathematization, (b) interpretation, (c) technical mathematical operation. (c) is the only part in which "pure mathematics" is entered. Through this conceptualization of the modeling process, it should be emphasized that mathematics cannot be reduced to computational techniques in Pure Mathematics. Instead, there is a strong conceptual interdependence between mathematics and physics in tasks from physics. This interdependence is not well enough reflected in the conceptualization by Blum and Leiss (2007). That is why we will also suggest an adapted version of the conceptualization of mathematical modeling.

32.2.3 Conceptions of Problem-Solving

Our third perspective is mathematical problem solving by Polya (1949), who intended to advise students on how to solve mathematical problems as well as applied problems using mathematics. He divides solving processes into four phases:

1. understanding the problem
2. devising a plan
3. carrying out the plan
4. looking back.

Polya (1949, p. 93–94) also uses his conceptualization of problem-solving on applied mathematical problems and states: "Practical problems are different in various respects from purely mathematical problems, yet the principal motives and procedures of the solution are essentially the same. Practical engineering problems usually involve mathematical problems." In applied problems, the unknowns, conditions and data are more complex and less sharply formulated than in mathematical exercises. In his example, the construction of a dam, data refer to topography, geology, meteorology, economics, etc. As much of this data as possible should be considered in devising a plan, but others must be neglected. Therefore, it is difficult to answer the questions “Did you use all the data?” and “Did you use the whole condition?” Because of this, the solution to the problem is often approximate precisely because it makes sense to allow for minor inaccuracies in the calculations in favor of a simpler model.

32.2.4 Conceptualizations About the Use of Mathematics in Physics

For the analysis of actual solution processes of students, which are not part of this chapter, we use theoretical approaches developed by Redish and his working group, i.e., by Tuminaro and Redish (2007) and Bing (2008), in addition to the normative solution.

32.3 Synthesis of Frameworks with a View Towards the Electrotechnical Tasks

Regarding the exercises we will analyze, it pays to use a more straightforward structure and redefine the activities.

1. Mathematization: Understanding the task and setting up a mathematical-electrotechnical model
2. Mathematical-electrotechnical symbolic-conceptual manipulation and reasoning: solution of the exercise using a “mathematics of quantities.”
3. Validation: Critical review of the results

Mathematization (*1*). Unlike more open modeling problems, a task is presented where – in the terminology of thematical modeling community, cf. Blum and Leiss (2007) – the idealized real model is implicitly provided and has to be recognized by the students. A standard help for this is a conventionalized diagram. Students are not asked to make their own idealizations and simplifications. These are often implicit parts of the knowledge taught in the course on FoEE. The identification of relevant magnitudes and equations between them is mediated by the electrotechnical theory based on physics. This first phase is well described by the first stage of Polya’s applied problem-solving.

Mathematical-electrotechnical symbolic-conceptual manipulation and reasoning (*2*). Instead of entering the “world of mathematics”, students enter into a “mathematics of physical quantities”, i.e., numbers with units. (Differential) equations, integrals etc. are usually formulated with a formula where the symbols represent physical quantities. Exercises differ as to which degree further abstractions into mathematics are needed or are helpful. For instance, it is necessary to think of mathematical functions and variables instead of physical quantities when looking for a mathematical solution. Thus, we approach this phase more in the sense of Redish (2005), Bing (2008), Uhden et al. (2012), and Schürmann (2018a).

In a rough sketch, phase 1 can be characterized as setting up equations (with knowns and unknowns) and phase 2 using mathematical procedures to solve these equations.

Validation (*3*). In the modeling cycle, validation checks whether the models are adequate against situational knowledge or real data. It is not the task of students from FoEE to question fundamental assumptions of electrotechnical theory. Therefore, they are not asked and usually do not carry out a validation as described in the modeling cycle. Still, they review the calculations done in the mathematical-electrotechnical work critically. This step is similar to "looking back" in Polya's framework, however, it does not include the question of whether the result or the method is transferable to other exercises.

This structuring will be used as a theoretical framework for understanding the cognitive processes needed for solving the exercise.

32.4 Research Questions

Based on the four theoretical approaches, the research questions for our studies are:

1. How can we reconstruct the competencies and skills needed for solving these exercises from the perspective of electrical engineering teachers?
2. How can we reconceptualize these competencies and skills from the perspective of what kind of mathematics is needed, and how can we conceptualize the interface between mathematics and electrical engineering?

32.5 Methodology and Data Collection

32.5.1 Overview

To answer the first two research questions, we followed the following process

1. RQ1: Expert interviews were used to identify the explicit and implicit competence expectations.

   Typically, there is a division at German universities into lecture and small group tutorials in Electrical Engineering courses. The lecture is held by a professor who is responsible for the course. PhD-students supervise the small group tutorials at the EE institute. They prepare the exercises for the students in consultation with the lecturer and are familiar with the teaching tradition.

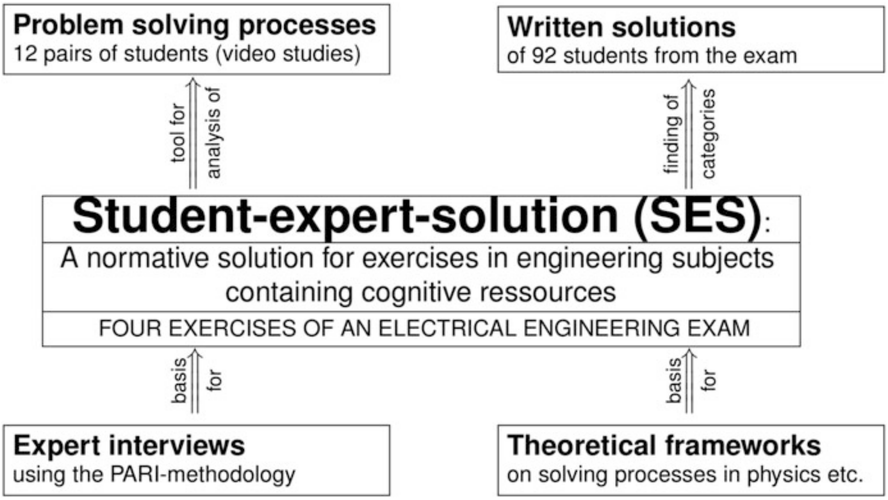

Fig. 32.1 Diagram of the connection of the different elements of our analyses

Therefore, they are experts in teaching practice, and for this reason, they are interviewed in our expert interviews. The interviews will provide information on typical mistakes, alternative solution paths, and validation methods.

2. RQ 2. A normative a priori analysis of the four exercises, the so-called "student-expert-solution" (SES), was developed using the theoretical approaches and expert interviews.

As shown in Fig. 32.1, the primary theoretical tool and result of our analysis is the so-called student expert solution (SES), a specific type of an a priori analysis, which builds on expert interviews and the four theoretical frameworks. The SES characterizes idealized solution processes, which we can expect from first-year students. Of course, students are not expected to write all this down in their real solution processes. Still, the SES identifies necessary background knowledge for solving the exercises, which students may have to activate and reason explicitly when their working process is observed. It depends on the course how much explanation and justification the students have to write down in a written exam. The development of the SES consists of two successive versions: SES1 and SES2. Our specific approach is to use expert interviews based on SES1 to develop the revised version of SES2. Both versions make an essential distinction between the object and the meta-level, being represented in an "object-column" and "meta-column" in the SESs. The "object-column" contains the solution steps on an object-level (for SES2 using results of phase 1 from the interview). The "meta-column" contains a meta-level where cognitive steps, resources, and anticipated obstacles were explicated. The conceptualizations on the meta-level are based on our theoretical framework and not just a repetition of the wordings of the interviewee.

1. For developing SES1, the brief *official solution outline* is first complemented on the object-level in the object-column. Then, intermediate steps for the calculations are added, and the underlying mathematical theory is pointed out by an expert of the teaching practice in MfES, which is similar to EE described above. Finally, the relevant physical mechanisms and laws are added from the perspective of FoEE. Usually, students are not required to explicate all these laws in their written solution, but they are expected to know them. The meta-level in the meta-column consists of the supplement of cognitive resources necessary as well as conceptualizations of the processes on the object-level using the previously mentioned theoretical frameworks.
2. For SES2, the data of the expert interviews are used to enhance and check the object and the meta-level of SES1. Moreover, we reconstruct the didactical aspects of an exercise and add them to the meta-level of SES2. We interview the experts to get a deeper insight into the interface between mathematics and EE. The SES1 is necessary for the interviewer to prepare questions, mainly aiming at the implicit competence expectations in the tasks. SES2 is a revised version of SES1, which also contains additional aspects.

The following section describes how the expert interviews were carried out.

32.5.2 Goals and Methods for Interviewing EE Experts About the Tasks

We asked the exercise designer and EE experts to solve the exercises from the perspective of the knowledge taught in the course, i.e., from what they expect students who have well understood the course contents would do. Afterward, we interviewed them concerning their solution processes, intending to reconstruct implicit reasoning processes. We asked for didactical aspects of the task and tried to determine how they conceived of the interface between mathematics and EE. We conducted the interviews building on the Precursor-Action-Result-Interpretation (PARI) method by Hall et al. (1995), an exercise-based interview technique aiming at reconstructing problem-solving skills of experts in troubleshooting situations. In the first phase of the interviews, the experts are asked to solve tasks while thinking aloud without interruptions. In a second phase, they are asked for the precursors of and the information required for their actions and the interpretation of their results. In the third phase, the interviewer asks for alternative results and interpretations, alternative actions and alternative precursors to collect data on additional ways to solve the task. We adapted this methodology for use in our research study, which requires new interpretations and changes in the last phase of the interview:

Our adapted version of the PARI-interview with its three phases is as follows:

1. *Documentation of the oral and written solution of the expert (imaging students' work)*. In the first phase, the experts are asked to solve the exercise themselves

with the knowledge and techniques of second-semester students and not based on their "expert knowledge" as advanced researchers in EE. They are initially provided only with the tasks but not the official solution outline (provided to the correctors of the exam). They are asked to think aloud to identify solution steps that they would execute otherwise without speaking during their solution processes. Unlike the original PARI interview, a written solution has to be also provided by the expert.

2. *Reconstructed reasons for the actions and used resources*. In phase two, the task solution process is to be gone through step by step to supplement the solution created in phase one. In the PARI methodology, this questioning is guided by the interviewer's task analysis. In our case, the guidance is based on the analysis in SES1. The aim is to determine what prior knowledge and resources have to be used by the students, and what else needs to be considered during the solution process. This step contains questions concerning reasons for the actions and the intermediate steps in the solution process, including the interpretation of the results.
3. *Reconstructed didactical aspects*. The goal of the third phase is to obtain more general information about the exercise type, which means all the considerations that an expert has taken or has to take into account when developing the task. With the help of these questions, a didactic reconstruction of the tasks' aims takes place. Roughly, these questions can be divided into three subphases corresponding to the additional information:

   (1) Acquisition of several possible ways for solving the tasks, usually expected hurdles for second semester students, alternative ways of solution with knowledge from higher semesters, sources of errors, and methods of critically checking the answers.
   (2) Analysis of the interface between mathematics and EE: focussing on the transitions between mathematics and EE in the problem-solving process and how they are intertwined.
   (3) Educational motives of the lecturers when designing the task and competencies expected from students when setting such a task: reasons for selecting the task, possible variations of the task.

   The specifications in 2. and 3. Stem from our specific research questions.

32.5.3 Methods for Developing the Student-Expert-Solution

The starting point for creating the SES was the official solution outline provided by the EE-staff as a basis for the correction of the written exams. These solutions are relatively brief, as they are written for experts in EE who have a good knowledge of the underlying theory, which has not to be made explicit.

We used the following procedure to develop the SES1 (step 1 to 4) and SES2 (step 5):

1. Supplementing the solution outline with intermediate steps (object-column): This step aims to close gaps in the existing solution. In the analyzed exercise, the solving process of the ordinary differential equation is part of this step.
2. Embedding in physical theory. (object-column): In the *official solution outline*, hardly any statements are made about the electrotechnical background or the physical processes considered in the exercises. We interpret this in the terminology of ATD that students are not required to reproduce parts of the theoretical block (justification of techniques used) of the lecture for justifying their actions and decisions in the written solutions. However, they might have to consider elements of this theoretical knowledge when searching for a problem solution. The source for this supplement was the lecture notes for the course the exam was taken.
3. Structuring the exercises and solutions according to the synthesis of theoretical approaches in the object-column. In this step, the solutions of the exercises are broken down into sub-steps and thus roughly structured, using the language of the modeling cycle and problem-solving. Such sub-steps can be, for example, the setting up of a formula and the calculation of a quantity with the help of the set-up formula, whereby the phases according to Polya (1949) allow a more refined structure.
4. Adding the cognitive resources (meta-column): In this step, the solutions obtained by the previous steps are supplemented by the cognitive resources, i.e., competencies to solve the task like reading the sketch or forming an equation.
5. Supplementing with data from expert interviews (object-column and meta-column): In this fifth step, the developers of the SES add further information from the expert interviews using the PARI-methodology described above. In this step, we aim to find differences in problem-solving processes by experts (rather than novices in the steps before) and to answer questions that arose while going through the previous steps.

Examples for complete student expert solutions on all four exercises of the first-year course on EE can be found in Kortemeyer (2019). The topics of the exercises are magnetic circuits, oscillating circuits and compensation processes (using ordinary differential equations, signal analysis (using integration in one variable), and complex alternating current (using complex numbers).

32.6 The Exercise on Oscillating Current as an Example and Its Solution Outline for the First Two Subtasks

This section presents the exercise on oscillating circuits, exercise B in the analyses. We first show the subtasks and *the official solution outline*. The translation was done by the authors.

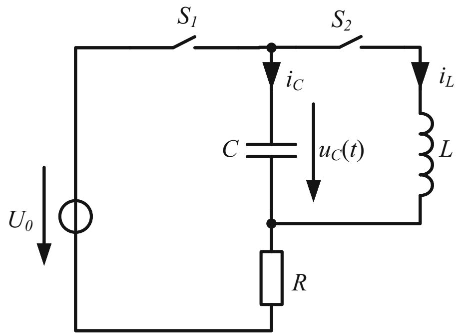

Fig. 32.2 Diagram of an oscillating circuit with two open switches

32.6.1 The Subtasks B1 and B2 and the Official Solution Outline

Remark on the used notation: In FoEE, 0^- stands for the limit from the left, analogously 0^+ stands for the limit from the right.

