A Generalized Multivariate Gamma Distribution

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Abstract In this chapter, we introduce a multivariate gamma distribution whose marginals are finite mixtures of gamma distributions and correlation between any pair of variables is negative. Several of its properties such as joint moments, correlation coefficients, moment generating function, Rényi and Shannon entropies have been derived. Simulation study have been conducted to evaluate the performance of the maximum likelihood method.

1 Introduction

Gamma distribution is an important continuous distribution in probability and statistics. Several distributions such as exponential, Erlang, and chi-square are special cases of this distribution. Several univariate generalizations of gamma distribution have also been studied. Gamma distribution and its variants have been applied in different disciplines to model continuous variables that are positive and have skewed distributions. Gamma distribution has been used to model amounts of daily rainfall (Aksoy [1]) and in neuroscience this model is often used to describe the distribution of inter-spike intervals (Robson and Troy [26]). The gamma distribution is widely used as a conjugate prior in Bayesian statistics. It also plays an important role in actuarial sciences (Furman [9]).

Several multivariate generalizations of univariate gamma distributions are also available in the literature. Mathai and Moschopoulos [20, 21] introduced two multivariate gamma models as the joint distribution of certain linear combinations/partial sums of independent three parameter gamma variables. All the components of their multivariate gamma vectors are positively correlated and have three parameter

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A. Bekker et al. (eds.), *Innovations in Multivariate Statistical Modeling*, Emerging Topics in Statistics and Biostatistics,
https://doi.org/10.1007/978-3-031-13971-0_12

gamma distributions. They have also indicated that their models have potential applications in stochastic processes and reliability. Furman [9] used the multivariate reduction technique to derive a multivariate probability model possessing a dependence structure and gamma marginals. Kowalczyk and Tyrcha [17] used a re-parameterized form of the gamma distribution to define a multivariate gamma vector and studied a number of properties of their distribution. Recently, Semenikhine, Furman and Su [28] introduced a multiplicative multivariate gamma distribution with gamma marginals and applied their results in actuarial science. They proved that the correlation coefficient between any pair of variables is positive and belongs to $(0, 1/2)$. Multivariate gamma distributions have been used in diverse fields like hydrology, space (wind modeling), reliability, traffic modeling, and finance. For further results on multivariate gamma distribution, the reader may consult articles by Balakrishnan and Ristić [4], Carpenter and Diawara [5], Dussauchoy and Berland [6], Gaver [10], Krishnaiah and Rao [18], Marcus [19], Pepas et al.[23], Royen [27], Vaidyanathan and Lakshmi [33], and an excellent text by Kotz, Balakrishnan and Johnson [16]. For a good review on bivariate gamma distributions, see Balakrishnan and Lai [3], Arnold, Castillo and Sarabia [2], Hutchinson and Lai [14], and Kotz, Balakrishnan and Johnson [16]. For a review on some recent work and applications the reader is referred to Rafiei, Iranmanesh, and Nagar [24] and references therein.

In this chapter, we introduce a multivariate gamma distribution whose marginals are finite mixtures of gamma distributions and correlation between any pair of variables is negative. We organize our work as follows: In Sect. 2, we introduce the new multivariate gamma distribution. In Sects. 3 and 4, results on marginal distributions and factorizations of the multivariate gamma distribution are derived. Sections 5–8 deal with properties such as joint moments, correlation, moment generating function, entropies and estimations. In Sect. 9, simulations of the new distribution are performed in different ways, and the results are provided to evaluate the performance of the maximum likelihood method. Section 10 contains the conclusion. Finally, the Appendix lists a number of results used in this chapter.

2 The Multivariate Gamma Distribution

Recently, Rafiei, Iranmanesh, and Nagar [24] have defined a bivariate gamma distribution with parameters α, β and k and the pdf

$$f(x_1, x_2; \alpha, \beta, k) = \frac{\Gamma(2\alpha)}{\beta^{2\alpha+k}\Gamma^2(\alpha)\,\Gamma(2\alpha+k)}(x_1x_2)^{\alpha-1}(x_1+x_2)^k \exp\left[-\frac{1}{\beta}(x_1+x_2)\right],$$

where $x_1 > 0, x_2 > 0, \alpha > 0, \beta > 0$, and $k \in \mathbb{N}_0$. A natural multivariate generalization of this distribution can be given as follows.

Definition 1 The random variables $X_1, \ldots, X_n$ are said to have a generalized multivariate gamma distribution, denoted as $(X_1, \ldots, X_n) \sim \text{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$, if their joint pdf is given by

$$f(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n; \beta, k) = C(\alpha_1, \ldots, \alpha_n; \beta, k) \prod_{i=1}^{n} x_i^{\alpha_i - 1} \left(\sum_{i=1}^{n} x_i \right)^k \times \exp\left(-\frac{1}{\beta} \sum_{i=1}^{n} x_i \right), \; x_i > 0, \; i = 1, \ldots, n, \tag{1}$$

where $\alpha_1 > 0, \ldots, \alpha_n > 0$, $\beta > 0$, $k \in \mathbb{N}_0$ and $C(\alpha_1, \ldots, \alpha_n; \beta, k)$ is the normalizing constant.

By integrating the joint density of $X_1, \ldots, X_n$ over its support set, the normalizing constant is derived as

$$\begin{aligned}[C(\alpha_1, \ldots, \alpha_n; \beta, k)]^{-1} &= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} x_i^{\alpha_i - 1} \left(\sum_{i=1}^{n} x_i \right)^k \exp\left(-\frac{1}{\beta} \sum_{i=1}^{n} x_i \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \beta^{\sum_{i=1}^{n} \alpha_i + k} \left[\prod_{i=1}^{n} \Gamma(\alpha_i) \right] (\alpha_1 + \cdots + \alpha_n)_k \,,\end{aligned}$$

where the last line has been obtained by using Lemma 2. Finally, from the above expression

$$C(\alpha_1, \ldots, \alpha_n; \beta, k) = \frac{\Gamma\left(\alpha_1 + \cdots + \alpha_n\right)}{\beta^{\sum_{i=1}^{n} \alpha_i + k} \left[\prod_{i=1}^{n} \Gamma(\alpha_i) \right] \Gamma\left(\alpha_1 + \cdots + \alpha_n + k\right)}. \tag{2}$$

For $k = 0$, the multivariate gamma density simplifies to the product of n independent univariate gamma densities with common scale parameter β. For $k = 1$, the multivariate gamma density can be written as

$$\sum_{j=1}^{n} \left(\frac{\alpha_j}{\sum_{i=1}^{n} \alpha_i} \right) \frac{x_j^{\alpha_j} \exp(-x_j/\beta)}{\beta^{\alpha_j + 1} \Gamma(\alpha_j + 1)} \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{x_i^{\alpha_i - 1} \exp(-x_i/\beta)}{\beta^{\alpha_i} \Gamma(\alpha_i)}, \; x_1 > 0, \ldots, x_n > 0. \tag{3}$$

For $n = 2$ in (1), the bivariate gamma density is obtained as

$$C(\alpha_1, \alpha_2; \beta, k) x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} (x_1 + x_2)^k \exp\left[-\frac{1}{\beta} (x_1 + x_2) \right], \; x_1 > 0, \; x_2 > 0, \tag{4}$$

where

$$C(\alpha_1, \alpha_2; \beta, k) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\beta^{-(\alpha_1 + \alpha_2 + k)}}{\Gamma(\alpha_1 + \alpha_2 + k)}.$$

In a recent article, Rafiei, Iranmanesh, and Nagar [24] have studied the above distribution for $\alpha_1 = \alpha_2$. Substituting $n = 2$ in (3) or $k = 1$ in (4), the generalized bivariate gamma density takes the form

$$\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{x_1^{\alpha_1}x_2^{\alpha_2-1}\exp[-(x_1+x_2)/\beta]}{\beta^{\alpha_1+\alpha_2+1}\Gamma(\alpha_1+1)\Gamma(\alpha_2)} + \frac{\alpha_2}{\alpha_1+\alpha_2}\frac{x_1^{\alpha_1-1}x_2^{\alpha_2}\exp[-(x_1+x_2)/\beta]}{\beta^{\alpha_1+\alpha_2+1}\Gamma(\alpha_1)\Gamma(\alpha_2+1)},\ x_1>0,\ x_2>0,$$

which yields the marginal density of X_1 as

$$\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{x_1^{\alpha_1}\exp(-x_1/\beta)}{\beta^{\alpha_1+1}\Gamma(\alpha_1+1)} + \frac{\alpha_2}{\alpha_1+\alpha_2}\frac{x_1^{\alpha_1-1}\exp(-x_1/\beta)}{\beta^{\alpha_1}\Gamma(\alpha_2+1)},\ x_1>0.$$

Clearly, the marginal density of X_1 is a mixture of two gamma densities indicating that, in general, marginal density of any subset of $X_1, \ldots, X_n$ is not a generalized multivariate gamma.

It may be noted here that the multivariate gamma distribution defined above belongs to the Liouville family of distributions (Sivazlian [30], Gupta and Song [12], Gupta, and Richards [13], Song and Gupta [31]). Because of mathematical tractability, this distribution further enriches the class of multivariate Liouville distributions and may serve as an alternative to many existing distributions belonging to this class.

3 Marginal Distributions

In this section, we derive results on marginal distributions of the generalized multivariate gamma distribution defined in this chapter. By using multinomial expansion of $\left(\sum_{i=1}^n x_i\right)^k$, namely,

$$\left(\sum_{i=1}^n x_i\right)^k = \sum_{k_1+k_2+\cdots+k_n=k}\binom{k}{k_1,k_2,\ldots,k_n}x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$$

in (1), the joint density of $X_1, \ldots, X_n$ can be restated as

$$C(\alpha_1,\ldots,\alpha_n;\beta,k)\sum_{k_1+k_2+\cdots+k_n=k}\binom{k}{k_1,k_2,\ldots,k_n}\prod_{i=1}^n x_i^{\alpha_i+k_i-1}\exp\left(-\frac{1}{\beta}\sum_{i=1}^n x_i\right),$$

where $x_i > 0, i = 1, 2, \ldots, n$. Thus, the generalized multivariate gamma distribution is a finite mixture of product of independent gamma densities.

