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Estelle Basor
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Toeplitz Operators and Random Matrices

In Memory of Harold Widom

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Preface

Harold Widom, born on September 23, 1932, passed away on January 20, 2021. We have lost a good friend and colleague, an inimitable teacher, and an outstanding mathematician. Harold has enriched mathematics with his ideas and groundbreaking work since the 1950s until the present time. His personality has left its imprint on all those who accompanied him some period or met him only occasionally.

This volume is dedicated to his memory. It contains a biography of Harold Widom and personal notes written by his former students or colleagues. We are at the same time sad and proud to publish also his last paper, *Domain walls in the Heisenberg-Ising Spin-*$\frac{1}{2}$ *chain*, which he started jointly with Axel Saenz and one of us but could not see it accomplished. Harold's most famous contributions were made to Toeplitz operators, random matrices, and the asymmetric simple exclusion process. While his work on the last two topics is part of almost all the present-day research activities in these fields, his work in Toeplitz operators and matrices was done mainly before 2000, and we therefore included an article which describes his achievements in just this area.

The volume contains several invited and refereed research and expository papers. These present new results or new perspectives on topics related to Harold's work. We are very grateful to all the authors for their effort to make this volume a highly deserved tribute to Harold Widom.

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July 2022	

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Part I
Harold Widom's Life, Work, and Last Paper

Biography of Harold Widom

Estelle Basor, Albrecht Böttcher, Torsten Ehrhardt, and Craig A. Tracy

Harold Widom was born September 23, 1932, in Newark, New Jersey, during the heart of the Depression. His parents were born in eastern Europe, and they came to the United States in 1914, when his mother was 15 years old and his father 22. They met in New York and were married there in 1924.

Harold was only eight when his father died. He had not seen him in the preceding three years, since his father, a dentist who contracted tuberculosis while serving in the US army in the First World War, had been in a tuberculosis sanitarium in Arizona and then in Colorado. In 1939, Harold, his brother, and their mother moved to Brooklyn.

Harold went to Stuyvesant High School in Manhattan. There he was captain of the math team. Coincidentally, the captain of the rival team at the Bronx High School of Science was Henry Landau, who became a long-time friend and colleague of Harold's. Al Kelley and Tony Tromba [1] write that the Stuyvesant team included also "two other famous twentieth century mathematicians, Elias Stein of Princeton and Paul Cohen of Stanford, who would all ultimately specialize in the field of mathematical analysis. Elias was one year older than Harold, and Paul two years

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younger. Paul, who would go on to win a Fields Medal in Mathematics in 1962 in recognition of his path-breaking solution of Hilbert's first problem, was generously tutored by Harold for several years in high school. All three remained together to study analysis under the guidance of Antoni Zygmund and Alberto Calderón in graduate school at the University of Chicago. Reflecting on his life in a speech at Stanford in 2001, Paul thanked Harold for the profound influence he had on his early mathematical career."

From 1949 to 1952, Harold attended the City College of New York, and in 1952 he moved to the University of Chicago, where he became a Ph.D. student of Irving Kaplansky and defended his Ph.D. thesis, *Embedding of AW^*-algebras*, in 1955. Irving Kaplansky [5] beautifully characterizes the spirit of those times and Harold's place in them as follows. "In 1946 Marshall Stone left Harvard to accept the chairmanship of the Department of Mathematics at the University of Chicago. There followed quickly a series of stellar appointments that raised the department to a very high level. (I can say this without being self-serving; John Kelley and I were the last appointments made before the "Stone Age".) It was an exciting time to be at Chicago. But it was not only the faculty that created the excitement—a stream of superb students arrived. I was lucky enough to attract my fair share, and that included Harold. His thesis was on AW^*-algebras . . . His bibliography shows three fine papers on the topic and then shifts. (With the shift, his output moved to a different part of Mathematical Reviews). I understand that the shift can be attributed to the influence of Mark Kac at Cornell and one could not ask for a better source of inspiration. I am proud and happy about what Harold added to the theory of AW^*-algebras, and equally proud and happy about what he has accomplished since then."

In 1955, Harold began his academic career as an instructor at Cornell University where he rose through the ranks to become full professor in 1965. At Cornell, he came under the influence of Mark Kac, who persuaded him to embark on the asymptotic behavior of the spectra of operators, especially Toeplitz operators. Harold then proved many of the early beautiful theorems about Toeplitz operators. More about this can be found in the article [2]. Shortly before 1968, he spent one year at Stanford University, and although at Cornell he had Mark Kac and his brother Benjamin (on the Cornell faculty of chemistry) around him, he then felt, as a rumor says, that the California weather is preferable to the Ithaca winters.

In the fall of 1968, Harold accepted an offer from University of California at Santa Cruz to become a founding member of the Mathematics Department. He served the department 26 years, with 3 years as the chairman, until 1994, when he used the opportunity for early retirement. Every topic has its time. As for Toeplitz and related operators, the late 1950s and 1960s may be regarded as the years of gold rush. However, the period between the 1970s and the late 1990s was the true Golden Age (or Belle Époque, as Nikolai Nikolski once called it) of research into Toeplitz and Wiener-Hopf operators as well as into pseudodifferential operators. It was not only fortunate conicidence that Harold's work in Santa Cruz fell into this age. In fact, Harold was one of the principal figures in this development, and it was just he who made some of the brightest contributions to the blossoming of the field. We refer again to [2] for a more detailed description of his tremendous achievements in this period.

Harold's self-chosen early retirement in 1994 was truly a huge loss for the UCSC Mathematics Department. Tony Tromba always joked that when Harold retired, the department entered a completely new chapter, Chapter 11. However, for Harold it was the right decision. It was the beginning of his joint and fruitful work with the fourth of us on random matrices and asymmetric simple exclusion processes, which had lasted 30 years until Harold's death in 2021. The discovery of what is now called the Tracy-Widom distribution brought him wide international recognition. Al Kelly and Tony Tromba [1] write "The densities of the Tracy-Widom distributions are on the cover of each issue of the journal *Random Matrices: Theory and Applications*, a rare tribute to someone's work." We refer to [4] for a profound exposition of Harold's work on random matrices and on the asymptotic behavior for the asymmetric simple exclusion process.

As of July 2022, MathSciNet lists 167 publications by Harold with about 4000 citations by nearly 1800 authors. Solely the paper *Level-spacing distributions and the Airy kernel* in Comm. Math. Phys. 159, 151–174 (1994), received more than 600 citations. Harold wrote three books: the Springer Lecture Notes volume *Asymptotic expansions for pseudodifferential operators on bounded domains*, which was published in 1985, and the two beautiful short books *Lectures on Integral Equations* and *Lectures on Measure and Integration* for students, based on lectures he gave at Cornell. The latter two resulted in part from notes written by David Drazin and Anthony Tromba, both students in his classes at the time. They were first published by Van Nostrand in 1969 and later by Dover. His probably last paper, *Domain walls in the Heisenberg-Ising Spin-*$\frac{1}{2}$ *chain*, jointly with Axel Saenz and one of us, is published in this volume.

Harold received numerous awards. In 2002, he was awarded the George Pólya Prize. In 2006, he received the Norbert Wiener Prize in Applied Mathematics and then in 2020, the American Mathematical Society's Steele Prize for Seminal Research. The fourth of us has the privilege to share these three prizes with Harold. In 2006, Harold was elected to the American Academy of Arts and Sciences.

Harold successfully guided 8 Ph.D. students:

Lidia Luquet, 1972, p-Norm inequalities for entire functions,

Estelle Basor, 1975, Asymptotic formulas for Toeplitz determinants,

Ray Roccaforte, 1982, Asymptotic expansions of traces for certain convolution operators,

Richard Libby, 1990, Asymptotics of determinants and eigenvalue distributions for Toeplitz matrices associated with certain discontinuous symbols,

Xiang Fu, 1991, Asymptotics of Toeplitz matrices with symbols of bounded variations,

Shuxian Lou, 1992, The second order asymptotics of a class of integral operators with discontinuous symbols, Bobette Thorsen, 1992, An asymptotic expansion for the trace of certain integral operators,

Bin Shao, 1993, Second order asymptotics for the discrete analogue of a class of pseudodifferential operators.

Harold had many interests outside of mathematics. He played the violin as a child and was part of the UCSC orchestra for several years. He especially loved hiking. Al Kelly and Tony Tromba [1] write "For over 15 years, the three of us hiked almost every week. We thoroughly enjoyed being together and having extended conversations on almost any topic, mathematical, political, or simply campus and departmental issues. After some time we only hiked every other week or so, and then finally much less often. One favorite (and most spectacular) hike was to go from Twin Gates on Empire Grade down to Wilder Ranch."

Harold remained mathematically active until his last months. He maintained a blackboard both at his home and at his university office which ought not to be erased and which captured the problems on which he was currently working. When a colleague of ours approached us in the Fall of 2019 with an intricate asymptotic question, it was Harold who came up with the correct answer first. Harold kept teaching until he was 79 years old. He was known to the students to enter the classroom with at most a tiny piece of paper and deliver his lecture easily and elegantly on the blackboard. He seemed in good spirits on his 88th birthday in September 2020. He had recently broken a hip but had been recovering. Sadly, he fell seriously ill a few months later.

Harold passed away on January 20, 2021. He is survived by his wife Linda Larkin, former wife Lois Widom, brother Benjamin Widom, daughter Barbara Widom, daughter Jennifer Widom, son Steven Widom, and four grandchildren. Harold's brother Benjamin is five years older. He is the Goldwin Smith Professor of Chemistry at Cornell University and was awarded the Boltzmann Medal in 1998 for his achievements in physical chemistry and statistical mechanics. Harold's daughter Barbara Widom is an endocrinologist in Fort Collins, and his daughter Jennifer Widom is the Frederick Emmons Terman Dean of Engineering at Stanford University. Harold's son Steven Widom is a software engineer.

Credits The present article is in part based on [1, 3, 5, 6].

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Domain Walls in the Heisenberg-Ising Spin-$\frac{1}{2}$ Chain

Axel Saenz, Craig A. Tracy, and Harold Widom

Abstract In this chapter we obtain formulas for the distribution of the left-most up-spin in the Heisenberg-Ising spin-1/2 chain with anisotropy parameter Δ, also known as the XXZ spin-1/2 chain, on the one-dimensional lattice $\mathbb{Z}$ with domain wall initial conditions. We use the Bethe Ansatz to solve the Schrödinger equation and a recent antisymmetrization identity of Cantini, Colomo, and Pronko to simplify the marginal distribution of the left-most up-spin. In the $\Delta = 0$ case, the distribution F_2 arises. In the $\Delta \neq 0$ case, we propose a conjectural series expansion type formula based on a saddle point analysis. The conjectural formula turns out to be a Fredholm series expansion in the $\Delta \to 0$ limit and recovers the result for $\Delta = 0$.

Keywords Heisenberg-Ising Spin Chain · XXZ · Bethe Ansatz · Saddle Point Analysis

1 Introduction

We consider the dynamics of the Heisenberg-Ising spin-1/2 chain with anisotropy parameter Δ, also known as the XXZ spin-1/2 chain, on the one-dimensional lattice $\mathbb{Z}$ with domain wall initial conditions. We start with an initial state of N up-spins at the sites $\{1, 2, \ldots, N\}$ in a sea of down-spins; and by utilizing ideas from coordinate

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Bethe Ansatz [3, 13, 26, 37] to solve the Schrödinger equation, we find the quantum state $\Psi_N(t)$ at time t is

$$\Psi_N(t) = \sum_X \psi_N(X,t) e_X,$$

where the sum is over all $X = \{x_1 < x_2 < \cdots < x_N\}$ and e_X denotes the state with up-spins at X. Alternatively we can view a spin up at site x_j as a particle and a spin down as an empty lattice site or hole. The "Bethe-coordinates" $\psi_N(X,t)$ are given below in Theorem 2.[1] They have the interpretation that $|\psi_N(X;t)|^2$ is the probability the system is in state X at time t. Observe that the $\psi_N(X,t)$ have the standard Bethe Ansatz structure as a sum over the permutation group $\mathcal{S}_N$; where now, each term in the summand is an N-dimensional contour integral.

1.1 One-Point Functions

If $X_1(t)$ denotes the position of the left-most particle at time t, then

$$\mathbb{P}_N(X_1(t) = x) = \sum_{X, x_1 = x} |\psi_N(X,t)|^2$$

where the sum is over all $X = \{x_1 = x < x_2 < \cdots < x_N\}$. In ASEP the analogous quantity involves a *single sum* over $\mathcal{S}_N$ where as now we have a *double sum* over $\mathcal{S}_N$. In [29] an identity involving the sum over the permutation group[2] was used to reduce the sum to a single N-dimensional integral. Cantini, Colomo, and Pronko [9] have generalized the single sum permutation identity to a double sum permutation identity, which also generalize to the (spin) Hall-Littlewood functions [20, 36]. Employing this new identity reduces the expression for $\mathbb{P}_N(X_1(t) = x)$ to a single $2N$-dimensional integral whose integrand involves the famous Izergin-Korepin determinant [16, 18]. The resulting expression is given in Theorem 3. This part of the paper overlaps the recent work of J. M. Stéphan [23, 25].

For the special case $\Delta = 0$, the analysis simplifies considerably. Using Toeplitz operators and their determinants, we show the $N \to \infty$ limit can be taken resulting in the representation

$$\lim_{N\to\infty} \mathbb{P}_N(X_1(t) \geq x) = \det(I - L)$$

[1] Since the Hamiltonian H_Δ of the Heisenberg-Ising model is a (non-unitary) similarity transformation of the Markov generator of the ASEP [15], the results in [29] give immediately the Bethe coordinates of Theorem 2 once an identification of parameters is made (see Sect. 3.3).

[2] See equation (1.6) in [29].

where L is an integral operator whose kernel is the *discrete Bessel kernel* [4, 6, 17] (see also Chapter 8 in [2]). This makes connections to the distribution of the length of the longest increasing subsequence in a random permutation [1, 2]. See Theorem 4 below. From this identification it follows that

$$\lim_{t\to\infty} \lim_{N\to\infty} \mathbb{P}_N \left(\frac{X_1(t) + 2t}{t^{1/3}} \geq -y \right) = F_2(y)$$

where F_2 is the TW_2 distribution [27, 28]. This last result appears to be well-known in the physics literature since the case $\Delta = 0$ is reducible to a "free fermion" model [12, 21, 23, 34].

1.2 Contour Deformations and a Conjecture

Taking the contour integral functions for the one-point function to the infinite time statistics is another major challenge. In the case of the ASEP, this was achieved by Tracy-Widom [30] by deforming the contours to obtain a Fredholm determinant. Then, in a later work by the same authors [31], the Fredholm determinant was further analyzed by deforming the kernels to obtain the Tracy-Widom distribution. We also deform the contour integrals for our one-point function, in Sect. 7, to obtain a type of series expansion.

Theorem 1 *Let $X_1(t)$ be the location of the left-most particle in the Heisenber-Ising spin-1/2 chain with N particles, initial conditions $Y = (y_1 < y_2 < \cdots < y_N)$, and $\Delta \in \mathbb{R}$ so that $\Delta \neq 0$. Then, $\mathbb{P}(X_1(t) \geq x)$ is equal to*

$$\sum_{n=0}^{N} \sum_{\tau\in\mathcal{T}_n} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \oint_{\mathcal{C}_{R'}} \cdots \oint_{\mathcal{C}_{R'}} I_N(\xi, \zeta; \tau) f(\xi, \zeta; \tau) \left(\prod_{j\in J} d\zeta_j \right) d^N\xi \tag{1}$$

where the integrand is given by (67), the summation is take over the set of maps $\mathcal{T}_n$ given by (59), and the contours $\mathcal{C}_R$ and $\mathcal{C}_{R'}$ are circles centered at zero with radii $R, R' > 0$ that satisfy the following inequalities $\max\{2|\Delta|^{-1}, 2(1 + 2|\Delta|)\} < R < \max\{4|\Delta|^{-1}, 4(1 + 2|\Delta|)\} < R'/2$.

We expect this series expansion to to give rise to a series expansion of a Fredholm determinant in the infinite time limit. In fact, we may deform the contours in the previous formula to the steepest descent to contours in an effort to obtain the infinite time limit by a saddle point analysis. The result is given by our Conjecture 1. Aside from technical details of certain bounds and approximations, there are some terms that we still can't control after the saddle point analysis. Recent results [8, 10, 23, 24], based on numerical, hydrodynamic and analytical arguments are inconclusive in the appropriate scaling, i.e. $t^{1/2}$ versus $t^{1/3}$, for the fluctuations of the one-point function in the infinite time limit. Based on our conjecture, we expect the location of the left-most particle to be at $-2t$ with fluctuations on the order of $t^{1/3}$ but the limiting distribution is still unclear.

2 XXZ Quantum Spin-$\frac{1}{2}$ Hamiltonian

The definition of the quantum spin chain Hamiltonian on the infinite lattice $\mathbb{Z}$ requires some explanation since there is the problem of making sense of infinite tensor products in the construction of a Hilbert space of states. The general construction uses the Gelfand-Naimark-Segal (GNS) construction; but in the case considered here, there is an elementary treatment [19] which we now describe.

Let $\mathcal{H}_0 = \mathbb{C}$. For each positive integer N we define

$$\mathcal{X}_N := \left\{ X = \{x_1, \ldots, x_N\} \in \mathbb{Z}^N : x_1 < \cdots < x_N \right\}$$

and

$$\mathcal{H}_N := \ell^2(\mathcal{X}_N).$$

The Hilbert space of states is

$$\mathcal{H} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N.$$

The normalized state $\Omega = 1 \in \mathcal{H}_0$ is the ground state of all spins down. In physicists' notation

$$\Omega = |\cdots \downarrow \cdots \downarrow \cdots \downarrow \cdots\rangle.$$

Given $N \in \mathbb{Z}^+$ and $X = \{x_1, \ldots, x_N\} \in \mathcal{X}_N$, define $e_X \in \mathcal{H}_N$ by

$$e_X(Y) = \delta_{X,Y}.$$

The set $\{e_X\}_{X \in \mathcal{X}_N}$ defines a natural orthonormal basis of $\mathcal{H}_N$. The physical interpretation of e_X is the state with up spins at $x_1 < \cdots < x_N$ in a sea of down spins:

$$e_X = |\cdots \underset{x_1}{\uparrow} \cdots \underset{x_2}{\uparrow} \cdots \underset{x_N}{\uparrow} \cdots\rangle.$$

This is a model of a quantum lattice gas (see, for example, §6.1.6 of [26]). We will frequently use this particle interpretation.

We introduce the *Pauli operators* σ_j^α, $j \in \mathbb{Z}$, $\alpha = 3, \pm$.

$$\sigma_j^3 e_X = \begin{cases} \;\; e_X \text{ if } \; j \in X = \{x_1, \ldots, x_N\}, \\ -e_X \text{ otherwise.} \end{cases} \tag{2}$$

$$\sigma_j^+ e_X = \begin{cases} 0 \quad \text{if } \; j \in \{x_1, \ldots, x_N\}, \\ e_{X^+} \text{ where } \; X^+ = \{x_1, \ldots, x_k, j, x_{k+1}, \ldots, x_N\}, \; x_k < j < x_{k+1} \end{cases} \tag{3}$$

$$\sigma_j^- e_X = \begin{cases} 0 \quad \text{if } \; j \notin X = \{x_1, \ldots, x_N\} \\ e_{X^-} \text{ where } \; X^- = \{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N\}, \quad j = x_k \end{cases} \tag{4}$$

In words, $\sigma_j^+ : \mathcal{H}_N \to \mathcal{H}_{N+1}$ acts as the identity except at the site j where it takes $\downarrow\rightsquigarrow\uparrow$ and annihilates a $\uparrow$ state. Similarly, $\sigma_j^- : \mathcal{H}_N \to \mathcal{H}_{N-1}$ acts as the identity except at the site j where it takes $\uparrow\rightsquigarrow\downarrow$ and annihilates a $\downarrow$ state. By definition $\sigma_j^3\Omega = -\Omega$, $\sigma_j^-\Omega = 0$ and $\sigma_j^+\Omega = e_{\{j\}}$. We also recall the Pauli operators $\sigma_j^1 = \sigma_j^+ + \sigma_j^-$ and $\sigma_j^2 = -i\sigma_j^+ + i\sigma_j^-$. Define

$$\begin{aligned} h_{j,j+1} &= \frac{1}{2}\left(\sigma_j^1\sigma_{j+1}^1 + \sigma_j^2\sigma_{j+1}^2 + \Delta(\sigma_j^3\sigma_j^3 - 1)\right) \\ &= \sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+ + \frac{\Delta}{2}(\sigma_j^3\sigma_{j+1}^3 - 1) \end{aligned}$$

and

$$H_{XXZ} = \sum_{j\in\mathbb{Z}} h_{j,j+1}. \tag{5}$$

The operator H_{XXZ} is the *Heisenberg-Ising spin-$\frac{1}{2}$ chain Hamiltonian*; or more briefly, the XXZ spin Hamiltonian. It's clear from the above definitions that $H_{XXZ} : \mathcal{H}_N \to \mathcal{H}_N$. Since the number of particles is conserved under the dynamics of H_{XXZ}, we can work in a sector $\mathcal{H}_N$.

A state $\Psi_N = \Psi_N(t) \in \mathcal{H}_N$ can be represented by

$$\Psi_N(t) = \sum_{X\in\mathcal{X}_N} \psi_N(X,t)e_X. \tag{6}$$

The initial condition is $\Psi(0) = e_Y$, $Y = \{y_1, \ldots, y_N\} \in \mathcal{X}_N$, so that $\psi_N(X;0) = \delta_{X,Y}$. The dynamics is determined by the Schrödinger equation

$$\mathrm{i}\,\frac{\partial\Psi_N}{\partial t} = H_{XXZ}\Psi_N. \tag{7}$$

The Hamiltonian H_{XXZ} is self-adjoint and so by Stone's theorem there exists a unitary operator
$U = \exp(-\mathrm{i}tH_{XXZ})$ such that $\Psi_N(t) = U(t)\Psi_N(0)$. We have

$$\langle \Psi_N(t), \Psi_N(t)\rangle = \sum_{X\in\mathcal{X}_N} |\psi_N(X;t)|^2 = 1.$$

The goal is to describe the dynamics $\Psi_{DW}(t)$ starting from the *domain wall* (DW) initial state

$$e_{\mathbb{N}} = |\cdots \downarrow\downarrow\underset{0}{\downarrow}\underset{1}{\uparrow}\uparrow\uparrow \cdots\rangle.$$

One immediately sees the difficulty in that $e_{\mathbb{N}}$ is not an element of $\mathcal{H}_N$ for any N.[3] If $X_m(t)$ denotes the position of the mth particle on the left, we define

$$\mathbb{P}_{\mathbb{N}}(X_m(t) = x) = \lim_{N\to\infty} \mathbb{P}_{\{1,\ldots,N\}}(X_m(t) = x).$$

3 Bethe Ansatz Solution $\Psi_N(t)$

This section closely follows [29, 37] and additional details may be found on the arXiv version of this paper [22]. We first note that

$$h_{j,j+1}|\cdots \underset{j}{\uparrow}\underset{j+1}{\uparrow} \cdots\rangle = 0, \tag{8}$$

$$h_{j,j+1}|\cdots \underset{j}{\downarrow}\underset{j+1}{\downarrow} \cdots\rangle = 0, \tag{9}$$

$$h_{j,j+1}|\cdots \underset{j}{\uparrow}\underset{j+1}{\downarrow} \cdots\rangle = -\Delta|\cdots \underset{j}{\uparrow}\underset{j+1}{\downarrow} \cdots\rangle + |\cdots \underset{j}{\downarrow}\underset{j+1}{\uparrow} \cdots\rangle, \tag{10}$$

$$h_{j,j+1}|\cdots \underset{j}{\downarrow}\underset{j+1}{\uparrow} \cdots\rangle = -\Delta|\cdots \underset{j}{\downarrow}\underset{j+1}{\uparrow} \cdots\rangle + |\cdots \underset{j}{\uparrow}\underset{j+1}{\downarrow} \cdots\rangle. \tag{11}$$

Define [37] (the Yang-Yang S-matrix)

$$S_{\beta\alpha}(\xi_\beta, \xi_1) = -\frac{1 + \xi_\alpha\xi_\beta - 2\Delta\xi_\beta}{1 + \xi_\alpha\xi_\beta - 2\Delta\xi_\alpha}, \tag{12}$$

for $\alpha, \beta = 1, \ldots, N$ and $\xi_\alpha, \xi_\beta \in \mathbb{C}$.

[3] Presumably, one could construct a domain wall Hilbert space $\mathcal{H}_{DW}$ by replacing the state Ω by $e_{\mathbb{N}}$. Unfortunately, we do not know how to proceed with a Bethe Ansatz solution in this space.

The generator of the finite N asymmetric simple exclusion process (ASEP) is a similarity transformation (*not* a unitary transformation!) of the Heisenberg-Ising Hamiltonian. Because of this the Schrödinger equation (7) for the quantum spin chain is essentially identical to the master equation (Kolmogorov forward equation) for the Markov process ASEP assuming the identification of parameters

$$\xi_i = \xi_i'/\sqrt{\tau},\ \ \tau = \frac{p}{q},\ 2\Delta = \frac{1}{\sqrt{pq}},$$

$$S_{\beta\alpha}^{XXZ}(\xi_\beta, \xi_\alpha) = S_{\beta\alpha}^{ASEP}(\xi_\beta', \xi_\alpha'),\ \varepsilon^{XXZ}(\xi) = \frac{1}{\sqrt{pq}}\varepsilon^{ASEP}(\xi').$$

Thus given the ASEP result [29, 32] and the above identifications, we have

Theorem 2 *For $\sigma \in \mathcal{S}_N$, define*

$$A_\sigma(\xi) = \prod \left\{ S_{\beta\alpha}(\xi_\beta, \xi_\alpha) : \{\beta, \alpha\} \text{ is an inversion in } \sigma \right\}, \tag{13}$$

then the solution to (7) satisfying the initial condition $\psi_N(X;0) = \delta_{X,Y}$ is

$$\psi_N(X;t) = \sum_{\sigma\in\mathcal{S}_N} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i-1}\, e^{-it\varepsilon(\xi_i)} \right) d\xi_1 \cdots d\xi_N \tag{14}$$

where $\mathcal{C}_r$ is a circle centered at zero with radius r so small that all the poles of A_σ lie outside of $\mathcal{C}_r$.

Additionally, we have a contour integral formula with large contours instead of small contours as above in Theorem 2. Below, we will use a combination of the small and large contour formulas.

Theorem 2a *For $\sigma \in \mathcal{S}_N$, define*

$$A_\sigma(\xi) = \prod \left\{ S_{\beta\alpha}(\xi_\beta, \xi_\alpha) : \{\beta, \alpha\} \text{ is an inversion in } \sigma \right\},$$

then the solution to (7) satisfying the initial condition $\psi_N(X;0) = \delta_{X,Y}$ is

$$\psi_N(X;t) = \sum_{\sigma\in\mathcal{S}_N} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i-1}\, e^{-it\varepsilon(\xi_i)} \right) d\xi_1 \cdots d\xi_N \tag{15}$$

where $\mathcal{C}_R$ is a circle centered at zero with radius R so large that all the poles of A_σ lie inside of $\mathcal{C}_R$.

The arguments for the proof of this statement are almost verbatim to the arguments of the proof of Theorem 1 given in [29]. In this case, one would need to expand contours to infinity instead of shrinking them to zero as it was done in [29]. The arguments are then adjusted mutatis mutandis; the details may be found in Appendix A of the arXiv version of this paper [22]. We skip the details here for conciseness sake.

4 Probability $\mathcal{P}_Y(x, m; t)$

If the initial state is $e_Y \in \mathcal{H}_N$, $Y \in \mathcal{X}_N$, then at time t the system is in state $\Psi_N(t) = \sum_{X \in \mathcal{X}_N} \psi_N(X; t) e_X$ where $\psi_N(X; t)$ is given by (14) or (15). The quantity

$$|\langle e_X, \Psi_N(t)\rangle|^2 = |\psi_N(X; t)|^2, \quad X \in \mathcal{X}_N,$$

is the probability that the system is in state e_X at time t.

Denote by $\mathcal{P}_Y(x, m; t)$ the probability that at time t the state has the mth particle from the left at position x given initially the state is Y. Let $X = \{x_1, x_2, \ldots, x_N\} \in \mathcal{X}_N$, $1 \le m \le N$, and define the projection operator

$$P_{x,m} e_X = \begin{cases} e_X & \text{if } x_m = x, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Then the outcome of the measurement yielding "the mth spin from the left is at position x at time t" is that the system is now in state

$$\Psi_N(x, m; t) := P_{x,m} \Psi_N(t) = \sum_{\substack{X \in \mathcal{X}_N \\ x_m = x}} \psi_N(X; t) e_X.$$

Thus the probability of this outcome is

$$\mathcal{P}_Y(x, m; t) := \langle \Psi_N(x, m; t), \Psi_N(x, m; t)\rangle = \sum_{\substack{X \in \mathcal{X}_N \\ x_m = x}} |\psi_N(X; t)|^2. \tag{17}$$

4.1 Distribution of Left-Most Particle

We now restrict to the case $m = 1$, i.e. $\mathcal{P}_Y(x, 1; t)$. Let

$$x_1 = x,\ x_2 = x + v_1,\ \ldots, x_N = x + v_1 + v_2 + \cdots + v_{N-1},\ v_i \ge 1,$$

and note that $\overline{\Psi_N(x; t)} = \Psi_N(x; -t)$. Then, using (14) for $\Psi(x; t)$ and (15) for $\Psi(x; -t)$ with $R\, r < 1$, followed by performing the geometric sums (since $R\, r < 1$, the summations may be brought inside)

$$
\begin{aligned}
\mathcal{P}_Y(x,1;t) &= \sum_{\substack{X\in\mathcal{X}_N\\ x_1=x}} \psi_N(X;t)\psi_N(X;-t)\\
&= \sum_{\sigma,\mu\in\mathcal{S}_N} \int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r} \sum_{v_i\geq 1} \Big(A_\sigma(\xi)A_\mu(\zeta)\\
&\qquad \times (\xi_{\sigma(2)}\zeta_{\mu(2)})^{v_1}(\xi_{\sigma(3)}\zeta_{\mu(3)})^{v_1+v_2}\cdots(\xi_{\sigma(N)}\zeta_{\mu(N)})^{v_1+\cdots+v_{N-1}}\Big)\\
&\qquad \times \prod_j (\xi_j\zeta_j)^{x-y_j-1}\mathrm{e}^{-\mathrm{i}t(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d\zeta_1\cdots d\zeta_N d\xi_1\cdots d\xi_N\\
&= \sum_{\sigma,\mu\in\mathcal{S}_N} \int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r} A_\sigma(\xi)A_\mu(\zeta)\,\frac{\xi_{\sigma(2)}\zeta_{\mu(2)}\xi^2_{\sigma(3)}\zeta^2_{\mu(3)}\cdots\xi^{N-1}_{\sigma(N)}\zeta^{N-1}_{\mu(N)}}{\prod_{j=2}^N\big(1-\xi_{\sigma(j)}\zeta_{\mu(j)}\cdots\xi_{\sigma(N)}\zeta_{\mu(N)}\big)}\\
&\qquad \times \prod_j (\xi_j\zeta_j)^{x-y_j-1}\mathrm{e}^{-\mathrm{i}t(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d\zeta_1\cdots d\zeta_N d\xi_1\cdots d\xi_N
\end{aligned}
$$

In the formulas above, we have $2N$ contour integrals with the contour $\mathcal{C}_r$ for the first N contours and the contours $\mathcal{C}_R$ for the following N contours. Now, at the analogous step in ASEP, an identity[4] was derived that simplified the sum over $\mathcal{S}_N$ resulting in a single multidimensional integral.[5] Now we have a *double sum* over $\mathcal{S}_N$ and we need a new identity. Fortunately such an identity has been discovered by Cantini, Colomo, and Pronko [9]. Let

$$
d(x,y) := \frac{1}{(1-x\,y)(x+y-2\Delta\,x\,y)} \quad\text{and}\quad D_N(\xi,\zeta) = \det\big(d(\xi_i,\zeta_j)|_{1\leq i,j\leq N}\big)\,, \tag{18}
$$

then

$$
\begin{aligned}
&\sum_{\sigma,\mu\in\mathcal{S}_N} A_\sigma(\xi)A_\mu(\zeta)\frac{\xi_{\sigma(2)}\zeta_{\mu(2)}\xi^2_{\sigma(3)}\zeta^2_{\mu(3)}\cdots\xi^{N-1}_{\sigma(N)}\zeta^{N-1}_{\mu(N)}}{\prod_{j=2}^N\big(1-\xi_{\sigma(j)}\zeta_{\mu(j)}\cdots\xi_{\sigma(N)}\zeta_{\mu(N)}\big)}\\
&\quad = \frac{\big(1-\prod_j\xi_j\zeta_j\big)\prod_{i,j=1}^N(\xi_i+\zeta_j-2\Delta\xi_i\zeta_j)}{\prod_{i<j}(1+\xi_i\xi_j-2\Delta\xi_i)(1+\zeta_i\zeta_j-2\Delta\zeta_i)}\,D_N(\xi,\zeta)
\end{aligned} \tag{19}
$$

[4] See (1.6) in [29].

[5] See Theorem 3.1 in [29].

Remarks

- The identity (19) is Proposition 6 of [9] (with a change of notation). The identity (19) also appears in a more general setting of (spin) Hall-Littlewood functions in [20, 36], which specializes to the ASEP case as shown in Corollary 7.1 in [20].
- In Appendix B of [9], the authors show that (19) reduces to (1.6) of [29] in the limit $\xi_j \to \sqrt{\frac{q}{p}}\,\xi_j$ and $\zeta_j \to \sqrt{\frac{p}{q}}$.
- The determinant $D_N(\xi,\zeta)$ "is nothing but the well-known *Izergin-Korepin determinant* [16, 18] in disguise" [35].

We thus have

$$\mathcal{P}_Y(x,1;t) = \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_r} \frac{(1-\prod_j \xi_j\zeta_j)\prod_{i,j=1}^N(\xi_i+\zeta_j-2\Delta\xi_i\zeta_j)}{\prod_{i<j}(1+\xi_i\xi_j-2\Delta\xi_i)(1+\zeta_i\zeta_j-2\Delta\zeta_i)} \times D_N(\xi,\zeta)\prod_j(\xi_j\zeta_j)^{x-y_j-1}\mathrm{e}^{-\mathrm{i}t(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d^N\zeta\,d^N\xi \tag{20}$$

The factor $(1-\prod_j \xi_j\zeta_j)$ is eliminated if we consider

$$\mathcal{F}_N(x,t) := \mathbb{P}_Y(X_1(t)\ge x) = \sum_{n=x}^{\infty}\mathcal{P}_Y(n,1;t) \tag{21}$$

From [9]

$$\prod_{1\le j,k\le N}(\xi_j+\zeta_k-2\Delta\xi_j\zeta_k)\cdot D_N(\xi,\zeta) = \frac{\Delta_N(\xi)\Delta_N(\zeta)}{\prod_{j,k}(1-\xi_j\zeta_k)}Q_N(\xi,\zeta) \tag{22}$$

where Q_N is a "polynomial of degree $N-1$ in each variable, separately symmetric under permutations of the variables within each set" [9, 35].[6] Here $\Delta_N(\xi)$ is the Vandermonde product $\prod_{1\le j<k\le N}(\xi_k-\xi_j)$ (not to be confused with the constant Δ). It's useful to define

$$U(\xi,\xi') := \frac{1+\xi\xi'-2\Delta\xi}{\xi'-\xi}.$$

[6] For example

$$\begin{aligned} Q_1(\xi,\zeta) &= 1,\\ Q_2(\xi,\zeta) &= 4\Delta^2\zeta_1\zeta_2\xi_1\xi_2 - 2\Delta\zeta_1\zeta_2\xi_1 - 2\Delta\zeta_1\zeta_2\xi_2 - 2\Delta\zeta_1\xi_1\xi_2 - 2\Delta\zeta_2\xi_1\xi_2\\ &\quad +\zeta_1\zeta_2\xi_1\xi_2 + \zeta_1\zeta_2 + \xi_1\xi_2 + 1, \end{aligned}$$

Q_3 in expanded form has 459 terms, and Q_4 has 60,820 terms.

The identity (19) can be rewritten as

$$\sum_{\sigma,\mu}\prod_{i<j} U(\xi_{\sigma(i)},\xi_{\sigma(j)})U(\zeta_{\mu(i)},\zeta_{\mu(i)})\,\frac{\xi_{\sigma(2)}\zeta_{\mu(2)}\xi^2_{\sigma(3)}\zeta^2_{\mu(3)}\cdots\xi^{N-1}_{\sigma(N)}\zeta^{N-1}_{\mu(N)}}{\prod_{j=2}^{N}\left(1-\xi_{\sigma(j)}\zeta_{\mu(j)}\cdots\xi_{s(N)}\zeta_{\mu(N)}\right)}$$
$$=\frac{1-\prod_j\xi_j\zeta_j}{\prod_{j,k}(1-\xi_j\zeta_k)}\,Q_N(\xi,\zeta) \tag{23}$$

The close relationship of (23) to (1.6) of [29] (see also Identity 1_L in [33]) is now clearer. We have proved

Theorem 3 $\mathcal{F}_N(x,t)=\mathbb{P}_Y(X_1(t)\geq x)$ *equals*

$$\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r}\frac{\prod_{j,k}(\xi_j+\zeta_k-2\Delta\xi_j\zeta_k)}{\prod_{j<k}(1+\xi_j\xi_k-2\Delta\xi_j)(1+\zeta_j\zeta_k-2\Delta\zeta_j)}\,D_N(\xi,\zeta)$$
$$\times\prod_j(\xi_j\zeta_j)^{x-y_j-1}e^{-it(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d^N\zeta\,d^N\xi \tag{24}$$
$$=\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r}\frac{\Delta_N(\xi)\Delta_N(\zeta)}{\prod_{j<k}(1+\xi_j\xi_k-2\Delta\xi_j)(1+\zeta_j\zeta_k-2\Delta\zeta_j)}\,\frac{Q_N(\xi,\zeta)}{\prod_{j,k}(1-\xi_j\zeta_k)}$$
$$\times\prod_j(\xi_j\zeta_j)^{x-y_j-1}e^{-it(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d^N\zeta\,d^N\xi \tag{25}$$

where $\mathcal{C}_r$ (resp. $\mathcal{C}_R$) is a circle centered at zero with radius r (resp. R) so small (resp. large) that all the poles of the integrand except for the the poles at the origin (resp. infinity) lie outside $\mathcal{C}_r$ (resp. inside $\mathcal{C}_R$) and $R\,r<1$.