The following network with the resistor R, the inductor L, the capacitor C, and the ideal voltage source U_0 is given (Fig. 32.2).

The switches S_1 and S_2 are open for $t \leq 0$s, and the inductor and the capacitor are fully discharged. At the time $t = 0$, switch S_1 is closed. The switch S_2 remains open.

Subtask B1 Give the values of $u_C(t = 0^-)$, $i_L(t = 0^-)$, and $i_C(t = 0^-)$. Give reasons for your answers.

Solution

$u_C(t = 0^-) = 0$,	Since according to the task: completely discharged.
$i_L(t = 0^-) = 0$,	Since capacitor and inductor are completely discharged according to the task.
$i_C(t = 0^-) = 0$,	Since capacitor and inductor are completely discharged according to the task.

Subtask B2 Give the values of $u_C(t = 0^+)$, $i_L(t = 0^+)$ and $i_C(t = 0^+)$ and give reasons for your answers.

Solution

$u_C(t = 0^+) = 0$,	Since the voltage at the capacitor cannot change by leaps.
$i_L(t = 0^+) = 0$,	Since the current through an inductor cannot change by leaps.
$i_C(t = 0^+) = U_0/R$,	Since the total voltage U_0 is completely applied to R (because $u_C(t = 0^+) = 0$) and $i_L(t = 0^+) = 0$.

32.6.2 Summary of the SES of Subtasks B1 and B2

We only summarize the SES on B1 and B2, as both subtasks do not include mathematical competencies in their solving processes. Still, physical argumentation is required: As both switches are open initially, all components are discharged as there is no possibility of charging them with a network not being closed. Therefore, all quantities are zero at 0^-. When closing S_1, the voltage at a capacitor, $u_C(t)$, and the current at an inductor, $i_L(t)$, cannot change by leaps. That is knowledge from the lecture. However, because of the open switch S_2, the inductor is not even part of the setting. Following Ohm's law, the current at the ohmic resistance is given by $i_R(t^+) = U_0/R$. It is not addressed in the lecture to what extent this discontinuity is an idealization that is adequate for the (macroscopic) situation.

32.7 Development of the Student Expert Solution for Exercise B3: Setting Up the Differential Equation)

This section and the following Sect. 32.7 show how the described methodology is used to create a normative solution for answering research questions 1 and 2, i.e., to reconstruct the competencies and skills needed to solve the exercise as well as to conceptualize the interface between mathematics and EE by using the expert interview. Section 32.6 deals with forming an ordinary differential equation (subtask B3) and Sect. 32.7 with its solving (subtask B4).

32.7.1 Official Solution Outline of Subtask B3

In the following, we have added letters in square brackets for reference purposes:

Subtask B3 Deduce the differential equation for $u_C(t)$ for $t \geq 0$.

Solution
Component equations:

$$u_R(t) = i_R(t)R \quad [\mathrm{A}]$$
$$C\dot{u}_C(t) = i_C(t) \quad [\mathrm{B}]$$

Mesh equation:

$$
\begin{aligned}
&U_0 = u_R(t) + u_C(t) \quad [\mathrm{C}]\\
&i_C(t) = \frac{u_R(t)}{R} = \frac{U_0 - u_C(t)}{R}\\
&C\dot{u}_C(t) = \frac{U_0 - u_C(t)}{R}\\
&RC\,\dot{u}_C(t) + u_C(t) = U_0
\end{aligned}
$$

32.7.2 SES1 Object-Level: Extended Structured Solution Outline, Knowledge from EE-Theory Relevant for the Solving Process

The challenge for the students can be reconstructed as

1. Mathematization: Understanding the task and setting up a mathematical-electrotechnical model

In this context, this means to remember and activate the equations.

(1) Equations of resistor [A] and capacitor [B].
(2) Mesh equation [C].

To arrive at the requested solution, a further equation has to be used, using physical knowledge.

(3) $i_R = i_C$ (constancy of current) – [D].

Here knowledge from FoEE is required (see below).

The next step is

2. Mathematical-electrotechnical symbolic-conceptual manipulation and reasoning: solution of the exercise using a "mathematics of quantities."

The situation is characterized by four unknown functions $u_R(t)$, $i_R(t)$, $u_C(t)$, $i_C(t)$. The student has to find relations between these functions (and their derivatives) aiming at one single (differential) equation with only the unknown function $u_C(t)$.

What now has to follow is what we call "equation management". [A], [B] = > $u_R(t) = C\dot{u}_C(t)R$ [E].

As the equation on the right still contains two unknowns, u_R and u_C, students have to use another equation, the mesh-equation [C], giving:

$$[\mathrm{E}],[\mathrm{C}] => U_0 - u_C(t) = C\dot{u}_C(t)\ \ R => CR\dot{u}_C(t) + u_C(t) = U_0$$

Students can arrive at these three equations when analyzing the idealized physical situation. After the closing of switch S_1, students work with a network containing a capacitor C, an ohmic resistance R, and an ideal voltage source U_0. The mathematization works by translating the components capacitor and resistance into their component equations known from the FoEE-course. They also need to consider the experimental set-up, as after closing S_1, the left part forms a so-called mesh, in which Kirchhoff's voltage law can be applied to get another equation.

To consider the physical background, it is essential to know that a capacitor stores an electric charge q, and the two electrodes each carry charges of opposite signs. In this case, the equation $q(t) = Cu(t)$ holds (with a capacity C), and using $i(t)=\dot{q}(t)$, we get the component equation for the capacitor [B].

Kirchhoff's rule of meshes must be applied to take the experimental set-up into account. It states that in a closed part of a network, a so-called mesh, the sum of all directed voltages is zero. The left part of the diagram becomes a mesh when the switch S_1 is closed. The rule of meshes uses the idealization that the voltage is not depending on the current going through the ideal voltage source U_0, i.e., the voltage source does not have an inner resistance. Using the physical knowledge that $i_L(t)\equiv 0$ implies [D] according to Kirchhoff's law of nodes (stating the identity $i_R = i_C. + i_L$).

32.7.3 SES1 Meta-Level: Viewing the Solution According to the Theoretical Approaches and Identifying Cognitive Resources

Mathematization In the perspective of the modeling cycle, the exercise shows the characteristics mentioned above. However, in contrast to more open modeling problems, an idealized real model is provided in the form of a conventionalized diagram. Thus, the students do not have to make their own simplifications and work with an idealized model without the necessity to know which idealizations are used in the exercise. Instead, they have to extract all relevant information from the diagram to get enough equations to set up a solvable (differential) equation by equation management.

From the view of problem-solving, the conventionalized diagram is used to clarify the problem in the form of an informative figure. It represents the components and how the setting is arranged, which helps to mathematize it. At the starting point of exercise B, the students have to be able to read the conventionalized diagram of the network containing an oscillating circuit. They require to know the technical terms and the physical properties of the components. For the determination of the values for $u_C(t)$, $i_L(t)$ and $u_R(t)$, they must read circuit diagrams and apply general physical knowledge on circuits as well as the voltage curve and current course of capacitors and inductors.

The students saw similar problems in the lectures and exercise classes, so they do not have to create new heuristics for solving this exercise but can rely on their experience from similar problems.

Mathematical-Electrotechnical Symbolic-Conceptual Manipulation and Reasoning
In the second step, a "world of mathematics" is not entered because the mathematical electrotechnical work uses quantities instead of just numbers. With this extension of mathematics by units, students can nevertheless apply formula manipulation skills. However, what we call "equation management skills" is not something that was explicitly taught in the mathematics courses and has the following characteristics:

- Students must regard functions as objects and unknowns in an equation and interpret formulas according to known and unknown functions and quantities (similar to Polya's applied problem solving).
- Students need skills in manipulating a set of equations so that one equation remains, where only one function and its derivative are unknown. No straightforward method for solving "systems of equations" exists.

In an actual solution process, it can happen that students have set up more or fewer equations or other equations than the three equations above and have to go back to the physical situation and their knowledge to add further equations. In other words, the mathematization and the symbolic manipulation steps can be intertwined.

32.7.4 Developing SES2 of B3 Based on the Expert Interviews

The PARI based interview, cf. Hall et al. (1995) and Sect. 32.5.2, was conducted with an EE-expert, who was, in this case, not the creator of the exercise but had a similar background and function in teaching and organizing EE courses. In the following, we cannot reproduce the three PARI-phases (see Sect. 32.5.2) but exemplarily summarize how we transformed the SES 1 into SES 2 based on the results of the PARI interview.

Remark In the second part (subtasks B5 to B7), the switch S_1 is opened at the time $t = t_1$. The transient is defined as totally finished at this time. Furthermore, switch S_2 is closed.

We focus in the following on possible misunderstandings in the mathematization phase that we asked the expert for. They concern:

1. The notation 0^- resp. 0^+
2. The understanding of the conventionalized diagram
3. The dealing with implicit idealizations

The notation 0^- (resp. 0^+) is important to note, as it is not part of the notations in MfES. Mathematically speaking, it stands for the limit to 0 from the left (right) of the respective function. In order words, the functions could have a discontinuity at 0. The FoEE terminology avoids the mathematical language of limits and continuity.

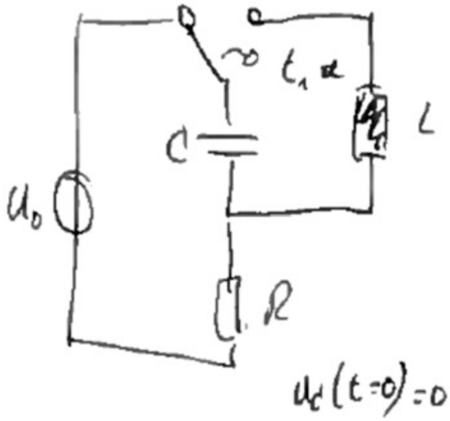

Fig. 32.3 Alternative circuit, sketched by the interviewed expert

In the second part, the expert thinks that students might read over that S_1 is reopened when S_2 is closed, giving them a more complicated situation of an RLC circuit. However, this possible hurdle can be clarified by working with another type of switch, the so-called changeover switch, which adds further information to the diagram and clarifies that both switches are closed at no time (Fig. 32.3).

The expert sees a further possibility of misunderstandings due to other implicit idealizations: "I also don't think it's very clever here to talk about an inductor. I would find inductance a little more suitable because it would symbolize at least an ideality to me. So, it specifies that there is no ohmic component in the inductor. When I talk about an inductor, I mean an inductor that is practically realizable, and this inductor is not practically feasible. That's why I would find it more pleasant if it said inductance and not inductor. The same with capacitor and capacity."

This quote shows some variability in dealing with implicit idealizations in the context of FoEE (remember that our expert was different from the task designer). Obviously, the inductor in the diagram stands for an idealized inductor with no ohmic resistance. The wording of the task assumes that this is "clear", but of course, this may be a source of problems if students interpret the diagram as showing a real inductor.

Given these three aspects, the interview extended the SES1 to a SES2 by remarks on possible misunderstandings, which can be on different levels, e.g. the notation, the diagram or the interpretation of used words. It deepens the SES1 by remarks, in which way information is compressed in the exercise presentation of FoEE and which effects small changes can have.

32.8 Development of the Student Expert Solution for Exercise B4 (the Solving of the Differential Equation)

32.8.1 Official Solution Outline

Subtask B4 Solve the differential equation. Which is the significant time constant of this circuit?

Solution
Solution of the homogeneous part with $u_{ch}(t) = U_{ch0}e^{-\frac{t}{\tau}}$

$$\begin{aligned} &RC\left(-\frac{1}{\tau}\right)U_{ch0}e^{-\frac{t}{\tau}} + U_{ch0}e^{-\frac{t}{\tau}} = 0 \\ &RC\left(-\frac{1}{\tau}\right) = -1 \\ &\tau = RC \end{aligned}$$

Solution of the inhomogeneous part:

$$u_{ci}(t) = U_0 \text{ for } t \to \infty$$

From this, the general solution follows:

$$\begin{aligned} u_C(t) &= u_{ch}(t) + u_{ci}(t) \\ u_C(t) &= U_{ch0}e^{-\frac{t}{\tau}} + U_0 \end{aligned}$$

Substitution of the initial values:

$$\begin{aligned} u_C(t = 0) &= 0 \\ u_C(t) &= U_{ch0}e^{-0} + U_0 = 0 \\ U_{ch0} &= -U_0 \end{aligned}$$

From this, the solution follows:

$$u_c(t) = -U_0e^{-\frac{t}{RC}} + U_0 = U_0\left(1 - e^{-\frac{t}{RC}}\right)$$

32.8.2 SES1 Object-Level: Extended Structured Solution Outline, Knowledge from EE-Theory Relevant for the Solving Process

Mathematical-Electrotechnical Symbolic-Conceptual Manipulation and Reasoning The equation set up in subtask B3 is an inhomogeneous ODE of order one in u_C. The solution process in MfES and the solution from FoEE have in common that they both split up the solving process into the solving of the homogenized ODE and finding a solution of the inhomogeneous ODE. In the perspective of MfES-courses, students can solve such equations as follows: At first, they have to homogenize, which is done in this case by omitting the U_0-term, as this term neither contains $u_C(t)$ nor its derivatives. This leads to $RC\dot{u}_C(t) + u_C(t) = 0$. The solving can be done in two ways:

1. The knowledge that all solutions of ordinary differential equations of the form $\dot{y}(t) - ay(t) = 0$ have the form $y = ce^{at}$, $c \in \mathbb{R}$, as the conditions of the Picard-Lindelöf-theorem hold for the function $y(t)$. I.e., as $ay(t)$ is uniformly Lipschitz continuous in y and continuous in t, there is a unique solution to the initial value problem for each choice of $t \in \mathbb{R}$, see Bruckner et al. (2001) for more details
2. Using the "separation of variables" as the general method for homogenized ODE of order 1.

The use of these resources would yield for the constant $a = 1/RC$ and that $c = y(0)$, so the solution in the last line of the official solution outline. In MfES, properties of function y (differentiability etc.) have to be provided. Students have to check whether the conditions for applying the mathematical theorems are fulfilled to justify their solution referring to the mathematical technologies in the sense of ATD.