In the remaining part of this section and the next section, we derive marginal distributions, distribution of partial sums and several factorizations of the generalized multivariate gamma distribution.

Theorem 1 *Let* $(X_1, \ldots, X_n) \sim \text{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Then, for* $1 \le s \le n-1$, *the marginal density of* $X_1, \ldots, X_s$ *is given by*

$$C\left(\alpha_1, \ldots, \alpha_s, \sum_{i=s+1}^{n} \alpha_i; \beta, k\right) \prod_{i=1}^{s} x_i^{\alpha_i - 1} \exp\left(-\frac{1}{\beta}\sum_{i=1}^{s} x_i\right) \left(\sum_{i=1}^{s} x_i\right)^k$$
$$\times \beta^{\sum_{i=s+1}^{n}\alpha_i} \sum_{j=0}^{k} \binom{k}{j} \Gamma\left(\sum_{i=s+1}^{n} \alpha_i + j\right) \left(\frac{\sum_{i=1}^{s} x_i}{\beta}\right)^{-j}, \; x_i > 0, \; i = 1, \ldots, s.$$

Proof Integrating out $x_{s+1}, \ldots, x_n$ in (1), the marginal density of $X_1, \ldots, X_s$ is derived as

$$C(\alpha_1, \ldots, \alpha_n; \beta, k) \prod_{i=1}^{s} x_i^{\alpha_i - 1} \exp\left(-\frac{1}{\beta}\sum_{i=1}^{s} x_i\right)$$
$$\times \int_0^\infty \cdots \int_0^\infty \prod_{i=s+1}^{n} x_i^{\alpha_i - 1} \left(\sum_{i=1}^{s} x_i + \sum_{i=s+1}^{n} x_i\right)^k \exp\left(-\frac{1}{\beta}\sum_{i=s+1}^{n} x_i\right) \prod_{i=s+1}^{n} \mathrm{d}x_i$$
$$= C(\alpha_1, \ldots, \alpha_n; \beta, k) \prod_{i=1}^{s} x_i^{\alpha_i - 1} \exp\left(-\frac{1}{\beta}\sum_{i=1}^{s} x_i\right)$$
$$\times \frac{\prod_{i=s+1}^{n} \Gamma(\alpha_i)}{\Gamma(\sum_{i=s+1}^{n} \alpha_i)} \int_0^\infty x^{\sum_{i=s+1}^{n} \alpha_i - 1} \left(\sum_{i=1}^{s} x_i + x\right)^k \exp\left(-\frac{1}{\beta} x\right) \mathrm{d}x \tag{5}$$

where the last line has been obtained by using (16). Substituting $x / \sum_{i=1}^{s} x_i = z$ in (5), the marginal density of $X_1, \ldots, X_s$ is rewritten as

$$C\left(\alpha_1, \ldots, \alpha_s, \sum_{i=s+1}^{n} \alpha_i; \beta, k\right) \prod_{i=1}^{s} x_i^{\alpha_i - 1} \exp\left(-\frac{1}{\beta}\sum_{i=1}^{s} x_i\right) \left(\sum_{i=1}^{s} x_i\right)^{\sum_{i=s+1}^{n}\alpha_i + k}$$
$$\times \int_0^\infty z^{\sum_{i=s+1}^{n} \alpha_i - 1} (1+z)^k \exp\left[-\frac{1}{\beta}\left(\sum_{i=1}^{s} x_i\right) z\right] \mathrm{d}z. \tag{6}$$

Now, writing $(1+z)^k$ using binomial theorem and integrating z in (6), the marginal density of $X_1, \ldots, X_s$ is derived. □

Alternately, the density of $X_1, \ldots, X_s$ given in (7) can be written as

$$C\left(\alpha_1, \ldots, \alpha_s, \sum_{i=s+1}^{n} \alpha_i; \beta, k\right) \beta^{\sum_{i=s+1}^{n} \alpha_i + k} \prod_{i=1}^{s} x_i^{\alpha_i - 1} \exp\left(-\frac{1}{\beta} \sum_{i=1}^{s} x_i\right)$$
$$\times \sum_{j=0}^{k} \binom{k}{j} \Gamma\left(\sum_{i=s+1}^{n} \alpha_i + k - j\right) \left(\frac{\sum_{i=1}^{s} x_i}{\beta}\right)^j, \; x_i > 0, \; i = 1, \ldots, s.$$

Corollary 1 *The marginal density of X_1 is given by*

$$C\left(\alpha_1, \sum_{i=2}^{n} \alpha_i; \beta, k\right) \beta^{\sum_{i=2}^{n} \alpha_i + k} x_1^{\alpha_1 - 1} \exp\left(-\frac{1}{\beta} x_1\right)$$
$$\times \sum_{j=0}^{k} \binom{k}{j} \Gamma\left(\sum_{i=2}^{n} \alpha_i + k - j\right) \left(\frac{x_1}{\beta}\right)^j, \; x_1 > 0.$$

Corollary 2 *The marginal density of X_1 and X_2 is given by*

$$C\left(\alpha_1, \alpha_2, \sum_{i=3}^{n} \alpha_i; \beta, k\right) \beta^{\sum_{i=3}^{n} \alpha_i + k} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \exp\left[-\frac{1}{\beta}(x_1 + x_2)\right]$$
$$\times \sum_{j=0}^{k} \binom{k}{j} \Gamma\left(\sum_{i=3}^{n} \alpha_i + k - j\right) \left(\frac{x_1 + x_2}{\beta}\right)^j, \; x_1 > 0, x_2 > 0.$$

Substituting $u = z/(1+z)$ with $dz = (1-u)^{-2} du$ in (6), one gets

$$C\left(\alpha_1, \ldots, \alpha_s, \sum_{i=s+1}^{n} \alpha_i; \beta, k\right) \prod_{i=1}^{s} x_i^{\alpha_i - 1} \exp\left(-\frac{1}{\beta} \sum_{i=1}^{s} x_i\right) \left(\sum_{i=1}^{s} x_i\right)^{\sum_{i=s+1}^{n} \alpha_i + k}$$
$$\times \int_0^1 u^{\sum_{i=s+1}^{n} \alpha_i - 1} (1-u)^{-(\sum_{i=s+1}^{n} \alpha_i + k + 1)} \exp\left[-\frac{\left(\sum_{i=1}^{s} x_i\right) u}{\beta(1-u)}\right] \mathrm{d}u. \tag{7}$$

Now, writing

$$(1-u)^{-(\sum_{i=s+1}^{n} \alpha_i + k + 1)} \exp\left[-\frac{\left(\sum_{i=1}^{s} x_i\right) u}{\beta(1-u)}\right] = \sum_{j=0}^{\infty} u^j L_j^{(\sum_{i=s+1}^{n} \alpha_i + k)} \left(\frac{\sum_{i=1}^{s} x_i}{\beta}\right)$$

in (7) and integrating u, the marginal density of $X_1, \ldots, X_s$, in series involving generalized Laguerre polynomials, is derived as

$$C\left(\alpha_1,\ldots,\alpha_s,\sum_{i=s+1}^{n}\alpha_i;\beta,k\right)\prod_{i=1}^{s}x_i^{\alpha_i-1}\exp\left(-\frac{1}{\beta}\sum_{i=1}^{s}x_i\right)\left(\sum_{i=1}^{s}x_i\right)^{\sum_{i=s+1}^{n}\alpha_i+k}$$
$$\times\sum_{j=0}^{\infty}\frac{1}{\sum_{i=s+1}^{n}\alpha_i+j}L_j^{(\sum_{i=s+1}^{n}\alpha_i+k)}\left(\frac{\sum_{i=1}^{s}x_i}{\beta}\right).$$

Theorem 2 *Let* $(X_1,\ldots,X_n)\sim \text{GMG}(\alpha_1,\ldots,\alpha_n;\beta,k)$. *Then, for* $2\le r\le n$, *the marginal density of* $X_r,\ldots,X_n$ *is given by*

$$C\left(\sum_{i=1}^{r-1}\alpha_i,\alpha_r,\ldots,\alpha_n;\beta,k\right)\beta^{\sum_{i=1}^{r-1}\alpha_i+k}\prod_{i=r}^{n}x_i^{\alpha_i-1}\exp\left(-\frac{1}{\beta}\sum_{i=r}^{n}x_i\right)$$
$$\times\sum_{\ell=0}^{k}\binom{k}{\ell}\Gamma\left(\sum_{i=1}^{r-1}\alpha_i+k-\ell\right)\left(\frac{\sum_{i=r}^{n}x_i}{\beta}\right)^{\ell},\ x_i>0,\ i=r,\ldots,n.$$

Proof Similar to the proof of Theorem 1. □

Corollary 3 *The marginal density of* X_n *is given by*

$$C\left(\sum_{i=1}^{n-1}\alpha_i,\alpha_n;\beta,k\right)\beta^{\sum_{i=1}^{n-1}\alpha_i+k}x_n^{\alpha_n-1}\exp\left(-\frac{1}{\beta}x_n\right)$$
$$\times\sum_{\ell=0}^{k}\binom{k}{\ell}\Gamma\left(\sum_{i=1}^{n-1}\alpha_i+k-\ell\right)\left(\frac{x_n}{\beta}\right)^{\ell},\ x_n>0.$$

Theorem 3 *Let* $(X_1,\ldots,X_n)\sim \text{GMG}(\alpha_1,\ldots,\alpha_n;\beta,k)$. *Then, for* $r=1,\ldots,n$, *the marginal density of* X_r *is given by*

$$C\left(\sum_{i(\neq r)=1}^{n}\alpha_i,\alpha_r;\beta,k\right)\beta^{\sum_{i(\neq r)=1}^{n}\alpha_i+k}x_r^{\alpha_r+k-1}\exp\left(-\frac{x_r}{\beta}\right)$$
$$\times\sum_{j=0}^{k}\binom{k}{j}\Gamma\left(\sum_{i(\neq r)=1}^{n}\alpha_i+k-j\right)\left(\frac{x_r}{\beta}\right)^{j},\ x_r>0.$$

4 Factorizations

This section deals with several factorizations of the multivariate gamma distribution defined in Sect. 2.