5 Special Case $\Delta = 0$

When $\Delta=0$, (25) reduces to

$$\mathcal{F}_N(x,t)\big|_{\Delta=0}$$
$$=\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r}\frac{\Delta_N(\xi)\Delta_N(\zeta)}{\prod_{j,k}(1-\xi_j\zeta_k)}\prod_j(\xi_j\zeta_j)^{x-y_j-1}\mathrm{e}^{-\mathrm{i}t(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d^N\zeta\,d^N\xi \tag{26}$$
$$=\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r}\det\left(\frac{1}{1-\xi_j\zeta_k}\right)\prod_j(\xi_j\zeta_j)^{x-y_j-1}\mathrm{e}^{-\mathrm{i}t(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\,d^N\zeta\,d^N\xi \tag{27}$$

since

$$\lim_{\Delta\to 0}\frac{Q_N(\xi,\zeta)}{\prod_{j<k}(1+\xi_j\xi_k-2\Delta\xi_j)(1+\zeta_j\zeta_k-2\Delta\zeta_j)}=1.$$

and

$$\det\left(\frac{1}{1-\xi_j\zeta_k}\right)=\frac{\Delta_N(\xi)\Delta_N(\zeta)}{\prod_{j,k}(1-\xi_j\zeta_k)}.$$

More directly since $A_\sigma\big|_{\Delta=0}=\text{sgn}(\sigma)$, we use the identity

$$\sum_{\sigma,\mu}\text{sgn}(\sigma)\text{sgn}(\mu)\frac{\xi_{\sigma(2)}\zeta_{\mu(2)}\xi^2_{\sigma(3)}\zeta^2_{\mu(3)}\cdots\xi^{N-1}_{\sigma(N)}\zeta^{N-1}_{\mu(N)}}{\prod_{j=2}^N\left(1-\xi_{\sigma(j)}\zeta_{\mu(j)}\cdots\xi_{\sigma(N)}\zeta_{\mu(N)}\right)}=\det\left(\frac{1}{1-\xi_j\zeta_k}\right) \tag{28}$$

5.1 *Fredholm Determinant Representation*

Define

$$\phi_j(\xi)=\xi^{x-y_j-1}\mathrm{e}^{-\mathrm{i}t\varepsilon(\xi)},\ \ \psi_j(\zeta)=\zeta^{x-y_j-1}\mathrm{e}^{\mathrm{i}t\varepsilon(\zeta)}$$

and

$$K(j,k)=\frac{\phi_j(\xi_j)\psi_k(\zeta_k)}{1-\xi_j\zeta_k} \tag{29}$$

Thus

$$\begin{aligned}\mathcal{F}_N(x,t)\big|_{\Delta=0}&=\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_r}\det(K)\,d^N\zeta\,d^N\xi\\&=\int_{\mathcal{C}_r}\cdots\int_{\mathcal{C}_r}\det(K)\,d^N\zeta\,d^N\xi\end{aligned} \tag{30}$$

For the second identity, we deformed the contours from $\mathcal{C}_R$ to $\mathcal{C}_r$ for all the ζ-variables. When we deform the contours, we don't cross any poles since the poles, given by $1-\xi_j\zeta_k=0$, are located outside of the contour $\mathcal{C}_R$ since we have taken $Rr<1$. Additionally, note that the variable ξ_j appears only in row j and ζ_k appears only in column k. It follows that the multiple integral is gotten by integrating each $K(j,k)$ with respect to ξ_j, ζ_k. Therefore the multiple integral (30) equals the determinant with j,k entry

$$K_N(j,k) = \int_{\mathcal{C}_r}\int_{\mathcal{C}_r} \frac{\phi_j(\xi)\psi_k(\zeta)}{1-\xi\zeta}\, d\zeta\, d\xi$$

We consider step initial condition, so that $y_j = j$. In preparation for taking the limit as $N \to \infty$, we make the replacements $j \to j+1$, $k \to k+1$, so that the indicies run for 0 to $N-1$ rather than 1 to N. Then, in preparation for eventual steepest descent, we make the substitutions $\xi \to \mathrm{i}\xi$, $\zeta \to \zeta/\mathrm{i}$. Aside from the factor $\mathrm{e}^{\mathrm{i}\pi(j-k)/2}$, which will not affect the determinant, the kernel becomes

$$L_N(j,k) = \int_{\mathcal{C}_r}\int_{\mathcal{C}_r} \frac{\xi^{x-j-2}\,\zeta^{x-k-2}}{1-\xi\zeta}\mathrm{e}^{t(\theta(\xi)+\theta(\zeta))}\, d\zeta\, d\xi,$$

where we have set $\theta(\xi) = \xi - 1/\xi$. We write the above as

$$\sum_{\ell=0}^{\infty}\int_{\mathcal{C}_r}\int_{\mathcal{C}_r} \xi^{x-j+\ell-2}\zeta^{x-k+\ell-2}\,\mathrm{e}^{t(\theta(\xi)+\theta(\zeta))}\, d\zeta\, d\xi.$$

We may take all integrations over the unit circle $\mathcal{C}_1$ and in the ζ-integral make the substitution $\zeta \to 1/\zeta$. We obtain

$$L_N(j,k) = \sum_{\ell=0}^{\infty}\int_{\mathcal{C}_1}\int_{\mathcal{C}_1} \xi^{x-j+\ell-2}\zeta^{-x+k-\ell}\,\mathrm{e}^{t(\theta(\xi)-\theta(\zeta))}\, d\zeta\, d\xi.$$

In Toeplitz terms this is the operator

$$P_N T(a) T(a^{-1}) P_N,$$

where P_N is the projection from $\ell^2(\mathbb{Z}^+)$[7] to $\ell^2([0,\ldots,N-1])$ and where a is the symbol

$$a(\xi) = \xi^{x-1}\mathrm{e}^{t\,\theta(\xi)}.$$

It it known (see, e.g. §5.1 in [7]) that $T(a)T(a^{-1})$ is of the form I+trace class and so $\det(K_N)$ has the limit $\det(T(a)T(a^{-1}))$ on $\ell^2(\mathbb{Z}^+)$.[8]

[7] $\mathbb{Z}^+$ denotes the set of nonnegative integers.

[8] One can show that for $x > 1$ the determinant of the product is zero.

By a well-known identity, $T(a)T(a^{-1}) = I - H(a)H(\tilde{a}^{-1})$, where $H(a)$ denotes the Hankel operator and $\tilde{a}(\xi) = a(\xi^{-1})$. In this case $\tilde{a} = a^{-1}$ and the square of $H(a)$ has kernel[9]

$$L(j,k) = \sum_{\ell=0}^{\infty} \int_{\mathcal{C}_1} \int_{\mathcal{C}_1} \xi^{x-j-\ell-3} \zeta^{x-k-\ell-3} \, \mathrm{e}^{t(\theta(\xi)+\theta(\zeta))} \, d\zeta \, d\xi,$$

and we are interested in $\det(I - L)$. The substitutions $\xi \to 1/\xi$, $\zeta \to 1/\zeta$ give

$$L(j,k) = \sum_{\ell=0}^{\infty} \int_{\mathcal{C}_1} \int_{\mathcal{C}_1} \xi^{-x+j+\ell+1} \zeta^{-x+k+\ell+1} \, \mathrm{e}^{-t(\theta(\xi)+\theta(\zeta))} \, d\zeta \, d\xi. \tag{31}$$

If we take our integrals over $\mathcal{C}_r$ and sum we obtain

$$L(j,k) = \int_{\mathcal{C}_r} \int_{\mathcal{C}_r} \frac{\xi^{-x+j+1} \zeta^{-x+k+1} \, \mathrm{e}^{-t(\theta(\xi)+\theta(\zeta))}}{1 - \xi\zeta} \, d\zeta \, d\xi \tag{32}$$

The kernel $L(j,k)$ is known as the *discrete Bessel kernel* [4] (see also Chapter 8 in [2]) due to the following representation. Using the Bessel generating function

$$\exp(t\theta(\xi)) = \sum_{n=-\infty}^{\infty} \xi^n J_n(2t)$$

in (31) and the identity, $\nu \neq \mu$,

$$\sum_{n=0}^{\infty} J_{\nu+n}(t) J_{\mu+n}(t) = \frac{t}{2(\nu-\mu)} \left[J_{\nu-1}(t) J_{\mu}(t) - J_{\nu}(t) J_{\mu-1}(t) \right] \tag{33}$$

we find

$$L(j,k) = t \, \frac{J_{j-x+1}(2t) J_{k-x+2}(2t) - J_{j-x+2}(2t) J_{k-x+1}(2t)}{j-k}$$

For $j = k$ one lets $\mu \to \nu$ in (33) to find

$$\begin{aligned} L(j,j) &= \sum_{n=0}^{\infty} J_{\nu+n}(2t)^2 \\ &= t \left[J_{\nu}(2t) \frac{\partial J_{\mu}}{\partial \mu}\Big|_{\mu=\nu-1} - J_{\nu-1}(2t) \frac{\partial J_{\mu}}{\partial \mu}\Big|_{\mu=\nu-1} \right], \quad \nu = -x + j + 1. \end{aligned}$$

[9] Recall that the i, j-entry of $H(f)$ is $f_{i+j+1} = \int \xi^{-i-j-2} f(\xi)\, d\xi$.

For $x \leq 1$ and domain wall initial condition $Y = \mathbb{N}$, we have the Toeplitz representation

$$\begin{aligned}\mathbb{P}_{\mathbb{N}}(X_1(t) \geq x)\big|_{\Delta=0} &= \det(I - L)_{\ell^2(\{1-x,2-x,\dots\})} \\ &= \mathrm{e}^{-t^2} \det\left(I_{j-k}(2t)\right)\Big|_{j,k=0,\dots,-x}\end{aligned}$$

where the last equality[10] was proved in [5].

If $\mathcal{L}(t)$ denotes the length of the longest increasing subsequence of a random permutation of size $\mathcal{N}$ where $\mathcal{N}$ is a Poisson random variable with parameter t^2, then [1, 2, 14]

$$\mathbb{P}(\mathcal{L}(t) \leq n) = \mathrm{e}^{-t^2} \det(I_{j-k}(2t))_{j,k=0,\dots,n-1}$$

Theorem 4 *For $x \leq 1$ and domain wall initial conditions $Y = \mathbb{N}$, we have*

$$\mathbb{P}_{\mathbb{N}}(X_1(t) \geq x)\big|_{\Delta=0} = \mathbb{P}(\mathcal{L}(t) \leq 1 - x) \tag{34}$$

where $\mathcal{L}(t)$ denotes the length of the longest increasing subsequence of a random permutation of size $\mathcal{N}$ so that $\mathcal{N}$ is a Poisson random variable with parameter t^2.

5.2 Asymptotics

From the classic work of Baik, Deift, and Johansson [1] (see also Chapter 9 in [2]), we know that the limiting distribution of $\mathcal{L}(t)$ is

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\mathcal{L}(t) - 2t}{t^{1/3}} \leq x\right) = F_2(x) \tag{35}$$

where F_2 is the $\beta = 2$ TW distribution [27, 28]. In the present problem, $\Delta = 0$, we can therefore conclude that the left-most particle for domain wall initial condition $Y = \mathbb{N}$ has the limiting distribution

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{X_1(t) + 2t}{t^{1/3}} \geq -y\right) = F_2(y). \tag{36}$$

[10] $I_\nu(z)$ is the modified Bessel function of order ν.

6 Steepest Descent Curve

6.1 Spectral Functions

We introduce a pair of functions

$$G(\xi) = x \log \xi - it(\xi + \xi^{-1}), \quad H(\zeta) = -x \log \zeta - it(\zeta + \zeta^{-1}), \tag{37}$$

which we call the *spectral functions*. Note that the spectral functions appear in the integrand of the formula for $\mathcal{F}_N(x, t)$ given by (51). In particular, we have

$$(\xi_j \zeta_j)^x e^{-it(\varepsilon(\xi_j) - \varepsilon(\zeta_j))} = \exp\left\{G(\xi_j) - H(\zeta_j)\right\}. \tag{38}$$

In the following, we will deform the contours in the contour integral formula for $\mathcal{F}_N$ given by (51) so that the real part of the difference of the spectral function is negative, $\mathrm{Re}(G - H) < 0$. Thus, making $\mathcal{F}_N$ suitable for asymptotic analysis. Some more details for this section are given in the arXiv version of this paper [22].

6.2 Critical Points

The steepest descent contours in the contour integral formula $\mathcal{F}_N$ given by (24) are determined by the critical points of the spectral functions. We have

$$G'(\xi) = \frac{-it\xi^2 + x\xi + it}{\xi^2}, \quad H'(\zeta) = \frac{-it\zeta^2 - x\zeta + it}{\zeta^2}. \tag{39}$$

so that the critical points are given by

$$\xi = \frac{x \pm \sqrt{x^2 - 4t^2}}{2it}, \quad \zeta = \frac{-x \pm \sqrt{x^2 - 4t^2}}{2it}. \tag{40}$$

Note that each function, G and H, has a double critical point when $x = \pm 2t$ and the critical point are

$$\xi_0 = \begin{cases} -i, & x = 2t \\ i, & x = -2t \end{cases}, \quad \zeta_0 = \begin{cases} i, & x = 2t \\ -i, & x = -2t \end{cases}, \tag{41}$$

respectively. Physically, we expect the point $x = -2t$ to correspond to the left-edge of the up-spins and the point $x = 2t$ to correspond to the right-edge of the up-spins. Thus, we restrict our attention to the critical point given by $x = -2t$ and take $(\xi_0, \zeta_0) = (i, -i)$.

6.3 Steep Descent Curves

We introduce the *steep descent contours* given by three segments on three regions: in the region near the critical points, we take straight lines emanating from the critical point at angles $\pm\pi/6$ and $\pm 5\pi/6$; in an intermediate region, we take horizontal lines emanating from the end points of the straight lines in region near the critical point; in the region far away from the critical point, we take a segment of a large circle that connects with the horizontal lines. We use these contours so that we may explicitly determine the location of the poles when we deform to these *steep descent contours*. Although these contours don't follow the path of steepest descent for the real part of the spectral function, we show below that we still have the main property that $Re\{G(\xi) - G(\xi_0)\} \leq 0$ and $Re\{H(\zeta) - H(\zeta_0)\} \geq 0$ along these steep descent contours.

We now give a precise definition for the steep descent contours. We give a piecewise description based on the proximity to the critical points. Let $\mathcal{B}(z, r)$ be a ball centered at $z \in \mathbb{C}$ of radius $r > 0$ and $\mathcal{B}(z, r)^c$ be its complement. Then, we take the components

$$\begin{aligned}
\Gamma_{\pm}^{(1)} &= \{\pm i + x e^{\pm\pi i/6} \mid 0 \leq x\} \cap \mathcal{B}(\pm i, 1), \\
\Gamma_{\pm}^{(2)} &= \{\pm i + x e^{\pm 5\pi i/6} \mid 0 \leq x\} \cap \mathcal{B}(\pm i, 1) \\
\Gamma_{\pm}^{(3)} &= \{\pm i + e^{\pm\pi i/6} + x \mid 0 \leq x\} \cap \mathcal{B}(\pm i, 1)^c \cap \mathcal{B}(0, R_{\pm}),, \\
\Gamma_{\pm}^{(4)} &= \{\pm i + e^{\pm 5\pi i/6} - x \mid 0 \leq x\} \cap \mathcal{B}(\pm i, 1)^c \cap \mathcal{B}(0, R_{\pm}) \\
\Gamma_{\pm}^{(5)} &= \mathcal{C}_R \cap \{z \in \mathbb{C} \mid Im\{z\} \leq (\pm 1) Im\{\pm i + e^{\pm\pi i/6}\}\}
\end{aligned} \tag{42}$$

with radii $R_{\pm} > \sqrt{3}$. The bound on the radii is chosen so that the horizontal segments of the contours are non-trivial. Then, the steep descent contours are given by

$$\Gamma_k = \Gamma_k^{(1)} \cup \Gamma_k^{(2)} \cup \Gamma_k^{(3)} \cup \Gamma_k^{(4)} \cup \Gamma_k^{(5)} \tag{43}$$

for $k = \pm$. See Figs. 1 and 2.

Lemma 1 *Let $x = -2t$ and take the contours Γ_k, $k = \pm$, given by (42) and (43). Additionally, take $t^{-\alpha} \leq T \ll 1$ with $1/4 < \alpha < 1/3$. Then, we have*

$$Re\{G(\xi) - G(\xi_0)\} \leq 0, \quad Re\{H(\zeta) - H(\zeta_0)\} \geq 0 \tag{44}$$

if $\xi \in \Gamma_+$ and $\zeta \in \Gamma_-$. Moreover, if $\xi \in \Gamma_+ \cap \mathcal{B}(i, t^{-\alpha})^c$ and $\zeta \in \Gamma_- \cap \mathcal{B}(-i, t^{-\alpha})^c$, we have

$$Re\{G(\xi) - G(\xi_0)\} < -c_1(T)\, t^{1-3\alpha}, \quad Re\{H(\zeta) - H(\zeta_0)\} > c_2(T)\, t^{1-3\alpha}, \tag{45}$$

for some constants $c_1(T), c_2(T) > 0$ that depend only on T.

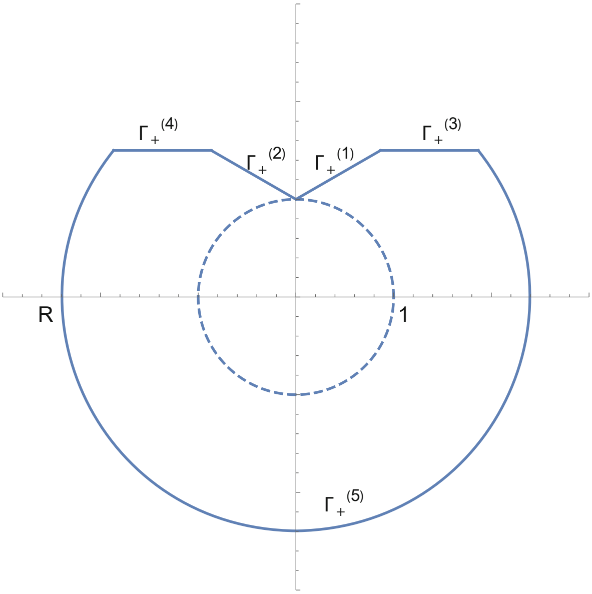

Fig. 1 The components of the Γ_+ contour.

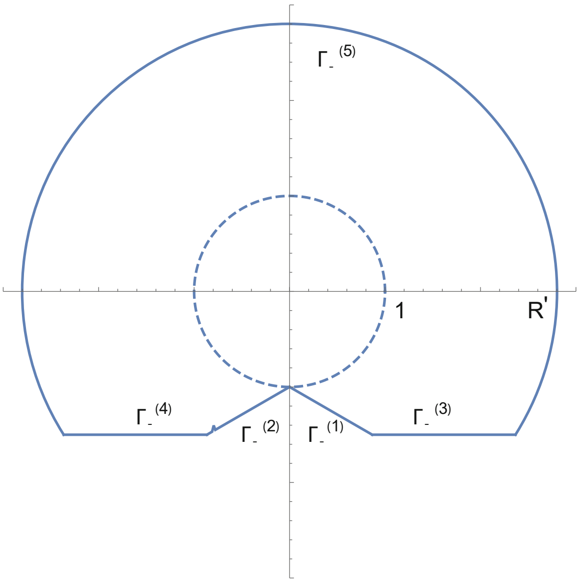

Fig. 2 The components of the Γ_- contour.

Proof We prove the bounds by showing that derivative of the real part of the functions are monotone along the different segments of the contours $\Gamma_\pm$ as parameterized in (42). Since $G(\xi) - G(\xi_0) = 0$ for $\xi = \xi_0$ and $H(\zeta) - H(\zeta_0) = 0$ for $\zeta = \zeta_0$, the first bounds (44) then follow by monotonicity. Moreover, since the real part of the functions are monotone, we establish the bounds (45) by bounding the real part of the functions on the boundary of the segment $\Gamma_{\pm} \cap \mathcal{B}(\pm \mathrm{i}, t^{-\alpha})$.

The arguments for both functions are the same, except for some negative signs here and there. So, we focus solely on the case for the G function. Additionally, the arguments are fairly routine and standard. So, we just sketch the main idea needed for the bounds.

Take $\xi \in \Gamma_+^{(1)} \cup \Gamma_+^{(2)}$. In this case, we have $\xi = \mathrm{i} + xe^{\pi i/6}$ or $\xi = \mathrm{i} + xe^{5\pi i/6}$, with $0 \leq x \leq 1$ since $\Gamma_+^{(1)} \cup \Gamma_+^{(2)} \subset \mathcal{B}(i, 1)$. Then, we may write the real part of the G function explicitly and show that it is monotone by taking its derivative. For instance, we have

$$\frac{d}{dx} Re\,\{G(i + xe^{\pi i/6}) - G(i)\} = \frac{t}{2}\left(1 - \frac{2+4x}{1+x+x^2} + \frac{1+4x+x^2}{(1+x+x^2)^2}\right). \tag{46}$$

One may now check that the derivative is zero when $x = 0$ and negative if $0 < x < 1 + \sqrt{3}$. Thus, the bound (44) follows for this segment.

Take $\xi \in \Gamma_+^{(3)} \cup \Gamma_+^{(4)}$. In this case, we have $\xi = i + e^{\pi i/6} + x$ or $\xi = i + e^{5\pi i/6} - x$, with x non-negative and bounded since $\Gamma_+^{(3)} \cup \Gamma_+^{(4)} \subset \mathcal{B}(0, R_+)$. Then, we may write the real part of the G function explicitly and show that it is monotone by taking its derivative. For instance, we have

$$\frac{d}{dx} Re\,\{G(i + e^{\pi i/6} + x) - G(i)\} = -t\left(1 - \frac{3(\sqrt{3}+2x)}{2(3+\sqrt{3}x+x^2)^2}\right). \tag{47}$$

Form this, one may show that the derivative is strictly negative for all $x \geq 0$. The bound (44) follows for this segment.

Take $\xi \in \Gamma_+^{(5)}$. In this case, we have $\xi = R_+ e^{i\theta}$, with $-\pi/2 \leq \theta \leq \phi_1 < \pi/2$ and $\pi/2 < \phi_2 \leq \theta \leq 3\pi/2$ for some constants ϕ_1 and ϕ_2 since $\Gamma_+^{(5)} \subset \{z \in \mathbb{C} \mid Im\,\{z\} \leq Im\,\{i + e^{\pi i/6}\}\}$. In this case, we have

$$Re\,\{G(\xi) - G(i)\} = -2t\,\log R_+ + t(R_+ + R_+^{-1})\sin\theta. \tag{48}$$

Since $R_+ > 1$, one may then show that this function is monotone on θ for each of the segments $-\pi/2 \leq \theta \leq \phi_1 < \pi/2$ and $\pi/2 < \varphi_2 \leq \theta \leq 3\pi/2$. The bound (44) follows for this segment.

The bound (45), now that we have established that the function is monotone along all the segments of the contours, follows by evaluating the function on the boundary of the segment $\Gamma_+ \cap \mathcal{B}(i, t^{-\alpha})$. That is, we evaluate the function at the points $\xi = \xi_0 + t^{-\alpha} e^{\pi \mathrm{i}/6}$ and $\xi = \xi_0 + t^{-\alpha} e^{5\pi \mathrm{i}/6}$. In particular, we use the Taylor expansion

$$G(\xi) - G(\xi_0) = -\frac{1}{3}x^3 t^{1-3\alpha} + \mathcal{O}(t^{1-4\alpha}) \tag{49}$$

to approximate the function at the desired points. Since $t^{-\alpha} < T \ll 1$, we obtain the bound (45). □

7 Contour Deformations

7.1 Small to Large Contour deformations

We deform the contours in the probability function for the left-most particle given by (24). In particular, we deform the contours $\mathcal{C}_r$, for the ζ-variables, to some contour $\mathcal{C}_{R'}$ with a large radius $R' > 0$. Let

$$\Omega(\xi) := \mathcal{C}^{(0)} \cup -\mathcal{C}^{(1)} \cup -\mathcal{C}^{(2)} \cup \cdots \cup -\mathcal{C}^{(N)} \tag{50}$$

be the union of $(N+1)$ circles so that $-\mathcal{C}^{(j)}$, for $j = 1, \ldots, N$, is a negatively oriented circle centered at ξ_j^{-1} with radius $r' > 0$ and $\mathcal{C}^{(0)}$ is a positively oriented circle centered at the origin with radius $R' > 0$. We give precise conditions on the radii in the statement of Lemma 2 below. Then, as we deform the $\mathcal{C}_r$ contour, we will encounter poles at $\zeta_i = \xi_j^{-1}$ for $i, j = 1, \ldots, N$. As a result, we obtain the contour $\Omega(\xi)$ when we deform the contour $\mathcal{C}_r$ to $\mathcal{C}_{R'}$. This result and the proof for the contour deformations, given by Lemma 2 below, is similar to the contour deformation in [11].

Lemma 2 *For $\Delta \neq 0$, $\mathcal{F}_N(x,t) = \mathbb{P}_Y(X_1(t) \geq x)$ equals*

$$\int_{\mathcal{C}_R} \cdots \int_{\Omega(\xi)} \frac{\prod_{j,k}(\xi_j + \zeta_k - 2\Delta\xi_j\zeta_k)}{\prod_{j<k}(1+\xi_j\xi_k - 2\Delta\xi_j)(1+\zeta_j\zeta_k - 2\Delta\zeta_j)} D_N(\xi,\zeta) \times \prod_j (\xi_j\zeta_j)^{x-y_j-1} e^{-it(\varepsilon(\xi_j)-\varepsilon(\zeta_j))}\, d^N\zeta\, d^N\xi \tag{51}$$

where the contour $\mathcal{C}_R$ for the ξ-variables is a circle centered at zero with radius $R > 0$ and the contour $\Omega(\xi)$ for the ζ-variables is given by (50) with radii $R' > 0$ and $r' = 1/(2R)$, so that the radii satisfy the following inequalities $\max\{2|\Delta|^{-1}, 2(1+2|\Delta|)\} < R < \max\{4|\Delta|^{-1}, 4(1+2|\Delta|)\} < R'/2$.

Proof We take formula (24) with radius R as given in the conditions in the Lemma and radius $r > 0$ so that $\max\{4|\Delta|^{-1}, 4(1+2|\Delta|)\} < r^{-1} < R'/2$. Note that the conditions on the contours $\mathcal{C}_R$ and $\mathcal{C}_r$ given in Theorem 3 (i.e. all the poles lie inside/outside of the contours) are satisfied for our choice of radii. Then, we deform the contour in (24) for the ζ-variables to a large radius $R' > 0$, with R' satisfying the conditions given in the Lemma. We begin by deforming the contour for ζ_N, then the contour for ζ_{N-1}, and continue successively until we deform the contour for ζ_1. When we deform the contour for the ζ_n variable, we encounter three types of poles

$$(a)\; 1 - \xi_i\zeta_n = 0; \quad (b)\; 1 + \zeta_i\zeta_n - 2\Delta\zeta_i = 0,\; i < n; \quad (c)\; 1 + \zeta_n\zeta_j - 2\Delta\zeta_n,\; n < j \tag{52}$$

for any $i, j = 1, \ldots, N$. The contribution for a type (a) pole is given by the contour integral with respect to the variable ζ_n with contour $-\mathcal{C}^{(i)}$, i.e. a negatively oriented circle centered at ξ_i^{-1} with radius $r' > 0$ as given in the conditions of the Lemma. Note that the only pole, with respect to the variable ζ_n, inside the contour $-\mathcal{C}^{(i)}$ is given by $\zeta_n = \xi_i^{-1}$ because r' is chosen to be small enough. The result then follows by showing that the type (b) and (c) poles contribute no residue.

Assume we have already deformed the ζ_j variables for $j > n$ so that $\zeta_i \in \mathcal{C}_r$ for $i < n$ and $\zeta_j \in \Omega(\xi)$ for $j > n$. We then deform the contour for the ζ_n variable. Below, we consider the residue contribution from the type (b) and (c) poles.

Case (b). We compute the residue at

$$\zeta_n = (2\Delta\zeta_\ell - 1)/\zeta_\ell \tag{53}$$

for $\ell < n$. The result is a $(2N-1)$-fold contour integral with the same integrand, say $I_N(\xi, \zeta; t)$, except that the term $1 + \zeta_\ell\zeta_n - 2\Delta\zeta_\ell$ is replaced by ζ_ℓ and the variable ζ_n is evaluated at $(2\Delta\zeta_\ell - 1)/\zeta_\ell$ for the rest of the terms.

We then compute the integral with respect to the ζ_ℓ variable for the resulting residue term. The integral is computed by analyzing the poles and residues inside the contour $\mathcal{C}_r$ for ζ_ℓ. The possible poles are given by

$$\begin{aligned}
&1 - \xi_k\zeta_n = 0, && k = 1, \ldots, N\\
&1 + \zeta_\ell\zeta_j - 2\Delta\zeta_\ell = 0, && \ell < j,\ \zeta_j \in \Omega(\xi)\\
&1 + \zeta_i\zeta_\ell - 2\Delta\zeta_i = 0, && i < \ell,\ \zeta_i \in \mathcal{C}_r\\
&1 + \zeta_i\zeta_n - 2\Delta\zeta_i = 0, && i < n,\ \zeta_i \in \mathcal{C}_r\\
&1 + \zeta_n\zeta_j + 2\Delta\zeta_n = 0, && n < j,\ \zeta_j \in \Omega(\xi)\\
&\zeta_\ell^{x-y_j-1}\zeta_n^{x-y_n-1} = 0, && j \neq n.
\end{aligned} \tag{54}$$

In particular, the location of the possible poles is given by the following

$$\begin{aligned}
\zeta_\ell &= \frac{\xi_k}{2\Delta\xi_k - 1} && \Rightarrow && \left|\frac{\xi_k}{2\Delta\xi_k - 1}\right| > r\\
\zeta_\ell &= \frac{1}{2\Delta - \zeta_j} && \Rightarrow && \zeta_n = 2\Delta - \zeta_\ell^{-1} = \zeta_j\\
\zeta_\ell &= 2\Delta - \zeta_i^{-1} && \Rightarrow && \left|2\Delta - \zeta_i^{-1}\right| > r\\
\zeta_n &= 2\Delta - \zeta_i^{-1} && \Rightarrow && \zeta_\ell = \zeta_i\\
\zeta_\ell &= \frac{2\Delta - \zeta_j}{4\Delta^4 - 2\Delta\zeta_j - 1} && \Rightarrow && \left|\frac{2\Delta - \zeta_j}{4\Delta^2 - 2\Delta\zeta_j - 1}\right| > r\\
&\zeta_\ell^{y_n - y_\ell} && \Rightarrow && y_n - y_\ell \geq 1.
\end{aligned} \tag{55}$$

We use the assumptions on the radii $R, R', r' > 0$ given in the statement of the Lemma and the condition on the radius $r > 0$ fixed at the beginning of the proof to establish the inequalities above. For the first two inequalities, it suffices to have $R, r^{-1} > 1 + 2|\Delta|$. For the third inequality, we have to consider two cases $\zeta_j \in \mathcal{C}^{(0)}$ or $\zeta_j \in \mathcal{C}^{(k)}$ with $k \neq 0$. In the first case when $\zeta_j \in \mathcal{C}^{(0)}$, we have that $|\zeta_j| = R'$ and we use the bounds $R' > 16|\Delta|$ and $R' > 8|\Delta|^{-1}$ that follow from the condition on the statement of the Lemma. In the second case when $\zeta_j \in \mathcal{C}^{(k)}$ with $k \neq 0$, we have that $|\zeta_j| \leq (3/2)R^{-1}$ and we use the bound $(3/2)R^{-1} < |\Delta|$ that follows from the statement of the Lemma. Then, in all the cases above except for the second and fourth case, the poles lie outside the contour $\mathcal{C}_r$, meaning that there is no residue contribution. In the second and fourth cases, the determinant term $D_N(\xi, \zeta)$ vanishes because two columns in the matrix of the determinant are equal to each other since two ζ variables are equal to each other. In the last case, there is no pole since the exponent is positive. Then, the pole from the denominator and the zero from the determinant cancel out, meaning that these cases don't produce a residue.

Therefore, by computing the integral with respect to the ζ_ℓ variable, we have that the residues from the type (b) poles vanish.

Case (c). We compute the residue at

$$\zeta_n = \frac{1}{2\Delta - \zeta_\ell} \tag{56}$$

with $n < \ell$. The result is a $(2N - 1)$-fold contour integral with the same integrand, say $I_N(\xi, \zeta; t)$, except that the term $1 + \zeta_n\zeta_\ell - 2\Delta\zeta_n$ is replaced by $2\Delta - \zeta_\ell$ and the variable ζ_n is evaluated at $1/(2\Delta - \zeta_\ell)$ for the rest of the terms.

In this case, we have have $\zeta_\ell \in \Omega(\xi)$ since $\ell > n$. Thus, we have two possibilities: (i) $\zeta_\ell \in \mathcal{C}^{(0)} = \mathcal{C}_{R'}$, or (ii) $\zeta_\ell \in -\mathcal{C}^{(k)}$ for some $k = 1, \ldots, N$ (i.e. a negatively oriented small circle of radius r' centered at ξ_k^{-1}). In the first case, we will not cross a pole in the contour deformation and there will be no residue to consider. In the second case, the pole will cancel out with a zero from the numerator and, again, there will be no residue to consider. We give more details below.

In the first case, when $\zeta_\ell \in \mathcal{C}_{R'}$, we have $\zeta_n = 1/(2\Delta - \zeta_\ell)$. This pole lies inside the contour $\mathcal{C}_r$ since $R'r > 2$ and $r < (1 + 2|\Delta|)^{-1}$. Thus, we don't cross this pole when we deform the $\mathcal{C}_r$ contour to $\mathcal{C}^{(0)} = \mathcal{C}_{R'}$.

In the second case, when $\zeta_\ell \in -\mathcal{C}^{(k)}$, we first compute the residue at $\zeta_\ell = \xi_k^{-1}$. We obtain an $(2N-1)$-fold contour integral with the same integrand, say $I_N(\xi, \zeta; t)$, except that the determinant $D_N(\xi, \zeta)$ is replaced the same determinant with the k^{th} row and the ℓ^{th} removed and multiplied by the factor $(1 + \xi_k^2 - 2\Delta\xi_k)^{-1}$, and the rest of the terms are the same with the variable ζ_ℓ evaluated at ξ_k^{-1}.

We now deform the contour for ζ_n to the contour $\mathcal{C}^{(0)}$. After taking the $\zeta_\ell = \xi_k^{-1}$ residue, it turns out that the terms giving rise to the pole $\zeta_n = 1/(2\Delta - \zeta_\ell)$ becomes

$$(1 + \zeta_n\zeta_\ell - 2\Delta\zeta_n) = \xi_k^{-1}(\xi_k + \zeta_n - 2\Delta\xi_k\zeta_n). \tag{57}$$

Note that this term also appears in the numerator of the integrand, meaning that this term cancels out and there is no residue in this case.

Therefore, when we deform the contour for the ζ_n to infinity, we don't cross any type (c) poles. Moreover, this, along with the argument for the type (b) poles, means that we only cross the poles due to the type (a) poles. This establishes the result.

□

7.2 Series Expansion

We write the contour formula (51) as a summation by expanding the integrals over the contour $\Omega(\xi)$, given by (50), as a summations of $N+1$ integrals. We introduce some notation to encode the different terms in the summation.