However, the solution outline shows a different approach, rooted in the FoEE way of solving differential equations in the context of electrical circuits (a praxeology different from that of MfES). The concept of the time constant τ is a characteristic quantity of electric networks. This, in a sense, assumes that the ODE has already been solved. For the determination of τ, the approach $u_{ch}(t) = U_{ch0}e^{-\frac{t}{\tau}}$ is taught in the lecture. Instead of an approach like in Castela (2017) (see above), the time constant τ is introduced as an additional factor of t in the exponent with an EE-meaning. As long as $\tau \neq 0$, this does not influence the solvability of the homogenized ODE, as τ is constant. As $U_{ch0} \neq 0$, the insertion in the homogenized ODE results in the equation $\tau = RC$ (u_{ch} stands for the homogeneous solution, u_{ci} for the inhomogeneous one)

To find a solution to the inhomogeneous ODE, besides using the method of separation of variables, i.e. [2], the MfES-course presented the technique:

> Finding a special solution of the inhomogeneous ODE: in this case, this method can be done fast, as $u_{Ci}(t) = U_0$ solves the inhomogeneous ODE $CR\dot{u}_{Ci}(t) + u_{Ci}(t) = U_0$.

However, the solution outline from FoEE shows different reasoning, namely suggesting that $u_{Ci}(t) = U_0$ *is a solution when "*$t \to \infty$". This is a typical problem that we clarified with interviewing the EE expert. Below, we will point out that this is an instance of hybrid reasoning combining arguments from mathematics and FoEE.

Using the initial value $u_C(0) = 0$, students can calculate the value U_{ch0}, the particulate solution of the initial value problem. The method is the same except for choosing the constant's name, which can be freely chosen in MfES. The adding of the homogeneous and the particular solution, the complete solution is,

$$u_c(t) = -U_0 e^{-\frac{t}{RC}} + U_0 = U_0\left(1 - e^{-\frac{t}{RC}}\right).$$

The EE-theory is used in two aspects in the solution of B4:

1. The negligence of the solution $U_{ch0} = 0$.
2. The critical review of the results.

From the FoEE-course, it is known that the capacitor loads up to the value of the voltage source, U_0, via an exponential function. If U_{ch0} were 0, the solution of the homogenized ODE would be *0*. No exponential function would be part of the solution, as no exponential function could be a particular solution of the inhomogeneous ODE. This shows aspects of transposed praxeologies using pragmatic relations to the MfES-praxeology, see Castela (2017). Additionally, as the capacitor loads up to U_0, students know that this is the limit value for t towards infinity.

The students can review their solutions critically by looking at the physical situation. The function $u_c(t)$ starts at *0* and converges to U_0. This behavior corresponds to the physical behavior of a capacitor that charges up to the value of an ideal voltage source in the network.

32.8.3 SES1 Meta-Level: Structuring the Solution According to the Theoretical Approaches and Identifying Cognitive Resources

The enhanced solution sketch shows well that no complete entering into an abstract world of mathematics occurs. The symbols keep their EE-meaning, and even the reasoning in solving the inhomogeneous ODE contains hybrid arguments from mathematics and EE. The usual conceptualization of the modeling process suggests that one can separate the two processes: working in the world of mathematics and reason with mathematical means of reasoning only and interpreting the mathematical results later in the context. Hybrid reasoning is suggested. Moreover, mathematical justifications are not explicitly asked for, e.g., if the function $u_C(t)$ fulfills the conditions of the mathematical theorem that specifies the solution space of the ODE. Students however may have difficulties when both resources from FoEE and MFES are activated in their problem-solving process.

Moreover, students have further qualitative resources to validate the solution of the ODE. For example, the answer, $u_c(t) = U_0\left(1 - e^{-\frac{t}{RC}}\right)$, describes a function $u_C(t)$, which grows exponentially with an asymptote at $u_C(t) = U_0$. This corresponds to the charging of the capacitor until it reaches the maximal value U_0 in the experimental set-up.

The students saw similar problems in the lectures and exercise classes and know that this qualitative behavior of the system is considered correct. However, the qualitative knowledge that the solution is an increasing function with an asymptote does not imply that this function needs to be exponential. Moreover, although – mathematically speaking – U_0 is not reached at any finite time, it is interpreted in FoEE that the system practically reaches the maximum after a finite time, which however cannot be theoretically calculated. In similar cases, this practical interpretation of a limit is generally done in physics. Students are not asked to validate the results against experimental data from a real circuit that could yield knowledge about when the voltage is practically identical to 0.

These activities would be something that would be suggested if we see the example from the perspective of mathematical modeling, and students may be asked to do this in EE practical courses later in their studies. The conceptualization of mathematical modeling may better fit such courses.

32.8.4 Developing SES2 to B4, Based on the Expert Interviews

The following focuses on possible problems in the hybrid ways of solving ODEs using arguments from MfES and FoEE. The expert said concerning solving a differential equation: "For me, it is not necessary now to solve this differential equation formally because I see directly what form it will have. In the end, the capacitor will be charged to U_0. In the beginning, it has the value 0. Since it is only one energy store, a balancing process with an exponential function will occur between the two quantities." The "now" in his wording highlights his use of expert knowledge. He knows the behavior of such test arrangements and chooses to adjust the parameters to fit the given situation. He supports this statement with a drawing of a graph showing the qualitative increase of the voltage $u_C(t)$. With those arguments, he writes down the correct solution of the ODE and reviews it critically by inserting minimal and maximal values of t, i.e., $t = 0$ resp. $t = \infty$. This practice is purposeful in the context of EE, as it leads to a correct solution. He uses values of the function at two points to determine the constant k in $u_c(t) = k\left(1 - e^{-\frac{t}{RC}}\right)$. For this purpose, he inserts t equal to ∞, which gives $k = U_0$ (mathematically speaking, a limit of $t \to \infty$ was taken). So, in contrast to determining a homogeneous and inhomogeneous ODE solution in mathematical praxeology, the expert works with his physical knowledge. We regard this as a possible interpretation of the shorthand "$t \to \infty$" at $u_{ci}(t)$ in the solution outline for correctors, which makes no sense in terms of the mathematical praxeology.

Being asked for the learning goals of the exercise, the expert states that there are two goals: (1) solving differential equations, (2) working with principles of capacity and inductance, e.g., that both cannot change by leaps. The expert says that the treatment of ODEs in MfES is not sufficient for practical applications, which calls for contextualized solving strategies, and FoEE fills this gap.

To sum up: The interview extended the SES1 to a SES2 by remarks on hybrid strategies using both mathematical and EE arguments. By knowing physical mechanisms, the charging behavior of a capacitor (known by experiments) and determining unknown constants of functional expressions by physical ideas, the function $u_c(t)$ can also be set up. This alternative practice supplements the SES1 we had developed for B4.

32.9 Summary and Outlook

We presented and exemplified a methodology for developing theory-based "student expert solutions". They can be used to express competence expectations related to the application of mathematics in EE.

The developed methodology of the SES1 and SES2 gives a reconstruction of competencies and skills required in the EE exercises, i.e., it answers RQ1. The SES1 has two levels: The institute's solution is mathematically and physically augmented at the object-level. On the meta-level, a classification into theoretical frameworks and the addition of cognitive resources takes place. Then, through expert interviews, both levels of the SES1 are extended to the SES2, which includes implicit competence expectations. This is illustrated in Chaps. 3 and 4, starting with a SES1, which is theoretically enhanced by expert interviews to a SES2.

Sects. 32.7.4 and 32.8.4 present the reconceptualization of competencies and skills and provide a closer description of the interface between mathematics and electrical engineering, i.e., an answer to RQ2. In dealing with ordinary differential equations, two techniques from MfES can be applied: the so-called separation of variables or the superposition of a general homogeneous and a particular inhomogeneous solution. Examples of the interface are the translation of conventionalized sketches to set up mathematical formulas and hybrid methods like curve fitting to solve ODEs using specific values known from the experimental set-up.

We have used the SES to analyze related empirical studies in which students do the presented exercises (Kortemeyer, 2019). Our goal was to determine how students actually solve the exercises and what challenges, errors, and solution strategies occur. In addition to this use in empirical studies of students' solution processes, we see our paper making a twofold contribution. It is a methodological contribution to doing an a priori analysis of a task from an authentic teaching context supported by interviews of designers of these tasks. Last but not least, we see our analysis as support for aiming at an alternative conceptualization of the mathematical modeling cycle suitable for describing the mathematical practices in courses on electrical engineering or physics in general.

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Chapter 33
Tertiary Mathematics Through the Eyes of Non-specialists: Engineering Students' Experiences and Perceptions

Eva Jablonka and Christer Bergsten

Abstract This chapter brings together insights from studies of students' experiences from a position that takes into account the social and cultural conditions and the institutional context of mathematics for non-specialists. It complements and expands our earlier analyses of interview data from first-year engineering students in Sweden, with a focus on their appreciation of specificities of mathematical discourse encountered in the core mathematics, their perceptions of the usefulness of mathematics, and their experiences of studying mathematics as compared to other subjects. Drawing on Bourdieu's notions of field and habitus, we consider the control of content and pedagogy of mathematics as a service-course as an element of a larger symbolic struggle. This puts engineering students in a social position where they might be confronted with conflicting 'rules for the game' in core mathematics as compared to mathematics in the engineering sciences. Our findings reflect that success in the service-courses depends on recognising the criteria of pure mathematics rather than mathematical applications or modelling. We also reconstructed four different modes of perceived usefulness of mathematics. Further, we grouped students' perceptions of relations between mathematics and other subjects into three major dimensions, considering if and how hierarchies between these subjects were produced.

Keywords Engineering students´ appreciation of mathematics · Symbolic struggle in undergraduate mathematics · Qualitative interviews with engineering students · University mathematics discourse · Usefulness of mathematics · Pure versus applied mathematics

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33.1 Introduction – Students' Perceptions and the Curriculum

It has been well established that mathematics is a stumbling block for many students who are enrolled in "service-courses" for non-specialists with institutionalised curricula and teaching practices, for which mathematics faculty staff is responsible (Hochmuth, 2020). Major concerns in this context are low pass-rates and the resulting impact of failing a mathematics course on engineering graduation numbers (Faulkner et al., 2019; Heublein, 2014).

As some of these problems have been attributed to the perceived irrelevance of the course content, authentic course tasks that appear more significant for the targeted engineering knowledge have been designed (e.g., Alpers, 2017; Schmidt & Winsløw, 2021; Wolf & Biehler, 2016). However, in some places, as Faulkner et al. (2019, p. 97) observe, "engineering departments are increasingly looking at drastic options of taking students out of mathematics courses and teaching students mathematics themselves". In return, such a move might be perceived as a threat as the budget of mathematics departments often depends on the number of students enrolled in service-courses (Rota, 1997, cited in Faulkner et al., 2019, p. 98).

In this context, students´ experiences and perceptions have been investigated from a range of vantage points, mostly aiming at identifying motives, (self-)beliefs and specific needs of different groups of students, which, if they were taken into account by mathematics faculty, eventually would help to overcome the problems. For example, from the perspective of educational psychology, attempts have been made to relate differences in motivation, self-regulation and beliefs about the nature of mathematics to the students' mathematics performance (e.g., Berkaliev & Kloosterman, 2009; Code et al., 2016). Self-efficacy beliefs regarding (specific) mathematical capabilities have also been studied (Lent et al., 1991; Matsui et al., 1990; Zakariya, 2021) with the aim to establish "mechanisms" regarding sources of these beliefs and relations to programme choices or performance. These types of investigations, however, tend to essentialise individual differences and thereby construct fixed categories of more or less "problematic students". Hence, these studies ignore both the experiences in which students' perceptions originate and the social and cultural conditions of curriculum and teaching practices.

Kleanthous and Williams (2013) also investigated mathematics self-efficacy beliefs, which they conceive of as part of a "mathematical habitus" (as conceptualised by Zevenbergen, 2005) that is shaped by their family and their experiences at school, integrating constructs from social cognitive theory with aspects of Bourdieu's sociology of cultural reproduction. Their quantitative modelling suggests that the family influence concerns choice of (core or advanced) mathematics at school and has "inculcated an inclination towards mathematics that mediates students' dispositions towards studying mathematically-demanding

HE [higher education] courses" (p. 64). With regard to beliefs about mathematics and its usefulness, Gainsburg (2015) offers a taxonomy of "levels" of epistemological views of mathematics by engineering students, of which she sees a progression towards a more sophisticated understanding of its contribution to engineering practices as the wished outcome. The "highest level" includes an "increasingly expert-like knowledge of when to enact relativistic thinking or sceptical reverence" (p. 160). This points to a shift in faith in the significance of mathematical considerations as a basis for judgement. It does, however, obviously not develop at university, as the few students who exhibited it in Gainsburg's interviews were drawing on their own or a parent's experience of work as a civil engineer. Further, regarding the entire curriculum for engineering education, a growing discrepancy between demands from new professional practices and university education has been diagnosed (Adams & Forin, 2014; Crawley, 2001; Trevelyan, 2014). Consequently, the perceived difference between "theory" exposed to in their studies and the future professions the students aim at, might lead to questioning the rationale for identifying the knowledge base in terms of academic subjects on the whole, not only for core mathematics.

Altogether, studies that attend to the social and cultural conditions of the students' and teachers' activities are still rare. They have the potential of contributing to understanding differences in (emerging) identities between students specialising in mathematics and others who study mathematics as a service subject (e.g., Bergsten & Jablonka, 2013; Black et al., 2010). While in the context of the "socio-cultural turn" in research on university mathematics education (cf. Artigue, 2021), the cultural dimension (regarding specific mathematical cultures) is well theorised by the ATD, researchers interested in the social and political dimensions mostly draw on theories external to mathematics education (e.g., Adiredja & Andrews-Larson, 2017; Anastasakis et al., 2020; Bergsten & Jablonka, 2019; Jablonka et al., 2013; Williams & Choudry, 2016).

The investigation presented in this chapter complements and expands earlier analyses of data from interviews with first-year engineering students conducted in the context of a larger study of the transition between school and tertiary mathematics education in Sweden (cf. Bergsten et al., 2015). We first outline the theoretical horizon for our analysis, before we explicate the research questions and briefly present the context and methods, which we explain in more detail in the findings. These are structured in three parts, which concern (i) the students' appreciation of specificities of mathematical discourse encountered in the core mathematics, (ii) their perceptions of the usefulness of mathematics, and (iii) their experiences of studying mathematics as compared to other subjects. Finally, we discuss the outcomes.

33.2 Theoretical Horizon – Mathematics as a Service Subject, Disintegrated Practice and the Position of Engineering Students

From the perspective of Bourdieu, a *field* is “a relatively autonomous domain of activity that responds to rules of functioning [. . .] specific to it and which define the relations among the agents” (Hilgers & Mangez, 2014, p. 5). The academic field is characterised by similar effects as other social fields, in which we find,

> multiple material and symbolic positions occupied by purposive subjects, who, within their relationally structured spaces of action and interaction, compete over access to resources, influence, status and – ultimately – power (Susen, 2016, p. 4).