In the next theorem, we give the joint distribution of partial sums of random variables distributed jointly as generalized multivariate gamma.

Let $n_1, \dots, n_\ell$ be non-negative integers such that $\sum_{i=1}^{\ell} n_i = n$ and define

$$\alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j, \quad n_0^* = 0, \quad n_i^* = \sum_{j=1}^{i} n_j, \quad i = 1, \dots, \ell.$$

Theorem 4 *Let* $(X_1, \dots, X_n) \sim \mathrm{GMG}(\alpha_1, \dots, \alpha_n; \beta, k)$. *Define* $Z_j = X_j/X_{(i)}$, $j = n_{i-1}^* + 1, \dots, n_i^* - 1$ *and* $X_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} X_j$, $i = 1, \dots, \ell$. *Then,*

(i) $(X_{(1)}, \dots, X_{(\ell)})$ *and* $(Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1})$, $i = 1, \dots, \ell$, *are independently distributed,*

(ii) $(Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1}) \sim \mathrm{D1}(\alpha_{n_{i-1}^*+1}, \dots, \alpha_{n_i^*-1}; \alpha_{n_i^*})$, $i = 1, \dots, \ell$, *and*

(iii) $(X_{(1)}, \dots, X_{(\ell)}) \sim \mathrm{GMG}(\alpha_{(1)}, \dots, \alpha_{(\ell)}; \beta, k)$.

Proof Substituting $x_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} x_j$ and $z_j = x_j/x_{(i)}$, $j = n_{i-1}^* + 1, \dots, n_i^* - 1$, $i = 1, \dots, \ell$ with the Jacobian

$$\begin{aligned} &J(x_1, \dots, x_n \to z_1, \dots, z_{n_1-1}, x_{(1)}, \dots, z_{n_{\ell-1}^*+1}, \dots, z_{n-1}, x_{(\ell)}) \\ &= \prod_{i=1}^{\ell} J(x_{n_{i-1}^*+1}, \dots, x_{n_i^*} \to z_{n_{i-1}^*+1}, \dots, z_{n_i^*-1}, x_{(i)}) \\ &= \prod_{i=1}^{\ell} x_{(i)}^{n_i-1}. \end{aligned}$$

in the density of $(X_1, \dots, X_n)$ given by (1), we get the joint density of $Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1}, X_{(i)}$, $i = 1, \dots, \ell$ as

$$\begin{aligned} &C(\alpha_1, \dots, \alpha_n; \alpha, \beta) \prod_{i=1}^{\ell} x_{(i)}^{\alpha_{(i)}-1} \left(\sum_{i=1}^{\ell} x_{(i)}\right)^k \exp\left(-\frac{1}{\beta} \sum_{i=1}^{\ell} x_{(i)}\right) \\ &\times \prod_{i=1}^{\ell} \left[\left(\prod_{j=n_{i-1}^*+1}^{n_i^*-1} z_j^{\alpha_j-1}\right) \left(1 - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} z_j\right)^{\alpha_{n_i^*}-1}\right], \end{aligned} \tag{8}$$

where $x_{(i)} > 0$, $i = 1, \dots, \ell$, $z_j > 0$, $j = n_{i-1}^* + 1, \dots, n_i^* - 1$, $\sum_{j=n_{i-1}^*+1}^{n_i^*-1} z_j < 1$, $i = 1, \dots, \ell$. From the factorization in (8), it is easy to see that $(X_{(1)}, \dots, X_{(\ell)})$ and $(Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1})$, $i = 1, \dots, \ell$, are independently distributed. Further $(X_{(1)}, \dots, X_{(\ell)}) \sim \mathrm{GMG}(\alpha_{(1)}, \dots, \alpha_{(\ell)}; \beta, k)$ and $(Z_{n_{i-1}^*+1}, \dots, Z_{n_i^*-1}) \sim \mathrm{D1}(\alpha_{n_{i-1}^*+1}, \dots, \alpha_{n_i^*-1}; \alpha_{n_i^*})$, $i = 1, \dots, \ell$. □

Corollary 4 *Let* $(X_1, \dots, X_n) \sim \mathrm{GMG}(\alpha_1, \dots, \alpha_n; \beta, k)$. *Define* $Z_i = X_i/Z$, $i = 1, \dots, n-1$, *and* $Z = \sum_{j=1}^{n} X_j$. *Then,* $(Z_1, \dots, Z_{n-1})$ *and* Z *are independent,* $(Z_1, \dots, Z_{n-1}) \sim \mathrm{D1}(\alpha_1, \dots, \alpha_{n-1}; \alpha_n)$ *and* $Z \sim \mathrm{G}\left(\sum_{i=1}^{n} \alpha_i + k, \beta\right)$.

Corollary 5 *If* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$, *then* $\sum_{j=1}^n X_j$ *and* $\frac{\sum_{i=1}^s X_i}{\sum_{i=1}^n X_i}$ *are independent. Further*

$$\frac{\sum_{i=1}^s X_i}{\sum_{i=1}^n X_i} \sim \mathrm{B1}\left(\sum_{i=1}^s \alpha_i, \sum_{i=s+1}^n \alpha_i\right), \ s < n.$$

Theorem 5 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $W_j = X_j / X_{n_i^*}$, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$ *and* $X_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} X_j$, $i = 1, \ldots, \ell$. *Then,*

(i) $(X_{(1)}, \ldots, X_{(\ell)})$ *and* $(W_{n_{i-1}^*+1}, \ldots, W_{n_i^*-1})$, $i = 1, \ldots, \ell$, *are independently distributed,*

(ii) $(W_{n_{i-1}^*+1}, \ldots, W_{n_i^*-1}) \sim \mathrm{D2}(\alpha_{n_{i-1}^*+1}, \ldots, \alpha_{n_i^*-1}; \alpha_{n_i^*})$, $i = 1, \ldots, \ell$, *and*

(iii) $(X_{(1)}, \ldots, X_{(\ell)}) \sim \mathrm{GMG}(\alpha_{(1)}, \ldots, \alpha_{(\ell)}; \beta, k)$.

Corollary 6 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $W_i = X_i / X_n$, $i = 1, \ldots, n-1$ *and* $Z = \sum_{j=1}^n X_j$. *Then,* $(W_1, \ldots, W_{n-1})$ *and* Z *are independent,* $(W_1, \ldots, W_{n-1}) \sim \mathrm{D2}(\alpha_1, \ldots, \alpha_{n-1}; \alpha_n)$ *and* $Z \sim \mathrm{G}(\sum_{i=1}^n \alpha_i, \beta, k)$.

Corollary 7 *If* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$, *then* $\sum_{j=1}^n X_j$ *and* $\frac{\sum_{i=1}^s X_i}{\sum_{i=s+1}^n X_i}$ *are independent. Further*

$$\frac{\sum_{i=1}^s X_i}{\sum_{i=s+1}^n X_i} \sim \mathrm{B2}\left(\sum_{i=1}^s \alpha_i, \sum_{i=s+1}^n \alpha_i\right), \ s < n.$$

In next six theorems, we give several factorizations of the generalized multivariate gamma density.

Theorem 6 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $Y_n = \sum_{j=1}^n X_j$ *and* $Y_i = \sum_{j=1}^i X_j / \sum_{j=1}^{i+1} X_j$, $i = 1, \ldots, n-1$. *Then,* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \mathrm{B1}(\sum_{j=1}^i \alpha_j, \alpha_{i+1})$, $i = 1, \ldots, n-1$, *and* $Y_n \sim \mathrm{G}(\sum_{i=1}^n \alpha_i + k, \beta)$.

Proof Substituting $x_1 = y_n \prod_{i=1}^{n-1} y_i, x_2 = y_n(1 - y_1) \prod_{i=2}^{n-1} y_i, \ldots, x_{n-1} = y_n(1 - y_{n-2}) y_{n-1}$ and $x_n = y_n(1 - y_{n-1})$ with the Jacobian $J(x_1, \ldots, x_n \to y_1, \ldots, y_n) = \prod_{i=2}^n y_i^{i-1}$ in (1) we get the desired result. □

Theorem 7 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $Z_n = \sum_{j=1}^n X_j$ *and* $Z_i = X_{i+1} / \sum_{j=1}^i X_j$, $i = 1, \ldots, n-1$. *Then,* $Z_1, \ldots, Z_n$ *are independent,* $Z_i \sim \mathrm{B2}(\alpha_{i+1}, \sum_{j=1}^i \alpha_j)$, $i = 1, \ldots, n-1$, *and* $Z_n \sim \mathrm{G}(\sum_{j=1}^n \alpha_j + k, \beta)$.

Proof The desired result follows from Theorem 6 by noting that $(1 - Y_i)/Y_i \sim \mathrm{B2}(\alpha_{i+1}, \sum_{j=1}^i \alpha_j)$. □

Theorem 8 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $W_n = \sum_{j=1}^n X_j$ *and* $W_i = \sum_{j=1}^i X_j / X_{i+1}$, $i = 1, \ldots, n-1$. *Then,* $W_1, \ldots, W_n$ *are independent,* $W_i \sim \mathrm{B2}(\sum_{j=1}^i \alpha_j, \alpha_{i+1})$, $i = 1, \ldots, n-1$, *and* $W_n \sim \mathrm{G}(\sum_{i=1}^n \alpha_i + k, \beta, k)$.

Proof The result follows from Theorem 7 by noting that $1/Z_i \sim \mathrm{B2}(\sum_{j=1}^{i} \alpha_j, \alpha_{i+1})$. □

Theorem 9 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $Y_n = \sum_{j=1}^{n} X_j$ *and* $Y_i = X_i / \sum_{j=i}^{n} X_j$, $i = 1, \ldots, n-1$. *Then,* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \mathrm{B1}(\alpha_i, \sum_{j=i+1}^{n} \alpha_j)$, $i = 1, \ldots, n-1$, *and* $Y_n \sim \mathrm{G}(\sum_{i=1}^{n} \alpha_i + k, \beta)$.