Take the set of all maps from the index set $\{1,\ldots,N\}$ to the set $\{0,1,\ldots,N\}$ and denote it by

$$\mathcal{T} := \{\tau : \{1,\ldots,N\} \to \{0,1,\ldots,N\}\} = \mathrm{Hom}\,(\{1,\ldots,N\},\{0,1,\ldots,N\})\,. \tag{58}$$

In the following, a map $\tau \in \mathcal{T}$ will correspond to a term with contours $\mathcal{C}^{(\tau(k))}$, given by (50), for the ζ_k variable and $k = 1,\ldots,N$. Moreover, we will show that some contour integrals will vanish for certain $\tau \in \mathcal{T}$. We consider the set of maps that map injectively to the elements $\{1,\ldots,N\}$ in the and the cardinality of the preimage $\sigma^{-1}(0)$ is fixed;

$$\mathcal{T}_n := \{\tau \in \mathcal{T} \mid |\tau^{-1}(0)| = n;\, |\tau^{-1}(k)| \le 1,\ k = 1,\ldots,N\}. \tag{59}$$

Lemma 3 *For $\Delta \neq 0$, $\mathcal{F}_N(x,t) = \mathbb{P}_Y(X_1(t) \ge x)$ equals*

$$\sum_{n=0}^{N}\sum_{\tau\in\mathcal{T}_n}\oint_{\mathcal{C}_R}\cdots\oint_{\mathcal{C}_R}\oint_{\mathcal{C}^{(\tau(1))}}\cdots\oint_{\mathcal{C}^{(\tau(N))}} I_N(\xi,\zeta;x,t)\,d^N\zeta\,d^N\xi \tag{60}$$

where the integrand $I_N(\xi,\zeta;x,t)$ is the same integrand as in (51), the summation is take over the set of maps $\mathcal{T}_n$ given by (59), the contour $\mathcal{C}_R$ is a circle centered at zero with radius $R > 0$, the contours $\mathcal{C}^{(\tau(k))}$ are given by (50) with radii $r' = R^{-1}/2$, $R' > 0$ so that the radii satisfy the bounds $\max\{2|\Delta|^{-1}, 2(1+2|\Delta|)\} < R < \max\{4|\Delta|^{-1}, 4(1+2|\Delta|)\} < R'/2$

Proof We take the contour formula (51) from Lemma7.1. We then expand the integrals over the contours $\Omega(\xi)$ as a sum of $N+1$ integrals with contours given by the right side of (50). The result is a summation over the set of maps $\mathcal{T}$ given by (58),

$$\mathcal{F}_N(x,t) = \sum_{\tau\in\mathcal{T}}\oint_{\mathcal{C}_R}\cdots\oint_{\mathcal{C}_R}\oint_{\mathcal{C}^{(\tau(1))}}\cdots\oint_{\mathcal{C}^{(\tau(N))}} I_N(\xi,\zeta;x,t)\,d^N\zeta\,d^N\xi. \tag{61}$$

The result of this lemma follows by showing that some terms vanish, i.e. if $\tau \notin \mathcal{T}_n$ the corresponding contour integral will vanish. Below, we show that a term in the summation vanishes if $\tau(j) = \tau(k) > 0$ with $j \neq k$.

Take $\tau \in \mathcal{T}$ with $\tau(j') = \tau(k') = \ell > 0$ with $n \neq m$ and $j', k' = 1, \ldots, N$. We show that the term in the summation (61) with this $\tau \in \mathcal{T}$ vanishes by taking the integrals with respect to the variables $\zeta_{j'}$ and $\zeta_{k'}$. We take the integral with respect to the $\zeta_{j'}$ and $\zeta_{k'}$ variables by taking the residues at the poles given by $\zeta_{j'} = \xi_\ell^{-1}$ and $\zeta_{k'} = \xi_\ell^{-1}$. Note that the poles given by $\zeta_{j'} = \xi_\ell^{-1}$ and $\zeta_{k'} = \xi_\ell^{-1}$ correspond to the (ℓ, j')-entry and the (ℓ, k')-entry of the matrix for the $D_N(\xi, \zeta)$ determinant. First, we take the residue at $\zeta_{j'} = \xi_\ell^{-1}$, the determinant transforms as follows

$$
\begin{aligned}
D_N(\xi, \zeta) &= \det\left(\frac{1}{(1-\xi_j\zeta_k)(\xi_j + \zeta_k - 2\Delta\xi_j\zeta_k)}\right)_{j,k=1}^{N} \\
&\longrightarrow \frac{(-1)^{\tau(j')-j'-1}}{1+\xi_\ell^2 - 2\Delta\xi_\ell} \det\left(\frac{1}{(1-\xi_j\zeta_k)(\xi_j + \zeta_k - 2\Delta\xi_j\zeta_k)}\right)_{j\neq\ell, k\neq j'}.
\end{aligned}
\tag{62}
$$

For the rest of the factors in the integrand, one evaluates $\zeta_{j'} = \xi_\ell^{-1}$ when we take the residue at $\zeta_{j'} = \xi_\ell^{-1}$. One may check that this doesn't introduce any poles with respect to the $\zeta_{j'}$ variable inside the $\mathcal{C}^{(\ell)}$ contour. Then, the residue at $\zeta_{j'} = \xi_\ell^{-1}$ doesn't have a pole at $\zeta_{k'} = \xi_\ell^{-1}$ since the pole at $\zeta_{k'} = \xi_\ell^{-1}$ is removed when we take the residue and no other pole is introduced. Thus, by taking the residue at $\zeta_{k'} = \xi_\ell^{-1}$ after taking the residue at $\zeta_{j'} = \xi_\ell^{-1}$, we have that the term vanishes. That is,

$$
\oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \oint_{\mathcal{C}^{(\tau(1))}} \cdots \oint_{\mathcal{C}^{(\tau(N))}} I_N(\xi, \zeta; x, t)\, d^N\zeta\, d^N\xi = 0 \tag{63}
$$

if $\tau(n) = \tau(m) = \ell > 0$ with $n \neq m$ and $n, m = 1, \ldots, N$.

The result of the lemma then follows by taking the summation representation given by (61) and noting that the terms with $\tau \notin \mathcal{T}_n$ for some $n = 0, 1, \ldots, N$ vanish due to the identities (63). □

7.3 Residue Computations

We compute the contour integrals with respect to the ζ_k variables with $\tau(k) \neq 0$ for each of the terms in the series expansion given by (60). First, we introduce some notation to represent the resulting integrand after the residue computations.

Fix $\tau \in \mathcal{T}_{N-M}$ with $0 \leq M \leq N$ and $\mathcal{T}_{N-M}$ given by (59). Then, define the following sets

$$\begin{aligned} K_1 &:= \tau^{-1}(0) = \{\overline{k}_1 < \cdots < \overline{k}_{N-M}\}, \\ K_2 &:= K_1^c = \{k_1 < \cdots < k_M\}, \\ J_2 &:= \tau\,(K_2) = \{\tau_1 = \tau(k_1), \ldots, \tau_M = \tau(k_M)\}, \\ J_1 &:= J_2^c = \{j_1 < \cdots < j_{N-M}\}. \end{aligned} \tag{64}$$

We let $\pi : \mathcal{T}_{N-M} \hookrightarrow \mathcal{S}_N$ be an injection given by

$$\pi(\tau) = \begin{cases} k_m \mapsto \tau_m, & m = 1, \ldots, M \\ \overline{k}_n \mapsto j_n, & n = 1, \ldots, N-M \end{cases}. \tag{65}$$

Also, we take the set of permutations that fix every element of the set J_2, denoted as follows

$$\mathcal{S}_N(J_2) := \{\sigma \in \mathcal{S}_N \mid \sigma(j) = j,\ j \in J_2\}. \tag{66}$$

Lastly, we introduce the following functions

$$\begin{aligned} I_N(\xi, \zeta; \tau) &= \frac{\prod_{j\in J_1, k\in K_1}(\xi_j + \zeta_k - 2\Delta\xi_j\zeta_k) D_N(\xi, \zeta; \tau)}{\prod_{\substack{j<k \\ j,k\in J_1}} (1 + \xi_j\xi_k - 2\Delta\xi_j) \prod_{\substack{j<k \\ j,k\in K_1}} (1 + \zeta_j\zeta_k - 2\Delta\zeta_j)} \\ &\quad \times \prod_{j\in J_1} \xi_j^{x-y_j-1} e^{-\mathrm{i}t\epsilon(\xi_j)} \prod_{k\in K_1} \zeta_k^{x-y_k-1} e^{\mathrm{i}t\epsilon(\zeta_k)} \\ D_N(\xi, \zeta; \tau) &= (-1)^{|\pi(\tau)|} \det\big(d(\xi_j, \zeta_k)\big)_{j\in J_1, k\in K_1} \\ &= (-1)^{|\pi(\tau)|} \sum_{\gamma\in\mathcal{S}_N(J_2)} (-1)^{|\gamma|} \prod_{k\in K_1} d(\xi_{\gamma(k)}, \zeta_k) \\ f(\xi, \zeta; \tau) &= \prod_{\ell=1}^{M} \Bigg(\prod_{\substack{\tau_\ell<k \\ k\neq\tau_{\ell+1},\ldots,\tau_M}} \left(\frac{1 + \xi_{\tau_\ell}\xi_k - 2\Delta\xi_k}{1 + \xi_{\tau_\ell}\xi_k - 2\Delta\xi_{\tau_\ell}}\right) \\ &\quad \times \prod_{\substack{k_\ell<k \\ k\neq k_{\ell+1},\ldots,k_M}} \left(\frac{\xi_{\tau_\ell} + \zeta_k - 2\Delta\xi_{\tau_\ell}\zeta_k}{\xi_{\tau_\ell} + \zeta_k - 2\Delta}\right)\Bigg) \prod_{\ell=1}^{M} \xi_{\tau_\ell}^{y_{k_\ell}-y_{\tau_\ell}-1} \end{aligned} \tag{67}$$

where the function $d(\xi, \zeta)$ is given by (18), $(-1)^{|\sigma|}$ denotes the signature of a permutation for any $\sigma \in \mathcal{S}_N$, and the sets K_1, J_1 are given by (64) and $M =$

$N - |\tau^{-1}(0)|$. Note that $I_N(\xi, \zeta; \tau)$ is equal to the integrand of contour integrals (24), (51) and (60) if $|\tau^{-1}(0)| = N$.

Lemma 4 *Fix $\tau \in \mathcal{T}_{N-M}$, with $0 \le M \le N$, and take the notation from (64). Then, for $\Delta \neq 0$, we have*

$$\oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \oint_{\mathcal{C}^{\tau(1)}} \cdots \oint_{\mathcal{C}^{\tau(N)}} I_N(\xi, \zeta) d^N\zeta\, d^N\xi$$
$$= \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \oint_{\mathcal{C}_{R'}} \cdots \oint_{\mathcal{C}_{R'}} I_N(\xi, \zeta; \tau) f(\xi, \zeta; \tau) \left(\prod_{k \in K_1} d\zeta_j \right) d^N\xi \tag{68}$$

where the integral on the left side is a $2N$-fold contour intergal and the integral on the right side is a $(N + |K_1|)$-fold contour integral, the integrand on the left side is equal to the integrand in (51) and the integrand on the right side is given by (67), and the contours are the same as in the statement of Lemma 3 so that the ζ variables are integrated with respect to $\mathcal{C}_{R'}$ contours.

Proof We obtain the identity in this lemma by computing the integrals with respect to the ζ_{k_ℓ} variables with $k_\ell \in K_2$. In particular, the contours are given by $-\mathcal{C}^{(\tau_\ell)}$, which are negatively oriented circles of radius $r' = 1/(2R)$ and centered at $\xi_{\tau_\ell}^{-1}$, for the integrals with respect to ζ_{k_ℓ} and $k_\ell \in K_2$. Then, we compute the integrals by taking the residues at $\zeta_{k_\ell} = \xi_{\tau_\ell}^{-1}$. We start by taking the residue at $\zeta_{k_M} = \xi_{\tau_M}^{-1}$ and continue successively until we take the residue at $\zeta_{k_1} = \xi_{\tau_1}^{-1}$.

Let's take the residue with respect to $\zeta_{k_M} = \xi_{\tau_M}^{-1}$. Note that the pole corresponding to this residue comes from the (τ_M, k_M)-entry of the matrix of the $D_N(\xi, \zeta)$ determinant. Then, when we take the residue, the determinant is replaced by a determinant of the same matrix with the τ_M-row and k_M-column removed and a prefactor $(-1)^{\tau_M - k_M}(1 + \xi_{\tau_M}^2 - 2\Delta\, \xi_{\tau_M})^{-1}$. That is,

$$\det\left(\frac{1}{(1 - \xi_i\zeta_j)(\xi_i + \zeta_j - 2\Delta\xi_i\zeta_j)} \right)_{i,j=1}^{N}$$
$$\longrightarrow \frac{(-1)^{\tau_M - k_M - 1}}{1 + \xi_{\tau_M}^2 - 2\Delta\xi_{\tau_M}} \det\left(\frac{1}{(1 - \xi_i\zeta_j)(\xi_i + \zeta_j - 2\Delta\xi_i\zeta_j)} \right)_{i \neq \tau_M, j \neq k_M}. \tag{69}$$

The other terms of the integrand, when we compute the residue, transform by evaluating $\zeta_{k_M} = \xi_{\tau_M}^{-1}$. Then, the result after taking the residue is

$$\begin{aligned}&\frac{\prod_{j\neq\tau_M,k\neq k_M}(\xi_j+\zeta_k-2\Delta\xi_j\zeta_k)}{\prod_{\substack{j<k\\ j,k\neq\tau_M}}(1+\xi_j\xi_k-2\Delta\xi_j)\prod_{\substack{j<k\\ j,k\neq k_M}}(1+\zeta_j\zeta_k-2\Delta\zeta_j)}\\&\times\prod_{j\neq\tau_M}\xi_j^{x-y_j-1}e^{-\mathrm{i}t\epsilon(\xi_j)}\prod_{k\neq k_M}\zeta_k^{x-y_k-1}e^{\mathrm{i}t\epsilon(\zeta_k)}\\&\times(-1)^{\tau_M-k_M}\det\left(\frac{1}{(1-\xi_j\zeta_k)(\xi_j+\zeta_k-2\Delta\xi_j\zeta_k)}\right)_{j\neq\tau_M,k\neq k_M}\\&\times\xi_{\tau_M}^{y_{k_M}-y_{\tau_M}-1}\prod_{\tau_M<k}\left(\frac{1+\xi_{\tau_\ell}\xi_k-2\Delta\xi_k}{1+\xi_{\tau_\ell}\xi_k-2\Delta\xi_\ell}\right)\prod_{k_M<k}\left(\frac{\xi+\zeta_k-2\Delta\xi_\ell\zeta_k}{\xi_\ell+\zeta_k-2\Delta}\right)\end{aligned}\quad. \tag{70}$$

The sign infront of the determinant changed by negative one since we are taking the integral over a negatively oriented circle.

We continue taking the integrals with respect to the variables ζ_{k_ℓ}, successively with ℓ decreasing, and evaluating the residues at $\zeta_{k_\ell}=\xi_{\tau_\ell}^{-1}$. The computations are similar to the base case $\zeta_{k_M}=\xi_{\tau_M}^{-1}$. In particular, the pole giving rise to residue comes from the (τ_ℓ,k_ℓ)-entry of the determinant. Then, when we take the residue, the determinant transforms by removing the τ_ℓ-row and the k_ℓ-column and adding a prefactor. The other terms in the integrand transform by evaluating $\zeta_{k_\ell}=\xi_{\tau_\ell}^{-1}$. We skip the details here since the computations are very similar to the base case. The result follows by computing all the integrals with respect to the ζ_{k_ℓ} variable with $k_\ell\in K$. □

Theorem 5 *For $\Delta\neq 0$, $\mathcal{F}_N(x,t)=\mathbb{P}_Y(X_1(t)\geq x)$ equals*

$$\sum_{n=0}^{N}\sum_{\tau\in\mathcal{T}_n}\oint_{\mathcal{C}_R}\cdots\oint_{\mathcal{C}_R}\oint_{\mathcal{C}_{R'}}\cdots\oint_{\mathcal{C}_{R'}}I_N(\xi,\zeta;\tau)f(\xi,\zeta;\tau)\left(\prod_{k\in K_1}d\zeta_k\right)d^N\xi \tag{71}$$

where the integrand is given by (67), the summation is take over the set of maps $\mathcal{T}_n$ given by (59), and the contours $\mathcal{C}_R$ and $\mathcal{C}_{R'}$ are circles centered at zero with radii $R,R'>0$ so that $\max\{2|\Delta|^{-1},2(1+2|\Delta|)\}<R<\max\{4|\Delta|^{-1},4(1+2|\Delta|)\}<R'/2$.

Proof The result is a direct consequence of Lemmas 3 and 4. □

7.4 *Deformation to Steep Descent Contours*

We take the series expansion formula (71) and deform the contours to the steep descent contours given by (43).

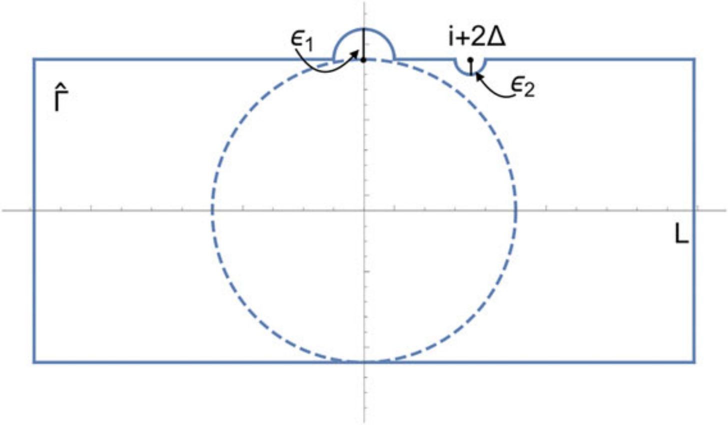

Fig. 3 The contour $\widehat{\Gamma}$.

Let $\widehat{\Gamma}$ be a positive oriented rectangle centered at zero, with length equal to $2L = 2\sqrt{R^2-1}$ and height equal to 2, and two half-circle bumps as indicated on Fig. 3. The bump centered at i has radius ϵ_1 and the bump centered at $\mathrm{i} + 2\Delta$ has radius ϵ_2 so that $0 < \epsilon_2 \ll \epsilon_1 \ll 1$.

Lemma 5 *Fix* $\tau \in \mathcal{T}_{N-M}$, *with* $0 \le M \le N$, *take the notation from (64). Then, for* $\Delta \neq 0$, *we have*

$$\begin{aligned}&\oint_{\mathcal{C}_R}\cdots\oint_{\mathcal{C}_R}\oint_{\mathcal{C}_{R'}}\cdots\oint_{\mathcal{C}_{R'}} I_N(\xi,\zeta;\tau) f(\xi,\zeta;\tau)\, d^J\zeta\, d^N\xi \\ &= \oint_{\Gamma_+}\cdots\oint_{\Gamma_-} I_N(\xi,\zeta;\tau)\left(\oint_{\widehat{\Gamma}}\cdots\oint_{\widehat{\Gamma}} f(\xi,\zeta;\tau) d^{J_2}\xi\right) d^{K_1}\zeta\, d^{J_1}\xi\end{aligned} \tag{72}$$

where the integrand is given by (67), the differentials $d^S\xi$ *or* $d^S\zeta$ *are* $|S|$*-fold differential over the variables* ξ_s *or* ζ_s *with* $s \in S$, *the contours* $\Gamma_\pm$ *are given by (43) with* $R_+ = R$ *and* $R_- = R'$ *so that* $\xi_j \in \Gamma_+$ *and* $\zeta_k \in \Gamma_-$ *for* $j \in J_1$ *and* $k \in K_1$, *the contour* $\widehat{\Gamma}$ *is given by Fig. 3, the contours* $\mathcal{C}_R$ *and* $\mathcal{C}_{R'}$ *are circles centered at zero with radii* $R, R' > 0$ *so that* $\max\{2|\Delta|^{-1}, 2(1+2|\Delta|)\} < R < \max\{4|\Delta|^{-1}, 4(1+2|\Delta|)\} < R'/2$.

Proof We obtain the result by deforming the contours and showing that we don't cross any poles. We begin by deforming the contour, for the ξ_j variables with $j \in J_2$, from $\mathcal{C}_R$ to $\widehat{\Gamma}$. Then, for the ξ_j variables with $j \in J_1$, we deform the contours $\mathcal{C}_R$ to the contours Γ_+. Finally, for the ζ_k variables, we deform the contours $\mathcal{C}_{R'}$ to the contours Γ_-.

Consider the integral with respect to $\xi_\ell \in \mathcal{C}_R$ and $\ell \in J_2$. We deform the contour $\mathcal{C}_R$ to the contour $\widehat{\Gamma}$. Note that the factor $I_N(\xi, \zeta; \tau)$ is independent of the ξ_ℓ variable. Then, the only possible poles are given by

$$1 + \xi_\ell \xi_k - 2\Delta\xi_\ell = 0, \quad \xi_\ell + \zeta_k - 2\Delta = 0. \tag{73}$$

In the first case of (73), the location of the pole is given by $(2\Delta - \xi_k)^{-1}$ with $\xi_k \in \mathcal{C}_R$ or $\xi_k \in \widehat{\Gamma}$, depending on the index and if the contour for the variable has been deformed. If $\xi_k \in \mathcal{C}_R$, the location of the pole $(2\Delta - \xi_k)^{-1}$ clearly lies inside the unit circle since $R > 1 + 2|\Delta|$. In particular, we don't cross this pole when we deform from the contour $\mathcal{C}_R$ to the contour $\widehat{\Gamma}$, since the contour $\widehat{\Gamma}$ lies outside the unit circle. If $\xi_k \in \widehat{\Gamma}$, we note that the location of the pole $(2\Delta - \xi_k)^{-1}$ also lies inside the unit circle except for the region with the small half-circle bump of radius ϵ_2. We then consider ξ_k lying on the small half-circle bump of $\widehat{\Gamma}$ and we write $\xi_k = \mathrm{i} + 2\Delta + \epsilon_2\, e^{\mathrm{i}\phi}$. Then, the location of the pole is given by

$$(2\Delta - \xi_k)^{-1} = (-\mathrm{i} - \epsilon_2\, e^{\mathrm{i}\phi})^{-1} = \mathrm{i} - \epsilon_2\, e^{\mathrm{i}\phi} + \mathcal{O}(\epsilon_2^2), \tag{74}$$

where the last equality follows from $0 < \epsilon_2 \ll 1$. Moreover, since $\epsilon_2 \ll \epsilon_1$, we have that the location of the pole $(2\Delta - \xi_k)^{-1}$ lies inside the large bump of the contour $\widehat{\Gamma}$, when ξ_k lies on the small bump. Then, we have that the pole $(2\Delta - \xi_k)^{-1}$ lies inside the unit circle if ξ_k doesn't lie on the small bump, and the pole lies inside the large bump if ξ_k lies on the small bump. In particular, if $\xi_k \in \mathcal{C}_R \cup \widehat{\Gamma}$, the location of the pole lies inside the contour $\widehat{\Gamma}$ and we don't cross any poles, given by the first case of (73), when we deform form the contour $\mathcal{C}_R$ to the contour $\widehat{\Gamma}$.

In the second case of (73), the location of the pole is given by $2\Delta - \zeta_k$. Additionally, we have that $\zeta_k \in \mathcal{C}_{R'}$. Given the conditions on the radii $R, R' > 0$, it follows that $R < R' - 2|\Delta|$. Then, the pole given by $2\Delta - \zeta_k$ lies outside the contour $\mathcal{C}_R$. In particular, we don't cross the pole when we deform the contour form $\mathcal{C}_R$ to $\widehat{\Gamma}$. Thus, we don't cross any poles, given by the second case of (73), when we deform the contours from $\mathcal{C}_R$ to $\widehat{\Gamma}$.

Consider now the integral with respect to $\xi_\ell \in \mathcal{C}_R$ with $\ell \in J_1$. We deform the contour $\mathcal{C}_R$ to the contour Γ_+ with $\xi_j \in \widehat{\Gamma}$ for $j \in J_2$. The location of the possible poles are given by

$$(2\Delta - \xi_j)^{-1}, \quad 2\Delta - \xi_j^{-1}, \quad \zeta_k^{-1}. \tag{75}$$

In the first case of (75), the variable ξ_j may lie on the the contours Γ_+ or $\mathcal{C}_R$, depending on the index. In particular, if $\xi_j \in \widehat{\Gamma}$, then $j = \tau_k$ for some k, see (64). Moreover, $I_N(\xi, \zeta; \tau)$ is independent of $\xi_j = \xi_{\tau_k}$ and the pole due to the $f(\xi, \zeta; \tau)$ function is of the form $(2\Delta - \xi_{\tau_k})^{-1}$; see (67). Thus, for the first case, ξ_j will never lie on the contour $\widehat{\Gamma}$ and only lie on the contours $\mathcal{C}_R$ or Γ_+. If $\xi_j \in \mathcal{C}_R$, the location of the pole $(2\Delta - \xi_j)^{-1}$ clearly lies inside the unit circle since $R - 2|\Delta| > 1$. In particular, we don't cross this pole when we deform from the contour $\mathcal{C}_R$ to the contour Γ_+, since the contour Γ_+ lies outside the unit circle. If $\xi_j \in \Gamma_+$, the location of the pole $(2\Delta - \xi_j)^{-1}$ will also lie outside the unit circle. This due to the fact the Δ is a real number and $R - 2|\Delta| > 1$. In particular, if $\xi_j \in \mathcal{C}_R \cup \Gamma_+$, we don't cross a pole, given by the first case of (75) when we deform from the contour $\mathcal{C}_R$ to the contour Γ_+.

In the second case of (75), the variable ξ_j may lie on the the contours Γ_+, $\mathcal{C}_R$, or $\widehat{\Gamma}$, depending on the index. In all three cases, we have that the $-\xi_j^{-1}$ point lies inside the unit circle since the contours lie outside the unit circle. Then, the pole $2\Delta - \xi_j^{-1}$ will lie inside Γ_+ since Δ is a real number and $2(1 + 2|\Delta|) < R$. In particular, if $\xi_j \in \mathcal{C}_R \cup \Gamma_+ \cup \widehat{\Gamma}$, we don't cross a pole, given by the second case of (75), when we deform from the contour $\mathcal{C}_R$ to the contour Γ_+.

In the third case of (75), we have $\zeta_k \in \mathcal{C}_{R'}$. Then, the location of the pole ζ_k^{-1} lies completely inside the unit circle. Then, since Γ_+ lies outside the unit circle, we don't cross a pole when we deform the contour $\mathcal{C}_R$ to the contour Γ_+.

Lastly, consider the integral with respect to $\zeta_\ell \in \mathcal{C}_{R'}$ with $\ell \in K_1$. We deform the contour $\mathcal{C}_{R'}$ to the contour Γ_-. The location of the possible poles is given by

$$(2\Delta - \zeta_j)^{-1}, \quad 2\Delta - \zeta_j^{-1}, \quad 2\Delta - \xi_k, \quad \xi_k^{-1} \tag{76}$$

where the variables may lie on different contours depending on the indexes.

In the first case of (76), the variable ζ_j may lie on the contour $\mathcal{C}_{R'}$ or on the contour Γ_-. In either case, the location of the pole lies completely inside the unit circle. When $\zeta_j \in \mathcal{C}_{R'}$, this follows from the bound $R > 2(1+2|\Delta|)$. When $\zeta_j \in \Gamma_-$, in addition the bound $R > 2(1+2|\Delta|)$, we also need the fact that Δ is a real number, which means that $(2\Delta - \zeta_j)$ lies outside the unit circle for $\zeta_j \in \Gamma_-$. Then, we have that the location of the pole $(2\Delta - \zeta_j)^{-1}$ lies completely inside the unit circle and we don't cross any poles when we deform the contour $\mathcal{C}_{R'}$ to the contour Γ_-.

In the second case of (76), the variable ζ_j may lie on the contour $\mathcal{C}_{R'}$ or on the contour Γ_-. In either case, we know that ζ_j^{-1} lies inside the unit circle since $\mathcal{C}_{R'}$ and Γ_- lie outside the unit circle. Then, since Δ is a real number and $4(1 + 2|\Delta|) < R'$, we have that the location of the pole $2\Delta - \zeta_j^{-1}$ lies completely inside the contour Γ_-. Thus, we don't cross any poles when we deform the contour $\mathcal{C}_{R'}$ to the contour Γ_-.

In the third case of (76), the variable ξ_k may lie on $\widehat{\Gamma}$ since this pole is due to the $f(\xi, \zeta; \tau)$ factor in the integrand; see (67). In this case, the location of the pole $2\Delta - \xi_k$ lies completely inside the contour Γ_- due to the bumps of the contour $\widehat{\Gamma}$. Since $0 < \epsilon \ll 1$, the large bump of the contour $\widehat{\Gamma}$ lies completely above the horizontal section of the contour Γ_-. Since the small bump in the contour $\widehat{\Gamma}$ lies inside the rectangle, the small bump will also lie completely above the V-section of the Γ_- contour. Additionally, since $R + 2|\Delta| < R'$, the rest of the contour $\widehat{\Gamma}$ will lie completely inside the contour Γ_-. Then, we don't cross any poles when we deform the contour $\mathcal{C}_{R'}$ to the contour Γ_-.

In the fourth case of (76), we have $\xi_k \in \Gamma_+$. Then, the location of the pole ξ_k^{-1} lies completely inside the unit circle, since the contour Γ_+ lies outside the unit circle. Then, since Γ_- lies outside the unit circle, we don't cross a pole when we deform the contour $\mathcal{C}_{R'}$ to the contour Γ_-.

We have now shown that we don't cross any poles in any case when we deform the contours. Thus, the result follows.

□

Proposition 1 *For $\Delta \neq 0$, $\mathcal{F}_N(x,t) = \mathbb{P}_Y(X_1(t) \geq x)$ equals*

$$\sum_{n=0}^{N} \sum_{\tau \in \mathcal{T}_n} \oint_{\Gamma_+} \cdots \oint_{\Gamma_-} I_N(\xi, \zeta; \tau) \left(\oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} f(\xi, \zeta; \tau) d^{J_2}\xi \right) d^{K_1}\zeta \, d^{J_1}\xi \tag{77}$$

where the integrand is given by (67), the sets J_1, J_2, K_1, K_2 are given by (64), the summation is take over the set of maps $\mathcal{T}_n$ given by (59), and the contours $\Gamma_\pm$ and $\widehat{\Gamma}$ are given by (50) and Fig. 3 with $R_+ = R$, $R_- = R'$ so that $\max\{2|\Delta|^{-1}, 2(1+2|\Delta|)\} < R < \max\{4|\Delta|^{-1}, 4(1+2|\Delta|)\} < R'/2$.

Proof The result is a direct consequence of Proposition 7.4 and Lemma 5. □

8 Asymptotic Analysis, a Conjecture

We believe that the formula for the probability of the left-most particle given by (71) in Theorem 7.6 may be suitable for asymptotic analysis when $t \ll N \to \infty$. Note that we have decomposed the integrand into two factors, $I_N(\xi, \zeta; \tau)$ and $f(\xi, \zeta; \tau)$. In particular, note that that the factor $f(\xi, \zeta; \tau)$ is independent of time t. Additionally, for the variables of the term $I_N(\xi, \zeta; \tau)$, we have deformed the contours to steepest descent paths. Thus, in the asymptotic limit, we expect the main contribution for the $I_N(\xi, \zeta; \tau)$ term to come from the saddle point $(\xi_0, \zeta_0) = (i, -i)$. Moreover, we expect the asymptotic limit of $I_N(\xi, \zeta; \tau)$ to be given by the Airy kernel. We give some details of the computation below but, unfortunately, we don't give all the technical details here. The arguments below need more careful consideration.

Fix $\tau \in \mathcal{T}_n$ and let's consider the contribution of the contour integrals near the saddle point. We use the following notation for the index sets:

$$K_1 := \tau^{-1}(0), \quad K_2 := (K_1)^c, \quad J_1 := \tau(K_2)^c, \quad J_2 := \tau(K_2) \tag{78}$$

The sets K_1 and K_2 will be used to index the ζ-variables and the sets J_1 and J_2 will be used to index the ξ-variables. In particular, variables with index from the sets K_1 and J_1 will lie on the contours $\Gamma_\pm$, respectively, and the variables with index from the set J_2 will lie on the contour $\widehat{\Gamma}$. There are no variables with index from the set K_2 because these variable have been integrated out, but nonetheless, this index set will appear in our formulas. Note $K_1 \cup K_2 = J_1 \cup J_2 = \{1, \ldots, N\}$.

Recall that the spectral functions G and H, given in (37), have a double critical point at $\xi = i$ and $\zeta = -i$, respectively, when $x = -2t$. Let $\mathcal{B}(z, r)$ be an open ball centered at $z \in \mathbb{C}$ of radius $r > 0$ and $\mathcal{B}(z, r)^c$ be its complement. Then, we take the following scaling

$$x = -2t - st^{1/3}, \quad \xi = \mathrm{i} + \mathrm{i}\,\tilde{\xi}\, t^{-1/3}, \quad \zeta = -\mathrm{i} + \mathrm{i}\,\tilde{\zeta}\, t^{-1/3}, \quad y_j + 1 = v_j\, t^{1/3} \tag{79}$$

if $\xi \in \mathcal{B}(i, t^{-\alpha})$ and $\zeta \in \mathcal{B}(-i, t^{-\alpha})$ with $1/4 < \alpha < 1/3$.

We also have that the integrand $I_N(\xi, \zeta; \tau)$ is exponentially small if $\xi_j \in \mathcal{B}(i, t^{-\alpha})^c$, for $j \in J_1$, or $\zeta_j \in \mathcal{B}(-i, t^{-\alpha})^c$, for $j \in K_1$. This follows from Lemma 1. Additionally, we may uniformly bound the factor $f(\xi, \zeta; \tau)$, independently of t, on all the ξ and ζ variables. Then, we may restrict the contourss $\Gamma_\pm$ to the a neighborhood around the saddle points and only lose an exponentially small term. That is,

$$\begin{aligned}
&\oint_{\Gamma_+} \cdots \oint_{\Gamma_-} I_N(\xi, \zeta; \tau) \left(\oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} f(\xi, \zeta; \tau) d^{J_2}\xi \right) d^{K_1}\zeta\, d^{J_1}\xi = \\
&\oint_{\Gamma_+ \cap \mathcal{B}(\mathrm{i}, t^{-\alpha})} \cdots \oint_{\Gamma_- \cap \mathcal{B}(-\mathrm{i}, t^{-\alpha})} I_N(\xi, \zeta; \tau) \left(\oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} f(\xi, \zeta; \tau) d^{J_2}\xi \right) d^{K_1}\zeta\, d^{J_1}\xi \\
&\qquad + \mathcal{O}(e^{-Ct^{1-3\alpha}})
\end{aligned} \tag{80}$$

for some positive constant $C > 0$, based on Lemma 1, and $1/4 < \alpha < 1/3$.

Let us now approximate the integrands $I_N(\xi, \zeta; \tau)$ and $f(\xi, \zeta; \tau)$ when $\xi_j \in \Gamma_+ \cap \mathcal{B}(i, t^{-\alpha})$, for $j \in J_1$, and $\zeta_k \in \Gamma_- \cap \mathcal{B}(-i, t^{-\alpha})$, for $k \in K_1$. In particular, we take the scaling (79) for the variables with indexes in the sets J_1 and K_1, for the ξ-variables and ζ-variables respectively.

Note that $I_N(\xi, \zeta; \tau)$ only depends on the variables with indexes from the sets K_1 and J_1. Then, we have

$$\begin{aligned}
I_N(\xi, \zeta; \tau) = (-1)^{|J_1| + |\pi(\tau)|} &\sum_{\gamma \in \mathcal{S}_N(J_2)} (-1)^{|\gamma|} \prod_{k \in \mathcal{Z}_1} \Bigg((-\mathrm{i})^{y_{\gamma(k)} - y_k} \\
&\times g(\tilde{\xi}_{\gamma(k)}, \tilde{\zeta}_k; v_{\gamma(k)}, v_k)\, t^{n/3} \Bigg) + \mathcal{O}(t^{(n-1)/3}) \\
g(\xi, \zeta; x, z) = \frac{\exp\left(\frac{1}{3}\xi^3 - \frac{1}{3}\zeta^3 - (s + x)\,\xi + (s + z)\,\zeta \right)}{(\xi - \zeta)}&,
\end{aligned} \tag{81}$$

where $\pi : \mathcal{T}_n \hookrightarrow \mathcal{S}_N$ is given by (65) and $\mathcal{S}_N(J_2)$ is given by (66). This approximation is obtained by expanding the determinant in the term $I_N(\xi, \zeta; \tau)$, given by (67) and taking the scaling (79). More details regarding this approximation are given in Appendix B of the arXiv version of this paper [22].

Now, consider the approximation of the term $f(\xi, \zeta; \tau)$ when $\xi_j \in \Gamma_+ \cap \mathcal{B}(i, t^{-\alpha})$, for $j \in J_1$, and $\zeta_k \in \Gamma_- \cap \mathcal{B}(-i, t^{-\alpha})$, for $k \in K_1$. We introduce the following function

$$B(\xi; \tau) = \prod_{\substack{j<k,\, j,k\in K_2 \\ \tau(k)<\tau(j)}} \left(\frac{1 + \xi_{\tau(k)}\xi_{\tau(j)} - 2\Delta\xi_{\tau(j)}}{1 + \xi_{\tau(k)}\xi_{\tau(j)} - 2\Delta\xi_{\tau(k)}} \right), \tag{82}$$

with the indexes $j, k \in K_2$ and $\tau(k), \tau(j) \in J_2$. Also, let us denote the number of inversions of the τ map as follows,

$$\begin{aligned} \nu_1(j; \tau) &:= \#\{ j' \in K_2 \mid j' < j, \quad \tau(j') > \tau(j) \} \\ \nu_2(j; \tau) &:= \#\{ j' \in K_2 \mid j < j', \quad \tau(j) > \tau(j') \}. \\ \nu(j; \tau) &:= j - \tau(j) + \nu_2(j; \tau) - \nu_1(j; \tau) \end{aligned} \tag{83}$$

Note that, in the case $K_2 = \{1, 2, \ldots, N\}$, we have $\nu(j; \tau) = 0$ for $j = 1, \ldots, N$. Then, by taking the scaling (79), we obtain

$$\begin{aligned} & f(\xi, \zeta; \tau) \\ & = B(\xi; \tau) \prod_{j\in K_2} \left(\frac{\xi_{\tau(j)} - (2\Delta + \mathrm{i})}{(2\mathrm{i}\Delta + 1)\xi_{\tau(j)} - \mathrm{i}} \right)^{\nu(j;\tau)} \prod_{j\in K_2} \xi_{\tau(j)}^{y_j - y_{\tau(j)} - 1} + \mathcal{O}(t^{-1/3}). \end{aligned} \tag{84}$$

This approximation is obtained by applying the scaling (79) and taking the leading term in the $t^{-1/3}$ expansion of the $f(\xi, \zeta; \tau)$ function. More details regarding this approximation are given in Appendix B of the arXiv version of this paper [22].