Academic fields are (re)produced “by agents with the *habitus* needed to make them work” (Bourdieu, 2013, p. 67), who acquire the cultural capital embodied in this habitus through (long term) pedagogic processes (in the widest sense). Central to Bourdieu’s notion of habitus is the dialectic between experiences and perceptions, as the habitus “ensures the active presence of past experiences, which, deposited in each organism in the form of schemes of perceptions, thought and action, tend to guarantee the ´correctness´ of practices and their constancy over time” (Bourdieu, 2013, p. 54).

The synchronisation between the professional habitus and the field defines the scope of possible actions and the way these are performed. Mastery includes automatised enactment of operational schemes, for example manipulating of formulae and instruments (Bourdieu, 1993; Bourdieu, 2013; Moore, 2014). The professional habitus of an established member is tuned to the current definition of what constitutes a problématique worth researching and includes mastery of specific techniques and knowledge based on a shared faith in the importance of the whole endeavour.

In engineering education, the relation between pure mathematics that aims at its own development and applications of mathematics in engineering practices has a complex and long history (Karp & Schubring, 2014). This history indeed reflects changing conceptions of the epistemological value of mathematics. Conflicting conceptions related to “classicality” versus “modernity” were in some places institutionalised in different tertiary institutions, for example, in Germany and Sweden (Schubring, 2014; Heinonen, 2006). Hierarchies between theoretical and applied mathematical subjects have been established in particular by the association between the level of abstraction with a concomitant level of alleged epistemic certainty (Oki, 2014).

In Bourdieu’s view, the outcomes of such struggles about classifications and ranks outlast the collective reflective memory of what has been at stake. As a consequence, the professional habitus is not only constituted by current characteristic repertoires of techniques, schemes of perception and convictions, but also reflects the historically evolved status of the discipline in relation to others (Bourdieu, 1993). Any hierarchising between pure mathematics and applications of mathematics in engineering sciences would then have survived in faculty habitus.

Bergsten et al. (2010) illustrated this claim by means of statements from interviews with mathematics faculty members with regard to the teaching of the fundamental theorem of calculus in core mathematics. Similarly, in a study about mathematics lecturers teaching at different departments, Bingolbali and Ozmantar (2009) found that "they consciously privileged different aspects of mathematics" (p. 597), depending on whether mechanical engineering students or mathematics students were their audience. Yet, those views contrast with these of engineers from a study in South Africa and Sweden, who deemed "conceptual understanding" in mathematics as important, in particular regarding the use of mathematical technology (Engelbrecht et al., 2017).

Altogether, this points to a disintegration of mathematical practices, in particular regarding mathematics for engineering. In a view based on Bourdieu's notion of field, the cultural capital needed for acquiring the feeling for the 'rules of the game' and the associated symbolic capital (such as distinctions of disciplines as rigorous or inexact, evidence-based or unscientific, hard or soft etc.) as well as economic resources are a focus of a symbolic struggle, in which members strive for establishing a superiority of their own distinctive features and an official sanction for these. The resulting classifications undergo a process of naturalisation. In the context of academic research (in mathematics and beyond), external funding has increasingly become what is at stake in exchange for symbolic capital. In this context, Malek-Madani and Saxe (2019) observe that applied and computational mathematics receive comparatively more grants. Viewed from the perspective of Bourdieu, this might have increased the status and influence of researchers who work in these subfields. On the other hand, in particular in the context of "shaping and selecting" new entrants, (rites of passage, examinations etc.)" orthodox approaches dominate (Bourdieu, 2013, p. 68), which we see reflected in the examples above from first-year core mathematics courses.

As a whole, these considerations help us sharpening the perception of the social position of students who study mathematics in service-courses, acknowledging that the control of the content and pedagogy of such courses is an element of a larger symbolic struggle. From this point of view, the engineering students are in a precarious position. They are not just relative mathematical newcomers who after investment of time and effort have the prospect of becoming insiders (like the mathematics majors with whom they share the introductory courses in the contexts we have investigated). Moreover, they will perhaps occupy a position as permanent outsiders whose habitus will not become synchronised with the field. That is, they do not and will not share with their teachers and fellow students the "schemes of perception, thought and action" (Bourdieu, 2013, p. 54) including their epistemic ideals and fundamental faith in the importance of the endeavour that defines the habitus of future mathematicians. In addition, they might be confronted with conflicting 'rules for the game' in core mathematics and mathematics in the engineering sciences.

33.3 Research Questions, Context and Data

Based on the considerations presented in the previous section, our investigation was guided by the following questions:

- *How will the engineering students acquire the 'rules of the game' in mathematics?*
- *How do they construct the importance of mathematics for their chosen engineering fields and their career?*
- *How do they develop a motive for investing time and effort, in particular in comparison to other courses from their chosen programmes of study?*

The study took place at two Swedish universities. We interviewed 60 students enrolled in five-year engineering education ("civil engineering"). These students were purposefully selected from five study programmes (Table 33.1) and different achievement levels in mathematics (in a diagnostic entrance test) in order to allow for differentiated insights, in particular as the programmes are not similar regarding the alleged or imagined usefulness of mathematics.

The mathematics courses attended by these students were delivered by mathematics faculty and were also attended by mathematics majors. Neither this practice nor the content of the courses has undergone substantial changes since we conducted the interviews in 2011. The interviews were semi-structured and individual, conducted in the middle and at the end of the first year of study.[1] They were audio-taped, transcribed and analysed in the original language.

The specific prompts used in the interviews to pursue the research questions and further details regarding methodology are integrated in the next sections.

Table 33.1 Study programmes and number of students interviewed at the two universities

Study programme	University 1	University 2	Total
Computer technology (**C**)	9	1	10
Energy and environment (**E**)	10	1	11
Industrial economy (**I**)	8	2	10
Mechanical engineering (**M**)	11	5	16
Technical physics and electric engineering (**T**)	9	4	13
Total	47	13	60

[1] During the first year the programmes included a foundation course in mathematics and courses in linear algebra and calculus in one or (for two programmes) several variable(s). The programme C included discrete mathematics.

33.4 Findings

33.4.1 Recognising a "Mathematics Text"

In the interview with the students after the first series of exams, we were interested in the extent to which students know what counts as a legitimate mathematical activity and how they recognise and articulate this (Jablonka et al., 2017). The characteristics of university mathematics, as delivered by staff from mathematics faculty, have commonly been described in terms of rigour, high level of abstraction, and formalisation (e.g., Gueudet, 2008), which not only contrasts with school mathematics but also with other forms of more applied mathematics found in some of the engineering core subjects. We were also interested in the students' achievement, because – at least theoretically – the course examinations test whether new modes of producing and communicating mathematical 'truths' have become part of the students' habitus. The achievement we categorized in three levels, based on their grades on all mathematics courses during the first year.[2]

In order to investigate the students' recognition of criteria we (i) asked the students to compare and rank four excerpts from different calculus textbooks in terms of which they felt were "more mathematical"; (ii) showed four authentic calculus exam tasks with (fictitious) student solutions and asked to mark these and provide reasons. The selected textbooks represented a variety of expositions of introductory calculus, which we analysed in terms of stronger or weaker "classification" of the knowledge and "framing" (Bernstein, 1996) of the pedagogic relation between author-teacher and reader-student. Based on these differences, we then ranked the four textbooks. We also asked mathematics faculty staff (who teach the core mathematics) in a focus group interview for their ranking, which accorded with our theoretically derived one (see Jablonka et al., 2017).

We expected a correspondence between students' recognition of realisation principles for the text that was considered 'most mathematical' and their success in the exams. Our findings confirmed this expectation (for details, see Jablonka et al., 2017). More interestingly, the arguments for the rankings provided by students at different achievement levels differed in terms of focus. Academically successful students pertained to the content, level of technicality and coherence as characterisations of mathematics texts, as for example:

> these here now [the strongly classified texts] deal more with the mathematics itself... describe things within the mathematics (**1E3-H**).[3]
>
> much palaver about greater than zero and such stuff (**1M7-H**).
>
> this is proof ...with lots of intervals and continuous (**1E4-H**).

[2] We denoted these levels L (low), M (middle) and H (high). See Jablonka et al. (2017, p. 81) for a description of how the levels were constructed.

[3] The code 1E3-H for identifying this student indicates that he/she studied at university 1 and followed the programme E (see Table 33.1) listed as number 3, with achievement level H.

In contrast, academically less successful students often talked about how they experienced the accessibility of the texts and provided a different ranking:

> this one [a weakly classified text] I think feels clear and good... this one I like... structured and such... most academic possibly... doesn't mix so much letters and numbers but partitions it like this... so that the brain can more easily register if each stands in its own line (**1I12-M**).
>
> simply harder to understand [a strongly classified text]... here they assume things all the time... very very much theory (**1M1-L**).

As to the second interview prompt, the high-achieving students tended to focus more clearly on the mathematical content in the solutions than the others, which in Ashjari's (2013) interpretation indicates awareness of a more developed mathematical praxeology (e.g., Chevallard, 1999). Moreover, we also observed differences in certainty regarding the level of detail required in written solutions.

33.4.2 The Usefulness and Role of Mathematics

In the last interview towards the end of the first year, one prompt concerned how the students thought about the usefulness of mathematics for their (imagined) future workplace or other potential gains from studying (some) mathematics. In the following we focus on this part of the interview and present a new analysis of the full set of data, which we have only partly analysed previously (Bergsten & Jablonka, 2013). We started with a question about concrete uses of mathematics at work places associated with their programme of study, followed up by questions about applications in other courses or possible other gains, if any. The latter also was intended to allow reference to the symbolic capital of high marks in mathematics (e.g., for finding a job); something, as it turned out, none of the students mentioned.

In the analysis, our initial attention focussed on whether the answers referred to an embodied mental schema or to examples of useful mathematics as a more specific professional (material or thinking) tool. From this starting point, in an open process of coding, we distinguished seven modes of the usefulness of mathematics in their ideas.

33.4.2.1 General Mathematico-Logical Thinking

In this category we included references to general "ways of thinking" acquired though mathematical activity or to a particular mathematical "gaze" and other forms of embodied changes of the epistemological functioning of the self that the students saw as an outcome of mathematical learning, such as in the following examples:

> It is more like an understanding... a mathematical gaze that you have (**1M3-M**).
>
> developing the head a little... thinking logically (**2I2-M**).

to bring in the mathematical thinking is always good whatever you're up to do... to acquire a logical thinking... not solving equations and such things (**1M1-L**).

maths I think is something that is always good 'cause you develop a special way of thinking and the brain develops in a good way (**1I7-M**).

you learn to take in... to understand mathematical relations... then you learn to understand relations kind of generally (**1C6-L**).

33.4.2.2 Schema for Learning

Under this category we subsumed statements about changing or expanding ways of learning through the activity of learning mathematics, such as:

to learn how to learn (**1E8-M**).

learn how to learn new systems (**1I7-M**).

learn to understand relationships in general... learn to learn... easier to understand new things... (**1C5-H**).

33.4.2.3 Ways of Thinking for Systematic Problem-Solving

In contrast to the acquisition of mental schemes that were directed towards perception, insight and learning, we found many explanations of "ways of thinking" that referred to the activity of solving problems. Many of the students described changes in common terms associated with analytical thinking as in some of the examples below:

one does learn a certain way of thinking and to break up a problem into smaller parts and then solve each part kind of separately... and so finally solve the whole thing (**1M9-H**).

one always has a structured way... in maths to solve a problem and then one can bring that way of thinking into many other things in life (**2I2-M**).

it is to acquire this problem-solving-thinking and that is I guess what it means to be an engineering masters... to be a problem solver (**2E1-L**).

you more easily see through the problem... you can put it into small pieces somehow so that it doesn't get so big [...] I already feel that one has kind of changed as a person by the maths (**1I12-L**).

if not doing the calculations it is more this way of thinking... the problem-solving-ability (**1I9-H**).

33.4.2.4 Understanding of the Mathematical Underpinnings of Activities at Work Place

In a few statements students referred to activities at (imagined) work places related to their programmes of study. Rather than giving concrete examples of mathematical techniques that could be useful tools, their explanations indicated that they referred to understanding the principles of an activity.

the calculations are done by the computer but the very understanding of what you are actually doing... the process... is perhaps important but not calculation (**2M1-L**).

you have to understand the basis... otherwise there is no purpose... if you don't understand what you are doing you can't understand if something gets wrong (**2M5-M**).

will be better programmer if you can do [mathematics]... but not necessary (**1C4-H**).

well... understand what the computer does... reasonableness (**2T3-H**).

33.4.2.5 Understanding of Mathematical Underpinnings of Other Academic Subjects

As in the category above, the statements refer to the foundational, theoretical character of mathematics, that is to "understanding" of other subjects, rather than to listing concrete mathematical techniques they identified as useful.

you can only see from the courses we read that are not mathematics that most of them are based on some kind of mathematics that we have learned... linear algebra in mechanics... even integrals (**1M6-H**).

very foundational for physics as we have started now so you need the mathematical knowledge (**1T1-M**).

a lot is based on it [mathematics]... only if you are going to have it for some courses you have... then build on... for the understanding... have the foundation before getting deeper into it (**1T8-H**).

33.4.2.6 Applications of Mathematics at Workplace

Only very few statements we interpreted as referring to concrete applications of mathematical techniques in problem contexts that might emerge in their future professions.

we as engineers are supposed to develop the new technology and for that we need mathematics (**1T1-M**).

you can make calculations on pollution and... there is mathematics everywhere... there is so much in technical solutions (**1E4-H**).

mathematics is in the programs... the closer to the hardware the more you have to do [mathematics] (**1C10-M**).

33.4.2.7 Applications of Mathematics in Other Academic Subjects

Very few students talked about applications of specific mathematical techniques in their other academic subjects, but many assumed that this will happen in the course of their studies.

software courses... some applications (**1C2-H**).

the few courses we have read we have still used in thermodynamics perhaps and in mechanics (**1M10-M**).

good for the courses that come later (**1I7-M**).

the benefit does not come out in the [maths] courses... but in the later courses you see it [...] depends on what you choose next (**1M6-H**).

the differential equation bit is quite useful... integrals and derivatives a lot in physics (**1E11-H**).