Proof Substituting $x_1 = y_n y_1, x_2 = y_n y_2(1 - y_1), \ldots, x_{n-1} = y_n y_{n-1}(1 - y_1) \cdots (1 - y_{n-2})$, and $x_n = y_n(1 - y_1) \cdots (1 - y_{n-1})$ with the Jacobian $J(x_1, \ldots, x_n \to y_1, \ldots, y_n) = y_n^{n-1} \prod_{i=1}^{n-2} (1 - y_i)^{n-i-1}$ in (1), we get the desired result. □

Theorem 10 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $Z_n = \sum_{j=1}^{n} X_j$ *and* $Z_i = X_i / \sum_{j=i+1}^{n} X_j$, $i = 1, \ldots, n-1$. *Then,* $Z_1, \ldots, Z_n$ *are independent,* $Z_i \sim \mathrm{B2}(\alpha_i, \sum_{j=i+1}^{n} \alpha_j)$, $i = 1, \ldots, n-1$, *and* $Z_n \sim \mathrm{G}(\sum_{i=1}^{n} \alpha_i + k, \beta)$.

Proof The result follows from Theorem 9 by observing that $Y_i/(1 - Y_i) \sim \mathrm{B2}(\alpha_i, \sum_{j=i+1}^{n} \alpha_j)$. □

Theorem 11 *Let* $(X_1, \ldots, X_n) \sim \mathrm{GMG}(\alpha_1, \ldots, \alpha_n; \beta, k)$. *Define* $W_n = \sum_{j=1}^{n} X_j$ *and* $W_i = \sum_{j=i+1}^{n} X_j / X_i$, $i = 1, \ldots, n-1$. *Then,* $W_1, \ldots, W_n$ *are independent,* $W_i \sim \mathrm{B2}(\sum_{j=i+1}^{n} \alpha_j, \alpha_i)$, $i = 1, \ldots, n-1$, *and* $W_n \sim \mathrm{G}(\sum_{i=1}^{n} \alpha_i + k, \beta)$.

Proof The result follows from Theorem 10 by noting that $1/W_i \sim \mathrm{B2}(\sum_{j=i+1}^{n} \alpha_j, \alpha_i)$. □

5 Joint Moments

By definition

$$
\begin{aligned}
E(X_1^{r_1} \cdots X_n^{r_n}) &= C(\alpha_1, \ldots, \alpha_n; \beta, k) \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} x_i^{\alpha_i + r_i - 1} \left(\sum_{i=1}^{n} x_i \right)^k \\
&\qquad \times \exp\left(-\frac{1}{\beta} \sum_{i=1}^{n} x_i \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \frac{C(\alpha_1, \ldots, \alpha_n; \beta, k)}{C(\alpha_1 + r_1, \ldots, \alpha_n + r_n; \beta, k)}.
\end{aligned}
$$

Now, simplifying the above expression by using (2), one gets

$$
E(X_1^{r_1} \cdots X_n^{r_n}) = \beta^r \frac{\Gamma(\alpha)\Gamma(\alpha + r + k)}{\Gamma(\alpha + k)\Gamma(\alpha + r)} \prod_{i=1}^{n} \frac{\Gamma(\alpha_i + r_i)}{\Gamma(\alpha_i)},
$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$ and $r = \sum_{i=1}^{n} r_i$.

Further, substituting appropriately in the above expression, one gets

$$E(X_\ell^{r_\ell} X_m^{r_m}) = \beta^{r_\ell + r_m} \frac{\Gamma(\alpha)\Gamma(\alpha + r_\ell + r_m + k)}{\Gamma(\alpha + k)\Gamma(\alpha + r_\ell + r_m)} \frac{\Gamma(\alpha_\ell + r_\ell)\Gamma(\alpha_m + r_m)}{\Gamma(\alpha_\ell)\Gamma(\alpha_m)},$$

$$E(X_\ell X_m) = \beta^2 \frac{\alpha_\ell \alpha_m (\alpha + k)(\alpha + k + 1)}{\alpha(\alpha + 1)},$$

$$E(X_j) = \beta \frac{\alpha_j (\alpha + k)}{\alpha},$$

and

$$E(X_j^2) = \beta^2 \frac{\alpha_j (\alpha_j + 1)(\alpha + k)(\alpha + k + 1)}{\alpha(\alpha + 1)}.$$

Finally, by using appropriate definitions, we get

$$\operatorname{var}(X_j) = \beta^2 \frac{\alpha_j (\alpha + k)[\alpha(\alpha + 1) + (\alpha - \alpha_j)k]}{\alpha^2(\alpha + 1)},$$

$$\operatorname{cov}(X_\ell, X_m) = -k\beta^2 \frac{\alpha_\ell \alpha_m (\alpha + k)}{\alpha^2(\alpha + 1)},$$

$$\operatorname{corr}(X_\ell, X_m) = -k \sqrt{\frac{\alpha_\ell \alpha_m}{[\alpha(\alpha + 1) + (\alpha - \alpha_\ell)k][\alpha(\alpha + 1) + (\alpha - \alpha_m)k]}}.$$

6 Moment Generating Function

By definition, the joint mgf of $X_1, \ldots, X_n$ is given by

$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = C(\alpha_1, \ldots, \alpha_n; \beta, k) \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n x_i^{\alpha_i - 1} \left(\sum_{i=1}^n x_i \right)^k \times \exp\left(\sum_{i=1}^n t_i x_i - \frac{1}{\beta} \sum_{i=1}^n x_i \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{9}$$

Substituting $x_1 = r_1 s, \ldots x_{n-1} = r_{n-1} s$ and $x_n = s(1 - \sum_{i=1}^{n-1} r_i)$ in (9) with the Jacobian $J(x_1, \ldots, x_{n-1}, x_n \to r_1, \ldots, r_{n-1}, s) = s^{n-1}$ and integrating s, we get

$$
\begin{aligned}
&M_{X_1,\ldots,X_n}(t_1,\ldots,t_n)\\
&= C(\alpha_1,\ldots,\alpha_n;\beta,k)\beta^{\sum_{i=1}^{n}\alpha_i+k}\Gamma\left(\sum_{i=1}^{n}\alpha_i+k\right)\underset{\substack{r_1+\cdots+r_{n-1}<1\\ 0<r_i,\, i=1,\ldots,n-1}}{\int\cdots\int}\prod_{i=1}^{n-1}r_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n-1}r_i\right)^{\alpha_n-1}\\
&\quad\times\left[\sum_{i=1}^{n-1}(1-\beta t_i)r_i+(1-\beta t_n)\left(1-\sum_{i=1}^{n-1}r_i\right)\right]^{-(\sum_{i=1}^{n}\alpha_i+k)}\mathrm{d}r_1\cdots\mathrm{d}r_{n-1},
\end{aligned}
\tag{10}
$$

where $1-t_i\beta>0$, $i=1,\ldots,n$. Now, writing

$$
\left[\sum_{i=1}^{n-1}(1-\beta t_i)r_i+(1-\beta t_n)\left(1-\sum_{i=1}^{n-1}r_i\right)\right]^{-(\sum_{i=1}^{n}\alpha_i+k)}
$$

$$
\begin{aligned}
&= (1-t_n\beta)^{-(\sum_{i=1}^{n}\alpha_i+k)}\left[1-\sum_{i=1}^{n-1}r_i\left(1-\frac{1-t_i\beta}{1-\beta t_n}\right)\right]^{-(\sum_{i=1}^{n}\alpha_i+k)},\\
&\frac{1-t_i\beta}{1-t_n\beta}<1,\quad i=1,\ldots,n-1
\end{aligned}
$$

in (10) and integrating r, we get

$$
\begin{aligned}
&M_{X_1,\ldots,X_n}(t_1,\ldots,t_n)\\
&= C(\alpha_1,\ldots,\alpha_n;\beta,k)\beta^{\sum_{i=1}^{n}\alpha_i+k}\Gamma\left(\sum_{i=1}^{n}\alpha_i+k\right)(1-t_n\beta)^{-(\sum_{i=1}^{n}\alpha_i+k)}\\
&\quad\times\underset{\substack{r_1+\cdots+r_{n-1}<1\\ 0<r_i,i=1,\ldots,n-1}}{\int\cdots\int}\prod_{i=1}^{n-1}r_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n-1}r_i\right)^{\alpha_n-1}\left[1-\sum_{i=1}^{n-1}r_i\frac{\beta(t_i-t_n)}{1-\beta t_n}\right]^{-(\sum_{i=1}^{n}\alpha_i+k)}\mathrm{d}r_1\cdots\mathrm{d}r_{n-1}\\
&= C(\alpha_1,\ldots,\alpha_n;\beta,k)\beta^{\sum_{i=1}^{n}\alpha_i+k}\Gamma\left(\sum_{i=1}^{n}\alpha_i+k\right)(1-t_n\beta)^{-(\sum_{i=1}^{n}\alpha_i+k)}\frac{\prod_{i=1}^{n}\Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{n}\alpha_i)}\\
&\quad\times F_D^{(n-1)}\left(\sum_{i=1}^{n}\alpha_i+k,\alpha_1,\ldots,\alpha_{n-1};\sum_{i=1}^{n}\alpha_i;\frac{\beta(t_1-t_n)}{1-\beta t_n},\ldots,\frac{\beta(t_{n-1}-t_n)}{1-\beta t_n}\right),
\end{aligned}
$$

where the last line has been obtained by using the integral representation of the fourth hypergeometric function of Lauricella given in (15). Finally, substituting for $C(\alpha_1,\ldots,\alpha_n;\beta,k)$ and simplifying, we get

$$M_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = (1-t_n\beta)^{-(\sum_{i=1}^n \alpha_i+k)} \times F_D^{(n-1)}\left(\sum_{i=1}^n \alpha_i + k, \alpha_1,\ldots,\alpha_{n-1}; \sum_{i=1}^n \alpha_i; \frac{\beta(t_1-t_n)}{1-\beta t_n},\ldots,\frac{\beta(t_{n-1}-t_n)}{1-\beta t_n}\right).$$

For $t_1 = \cdots = t_n = t$, we have

$$M_{X_1,\ldots,X_n}(t,\ldots,t) = M_{X_1+\cdots+X_n}(t) = (1-t\beta)^{-(\sum_{i=1}^n \alpha_i+k)}$$

which is the mgf of a gamma random variable with shape parameter $\sum_{i=1}^n \alpha_i + k$ and scale parameter β.