We now combine the approximations (80), (81) and (84), given above. Note that the leading term of the approximation (84) is independent of the $\tilde{\xi}$ and $\tilde{\zeta}$ variables. We then introduce the term

$$\begin{aligned} F(\tau) = (-\mathrm{i})^{|K_2|} \oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} B(\xi; \tau) \prod_{j\in K_2} \left(\frac{\xi_{\tau(j)} - (2\Delta + \mathrm{i})}{(2\mathrm{i}\Delta + 1)\xi_{\tau(j)} - \mathrm{i}} \right)^{\nu(j;\tau)} \\ \times \prod_{j\in K_2} (-\mathrm{i}\, \xi_{\tau(j)})^{y_j - y_{\tau(j)} - 1} d^{J_2}\xi, \end{aligned} \tag{85}$$

where we have taken the leading term of the $f(\xi, \zeta; \tau)$ function and also incorporated the $(\mathrm{i})^{y_{\gamma(k)}-y_k}$ term from the approximation of $I_N(\xi, \zeta; \tau)$ given by (81), noting that $\sum_{k\in K_1} y_{\gamma(k)} - y_k + \sum_{k\in K_2} y_{\tau(k)} - y_k = 0$. Then, for fixed $\tau \in \mathcal{T}_n$, we obtain the following approximation near the saddle point

$$\begin{aligned}
&\oint_{\Gamma_+} \cdots \oint_{\Gamma_-} I_N(\xi, \zeta; \tau) \left(\oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} f(\xi, \zeta; \tau) d^{J_2}\xi \right) d^{K_1}\zeta \, d^{J_1}\xi \\
&= t^{-n/3}(-1)^{|J_1|+|\pi(\tau)|} \sum_{\gamma\in\mathcal{S}_N(J_2)} \left(F(\tau)(-1)^{|\gamma|} \prod_{k\in K_1} \mathbf{K}_{Ai}\left(s + v_{\gamma(k)}, s + v_k\right) \right) \\
&\qquad + \mathcal{O}(t^{(1-n)/3}) + \mathcal{O}(e^{-Ct^{1-3\alpha}}).
\end{aligned} \tag{86}$$

The $t^{-n/3}$ term and the Airy kernel $\mathbf{K}_{Ai}$ are obtained by taking the change of variables (79) and the following expression for the Airy kernel

$$\mathbf{K}_{Ai}(x, z) = \int_{\infty e^{-2\pi \mathrm{i}/3}}^{\infty e^{2\pi \mathrm{i}/3}} \int_{\infty e^{-\pi \mathrm{i}/3}}^{\infty e^{\pi \mathrm{i}/3}} \frac{\exp\left(\frac{1}{3}\xi^3 - \frac{1}{3}\zeta^3 - x\,\xi + z\,\zeta\right)}{\xi - \zeta} d\xi \, d\zeta, \tag{87}$$

where the contours for the ξ (resp. ζ) variable starts at $\infty\, e^{-\pi i/3}$ (resp. $\infty\, e^{-2\pi i/3}$) goes through the origin and ends at $\infty\, e^{\pi i/3}$ (resp. $\infty\, e^{2\pi i/3}$).

Let's now consider the formula (77) and, in particular, the summation over $\mathcal{T}_n$ and n. We substitute the term in the summation by the right side of the approximation (86). The result is a summation over $\mathcal{T}_n$, n, and injective maps $\gamma : K_1 \to J_1$. More precisely, the summation is over a pair of bijective maps

$$\tau : K_2 \to J_2, \quad \gamma : K_1 \to J_1, \tag{88}$$

where $K_1 \cup K_2 = J_1 \cup J_2 = \{1, 2, \ldots, N\}$. This means that we may write the summation, over $\mathcal{T}_n$, n and the injective maps $\gamma : K_1 \to J_1$ and $\tau : K_2 \to J_2$, as the summation over permutations of the set $[N] = \{1, 2, \ldots, N\}$. In particular, we may uniquely identify a pair of bijective maps (τ, γ) with a permutation $\sigma \in \mathcal{S}_N$ and a subset $S \subset [N]$ so that $(\tau, \gamma) = (\sigma|_{S^c}, \sigma|_S)$, where the right side are restrictions of the permutation to the indicated sets. Then, under this identification, we rewrite some the notation introduced earlier. For (σ, S) with $\sigma|_{S^c} = \tau$, we have

$$B(\xi; \sigma, S) = B(\xi; \tau) = \prod_{\substack{j,k\in S^c, j<k \\ \sigma(j)>\sigma(k)}} \left(\frac{1 + \xi_{\sigma(k)}\xi_{\sigma(j)} - 2\Delta\xi_{\sigma(j)}}{1 + \xi_{\sigma(k)}\xi_{\sigma(j)} - 2\Delta\xi_{\sigma(k)}} \right). \tag{89}$$

Additionally, for (σ, S) with $\sigma|_{S^c} = \tau$, we write the inversion sets as follows,

$$\begin{aligned} \nu_1(j;\sigma,S) &= \nu_1(j;\tau) = \#\{j' \in S^c \mid j' < j, \quad \sigma(j') > \sigma(j)\,\} \\ \nu_2(j;\sigma,S) &= \nu_2(j;\tau) = \#\{j' \in S^c \mid j < j', \quad \sigma(j) > \sigma(j')\,\}. \\ \nu(j;\sigma,S) &= \nu(j;\tau) = j - \sigma(j) + \nu_2(j;\sigma,S) - \nu_1(j;\sigma,S). \end{aligned} \tag{90}$$

Lastly, for (σ, S) with $\sigma|_{S^c} = \tau$, we write

$$\begin{aligned} &F(\sigma,S) = F(\tau) \\ &= (-\mathrm{i})^{|S^c|} \oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} B(\xi;\sigma,S) \prod_{j\in S^c} \left(\frac{\xi_{\tau(j)} - (2\Delta + \mathrm{i})}{(2\mathrm{i}\Delta + 1)\xi_{\tau(j)} - \mathrm{i}} \right)^{\nu(j;\sigma,S)} \\ &\qquad\qquad \times \prod_{j\in S^c} (-\mathrm{i}\,\xi_{\tau(j)})^{y_j - y_{\tau(j)} - 1} d^{\sigma(S^c)}\xi. \end{aligned} \tag{91}$$

Then, under the identification of the pair of injective maps and the permutations, we have

$$\begin{aligned} &\sum_{n=0}^{N} \sum_{\tau\in\mathcal{T}_n} \oint_{\Gamma_+} \cdots \oint_{\Gamma_-} I_N(\xi,\zeta;\tau) \left(\oint_{\widehat{\Gamma}} \cdots \oint_{\widehat{\Gamma}} f(\xi,\zeta;\tau) d^{J_2}\xi \right) d^{K_1}\zeta\, d^{J_1}\xi \\ &= \sum_{\sigma\in\mathcal{S}_N} (-1)^{\sigma} \sum_{S\subset[N]} (-1)^{|S|} t^{-|S|/3} \Bigg(F(\sigma,S) \\ &\quad \times \prod_{k\in S} \mathbf{K}_{Ai} \left(s + \frac{y_{\sigma(k)}+1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) + \mathcal{O}\left(t^{-1/3}\right) + \mathcal{O}\left(e^{-Ct^{1-3\alpha}}\right) \Bigg) \end{aligned} \tag{92}$$

Assuming that the error terms don't contribute in the limit, we have the following conjecture.

Conjecture 1 As $t \ll N \to \infty$, $\mathcal{F}_N(x,t) = \mathbb{P}_Y(X_1(t) \geq x)$, with $x = -2t - s\,t^{-1/3}$ and $y_j + 1 = v_j\, t^{1/3}$, is equal to the limit of

$$\sum_{\sigma\in\mathcal{S}_N} (-1)^{\sigma} \sum_{S\subset[N]} (-1)^{|S|} t^{-|S|/3} F(\sigma,S) \prod_{k\in S} \mathbf{K}_{Ai}\left(s + v_{\sigma(k)}, s + v_k\right) \tag{93}$$

where F is given by (85) and the Airy kernel $\mathbf{K}_{Ai}$ is given by (87).

At the moment, we are not able to control the limit of (93) when $t \ll N \to \infty$. The main obstacle is the term $F(\sigma, S)$ on (93). However, under some assumptions, we may simplify (93) as a determinant of the difference of two kernels. For instance, assume

$$F(\sigma, S) = \prod_{j \in S^c} \mathbf{Q}(\sigma(j), j) \tag{94}$$

for some kernel $\mathbf{Q}$ on the set $\{1, \ldots, N\}$. Then, we have

$$\begin{aligned} &\sum_{\sigma \in \mathcal{S}_N} (-1)^\sigma \sum_{S \subset [N]} (-1)^{|S|} t^{-|S|/3} F(\sigma, S) \prod_{k \in S} \mathbf{K}_{Ai} \left(s + \frac{y_{\sigma(k)} + 1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) \\ &= \sum_{\sigma \in \mathcal{S}_N} (-1)^\sigma \sum_{S \subset [N]} (-1)^{|S|} t^{-|S|/3} \prod_{j \in S^c} \mathbf{Q}(\sigma(j), j) \\ &\qquad \times \prod_{k \in S} \mathbf{K}_{Ai} \left(s + \frac{y_{\sigma(k)} + 1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) \qquad (95) \\ &= \sum_{\sigma \in \mathcal{S}_N} (-1)^\sigma \prod_{k=1}^{N} \left(\mathbf{Q}(\sigma(k), k) - t^{-1/3} \mathbf{K}_{Ai} \left(s + \frac{y_{\sigma(k)} + 1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) \right) \\ &= \det \left(\mathbf{Q}(j, k) - t^{-1/3} \mathbf{K}_{Ai} \left(s + \frac{y_j + 1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) \right)_{j,k=1}^{N}, \end{aligned}$$

given the assumption (94). In fact, when $\Delta = 0$, one may check the assumption to be true and we have

$$F(\sigma, S) = \mathbb{1}\left(\sigma|_{S^c} = \mathrm{Id}_{S^c}\right) = \prod_{j \in S^c} \mathbb{1}(\sigma(j) = j), \tag{96}$$

where the functions with $\mathbb{1}$ are indicator functions. This identity is easy to check since the first two terms in the intergand for $F(\sigma, S)$, given by (85), are identically equal to one when $\Delta = 0$. Then, we have

$$\begin{aligned} &\det \left(\mathbf{Id}(j, k) - t^{-1/3} \mathbf{K}_{Ai} \left(s + \frac{y_j + 1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) \right)_{j,k=1}^{N} \\ &= \sum_{\sigma \in \mathcal{S}_N} (-1)^\sigma \sum_{S \subset [N]} \left((-1)^{|S|} t^{-|S|/3} F(\sigma, S) \right. \\ &\qquad \left. \times \prod_{k \in S} \mathbf{K}_{Ai} \left(s + \frac{y_{\sigma(k)} + 1}{t^{1/3}}, s + \frac{y_k + 1}{t^{1/3}} \right) \right) \end{aligned} \tag{97}$$

when $\Delta = 0$. This means that Conjecture 1 is true when $\Delta = 0$. Moreover, if $\Delta = 0$ and $y_j = j$, we may take the limit $t \ll N \to \infty$. The right side becomes a sum of Riemann integrals, corresponding to the series expansion of a Fredholm determinant. Then, we have

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}_Y\left(\frac{X_1(t)+2t}{t^{1/3}} \geq -s\right)\\
&= \lim_{t\ll N\to\infty} \sum_{\sigma\in\mathcal{S}_N}\sum_{S\subset[N]} t^{-|S|/3} \det\left(\mathbf{K}_{Ai}\left(s+\frac{j+1}{t^{1/3}}, s+\frac{k+1}{t^{1/3}}\right)\right)_{j,k\in S} \qquad (98)\\
&= \det\left(\mathbf{Id}-\mathbf{K}_{Ai}\right)_{L^2(s,\infty)}\\
&= F_2(s).
\end{aligned}$$

This matches the earlier result (36) for $\Delta = 0$.

We also may compute the terms in (93) when $S = \emptyset$ and $S^c = \{1, \dots, N\} = [N]$. In that case, the formula for $F(\sigma, \emptyset)$ simplifies as follows

$$F(\sigma,\emptyset) = \oint_{\widehat{\Gamma}}\cdots\oint_{\widehat{\Gamma}} \prod_{\substack{j<k\\ \sigma(k)<\sigma(j)}} \left(\frac{1+\xi_{\tau(k)}\xi_{\tau(j)}-2\Delta\xi_{\tau(j)}}{1+\xi_{\tau(k)}\xi_{\tau(j)}-2\Delta\xi_{\tau(k)}}\right) \prod_{j=1}^{N} \xi_{\tau(j)}^{y_j-y_{\tau(j)}-1} d^N\xi, \tag{99}$$

where $i, j = 1, 2, \dots, N$ on the first product of the integrand. Additionally, we may deform the contours $\widehat{\Gamma}$ to arbitrarily large circles centered at the origin. Note that $(-1)^\sigma F(\sigma, \emptyset)$ is equal to the integral inside the sum of (15) with $x_i = y_i$, for $i = 1, \dots, N$, and $t = 0$. Then, by Theorem 2a, we have

$$\sum_{\sigma\in\mathcal{S}_N} (-1)^\sigma F(\sigma,\emptyset) = 1 \tag{100}$$

for any $N > 0$.

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Harold Widom's Contributions to the Spectral Theory and Asymptotics of Toeplitz Operators and Matrices

Estelle Basor, Albrecht Böttcher, and Torsten Ehrhardt

Abstract This is a survey of Harold Widom's work in the spectral theory of Toeplitz and Wiener-Hopf operators and on asymptotic problems for truncations of these operators as the truncation parameter goes to infinity. The asymptotic problems include Toeplitz and Wiener-Hopf determinants, extreme eigenvalues, and collective eigenvalue distribution. Harold Widom has made groundbreaking contributions to all these topics.

Keywords Toeplitz operators · Toeplitz matrices · Wiener-Hopf operators

1 Toeplitz Matrices and Operators

An $n \times n$ Toeplitz matrix is an $n \times n$ matrix that is constant along its diagonals, that is, a matrix of the form

$$(a_{j-k})_{j,k=1}^{n} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \dots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \dots & a_{-(n-3)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}_{n\times n}.$$

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The entries are complex numbers. If the entries are $N \times N$ matrices, and hence the matrix actually has dimension nN, one speaks of block Toeplitz matrices. An infinite Toeplitz matrix is a matrix of the form

$$(a_{j-k})_{j,k=1}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

We may think of the infinite matrix as a linear operator acting on $\ell^2 := \ell^2(\mathbf{N})$, and the first question that arises is to characterize the sequences $\{a_k\}_{k\in\mathbf{Z}}$ for which this operator is bounded. This question was answered by Otto Toeplitz in (a footnote of) his 1911 paper [43]. He showed that the infinite matrix induces a bounded linear operator on ℓ^2 if and only if the numbers a_k are the Fourier coefficients of a function $a \in L^\infty$ on the complex unit circle $\mathbf{T}$:

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta}\, d\theta \quad (k \in \mathbf{Z}).$$

If such a function a exists, it is unique. We denote the infinite matrix as well as the bounded linear operator it induces on ℓ^2 by $T(a)$ and the finite matrix, which may be regarded as the principal $n \times n$ truncation of the infinite matrix, by $T_n(a)$. The function a is in this context referred to as the symbol. Clearly, we may even chose a from L^1 on the unit circle, take the Fourier coefficients, and built the matrices $T_n(a)$ and $T(a)$, but in that case $T(a)$ need not generate a bounded operator.

The next question to ask after boundedness is about the spectrum of the operator $T(a)$. Since

$$T(a) - \lambda I = T(a - \lambda),$$

this question essentially amounts to finding invertibility criteria for Toeplitz operators. A simpler problem is to study invertibility modulo compact operators, which is the task of Fredholm theory for Toeplitz operators. These topics have been investigated for a century, and the seminal contributions made to them by Harold Widom will be the subject of this paper.

Questions about explicitly given finite Toeplitz matrices $T_n(a)$ may nowadays quickly be answered by the computer if n is of moderate size. Things become mathematically interesting if the matrix dimension n is large or unspecified or if the matrix involves parameters. It was only a few years after Toeplitz when Gábor Szegő came across the problem of describing the asymptotic behavior of the determinants of $T_n(a)$ as n goes to infinity. Further questions have led into the study of the behavior of inverses and of the eigenvalues of $T_n(a)$ for large n or for n tending to infinity. It was again Harold Widom who brought us fundamental insights and results in this connection. These will be described in the following pages.

The present paper is in part based on our article [2].

2 Toeplitz Operators with Continuous Symbols

As for invertibility and Fredholmness, it was not primarily Toeplitz operators but rather their relatives that were studied in the first half of the previous century. The main relatives are the operators coming from the Riemann-Hilbert boundary value problem, singular integral operators, and Wiener-Hopf integral operators. Many mathematicians, including F. Noether, J. Plemelj, S. G. Mikhlin, G. Fichera, and T. Carleman, studied singular integral operators with continuous coefficients. Stated in terms of the Toeplitz operator $T(a)$, their results say that if a is a continuous function on $\mathbf{T}$, then for for $T(a)$ to be Fredholm it is sufficient that a have no zeros on $\mathbf{T}$ and for $T(a)$ to be invertible it is sufficient that a have no zeros on $\mathbf{T}$ and that the winding number of a about the origin be zero. Only in 1952, Israel Gohberg, by an ingenious application of Gelfand theory of Banach algebras, was able to prove that these sufficient conditions are also necessary.

Inversion of Toeplitz operators or the description of the kernel and co-kernel in the case where the operator is not invertible were then mainly tackled by versions of what is now called Wiener-Hopf factorization. The first variant of such a factorization appeared in a 1931 paper by Norbert Wiener and Eberhard Hopf. A complete understanding of that method was gained only in the works of F. D. Gakhov in 1949 and I. Gohberg and Mark Krein in the 1950s. In the language of Toeplitz operators, this factorization amounts to factoring

$$a(t) = a_-(t)t^\kappa a_+(t) \quad (t \in \mathbf{T})$$

with invertible analytic and anti-analytic factors a_+ and a_- and with $\kappa \in \mathbf{Z}$. This gives the representation

$$T(a) = T(a_-)T(t^\kappa)T(a_+)$$

with the upper triangular Toeplitz matrix $T(a_-)$, the lower triangular Toeplitz matrix $T(a_+)$, and a "middle factor" $T(t^\kappa)$, which is the Toeplitz matrix whose κth diagonal consists of ones and the remaining diagonals of which are zero. The matrix $T(a)$ generates an invertible operator if and only if $\kappa = 0$, that is, if and only if the middle factor is absent. In that case the inverse $T^{-1}(a) := [T(a)]^{-1}$ is given by $T^{-1}(a) = T(a_+^{-1})T(a_-^{-1})$.

Harold Widom entered the Toeplitz operators scene with his 1959 paper [20], jointly with Alberto Calderón and Frank Spitzer. This paper deals with Toeplitz operators $T(a)$ generated by symbols a in the Wiener algebra, that is, by symbols a satisfying $\sum |a_k| < \infty$. Clearly, such symbols are continuous. The point is that for functions in Wiener algebra one can write down explicit formulas for the Wiener-Hopf factors. Namely, if a has no zeros on $\mathbf{T}$ and $\kappa \in \mathbf{Z}$ is the winding number about the origin, then $a(t)t^{-\kappa} = e^{b(t)}$ with a function b in the Wiener algebra, which gives $a(t) = a_-(t)t^\kappa a_+(t)$ with

$$a_-(t) = \exp \sum_{k<0} b_k t^k, \quad a_+(t) = \exp \sum_{k\geq 0} b_k t^k.$$

Using this factorization, Calderón, Spitzer, and Widom considered $T(a)$ as an operator on ℓ^∞ and on ℓ^2, and they show that in both contexts $T(a)$ is invertible if and only if a has no zeros on $\mathbf{T}$ and winding number zero about the origin. The paper was submitted in May 1958, and in a note added in proof, the authors remark that a substantial part of their results are also in a 1958 paper by M. Krein.

3 Toeplitz Operators with L^∞ Symbols

Fortunately, one theorem of [20] was not in Krein's paper: it replaces the condition $\sum |a_k| < \infty$ by the sole requirement that $a \in L^\infty(\mathbf{T})$ and says that for $T(a)$ to be invertible on ℓ^2 it is sufficient that a is invertible in $L^\infty(\mathbf{T})$ and $a/|a| = \exp(i\,\widetilde{v})$ with a real-valued $v \in L^\infty(\mathbf{T})$ and with $\widetilde{v}$ denoting the conjugate function of v. What a great first step into the depth of L^∞! For example, if ω is a conformal map of the open unit disk onto the region $\{x + iy : y > |\tan x|, -\pi/2 < x < \pi/2\}$, then, considering the boundary values of ω, the real part $v = \operatorname{Re}\omega$ is in L^∞ while the imaginary part $\widetilde{v} = \operatorname{Im}\omega$ is unbounded, so that $T(e^{i\,\widetilde{v}})$ is an invertible operator with a heavily oscillating symbol.

In August 1958, Widom submitted his 1960 paper [45], which laid the foundations for the invertibility theory of Toeplitz operators on ℓ^2. The paper has four theorems. In Theorems II and III, unaware of previous work by A. Wintner (1929) and P. Hartman and A. Wintner (1954), he rediscovered their invertibility criteria for triangular and Hermitian Toeplitz matrices. Theorem I was a real breakthrough. It states that for $T(a)$ to be invertible it is necessary and sufficient that $a = a_-a_+$ with $a_-^{\pm 1} \in L^2_-(\mathbf{T})$, $a_+^{\pm 1} \in L^2_+(\mathbf{T})$ such that the operator $f \mapsto a_+^{-1} P a_-^{-1} f$ is bounded on $L^2(\mathbf{T})$. Here $L^2_\pm(\mathbf{T})$ are the usual Hardy spaces and P is the orthogonal projection of $L^2(\mathbf{T})$ onto $L^2_+(\mathbf{T})$. Note that $P = (I+S)/2$ where S is the Cauchy singular integral operator. He understood that this is a question about the weights w for which S is bounded on $L^p(\mathbf{T}, w)$. It was a lucky tie of events that just at that time, in 1960, H. Helson and G. Szegő were able to characterize these weights. Combining his Theorem I and the Helson-Szegő theorem, Widom arrived at the conclusion that $T(a)$ is invertible if *and only if* a is invertible in L^∞ and

$$a/|a| = \exp(i(c + u + \widetilde{v})),$$

where c is a real constant, u and v are two real-valued functions in $L^\infty(\mathbf{T})$, and $\|u\|_\infty < \pi/2$. This beautiful result, which was published in 1960 by Widom in

[46] and was rediscovered by Allen Devinatz in 1964, entered the text books as the Widom-Devinatz theorem.

We should mention that an essential generalization of Widom's Theorem I, namely, its extension to Toeplitz operators with matrix-valued symbols on the Hardy spaces $L^p_+(\mathbf{T})$ was independently discovered by Igor Simonenko in 1961. In fact several basic results on Toeplitz operators which nowadays appear on the first pages of the textbooks were established just around 1960 and tracing back to the sources of these results is a subtle matter. Those years were turbulent times. For example, the Brown-Halmos theorem, according to which the spectrum of $T(a)$ is a subset of the convex hull of the essential range of a, though explicitly published for the first time by P. Halmos and A. Brown in 1963, was known to at least Widom and Igor Simonenko already in 1960. As for Widom, the theorem is in his article [51], which is based on lectures at the IAS in 1960. We also remark that in the very early 1960s, Simonenko [41, 42] already had the results of [26] on locally sectorial symbols and the theorem that a Toeplitz operator is invertible if and only if it is Fredholm of index zero, which was published by Lewis Coburn in 1967 and is known as Coburn's lemma since then.

4 Toeplitz Operators with Piecewise Continuous Symbols

Bounded piecewise continuous functions are in L^∞ and hence covered by the previous section. So why a new section about them? The point is that we were cheating in Sect. 1 when saying that due to the equality $T(a) - \lambda I = T(a - \lambda)$, the description of the spectrum of $T(a)$ essentially amounts to finding invertibility criteria for Toeplitz operators. The results of Sect. 3 solve the invertibility problem for Toeplitz operators with arbitrary L^∞ symbols completely but in *analytical language*. In contrast to this, the nice index zero condition quoted in Sect. 2 gives an answer not only to invertibility but also a description of the spectrum in purely *geometric language*: if a is continuous, then the spectrum of $T(a)$ is the union of the curve $a(\mathbf{T})$ and of all points in the plane that are encircled by this curve with nonzero winding number.

Let's come back to Widom's paper [45]. Theorem IV in it concerns the case where $a \in L^\infty$ is piecewise continuous with at most finitely many jumps. Consider the continuous and naturally oriented curve in the plane that arises from the essential range of a by filling in line segments between the endpoints $a(t-0)$ and $a(t+0)$ of each jump. Widom proved that $T(a)$ is invertible on ℓ^2 if and only if this curve does not contain the origin and has winding number zero about the origin. This delivers a geometrical description of the spectrum of $T(a)$ analogous to the case of continuous symbols, the only difference being that instead of the curve $a(\mathbf{T})$ one has to take the curve that arises from the essential range after filling in line segment between the endpoints of the jumps. How beautiful!

In fact, Theorem IV of [45] was the very beginning of a long and fascinating story. The first chapter of this story was written by none other than Widom himself in [47]. The space ℓ^2 may be naturally identified with the Hardy space $H^2 = L^2_+$ of the unit circle (equivalently, of the unit disk). Consequently, the ℓ^2 theory of Toeplitz operators bifurcates into the ℓ^p and L^p theories for $1 < p < \infty$. The latter two theories are based on completely different techniques although, and this is something of a mystery, in the case of piecewise continuous symbols the final results are almost identical. In [47], Widom studied Toeplitz operators $T(a)$ with piecewise continuous symbols a on the Hardy space $L^p_+(\mathbf{R})$ of the upper half-plane. These operators are defined by $f \mapsto P(af)$ where $P = (I + S)/2$ and S is the Cauchy singular integral operator on $L^p(\mathbf{R})$. (One could equally well work on the Hardy space $H^p = L^p_+(\mathbf{T})$, the differences being only technical and psychological.) Widom again arrived at the boundedness of $f \mapsto a_+^{-1} P a_-^{-1} f$ on $L^p_+(\mathbf{R})$. This time it is the question about the weights w for which S is bounded on $L^p(\mathbf{R}, w)$. He showed that S is bounded if $w(x) = (1 + |x|)^\alpha \prod_{k=1}^m |x - x_k|^{\alpha_k}$ with

$$-1/p < \alpha_k < 1/q \quad \text{and} \quad -1/p < \alpha + \sum_{k=1}^m \alpha_k < 1/q,$$

where $1/p + 1/q = 1$. Using this insight, he was able to prove that $T(a)$ is invertible on $L^p_+(\mathbf{R})$ if and only if a certain curve does not contain the origin and has winding number zero about the origin. This curve results from the essential range of a by filling in certain circular arcs $\mathcal{A}_p(a(x-0), a(x+0))$ depending on p between the endpoints of the jumps at $x \in \mathbf{R}$ and the arc $\mathcal{A}_q(a(+\infty), a(-\infty))$ for the jump at infinity. Here, for two distinct points $\alpha, \beta \in \mathbf{C}$ and a number $r \in (1, \infty)$, we denote by $\mathcal{A}_r(\alpha, \beta)$ the circular arc at the points of which the line segment $[\alpha, \beta]$ is seen at the angle $2\pi/\max\{r, s\}$, where $1/r + 1/s = 1$, and which lies on the right (resp. left) of the oriented line passing first α and then β if $1 < r < 2$ (resp. $2 < r < \infty$). For $r = 2$, $\mathcal{A}_r(\alpha, \beta)$ is simply the line segment $[\alpha, \beta]$. In formulas,

$$\mathcal{A}_r(\alpha, \beta) = \{\alpha, \beta\} \cup \left\{ z \neq \alpha, \beta : \frac{1}{2\pi} \arg \frac{z-a}{z-b} \in \frac{1}{r} + \mathbf{Z} \right\}.$$

A parametric representation of $\mathcal{A}_r(\alpha, \beta)$ is

$$z(\mu) = \alpha + \sigma_r(\mu)(\beta - \alpha), \quad 0 \le \mu \le 1,$$

where $\sigma_r(\mu) = \mu$ for $r = 2$ and

$$\sigma_r(\mu) = \frac{\sin(\theta\mu)\exp(i\theta\mu)}{\sin(\theta)\exp(i\theta)} \text{ with } \theta = \pi\left(\frac{1}{r} - \frac{1}{s}\right)$$

for $r \neq 2$. For example, if $a(x) = \operatorname{sign} x$, then we have two circular arcs $\mathcal{A}_p(-1, 1)$ and $\mathcal{A}_q(1, -1)$, and since $\mathcal{A}_q(1, -1) = \mathcal{A}_p(-1, 1)$, it follows that $T(\text{sign})$ is invertible if and only if $p \neq 2$. Widom also computed the kernel and co-kernel dimensions of the operators if the curve has nonzero winding number. Overall, paper [47] contained the full Fredholm theory of Toeplitz operators with piecewise continuous symbols on $L^p_+(\mathbf{R})$, including an index formula.

In different language, particular cases of the Fredholm results of [47] were already evident in papers by B. V. Hvedelidze since 1947. The characterization of the weights w for which S is bounded on $L^p(\Gamma, w)$ has a long history, starting with G. H. Hardy and J. E. Littlewood and culminating with work by R. Hunt, B. Muckenhoupt, R. Wheeden (1973), A. Calderón (1977), G. David (1984). In the late 1960s and the 1970s, I. Gohberg and N. Krupnik introduced their local principle by means of which they could not only give a simpler proof of Widom's result but also consider Lyapunov curves Γ with power weights w, the case of matrix-valued symbols, and Banach algebras generated by Toeplitz operators with piecewise continuous symbols. In 1972, R. Duduchava settled matters for Toeplitz operators on ℓ^p. The theory reached a certain final stage only in the 1990s by work of I. Spitkovsky (general weights w) and Yu. I. Karlovich and the second author (general curves Γ and general weights w). In these more general situations, Harold Widom's circular arcs undergo a metamorphosis into horns, logarithmic double-spirals, spiralic horns, and eventually into leaves with a halo. See the book [11].

The results on Toeplitz operators with continuous or piecewise continuous symbols we have cited imply that their spectrum and essential spectrum are connected sets. (The essential spectrum of an operator T is the set of all complex λ for which $T - \lambda I$ is not Fredholm, that is, not invertible modulo compact operators. In the case of a continuous symbol a, the essential spectrum of $T(a)$ is simply the curve $a(\mathbf{T})$, and for a piecewise continuous symbol, it is the essential range with connected sets filled in between the endpoints of the jumps.) In 1963, Paul Halmos posed the question whether the spectrum of $T(a)$ is connected for every $a \in L^\infty(\mathbf{T})$. In [50], submitted in April 1963, Widom proved that the answer is *Yes* for the spectrum of Toeplitz operators on ℓ^2, and in his paper [52] of 1966, he performed the same feat for Toeplitz operators on $L^p_+(\mathbf{T})$. In 1972, Ronald Douglas established the connectedness of the essential spectrum of Toeplitz operators on ℓ^2, and only in 2009, A. Yu. Karlovich and I. Spitkovsky [35] were able to prove that both the spectrum and the essential spectrum of Toeplitz operators are always connected on $L^p_+(\Gamma, w)$ for $1 < p < \infty$ and general curves Γ and weights w. In Fig. 1 we see some of the mathematicians whose names we encountered above. Figs. 2, 3, 4, scattered over the rest of this paper, show Harold Widom in the years 1969, 1985, 2002.

5 Asymptotics of Extreme Eigenvalues

Extreme eigenvalues of Hermitian Toeplitz matrices have been studied at least since Kac, Murdock, and Szegő's work in the 1950s. In the 1960s, Seymor Parter undertook the matter a thorough analysis and established a series of deep results. As Harold told us, there was an agreement between Parter and him that Parter should focus on the Toeplitz case while he would embark on integral operators, that is, on the Wiener-Hopf case.

Fig. 1 Joint German-Israeli workshop "Linear and one-dimensional singular integral equations" in Tel Aviv in March 1995. From the left to the right: Bernd Silbermann, Harold Widom, Joe Ball, Amelia Ball, Ludmilla Meister, Erhard Meister, Asya Moiseyevna Vishik, Albrecht Böttcher, Mark Vishik, Lothar von Wolfersdorf, unknown, Yuri Karlovich, Luise Blank, Uri Toeplitz (a son of Otto Toeplitz), Naum Krupnik, Elias Wegert, Victor Katsnelson, Steffen Roch, Israel Gohberg, Rien Kaashoek, Efim Spigel, Asher Ben-Artzi, Israel Feldman, Ilya Spitkovsky, Yan Zucker, Johannes Elschner, Victor Vinnikov, Vladimir A. Marchenko.

Consider the integral operators W_τ given by

$$(W_\tau f)(x) = \int_0^\tau k(x-y)f(y)\,dy, \quad x \in (0,\tau),$$

on $L^2(0,\tau)$. These operators are the continuous analogue of finite Toeplitz matrices. We may think of W_τ as the compression to $L^2(0,\tau)$ of the Wiener-Hopf integral operator defined by

$$(Wf)(x) = \int_0^\infty k(x-y)f(y)\,dy, \quad x \in (0,\infty),$$

on $L^2(0,\infty)$. Clearly, W is the continuous analogue of an infinite Toeplitz matrix. The symbol of the operators at hand is the Fourier transform of the function k,

$$\hat{k}(\xi) := \int_{-\infty}^\infty k(x)e^{i\xi x}\,dx, \quad \xi \in \mathbf{R}.$$

Of interest is the case in which the function k is real-valued and even and in $L^1(\mathbf{R})$. In that case W_τ is a compact Hermitian operator and we may label the upper eigenvalues as $\lambda_1(W_\tau) \geq \lambda_2(W_\tau) \geq \dots$. As predicted by Kac, Murdock, Szegő, and Parter in the discrete case, the asymptotic behavior of $\lambda_j(W_\tau)$ for fixed j and for τ going to infinity depends heavily on the behavior of the symbol $\hat{k}$ near its maximum. Suppose that the maximal value is 1 and that it is attained at $\xi = 0$ and only there. Under the assumption that

$$\hat{k}(\xi) = 1 - c|\xi|^\alpha + o(|\xi|^\alpha) \ \text{ as } \ \xi \to 0$$

and that some more minor technical conditions are satisfied, Widom proved that

$$\lambda_j(W_\tau) = 1 - \frac{c}{\mu_{j,\alpha}} \frac{1}{\tau^\alpha} + o\left(\frac{1}{\tau^\alpha}\right) \ \text{ as } \ \tau \to \infty,$$

where the $\mu_{j,\alpha}$ are certain constants. For $\alpha = 2$, this was done in his 1958 paper [44], where he even improved the $o(1/\tau^2)$ to $\nu_{j,\alpha}/\tau^3 + o(1/\tau^3)$. Papers [48] and [49] of 1961 are for general $\alpha \in (0, \infty)$. The constants $\mu_{j,\alpha}$ are shown to be the eigenvalues of a certain positive definite integral operator with some kernel $K_\alpha(x, y)$ on $L^2(-1, 1)$. If $\alpha = 2k$ is an even natural number, then $K_\alpha(x, y)$ is Green's function of the differential operator $u \mapsto (-1)^k u^{(2k)}$ on $(-1, 1)$ with the boundary conditions $u^{(\ell)}(-1) = u^{(\ell)}(1) = 0$ for $0 \leq \ell \leq k - 1$.

To prove these results, Widom derives a formula for the determinants of banded Toeplitz matrices and some kind of an analogue of this formula for integral operators. Subtracting λI and putting the resulting determinants zero, he gets the eigenvalues, and a clever approximation argument then yields the desired result.