Table 33.2 Modes of perceived usefulness of mathematics

	Orientation for perception	Orientation for action
Useful mental schema	General mathematico-logical thinking Schema for learning	Ways of thinking for systematic problem-solving
Useful tool (material or mental)	Understanding of underpinnings of activities at workplace Understanding of underpinnings of other academic subjects	Applications of mathematics at workplace Applications of mathematics in other academic subjects

We organised the dimensions as depicted in Table 33.2. An interpretation of the usefulness of mathematics as providing an orientation for action when faced with tasks at work place for which a specific mathematical tool was available (lower right cell), would clearly define it as an applied subject. This was not reflected in the students' comments. We rather found the usefulness of mathematics seen as providing a general orientation for action via acquisition of a mental schema for problem-solving (upper right cell). This view was often complemented by comments on its foundational nature for other subjects or activities related to engineering work places (lower left cell).

We observed a stark contrast between the sparseness of concrete examples of useful mathematical tools and the repeated articulation of the idea of using a particular way of thinking acquired in mathematics for solving non-mathematical problems, not necessarily only in engineering contexts. Regarding examples of applications at workplace, there were only very few general ones mentioned. Most students were not sure about the concrete usefulness of particular techniques, which for them appeared to be compensated by having acquired general schemes for thinking and problem solving; only very few said they considered mathematics as useless. For some, the application of mathematics appeared as mediated through other academic subjects, which in turn rely on mathematical principles or directly employ specific mathematical techniques. Mathematical modelling was not mentioned.

33.4.3 Ways of Studying Mathematics Compared to the Other Subjects

This section presents new findings from the same interview as the previous one, here concerning comparisons of mathematics and other subjects. The question was posed as, "How is it to study mathematics in comparison with the other subjects you have?"

In their answers, the students referred to a range of aspects. In our coding we grouped these into three major dimensions, which are relevant in the theoretical framing of the study. For the presentation in this chapter, we looked at the data with a particular sensitivity towards if and how hierarchies between subjects were produced. The following examples illustrate the rationale for our grouping; excerpts that appear to fit into more than one group were checked against the context of the statement. As obviously the dimensions are related and contextualise each other, this was not always possible.

33.4.3.1 Knowledge Structures, Criteria for Accomplishment and Intellectual Demands

The knowledge structure of mathematics in relation to other subjects was described in different ways, but some commonality relates to a form of verticality:

> I think maths is the most structured to study… that's what's easiest really now (**1T8-H**).
>
> with maths [you need to deal] all the time… if you are lagging behind in something (**1D5-H**).

Hence, in mathematics it is "much more important to keep up" (**1D9-H**) in contrast to, for example, programming where "you like sit there and try" (**1T7-H**), and "there are a lot of propositions and axioms to remember", while economics is "more pure facts and then not as important if you forget something" (**1I1-L**).

Some described mathematics as a collection of well-defined methods with typical problems solved by following "a specific template to arrive at the answer" in contrast to "a performance to present a finished result" (**1D3-H**) such as in programming. In contrast to mathematics, in other subjects "you can take a little different paths" (**1M5-M**). This amounts to contrasting "that you know the type problems and then you can do it" with having "to work in a completely different way with the understanding" (**1T9-H**).

Precision, clarity and exactness was felt as being important in mathematics, often mentioned in sharp contrast to economics, as "in economics you can happen to be careless and it can be good anyway" (**1I8-H**), or:

> economics feels very slack… a little more fuzzy […] it can be right in so many ways… math is usually one way that is right and the others you try to come up with are wrong… so it is more black and white (**1M6-H**).
>
> it's like night and day… so economics lectures […] it feels like they do not have anything real… it feels quite unsound […] the economics book I read almost like a fiction book (**1I12-L**).

The perceived precision and coherence also entailed a clarity of criteria for accomplishment in contrast to other "more fuzzy" subjects where "you do not really know what to do" (**1T9-H**), while in mathematics:

> [you] get a problem [check whether] it is wrong… it is quickly okay so I can go ahead and try to solve some new… in other courses it may be that you are completely lost and do not know what to do (**1I11-H**).
>
> on the whole I think it is more fun… that it is exciting to solve problems... there is only one solution it is not something that you can discuss about […] it is nice (**1I7-M**)

This difference in clarity also applies to exams. As mathematics is “more theoretically… controlled and very strict […] the exams also feel more controlled”, while “in the other courses it can be a little harder to know” (**1M5-M**). In economics “you can only argue for it so you can be right whatever you say… so it is also on the exams the math is so all-out exact” (**1M6-H**).

The perceived abstract nature of mathematics for some felt more demanding, for example:

> [other subjects] are more practical… create programs and calculate circuits… the maths is the more theoretical […] practical things are always easier you can see what’s happening (**1D7-L**).

Higher intellectual demands were also described as mathematics being “heavy” (**1E7-M**) or needing more “focus” (**1T7-H**) or making you “exhausted” (**1M3-M**).

Some other subjects were recognised as “mixed subjects” in relation to mathematics, like mechanics and thermodynamics (**1M8-H**), physics and programming (**1T10-H**), statics (**1M3-M**) and others, where additional subject specific features were perceived as either theoretical or practical:

> it [thermodynamics and chemistry] is reminiscent of math but there is much more theory behind… easier to calculate […] (**1E4-H**).
>
> waves physics is quite similar to calculations but you have it more in a practical way (**1T1-M**).

Table 33.3 summarises these contrasts, most of which are represented in the selected excerpts above.

Table 33.3 Contrasts between mathematics and other subjects related to *knowledge structure*, *criteria for accomplishments* and *intellectual demands*

Mathematics	(Some) other subjects
Logical structure Connected Closed range of solutions and methods	Collection of wide-ranging facts Independent areas Open range of solutions and methods
Types of tasks and legitimate methods Explicit methods and criteria Precision needed Control of solutions needed	General understanding implicit criteria in discussion and argument Precision not needed Control not needed
Less demanding because of - clear criteria and coherence More demanding because of - abstractness, intensity and required focus	More demanding because of - lack of clear criteria and unconnected nature Less demanding because of - practical nature, lack of rigour and less needed focus
Mixed subjects (e.g., thermodynamics, mechanics, electronics)	
Similar to mathematics but more subject-related theory Similar to mathematics but more practical	

33.4.3.2 Forms and Habits of Working and Thinking, and Their Appreciation

Differences in forms and habits of working and thinking were by some directly related to differences in knowledge structure and also were a source of positive appreciation:

> [in other subjects it is] just reading and take in lots of facts. . . maths is more calculation and a little more about understanding [. . .] it is nice to get the maths and. . . start thinking like more logic (**1E6-H).**
>
> economics is not logical the way that maths is [. . .] economics you can cram [. . .] a maths course you have to understand more [. . .] (**1I6-M**)
>
> you have to be formal you should not be careless. . . it becomes somehow more interesting . . .you have got so many connections so you see that things are connected in all possible ways . . .strange ways (**1M7-H**).

On the other hand, the relative closure of the subject, in which there is not much to discuss about (see above), also featured as a source for positive appreciation.

Differences in ways of working through examples featured as a comparison, which made mathematical work less cumbrous:

> like maths you learn how to do and so you do so because if you do not do so it does not work. . . electronics is difficult to get a grip on. . . it is wrong somewhere along the way and you do not know where. . . maths runs pretty well (**1D6-L**).
>
> maths is quite nice it is. . . against for example thermodynamics [. . .] chemistry [. . .] which is quite messy. . . very broad facts where you like have no further help [. . .] it is not that hard then in math. . . more often you can just sit down and calculate a bit (**1E11-H**).

Differences in the ways of working were also framed by a contrast between extracting information from texts and dealing with numbers:

> environmental engineering is quite different. . . there is more sitting and grinding in [sic] books than sitting and grinding in [sic] numbers (**1E2-H**).
>
> practice practice practice [and in the other subjects] read. . . take in information yes (**1I9-H**).
>
> [maths]. . . is more fun and easier. . . not so much fun to just sit and read read and read. . . you avoid it. . . here you sit and calculate and it's well practical in that sense (**1I4-M**).

A lack of applicability was for some a source for lack of positive appreciation or interest:

> to sit and study maths one night it is not as tempting. . . [programming and logical design] are the applications you immediately feel that this is something I can enjoy. . . in maths. . . it is abstract (**1D5-H**).
>
> more interesting. . . the economics. . . that you can follow more in the news and stuff (**1I10-M**).
>
> I'm more interested in technology. . . it feels a bit dry I think with maths. . . it feels like you do a lot of things but. . . it does not feel that you can use it just now (**1T5-H**).

The different comparisons illustrated above are summarised in Table 33.4.

Table 33.4 Contrasts between mathematics and other subjects related to *forms of habits of working and thinking*, and *their appreciation*

Mathematics	(Some) other subjects
Exciting to solve problems with one solution Clear methods and smooth solutions Easy, fun and more practical to calculate Calculating and understanding Easy to calculate and check	Just discuss about issues Difficult to oversee complex facts Just sitting and reading Just remembering lots of facts Difficult to identify errors
Abstract and not tempting to deal with Dry feeling Less interesting	Applications immediately enjoyable Feeling of usefulness Helps following the news
Intense practicing Continuous understanding	Intense reading Cramming before the exam

33.4.3.3 Investment of Time and Effort, and the Worth of Credit Points

The need for more investment of time and effort was established as an immediate consequence of the difference in knowledge structure, intellectual demands or related to perceived importance:

> maths is more... pure... difficult... maths is simply another way of thinking... so I have to spend more time (**1M9-H**).
>
> maths needs almost more time it takes a while before you get into it all (**1T1-M**).
>
> the other subjects are less demanding [...] [for maths] you probably have to spend more time [...] it feels like the maths subjects are very important for you to understand the other subjects... it feels fundamental [...] you apply mathematics in all subjects (**1M10-M**).

The organisation in the form of lectures as compared to projects and labs leads to more continuous investment of time:

> [in contrast to project-based work in environment and energy]... chemistry and thermodynamics are probably more like maths that you work with all the time (**1E5-M**).
>
> [In economics it] is more to sit in a group and discuss... in programming no extra work... just delivering the product (**1M1-L**)

These differences also have a consequence for passing:

> [in economics] you can often get a few points lab like... in math everything is on the exam (**1E9-L**).

As a consequence of the investment, the worth of credit points in mathematics and other subjects differs:

> two points in maths are worth more than two points in economics (**1I8-H**).
>
> the four-point course in economics... it is half... if even that... as difficult as the maths courses of four points (**1I12-L**).
>
> the economics course... is very, very simple compared to maths... it is the same with the environmental courses that have been... it is maybe 10 or 20 times more you spend on the maths course (**1E8-M**).

In Table 33.5 the students' comparisons of investment are summarised.

Table 33.5 Contrasts between mathematics and other subjects related to *investment of time*, *effort*, and *the worth of credit points*

Mathematics	(Some) other subjects
More time needed because - special difficult way of thinking - slower process to get into it - more demanding	Less time needed
More continuous work needed because - organisation in lectures More difficult to pass because - only one exam	Less continuous work needed for - labs and project-based work Less difficult to pass because - variety of forms of examination
More investment because - more important and foundational - more respected subject - more demanding and more fun - more intense and extensive	Less investment needed
More work, time and effort per credit	Less work, time and effort per credit

In summary, the students described the knowledge structure of mathematics in different ways, ranging from connections between axioms and theorems to a collection of methods. The verticality of the subject, also reflected in its organisation in the form of regular lectures, amounted to a more continuous engagement with the subject. Many forms and habits of working, which are related to the knowledge structure, were appreciated by the students in contrast to other subjects, which they perceived as less coherent, unprincipled or lacking clarity of criteria. Doing calculations and obtaining solutions was described as more enjoyable than extracting information from texts.

There were only a few who expressed a lack of appreciation in relation to intellectual demands, abstractness and theoretical outlook. Other mathematics-heavy subjects, “mixed subjects”, were seen as either more theoretical (in terms of conceptual structure of the other subject) or more practical (in terms of applications).

The perceived intellectual demand, the amount of material in combination with the pace, and the perceived importance and status of mathematics led to more investment of time or effort. As an outcome, a symbolic economy of credit points emerged, with different exchange rates between mathematics and economics. Hierarchies were also established in terms of the amount of the matter covered, with mathematics ranking over or similar with electronics, and over programming. Also, the stronger coherence, its function as basis for other subjects, and more clear criteria for a legitimate knowledge production contributed to a higher ranking of mathematics as compared to for example chemistry, economics and environmental engineering.

33.5 Discussion

The framing and findings of our investigation of students' perceptions and experiences of mathematics in the form of service-courses alert to the institutional environment, in which these are situated. As we outlined in the second section, without overusing the idea of a profession-specific habitus of mathematicians in distinction from engineers, the concept still might help to draw attention to processes of how a stratified organisation of the academic field is (re)produced at the level of interaction at the micro-level in emerging hierarchies of subjects and students. Other examples pertain to mathematics service-courses in teacher education regarding the future careers that separate groups of students (Hanke & Schäfer, 2018) or the transition from school to university (Stender & Stuhlmann, 2018). A distinction between types of students is most salient in concerns about support-centres for statistics, in particular for students from the 'soft' sciences (Lawson et al., 2020).

While it might be that "[t]he 'pure and applied' dichotomy is just a false opposition" (Alsina, 2001, p. 10), we have argued with reference to Bourdieu that it still might be inherited via a habitus that reflects the socio-historical background of the organisation of the field, embedded in a struggle for status, resources, the 'best' students and influence on the curriculum. Our findings concerning the recognition of what counts as a mathematics text proper, in which students as well as mathematicians ranked textbook-excerpts, reflect that the success in the service-courses depends on recognising the criteria of pure mathematics, and not of mathematical applications or modelling in the engineering sciences. Hence, in the context we have investigated, the distinction between 'pure' and 'applied' mathematics still (re)-produces in the socio-cultural practice, in which service-courses are embedded, a specific social hierarchy based on implicit rules and the values and schemes of action embodied by its members.

Regarding the usefulness of mathematics, the students indeed recognised some mathematics in other courses, despite the incongruency of knowledge criteria in the core mathematics with what counts as mathematics in the programme-related core courses (such as mechanics, electronics or programming). The idea of mathematical modelling (either in the form of theoretical models or through the use of mathematics as an empirical language of description) did not surface, just as little as any concrete examples of useful mathematical techniques. However, some alerted to the importance of knowing mathematics for understanding outputs of mathematical black-boxes. As Artigue et al. (2007) noted in the context of (mis)recognition of mathematics by engineers, it is the definition of mathematics that is at stake. The characterisation of different mathematical cultures in terms of "praxeologies" (Hochmuth & Schreiber, 2015; Job & Schneider, 2014; Winsløw et al., 2014), in which the concomitant knowledge norms emerge from socialisation in institutionalised practices, certainly relates to the idea of a professional habitus that follows a repertoire of practical schemes as per the writings of Bourdieu.