7 Entropies

In this section, exact forms of Rényi and Shannon entropies are derived for the multivariate gamma distribution defined in this article.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf f associated with $\mathcal{P}$, dominated by $\sigma-$finite measure μ on $\mathcal{X}$. Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon [29]. It is define by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \log f(x)\, \mathrm{d}\mu. \tag{11}$$

One of the main extensions of the Shannon entropy was defined by Rényi [25]. This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\log G(\eta)}{1-\eta} \qquad (\text{for } \eta > 0 \text{ and } \eta \neq 1), \tag{12}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^\eta d\mu.$$

The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η, while Shannon entropy (11) is obtained from (12) for $\eta \uparrow 1$. For details see Nadarajah and Zografos [22], Zografos and Nadarajah [36] and Zografos [35].

Theorem 12 *For the generalized multivariate gamma distribution defined by the pdf (1), the Rényi and the Shannon entropies are given by*

$$
\begin{aligned}
H_R(\eta, f) = \frac{1}{1-\eta}\Bigg[& \eta \ln C(\alpha_1, \ldots, \alpha_n; \beta, k) + \left[\eta \sum_{i=1}^{n} (\alpha_i - 1) + n + \eta k\right] \ln\left(\frac{\beta}{\eta}\right) \\
& + \sum_{i=1}^{n} \ln \Gamma[\eta(\alpha_i - 1) + 1] + \ln \Gamma\left[\eta \sum_{i=1}^{n} (\alpha_i - 1) + n + \eta k\right] \\
& - \ln \Gamma\left[\eta \sum_{i=1}^{n} (\alpha_i - 1) + n\right]
\end{aligned}
$$

and

$$
\begin{aligned}
H_{SH}(f) = & - \ln C(\alpha_1, \ldots, \alpha_n; \beta, k) - \Bigg[\left(\sum_{i=1}^{n} \alpha_i + k - n\right) \ln \beta - \left(\sum_{i=1}^{n} \alpha_i + k\right) \\
& + \sum_{i=1}^{n} (\alpha_i - 1)\psi(\alpha_i) + \left(\sum_{i=1}^{n} \alpha_i + k - n\right)\psi\left(\sum_{i=1}^{n} \alpha_i + k\right) \\
& - \left(\sum_{i=1}^{n} \alpha_i - n\right)\psi\left(\sum_{i=1}^{n} \alpha_i\right)\Bigg],
\end{aligned}
$$

respectively, where $\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{1}{\Gamma(z)} \frac{d}{dz} \Gamma(z)$ *is the digamma function.*

Proof For $\eta > 0$ and $\eta \neq 1$, using the joint density of $X_1, \ldots, X_n$ given by (1), we have

$$
\begin{aligned}
G(\eta) &= \int_0^\infty \cdots \int_0^\infty f^\eta(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_n; \beta, k) \prod_{i=1}^{n} \mathrm{d}x_i \\
&= [C(\alpha_1, \ldots, \alpha_n; \beta, k)]^\eta \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} x_i^{\eta(\alpha_i - 1)} \left(\sum_{i=1}^{n} x_i\right)^{\eta k} \\
&\quad \times \exp\left(-\frac{\eta}{\beta} \sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} \mathrm{d}x_i \\
&= [C(\alpha_1, \ldots, \alpha_n; \beta, k)]^\eta \frac{\prod_{i=1}^{n} \Gamma[\eta(\alpha_i - 1) + 1]}{\Gamma[\eta \sum_{i=1}^{n} (\alpha_i - 1) + n]} \\
&\quad \times \int_0^\infty x^{\eta \sum_{i=1}^{n} (\alpha_i - 1) + n + \eta k - 1} \exp\left(-\frac{\eta x}{\beta}\right) \mathrm{d}x,
\end{aligned}
$$

where the last line has been obtained by using (16). Finally, evaluating the above integral by using gamma integral and simplifying the resulting expression, we get

$$G(\eta) = [C(\alpha_1, \ldots, \alpha_n; \beta, k)]^{\eta} \frac{\prod_{i=1}^{n} \Gamma[\eta(\alpha_i - 1) + 1]}{\Gamma[\eta \sum_{i=1}^{n}(\alpha_i - 1) + n]} \Gamma\left[\eta \sum_{i=1}^{n}(\alpha_i - 1) + n + \eta k\right]$$
$$\times \left(\frac{\beta}{\eta}\right)^{\eta \sum_{i=1}^{n}(\alpha_i - 1) + n + \eta k}.$$

Now, taking logarithm of $G(\eta)$ and using (12) we get $H_R(\eta, f)$. The Shannon entropy is obtained from $H_R(\eta, f)$ by taking $\eta \uparrow 1$ and using L'Hopital's rule. □

8 Estimation

Let $(X_{11}, \ldots, X_{1n}), \ldots, (X_{N1}, \ldots, X_{Nn})$ be a random sample from GMG$(\alpha_1, \ldots, \alpha_n; \beta, k)$. The log-likelihood function, denoted by $l(\alpha_1, \ldots, \alpha_n; \beta)$, is given by

$$l(\alpha_1, \ldots, \alpha_n; \beta) = N\left[\ln \Gamma(\alpha) - (\alpha + k) \ln \beta - \sum_{i=1}^{n} \ln \Gamma(\alpha_i) - \ln \Gamma(\alpha + k)\right]$$
$$+ \sum_{h=1}^{N} \sum_{i=1}^{n} (\alpha_i - 1) \ln x_{hi} + k \sum_{h=1}^{N} \ln\left(\sum_{i=1}^{n} x_{hi}\right) - \frac{1}{\beta} \sum_{h=1}^{N} \sum_{i=1}^{n} x_{hi},$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$. Now, differentiating $l(\alpha_1, \ldots, \alpha_n; \beta)$ w.r.t. α_i, we get

$$\frac{\partial l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i} = N[\psi(\alpha) - \ln \beta - \psi(\alpha_i) - \psi(\alpha + k)] + \sum_{h=1}^{N} \ln x_{hi}.$$

Further,

$$\frac{\partial l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \beta} = -\frac{N(\alpha + k)}{\beta} + \frac{1}{\beta^2} \sum_{h=1}^{N} \sum_{i=1}^{n} x_{hi},$$

$$\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i \partial \alpha_\ell} = N[\psi_1(\alpha) - \psi_1(\alpha + k)], \quad 1 \le i \ne \ell \le n,$$

$$\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i^2} = N[\psi_1(\alpha) - \psi_1(\alpha_i) - \psi_1(\alpha + k)],$$

where $\psi_1(z)$ is the trigamma function defined as the derivative of the digamma function, $\psi_1(z) = \frac{d}{dz}\psi(z)$,

$$\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i \partial \beta} = -\frac{N}{\beta},$$

$$\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \beta^2} = \frac{N(\alpha + k)}{\beta^2} - \frac{2}{\beta^3} \sum_{h=1}^{N} \sum_{i=1}^{n} x_{hi}.$$

Now, noting that $\sum_{i=1}^{n} X_i \sim \mathrm{G}(\alpha + k, \beta)$ and the expected value of a constant is the constant itself, we obtain

$$\theta_{i\ell} = \theta_{\ell i} = E\left[\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i \partial \alpha_\ell}\right] = N\psi_1(\alpha) - N\psi_1(\alpha + k),\ 1 \le i \ne \ell \le n,$$

$$\theta_{i\,n+1} = \theta_{n+1\,i} = E\left[\frac{\partial l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i \partial \beta}\right] = -\frac{N}{\beta},\ 1 \le i \le n,$$

$$\theta_{ii} = E\left[\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i^2}\right] = N\psi_1(\alpha) - N\psi_1(\alpha_i) - N\psi_1(\alpha + k),\ 1 \le i \le n,$$

$$\theta_{n+1\,n+1} = E\left[\frac{\partial^2 l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \beta^2}\right] = -\frac{N(\alpha + k)}{\beta^2}.$$

The Fisher information matrix for the multivariate gamma distribution given by the density (1) is defined as

$$-\begin{pmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1n} & \theta_{1\,n+1} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{1n} & \theta_{2\,n+1} \\ \vdots & & & & \vdots \\ \theta_{n1} & \theta_{n2} & \cdots & \theta_{nn} & \theta_{n\,n+1} \\ \theta_{n+1\,1} & \theta_{n+1\,2} & \cdots & \theta_{n+1\,n} & \theta_{n+1\,n+1} \end{pmatrix}.$$

Further

$$\frac{\partial l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \beta} = -\frac{N(\alpha + k)}{\beta} + \frac{1}{\beta^2} \sum_{h=1}^{N} \sum_{i=1}^{n} x_{hi} = 0$$

gives

$$(\alpha + k)\beta = \sum_{i=1}^{n} \bar{x}_i \tag{13}$$

and

$$\frac{\partial l(\alpha_1, \ldots, \alpha_n; \beta)}{\partial \alpha_i} = N\left[\psi(\alpha) - \ln\beta - \psi(\alpha_i) - \psi(\alpha + k)\right] + \sum_{h=1}^{N} \ln x_{hi} = 0$$

gives

$$\psi(\alpha + k) - \psi(\alpha) + \ln\beta + \psi(\alpha_i) = \ln\tilde{x}_i,\ i = 1, \ldots, n,$$

where $\tilde{x}_i = \prod_{h=1}^{N} x_{hi}^{1/N}$, $i = 1, 2, \ldots, n$. Further, using

$$[\psi(z + m) - \psi(z) = \sum_{j=0}^{m-1} \frac{1}{z + j}$$

we have

$$\sum_{j=0}^{k-1} \frac{1}{\alpha + j} + \ln\beta + \psi(\alpha_i) = \ln\tilde{x}_i, \quad i = 1, \ldots, n. \tag{14}$$

Thus, by solving numerically (13) and (14), the MLEs of α_i and β can be obtained.