Widom's formula for the determinants of banded Toeplitz matrices is of interest by itself. So let us cite the formula here in the form presented by Schmidt and Spitzer in [38]. The formula along with a full proof is also contained as Theorem 2.8 in the book [10]. Let

$$a(t) = \sum_{j=-r}^{s} a_j t^j \quad (t \in \mathbf{T})$$

with $r \geq 1$, $s \geq 1$, $a_{-r} \neq 0$, $a_s \neq 0$, and let $z_1, \dots, z_{r+s}$ be the zeros of the polynomial $z^r a(z) = a_{-r} + a_{-r+1} z + \dots + a_s z^{r+s}$. If these zeros are pairwise distinct, then

$$\det T_n(a) = \sum_M C_M w_M^n$$

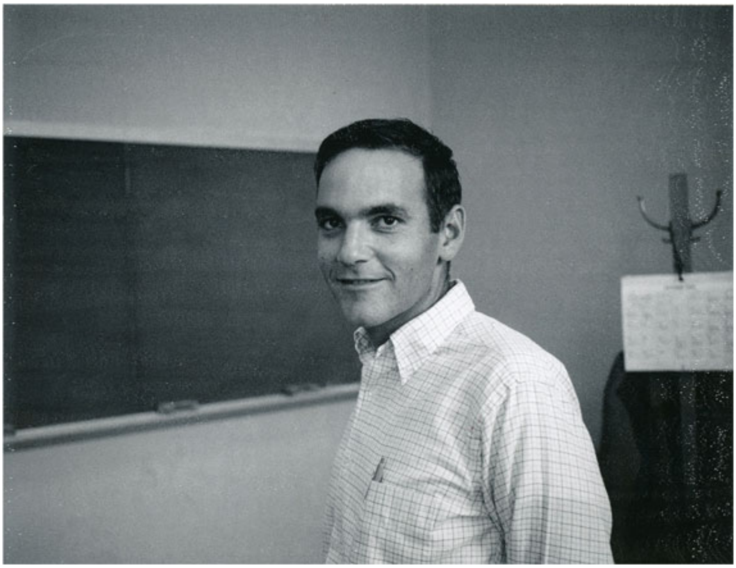

Fig. 2 Harold Widom in 1969. (Photo by Paul Halmos.).

for every $n \geq 1$, where the sum is taken over all $\binom{r+s}{s}$ subsets $M \subset \{1, 2, \dots, r+s\}$ of cardinality $|M| = s$ and

$$w_M = (-1)^s a_s \prod_{j \in M} z_j, \quad C_M = \prod_{j \in M} z_j^r \prod_{j \in M, k \notin M} (z_j - z_k)^{-1}.$$

This formula came to full effect in Schmidt and Spitzer's paper [38]. We here confine ourselves to a nice application of the formula in connection with a periodicity phenomenon for Toeplitz determinants. Consider the Toeplitz matrices with the symbol

$$a(t) = \sum_{k=-r}^{s} t^k = t^{-r} \frac{t^{r+s+1} - 1}{t - 1}.$$

For sufficiently large n, $T_n(a)$ has $r + 1$ ones followed by zeros in the first row and $s + 1$ ones followed by zeros in the first column. Since $z_j^{n+r+s+1} = z_j^n$ for the roots of the polynomial $z^r a(z)$, Widom's formula immediately yields

$$\det T_{n+r+s+1}(a) = (-1)^{s(r+s+1)} \det T_n(a) = (-1)^{rs} \det T_n(a).$$

It follows that $\det T_n(a)$ has the period $r + s + 1$ if r or s is even and that the period is $2(r + s + 1)$ if r and s are odd (with merely a sign change after $r + s + 1$ steps in the latter case). This is an explanation for the period 4 detected in [1] in the case $r = 1, s = 2$.

6 Asymptotics for Collective Eigenvalue Distribution

In 1915, Gábor Szegő established his celebrated first limit theorem, which states that if a is positive on $\mathbf{T}$, then the quotient $\det T_n(a)/\det T_{n-1}(a)$ converges to

$$G(a) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log a(e^{i\theta})\, d\theta\right)$$

as $n \to \infty$. This theorem implies that if a is real-valued, in which case the matrices $T_n(a)$ are all Hermitian, and if we denote by $\lambda_1(T_n(a)) \le \ldots \le \lambda_n(T_n(a))$ the eigenvalues of $T_n(a)$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \varphi(\lambda_j(T_n(a))) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(a(e^{i\theta}))\, d\theta$$

for every "test function" $\varphi \in C(\mathbf{R})$. This is a first order asymptotic result for the collective eigenvalue distribution of Toeplitz matrices. In 1952, eventually motivated by Lars Onsager's formula for the spontaneous magnetization of the two-dimensional Ising model, Szegő improved the result to a second order asymptotic formula, which is now called Szegő's strong limit theorem. We refer to the article [23] for an exhaustive treatment of this story.

Widom made several fundamental contributions to the collective eigenvalue distribution of truncated Toeplitz and Wiener-Hopf operators and their generalizations, such as pseudodifferential operators. In this section, we focus our attention on two of his papers on this topic.

In his 1980 paper [36] with Henry Landau, he investigated the positive definite operator given on $L^2(-\tau, \tau)$ by

$$(C_\tau f)(x) = \frac{\gamma}{2\pi i}\int_{-\tau}^{\tau} \frac{e^{-i\alpha(x-y)} - e^{-i\beta(x-y)}}{x-y} f(y)\, dy, \quad x \in (-\tau, \tau).$$

This operator is of crucial interest in random matrix theory and in laser theory. For example, as observed by H. Brunner, A. Iserles, and S. Nørsett [19], if $\gamma = \pi$, $\alpha = -2$, $\beta = 2$, in which case the operator is convolution by $\sin(2t)/t$, the eigenvalues of C_τ are the singular values of the famous Fox-Li operator. After changing integration over $(-\tau, \tau)$ to integration over $(0, 2\tau)$, the operator C_τ becomes a Wiener-Hopf integral operator with the symbol $\gamma\chi_{(\alpha,\beta)}$, which has two jumps. No general result of the type of Szegő's strong limit theorem delivered a second order trace formula in this situation. By an extremely ingenious argument,

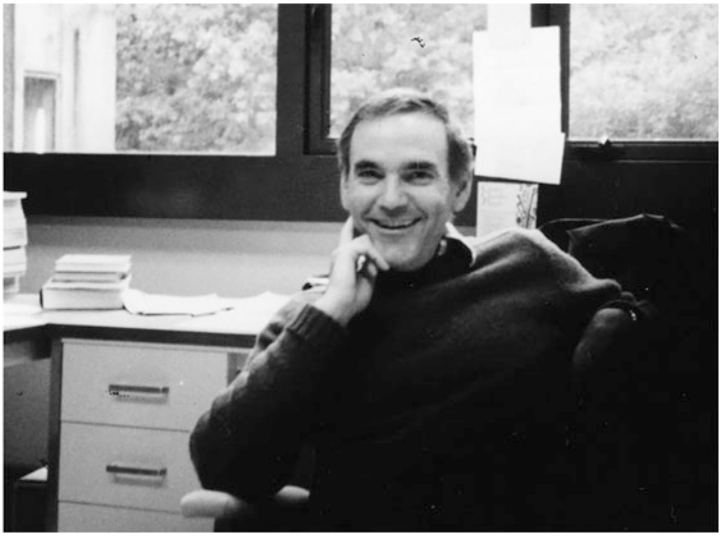

Fig. 3 Harold Widom in 1985. (Photo by Paul Halmos.).

Landau and Widom nevertheless succeeded in establishing a second order result for the eigenvalues, which confirmed a conjecture by D. Slepian of 1965. The result says that if φ is in $C^\infty(\mathbf{R})$ and $\varphi(0) = 0$, then

$$\sum_{j=1}^{\infty} \varphi(\lambda_j(C_\tau)) = \tau \, \frac{\varphi(\gamma)(\beta - \alpha)}{\pi} + \frac{\log(2\tau)}{\pi^2} \int_0^\gamma \frac{\gamma\varphi(x) - x\varphi(\gamma)}{x(\gamma - x)} \, dx + O(1).$$

The other paper we want to emphasize here is [62] of 1990. One is tempted to think that the eigenvalues of the $n \times n$ Toeplitz matrices $T_n(a)$ somehow mimic the spectrum of the infinite Toeplitz matrix $T(a)$ as $n \to \infty$. This is indeed the case if a is real-valued, but already in 1960, P. Schmidt and F. Spitzer [38] showed that this is in general no longer true if a is a Laurent polynomial ($\Leftrightarrow T(a)$ is banded). On the other hand, it was known that if a is piecewise continuous with exactly one jump and this jump is not too large, then the spectrum of $T_n(a)$ converges to the essential range of a. So what could the overall picture be? In [62], Widom raised the brave conjecture that except in rare cases, the eigenvalues of $T_n(a)$ are, in a sense, asymptotically distributed as the values of a. Such a rare case takes place, for instance, if a extends analytically a little into the interior or the exterior of $\mathbf{T}$, which happens in particular if a is a Laurent polynomial. And Widom proved this conjecture for various classes of symbols. One of the results of [62] says that if a is continuous, the range $a(\mathbf{T})$ is a Jordan curve, a is C^1 with nonvanishing derivative on $\mathbf{T}\setminus\{1\}$ but not in C^1 on all of $\mathbf{T}$, then the eigenvalues asymptotically cluster along $a(\mathbf{T})$. The proof is based on a thorough analysis of the determinants $\det(T_n(a) - \lambda I)$. In the case at hand, the function $a - \lambda$ is nonvanishing but has nonzero winding number about the origin, and getting asymptotic formulas for such determinants is one of the most difficult problems in the Toeplitz determinants business.

Widom's paper [62] already contained aspects of the idea to tackle eigenvalue distribution via potential theory. This idea has subsequently led to enormous success in understanding the collective asymptotics of eigenvalues. See [24, 25, 27]. And for another topic, we refer to [9, 22] for recent developments concerning the asymptotic behavior of individual eigenvalues in the bulk of the spectrum of large Toeplitz matrices.

7 Szegő-Widom

The revolutionary contributions of Widom to Toeplitz determinants with "regular" symbols are in his papers [54], [55], [56], which appeared from 1974 to 1976. Szegő's strong limit theorem says that, under certain assumptions,

$$\det T_n(a)/G(a)^n$$

converges to a nonzero limit $E(a)$ as $n \to \infty$. The original positivity assumption needed by Szegő was over the years relaxed by many mathematicians, including G. Baxter, I. I. Hirshman, Jr., A. Devinatz, to the requirement that a satisfies some mild smoothness condition, has no zeros on $\mathbf{T}$ and has winding number zero about the origin. The constants $G(a)$ and $E(a)$ are then given by

$$G(a) = \exp(\log a)_0,$$
$$E(a) = \exp \sum_{k=1}^{\infty} k(\log a)_k(\log a)_{-k},$$

where $(\log a)_j$ denotes the jth Fourier coefficient of any continuous logarithm of a. Widom did two important things. First, he extended the theorem to block Toeplitz matrices, and secondly, he found a remarkably elegant operator theoretic proof with immense impact on subsequent research into the asymptotics of Toeplitz matrices. Due to these achievements, Szegő's theorem for block Toeplitz matrices is now usually referred to as the Szegő-Widom theorem.

In the block case, a is a function of $\mathbf{T}$ into $\mathbf{C}^{N\times N}$, the Fourier coefficients a_j are $N \times N$ matrices, and $T_n(a)$ is accordingly a matrix of order nN. Given a matrix function a on the unit circle $\mathbf{T}$, we define, following Widom [56], the matrix function $\widetilde{a}$ by $\widetilde{a}(t) = a(1/t)$ for $t \in \mathbf{T}$. This matrix function associated with a plays an important role in the block case. Note that in the scalar case ($N = 1$) the matrix $T(\widetilde{a})$ is simply the transpose of $T(a)$. This is in general not true in the block case. We also note that in the following the tilde always has the meaning just introduced and no longer stands for the conjugate function we encountered in Sect. 3.

In addition to the block Toeplitz operator $T(a)$, we need the block Hankel operator $H(a)$ defined by the infinite block Hankel matrix

$$H(a) = (a_{j+k-1})_{j,k=1}^{\infty} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & \dots & \\ a_3 & \dots & & \\ \dots & & & \end{pmatrix}$$

on the $\mathbf{C}^N$-valued ℓ^2. Widom's paper [56] contains the two beautiful identities

$$T(ab) = T(a)T(b) + H(a)H(\widetilde{b}),$$
$$H(ab) = T(a)H(b) + H(a)T(\widetilde{b}).$$

These identities had been known and used for a long time, for example in the form

$$PabP = PaPbP + PaQbP,$$
$$PabQ = PaPbQ + PaQbQ,$$

but writing them in the above form, with the Hankel operators, was one of Widom's strokes of genius.

Widom's smoothness assumption was that

$$\|a\|_K := \|a\|_\infty + \Big(\sum_{j=-\infty}^{\infty} |j| \, \|a_j\|^2 \Big)^{1/2} < \infty,$$

where $\|\cdot\|$ is the spectral norm on $\mathbf{C}^{N\times N}$. In 1966, Mark Krein showed that such matrix functions form a Banach algebra. They key observation of Krein was that $H(a)$ is Hilbert-Schmidt if and only if $\sum_{j=0}^{\infty} j \, \|a_j\|^2 < \infty$. Consequently, if $\|a\|_K < \infty$ and $\|b\|_K < \infty$, then $T(a)$ and $T(\widetilde{b})$ are bounded while $H(a)$ and $H(b)$ are Hilbert-Schmidt. The identity $H(ab) = T(a)H(b) + H(a)T(\widetilde{b})$ and its analogue for $H(\widetilde{ab})$ therefore immediately imply that $\|ab\|_K < \infty$, too.

The Szegő-Widom theorem is the theorem established in [55] and [56]. It states that if a satisfies the above smoothness condition and $T(a)$ and $T(\widetilde{a})$ are Fredholm operators of index zero, then $T(a)T(a^{-1}) - I$ is a trace class operator and

$$\lim_{n\to\infty} \frac{\det T_n(a)}{G(a)^n} = E(a)$$

where

$$G(a) = \exp(\log\det a)_0, \quad E(a) = \det T(a)T(a^{-1}).$$

Here $\det T(a)T(a^{-1})$ is an operator determinant. If K is a trace class operator and $\{\lambda_j\}$ denotes the collection of its eigenvalues counted with algebraic multiplicity, then the operator determinant $\det(I+K)$ is defined as $\prod_j(1+\lambda_j)$. That in the concrete case at hand $T(a)T(a^{-1})-I$ is a trace class operator follows from the identity

$$T(a)T(a^{-1})-I=T(a)T(a^{-1})-T(aa^{-1})=H(a)H(\widetilde{a}^{-1})$$

and the fact that the two Hankel operators are Hilbert-Schmnidt.

But why is $\det T(a)T(a^{-1})$ equal to Szegő's original constant in the scalar case? Widom observed that this follows from another remarkable identity, namely, the formula

$$\det(e^A e^B e^{-A} e^{-B})=e^{\operatorname{tr}(AB-BA)},$$

which holds whenever A, B are bounded Hilbert space operators such that $AB-BA$ is of trace class. This formula was established independently by J. D. Pincus in 1972 and J. W. Helton and R. E. Howe in 1973, and a simple proof was given by the third author [29] in 2003. Widom had to struggle with several subtle complications, and we here confine ourselves to citing his argument in the simple case where a has a Wiener-Hopf factorization $a=a_-a_+$. Then

$$\begin{aligned}\det T(a)T(a^{-1}) &= \det T(a_-)T(a_+)T(a_+^{-1})T(a_-^{-1})\\ &= \det e^{T(\log a_-)}e^{T(\log a_+)}e^{-T(\log a_+)}e^{-T(\log a_-)}\\ &= e^{\operatorname{tr}[T(\log a_-)T(\log a_+)-T(\log a_+)T(\log a_-)]}\\ &= e^{\operatorname{tr}[T(\log a_-\,\log a_+)-T(\log a_+)T(\log a_-)]}\\ &= e^{\operatorname{tr} H(\log a_+)H((\log a_-)^{\sim})}\end{aligned}$$

and since

$$\operatorname{tr} H(c)H(\widetilde{b})=\operatorname{tr}\begin{pmatrix} c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & \dots & \\ c_3 & \dots & & \\ \dots & & & \end{pmatrix}\begin{pmatrix} b_{-1} & b_{-2} & b_{-3} & \dots \\ b_{-2} & b_{-3} & \dots & \\ b_{-3} & \dots & & \\ \dots & & & \end{pmatrix}=\sum_{k=1}^{\infty} kc_k b_{-k},$$

it follows that

$$\operatorname{tr} H(\log a_+)H((\log a_-)^{\sim}) = \sum_{k=1}^{\infty} k(\log a_+)_k(\log a_-)_{-k}$$
$$= \sum_{k=1}^{\infty} k(\log a)_k(\log a)_{-k},$$

which gives Szegő's scalar case formula for the constant $E(a)$.

Thus, we know that Widom's constant $\det T(a)T(a^{-1})$ coincides with Szegő's constant in the scalar case. But where does the $\det T(a)T(a^{-1})$ come from? We first of all want to remark that all previous proofs of the Szegő strong limit theorem were very complicated and rather indirect and did not convincingly reveal the actual origin of the constant $E(a)$. This changed with Widom's operator theoretic proof. However, instead of embarking on this proof here, we go some 25 years ahead. In 2000, Alexei Borodin and Andrei Okounkov [7] established a formula which, in notation subsequently suggested by no-one but Widom, reads

$$\frac{\det T_n(a)}{G(a)^n} = \frac{\det(I - Q_n H(b)H(\widetilde{c})Q_n)}{\det(I - H(b)H(\widetilde{c}))}.$$

Here Q_n is projection onto the coordinates indexed by $n+1, n+2, \ldots$, a is assumed to have the Wiener-Hopf factorizations $a = u_-u_+ = v_+v_-$ (note that in the matrix case one has to distinguish between "left" and "right" Wiener-Hopf factorizations), and b, c are defined by $b = v_-u_+^{-1}$, $c = u_-^{-1}v_+$. Obviously, $bc = I$. Since $Q_n \to 0$ strongly and $H(b)H(\widetilde{c})$ is of trace class, it follows that the right-hand side converges to $1/\det(I - H(b)H(\widetilde{c}))$, and since

$$\begin{aligned} 1/\det(I - H(b)H(\widetilde{c})) &= 1/\det T(b)T(c) \\ &= 1/\det T(v_-)T(u_+^{-1})T(u_-^{-1})T(v_+) \\ &= \det T(v_+^{-1})T(u_-)T(u_+)T(v_-^{-1}) \\ &= \det T(u_-)T(u_+)T(v_-^{-1})T(v_+^{-1}) \\ &= \det T(u_-u_+)T(v_-^{-1}v_+^{-1}) = \det T(a)T(a^{-1}), \end{aligned}$$

we arrive at Szegő-Widom.

Something like the Borodin-Okounkov formula was asked for by P. Deift and A. Its in 1999, and later it turned out that J. Geronimo and K. Case [31] had a similar formula proved earlier in 1979. Borodin and Okounov's proof of their formula was very intricate. Simple operator theoretic proofs were subsequently given by Widom and two of the authors in [5], [18]. The simplest of these proofs is based on Jacobi's formula, which says that if K is a trace class operator on the $\mathbf{C}^N$-valued $\ell^2(\mathbf{Z}_+)$ such that $I - K$ is invertible, P_n denotes the canonical projection onto the first n

coordinates, and $Q_n = I - P_n$ is as above, then

$$\det P_n(I-K)^{-1}P_n = \frac{\det(I - Q_n K Q_n)}{\det(I-K)}$$

for all $n \geq 1$. Letting $K = H(b)H(\widetilde{c})$ we have the right-hand side of Borodin-Okounkov, and taking into account that

$$\begin{aligned} &P_n(I - H(b)H(\widetilde{c}))^{-1}P_n = P_n T^{-1}(c)T^{-1}(b)P_n \\ &= P_n T(v_+^{-1})T(u_-)T(u_+)T(v_-^{-1})P_n = T_n(v_+^{-1})T_n(a)T_n(v_-^{-1}) \end{aligned}$$

and $\det T_n(v_+^{-1})T_n(a)T_n(v_-^{-1}) = G(a)^{-n}\det T_n(a)$, we get the left-hand side. We refer to the monograph [40] for an exhaustive presentation of the topics touched in this section and for nearly everything around Szegő's strong limit theorem.

To mention at least one impact of Widom's proof in [56] on the research in the years to follow, we note that in [56] we also see the beautiful identity

$$T_n(a)T_n(b) = T_n(ab) - P_n H(a)H(\widetilde{b})P_n - W_n H(\widetilde{a})H(b)W_n$$

for the product of two finite Toeplitz matrices. As above, P_n is projection onto the first n coordinates. The operator W_n is P_n followed by reversal of the coordinates. We remark that Widom himself wrote Q_n instead of W_n. The W_n was introduced in [12] (which was written before [39] but appeared only after that paper), not only because Q_n is there used for $I - P_n$ but mainly to give merit to Widom. It was this eye-catching identity along with the observation that the products of the Hankel operators are compact if a or b is continuous which inspired Bernd Silbermann in 1980 to study the stability of the sequence $\{T_n(a)\}_{n=1}^{\infty}$ by embedding it into a Banach algebra of sequences in which sequences of the form

$$\{P_n K P_n + W_n L W_n + C_n\}_{n=1}^{\infty}$$

with compact K, L and $\|C_n\| \to 0$ form a closed two-sided ideal [39] and by subsequently applying a so-called local principle. (Stability of the sequence $\{T_n(a)\}_{n=1}^{\infty}$ means that the inverses $[T_n(a)]^{-1}$ exist and have uniformly bounded norms for all sufficiently large n.) Harold told us that, although his command of German language is very limited, he had read Silbermann's paper [39] with great enthusiasm. Since the early 1980s, Silbermann's idea has led to enormous progress in the foundation of plenty of approximation methods and numerical algorithms; see, e.g., [13, 33, 34, 37]. The article [8] contains a photo showing Widom's paper [56].

8 Fisher-Hartwig

Symbols with discontinuities, zeros, poles, or nonzero winding number are referred to as singular symbols. If one of these four evils happens, Szegő's limit theorem breaks down. The 1968 paper [30] by Michael Fisher and Robert Hartwig set a big ball rolling. They introduced the class of singular symbols given by

$$a(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} |e^{i\theta} - e^{i\theta_r}|^{2\alpha_r} \varphi_{\beta_r,\theta_r}(e^{i\theta})$$

where b is a nice function (smooth, nonvanishing on $\mathbf{T}$, and with winding number zero about the origin), $e^{i\theta_1}, \ldots, e^{i\theta_R}$ are distinct points on $\mathbf{T}$, and the functions $\varphi_{\beta_r,\theta_r}$ are defined by

$$\varphi_{\beta_r,\theta_r}(e^{i\theta}) = \exp(i\beta_r \arg(-e^{i(\theta-\theta_r)}))$$

with the argument taken in $(-\pi, \pi]$. The function $\varphi_{\beta_r,\theta_r}$ satisfies

$$\varphi_{\beta_r,\theta_r}(e^{i(\theta_r+0)}) = e^{-\pi i\beta_r}, \quad \varphi_{\beta_r,\theta_r}(e^{i(\theta_r-0)}) = e^{\pi i\beta_r},$$

and it is continuous on $\mathbf{T}\setminus\{e^{i\theta_r}\}$. Such symbols a may have zeros ($\mathrm{Re}\,\alpha_r > 0$), poles ($\mathrm{Re}\,\alpha_r < 0$), oscillating discontinuities ($\mathrm{Re}\,\alpha_r = 0$), jumps ($\beta_r \notin \mathbf{Z}$), and nonzero winding numbers ($\beta_r \in \mathbf{Z}$).

Hartwig and Fisher raised the conjecture that

$$\det T_n(a)/G(a)^n \sim C(a)\, n^{\sum(\alpha_r^2-\beta_r^2)}$$

with some nonzero constant

$$C(a) = C(b, \theta_1, \ldots, \theta_R, \alpha_1, \ldots, \alpha_R, \beta_1, \ldots, \beta_R),$$

where $x_n \sim y_n$ means that $x_n/y_n \to 1$ as $n \to \infty$. It is required that $\mathrm{Re}\,\alpha_r > -1/2$ for all r, which guarantees that a is in $L^1(0, 2\pi)$ and hence has well-defined Fourier coefficients. The assumption that $|\mathrm{Re}\,\beta_r| < 1/2$ for all r is a basic case of the conjecture. It avoids certain unpleasant ambiguities caused by larger exponents β_r, in particular by the situation where some of the numbers $\alpha_r \pm \beta_r$ are integers.

In special cases, the conjecture was confirmed by A. Lenard and by Fisher and Hartwig themselves. With his 1973 paper [53], Widom was the first to provide a rigorous proof of the conjecture in a sufficiently general case: he proved it under the assumption that $\beta_r = 0$ for all r, and this proof is a gigantic piece of mathematical analysis. Hirschman writes in his review MR0331107 (48#9441) "The present paper represents a jump of several quanta in depth and sophistication in an area which is not only of great interest to mathematicians, but to theoretical physicists as well."

Widom also proved the conjecture for $R = 1$, $\alpha_1 > -1/2$, $-1/2 < \beta_1 < 1/2$, however, without determining the constant $C(a)$ in this case.

The conjecture of Fisher and Hartwig was subsequently confirmed by the first author under the assumption that $\operatorname{Re}\beta_r = 0$ for all r (1978) or that $\alpha_r = 0$ and $|\operatorname{Re}\beta_r| < 1/2$ for all r (1979), by B. Silbermann and the second author in the case where $|\operatorname{Re}\alpha_r| < 1/2$ and $|\operatorname{Re}\beta_r| < 1/2$ for all r (1985), and by B. Silbermann and the third author for $R = 1$, $\operatorname{Re}\alpha_1 > -1/2$, $\beta_1 \in \mathbf{C}$ arbitrary (1996). In each case, the constant $C(a)$ was completely identified. If $|\operatorname{Re}\alpha_r| < 1/2$ and $|\operatorname{Re}\beta_r| < 1/2$ and b is in the Wiener algebra, the constant is

$$C(a) = E(b)\prod_{r=1}^{R}\left[\mathbf{G}_{\gamma_r,\delta_r} b_-(t_r)^{-\gamma_r} b_+(t_r)^{-\delta_r}\right] \prod_{1\le r\ne s\le t_r}\left(1-\frac{t_r}{t_s}\right)^{-\delta_r\gamma_s}$$

where $t_r = e^{i\theta_r}$, $\gamma_r = \alpha_r + \beta_r$, $\delta_r = \alpha_r - \beta_r$,

$$\mathbf{G}_{\gamma,\delta} = \frac{\mathbf{G}(1+\gamma)\mathbf{G}(1+\delta)}{\mathbf{G}(1+\gamma+\delta)}$$

with Barnes' double Gamma function $\mathbf{G}(z)$,

$$b_-(t) = \exp\sum_{k<0}(\log b)_k t^k, \quad b_+(t) = \exp\sum_{k>0}(\log b)_k t^k,$$

and $E(b) = \exp\sum_{k=1}^{\infty} k(\log b)_k(\log b)_{-k}$.

It was observed by several authors, for example by Silbermann and the second author already in 1981, that the conjecture is in general no longer true if $\alpha_r \pm \beta_r$ may assume values in $\mathbf{Z}\setminus\{0\}$. A new conjecture, which covers all possible cases, was formulated by Craig Tracy and the first author [3] in 1991. This new conjecture was proved by the third author [28] in 1997 in all cases in which it coincides with the original conjecture and by Percy Deift, Alexander Its, and Igor Krasovsky [21] in 2009 in full generality. The entire development from Fisher and Hartwig's 1968 paper up to the present has both demanded and produced great progress in operator theory for Toeplitz and related matrices.

The Fisher-Hartwig conjecture has a continuous analogue for Wiener-Hopf determinants. In the 1983 paper [4] by Widom and the first author, this conjecture was proved for piecewise continuous symbols with a continuous argument, that is, for the case where $\alpha_r = 0$ and $\operatorname{Re}\beta_r = 0$ for all r. The idea of the proof is that Wiener-Hopf determinants when discretized become Toeplitz determinants. Unfortunately, one is led to determinants of the form $\det T_n(a^{(n)})$ in this way. Thus, not only the order of the determinant but also the symbol depend on n. However, sufficiently precise asymptotic results for Toeplitz matrices and determinants eliminate this obstacle.

For general piecewise symbols, the continuous analogue of the Fisher-Hartwig conjecture was settled in 1994 in the papers [15] and [16] by Widom, Silbermann, and the second author. These papers are based on another idea. This time it is that Wiener-Hopf operators may be regarded as Toeplitz matrices with operator-valued entries by thinking of $L^2(0,\infty)$ as ℓ^2-space with values in $L^2(0,1)$ and thus interpreting a convolution integral operator on $L^2(0,\infty)$ as a Toeplitz matrix whose entries are integral operators on $L^2(0,1)$. This idea was motivated by papers [14, 32].

We consider Wiener-Hopf operators whose symbol σ is a (complex-valued) function in $L^\infty(\mathbf{R})$ such that $\sigma - 1 \in L^2(\mathbf{R})$. The corresponding Wiener-Hopf operator on $L^2(0,\infty)$ is defined by

$$(W(\sigma)f)(x) = f(x) + \int_0^\infty k(x-t)f(t)\,dt, \quad 0 < x < \infty,$$

where

$$k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\sigma(\xi)-1)e^{-i\xi x}\,d\xi$$

is the inverse Fourier-Plancherel tranform of $\sigma - 1$. For $\tau > 0$, let, as in Section 5, $W_\tau(\sigma)$ denote the compression of $W(\sigma)$ to $L^2(0,\tau)$. The assumption that $\sigma - 1$ be in $L^2(\mathbf{R})$ implies that so also is k, and hence $W_\tau(\sigma)$ is of the form I plus Hilbert-Schmidt operator for every (finite) $\tau > 0$. If even $\sigma - 1 \in L^1(\mathbf{R})$, then k is continuous and therefore $W_\tau(\sigma)$ is of the form I plus trace class operator. In the latter case the determinant $\det W_\tau(\sigma)$ is well-defined, in the former case we may consider the so-called second regularized determinant $\det_2 W_\tau(\sigma)$:

$$\det W_\tau(\sigma) = \prod_j (1+\lambda_j), \quad \det_2 W_\tau(\sigma) = \prod (1+\lambda_j)e^{-\lambda_j},$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of $W_\tau(\sigma) - I$ (counted up to algebraic multiplicity).

Suppose now that $\sigma \in L^\infty(\mathbf{R})$ is a piecewise smooth function with jumps at the points $\omega_1, \ldots, \omega_r \in \mathbf{R}$. There are uniquely defined complex numbers β_j such that

$$e^{2\pi i\beta_j} = \frac{\sigma(\omega_j + 0)}{\sigma(\omega_j - 0)}, \quad -1/2 < \operatorname{Re}\beta_j \le 1/2.$$

Note that β_j is purely imaginary or real, respectively, if and only if σ has a continuous argument or a continuous modulus at ω_j. In [15], it was proved that if

$$\sigma - 1 \in L^1(\mathbf{R})$$

and some index zero condition is satisfied (which is equivalent to the invertibility of $W(\sigma)$ and includes that $-1/2 < \operatorname{Re}\beta_j < 1/2$ for all j), then

$$\det W_\tau(\sigma) \sim G(\sigma)^\tau \tau^{-(\beta_1^2+\cdots+\beta_r^2)} E(\sigma) g(\beta_1) \cdots g(\beta_r)$$

as $\tau \to \infty$, where $G(\sigma)$, $E(\sigma)$ are explicitly given constants, and

$$g(\beta) = e^{(1+\gamma)\beta^2} \prod_{n=1}^{\infty} (1+\beta^2/n^2)^n e^{-\beta^2/n} = \mathbf{G}(1+\beta)\mathbf{G}(1-\beta)$$

with Euler's constant γ and Barnes' double Gamma function $\mathbf{G}(z)$. In the case of continuous arguments, that is, if $\operatorname{Re}\beta_j = 0$ for all j, this result had been established in [4] ten years before.

The requirement that $\sigma - 1$ be in $L^1(\mathbf{R})$ ruled out many standard piecewise continuous symbols. In particular, if σ is the "canonical" piecewise continuous function given by

$$\sigma(\xi) = \left(\frac{\xi - i}{\xi + i}\right)^\beta \quad (\xi \in \mathbf{R})$$

(with an appropriate branch of z^β), then $\sigma - 1$ is in $L^2(\mathbf{R})$ but not in $L^1(\mathbf{R})$. In [16], this final hurdle was taken. It was shown that if

$$\sigma - 1 \in L^2(\mathbf{R})$$

and the index zero condition is satisfied, then

$$\det_2 W_\tau(\sigma) \sim G_2(\sigma)^\tau \tau^{-(\beta_1^2+\cdots+\beta_r^2)} E(\sigma) g(\beta_1) \cdots g(\beta_r)$$

with some new constant $G_2(\sigma)$ and with $E(\sigma)$ as before. In the case of the "canonical" symbol one gets

$$\det_2 \left(W_\tau \left(\frac{\xi - i}{\xi + i} \right)^\beta \right) \sim h(\beta)^\tau \left(\frac{\tau}{2}\right)^{-\beta^2} g(\beta)$$

with

$$h(\beta) = \exp\left(\frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\sin \beta y}{\sin y} \right)^2 dy \right)$$

$-1/2 < \operatorname{Re}\beta < 1/2$.

Over the years it has become clear that the asymptotic behavior of Toeplitz and Wiener-Hopf determinants with several Fisher-Hartwig singularities can be

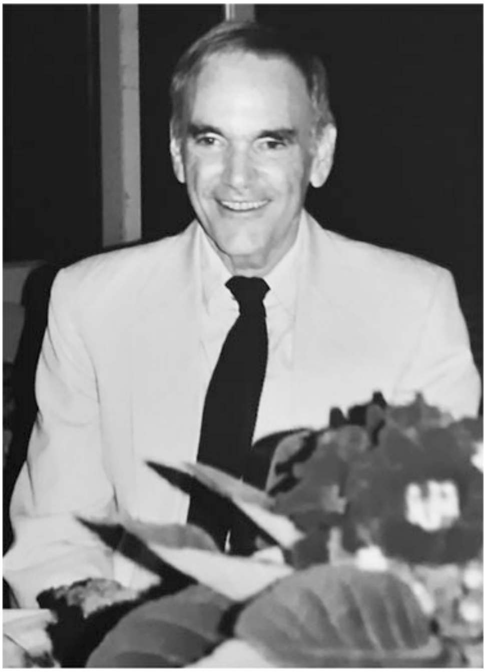

Fig. 4 Harold Widom at his 70th birthday in 2002.

determined by employing localization techniques provided one knows the asymptotics for at least one symbol with a single Fisher-Hartwig singularity. In the Toeplitz case, such a symbol is $(1-t)^\gamma(1-1/t)^\delta$ $(t \in \mathbf{T})$ because we have the factorizations

$$|t-1|^{2\alpha} = (1-t)^\alpha(1-1/t)^\alpha,$$
$$\varphi_{\beta,0}(t) = \exp(i\beta\arg(-t)) = (1-t)^\beta(1-1/t)^{-\beta},$$

which gives $(1-t)^\gamma(1-1/t)^\delta$ with $\gamma = \alpha+\beta$ and $\delta = \alpha-\beta$. Both exact and asymptotic formulas for the corresponding Toeplitz determinants were found in 1985 by Silbermann and the second author, and two elementary derivations of these formulas are also in the 2005 paper [17]. We remark in this connection that the symbols in the Fisher-Hartwig class may also be written in the form

$$a(t) = c(t)\prod_{r=1}^{R}\left(1-\frac{t}{t_r}\right)^{\gamma_r}\left(1-\frac{t_r}{t}\right)^{\delta_r} \quad (t \in \mathbf{T}),$$

where c is a nice function and $t_1, \ldots, t_R$ are points on $\mathbf{T}$. In that case the exponent $\sum(\alpha_r^2-\beta_r^2)$ becomes $\sum\gamma_r\delta_r$.

In the Wiener-Hopf case, things are dramatically more complicated. Only in 2004, in [6], Widom and the first author were able to prove the predicted asymptotic behavior for the Wiener-Hopf determinants with the symbol

$$\left(\frac{\xi + 0i}{\xi + i}\right)^{\gamma}\left(\frac{\xi - 0i}{\xi - i}\right)^{\delta} \quad (\xi \in \mathbf{R}),$$

still requiring that $\gamma = \alpha + \beta$ and $\delta = \alpha - \beta$ with the real parts of α, β in $(-1/2, 1/2)$. The proof is highly sophisticated. Roughly speaking, it is based on introducing a parameter to regularize the symbol, on applying the Wiener-Hopf analogue of the Borodin-Okounkov formula, which was established in 2003 by Y. Chen and the first author, on considering the quotient of the Wiener-Hopf determinant over $(0, R)$ and an appropriate $n \times n$ Toeplitz determinant, on taking the limit $n, R \to \infty$ with $n/R \to 1$, and on finally returning to the original symbol by passing to the limit that makes the regularization parameter disappear.

In his journey from eigenvalue distribution problems to Szegő's theorem and generalizations for singular symbols, Widom sometimes did an excursion into other more general classes of operators. In a series of papers in the late 1970s, [57–60], he proved a far-reaching extension of the classical Szegő theorem by developing a symbolic calculus for pseudodifferential operators. The context was general enough to include extensions with variable convolutions, higher dimensions, and general Riemannian manifolds. The applications ranged from the classical theorems in the Toeplitz case to heat expansions for Laplace-Beltrami operators. Many of these results entered his book [61].

Credits The two photos of Harold Widom in 1969 and 1985 are courtesy of the Paul R. Halmos Photograph Collection, The Dolph Briscoe Center for American History, The University of Texas at Austin. The photo of the German-Israeli workshop is the conference photo, and the photo of Harold Widom in 2002 is courtesy of the authors.

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Visualizations of Two Functions Emerging in Connection with Toeplitz Determinants

Elias Wegert

Abstract In this note we visualize the Barnes **G**-function and some related functions emerging in formulas for Toeplitz determinants, and discuss some of their properties using phase plots.

Keywords Toeplitz determinants · Barnes' double-Gamma function · Fisher-Hartwig conjecture · Phase plot

As shown in the two papers [1, 2] in this volume, the Barnes function $\mathbf{G}(z)$ is currently emerging in formulas for Toeplitz determinants. Albrecht Böttcher asked me to make a short contribution to this volume with visualizations of the related functions.