Despite the lack of perceived usefulness, the students in general did not resemble the often quoted dissatisfied group of customers of service-courses, which do not fit their needs. However, they did not conceive of mathematical methods as tools for solving profession-related problems either. The mathematical content featured as unrelated or as foundational for methods used in other disciplines ('understanding what one is doing'), reminiscent of the idea of many mathematicians that theory must precede applications.

Much more salient and held by most was the idea that mathematical activities develop their 'faculty of reason' (cf. Sect. 33.2), which we, however, further differentiated into descriptions of 'introversive' mental schemes (e.g., logical thinking) and 'extroversive' mental schemes in the form of approaches to solving problems. The latter were described in similar ways by many, even in a reified form as "problem-solving-thinking". Hence, a form of a not directly useful mathematics, in which nevertheless much time and effort was invested, was – perhaps as a compensation – interpreted as at least providing some orientation for action (in contrast to only for perception), which we might interpret as in line with an engineering habitus rather than with a general "mathematical habitus" (Kleanthous & Williams, 2013).

In the students' perceptions and experiences in comparison with other subjects, not only contrasts, but also hierarchies emerged between subjects. Mathematics enjoyed a high status based on a set of values related to perceived coherence, precision, clarity of criteria or associated higher intellectual demands. Lower ranked subjects were, for example, programming, chemistry, environmental engineering and economics. The latter featured as paradigmatic example of a subject lacking rigour, clarity and verifiable criteria for knowledge production, which obviously irritated the habitus of many students. Based on the greater amount of time and effort spent for work in mathematics, a symbolic economy of credit points emerged, with mathematics credits as the stronger currency.

Altogether the students' views reflect a hierarchy in two dimensions. One concerns the difference between foundational and applied subjects, the other between 'rigorous' and 'inexact' sciences. The disavowal of features attributed to 'soft' sciences can be interpreted as a socialisation into engineering professions, with a habitus that (mis)interprets these as detached from social or political contexts. It would be interesting to investigate whether the role of mathematics and the 'soft' sciences is seen differently by those who have chosen the programme Energy and Environment, for which we find some indication in our data.

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Chapter 34
Early Developments in Doctoral Research in Norwegian Undergraduate Mathematics Education

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Abstract This chapter reports from the early development of a Norwegian centre for excellence in (higher) education that was initially proposed to promote a vision of improved learning experiences in service-mathematics for students of other study programmes such as engineering, economics and science. The chapter considers the initial focus of the Centre's programme that sought to develop an international network of expertise and to learn from literature that exposed possible reasons for students' poor performance in mathematics. The second part of the chapter comprises seven brief reports from doctoral research promoted by the Centre. This research forms an important part of the Centre's effort to develop a programme of research into undergraduate mathematics education (RUME) in Norwegian higher education contexts, the scope of which is briefly described. The doctoral research reported offers an overview of inquiries into issues of concern central to the Centre's vision – active learning approaches and students' attitudes. The studies are mostly qualitative inquiries, these explore alternative teaching approaches such as "flipped classroom" online and blended learning, the use of digital simulations and modelling, competence development, and targeted provision for weaknesses in prior knowledge. Additionally, one study employs cutting edge statistical models to expose causal effects of students' self-efficacy on learning performance. As with all doctoral research, these studies make individual contributions to the international body of knowledge in the topics researched, together they represent an important foundation for RUME in Norway.

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Keywords Self-efficacy · Competency development · Mathematical modelling · Mathematics support · Online learning · Flipped classroom

The purpose of this chapter is to describe early developments of research that focuses on university mathematics education in Norway. Until the creation of a Centre for Excellence in Education (MatRIC) that addressed mathematics teaching and learning in Norwegian higher education institutions, there had been limited engagement in research in university mathematics education in the country. The mission of MatRIC is to improve the quality of teaching and learning through the sharing and dissemination of good practice, to encourage and support innovation in instruction and learning support, and to develop research into undergraduate mathematics education. This chapter does not report a single study, nor does it set out to inform about results that could influence instruction in other institutional or national contexts. Rather, the chapter describes some of the contributions to knowledge creation that form part of MatRIC's endeavour. More than anything, the chapter places a marker that lays out early developments of the research field and a statement of the present position. It is possible the true value of the chapter may lie in the future, as the foundations are evaluated and examined for characteristics that have advanced, or possibly obstructed development of the research field.

A major part of the chapter is composed of seven brief accounts of research pursued by doctoral fellows whose projects contribute to the knowledge creation endeavour of MatRIC, Centre for Research, Innovation and Coordination of Mathematics Teaching. The common ground for the research projects reported is their focus on teaching and learning mathematics as a service subject within Norwegian universities. The research projects adopt a variety of approaches to address concerns about teaching development (flipped classroom and blended learning approaches), shortcomings in students' prior knowledge, use of digital technology in learning, mathematical modelling, and exposing causal relationships between learning approaches and outcomes. The range of projects pursued is indicative (not encompassing) of the breadth of challenge confronting a Centre established with a vision that focuses on "students enjoying transformed and improved learning experiences of mathematics in higher education." Before introducing these brief accounts, the background of MatRIC and challenges faced by mathematics learners and teachers in Norwegian universities are described.

MatRIC is a Norwegian Centre for Excellence in Education (CEE), the CEE status was awarded at the beginning of 2014; each centre receives national funding for a period of 5 or 10 years. The CEEs are expected to disseminate excellence in education, and promote teaching and learning based on, and informed and enriched by, cutting edge research and development. CEE status is awarded on the basis of evidence of existing excellence in education together with a convincing action plan for achieving the national CEE programme's goals.

Initially, MatRIC focused on teaching and learning mathematics as a service subject, that is mathematics within other educational programmes such as economics, engineering, and science. This focus reflects the balance of mathematics education at the host institution, University of Agder (UiA). The university evolved from several smaller institutions that were especially concerned with professional education such as teacher education, engineering, health care, etc. Consequently, professions-oriented education is well-developed at UiA, and several hundred students study mathematics for economics and engineering. Although the proposal for MatRIC pointed to several evaluations of the university's educational provision that indicated excellence in mathematics teaching,[1] high failure rates at the end of students' first mathematics courses were endemic. Regularly around 40% of students, or more, would not achieve a pass grade resulting in unacceptably high repetition and dropout from programmes.

International research suggests there could be two fundamental reasons for the failure rates noted above – large classes and deficiency in prior knowledge. It must be admitted that much of the research relating to the effect of class size comes from school classes and is restricted to classes of less than 50 students (Englehart, 2007). The interpretation of the research is contested, it is used both to argue that class size has no effect and does have a significant effect on students' performance. However, the research to which the reader is directed here is the widely cited meta-study of Freeman et al. (2014). In this meta-study of 225 research reports relating to the positive effect of active learning approaches it is noted that the larger gains (by students in active-learning classes) were noted in classes with fewer than 50 students. At UiA, service mathematics for economics and engineering students could number between 250 and 500 students. Service mathematics classes of this size are not unusual in Norwegian universities.

Reasons for the inferior performance of students in very large classes may be hypothesised by extrapolating from research at school level mathematics and educational/psychological research. In the context of an under-developed research context, such as Norwegian research into university mathematics education, such extrapolation from related contexts – international, educational level and disciplinary knowledge – provides a useful starting point. The first issue we note is that with large groups it is difficult to provide timely and constructive feedback on individual student's work; the positive effect of feedback is well-known (e.g., Black & Wiliam, 1998; Hattie, 2008) and not contested. MatRIC addresses this issue by exploring alternative forms of assessment and feedback provision, in particular the use of computer aided assessment. Another consequence of the very large classes, perhaps not leading to high failure rates directly, but resulting in diminished student motivation and interest is that summative assessments tend to be based on routine tasks. Such tasks are favoured because they are easier and quicker to mark and apparently have greater statistical reliability than open problem-solving tasks that make greater demands on the markers' judgement. Students' preparation for assessment is likely to be extrinsically motivated and focus on procedural fluency rather than conceptual understanding, and extrinsic motivation is argued to reduce

[1] Most of these were programme or institutional evaluations conducted by the Norwegian Agency for Quality Assurance in Education (NOKUT).

engagement (Deci & Ryan, 2000). Motivation and interest are also likely to fall victim to the practice of forming large groups of students from several programmes of study, thus obstructing attempts to present the mathematics as relevant to individual students' interests. Furthermore, in an attempt to provide learning support to smaller groups of students, considerable use is made of student learning assistants (SLAs). At UiA these are drawn from cohorts in their later years of study on bachelor- and master-level courses. These students are committed and enthusiastic and MatRIC has developed a programme to provide them with some preparation for the task and on-going support. Nevertheless, SLAs lack the depth of knowledge and experience of course teachers.

As noted, students' poor performance could also arise from weaknesses in their prior knowledge and basic mathematical skills on joining courses. In Norway, many students embark on courses with weak grades in mathematics from their high school, or they may have followed a mathematics course at high school that does not provide the foundation required (Opstad et al. 2017). The Norwegian Mathematics Council has tested students at the beginning of their first semester biennially for over two decades, the tests consistently reveal around half the students registered on service mathematics courses are unable to cope with around 50% of the grade 10 mathematics curriculum.[2] MatRIC addresses this issue through the development of online video tutorials (in Norwegian) and Drop-in support centres following the pattern developed by the **sigma** network in the UK and Ireland (Grove et al. 2020).

In the UK an investigation into reasons for students dropping out of university programmes was undertaken for the Government (National Audit Office, 2007). There has not been a similar investigation in Norway, but the findings of the UK report may be relevant to demonstrate the above observations. Albeit apposite, the report may only expose a more complex situation than suggested in the foregoing. The UK study revealed that STEM[3] students "considered together, . . . are less likely to continue to a second year of study than students following other subjects" (National Audit Office, 2007, §1.25, p. 19.). However, dissatisfaction with the course or institution and lack of preparation were only two of several categories of reason for withdrawal, other factors were personal, lack of integration, wrong choice, financial, and to take up a more attractive opportunity (ibid., §1.28, p. 23).

Internationally, there already exists a substantial body of research that can be called upon to lead developments in teaching and learning in university mathematics reported in journals[4] and specialist conferences and topic study groups.[5] However, prior to the founding of MatRIC, research in undergraduate mathematics education

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[2]Reports from the tests, in Norwegian, are available at https://matematikkraadet.wixsite.com/matematikkraadet/forkunnskapstesten

[3]STEM: Science, Technology, Engineering, and Mathematics.

[4]E.g., Teaching Mathematics and its Applications, and International Journal of Research in Undergraduate Mathematics Education.

[5]E.g., The MAA SIG RUME Conference, INDRUM Conferences and CERME and ICME Topic Groups.

was under-developed in Norway. To convince teachers in Norwegian universities of the need to reflect critically on practices and consider alternative approaches to teaching and learning mathematics, MatRIC has sought to develop research-based evidence from Norwegian higher education settings to reveal the relevance of the corpus of international research to local conditions. MatRIC has approached this in several ways, such as using small "seed-corn" grants to promote lecturers' action research projects and supporting PhD fellows' research.

In Norway, PhD fellowships are usually funded by the Ministry of Education and Research. Fellowships include temporary employment at the university and academic costs; occasionally the latter may be subsidised from other funding sources. Fellowships are allocated to the university to distribute. MatRIC was provided with four fellowships to focus on researching undergraduate mathematics education (RUME). These created a small community of fellows that attracted others with open (not tied to a project or Centre) mathematics education fellowships to join the group. To date, around ten PhD projects have focused on RUME. The research pursued by the fellows tends to reflect opportunities rather than a sharply focused knowledge creation agenda, this may explain the rather disparate collection of brief reports that follows. Opening a broad front in this way is a means of making quick connections to existing international research, the downside is that it does not make a strong contribution to existing knowledge in the field. Although the fellowships have been distributed through UiA, the research has been conducted in several universities within Norway. Despite the variety of projects pursued, they all focus on the fundamental issues outlined above: characteristics of teaching and learning in large and small classes, relevance to the programme of study served by the mathematics studied, prior knowledge and use of technology to support learning. The chapter continues with these brief accounts. It is possible the reader may feel somewhat disappointed by the brevity of the accounts because they provide little more than an abstract of the research described. In most cases there has been an attempt to publish preliminary findings, in some cases where the research is more advanced several papers are in the public domain. The interested reader is encouraged to refer to these other published sources or contact the author directly.

34.1 Examples of Doctoral Research in RUME Supported by MatRIC

34.1.1 Researching Flipped Classroom Approaches by Helge Fredriksen

This research comprised a case-study of three consecutive cohorts of engineering students taught using the approach often referred to as *flipped classroom* (Bergmann & Sams, 2012). The qualitative research, based within a naturalistic paradigm (Moschkovich & Brenner, 2000), considered various aspects of mathematical learning when students are subject to this form of learner-centred teaching (Stephan, 2014). Research on flipped classroom approaches (FC) has increased substantially during the last decade. However, most studies consider mainly student satisfaction and performance comparisons between traditional lecture-based and FC teaching. As such, they provide little insight into the fundamental aspects of what makes the FC in tertiary mathematics education efficient or not. There is a notable lack of research providing qualitative socio-cultural studies of FC teaching and learning. A central aim of this study was to address these shortcomings in the research field in addition to a more general investigation of how FC teaching impacted on the learning of undergraduate mathematics. This goal led to various research questions that considered aspects of students' learning of mathematics in an FC environment. Special attention was given to how the knowledge gained from the videos became integrated into the mathematical discourse in the learning community of students and teachers.

The research design is framed in terms of an initial, an intermediate and a final study, each involving different cohorts of students. Data for this thesis were mostly collected through classroom filming of group work in addition to students' interviews. The methodological aspects of the thesis involved inductive coding informed by the various theoretical frameworks (Braun & Clarke, 2006; Patton, 2002), in addition to commognition.

The initial study showed that the transition towards FC teaching was not without obstacles for the students, leading to a consideration of dialectical contradictions in the activity system of the FC (Fredriksen & Hadjerrouit, 2020a). An important result from this initial study showed that students' participation in the mathematical discourse were crucial for the success of FC. Thus, the commognitive framework of Sfard (2008) became a reasonable choice to further analyse students' participation in mathematical problem-solving in-class, extending the leading discourse from out-of-class videos (Fredriksen & Hadjerrouit, 2020b).