9 Simulation

In this section, a simulation study for $p = 3$ is conducted to evaluate the performance of maximum likelihood method. For $p = 2$, see Rafiei, Iranmanesh, and Nagar [24]. Samples of size $n = 50, 200, 500$ from Equation (1) for selected values of parameters are generated by MCMC methods (Gibbs Metropolise, Markov Chain Monte Carlo Metropolise, Metropolise, Metropolise gaussian, random walk Metropolise and Metropolise-Hastings). We have performed the simulation for particular values of parameters, namely, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\beta = 2$, $k = 4, 8$, and $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 1$, $\beta = 2$, $k = 4, 8$. The results were similar for other choices. MLEs for parameters based on the numerical procedures were computed. This procedures was repeated five hundred times and $(\widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha_3}, \widehat{\beta})$, the average of biases (Ab) and the mean squared errors (MSE) were obtained by using Monte Carlo methods (the parameter k is an integer and the derivative method is not used to calculate its MLE).

Different packages such as MCMC, MCMCpack, gibbs.met, LearnBayes, MHadaptive, MetroHastings and walkMetropolis in R were used for simulation. After performing simulation using the above methods and comparing results, it was observed that the Gibbs sampling method provides better results. Therefore, the output of Gibbs method is presented in Tables 1, 2, 3, 4 and Figs. 1, 2, 3, 4 and 5. The MLEs of parameters and correlation cofficients are reported in Tables 1 and 2. The DEoptim package in R was used to calculate the MLEs. The average of biases and the mean squared errors of all the estimators are reported in Tables 3 and 4. In particular, biases for the maximum likelihood estimators of $\alpha_1, \alpha_2, \alpha_3$ and β are close to 0 and the mean squared errors of all estimators always decrease with increasing n.

Figure 1 shows 3D scatter plot of the simulation data for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$, $\beta = 2, k = 4, n = 50, 500$. Figure 2 shows 3D plot of the simulation data for $\alpha_1 = 1$, $\alpha_2 = 2, \alpha_3 = 3, \beta = 2, k = 4$. Figs. 3 and 4 show pairs style of the simulation data for $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 2$, $\beta = 2$, $k = 8$, $n = 50$ and $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 1$, $\beta = 2$,

Table 1 MLEs of parameters and correlation cofficients

α_1	α_2	α_3	β	k	n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\beta}$	$Corr(X_1, X_2)$	$Corr(X_1, X_3)$	$Corr(X_2, X_3)$
1	2	3	2	4	50	1.2315	2.6113	3.9812	1.5833	−0.3135	−0.3761	−0.1916
					200	1.0519	2.1541	3.1909	1.8805	0.0315	−0.2767	−0.3253
					500	1.0200	2.0488	3.0889	1.9368	−0.1351	−0.0087	−0.2359
1	2	3	2	8	50	1.4617	3.1957	4.7645	1.5884	0.3833	−0.2242	−0.6268
					200	1.0911	2.2651	3.4170	1.8736	−0.3302	−0.5510	−0.5064
					500	1.0332	2.0814	3.1645	1.9528	−0.1628	−0.1630	−0.3739

Table 2 MLEs of parameters and correlation cofficients

α_1	α_2	α_3	β	k	n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\beta}$	$Corr(X_1, X_2)$	$Corr(X_1, X_3)$	$Corr(X_2, X_3)$
2	2	1	2	4	50	2.5823	2.6287	1.2518	1.6702	0.0371	−0.3001	−0.0493
					200	2.2695	2.2684	1.1074	1.8370	−0.2307	−0.5114	0.1883
					500	2.0589	2.0384	1.0200	1.9507	−0.0746	−0.2110	−0.5513
2	2	1	2	8	50	3.1594	3.0801	1.4560	1.5814	−0.3203	−0.6072	−0.1092
					200	2.2567	2.3256	1.1060	1.9016	−0.3175	−0.2600	−0.1340
					500	1.0386	2.0907	3.1642	1.9729	−0.2865	−0.2239	−0.2340

Table 3 The average of biases and the mean squared errors of estimators

α_1	α_2	α_3	β	k	n	$Ab(\hat{\alpha}_1)$	$Ab(\hat{\alpha}_2)$	Ab($\hat{\alpha}_3$)	$Ab(\hat{\beta})$	$MSE(\hat{\alpha}_1)$	$MSE(\hat{\alpha}_2)$	$MSE(\hat{\alpha}_3)$	$MSE(\hat{\beta})$
1	2	3	2	4	50	0.2315	0.6112	0.9812	−0.4167	0.1969	1.3238	4.0505	0.1994
					200	0.0519	0.1541	0.1910	−0.1195	0.0276	0.2070	0.4197	0.0301
					500	0.0200	0.0488	0.0889	−0.0632	0.0094	0.0643	0.1793	0.0104
1	2	3	2	8	50	0.4617	1.1957	1.7635	−0.4116	0.6210	4.1107	8.8690	0.6040
					200	0.0911	0.2650	0.4170	−0.1264	0.0465	0.3430	1.0176	0.0458
					500	0.0332	0.0814	0.1644	−0.0472	0.0163	0.0930	0.3178	0.0160

Table 4 The average of biases and the mean squared errors of estimators

α_1	α_2	α_3	β	k	n	$Ab(\hat{\alpha}_1)$	$Ab(\hat{\alpha}_2)$	Ab($\hat{\alpha}_3$)	$Ab(\hat{\beta})$	$MSE(\hat{\alpha}_1)$	$MSE(\hat{\alpha}_2)$	$MSE(\hat{\alpha}_3)$	$MSE(\hat{\beta})$
2	2	1	2	4	50	0.5823	0.6287	0.2518	−0.3298	1.5689	1.6406	0.2319	1.4540
					200	0.2695	0.2684	0.1074	−0.1630	0.4725	0.4779	0.0697	0.4350
					500	0.0589	0.0384	0.0200	−0.0493	0.0640	0.0654	0.0108	0.0581
2	2	1	2	8	50	1.1594	1.0801	0.45560	−0.4186	4.2375	3.5325	1.0332	4.0178
				8	200	0.2567	0.3256	0.1060	−0.0984	0.3755	0.5124	0.0606	0.3522
				8	500	0.0386	0.0907	0.1642	−0.0271	0.0174	0.1017	0.3396	0.0173

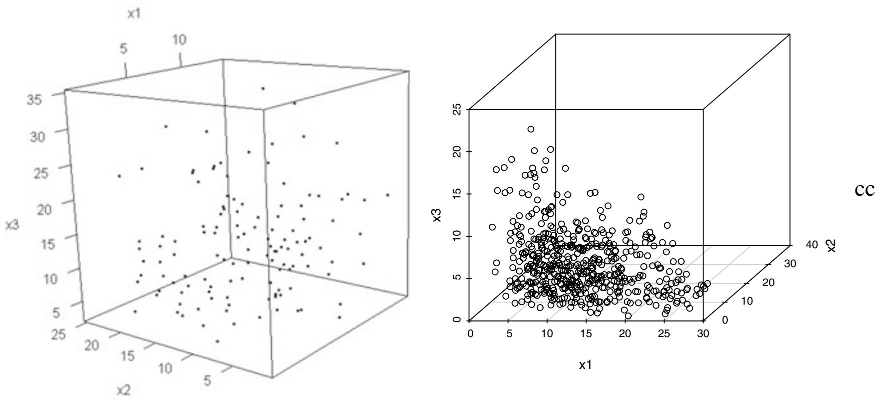

Fig. 1 3D scatter plot of simulation data with $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \beta = 2, k = 4, n = 50{,}500$

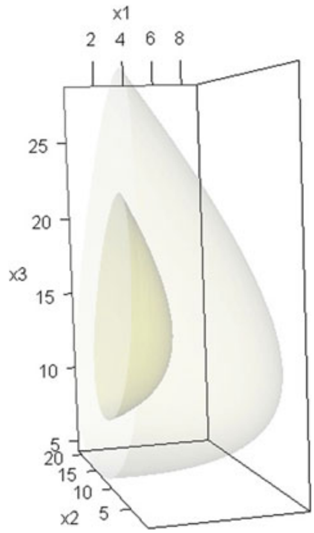

Fig. 2 3D plot for simulation data, $\alpha_1 = 1$, $\alpha_2 = 2, \alpha_3 = 3, \beta = 2$, $k = 4$

$k = 8, n = 500$, respectively. Figure 5 shows Trace plot for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$, $\beta = 2, k = 8, n = 500$. Finally, simulation points and 3D contour plot for different selected values of parameters are shown in Figs. 6, 7, 8 and 9.