The pure Fisher-Hartwig determinant is the determinant of the $n \times n$ Toeplitz matrix

$$T_n(a) = (a_{j-k})_{j,k=1}^n$$

generated by the Fourier coefficients of the function

$$a(e^{i\theta}) = (1 - e^{-i\theta})^\gamma (1 - e^{i\theta})^\delta.$$

The formula for the determinant given in [2] is

$$\det T_n(a) = \mathbf{G}(n+1)\, \frac{\mathbf{G}(\gamma+\delta+n+1)}{\mathbf{G}(\gamma+\delta+1)}\, \frac{\mathbf{G}(\gamma+1)}{\mathbf{G}(\gamma+n+1)}\, \frac{\mathbf{G}(\delta+1)}{\mathbf{G}(\delta+n+1)}, \tag{1}$$

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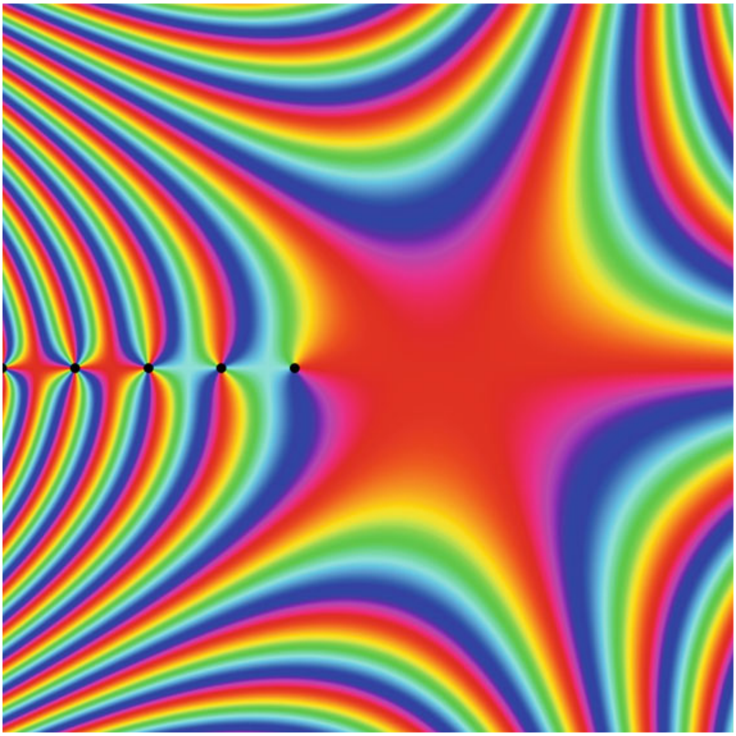

Fig. 1 Phase plot of the Barnes function $\mathbf{G}(z)$ in the domain $-4 < \operatorname{Re} z < 6$, $|\operatorname{Im} z| < 5$.

where $\mathbf{G}(z)$ is the Barnes function.[1] A phase plot of this function in the domain $-4 < \operatorname{Re} z < 6$, $|\operatorname{Im} z| < 5$ is shown in Fig. 1.

The points where all colors meet are zeros of $\mathbf{G}(z)$ (located at the points $0, -1, -2, \ldots$), their orders $1, 2, 3, \ldots$ correspond to the numbers of isochromatic lines (of one specific color) ending at these points. The isochromatic lines run in the direction of the gradient of the modulus of the function. The function grows (or decays) the faster the higher the density of the isochromatic lines is; parallel lines correspond to exponential growth. The "converging" lines towards the left upper and lower corners as well as in the direction of the positive real line indicate that $|\mathbf{G}(z)|$ grows even faster than exponential. On the other hand, the function decays quickly in the direction of the other two "red arms". For further information how to read phase plots we refer to [4] and the book [3].

[1] The function was introduced by Ernest Barnes (1874–1953) in a series of papers around 1900. In 1906, Barnes became John Littlewood's thesis advisor. Littlewood quickly solved the first problem Barnes gave him; the second problem posed by Barnes was the Riemann hypothesis. In 1915 Barnes left his job as a professional mathematician, and in 1924 he became Bishop of Birmingham. See [5].

Fig. 2 Truncated analytic landscape of the Barnes function $\mathbf{G}(z)$ with $-2 < \operatorname{Re} z < 6$, $|\operatorname{Im} z| < 4$.

The enormous growth of the Barnes function along the positive real line can also be seen from the functional equations $\mathbf{G}(z+1) = \Gamma(z)\,\mathbf{G}(z)$, $\Gamma(z+1) = z\,\Gamma(z)$, involving $\mathbf{G}(z)$ and the Euler Gamma function $\Gamma(z)$. The colored analytic landscape of $\mathbf{G}(z)$ depicted in Fig. 2 illustrates the behavior described above, though it is truncated at height 6.

The choices $(\gamma, \delta) = (\alpha, \alpha)$ and $(\gamma, \delta) = (-\beta, \beta)$ in formula (1) yield the pure modulus singularity

$$\omega_\alpha(e^{i\theta}) = (1 - e^{-i\theta})^\alpha (1 - e^{i\theta})^\alpha = |e^{i\theta} - 1|^{2\alpha}$$

and the canonical jump function

$$\varphi_\beta(e^{i\theta}) = (1 - e^{-i\theta})^{-\beta} (1 - e^{i\theta})^\beta = (-e^{i\theta})^\beta.$$

Note that, in general, both functions are complex-valued. Their singularities are located at $e^{i\theta} = 1$. At this point, the modulus $|\omega_\alpha(e^{i\theta})|$ has a zero if $\operatorname{Re}\alpha > 0$ and a pole if $\operatorname{Re}\alpha < 0$, and the argument $\arg \omega_\alpha(e^{i\theta})$ has a logarithmic singularity if

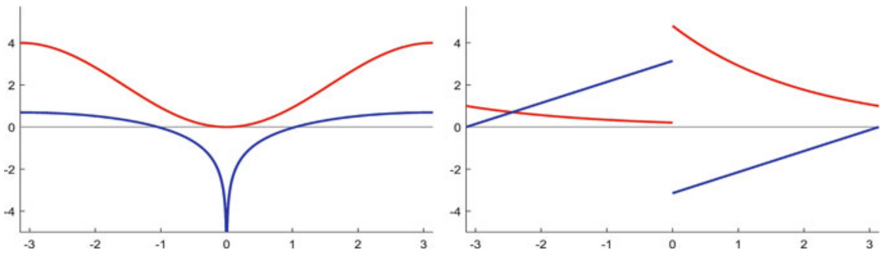

Fig. 3 Modulus (red) and argument (blue) of $\omega_\alpha(e^{i\theta})$ (left window) and $\phi_\beta(e^{i\theta})$ (right window) for $\alpha = \beta = 1 + i/2$.

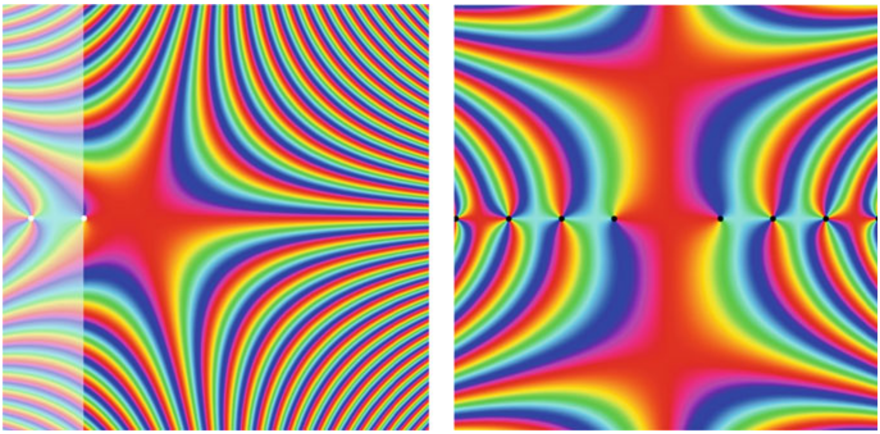

Fig. 4 Left picture: the function $\mathbf{G}(1+\alpha)^2/\mathbf{G}(1+2\alpha)$ for $-2 < \operatorname{Re}\alpha < 6$ and $|\operatorname{Im}\alpha| < 4$. Right picture: the function $\mathbf{G}(1+\beta)\mathbf{G}(1-\beta)$ for $|\operatorname{Re}\beta| < 4$, $|\operatorname{Im}\beta| < 4$.

$\operatorname{Im}\alpha \neq 0$. For the function $\varphi_\beta(e^{i\theta})$ jump singularities of modulus and argument are typical; see Fig. 3 for an illustration.

Combining formula (1) for the corresponding Toeplitz determinants with known asymptotic formulas for the Barnes function, one obtains the asymptotic formulas

$$\det T_n(\omega_\alpha) \sim \frac{\mathbf{G}(1+\alpha)^2}{\mathbf{G}(1+2\alpha)}\, n^{\alpha^2} \qquad (\operatorname{Re}\alpha > -1/2), \tag{2}$$

$$\det T_n(\varphi_\beta) \sim \mathbf{G}(1+\beta)\mathbf{G}(1-\beta)\, n^{-\beta^2} \qquad (\beta \notin \mathbb{Z}), \tag{3}$$

which have actually been known for decades.

Phase plots of the coefficients in (2) and (3) as functions of α and β are shown in Fig. 4. The saturated subdomain corresponds to $\operatorname{Re}\alpha > -1/2$ in which the asymptotic formula is valid. The function has poles at $\alpha = -1/2, -3/2, \ldots$, the white dots are two of them. Note that zeros and poles can be distinguished by the different orientations of colors in their neighborhood.

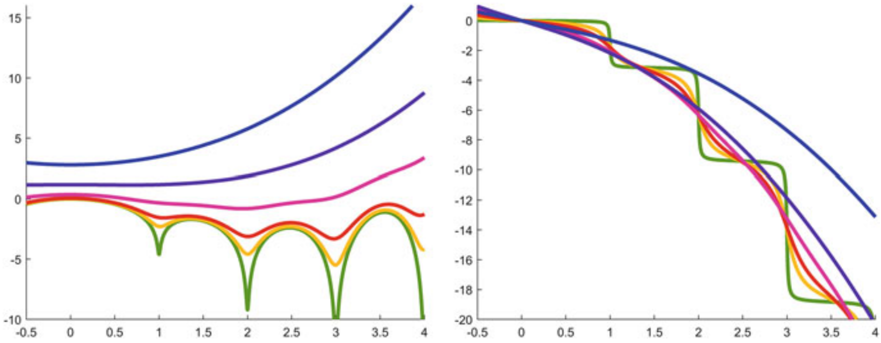

Fig. 5 The function $\mathbf{G}(1+\beta)\mathbf{G}(1-\beta)$ along the lines $\operatorname{Im}\beta = 0.01, 0.1, 0.2, 0.5, 1, 2$: logarithm of modulus on the left and argument on the right.

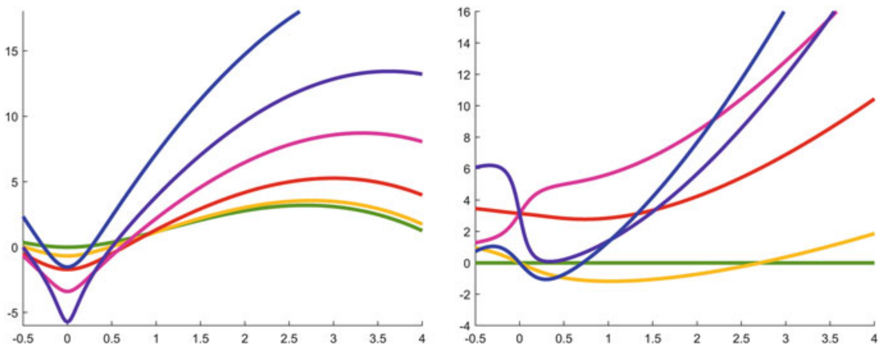

Fig. 6 The function $\mathbf{G}(1+\beta)\mathbf{G}(1-\beta)$ along the lines $\operatorname{Re}\beta = 0, 0.6, 1.4, 2.2, 2.9, 3.7$: logarithm of modulus on the left and argument on the right.

On the left of Fig. 5 the modulus of $\mathbf{G}(1+\beta)\mathbf{G}(1-\beta)$ along some horizontal lines in the β-plane is depicted using a logarithmic scale. The image on the right shows a continuous branch of the argument of these functions. The lines are $\operatorname{Im}\beta = 0, 0.1, 0.2, 0.5, 1, 2$. Figure 6 shows the corresponding functions along the vertical lines given by $\operatorname{Re}\beta = 0, 0.6, 1.4, 2.2, 2.9, 3.7$. For both figures the associated colors of the graphs are in this order: green, yellow, red, magenta, violet, and blue.

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Part II
Personal Notes

A Remarkable Advisor, Mentor, and Friend

Estelle Basor

Abstract This article contains some personal reflections of Harold Widom as my remarkable advisor, mentor, and friend.

Keywords Harold Widom · Personal history · Toeplitz · Determinants

In the Fall of 1968 I was a senior at the University of California, Santa Cruz (UCSC) and taking for the first time serious courses in mathematics. I was not by any means precocious as a mathematics student. I had started college thinking that I might go into social work, but mathematics seemed to be the only thing I was reasonably good at. I had had courses in calculus, linear algebra, differential equations, set theory, and combinatorics, but had little idea of what was to follow. When I look back at that year, I realize how lucky I had been to have had a course in abstract algebra taught by Nick Burgoyne, one of the pioneers in the simple group classification project, a functional analysis course taught by Robert Bonic, and the most influential for me, the undergraduate analysis course taught by Harold.

As other students will attest, Harold's lectures were captivating. He came to class with only a small piece of paper and with it produced exquisite lectures that revealed the deep inner core of analysis. His lectures were heuristic in a sense. He would motivate and outline steps and then fill in details, but always with a natural flow and often engaging the students to help.

The analysis sequence spanned two quarter terms and near the end of the second quarter, Harold approached me about staying at Santa Cruz for graduate school. The campus was only four years old at this time and many of the departments were just starting programs. I did not hesitate to agree. Thus I began graduate school in the Fall of 1969. The academic year 1969–1970 was a turbulent time for American universities and Santa Cruz was no exception. By the time Spring quarter had come around, most students were not attending class and protesting the American

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involvement in the Vietnam war. I was a teaching assistant and only meeting with students informally. Harold was chair of the department and this must have been an especially trying time for him.

Things settled down in the following years. Harold agreed to be my thesis advisor as I had hoped and my first task was to read his landmark paper, “Toeplitz determinants with singular generating functions.” I spent the better part of a year trying to understand the paper, which confirmed the determinant conjecture of Fisher and Hartwig in the first general cases.

In June of 1972 I attended my first ever mathematics conference at the University of Georgia. It was not the custom then for graduate students to go to conferences unless they were held nearby. The conference featured a series of talks by Ron Douglas on index theory for Toeplitz operators and there I also met Al Devinatz and Bill Helton. My trip was funded by Harold’s grant and we flew together to Atlanta. On the way, he suggested that we have martinis. (Dinner was offered free-of-charge on planes back then.) I had never had gin before, but it was not hard to appreciate. Over the years, Harold and I, along with Linda, his wife, and Kent, my husband, had many dinners together always starting with a gin martini.

By the summer of 1972, I had understood the singular determinant paper well enough so that Harold thought I might start working on a different, but related problem. The goal was to do something with singular symbols for Hankel matrices, relating them back to the techniques he had used. One minor hitch in the plan was that he was planning to spend the following year in France. There was no email then, no Skype, no Zoom and so if I made any progress I wrote a letter and he replied back. The truth was that everything I tried did not work and I was fairly discouraged by my lack of progress.

On a side note, just before the Widom family left for France, Kent and I were married with the Widom family at our wedding. After they arrived in France, they sent us a beautiful hand-painted souffle dish that I cherish.

When Harold came back from France, we worked through everything I had tried and finally both agreed the problem was not something that could be done—at least at that time. But one good thing that came out of my frustrating year is that I had learned a good many analysis techniques and I thought that perhaps I could push the Fisher-Hartwig results even further. And that was what became the main topic of my thesis.

After I graduated and started working at Cal Poly in San Luis Obispo I read with great care another one of his landmark papers, “Asymptotic behavior of block Toeplitz matrices and determinants,” which appeared in the Advances in Mathematics in 1975. I thought that perhaps some of the Fisher-Hartwig results could be redone using the operator theory methods. Of course the first person I told about this idea was Harold.

In the Fall of 1978, right around the November 11th Veterans Day holiday, Harold came to Cal Poly to give a colloquium. The next year (since that was always a day I did not teach) I visited him in Santa Cruz. So we made it a tradition for several years to always meet on the 11th, talk about mathematics and go out to lunch.

Over the years we often ended up at the same meetings and conferences. In 1985, we both attended the International Conference in Operator theory that was held in Bucharest, Romania. This was of course when the President of Romania was Ceausescu. The western mathematicians were housed in one hotel and ones from the east in another. The conference spanned two weeks and the organizers planned a non-optional bus tour for the participants for the weekend. On Saturday morning we went first to a very old orthodox monastery. Then we were taken to a hotel and then up to a dinner at an experimental farm that had been converted from a private estate. I actually have no idea where we really were, but when we arrived folk dancers greeted us. They attempted to get everyone to dance—all the mathematicians. Harold had a very funny look on his face. I knew he was not going to do this. One of the dancers grabbed my arm and I began to dance. Harold was left holding my purse. The next day we toured Dracula's castle in Transylvania.

The last meeting we attended together was an AIM workshop, Fisher-Hartwig asymptotics, Szegö expansions, and applications to statistical physics, held in March of 2017. Harold's talk was the highlight of the week.

It seems to me that when I look back, almost all the mathematics I know I learned from Harold. He was a master of analysis. He could change variables, integrate by parts, and in a flash transform something that could not be done into something doable. He was always rigorous, but his analysis was never tedious or dull. Working with him was just plain fun. I cherish every moment that I did and I miss him terribly.

Credits The photo in Fig. 1 is courtesy of the author, the photo in Fig. 2 is courtesy of Wolfgang Spitzer.

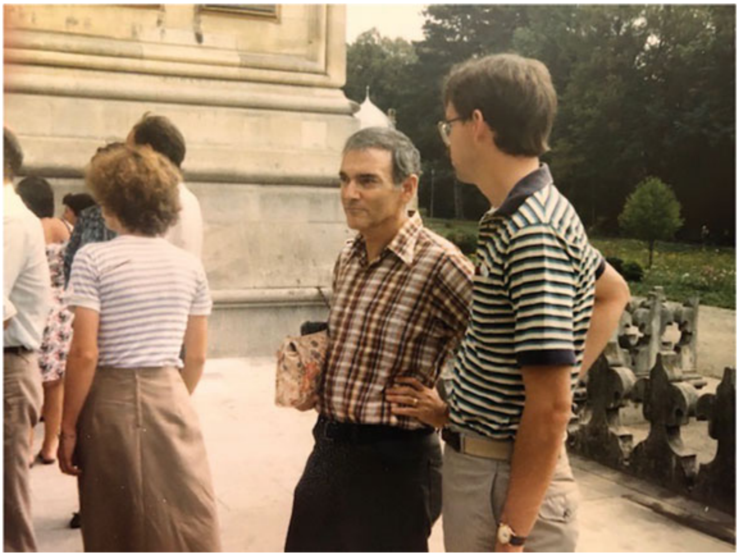

Fig. 1 Harold in Romania holding my purse.

Fig. 2 Harold, Craig Tracy and myself at the 2017 AIM workshop.

My Encounters with Harold Widom

Albrecht Böttcher

Abstract This is an essay containing personal reminiscences, including some photos, and describing a few selected topics of joint mathematical work of the author and Harold Widom.

Keywords Toeplitz operators · Toeplitz matrices · Wiener-Hopf operators

Let's begin with some prehistory. In the second half of the 1970s, I was a student of mathematics in Chemnitz, and in the second or third year I decided to go to Bernd Silbermann. I had attended his lecture courses Analysis I to III in the first three terms and felt that he was the right man under the guidance of whom I should continue the advanced terms of my study. Silbermann was a student of Siegfried Prössdorf, who left Chemnitz for Berlin in the mid of the 1970s, and when I approached Silbermann, he was still Dr. Silbermann. Only in 1979 he was appointed full professor.

Under the influence of Prössdorf, Silbermann had entered singular integral operators with so-called degenerate symbols. It had been known for a long time that certain operators are Fredholm if and only if a function associated with them, the so-called symbol, has no zeros. Degenerate symbols are those which have zeros, and in those years it was some kind of a business to understand what in the degenerate case happens. Prössdorf and Silbermann studied in particular projection methods for the solution of equations with degenerate symbols. In the course of these investigations large matrices emerge, their invertibility is one of the crucial questions, and hence it is no surprise that Silbermann came across theorems on Toeplitz determinants, in particular Widom's two papers [24, 25]. As a result, Silbermann made two major contributions to the topic [21].

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First, the assumption of Szegő's theorem is that the symbol is sufficiently smooth, has no zeros, and has winding number zero. Silbermann established asymptotic formulas for the Toeplitz determinants generated by symbols of the form

$$a(t) = b(t) \prod_{j=1}^{r} (t - t_j)^{\delta_j} \quad (t \in \mathbf{T}),$$

where $b(t)$ satisfies the assumptions of Szegő's theorem, $t_1, \ldots, t_r$ are points on the complex unit circle $\mathbf{T}$, and $\delta_1, \ldots, \delta_r$ are positive numbers. The points $t_1, \ldots, t_r$ are zeros and so $a(t)$ does not satisfy the assumptions of Szegő. I should notice that all these zeros are of "analytic" type, which simplifies things (from the perspective of today!). The actual challenge is symbols of the form

$$a(t) = b(t) \prod_{j=1}^{r} \left(1 - \frac{t_j}{t}\right)^{\gamma_j} (t - t_j)^{\delta_j} \quad (t \in \mathbf{T}),$$

where the zeros appear in both the "analytic" and the "anti-analytic" types. The famous Fisher-Hartwig conjecture of 1968 concerns the Toeplitz determinants generated by symbols of the latter form where, in addition, $b(t)$ is not assumed to be smooth but is allowed to make jumps (even at just the points t_j).

The second major contribution of Silbermann addressed the smoothness condition needed in Szegő's strong limit theorem. To state things in an easy case, Szegő's strong limit theorem holds if

$$\sum_{n=-\infty}^{\infty} |n|^{1/2} |a_n| < \infty,$$

where $\{a_n\}$ is the sequence of the Fourier coefficients of $a(t)$. Silbermann observed that one can relax the requirement on one half of the coefficients if at the same time the conditions on the other half is strengthened. For example, he proved Szegő under the assumption that

$$\sum_{n=-\infty}^{-1} |n|^{\alpha} |a_n| + \sum_{n=1}^{\infty} n^{\beta} |a_n| < \infty \quad \text{with} \quad \alpha > 0, \beta > 0, \alpha + \beta \geq 1.$$

In the late 1970s, Silbermann posed me the extension to block Toeplitz matrices of the latter result as the topic of my diploma paper. He gave me a photocopy of a paper on block Toeplitz matrices, and at this point Harold Widom stepped into my life. The paper was Widom's article [25]. I have kept it until now. Figure 1 shows my well thumbed copy. What resulted was my very first publication, [12], joint with Silbermann. One section of that paper has the title "An extension of Widom's arguments."

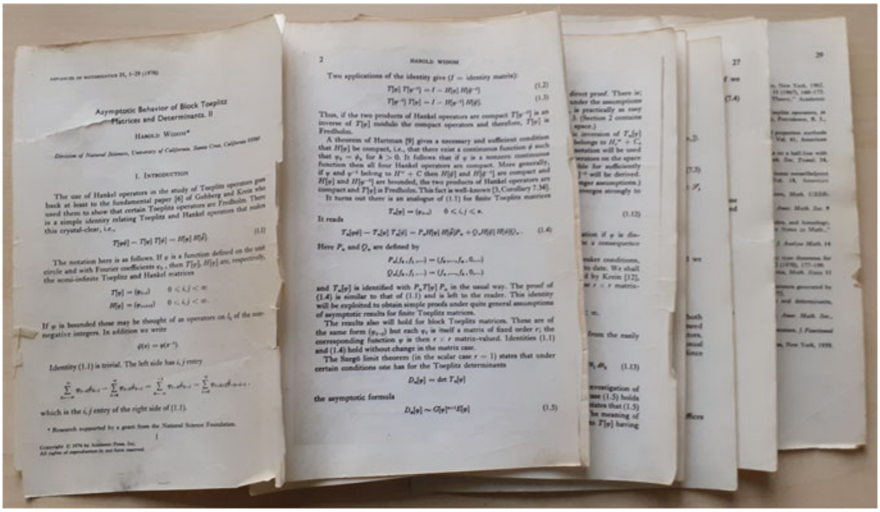

Fig. 1 My copy of paper [25]. I read it more than 40 years ago and have kept it since then.

In the 1980s, I embarked on several topics in Toeplitz operators and matrices. As for Toeplitz determinants, the Fisher-Hartwig conjecture was in the focus of my joint research with Silbermann. We read in particular with great admiration the papers by Estelle Basor [3, 4] and Harold Widom [24] and were eventually able to prove the Fisher-Hartwig conjecture in a basic case [13]. For more on this subject, I refer to [6, 14] and for the continuation in the 1990s to [19] (which is essentially Torsten Ehrhardt's dissertation, written under the guidance of Silbermann).

It was only in 1989 that I met Harold for the first time in person. At that time I knew many mathematicians by their name only, but things changed with the fall of the Iron Curtain. In 1989, Israel Gohberg, Rien Kaashoek, and Erhard Meister organized the (by now at least in our community legendary) Oberwolfach conference "Toeplitzoperatoren, Wiener-Hopf-Probleme und deren Anwendungen." There I made personal acquaintance with various of my mathematical heroes, including Israel Gohberg and, as said, Harold Widom. I received a true abundance of unforgettable impressions from this conference, meeting Harold being one of the highlights.

My second meeting with Harold was in 1992. I was invited to participate in the conference "Toeplitz and Wiener-Hopf Operators in Honor of Harold Widom," which was dedicated to Harold on his 60th birthday and took place in Santa Cruz. What an event! It was my very first trip across the Atlantic Ocean, the organizers had booked a rental car for me, and I experienced the joy of power steering and automatic transmission for the first time in my life. At the conference, I met in person many of my other mathematical heroes, including Estelle Basor, Ronald Douglas, and Donald Sarason. We all had lots of inspiring talks and discussions, an amazing birthday reception, and a wonderful dinner in Harold's house. Harold also

Fig. 2 Break at the conference in honor of Harold Widom on his 60th birthday in Santa Cruz in 1992. On the front table from the left to the right: Lidia Luquet, Cora Sadosky, Richard Libby, I, Israel Gohberg, Donald Sarason, Richard Rochberg, Ronald Douglas.

took me with his car on a half-day trip to Carmel Bay in the south of Santa Cruz. Figures 2, 3, and 4 are photos taken in those days of 1992. As I had a few more days after the conference, I made it with my rental car also to Yosemite and Lake Tahoe. Moreover, Estelle Basor invited me to a talk at the California Polytechnic State University in San Luis Obispo. So I enjoyed travelling on Highway 1 from Santa Cruz to San Luis Obispo. I remember with great pleasure the warm hospitality of Estelle and her husband Kent Morrison in their house and devouring the sunset behind the rock in Morro Bay with them.

After the reunion of Germany all professors of eastern universities lost their posts and had to apply anew. I remember that I took leave from Harold in Santa Cruz with the words that on my return in Germany I will find a letter on my desk beginning either with “We regret to inform you” or with “We are pleased to inform you”. Fortunately the latter happened and my professional life went into stable tracks. This enabled me to invite Harold to a visit in Germany, which he accepted in 1993.

We both liked everything connected with asymptotic eigenvalue distributions, and a fresh conjecture in those days was one raised by Anselone and Sloan [1]. They considered the truncated Wiener-Hopf integral operator given by

$$(W_\tau f)(x) = 2\int_0^x e^{t-x} f(t)\,dt + \int_x^\tau e^{x-t} f(t)\,dt, \quad 0 < x < \tau,$$

Fig. 3 Harold Widom and Estelle Basor at the birthday reception in Santa Cruz.

on the space $L^2(0, \tau)$ and conjectured on the basis of numerical computations that the spectrum of W_τ converges in the Hausdorff metric to the union of the circle $\{\lambda \in \mathbf{C} : |\lambda - 1/12| = 1/12\}$ and the line segment $[3/2 - \sqrt{2}, 3/2 + \sqrt{2}]$ as $\tau \to \infty$. During Harold's visit in Chemnitz in 1993, we understood that the symbol of W_τ is a rational function, $W_\tau = W_\tau(a)$ with

$$a(\xi) = \int_{-\infty}^{0} e^t e^{i\xi t}\, dt + \int_0^\infty 2e^{-t} e^{i\xi t}\, dt = \frac{3 + i\xi}{1 + \xi^2},$$

and that hence Anselone and Sloan's question is a particular case of the more general problem of establishing a Wiener-Hopf analogue of the famous results by P. Schmidt, F. Spitzer, and K. M. Day on the asymptotic eigenvalue distribution of large Toeplitz matrices with rational symbols. We were indeed able to solve the problem and so wrote [15]. The general result of this paper applied to the concrete situation at hand says that a nonzero point $\lambda \in \mathbf{C}$ is in the limiting set of the spectra if and only if the two zeros $\xi_1(\lambda)$ and $\xi_2(\lambda)$ in

$$1 - \frac{1}{\lambda} a(\xi) = 1 - \frac{1}{\lambda} \frac{3 + i\xi}{1 + \xi^2} = \frac{(\xi - \xi_1(\lambda))(\xi - \xi_2(\lambda))}{(\xi + i)(\xi - i)}$$

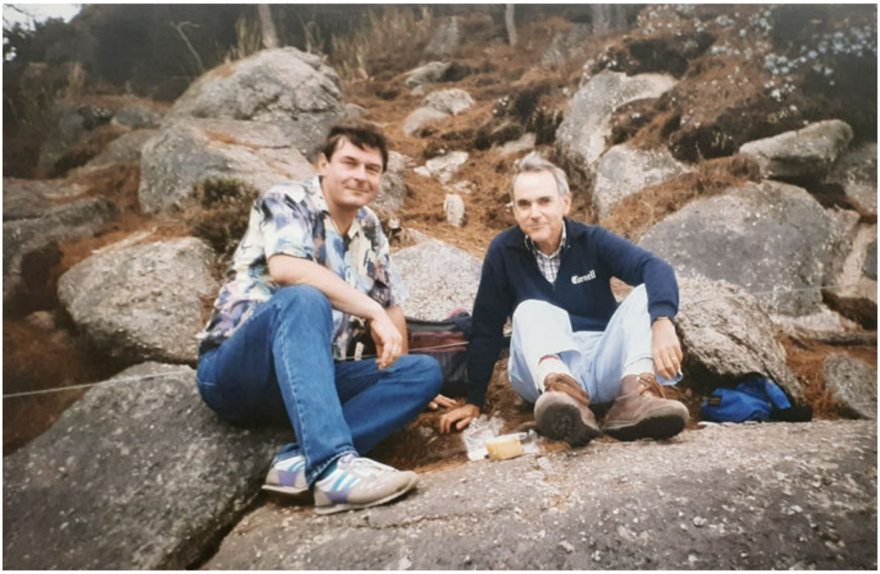

Fig. 4 With Harold at Point Lobos in the Carmel Bay in the south of Santa Cruz.

have equal imaginary parts. Since

$$\xi_{1/2}(\lambda) = \frac{i}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{-4\lambda^2 + 12\lambda - 1},$$

we obtain

$$\begin{aligned} &\operatorname{Im}\xi_1(\lambda) = \operatorname{Im}\xi_2(\lambda) \quad \Longleftrightarrow \quad \xi_1(\lambda) - \xi_2(\lambda) \text{ is real} \\ &\Longleftrightarrow \quad (\xi_1(\lambda) - \xi_2(\lambda))^2 \geq 0 \quad \Longleftrightarrow \quad -4 + 12(1/\lambda) - 1/\lambda^2 =: \delta \geq 0 \\ &\Longleftrightarrow \quad \lambda = 1/(6 + \sqrt{32 - \delta}) \text{ or } \lambda = 1/(6 - \sqrt{32 - \delta}) \text{ with } \delta \geq 0. \end{aligned}$$

The parameters $\delta \in [0, 32]$ give the two line segments

$$3/2 - \sqrt{2} \leq \lambda \leq 1/6 \quad \text{and} \quad 1/6 \leq \lambda \leq 3/2 + \sqrt{2},$$

while $\delta = 32 + \gamma^2$ with $\gamma \geq 0$ yields $\lambda = 1/(6 \pm i\gamma)$, which is readily seen to be a parametrization of the circle $|\lambda - 1/12| = 1/12$, exactly as conjectured by Anselone and Sloan.

Of course, when Harold visited me, we did not only mathematics. Some morning Harold came into my office and proudly reported "Yesterday I was in seven churches of Chemnitz!" And clearly, we also travelled with my car (without power steering and with gear shift) around Saxony, in part together with my family. Figure 5 is a remembrance of one of the short trips.

Fig. 5 In 1993, with Harold and my two children, Eva and Igor, on the Fichtelberg in the Ore Mountains, the highest summit in the eastern part of Germany.

In the years that followed, I met Harold on several occasions and we had of course been in permanent email contact. In 2000, Borodin and Okounkov published a formula that expresses Toeplitz determinants in a form that was asked for earlier by Deift and Its. See [8] and the article [6] in this volume. Basor and Widom [7] found a new proof of this formula, and the two papers [7, 8] landed on my desk from the Mathematical Reviews with the request to write a combined review. When reading them, I discovered still another way of proving the formula. I included my proof into the review (MR1780118 and MR1780119). I also communicated it to a few colleagues involved in business. Harold's wonderful reply was "So you can add yourself to the list of people who can kick themselves for not having found the formula when they were so close." Percy Deift wrote back that he had just finished a joint paper with Jinho Baik and Eric Rains, [2], and asked me whether I could also give proofs in my style for the determinant formulas found there. Fortunately, I was able to manage this.

This was around Christmas of 2000, and I enjoyed myself with the idea to post my proofs in the arXiv and to receive the arXiv identifier 0101001 for the first preprint of the new millennium. Thus, in the early morning of January 1, 2001, I got into my car, drove to my university office, and submitted the preprint (nowadays I could have done this from my computer at home). I was a little too late: the preprint, [9], received the identifier 0101008. The winners with 0101001 were Jinqiao Duan and Bjorn Schmalfuss.

Jacobi's formula says that if K is a trace class operator on $\ell^2(\mathbf{Z}_+)$ such that $I - K$ is invertible, P_n denotes the canonical projection onto the first n coordinates, and $Q_n = I - P_n$, then

$$\det P_n (I - K)^{-1} P_n = \frac{\det(I - Q_n K Q_n)}{\det(I - K)}$$

for all $n \geq 1$. It became clear quite quickly that this formula is at the heart of the Borodin-Okounkov formula. This is implicitly in [7] and explicitly in [9]. More about this can be found in the article [6] in this volume. A question of those days was whether one can also derive other results on Toeplitz determinants from Jacobi's formula, for example, results in the case where the underlying Toeplitz operator has nonzero Fredholm index. Opinions differed. In 2006, Harold and I felt we should save Jacobi and wrote our paper [17]. Its intention was to show that Jacobi's theorem on the minors of the inverse matrix remains one of the most comfortable tools for tackling the matter. We repeated my proof of the Borodin-Okounkov formula and thus of the strong Szegő limit theorem that is based on Jacobi's theorem. We then used Jacobi's theorem to derive exact and asymptotic formulas for Toeplitz determinants generated by functions with nonzero winding number. The latter derivation was new and completely elementary.

In 2002, I participated in the MSRI workshop on random matrix theory in Berkeley which was dedicated to Harold on his 70th birthday. As in 1992, I was overwhelmed by meeting in person so many mathematicians I had until that time known by their names only, for example, Alexei Borodin, Persi Diaconis, Freeman Dyson, Alice Guionnet, Kurt Johansson, Andrei Okounkov, Craig Tracy (in alphabetical order). I myself have always resisted the temptation to try my hands in random matrices, and in the course of this workshop I realized that indeed I had never reached the level of all these mathematical giants and that hence my resistance was very reasonable. So I left Berkeley with a good feeling.

Harold had multifarious mathematical interests, but Toeplitz determinants have never left him. Some day in 2003, I received a manuscript by him which contained an elementary proof of the pure Fisher-Hartwig determinant. This is the determinant of the $n \times n$ Toeplitz matrix

$$T_n(a) = (a_{j-k})_{j,k=1}^n$$

generated by the Fourier coefficients of the function

$$a(e^{i\theta}) = (1 - e^{-i\theta})^\gamma (1 - e^{i\theta})^\delta.$$

Note that the kth Fourier coefficient of a equals

$$(-1)^k \frac{\Gamma(\gamma + \delta + 1)}{\Gamma(\gamma + 1 + k)\Gamma(\delta + 1 - k)} \quad (k \in \mathbf{Z}).$$

The formula for the determinant is

$$\det T_n(a) = \mathbf{G}(n+1)\,\frac{\mathbf{G}(\gamma+\delta+n+1)}{\mathbf{G}(\gamma+\delta+1)}\,\frac{\mathbf{G}(\gamma+1)}{\mathbf{G}(\gamma+n+1)}\,\frac{\mathbf{G}(\delta+1)}{\mathbf{G}(\delta+n+1)},$$

where $\mathbf{G}(z)$ is the Barnes function. In our 1985 paper [13], Silbermann and I derived this formula from a factorization of the Toeplitz matrix $T_n(a)$ due to Roland Duduchava and Steffen Roch. Harold's proof was analogous to the usual derivation of the Cauchy determinant, and its philosophy was that the most elegant way to determine a rational function is to find its zeros and poles. It was self-contained and occupied nearly two pages. I wrote him that I have a proof of less than one page that is based on mere elementary row and column operations. I don't remember the exact wording of Harold's reply, but it was something like "Now that you say this, I remember that I also had such a proof, even before Silbermann and you. I have simply forgotten it. However, as I have never published that proof, this does not count." He invited me to record our two proofs in a short joint paper, and this led to the 4-pager [16]. I hope the reader will also enjoy [22].