The work on studying students' engagement in the mathematical discourse showed how important task design was for facilitating it. The task design heuristics of Realistic Mathematics Education (RME) was found to align well with FC principles of collaborative learning in-class. In the third paper, students' work at the situational level of RME (Gravemeijer & Doorman, 1999) was extended to formally include *pre-situational* activity through the out-of-class video-preparation (Fredriksen, 2021a). Finally, drawing on insights gained from the three studies, further consideration was given to the affordances and constraints students encounter in a flipped mathematics classroom (Fredriksen, 2021b). Based on the second-generation activity model of Leont'ev (1974), the study presented operational affordances out-of-class, action affordances at the mathematical task level, and finally activity affordances at the collective level.

A major finding from the research was the unveiling of contradictions in students' sense of autonomy and willingness to consider conceptual tasks in a collaborative learning environment. Moreover, alignment of out-of-class video content and in-class task design greatly impacted learning experiences, according to interviews and observations in-class. This latter finding emerged from both commognitive and RME perspectives. Under such circumstances, there was empirical evidence for students' reification of procedural content from videos during in-class work with mathematics. During the RME sessions, the teacher as well as the pre-situational video stage had an important impact on students' transitioning between modelling stages. A key finding from the study of affordances related to students' opportunities for interacting with the mathematical topics through various ways in a FC context, advantageous for retention purposes. Constraints for conceptual learning may emerge if activities in-class appear disconnected from out-of-class preparation. However, out-of-class videos could form an effective medium for procedural learning, preparing students for in-depth conceptual learning through in-class efforts on tasks facilitated through group-work.

34.1.2 Researching Online and Blended Learning Approaches in Mathematics for Engineering Students by Shaista Kanwal

In the present digital era, various possibilities of technology integration in mathematics education have given rise to online and blended instructions models (Borba et al., 2016). Many technological resources including internet-based automatic systems, online tools for graphing and visualising mathematical properties, online calculators, and programming are integrated in mathematics teaching and learning activities. However, little research has been done on students' interactions with the technology enhanced learning environments and their ultimate engagement with mathematics (Borba et al., 2016; Webel et al., 2017). This project set out to study the impact of the technology enhanced learning environments on the quality of students' engagement with mathematics. The collected findings (Kanwal, 2020) focus on undergraduate engineering students' learning activities in an online and in a blended learning environment, and illuminate the role of the factors from the environments in students' engagement with mathematics.

The project, based within a naturalistic paradigm, adopts case study research design. Two case studies were conducted in consecutive semesters of 2017 involving four participants from the same cohort of electronics engineering students in calculus courses. The first case study concerned the online environment, and the second case study concerned blended environment. The online environment involved administration of homework and assessments through an automated system (MyMathLab), lectures in the form of tutorial videos, and the final examination in digital format. In the blended environment, face-face lectures and students' group engagement in paper-based tasks were included in addition to the online homework and assessments. Data were collected in form of video-recorded observations of students' work, screen-recordings of the computer-based activity, and students' weekly journals about their use of different resources in their mathematics work.

Cultural historical activity theory (Engeström, 2014; Leont'ev, 1974) was used as an overarching theoretical framework. Engeström's (2014) model of activity was utilised to trace the contributing macro elements from the collective activity system.

Leont'ev's (1974) theoretical model of hierarchical layers was utilised to make sense of students' micro level interactions with tools and ultimate engagement in mathematics. In particular, the analysis of action and operation dynamics was central to micro-analysis of engagement with mathematics (Kanwal, 2019). In an earlier study (Kanwal, 2018), the documentational approach to didactics (Gueudet & Pepin, 2018) was used to study students' use of resources and their rationales for the selection and use of particular resources. Furthermore, students' mathematical reasoning processes were explored in technological and paper and pencil environment (paper under review).

With regards to students' activity in the online environment, the findings show that students incorporated several online resources including GeoGebra, WolframAlpha, and online calculators for solving posed tasks in MyMathLab (Kanwal, 2018). These tools offered wide action possibilities due to their different functionalities. Regarding the quality of engagement with mathematics, students' actions were focused on process of solving a task whereby engaging with mathematical properties at the one end and obtaining final solutions at the other end. The relationship between posed mathematical tasks and the available tools played a part in how effectively students made use of the tools. The tasks which required application of algorithms could be solved using online calculators and therefore did not require engaging with the solution process. On the other hand, the tasks which did not ask for direct application of algorithms led students to effectively use tools for exploring involved mathematical properties. Moreover, the internal conditions of MyMathLab also played a significant role in the sense that it had no mechanism for ensuring students' engagement with the process of solving the tasks as it could only evaluate the final answers. One of the reasons for students' selection of tool use was found to be the macro-condition of the online examination in the course as the students opted for the resources that they could use during final digital examination (Kanwal, 2019).

The quality of students' engagement with mathematics depended on how and which tools were used in relation to the posed tasks. In this regard, the research points to the need for ensuring integration of technology in such a manner that it facilitates students in exploring mathematical properties instead of bypassing the involved mathematics. This argument resembles that of Borba (2009), who asserts that in a technological environment, mathematical tasks require thoughtful considerations if we want students to engage with mathematics effectively.

34.1.3 Researching Learning with a Visualisation and Simulation Program by Ninni Marie Hogstad

This research, framed within commognitive theory (Sfard, 2008), sets out to study first-year engineering students' discourse about the mathematical object 'definite integral'. In the professional work life of engineers, mathematical objects are used

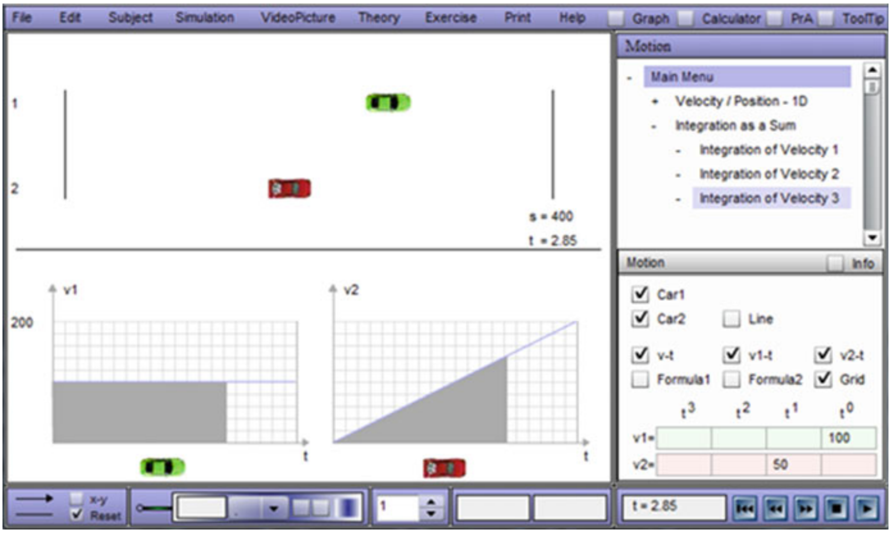

Fig. 34.1 The visual interface of Sim2Bil

and operated upon for specific practical goals in industry. Within engineering studies, students are introduced to and learn how to operate on several mathematical objects (e.g., functions, the derivative, and equations) where one of the most often applied key objects is the definite integral. Thus, it is important to investigate how engineering students make meaning of and communicate about this object as a foundation for its further applications.

Within engineering studies, students also meet different interactive digital tools designed for educational purposes. One such tool is Sim2Bil which has been designed at UiA for university students to realize the definite integral in an algebraic operator (∫), an area under a graph and an area-accumulation function. Each of these three realizations are characterized to be perceptually accessible and contribute to the students' meaning of the object 'definite integral'. The tool offers an animation of two cars travelling in a straight line from a start line to a finish line. Figure 34.1 shows the interface of this tool. The animation is shown in the upper left corner. The cars' behaviors are modelled by two velocity functions (see the lower right corner). Two functions are already typed in as a default: $v_1(t) = 100$ and $v_2(t) = 50t$. These functions make the cars run with different velocities and arrive at the finish line simultaneously after 4 s (see the description of the second assignment below given to the students). Students can insert other functions themselves. Also displayed are two graphs (lower left corner) one for each of the velocity functions and a shaded region beneath each of the graphs.[6]

[6] Sim2Bil is an application in SimReal and can be accessed here: https://simreal.no

The aim of the research is to gain insight into engineering students' mathematical discourse while using Sim2Bil. Four groups of first year engineering students working within four task situations participated in the study. This account focuses on two groups (six students in total) that had access to the digital tool. The assignments the students worked on were as follows:

- Press "Start" in the program and explain to each other what happens. What do the shaded areas represent?
- Determine other numbers in the table, so that the cars run with different velocities, and arrive at the finish line at the same time.
- What can you do to make the green car be only halfway when the red car reaches the finish line?
- Find the velocities of the green and the red car (v1 and v2), so that v2 is half of v1 when they reach the finish line simultaneously at 4 sec. Can you prove that your answer is correct?

The first research question concerns the engineering students' communication, while the second research question concerns the affordances and constraints of Sim2Bil experienced by the students when they engage in the group work. By following the students' journey in becoming "capable of agentive participation" (Lavie et al., 2019, p. 424) in the mathematical discourse, it is hoped to gain insights into how to adapt the teaching in order to prepare the engineering students for further studies. Within the analysis of discourse, the close relationship between thinking and communication is considered and features of the students' discourse such as word use, visual mediators operated upon, routines applied, and narratives established are studied.

Until now, different visual mediators operated upon by the students and the purposes for which they have been used have been identified. These mediators were within Sim2Bil, paper-based (such as sketches and symbols) and gestures of the cars' path and graphs (for identification of gesture as a mediator in students' discourse, see Ng (2016)). The students used these mediators to communicate about mathematical aspects and kinematical applications within different stages of the group work (Hogstad et al., 2016).

To scrutinize the interaction between students and Sim2Bil, the *Instrumental Approach* was chosen (Trouche, 2004). With this approach, the techniques the students applied for the different assignments with Sim2Bil have been analyzed while acknowledging both the mathematical and technical aspects. The analysis reveals that the students used both instrumented and pencil-and-paper techniques. When the students pressed the Start-button, the students were introduced to the functionalities of the tool and the conditions of the assignments. By starting the animation, the students were shown how the velocity functions provided the graphs and the running of the cars. Starting the animation was also used to check if their suggested velocity functions met the requirements in the assignments. Pencil-and-paper techniques were used to calculate the integral as an anti-derivative and to find equal areas beneath the graphs (Hogstad & Isabwe, 2017).

Currently a study of students' routine course of action is in progress. In this study, the relationship between students' different engagement is being investigated. To analyze the relationship between these different kinds of engagement, an adapted methodological lens by Nachlieli and Tabach (2019) will be used. The study derives from the study reported by Hogstad and Viirman (2017).

34.1.4 Researching Economics Students' Performance in Mathematics by Ida Landgärds

Poor performances and high failure rate in the compulsory mathematics course has endangered the continued inclusion of all students in the University Economics programme. Especially students who studied a practical mathematics route[7] in upper secondary school face difficulties (failure rates of about 40% are not unusual) (e.g., Busch, et al., 2017). A mathematics content gap between the national requirements for admission to the programme and the local expectations of prior student experience was identified and indicated unequal opportunities to access the content of the mathematics course (Landgärds, 2019).

For this reason, building on the educational philosophy of Carroll (1989, p. 30), that is: "we should seek mainly to achieve equality of opportunity for students," a blended-learning pre-course intervention was designed and implemented in 2018 at UiA. In order to give students, in Carroll's terms, the "opportunity to learn" topics that their school course had omitted, or students had not mastered, the intervention comprised a diagnostic test and a bridging course covering the crucial prerequisites.

It is generally assumed that remediation courses have a positive effect on students' mathematics skills and mitigate the heterogeneity of students' mathematics background on entry. However, the few European studies (Büchele, 2020a, b; De Paola & Scoppa, 2014; Di Pietro, 2014; Lagerlöf & Seltzer, 2009; Laging & Voßkamp, 2017) show no consensus about the effectiveness of remedial mathematics courses for raising student performance within the study of Economics. This study adds to the European literature on remedial course effectiveness. Research into the effect of the intervention aims to gain an understanding of the aspects of the pre-course actions that can contribute to equal opportunities and increased inclusivity of the Mathematics for Economists course, and consequently reduce the failure rate.

The research set out to investigate the relationship between students' participation in the bridging course and their subsequent performance in the Mathematics for Economists course. Data generation was integrated within the intervention actions, for example, students' participation was derived from students' attendance in

[7] "Practical Mathematics" is an optional route for students in Norwegian upper secondary/high schools that includes limited new theoretical abstractions beyond those encountered up to Grade 10 compulsory school.

workshops and learning analytics on students' use of training fields (online quizzes with help-options where the student can pause and learn more about how to solve such exercises through videos and written explanations before continuing with the exercise), written exercises with worked solutions, and written theory documents. Accordingly, the first research question focuses on process variables (derived from data about participation) that predict student achievement in the shorter-term (on a bridging course post-test) and in the longer-term (on the Mathematics for Economists course exam). The second research question, especially focusing on the group of at-risk students, is whether the least mathematically prepared students benefit from participation in the bridging course. To analyse relationships between process and outcome variables, ordinary-least-square regression (OLS) in several hierarchical stages with blockwise entry was used.

Results from the first intervention (Landgärds, 2021) indicate students' use of the bridging course online resource of 'training fields' is a significant predictor of exam performance. The training fields were intended to guide students in their learning, taking account of the students' prior knowledge. They took the form of training through self-assessment, but whenever the student felt the need for learning about the topic before answering a question, they could click on "show steps" where they found written explanations and/or a video about the particular task and topic. Hence, students themselves set their pace and decided on the content to be learned. The diversity of pathways is a key to promote equal opportunities to learn the basic mathematics needed. Individually, the other process variables did not prove good predictors of the achievement variables. Moreover, there was a significant positive effect of course participation (composite process variable) on the post-test score and on the examination score within the whole group of students. And importantly, participation was found particularly valuable for students who followed the practical mathematics route in school as the benefits of stronger participation were markedly higher levels of capability on the exam.

A new round of data collection is in progress. More data will enable further investigations of significant educational aspects of the bridging course. The goal is to further develop the design of the pre-course intervention to best support the students in their learning of basic mathematics skills needed in studies of economics to achieve equitable education.