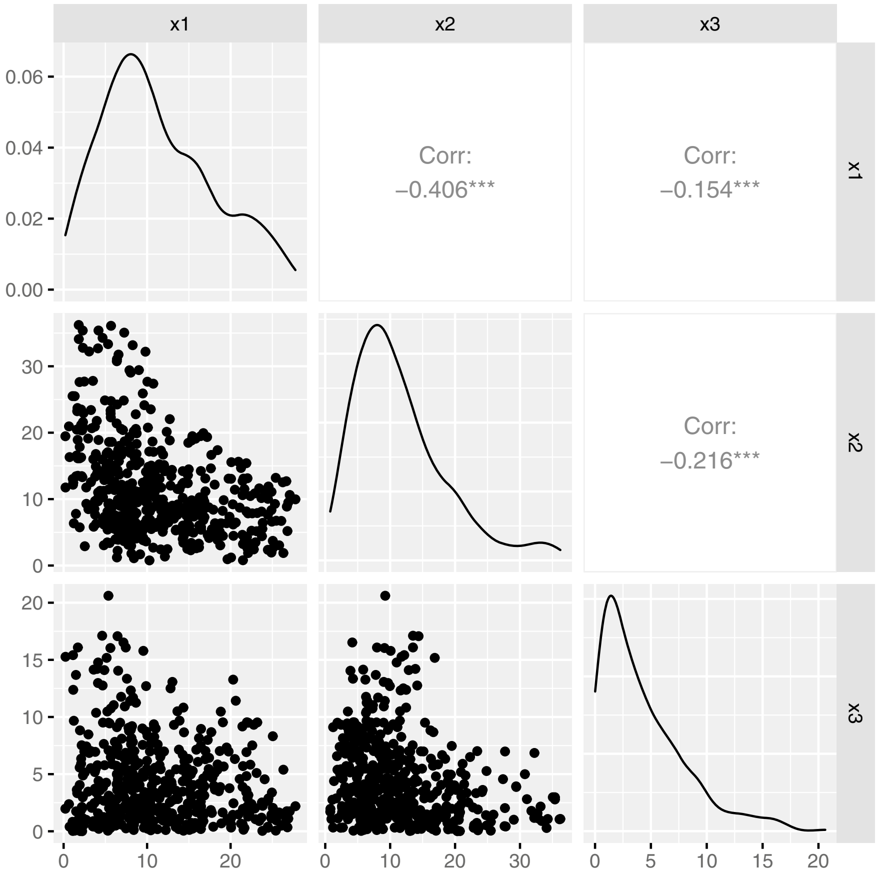

Fig. 3 Pairs plot for $\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 1, \beta = 2, k = 8, n = 50$

10 Conclusion

In this chapter, a new multivariate gamma distribution whose marginals are finite mixtures of gamma distributions is defined. It is shown that the correlation between any pair of variables is negative. Therefore, the newly introduced distribution could be suitable for fitting multivariate data with negative correlations. Several of its properties such as joint moments, correlation coefficients, moment generating function, Rényi and Shannon entropies have been derived. In Sect. 8, the method of MLE has been applied to estimate the parameters. Because the resulting likelihood equations are nonlinear, numerical methods have been used to solve them. Simulation studies have been conducted to evaluate the performance of the maximum likelihood method. Moreover, various tables and figures have been provided to confirm a proper simulation and results of the MLE method for estimating the parameters.

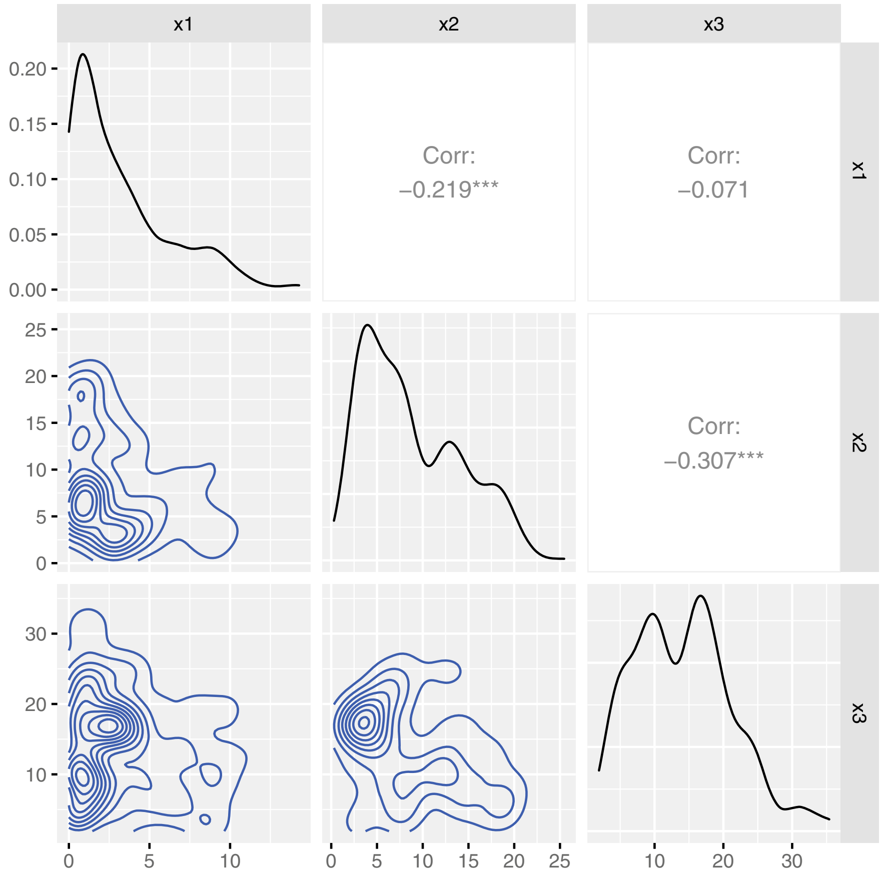

Fig. 4 Pairs plot for $\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 1, \beta = 2, k = 8, n = 500$

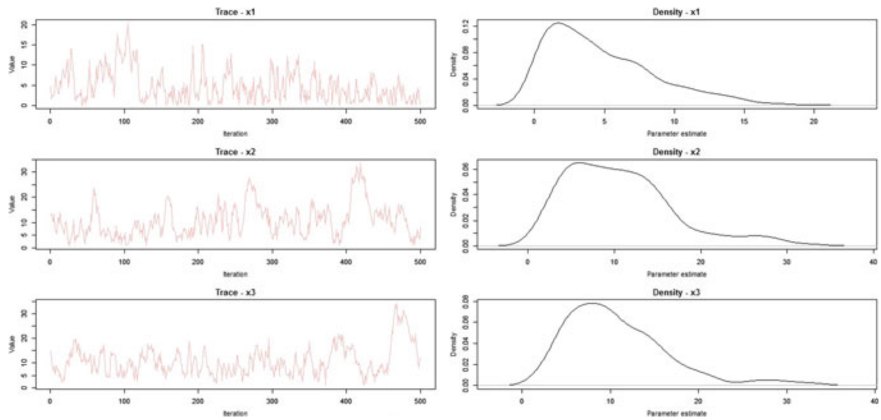

Fig. 5 Trace plots for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \beta = 2, k = 8, n = 500$

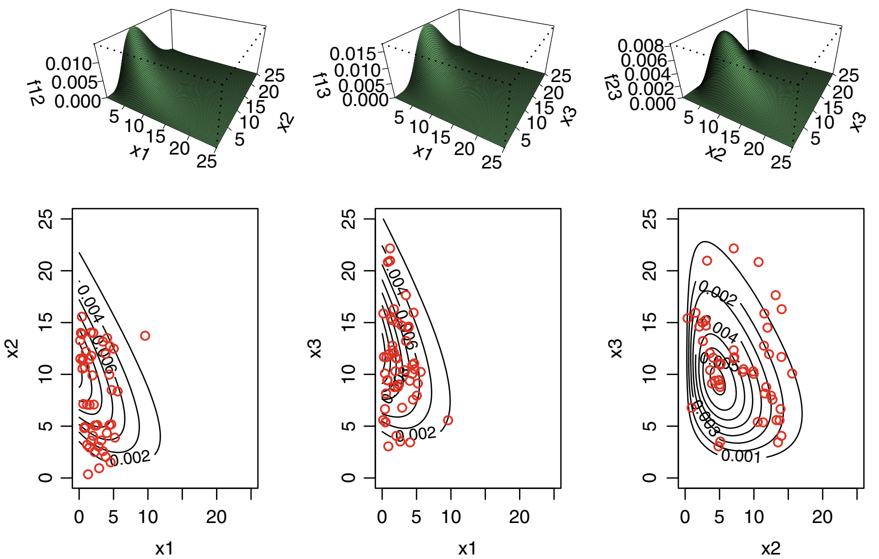

Fig. 6 Simulation points and 3D contour plot for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \beta = 2, k = 4, n = 50$

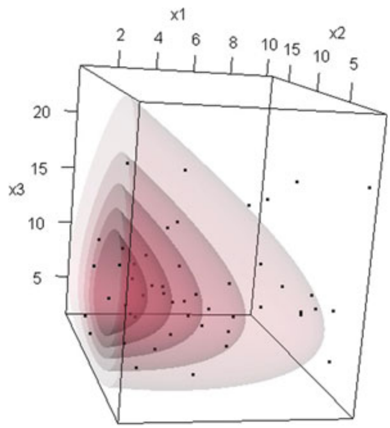

Fig. 7 Simulation points and contour plot for $\alpha_1 = 1$, $\alpha_2 = 2, \alpha_3 = 3, \beta = 2$, $k = 4, n = 50$

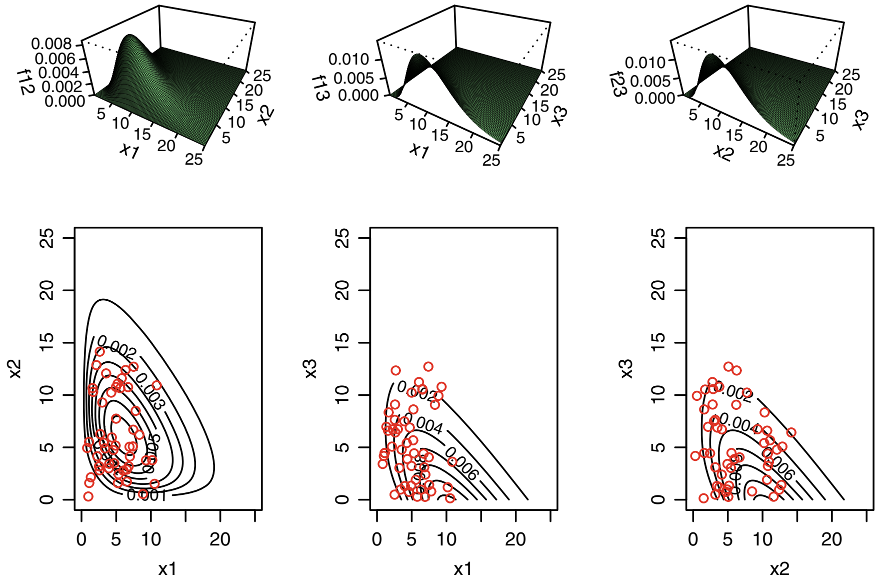

Fig. 8 Simulation points and contour plot for $\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 1, \beta = 2, k = 4, n = 50$

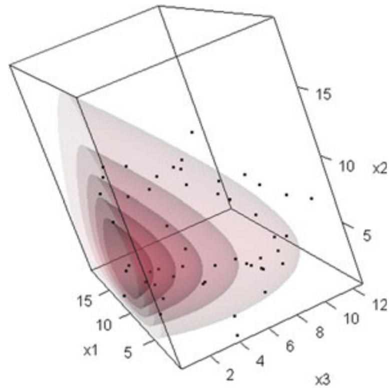

Fig. 9 Simulation points and 3D contour plot for $\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 1, \beta = 2, k = 4, n = 50$

Acknowledgements Authors are grateful to the worthy reviewers for their constructive and helpful comments and suggestions.