Another of my mathematical adventures connected with Harold is described in my contribution to [5]. Our paper [18] is a continuation of the story told in [5]. Let α be a natural number and consider the eigenvalue problem

$$\begin{aligned}&(-1)^\alpha u^{(\alpha)}(x) = \lambda u(x) \ \text{ for } \ x \in [0,1],\\ &u(0) = u'(0) = \ldots = u^{(\alpha-1)}(0) = 0, \quad u(1) = u'(1) = \ldots = u^{(\alpha-1)}(1) = 0.\end{aligned}$$

This problem has countably many eigenvalues, which are all positive and converge to infinity. Let $\lambda_{\min,\alpha}$ denote the smallest of them. In an earlier paper we proved that

$$\lambda_{\min,\alpha} = \sqrt{8\pi\alpha}\left(\frac{4\alpha}{e}\right)^{2\alpha}\left(1 + O\left(\frac{1}{\sqrt{\alpha}}\right)\right) \ \text{ as } \ \alpha \to \infty.$$

For $\alpha = 3$, the minimal eigenvalue $\lambda_{\min,3}$ is exactly equal to $(2\pi)^6$. We wanted to understand whether this coincidence is an accident or not. Paper [18] gives an answer. In the case $\alpha = 3$, it is convenient to start indexing the eigenvalues with $n = 2$, that is, to denote the eigenvalues by

$$\lambda_2\,(=\lambda_{\min,3}\,),\, \lambda_3,\, \lambda_4,\, \ldots.$$

We proved that $\lambda_n = (n\pi)^6$ if n is even and that $\lambda_n = (n\pi + \delta_n)^6$ if n is odd, where the δ_n's are *nonzero* numbers satisfying

$$\delta_n \sim 8(-1)^{[n/2]+1} e^{-(\pi\sqrt{3}/2)n} \ \text{ as } \ n \to \infty;$$

here $[n/2]$ is the integral part of $n/2$. Yes, Harold loved asymptotics! Let us write $\lambda_n = \mu_n^6$. Mark Embree computed the first five μ_n up to ten correct digits after the comma. The following list shows the values of the first five μ_n and of the first five $\Delta_n := 8(-1)^{[n/2]+1} e^{-(\pi\sqrt{3}/2)n}$.

$$\begin{aligned}
\mu_3 &= 9.4270555708 = 3\pi + 0.0022776101 & \Delta_3 &= +0.0022821082\\
\mu_5 &= 15.7079533785 = 5\pi - 0.0000098894 & \Delta_5 &= -0.0000098893\\
\mu_7 &= 21.9911486179 = 7\pi + 0.0000000428 & \Delta_7 &= +0.0000000428\\
\mu_9 &= 28.2743338821 = 9\pi - 0.0000000002 & \Delta_9 &= -0.0000000002\\
\mu_{11} &= 34.5575191894 = 11\pi + 0.0000000000 & \Delta_{11} &= +0.0000000000.
\end{aligned}$$

I met Harold for the last time in Edinburgh in 2007. However, our correspondence remained alive over the years. Let me finish with the last joint mathematical adventure with him.

In 2008, I received a (beautifully handwritten) letter from Peter Dörfler with the question whether I could help with the large n behavior of the maximal singular value (= spectral norm) of the $(n+1) \times (n+1)$ triangular Toeplitz matrices

$$T_n = (-1)^\nu \begin{pmatrix} 0 & \binom{0}{\nu-1} & \binom{1}{\nu-1} & \dots & \binom{n-1}{\nu-1} \\ & 0 & \binom{0}{\nu-1} & \dots & \binom{n-2}{\nu-1} \\ & & & \ddots & \vdots \\ & & & & \binom{0}{\nu-1} \\ & & & & 0 \end{pmatrix},$$

composed of binomial coefficients with an integer $\nu \geq 1$. The matrix T_n is the representation of the operator taking the νth derivative, $f \mapsto D^\nu f$, in the orthonormal basis of Laguerre polynomials in the space $\mathcal{P}_n$ of algebraic polynomials of degree at most n with the Laguerre norm given by $\|f\|^2 = \int_0^\infty |f(x)|^2 e^{-x}\,dx$. Thus, the norm $\|T_n\|$ is just the best constant for which the so-called Markov-type inequality $\|D^\nu f\| \leq M\|f\|$ holds for all $f \in \mathcal{P}_n$.

This question reminded me of an ingenious trick used in Harold's 1966 paper [23] (and employed independently also by Lawrence Shampine in [20]). Given an $n \times n$ matrix $A_n = (a_{jk})_{j,k=0}^{n-1}$, denote by H_n the integral operator on $L^2(0,1)$ with the piecewise constant kernel $h_n(x,y) = a_{[nx],[ny]}$, where $[\cdot]$ stands for the integral part. Widom and Shampine proved that $\|A_n\| = n\|H_n\|$. Thus, instead with having the matrices A_n on the sequence $\{\mathbf{C}^n\}$ of increasing spaces, we so can work with a sequence $\{H_n\}$ of operators in one and the same space $L^2(0,1)$. The goal is to show that after appropriate scaling the operators H_n converge in the operator norm to some nonzero operator H, that is, $n^{-\mu}H_n \to H$ in norm. This would imply that $n^{-\mu}\|H_n\| \to \|H\|$ and hence $\|A_n\| \sim \|H\| n^{\mu+1}$.

To compute $\|T_n\|$, we may ignore the factor $(-1)^\nu$ and the diagonal of zeros. In the resulting $n \times n$ matrix, the j,k entry is equal to $\binom{k-j}{\nu-1}$ for $j < k$. Thus, if $x < y$

then the kernel of the scaled integral operator $n^{-(\nu-1)}H_n$ is

$$\begin{aligned}\frac{1}{n^{\nu-1}}a_{[nx],[ny]} &= \frac{1}{n^{\nu-1}}\binom{[ny]-[nx]}{\nu-1}\\ &= \frac{1}{(\nu-1)!}\,\frac{[ny]-[nx]}{n}\,\frac{[ny]-[nx]-1}{n}\cdots\frac{[ny]-[nx]-\nu+2}{n},\end{aligned}$$

and this converges uniformly to $(y-x)^{\nu-1}/(\nu-1)!$ as $n\to\infty$. In the end we obtain the asymptotics $\|T_n\| = \|L_\nu\| n^\nu(1+o(1))$ where L_ν is the Volterra integral operator on $L^2(0,1)$ given by

$$(L_\nu f)(x) = \frac{1}{(\nu-1)!}\int_x^1 (y-x)^{\nu-1} f(y)\,dy.$$

Clearly, $L_\nu = L_1^\nu$ (= νth power of L_1) and $\|L_\nu\| = \|L_\nu^*\|$ with

$$(L_\nu^* f)(x) = \frac{1}{(\nu-1)!}\int_0^x (x-y)^{\nu-1} f(y)\,dy.$$

Note that it is well-known that $\|L_1\| = 2/\pi$. I refer to paper [10] for more on the subject and in particular for more about pieces of the amazing story around the norms of the Volterra operators L_ν.

After 2012, Holger Langenau was a PhD student (a Doktorand in German) of mine. He worked on best constants in Markov-type inequalities between spaces with different weights. In a large range of more general cases, things are not as simple as in the preceding paragraph, but we still encounter constants involving the operator norm of certain Volterra integral operators and the proofs can be based on the happy circumstance that these operators are Hilbert-Schmidt. The conjecture was and still is that in the remaining cases the same operators occur. A proof is outstanding. One (but not the only) step towards a proof is to show that certain operators are compact. Holger Langenau and I were able to prove the compactness of these operators, even their membership in certain Schatten classes, but for one of them the proof was extremely intricate and occupied many pages. We submitted the paper to Birkhäuser's OT volume containing the proceedings of the IWOTA 2014 in Amsterdam. As usual with the IWOTA proceedings, the submissions were strongly refereed. The report we received on our submission was positive but also contained an elegant argument that reduced the many pages we needed for the compactness of the one operator to about a single page. We asked the handling editor to ask the referee whether he or she would be willing to release anonymity and to join us as co-author. The referee agreed to the proposal and—you guess it—the referee was none other than Harold Widom. The 2016 paper [11] was the result of our joint effort. Holger Langenau was a great admirer of Harold Widom and was therefore full of joy and pride for having made it to a joint publication with him.

I am really glad and thankful for having had Harold as a partner and friend for decades. He is in the top of the many colleagues who have strongly influenced my interests and my way of doing mathematics. Now he has left us, but as the story with Holger Langenau reveals, I am sure that his name and achievements will live on and inspire future generations.

Credits The photos are courtesy of the author.

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Memories of Harold Widom

Richard A. Libby

Abstract This tribute shares personal memories of Harold Widom during the author's years in and after graduate school. These memories reflect on Harold's skills as a teacher and mathematician, but also on how his insights proved valuable in applications of mathematics in the author's professional work.

Keywords Harold Widom · UC Santa Cruz · Toeplitz operator

To begin a memorial tribute it is best to begin at the beginning. An invisible hand in this story is that of my grandfather, who gave me a large dose of academic career advice shortly before he passed away. He asserted that it was a mistake to get one's PhD from the same university as one's undergraduate degree. I was in my first year of a PhD program at UC San Diego, which had conferred my BA degree in mathematics the previous June. My father's father was a retired statistics professor from USC who had climbed his own career ladder into a successful administrative role and therefore probably knew the value of his advice, which I took, along with an MA in mathematics soon thereafter. (Among Harold's eight students Estelle Basor is a notable counterexample to my grandfather's advice.)

UC Santa Cruz as the choice for my eventual PhD came down to two ingredients: first, a known tenure dispute at the time in the Berkeley math department seemed too similar to my grandfather's stories of academic politics and so the second ingredient became a side trip to the UC Santa Cruz admissions office, where I received a very impressive brochure covering the accomplishments of its own faculty, including Harold's by then successful research in Toeplitz and Wiener-Hopf operators. I was duly admitted in the Fall of 1984.

My first interaction with Harold was in his core graduate analysis course using his lecture "Notes on Measure and Integration" as written up by David Drasin and Tony Tromba. As I write this memorial article I thumb through the pages of these notes conveniently bound with the back of each page left blank for the copious scribbled

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notes of additional course materials, often proofs of results using techniques of obvious general utility. Harold's lecturing style was quick and precise, interspersed with occasional comments more about style or strategy of proof than the specific content of the theorem at hand. I later took Harold's Functional Analysis course, which increased the quantity and latitude of good ideas worth copying onto the blank backs of pages in the bound lecture notes as well as the soul of wit evidenced in the brevity of Harold's lecturing style. In this course Harold would entertain speculative disruptions from some of the students as to the importance of various mathematicians or the sophistication of their techniques, but if a student asserted anything false in the way of mathematical content, Harold would pounce and rebut with solid and efficient reasoning. Out of respect for Harold I generally kept quiet during these side discussions except for one time when Harold wondered out loud, while lecturing on Cesàro summation and the Fejér kernel, where the accent went in spelling Lipót Fejér's last name. I offered a response based on my knowledge of French pronunciation while foolishly assuming Fejér was French and that we were pronouncing it as such. Harold gave me a funny look and I took it as a stroke of good luck on such a small matter, to only jump into these things in the future when I had as solid an understanding as Harold evidenced each day, and when I was positive I would be wasting no one's time in doing so. With this course I decided I would be even more foolish to not ask Harold to be my thesis advisor, given the quality of his work and what seemed my own mathematical predisposition to functional analysis and operator theory.

In all the courses and seminars where Harold either lectured or took part he always commanded a vast breadth of knowledge and could ask penetrating questions, the value of which I would sometimes only discover much later in life. When lecturing he would occasionally hit a snag of some kind, saying, "Hold on ... hold on!" as he studied the blackboard intently before delivering the missing detail or shifting direction slightly in the argument at hand. I must admit I acquired this technique and have discovered its principal value is in preserving the audience's attention across the pause in the stream of ideas, the momentum of which is a truly valuable commodity in the hands of any lecturer. I have made good use of another quote of Harold's showing his appreciation for self-referential paradox: "One thinks about mathematics to figure out why one does not need to think about mathematics." On another occasion in one seminar the presenter made use of the heat kernel, prompting Harold to ask a pair of rhetorical questions about the heat equation: was it not true the equation presumes information travels infinitely far in an infinitesimal amount of time? How would we modify it knowing nothing travels faster than the speed of light? Later in life I borrowed Harold's line of questioning for use against a different parabolic PDE used in finance, the Black-Scholes equation, which since its discovery has led to a number of financial losses for those who think of it as an exact answer rather than as an approximation.

One-on-one discussions with Harold during my later years in graduate school were a good way to polish one's argumentation style. The department had at the time the number theorist Sol Friedberg who introduced a few of us to the Circle Problem: how many integer lattice points are found inside a circle of radius R

centered at the origin? I poked at it a bit and came up with the estimate πR^2 with an error $O(R)$, basically duplicating the result Gauss found back in the early nineteenth century by putting the lattice points inside unit squares and then bounding the area above by a larger circle and bounding below by another, smaller one. The problem is of interest in part because the actual error seems to be quite smaller, perhaps $o(R^{1/2+\epsilon})$, an open problem. One day Harold brought up the Circle Problem on his own and asked me what the estimate should be. I said, "πR^2" and Harold immediately asked, "Why?" Now, Gauss' (and my) methods are true, but rather dull. Harold appeared more interested in why the intuition should be correct, so I gave him a different proof. I swapped the circle of radius R for one of unit radius and made the lattice points take integer values divided by R and sit within squares having side length $1/R$. I said, "Now, organize those little squares into columns and you have a picture you could show to undergraduate students demonstrating Riemann sums. The estimate holds because those little squares each have area $1/R^2$ and because, as we all know, Riemann integration works!" Harold said, "Right!"

The Circle Problem led to my learning of Harold's deep suspicion of all results computer generated. The Problem's conjectured error term was easily supported by computer evidence obtained in the department's computer lab, but Harold had no interest in seeing it. He did, however, ask me to calculate finite Toeplitz determinants for a symbol of interest to him. When I showed him the result, he was clearly pleased but absolutely did not want any more computer time spent on it. Around this time Harold received a batch of computer generated plots of eigenvalues for finite Toeplitz symbols related to the Fisher-Hartwig conjecture. He showed me one that was consistent with the conjecture and when I showed enthusiasm for the result he immediately showed me a second one that was not consistent. I knew enough about "machine arithmetic" to understand that potential rounding errors in the calculations showed the limitations of this kind of research.

Outside of the lectures, seminars and my one-on-one discussions with Harold of a mathematical nature, I had the opportunity to see Harold as Department Chair for several years. I succeeded another graduate student who, having taken the PhD, was no longer the student representative at faculty meetings and, with no other volunteers stepping forward, I took over the role. I found myself occasionally caught between a student's complaint about faculty decisions and the faculty who made them. I learned quickly how to represent a student without implying I was taking sides and found Harold was quite fluent in this valuable skill. Harold had a very precise sensibility about the importance of rules of order in meetings and would never tolerate the breaking of rules around confidentiality of certain information and similar items. With Harold running them, I found myself entering and exiting ongoing meetings based on what was about to be discussed.

It was during this time that the department received one of its periodic visits from the accreditation committee, who asked to interview a graduate student as part of the process. I found myself on a team of four students who spoke with the committee. Later on, Harold passed me a copy of the draft accreditation report that covered many topics and gave both compliments and criticisms. The report also contained a somewhat withering comment, “The graduate students seem to be a happy lot.” My sense is that we were simply charmed by the attention from a distinguished group of outside professors. Along with Harold’s occasional comments on how one conducts one’s research with the goal of attaining tenure, I found the report’s materials as efficient and precise an academic career guide as Harold’s coaching and teaching of mathematics.

It was also during this time that I fell in with a number of musicians and theater people, my amateur talents as a pianist having found a bit of a home among them. During a 1987 summer stock production of “Tomfoolery,” a London West End musical revue based on the comedy songs of Tom Lehrer, Harold took in a performance and gave me a great deal of encouragement afterwards. As it happened Tom Lehrer appeared on campus each spring to teach a theater arts course and a liberal arts introductory math course. Tom called the latter course “Math for Ribbon Clerks” and for one term I was his teaching assistant. In retrospect I appreciate Harold’s and the department’s tolerance for my side excursions into music, which preserved some of my sanity as the pressure to finish the PhD within a reasonable time naturally grew more urgent.

Graduate school is likely impossible without some form of setback. In my case my first attempt at an oral candidacy exam went awry over a rules challenge. A major theme of the 1980s was the cross fertilization of different areas of mathematics and I got the idea I should reflect this trend strategically by doing a candidacy exam on the Atiyah-Singer Index Theorem, an idea I succeeded in convincing a committee to undertake. Before the exam one professor bowed out and the replacement immediately objected on the grounds that the exam is supposed to be about a field of mathematics, not a theorem. As the committee chair Harold pointed out that the exam covered two fields, analysis and topology, but could not convince the new committee member to change his mind. After a few minutes of this impasse I could see in Harold’s eyes an idea had formed and he quickly and quietly brought about agreement we would cancel the exam. Shortly afterwards Harold shared his idea, that instead of the theorem I could do an exam on pseudodifferential operators and not only satisfy the rule but also cut the material needed to demonstrate mastery exactly in half. Unfortunately, Harold also mentioned he would soon be taking a year’s sabbatical, so instead of taking his suggestion, and in the interest of time, I reformed the committee on the more general topic of partial differential equations. In retrospect this setback was very fortunate for my later career in banking when doing battle with abuses of the Black-Scholes equation. I take the true lesson learned from the experience was that navigating graduate school should have been more tactical and less strategic.

In comparing stories with Harold’s other PhD students I have since discovered I was not the only one to tackle one thesis problem before switching to another. My first choice was to derive a Szegő theorem for spherical harmonics, a task that soon ended in a sea of Clebsch-Gordan coefficients having no discernible pattern. At this point the switch to a special case of the Fisher-Hartwig conjecture seemed a good choice given that I had an inkling as to how to do it. In writing this memorial I have Harold’s 1973 paper “Toeplitz Determinants with Singular Generating Functions” next to me as a reminder of which techniques were Harold's and where mine began. Fewer dissertation topics could have been a better fit. Most mathematicians learn of Cramer's rule in either high school or in their early college years and Jacobi's generalization to matrix minors therefore makes for quick study. The Euler-Maclaurin summation formula is also a quickly acquired skill by anyone who knows integration by parts. As it happens the coordination of these techniques with Harold's paper resulted in calculations that grew horrendously more complicated as the size of the matrix minor grew and the dissertation stopped with the two-by-two case of the matrix minor, extending the measured gap of a single discontinuity only from less than 1/2 in absolute value as found in Harold's paper, to less than 5/2, enough for a dissertation but nothing to shout from the rooftop of the mathematics department.

Acquiring a PhD in December 1990 rather than in June of any year meant my degree was in a sense out of season and I took a series of temporary jobs to pay bills while looking for a way to improve my dissertation result and look for more permanent work. Remembering Harold's words about understanding the structure of a result as well as the details, I could hypothesize how the asymptotic formulas in my work might extend themselves. Assuming this hypothesis I carried out a three-by-three case of the matrix minor without resorting to the detailed calculations of my thesis and got the result I was hoping for. Writing a paper extending the measured gap of the discontinuity to less than 7/2 in absolute value seemed an abuse of the "publish or perish" strategy graduate students learn about at an early stage of their career. The temporary jobs continued their iterations while I sought to apply Harold's advice to a more complete solution to this problem. At some point I realized that no matter how awful the calculations were, the choices as to what constituted each next step were only three in number, and each of the three added a term to the asymptotic expansion consistent with my hypothesis. By induction I therefore had a general theorem at hand and could apply it to the n-by-n case of the matrix minor. One brute force calculation remained evaluating an n-by-n determinant composed of entries having the demonstrated expansion and the result that came out fit like a jigsaw piece into the earlier results. The limitation on the size of the discontinuous gap was now removed.[1]

I shared these results with Harold and had the good fortune to be doing this work ahead of the conference organized in 1992 in honor of his 60th birthday. Harold's feedback was that he was convinced I was right, but if I was going to present these results at the conference, convincing the audience was going to be my job. By this time I was working in a bank solving operating errors and they consented to my taking time off to attend the conference and present these final results.

I saw Harold quite a few times in the years immediately following the PhD. His advisor Irving Kaplansky was still the head of MSRI and I attended a dinner with

[1] The expansion has a pattern that depends on the size of the discontinuous gap and the brute force calculation will in general not work in the case of two or more discontinuities with different sized gaps.

him and Harold in Berkeley where I could see how they shared the same spark for mathematical discovery. It was around this time my work in banking resulted in being offered a somewhat senior role in quantitative risk analysis that I could not turn down. My search for an academic role had coincided with the collapse of the Soviet Union and mathematics jobs were suddenly and then stubbornly hard to find. I saw less of Harold during my later banking years but the almost unreasonable effectiveness of all that he taught me was put to good use nonetheless.

Three ingredients in Harold's work that have had significant impact on my work in quantitative risk analysis are, first, linear operator theory, second, the use of projection operators and, third, the use of asymptotic methods where advantageous or appropriate. A portfolio of financial assets changes value over time in a process modeled approximately in terms of relative Brownian motion of incremental asset returns. A covariance matrix of asset return volatilities measures the uncertainty of the portfolio's value in the future. That the uncertainty is proportional to the square root of time under the Brownian motion model can be easily derived using multiple convolutions of the probability density with itself when time is considered in discrete intervals. The risk profile of a portfolio has an almost natural expression in terms of linear operators applied first to a Dirac distribution at time zero, the time at which the portfolio value is known in compete certainty, and then later to successive iterations. Projection operators appear in the analysis of optionality. We cut off any probability density via a projection operator in any part of the domain which results in the option expiring without value and these truncated probability densities may be included in the iterated convolutions already mentioned.

The value of knowing Harold's work in asymptotic analysis had a somewhat late appearance in my career, in the determining of the regulatory capital a bank needs to hold against its own operating errors, a capital requirement all banks have faced since the mid 2000s.[2] Quantitative finance has a certain love for Monte Carlo analysis, which works well modeling assets assumed to have reasonably stochastic returns, but which works poorly when modeling extreme events like tsunamis, earthquakes and the occasional upheaval in financial markets. I had made extensive use of Monte Carlo techniques in the first decade and a half of my banking career, but it was clearly a poor fit for modeling extreme events like large potential operating errors. Econometric modeling of the bank's operating error history suggested a "power law" distribution, such as a Pareto distribution having finite mean but infinite variance. Modeling the individual operational errors was straightforward but regulations required this analysis be done for cumulative errors over a year. With some digging a result by the statistician William Feller identified an asymptotic formula for the n-fold convolution of Pareto distributions, where n would be taken as the mean number of operating errors in a year. An analysis was now possible in the form of a spreadsheet instead of a massive computing exercise requiring technology consulting at considerable expense. Acquiring a PhD under Harold once again showed its value.

[2] The so-called "Basel Capital Accord," see www.bis.org for its history.

While the timing of my PhD coincided rather badly with a downturn in the academic job market I was offered not terribly long ago a temporary adjunct position substituting for a friend taking a yearlong sabbatical, teaching a masters level course in econometrics to two different batches of financial analysts in training. Being Harold's student gave me a model for how to lecture, how to hold the students' attention and how to scale the course content to the needs and abilities of my students. The only improvement possible was technological, with more slides and less chalk dust.

As I approach a traditional retirement age I have stepped up my involvement in both my undergraduate and graduate alumni associations and have targeted my support to things students need, in the spirit of the quality Harold consistently delivered when I was a student. In the world of banking, fewer financial meltdowns would have been the result of better technical training among risk managers, who as a rule know a bit of statistics and how to run an operations department, but not much on how often stochastic methods do not picture reality all that well. If Harold had opted for my line of work rather than academic research I have no doubt he would have been the terror of foolish optimists and practitioners of financial hubris. In the late 2010s I would occasionally visit the Santa Cruz campus on projects related to support for undergraduate education but Harold's health matters prevented spending time together. We did exchange emails on a number of things, including our joint admiration for the ragtime music of William Bolcom.

Harold lived a long and full life that nonetheless was cut short too early. He will be missed, but, more importantly, he will be remembered for the richness he brought to his students, to his family and to the mathematical community at large.

Personal Reflection on Harold Widom

Bin Shao

Abstract This essay contains a collection of memories of Harold Widom from the author's perspective as his Ph.D. student at UCSC and the years beyond. It shares a sketch of stories from a personal background and friendship, and covers a memorable view of Harold's life and work over a period of 25 year. Harold simply radiated boundless enthusiasm and respect for mathematics that has influenced many of his students and colleagues

Keywords Harold Widom · UC Santa Cruz · Toeplitz matrix · Random matrix · Wiener-Hopf operator · Pseudodifferential operator

1 The Unforgettable 1992

It was in Santa Cruz, in the mid-September of 1992, that I became a student of Harold Widom. It was also the time and place that a special conference on Toeplitz and Wiener-Hopf operators was held in celebration of his 60th birthday. Many active and prominent mathematicians worldwide were present, and the research presenters were full of praise for Harold's contributions to the mathematical community.

Harold had kindly convinced me to attend this conference as it could be beneficial to developing my thesis work. To show my support I gladly volunteered to arrange for the conference refreshments. Throughout the 3-day conference, it was an eye-opening experience to hear the level of stimulating conversations by a host of world class mathematicians. Indeed, like all participants, I felt excitement from the celebration of Harold's mathematical accomplishments. There were several very famous figures in the field of operator theory attending this meeting. During the coffee breaks, Harold introduced me to his dissertation advisor, Irving Kaplansky,—"your (academic) grandpa", as he kindly put it. Irving was the director of MSRI

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(Berkeley) at the time. He proudly spoke about the career of Harold's early research since the "Stone Age" at the University of Chicago.

2 UC Santa Cruz (1992–1994)

About a year prior to the 1992 Conference and after coming to Santa Cruz, I already knew that Harold was a world-class mathematician. He was also an erudite and cultivated person, who liked to read, listen to and play music and hike. Harold's musical talent was well known and I knew he had performed in chamber and music groups and orchestras at Cornell and UC Santa Cruz. A brilliant man of mathematics always attracts students' interest in the subject. Harold was one of a few professors for whom I had great admiration since the beginning of my student years at UCSC.

At the time when I took a course on functional analysis taught by Harold, I was deeply impressed by his style of lecturing, which frequently gave a vivid account of making a seemingly difficult concept abundantly clear with minimal wording. Written notes on less than a quarter page is all he needed to expand upon throughout the lecture time. His crafted board-work and articulate lectures have always been inspiring and influential on my academic career.

As a student, I benefitted from the extraordinary learning experience by Harold's art of lecturing and from his original approachability in mathematics teaching. His penetrating capacity in research and his ability to cast problems in a different light have always been a source of inspiration in my research activity. His insightful and constructive suggestions were invaluable and instrumental for the completion of my thesis work and stirred up my passion for mathematical research.

Despite his tight daily work schedule, Harold actively kept a close interaction with his students for their dissertation progress in all aspects. He used his coffee break to share ideas of doing mathematics. My favorite story was once asking for the motivation of his special proof for the connectedness of the spectra of Toeplitz operators, which he patiently gave me in bits of crucial ideas using the chalkboard in my compartment next to the coffee room across his office. That is certainly one of the most memorable and pleasant moments of mine at UCSC.

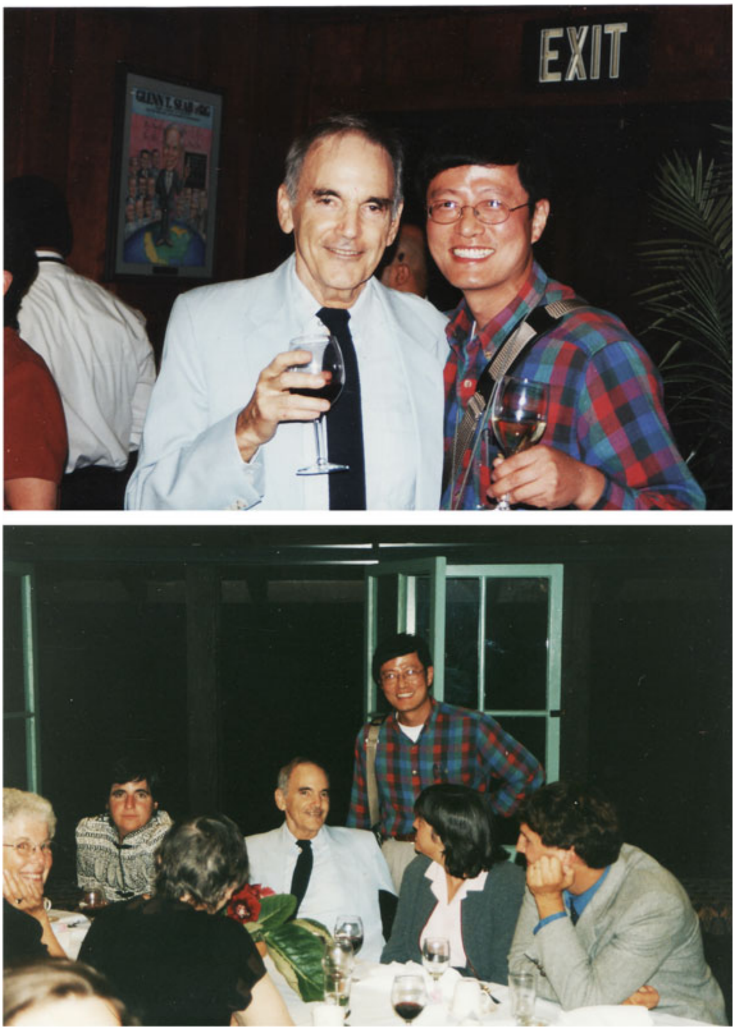
Going over the history of email correspondence with Harold, I recall that he made sure that I was financially secure during my thesis years. On several occasions, he wanted to know whether I had plans for improving myself over the summer. When he learned that I was attempting to study several topics on pseudo-differential operators and read various research papers to get ideas for my thesis, Harold managed to find some funds through a NSF grant to support my research in progress. That was a great help and lasted three summers, for which I have been eternally grateful. Another incidence is that he asked me to send a copy of my thesis to Persi Diaconis at Stanford and I did accordingly. He was so apologetic by not making it clear to let the department handle this outgoing mail, after he found out that I did it with a certified mail on my own. He insisted on covering the postage as he was trying to find out how much I paid for it. Harold always treated his students with generosity and thoughtfulness.

In the summer 1993 I actually got some results for my thesis using Harold's technique of localization for the asymptotics of trace class operators. Upon reading it, he confirmed that my result was correct and recognized my effort of making progress. I could not be more excited after receiving his encouraging comments. Of course having his assurance was such great news and he certainly sensed my joy. However, he kindly encouraged me to build confidence by performing a self-check for work in the future. Indeed, over the years I was always very appreciative of his encouragement.

3 Friendship

Harold and I remained in contact after completing my PhD under his supervision. He suggested several problems that I could work on, sending me a number of research papers. I felt very privileged to be treated as his colleague by his humble way of communicating through emails and postal mails. He was always precise and effective when it came to scholarly communications, which I deeply appreciated. In my heart he is always my beloved professor and a faithful friend. He encouraged me to attend research conferences while sharing his conference experience in Eastern Europe. For example, when he learned that I had trepidation to attend an invited conference in Bulgaria in 1999, he convinced me not to miss the opportunity. I ended up going there without regrets by the support of Santa Clara University and the work I presented also resulted in publication.

In the year of 2002, he was very pleased to see my presence, together with a group of prominent mathematicians as well as Harold's family members, celebrating his 70th birthday at UC Berkeley. I told him that I just attended ICM, Beijing, and presented a paper (accepted by IEOT) on the singular values of variable-coefficient Toeplitz matrices, extending one of his results in the Toeplitz case. I also told him that I attended the plenary lecture by Craig A. Tracy, who had been long collaborating with him, on their joint work on distribution functions for the largest eigenvalues of random matrices. This stimulated a memorable conversation which we enjoyed very much. During the celebration of Harold's 70th birthday, I got the message that everyone continued to be amazed and dazzled by the fact that Harold was still producing elegant theorems. The prediction was that he would continue to be doing so for at least another decade. This became undoubtedly true as can be seen from the records of his achievements in later years. I recall sending him a warm note of congratulation and a good wishes on his 80th birthday in 2012, which he incidentally spent at a conference at Banff, and he, of course, was very happy to hear from me in reply.



Celebration of Harold Widom's 70's birthday with his family, UC Berkeley.

4 Closing

Harold Widom could look back over the achievements of the past seven decades and find satisfaction in the acknowledged superiority of his methods in teaching and the extraordinary ability as a world-class research mathematician. I find myself very fortunate and feel honored to be his student. His quickness of casting a mathematical problem in a different light has been illuminating and inspiring. As one of the great heroes of the mathematical frontiers, Harold rightfully belongs to the world's greatest contributors to the progress in mathematics. Mathematics has lost one of its most articulate, original, and insightful minds. His kind, “Widom style”, does not come along often and will be dearly missed. His departure is a great loss for the mathematics community and his footprints will forever be seen in the world of mathematics.

Credits The two pictures of Harold Widom in 2002 are the courtesy of the author.

Part III
Invited Contributions

Loops in $SU(2)$ and Factorization, II

Estelle Basor and Doug Pickrell

Abstract In the prequel to this paper, we proved that for a $SU(2, \mathbb{C})$ valued loop having the critical degree of smoothness (one half of a derivative in the L^2 Sobolev sense), the following statements are equivalent: (1) the Toeplitz and shifted Toeplitz operators associated to the loop are invertible, (2) the loop has a unique triangular factorization, and (3) the loop has a unique root subgroup factorization. These equivalences hinge on factorization formulas for determinants of Toeplitz operators. The main point of this sequel is to discuss generalizations to measurable loops, in particular loops of vanishing mean oscillation. The VMO generalization hinges on an operator-theoretic factorization for Toeplitz operators, in lieu of factorization for determinants.

Keywords Toeplitz · Hankel · Vanishing mean oscillation · Factorization

Mathematics Subject Classification (2020) 22E67 (22E65, 30F20, 32A45, 32C36, 47A68, 47B35)

1 Introduction

This paper concerns the Polish topological groups of maps $W^{1/2}(S^1, SU(2))$, $\mathrm{VMO}(S^1, SU(2))$, and $\mathrm{Meas}(S^1, SU(2))$ (equivalence classes of $SU(2, \mathbb{C})$ valued loops which have one half of a derivative in the L^2 Sobolev sense, are of vanishing mean oscillation, and are Lebesgue measurable, respectively; the basic background—such as the Polish topologies of these groups—is recalled in Sect. 2). In an attempt to motivate the subject matter, we first consider a broader perspective.

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Suppose that K is a compact Lie group. The equatorial inclusions

$$S^0 \subset S^1 \subset S^2 \subset S^3 \subset \ldots \tag{1}$$

induce (down arrow) inclusions and (left to right arrow) trace homomorphisms of groups

$$\begin{array}{lllll}
\ldots \to & C^\infty(S^3, K) & \to C^\infty(S^2, K) & \to C^\infty(S^1, K) & \to C^\infty(S^0, K) \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\ldots \to & W^{3/2}(S^3, K) & \to W^1(S^2, K) & \to W^{1/2}(S^1, K) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\ldots \to & \mathrm{VMO}(S^3, K) & \to \mathrm{VMO}(S^2, K) & \to \mathrm{VMO}(S^1, K) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\ldots & \mathrm{Meas}(S^3, K) & \mathrm{Meas}(S^2, K) & \mathrm{Meas}(S^1, K) &
\end{array} . \tag{2}$$

The groups of smooth maps are Frechet Lie groups (see Sect. 3.2 of [10]), hence it is known what they look like locally, and their global topology can be analyzed using conventional methods of algebraic topology.

For the groups $W^{d/2}(S^d, K) \subset \mathrm{VMO}(S^d, K) \subset \mathrm{Meas}(S^d, K)$, generic group elements are not continuous mappings (Recall that $s = d/2$ is the critical L^2 exponent: the Sobolev embedding $W^{s,L^2}(S^d) \to C^0(S^d)$ holds for $s > d/2$ and marginally fails for $s = d/2$). The usual approach to understanding the local structure of continuous mapping groups is to fix a proper open coordinate neighborhood of $1 \in K$ (homeomorphic to $\mathbb{R}^n$, say) and consider the set of maps with image in this neighborhood. This fails in our context because generic group elements in this set are locally unbounded, and hence this set is not an open neighborhood of $1 \in W^{d/2}(S^d, K)$ (or VMO, or Meas). For similar reasons conventional methods of algebraic topology do not apply to understand the global topology. This is problematic, because it is important to understand the local and global topology of these (Polish) mapping groups; see [4], [5], [3], and references, for foundational work in this direction and further motivation. The simplest hypothesis—this is pure speculation—is that for all $d \geq 1$, $W^{d/2}(S^d, K)$ and $\mathrm{VMO}(S^d, K)$ are topological manifolds (they are definitely not smooth Lie groups as Polish topological groups), and the inclusions

$$C^\infty(S^d, K) \to W^{d/2}(S^d, K) \to \mathrm{VMO}(S^d, K) \tag{3}$$

are homotopy equivalences. This is exemplified by the existence of trace maps for VMO (see [5], and note we are considering an equatorial trace) and the nonexistence of trace maps for measurable maps in the above diagram. More directly relevant to this paper, in the elemental case $d = 1$, the global topology for the smooth loop space is intimately related to the map

$$C^\infty(S^1, K) \to \mathrm{Fred}(H_+) : g \to A(g) \tag{4}$$

where $A(g)$ is the Toeplitz operator with symbol g (see Chap. 6 of [10]); the point is that VMO(S^1, K) is the natural domain (see Proposition 1 below for a more precise statement).