34.1.5 Researching the Development of Mathematical Competency of Biology Students by Yannis Liakos

Evaluating competencies in education has been the focus of several studies (Hartig, et al., 2008). However, little has been done at the higher education level, with several studies pointing out the need for competence models of assessment (e.g., Blömeke et al., 2013a, b). Several instruments for assessing competencies are available (OECD, 2010; Blömeke et al., 2013a, b; Hill et al., 2005). However, the assessment of competencies at the undergraduate level is still an area under development, and further study for the underlying competence structures is needed.

This study aims to add knowledge to this area by exploring and assessing competency development in undergraduate studies and creating individual competency profiles for the students. The research pursues an interest in exploring the progress of students' competency development during a series of calculus sessions where they engage in solving non-routine mathematical tasks in the context of biology.

STEM studies are becoming increasingly interdisciplinary (e.g., Drake & Burns, 2004), and mathematical competencies are deemed necessary for all students. Smith and Karr-Kidwell (2000) conceptualize the interdisciplinary nature of STEM as "[a] holistic approach that links the [individual] disciplines so that learning becomes connected, focused, meaningful, and relevant to learners." (p. 24). Different disciplines may require different mathematical competencies without excluding the possibility that some mathematical competencies are considered necessary for most disciplines. Adding to this assumption, how students in biology departments develop their mathematical competencies will presumably differ from the development in other STEM fields. In the light of these assumptions, the following research questions are addressed in this project: (1) What is the progress of individual competency development over time for a student who participates in a series of calculus sessions working on non-routine mathematical tasks set in a biology context? What is the competence profile of each student after this series of calculus sessions? (2) Are the competencies interrelated and, if so, how?

This research includes a series of calculus sessions in a Norwegian university's biology department. Participants were first-year students following their main calculus course during autumn semester. They agreed to attend these sessions as part of an additional learning opportunity offered by their department. They would work on tasks that addressed four mathematical areas: periodic functions, exponential growth and regression, population dynamics, and integration and modelling.

For the data generation, traditional uses of observation methods were adopted, including video recordings, researcher's field notes, and the use of smartpens to secure the documentation of students' written work in digital form. The data analysis developed a competence framework that mainly built on an already existing one, namely, the KOM model (Niss, 2003), and can be thought of as an intersection of some of the literature's main sets of competencies (e.g., Niss & Højgaard, 2011; Maaß, 2006). This framework includes various sets of competencies (e.g., mathematical thinking and reasoning) believed to be necessary for undergraduate students in STEM programmes. Employing this framework and applying a thematic analysis on students' transcribed work, codes (corresponding to each of the mathematical competencies) were assigned to occurrences that were hypothesised to signify the activation of a particular competency(ies) while the students were working on their assigned tasks. Once the competency activation was identified, a scaling instrument constructed to assess (in levels) the quality of this activation, was employed to explore student's competency level. Quality of activation is an evaluation of a set of attributes that characterize a mathematical competency. These attributes concern the student's awareness of the task solution vision (T.S.V.), their use of

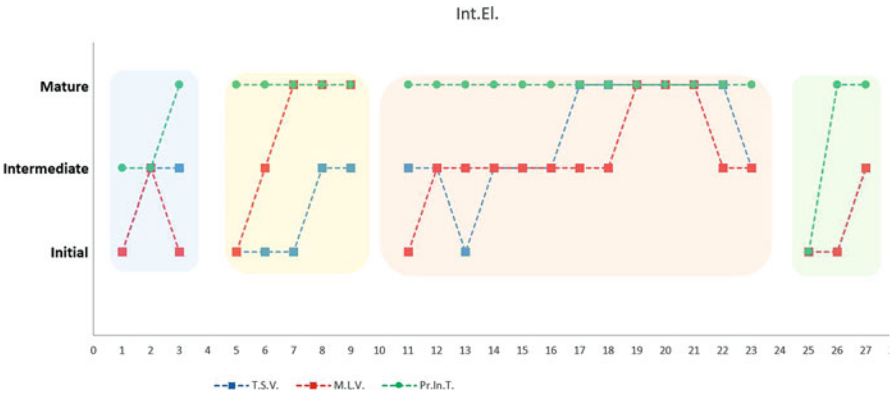

Fig. 34.2 Development graph for the interpret and translate elements of a model (Int.El.) competency aspect

mathematical language (M.L.V.), and how independently the student worked (Pr.In. T) to activate that competency. The analysis was accompanied by a graphical illustration (Fig. 34.2) of the competency development.

This research determined that the students displayed development of a set of mathematical competencies that varied from student to student and from one mathematical area to another. This variation was explored by employing the aforementioned scaling instrument. The students displayed evidence of competency development when working on the population dynamics sessions, and evidence was also found during the exponential growth and regression sessions. For example, Fig. 34.2, shows the profile of one student regarding the development of a particular aspect (competence to interpret and translate elements of a model, coded: Int.El.) of the mathematical modelling competency. Each shaded area represents one calculus unit (blue: periodic functions, orange: exponential growth and regression, grey: population dynamics, and green: integration and modelling). It can be seen that the most frequent activation occurred during the population dynamics unit.

The illustration in Fig. 34.2 provides additional information, besides the frequency of these activations, that gives an account of competency development. This study also inferred that particular pairs of competencies appeared to be activated concurrently more often than others and that the activation of a particular competency may exclude the possibility of activation of another competency.

34.1.6 Researching Biology Students' Use of Mathematics by Floridona Tetaj

The field of modern fisheries stock assessment is heavily dependent on mathematical models (MMs) which help fisheries biologists understand how fish populations respond to exploitation and harvesting. This project, focusing on a Norwegian graduate course called Ecosystem and Fisheries Assessment Models (EFAM) which introduces various MMs used in fisheries stock assessment, addresses the nature of students' engagement with MMs. The rationale for the study comes as a response to various calls for research aimed at understanding the mathematical needs of biology students (e.g., Steen, 2005; Scheaffer, 2011; Duran & Marshall, 2019). In the current literature, there is a lack of research that provides qualitative analysis of biology students' engagement with MMs and this study aims at contributing specifically towards this issue.

The research is an exploratory case-study of the course EFAM and the investigation focuses on the lecturer of the course and six students who engaged with different assignments that employed MMs. This qualitative study, located within a naturalistic paradigm (Lincoln, 2007), aims at exploring students' expected engagement with MMs from the lecturer's perspective and at analyzing students' performance. A commognitive framework (Sfard, 2008) was adopted as an overarching theoretical framework. Making use of the idea that each discipline (mathematics or biology) can be conceptualized and operationalized through the notion of discourses and their respective discursive properties (vocabulary, visual mediators, endorsed narratives and routines). Characteristics of mathematics discourse (MD), fisheries discourse and their interactions in the lecturer's and students' discourses were identified. For this purpose, the project was divided into two phases.

The first phase focused on exploring the type of participation that is expected from biology students with regard to mathematical discourse. This phase takes a case study approach, which uses multiple sources of evidence (Yin, 2014) such as the six assignments with which the students enrolled in EFAM engaged during one semester where the course took place. A semi-structured interview with the lecturer, and video recordings of the lectures where the lecturer discusses the solutions of the assignments were also used. The analysis set out to identify the characteristics of the mathematical discourse present in the task situations and to expose the relationships that this discourse has with the biological context. To this end, two separate analyses were undertaken.

The first analysis used a scheme presented in Tetaj (2021) that elaborates an analytical tool for investigating the characteristics of MD found in contextualized tasks. The characteristics of mathematical wordings, visual mediators, the nature of the expected mathematical routines and interactions between discourses present in the formulations and solutions of the six assignments were investigated. The analyses revealed that contrary to what is traditionally presupposed, biology students are expected to engage with a complex subsumed discourse that involves a variety of mathematical concepts, models, and methods. Moreover, the role that mediators play in shaping students' solving trajectories, were explored and thereby two types of transitions between discourses, named explicit and implicit transitions, were identified.

Next, using the work of Leont'ev (1974) on activities' levels, the exploration focused on the intentions of the lecturer regarding students' engagement with MD during the work on the assignments. Through the analysis of the interview and the video-recordings of his lectures, it was possible to identify assumptions, goals and chosen strategies of the lecturer for teaching MMs in EFAM. The results of this analysis indicated that the lecturer's motive for engaging students with the MD was to enable them become part of the fisheries community. This main motive was then observed in various goals such as the learning of applicability and limitations of mathematical models, giving students a sense of identity as future fisheries biologists and of belonging to the fisheries community, and various operations that the lecturer practiced during his teaching.

The second phase of the project focuses on analyzing students' work on the assignments. This phase is currently in progress and the data is yet to be analyzed. The goal of this phase is to characterize students' MD and identify challenges that students face when engaging with MD and transitioning to the biological context. The data collection for this case includes video-recordings of students while solving the assignments in groups. The plan of the project is to compare the results of two phases, hence, obtain a picture of EFAM students' mathematical needs.

34.1.7 Researching Relationships Between Prior Knowledge, Self-Efficacy and Approaches to Learning Mathematics of Engineering Students by Yusuf F. Zakariya

The motivation for this project is the recurrent poor performance of engineering students in a first-year calculus course at UiA. In the project, attempts are made to produce empirical evidence on the areas of concentration, as it concerns the student-source factors, to alleviate this problem. Previous studies suggest that prior mathematics knowledge, approaches to learning, and self-efficacy, among other student-source factors, serve crucial roles in fostering students' performance in mathematics. However, most of these studies are either old or correlational (e.g., traditional regression-based studies), which make it difficult to argue for the effectiveness of interventions on these factors as proxies to enhance performance (Zakariya, 2021). Thus, the aim of the project is to investigate the effects of prior mathematics knowledge, approaches to learning mathematics, and self-efficacy on students' performance in a first-year calculus course.

Two well-established psychological theories are combined to form the conceptual framework for justifying appropriateness and usefulness of chosen constructs under investigation coupled with hypothesised relationships among the constructs. These theories are student approaches to learning theory (Marton & Säljö, 2005) and self-efficacy theory (Bandura, 2012). A survey research design is adopted with a focus on first-year engineering students aimed at addressing three research questions. These research questions are formulated as follows:

1. Do approaches to learning mathematics differ with respect to the prevalence of deep and surface approaches among first-year engineering students?
2. Does self-efficacy influence adoption of either deep or surface approach to learning mathematics among first-year engineering students?
3. What are the direct and indirect effects of prior mathematics knowledge, approaches to learning, and self-efficacy on performance in mathematics among first-year engineering students?

The study is framed within the quantitative research paradigm and the data are generated by using questionnaires, a pre-test of students' basic mathematical knowledge and final examination scores in an introductory calculus course. Eight research hypotheses are formulated based on the postulates of the theories coupled with some insights from the literature. The hypotheses are tested by using some techniques of structural equation modelling.

The first set of results confirm some psychometric properties such as construct validity and reliability of a measure of approaches to learning (Zakariya, 2019; Zakariya et al., 2020a, b). Engineering students' approaches to learning mathematics are sufficiently characterised with deep and surface approaches. A novel measure of students' self-efficacy on calculus tasks is developed and validated. The results of the development/validation study confirm an acceptable construct validity, discriminant validity and reliability of the measure (Zakariya et al., 2019). Item qualities such as difficulty indices, discrimination indices, and item reliability of a measure of prior mathematics knowledge are also investigated and documented (Zakariya et al., 2020a, b, c). The results of the item analysis study shape the use of the measure in the main study.

The second set of results concern the research questions two and three. It is found that self-efficacy influences adoption of either deep or surface approaches to learning mathematics among the engineering students. (Zakariya et al., 2020d). It is also found that prior mathematics knowledge (test performance) has substantial negative and positive effects on surface approaches to learning and self-efficacy, respectively. However, its effect on performance was only significant when self-efficacy is not included in the model. The surface approaches to learning have a negative effect on students' performance in the course. Surprisingly, there was no substantial evidence to justify any considerable effect of the deep approaches to learning on students' performance (Zakariya, 2021; Zakariya et al., 2021). Accumulated evidence from this project points to the fact that self-efficacy (i.e., engineering students' convictions to solve first-year introductory calculus tasks successfully) has the most substantial effect on the students' performance in the course (Zakariya, 2021). Its effect outshines the effects of both (measured) prior mathematics knowledge and approaches to learning mathematics on students' performance in the course. Therefore, a major conclusion drawn from the findings of this project is the identification of self-efficacy as a prime factor whose interventions could enhance students' performance in the course.

34.2 Conclusion

Findings from the projects are disseminated through MatRIC events (conferences, workshops and seminars), research conferences and national subject councils. Locally, within UiA, projects are influential in programme developments. It will also be noticed from the foregoing accounts that the research has already resulted in several publications, one reason for this is the custom promoted in Scandinavia of producing a dissertation as a collection of published articles together with an embracing introduction.

There are many ways in which the research might be classified. The above brief descriptions indicate the breadth and scope of MatRIC-supported doctoral research against several classification schemes. For example, although most of the research takes a qualitative approach, Zakariya's study is intensely quantitative, and his analysis depends on advanced and cutting-edge statistical measurement/structural models. The research includes naturalistic inquiry, intervention research and data extraction approaches. Both exploratory and developmental research are represented, and all fit with the so-called "Pasteur's quadrant" of use-inspired basic research (Stokes, 1997). The research focuses on several teaching and learning contexts, from action research on flipped classroom approaches, outsider inquiry of blended learning, clinical approaches inquiring into students' use of visualisation, intervention research with economics and natural sciences. Mathematics as a service subject in engineering, economics and natural science are also included. Although MatRIC's research agenda has been intended to convince local teachers of the relevance of international research findings to their own practice it is evident from the individual accounts that the doctoral fellows are intent on making a substantive contribution to the corpus of international research. Our hope is that their efforts fix Norwegian RUME firmly in the global arena.

The foregoing accounts reveal considerable diversity in research design, theoretical framing and issues researched. It could be argued that the research field would have been provided with a more resilient and robust foundation if there had been an attempt to focus more sharply on fewer issues and ensure more coherence. The argument for the more open and free approach taken is that it has allowed individual research fellows, their supervisors and research groups to follow their own interests and thus, possibly, be better motivated in the research pursued. One positive consequence of the diversity has been the connections made with a wide international network of researchers in university mathematics education reflecting the variety of Norwegian interests. In the opening paragraph it was suggested that the value of the chapter may lie at a later moment when MatRIC's strategy to develop the research field can be evaluated. The questions that may be of interest could include whether the research field was sufficiently rooted – both nationally and internationally. Also, given that MatRIC's funding will cease at the end of 2023, whether researchers in the field within Norway have achieved the critical mass that ensures sustainability and continued development and growth in contributions to the international community.

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