Appendix

In this section, we give definitions and results that will be used in subsequent sections. Throughout this work we will use the Pochhammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots,$ and $(a)_0 = 1$.

The fourth hypergeometric function of Lauricella, denoted by $F_D^{(n)}$, in n variables $z_1, \ldots, z_n$ is defined by

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{(a)_{j_1+\cdots+j_n}(b_1)_{j_1}\cdots(b_n)_{j_n} z_1^{j_1}\cdots z_n^{j_n}}{(c)_{j_1+\cdots+j_n} j_1!\cdots j_n!},$$

where $|z_i| < 1$, $i = 1, \ldots, n$. An integral representation of $F_D^{(n)}$ in Exton [7, p. 49, Eq. (2.3.5)] is given as

$$\begin{aligned} F_D^{(m)}&(a, b_1, \ldots, b_m; c; z_1, \ldots, z_m) \\ &= \frac{\Gamma(c)}{\prod_{i=1}^n \Gamma(b_i)\Gamma(c - \sum_{i=1}^n b_i)} \\ &\quad \times \underset{\substack{\sum_{i=1}^n x_i < 1 \\ 0 < x_i,\, i=1,\ldots,n}}{\int \cdots \int} \frac{\prod_{i=1}^n x_i^{b_i-1}\left(1 - \sum_{i=1}^n x_i\right)^{c-\sum_{i=1}^n b_i - 1}}{\left(1 - \sum_{i=1}^n z_i t_i\right)^a} \mathrm{d}x_1 \cdots \mathrm{d}x_n. \end{aligned} \tag{15}$$

For further results and properties of this function the reader is referred to Exton [7] and Srivastava and Karlsson [32].

Let $f(\cdot)$ be a continuous function and $\alpha_i > 0$, $i = 1, \ldots, n$. The integral

$$D_n(\alpha_1, \ldots, \alpha_n; f) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n x_i^{\alpha_i - 1} f\left(\sum_{i=1}^n x_i\right) \prod_{i=1}^n \mathrm{d}x_i$$

is known as the Liouville-Dirichlet integral. Substituting $y_i = x_i/x$, $i = 1, \ldots, n-1$ and $x = \sum_{i=1}^n x_i$ with the Jacobian $J(x_1, \ldots, x_{n-1}, x_n \to y_1, \ldots, y_{n-1}, x) = x^{n-1}$ it is easy to see that

$$D_n(\alpha_1, \ldots, \alpha_n; f) = \frac{\prod_{i=1}^n \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^n \alpha_i)} \int_0^\infty x^{\sum_{i=1}^n \alpha_i - 1} f(x)\,\mathrm{d}x. \tag{16}$$

Finally, we define the beta type 1, beta type 2 and Dirichlet type 1 distributions. These definitions can be found in Wilks [34], Fang, Kotz and Ng [8], Johnson, Kotz and Balakrishnan [15], and Kotz, Balakrishnan and Johnson [16].

Definition 2 A random variable X is said to have the beta type I distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim \mathrm{B1}(a, b)$, if its pdf is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},\ 0 < x < 1.$$

Definition 3 A random variable X is said to have the beta type II distribution with parameters (a, b), denoted as $X \sim \mathrm{B2}(a, b)$, $a > 0$, $b > 0$, if its pdf is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1+x)^{-(a+b)},\ x > 0.$$

Definition 4 The random variables $U_1, \ldots, U_n$ are said to have a Dirichlet type 1 distribution with parameters $\alpha_1, \ldots, \alpha_n$ and α_{n+1}, denoted by $(U_1, \ldots, U_n) \sim \mathrm{D1}(\alpha_1, \ldots, \alpha_n; \alpha_{n+1})$, if their joint pdf is given by

$$\frac{\Gamma(\sum_{i=1}^{n+1}\alpha_i)}{\prod_{i=1}^{n+1}\Gamma(\alpha_i)} \prod_{i=1}^{n} u_i^{\alpha_i-1}\left(1-\sum_{i=1}^{n} u_i\right)^{\alpha_{n+1}-1},$$
$$0 < u_i,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} u_i < 1, \tag{17}$$

where $\alpha_i > 0$, $i = 1, \ldots, n+1$.

The Dirichlet type 1 distribution, which is a multivariate generalization of the beta type 1 distribution, has been considered by several authors and is well known in the scientific literature. By making the transformation $V_j = U_j/(1-\sum_{i=1}^{n} U_i)$, $j = 1, \ldots, n$, in (17), the Dirichelt type 2 density, which is a multivariate generalization of beta type 2 density, is obtained as

$$\frac{\Gamma(\sum_{i=1}^{n+1}\alpha_i)}{\prod_{i=1}^{n+1}\Gamma(\alpha_i)} \prod_{i=1}^{n} u_i^{\alpha_i-1}\left(1+\sum_{i=1}^{n} u_i\right)^{-\sum_{i=1}^{n+1}\alpha_i},\quad v_i > 0,\ i = 1, \ldots, n. \tag{18}$$

We will write $(V_1, \ldots, V_n) \sim \mathrm{D2}(\alpha_1, \ldots, \alpha_n; \alpha_{n+1})$ if the joint density of $V_1, \ldots, V_n$ is given by (18).

The matrix variate generalizations of beta type 1, beta type 2 and Dirichlet type 1 distributions have been defined and studied extensively. For example, see Gupta and Nagar [11].

Definition 5 Multinomial Theorem: For a positive integer k and a non-negative integer m,

$$(z_1 + \cdots + z_m)^k = \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \ldots, k_m} z_1^{k_1} \cdots z_m^{k_m},$$

where

$$\binom{k}{k_1, \ldots, k_m} = \frac{k!}{k_1! \cdots k_m!}.$$

The numbers appearing in the theorem are the multinomial coefficients. They can be expressed in numerous ways, including as a product of binomial coefficients of factorials:

$$\binom{k}{k_1, k_2, \ldots, k_m} = \frac{k!}{k_1!\,k_2! \cdots k_m!} = \binom{k_1}{k_1}\binom{k_1+k_2}{k_2} \cdots \binom{k_1+k_2+\cdots+k_m}{k_m}$$

Lemma 1 *For $a_1 > 0, \ldots, a_m > 0$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned} k! \sum_{k_1+\cdots+k_m=k} \frac{(a_1)_{k_1} \ldots (a_m)_{k_m}}{k_1! \cdots k_m!} &= (a_1+\cdots+a_m)_k \\ &= \frac{\Gamma(a_1+\cdots+a_m+k)}{\Gamma(a_1+\cdots+a_m)}. \end{aligned}$$

Proof Writing $(1-\theta)^{-(a_1+\cdots+a_m)}$ as $(1-\theta)^{-a_1} \cdots (1-\theta)^{-a_m}$ and using power series expansion, for $0 < \theta < 1$, we get

$$\begin{aligned} (1-\theta)^{-a_1} \cdots (1-\theta)^{-a_m} &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \frac{(a_1)_{k_1} \ldots (a_m)_{k_m}}{k_1! \cdots k_m!} \theta^{k_1+\cdots+k_n} \\ &= \sum_{k=0}^{\infty} \theta^k \sum_{k_1+\cdots+k_m=k} \frac{(a_1)_{k_1} \ldots (a_m)_{k_m}}{k_1! \cdots k_m!} \end{aligned}$$

and

$$(1-\theta)^{-(a_1+\cdots+a_m)} = \sum_{k=0}^{\infty} \frac{(a_1+\cdots+a_m)_k}{k!} \theta^k.$$

Now, comparing coefficients of θ^k, we get the desired result. □

Lemma 2 *Let*

$$g(a_1, \ldots, a_m; \beta, k) = \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{m} z_i^{a_i-1} \left(\sum_{i=1}^{m} z_i\right)^k \exp\left(-\frac{1}{\beta} \sum_{i=1}^{m} z_i\right) dz_1 \cdots dz_m, \tag{19}$$

where $a_1 > 0, \ldots, a_m > 0$ and $k \in \mathbb{N}$. Then

$$g(a_1, \ldots, a_m; \beta, k) = \beta^{\sum_{i=1}^{m} a_i + k} \left[\prod_{i=1}^{m} \Gamma(a_i)\right] (a_1+\cdots+a_m)_k$$

Proof Expanding $\left(\sum_{i=1}^{m} z_i\right)^k$ in (19) by using multinomial theorem and integrating $z_1, \ldots, z_m$, we obtain

$$g(a_1, \ldots, a_m; \beta, k) = \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \ldots, k_m} \prod_{i=1}^{m} \int_0^{\infty} z_i^{z_i+k_i-1} \exp\left(-\frac{1}{\beta} z_i\right) dz_i$$
$$= \beta^{\sum_{i=1}^{m} \alpha_i + k} \sum_{k_1+\cdots+k_m=k} \binom{k}{k_1, \ldots, k_m} \prod_{i=1}^{m} \Gamma(a_i + k_i).$$

Now, using Lemma 1, we get the desired result. □

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