Remark 1 Meas(S^d, K) is an outlier in this topological digression. Since its definition depends only upon the Lebesgue measure class of S^d, it is isomorphic to Meas($[0, 1], K$), and it is contractible.

In this paper $d = 1$, unless noted otherwise. In this case the claim about the homotopy equivalences basically follows from the Grassmannian model approach in Chap. 8 of [10] (with modifications). We are mainly interested in technology which is useful in understanding the local structure. We will focus on $K = SU(2)$ (see [9] for the general Lie theoretic framework). In the prequel to this paper, we showed that for $g \in W^{1/2}(S^1, SU(2))$, the following statements are equivalent: (1) the Toeplitz and shifted Toeplitz operators associated to g are invertible, (2) g has a unique triangular factorization, and (3) g has a unique root subgroup factorization (we will review this in Sect. 3). This is a statement about the (open) top stratum of the $W^{1/2}$ loop group, and there is a generalization to the finite codimensional lower strata. The key to the equivalence of (1)–(3), and in truth the more interesting point, is that there exists an explicit factorization for $\det(A(g)A(g^{-1}))$, akin to the Plancherel formula in linear Fourier analysis (see (32)). A corollary of this is that $W^{1/2}(S^1, SU(2))$ is a nonsmooth topological manifold modeled on l^2, and it is homotopy equivalent to the smooth loop group.

Remark 2 The scalar $\det(A(g)A(g^{-1}))$ appears prominently in Harold Widom's landmark paper [12], as the constant term in the expansion of determinants of block Toeplitz matrices for symbols that are bounded and in $W^{1/2}$. This paper not only gave the asymptotics in the block case, but paved the way for operator theory and Banach algebra approaches for the asymptotic expansions for determinants of structured operators. This constant is related to quantities that appear in the theory of tau-functions, dimer-models, random matrix theory, and other areas of mathematical physics and is now commonly called Widom's constant.

The main point of this paper is to investigate extensions of this theory to VMO (and more general Besov spaces which interpolate between $W^{1/2}$ and VMO), and some qualified extensions to the measurable (or L^2) context. In the VMO context, the Toeplitz operator $A(g)$ is Fredholm, the determinant $\det(A(g))$ makes sense as a section of a determinant line bundle, but the scalar expression $\det(A(g)A(g^{-1}))$ is identically zero in the complement of $W^{1/2}(S^1, SU(2))$. Roughly speaking the theory extends because, as we essentially observed in [1] (we will need a refinement), there is actually a factorization of $A(g)$, as an operator, in root subgroup coordinates.

1.1 Plan of the Paper

In Sect. 2 we establish basic notation and recall some background results, especially the operator theoretic realization of the topologies for the various spaces of loops.

In the first part of Sect. 3 we succinctly outline the main results from [8] for loops into $SU(2) := SU(2, \mathbb{C})$ with critical degree of smoothness in the L^2 Sobolev sense (the $W^{1/2}$ theory).

In Sect. 4 we consider measurable maps, which we refer to as the L^2 theory. Here we are probing the edge of deterministic results. For a measurable map into $SU(2)$, the Toeplitz operator is not in general Fredholm. Uniqueness in root subgroup factorization is lost because of the existence of singular inner functions.

In Sect. 5 we consider maps of vanishing mean oscillation, and more generally maps satisfying a Besov condition $B_p^{1/p}$ (which interpolates between $W^{1/2}$ and VMO).

For a more detailed version of this paper, see [2].

2 Notation and Background

If $f(z) = \sum f_n z^n$, then we will write

$$f = f_- + f_0 + f_+ \tag{5}$$

where $f_-(z) = \sum_{n<0} f_n z^n$ and $f_+(z) = \sum_{n>0} f_n z^n$, $f_{-0} = f_- + f_0$, $f_{0+} = f_0 + f_+$, and $f^*(z) = \sum (f_{-n})^* z^n$, where $w^* = \bar{w}$ is the complex conjugate of the complex number w. If the Fourier series is convergent at a point $z \in S^1$, then $f^*(z)$ is the conjugate of the complex number $f(z)$. If $f \in H^0(\Delta)$, then $f^* \in H^0(\Delta^*)$, where Δ is the open unit disk, Δ^* is the open unit disk at ∞, and $H^0(U)$ denotes the space of holomorphic functions for a domain $U \subset \mathbb{C}$.

We let $W^{1/2}(S^1, \mathbb{C})$ denote the Hilbert space of (equivalence classes of Lebesgue) measurable functions $f(z)$ which have half a derivative in the L^2 Sobolev sense; the precise form of the norm is not important, but one possibility is

$$|f|_{W^{1/2}} = \left(\sum_{n=-\infty}^{\infty} (1 + n^2)^{1/2} |\widehat{f}(n)|^2 \right)^{1/2} \tag{6}$$

where $\widehat{f}$ denotes the Fourier transform. Similarly $\text{VMO}(S^1)$ denotes the Banach space of (equivalence classes of Lebesgue) measurable functions which are of vanishing mean oscillation, or equivalently the closure of the subspace of continuous functions in BMO; again, the precise form of the norm is not important. $\text{Meas}(S^1, \mathbb{C})$ denotes equivalence classes of Lebesgue measurable functions with the topology corresponding to convergence in (Lebesgue) measure; this is induced by a complete separable metric, see below. Besov spaces $B_p^{1/p}$ which interpolate between $W^{1/2}$ and VMO for $2 \leq p \leq \infty$ will be used below and in Sect. 5 (see Chap. 6 and Appendix 2 of [7]). On the Fourier series side, $\mathbf{w}^{1/2}$ denotes the Hilbert space of complex sequences ζ such that $\sum_{k=1}^{\infty} k|\zeta_k|^2 < \infty$.

We let $L_{\text{fin}}SU(2)$ ($L_{\text{fin}}SL(2, \mathbb{C})$) denote the group consisting of functions $S^1 \to SU(2)$ ($SL(2, \mathbb{C})$, respectively) having finite Fourier series, with pointwise multiplication. For example, for $\zeta \in \mathbb{C}$ and $n \in \mathbb{Z}$, the function

$$S^1 \to SU(2) : z \to \mathbf{a}(\zeta) \begin{pmatrix} 1 & \zeta z^{-n} \\ -\bar{\zeta} z^n & 1 \end{pmatrix}, \tag{7}$$

where $\mathbf{a}(\zeta) = (1 + |\zeta|^2)^{-1/2}$, is in $L_{\text{fin}}SU(2)$.

As in the introduction, consider the groups

$$W^{d/2}(S^d, SU(2)) \subset \text{VMO}(S^d, SU(2)) \subset \text{Meas}(S^d, SU(2)). \tag{8}$$

In this paper we will always view these as topological groups with the complete separable (Polish) topologies induced by $W^{d/2}$, VMO, and convergence in measure, respectively. For measurable maps there is a well-known way to represent the topology using operator methods: the bijection

$$\text{Meas}(S^d, U(2)) \to \{\text{unitary multiplication operators on } L^2(S^d, \mathbb{C}^2)\} \tag{9}$$

is a homeomorphism with respect to the convergence in measure topology and the strong (or weak) topology for unitary multiplication operators (see Sect. 2 of [6]). For the other mapping groups, following [10], we will substitute restricted unitary groups (see below).

Now suppose that $d = 1$. In this setup the inclusions

$$L_{\text{fin}}SU(2) \subset C^{\infty}(S^1, SU(2)) \subset W^{1/2}(S^1, SU(2)) \subset \text{VMO}(S^1, SU(2)) \tag{10}$$

$\subset \text{Meas}(S^1, SU(2))$ are dense. The first three inclusions are homotopy equivalences (see subsection 2.2 of [2] for details which we are omitting in this paper). The fourth is a map into a contractible space.

Suppose that $g \in L^1(S^1, SL(2, \mathbb{C}))$. A triangular factorization of g is a factorization of the form

$$g = l(g)m(g)a(g)u(g), \tag{11}$$

where

$$l = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \in H^0(\Delta^*, SL(2,\mathbb{C})), \quad l(\infty) = \begin{pmatrix} 1 & 0 \\ l_{21}(\infty) & 1 \end{pmatrix},$$

l has a L^2 radial limit, $m = \begin{pmatrix} m_0 & 0 \\ 0 & m_0^{-1} \end{pmatrix}$, $m_0 \in S^1$, $a(g) = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$, $a_0 > 0$,

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in H^0(\Delta, SL(2,\mathbb{C})), \quad u(0) = \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix},$$

and u has a L^2 radial limit. Note that (11) is an equality of measurable functions on S^1.

As in [10], consider the polarized Hilbert space

$$H := L^2(S^1, C^2) = H_+ \oplus H_-, \tag{12}$$

where $H_+ = P_+ H$ consists of L^2-boundary values of functions holomorphic in Δ. If $g \in L^\infty(S^1, SL(2,\mathbb{C}))$, we write the bounded multiplication operator defined by g on H as

$$M_g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix} \tag{13}$$

where $A(g) = P_+ M_g P_+$ is the (block) Toeplitz operator associated to g and so on. If g has the Fourier expansion $g = \sum g_n z^n$, $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, then relative to the basis for H

$$...\epsilon_1 z, \epsilon_2 z, \epsilon_1, \epsilon_2, \epsilon_1 z^{-1}, \epsilon_2 z^{-1}, ... \tag{14}$$

where $\{\epsilon_1, \epsilon_2\}$ is the standard basis for $\mathbb{C}^2$, the matrix of M_g is block periodic of the form

$$\begin{array}{cccccccc}
 & . & . & . & . & . & . & . \\
.. & a_0 & b_0 & a_1 & b_1 & | & a_2 & b_2 & .. \\
.. & c_0 & d_0 & c_1 & d_1 & | & c_2 & d_2 & .. \\
.. & a_{-1} & b_{-1} & a_0 & b_0 & | & a_1 & b_1 & .. \\
.. & c_{-1} & d_{-1} & c_0 & d_0 & | & c_1 & d_1 & .. \\
- & - & - & - & - & - & - & - & - \\
.. & a_{-2} & b_{-2} & a_{-1} & b_{-1} & | & a_0 & b_0 & .. \\
.. & c_{-2} & d_{-2} & c_{-1} & d_{-1} & | & c_0 & d_0 & .. \\
 & . & . & . & . & . & . & .
\end{array} . \tag{15}$$

From this matrix form, it is clear that, up to equivalence, M_g has just two types of 'principal minors', the matrix representing $A(g)$, and the matrix representing the shifted Toeplitz operator $A_1(g)$, the compression of M_g to the closed subspace spanned by $\{\epsilon_i z^j : i = 1, 2, j > 0\} \cup \{\epsilon_1\}$.

Given the polarization $H = H_+ \oplus H_-$ and a symmetrically normed ideal $\mathcal{I} \subset \mathcal{L}(H)$, there is an associated Banach $*$-algebra, $\mathcal{L}_{(\mathcal{I})}$, which consists of bounded operators on H, represented as two by two matrices as in (13) such that $B, C \in \mathcal{I}$ with the norm

$$\left|\begin{pmatrix} A & \\ & D \end{pmatrix}\right|_{\mathcal{L}} + \left|\begin{pmatrix} & B \\ C & \end{pmatrix}\right|_{\mathcal{I}} \tag{16}$$

and the usual $*$-operation. The corresponding unitary group is

$$U_{(\mathcal{I})} = U(H) \cap \mathcal{L}_{(\mathcal{I})}; \tag{17}$$

it is referred to as a restricted unitary group in [10]. There are two standard topologies on $U_{(\mathcal{I})}$. The first is the induced Banach topology, and in this topology $U_{(\mathcal{I})}$ has the additional structure of a Banach Lie group. The second topology, the one we will always use, is the Polish topology for which convergence means that for $g_n, g \in U_{(\mathcal{I})}$, $g_n \to g$ if and only if $g_n \to g$ strongly and

$$\begin{pmatrix} & B_n \\ C_n & \end{pmatrix} \to \begin{pmatrix} & B \\ C & \end{pmatrix} \quad \text{in} \quad \mathcal{I}. \tag{18}$$

In the following proposition $\mathcal{L}_p$ refers to the Schatten ideal.

Proposition 1

(a) For the polarization (12) and $g \in L^\infty(S^1, \mathcal{L}(\mathbb{C}^2))$, $g \in \mathcal{L}_{(\mathcal{L}_p)}$ iff g belongs to the Besov space $B_p^{1/p}$ for $p < \infty$ and VMO *for $p = \infty$.*

(b) For $p < \infty$, $B_p^{1/p}(S^1, K) \to U_{(\mathcal{L}_p)}(H_+ \oplus H_-)$ is a homeomorphism onto its image; in particular

(b') $W^{1/2}(S^1, K) \to U_{(\mathcal{L}_2)}(H_+ \oplus H_-)$ is a homeomorphism onto its image.

(c) $\mathrm{VMO}(S^1, K) \to U_{(\mathcal{L}_\infty)}(H_+ \oplus H_-)$ *is a homeomorphism onto its image.*

(d) $U_{(\mathcal{L}_\infty)}(H_+ \oplus H_-) \to \mathrm{Fred}(H_+)$ *is a homotopy equivalence.*

Most of this is standard. For part (d) see Proposition 6.2.4 of [10].

Given a countably infinite dimensional Hilbert space such as H_+, Quillen constructed a holomorphic determinant line bundle $\mathrm{Det} \to \mathrm{Fred}(H_+)$ and a canonical holomorphic section det which vanishes on the complement of invertible operators. This induces a determinant bundle

$$A^*\mathrm{Det} \to \mathrm{VMO}(S^1, SU(2)) \tag{19}$$

(There is a discussion of this, and references, at the end of Sect. 7.7 of [10]). This is an elegant way to think about the following corollary, but there is also a simple proof using the operator-theoretic realization of the VMO topology.

Corollary 1 *For* $\mathrm{VMO}(S^1, SU(2))$ *the set of loops with invertible Toeplitz operators is defined by the equation* $\det(A(g)) \neq 0$*, hence is open. The same applies for the shifted Toeplitz operator.*

Proof Suppose that $g_n \in \mathrm{VMO}(S^1, SU(2))$ converges in VMO to g and $A(g)$ is invertible. We must show that $A(g_n)$ is invertible for large n.

$$A(g_n)A(g_n^{-1}) = 1 - B(g_n)C(g_n^{-1}) = 1 - B(g_n)B(g_n)^* . \tag{20}$$

By part (c) of the preceding proposition, this converges uniformly to $A(g)A(g^{-1}) = A(g)A(g)^* = 1 - B(g)B(g)^*$, which is invertible. This implies that $A(g_n)A(g_n^{-1})$ is invertible for large n, hence $A(g_n)$ is invertible for large n. □

Remark 3 For $\mathrm{Meas}(S^1, SU(2))$, or for its diagonal subgroup $\left\{ \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda(z)^{-1} \end{pmatrix} \right\}$, the set of loops with invertible Toeplitz operators is NOT open. To see this let $\lambda_n = \exp(f_n) : S^1 \to S^1$ be a continuous loop which rapidly winds once around the circle in the interval $[0, 1/n]$, and equals 1 otherwise (this is called a blip). This has degree one, hence the Toeplitz operator $\dot{A}(\lambda)$ has Fredholm index -1 and is not invertible for all n. Nonetheless $\lambda_n \to 1$ in measure.

This line of argument does not apply to $\mathrm{VMO}(S^1, S^1)$, because degree is well-defined, continuous and separates the group into path connected components - this is the main point of [4].

3 The $W^{1/2}$ Theory

The first part of this section is a succinct review of relevant results from [8]. The subsequent subsections describe some consequences.

Theorem 1 *Suppose that* $k_1 : S^1 \to SU(2)$ *is Lebesgue measurable. The following are equivalent:*

(I.1) $k_1 \in W^{1/2}(S^1, SU(2))$ *and is of the form*

$$k_1(z) = \begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix}, \quad z \in S^1, \tag{21}$$

where $a, b \in H^0(\Delta)$, $a(0) > 0$*, and* a *and* b *do not simultaneously vanish at a point in* Δ.

(I.2) *k_1 has a (root subgroup) factorization, in the sense that*

$$k_1(z) = \lim_{n\to\infty} \mathbf{a}(\eta_n)\begin{pmatrix} 1 & -\bar{\eta}_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} \dots \mathbf{a}(\eta_0)\begin{pmatrix} 1 & -\bar{\eta}_0 \\ \eta_0 & 1 \end{pmatrix} \tag{22}$$

for a.e. $z \in S^1$, where $(\eta_i) \in \mathbf{w}^{1/2}$ and the limit is understood in the $W^{1/2}$ sense.

(I.3) *k_1 has triangular factorization of the form*

$$\begin{pmatrix} 1 & 0 \\ y^*(z) & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_1 & 0 \\ 0 & \mathbf{a}_1^{-1} \end{pmatrix}\begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix}, \tag{23}$$

where $\mathbf{a}_1 > 0$, $y = \sum_{j=0}^{\infty} y_j z^j$ and $\alpha_1(z), \beta_1(z) \in W^{1/2}$.

Suppose that $k_2 : S^1 \to SU(2)$ is Lebesgue measurable. The following are equivalent:

(II.1) *$k_2 \in W^{1/2}(S^1, SU(2))$ and is of the form*

$$k_2(z) = \begin{pmatrix} d^*(z) & -c^*(z) \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1, \tag{24}$$

where $c, d \in H^0(\Delta)$, $c(0) = 0$, $d(0) > 0$, and c and d do not simultaneously vanish at a point in Δ.

(II.2) *k_2 has a (root subgroup) factorization of the form*

$$k_2(z) = \lim_{n\to\infty} \mathbf{a}(\zeta_n)\begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \dots \mathbf{a}(\zeta_1)\begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix} \tag{25}$$

for a.e. $z \in S^1$, where $(\eta_i) \in \mathbf{w}^{1/2}$ and the limit is understood in the $W^{1/2}$ sense.

(II.3) *k_2 has triangular factorization of the form*

$$\begin{pmatrix} 1 & x^*(z) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_2 & 0 \\ 0 & \mathbf{a}_2^{-1} \end{pmatrix}\begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix} \tag{26}$$

where $\mathbf{a}_2 > 0$, $x = \sum_{j=1}^{\infty} x_j z^j$, and $\gamma_2(z), \delta_2(z) \in W^{1/2}$.

Outline of the Proof For $k_2 \in L_{\text{fin}}SU(2)$, these correspondences are algebraic. To be more precise, given a sequence ζ as in II.2 with a finite number of nonzero terms, there are explicit polynomial expressions for x, α_2, β_2, γ_2 and δ_2, and

$$\mathbf{a}_2^2 = \prod_{k>0}(1 + |\zeta_k|^2). \tag{27}$$

Conversely, given k_2 as in II.1 or II.3, the sequence ζ can be recovered recursively from the Taylor expansion

$$(c_2/d_2)(z) = (\gamma_2/\delta_2)(z) = (-\overline{\zeta}_1)z + (-\overline{\zeta}_2)(1+|\zeta_1|^2)z^2 + \dots \quad . \tag{28}$$

The fact that these algebraic correspondences continuously extend to analytic correspondences depends on the following Plancherel-esque formulas (which explain the interest in root subgroup coordinates). For k_i as in Theorem 1,

$$\begin{aligned}\det(A(k_1)^*A(k_1)) &= \det(1 - C(k_1)^*C(k_1)) \\ &= \det(1 + \dot{B}(y)^*\dot{B}(y))^{-1} = \prod_{i\geq 1}(1+|\eta_i|^2)^{-i}\end{aligned} \tag{29}$$

and

$$\begin{aligned}\det(A(k_2)^*A(k_2)) &= \det(1 - C(k_2)^*C(k_2)) \\ &= \det(1 + \dot{B}(x)^*\dot{B}(x))^{-1} = \prod_{k\geq 1}(1+|\zeta_k|^2)^{-k}\end{aligned} \tag{30}$$

where in the third expressions, x and y are viewed as multiplication operators on $H = L^2(S^1)$, with Hardy space polarization. In (29), the first two terms are nonzero iff $k_1 \in W^{1/2}$, the third is nonzero iff $y \in W^{1/2}$, and the third is nonzero iff $\eta \in \mathbf{w}^{1/2}$.

Theorem 2 *Suppose $g \in W^{1/2}(S^1, SU(2))$. The following are equivalent:*

(i) The (block) Toeplitz operator $A(g)$ and shifted Toeplitz operator $A_1(g)$ are invertible.
(ii) g has a triangular factorization $g = lmau$.
(iii) g has a (root subgroup) factorization of the form

$$g(z) = k_1^*(z)\begin{pmatrix} e^{\chi(z)} & 0 \\ 0 & e^{-\chi(z)}\end{pmatrix}k_2(z) \tag{31}$$

where k_1 and k_2 are as in Theorem 1 and $\chi \in W^{1/2}(S^1, i\mathbb{R})$.

Outline of the Proof The equivalence of (i) and (ii) is standard (see also (34) below). Suppose that $g \in L_{\text{fin}}SU(2)$. If g has a root subgroup factorization as in (iii), one can directly find the triangular factorization (see Proposition 3 below), and from this explicit expression, one can see how to recover the factors η, χ, ζ (Incidentally, η and ζ have finitely many nonzero terms, but this is not so for χ, hence this calculation is not purely algebraic).

As was the case for Theorem 1, the fact that these correspondences extend to analytic correspondences depends on a number of Plancherel-esque identities. For

$g \in W^{1/2}(S^1, SU(2))$ satisfying the conditions in Theorem 2,

$$\det(A(g)^*A(g)) = \left(\prod_{i=0}^{\infty} \frac{1}{(1+|\eta_i|^2)^i}\right)\left(\prod_{j=1}^{\infty} e^{-2j|\chi_j|^2}\right)\left(\prod_{k=1}^{\infty} \frac{1}{(1+|\zeta_k|^2)^k}\right), \tag{32}$$

$$\det(A_1(g)^*A_1(g)) = \left(\prod_{i=0}^{\infty} \frac{1}{(1+|\eta_i|^2)^{i+1}}\right)\left(\prod_{j=1}^{\infty} e^{-2j|\chi_j|^2}\right)\left(\prod_{k=1}^{\infty} \frac{1}{(1+|\zeta_k|^2)^{k-1}}\right), \tag{33}$$

(where A_1 is the shifted Toeplitz operator)

$$a_0(g)^2 = \frac{\det(A_1(g)^*A_1(g))}{\det(A(g)^*A(g))} = \left(\prod_{i=0}^{\infty} \frac{1}{(1+|\eta_i|^2)}\right) \times \left(\prod_{k=1}^{\infty}(1+|\zeta_k|^2)\right). \tag{34}$$

Note that because g is unitary, i.e. $g^{-1} = g^*$ on S^1, parts (i) and (ii) are obviously inversion invariant, and this does not depend on the hypothesis that $g \in W^{1/2}$: if $g : S^1 \to SU(2)$ has the triangular factorization $g = lmau$, then $g^{-1} = g^*$ has triangular factorization $g^{-1} = u^*m^*al^*$. On the other hand part (iii), the existence of a root subgroup factorization, is not obviously inversion invariant.

Corollary 2 *Suppose $g \in W^{1/2}(S^1, SU(2))$. Then g has a root subgroup factorization (as in (iii) of Theorem 2) if and only if g^{-1} has a root subgroup factorization.*

We have used the hypothesis that $g \in W^{1/2}$ so that we can use the identities (32) and (33) to prove that the existence of a root subgroup factorization implies invertibility of the Toeplitz determinants. A central question related to the generalizations in the following sections is whether the hypothesis $g \in W^{1/2}$ is crucial for inversion invariance of root subgroup factorization.

Coordinates for $W^{1/2}(S^1, SU(2))$

Theorem 2 implies the following

Corollary 3 *$W^{1/2}(S^1, SU(2))$ is a topological Hilbert manifold modeled on the root subgroup parameters $\{((\eta_i)_{i\geq 0}, (\chi_j)_{j\geq 1}, (\zeta_k)_{k\geq 1}) \in l^2 \times l^2 \times l^2\} \times \{e^{\chi_0} \in S^1\}$ for the open set of loops with invertible A and A_1.*

As we noted in the introduction, it is not possible to use this (or any) coordinate to define a smooth structure which is translation invariant (because $W^{1/2}(S^1, su(2))$ is not a Lie algebra).

There are other coordinates, and this will be important when we consider VMO loops, because we will not be able to characterize VMO loops in terms of the coordinates η and ζ.

Theorem 3

(a) The maps

$$\{k_1 \textit{as in I.1-3 of Theorem 1}\} \to \{y = \sum_{n=0}^{\infty} y_n z^n \in W^{1/2}(S^1)\} : k_1 \to y \tag{35}$$

and

$$\{k_2 \textit{as in II.1-3 of Theorem 1}\} \to \{x = \sum_{n=1}^{\infty} x_n z^n \in W^{1/2}(S^1)\} : k_2 \to x \tag{36}$$

are bijections.

(b) (y, χ, x) *is a topological coordinate system for the open subset of* $W^{1/2}(S^1, SU(2))$ *with invertible* A *and* A_1.

Proof In the first part of the proof, we will prove a more general result for measurable loops, which we will exploit in the next section.

For part (a) we will use the Grassmannian model for the measurable loop group $\mathrm{Meas}(S^1, U(2))$, see Proposition (8.12.4) of [10], which describes the $\mathrm{Meas}(S^1, U(2))$ orbit of H_+ in the Grassmannian of $H = L^2(S^1, \mathbb{C}^2)$ (see (12)). Given $x(z) = \sum_{n=1}^{\infty} x_n z^n \in L^2(S^1)$, let W denote the smallest closed M_z-invariant subspace containing the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x^* \\ 1 \end{pmatrix}$. We claim that

$$\bigcap_{k \geq 0} z^k W = 0 \text{ and } \bigcup_{k \leq 0} z^{-k} W \text{ is dense in } H. \tag{37}$$

For the first condition, suppose that v is a point in the intersection. For each $N > 0$ it is possible to write $v(z) = \begin{pmatrix} z^N f_N(z) + z^N g_N(z) x^*(z) \\ z^N g_N(z) \end{pmatrix}$, where $f_N, g_N \in \dot{H}_+$ The second component of v has to be identically zero. This implies g_N has to be zero. Now the first component of v also has to vanish. The second condition is equivalent to showing that the subspace spanned by $\begin{pmatrix} s(z) + t(z) x^*(z) \\ t(z) \end{pmatrix}$, where s and t are finite Fourier series, is dense in $L^2(S^1)$. This is obvious.

This implies that W is in the Grassmannian in Proposition (8.12.4) of [10], and hence there exists $k_2 \in \text{Meas}(S^1, U(2))$ such that $k_2 H_+ = W$ (k_2 is obtained by taking an orthonormal basis for the two dimensional orthogonal complement of zW inside W, a Gram-Schmidt type process). This implies that $k_2^{-1} W = H_+$, hence $k_2^{-1} \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix}$ is holomorphic in the disk, and hence

$$k_2(z) = \lambda(z) \begin{pmatrix} d_2^*(z) & -c_2^*(z) \\ c_2(z) & d_2(z) \end{pmatrix} = \begin{pmatrix} 1 & x^*(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 & 0 \\ 0 & \mathbf{a}_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix} \tag{38}$$

where $\mathbf{a}_2 > 0$, $\lambda^2 = \det(k_2) : S^1 \to S^1$, $|c_2|^2 + |d_2|^2 = 1$ on S^1. From the second row of this equality, we see that λ extends to a holomorphic function in Δ. λ cannot vanish because γ_2 and δ_2 cannot simultaneously vanish. Thus λ is a constant; the normalizations in II.1-3 force $\lambda = 1$.

We now consider the hypothesis in part (a) of the theorem, i.e. $x \in W^{1/2}$. This implies that

$$\det(A(k_2)A(k_2^{-1})) = \det(1 - B(k_2)B(k_2)^*) = \det(1 + \dot{B}(x)\dot{B}(x)^*)^{-1} \tag{39}$$

is positive. Therefore $k_2 \in W^{1/2}$. The claim about k_1 and y is similar.

Part (b) follows from (a).

The preceding proof is abstract. In the next section (see Lemma 3) we will show how to solve for the unitary loop corresponding to a given $x = \sum_{n=1}^{\infty} x_n z^n \in L^2(S^1)$. Here we will simply state the result, which has a transparent meaning when $x \in W^{1/2}$.

Theorem 4 *Given $x = \sum_{n=1}^{\infty} x_n z^n \in W^{1/2}(S^1)$, the corresponding loop $k_2 \in W^{1/2}(S^1, SU(2))$ is determined by the identities*

$$\mathbf{a}_2^2 = \frac{1}{\langle 1|(1 + \dot{B}(x)\dot{B}(x)^*)^{-1}|1\rangle}, \tag{40}$$

$$\gamma_2^* = -\mathbf{a}_2^2(1 + \dot{B}(x)^*\dot{B}(x))^{-1}(x^*), \tag{41}$$

and

$$\delta_2 = \mathbf{a}_2^2(1 + \dot{B}(x)\dot{B}(x)^*)^{-1}(1). \tag{42}$$

4 The L^2 Theory

We now ask whether there are L^2 analogues of Theorems 1 and 2. Here is a naive L^2 analogue of Theorem 1 (we consider just the second set of equivalences):

Question 1 Suppose that $k_2 : S^1 \to SU(2)$ is Lebesgue measurable. Are the following equivalent:

(II.1) k_2 has the form

$$k_2(z) = \begin{pmatrix} d_2^*(z) & -c_2^*(z) \\ c_2(z) & d_2(z) \end{pmatrix}, \quad z \in S^1, \tag{43}$$

where $c_2, d_2 \in H^0(\Delta)$ do not simultaneously vanish, $c_2(0) = 0$ and $d_2(0) > 0$.

(II.2) There exists a unique $(\zeta_k) \in l^2$ such that

$$k_2(z) = \lim_{n \to \infty} \mathbf{a}(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \ldots \mathbf{a}(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix} \tag{44}$$

where the limit is understood in terms of convergence in measure.

(II.3) k_2 has triangular factorization of the form

$$\begin{pmatrix} 1 & \sum_{j=1}^{\infty} x_j^* z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 & 0 \\ 0 & \mathbf{a}_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix} \tag{45}$$

where $\mathbf{a}_2 > 0$.

For k_2 satisfying these conditions, we will see that

$$\mathbf{a}_2^2 = d_2(0)^{-2} = \prod_{k=1}^{\infty} (1 + |\zeta_k|^2) = |\gamma_2|^2 + |\delta_2|^2 \qquad (\text{on } S^1) \tag{46}$$

$$= 1 + \langle x | (1 + B(z^{-1}x) B(z^{-1}x)^*)^{-1} x \rangle_{L^2} = \frac{1}{\langle 1 | (1 + \dot{B}(x) \dot{B}(x)^*)^{-1} 1 \rangle_{L^2}} \tag{47}$$

(the meaning of the operators is explained in Lemma 3) and

$$|\alpha_2|^2 + |\beta_2|^2 = \mathbf{a}_2^{-2}(1 + |x|^2) \tag{48}$$

on S^1.

In the first part of this section, our goal is to explain how the various implications have to be qualified. One complication in this general context is the existence of singular inner functions.

Example 1 A simple non-example to bear in mind for (II.1) is

$$k_2(z) = \begin{pmatrix} d_2^*(z) & 0 \\ 0 & d_2(z) \end{pmatrix} \text{ where } d_2 = \frac{z-t}{1-tz} \tag{49}$$

and $0 < t < 1$. This does not satisfy the hypothesis that c_2 and d_2 are simultaneously nonvanishing, which is critical to show that the Toeplitz operator $A(k_2)$ is injective.

A complex example for (II.1) is a k_2 where $c_2(z) = \sqrt{t_1}C_2(z)$, $d_2(z) = \sqrt{t_2}D_2(z)$, C_2 and D_2 are inner functions which do not simultaneously vanish in Δ, and $t_1, t_2 > 0$, $t_1 + t_2 = 1$.

It is obvious that (II.3) implies (II.1). The important point is that the triangular factorization implies that c_2 and d_2 do not simultaneously vanish in Δ. For later use, notice that (II.3) and the special unitarity of k_2 imply ("the unitarity equations")

$$\mathbf{a}_2\alpha_2 + x^*\mathbf{a}_2^{-1}\gamma_2 = \mathbf{a}_2^{-1}\delta_2^*, \quad \mathbf{a}_2\beta_2 + x^*\mathbf{a}_2^{-1}\delta_2 = -\mathbf{a}_2^{-1}\gamma_2^* \tag{50}$$

and

$$\mathbf{a}_2^{-2}(\gamma_2^*\gamma_2 + \delta_2^*\delta_2) = 1. \tag{51}$$

These equations imply

$$\alpha_2 = -\mathbf{a}_2^{-2}x^*\gamma_2 + \mathbf{a}_2^{-2}\delta_2^* \quad \text{and} \quad \beta_2 = -\mathbf{a}_2^{-2}x^*\delta_2 - \mathbf{a}_2^{-2}\gamma_2^*. \tag{52}$$

Applying the $(\cdot)_{0+}$ projection to each of these, we obtain $\alpha_2 = 1 - (X^*\gamma_2)_+$ and $\beta_2 = -(X^*\delta_2)_{0+}$. Using (52) again, on S^1

$$|\alpha_2|^2 + |\beta_2|^2 = \mathbf{a}_2^{-4}((-x^*\gamma_2 + \delta_2^*)(-x\gamma_2^* + \delta_2) + (x^*\delta_2 + \gamma_2^*)(x\delta_2^* + \gamma_2)). \tag{53}$$

Expand this and use the obvious cancelations. Together with (51), this implies

$$|\alpha_2|^2 + |\beta_2|^2 = \mathbf{a}_2^{-2}(1 + |x|^2) \tag{54}$$

as claimed in the last part of Question 1.

Now assume (II.1). We can determine $\zeta_1, \zeta_2, \ldots$ using the Taylor series (28) for c_2/d_2 (note this is not identically zero, unlike the first loop in Example 1). Let

$$\begin{pmatrix} d_2^{(n)*}(z) & -c_2^{(n)*}(z) \\ c_2^{(n)}(z) & d_2^{(n)}(z) \end{pmatrix} = \mathbf{a}(\zeta_n)\begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \ldots \mathbf{a}(\zeta_1)\begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix}. \tag{55}$$

Because the polynomials $c_2^{(n)}(z)$ and $d_2^{(n)}(z)$ are bounded by 1 in the disk, given any subsequence, there exists a subsequence for which this pair will converge uniformly on compact subsets of Δ. The limits, denoted $\widetilde{c_2}(z)$ and $\widetilde{d_2}(z)$, are bounded by 1, hence will have radial boundary values. We will use the following elementary fact repeatedly.

Lemma 1 *Suppose that $f_n \in L^\infty H^0(\Delta)$ and f_n converges uniformly on compact subsets to $f \in L^\infty H^0(\Delta)$. Then there exists a subsequence f_{n_j} which converges pointwise a.e. on S^1 to f.*

Proof Because each f_j and f are essentially bounded, each f_j and f has radial limits, on a common subset E of S^1 of full Lebesgue measure. For each j there exists n_j such that $|f_{n_j} - f| < \frac{1}{j}$ on $(1-\frac{1}{j})S^1$. The subsequence f_{n_j} then converges pointwise on E to f. □

It follows that for some subsequence,

$$\widetilde{k_2}(\zeta)(z) := \lim_{j\to\infty} \begin{pmatrix} d_2^{(n_j)*}(z) & -c_2^{(n_j)*}(z) \\ c_2^{(n_j)}(z) & d_2^{(n_j)}(z) \end{pmatrix} \tag{56}$$

exists in the pointwise Lebesgue a.e. sense on the circle. Furthermore the sequence of zetas corresponding to $\widetilde{k_2}$ is $\zeta_1, \ldots$. Therefore using (28) $c_2/d_2 = \widetilde{c_2}/\widetilde{d_2}$. Together with unitarity and the simultaneous nonvanishing condition on c_2, d_2, this implies

$$\lambda := \frac{\widetilde{c_2}}{c_2} = \frac{\widetilde{d_2}}{d_2} \tag{57}$$

is a holomorphic function in Δ with radial boundary values and $|\lambda| = 1$ on S^1. Such a function has a unique factorization $\lambda = \lambda_b \lambda_s$, where λ_b is a Blaschke product and λ_s is a singular inner function, i.e.

$$\lambda_s(z) = \exp\left(\int_{S^1} \frac{z + e^{i\theta}}{z - e^{i\theta}} d\nu(\theta)\right) \tag{58}$$

where ν is a finite positive measure which is singular with respect to Lebesgue measure (see page 370 of [11]). The integral, as a holomorphic function of z is (up to a constant) usually referred to as the Caratheodory function of ν; because ν is singular, the Caratheodory function is not $W^{1/2}$, hence is forced to vanish when k_2 is $W^{1/2}$ (or more generally VMO). The simultaneous nonvanishing condition implies that $\lambda_b = 1$. Since $\widetilde{d_2}(0), d_2(0) > 0$, $\lambda(0) = 1$, and $d_2(0) = \prod_{k>0} \mathbf{a}(\zeta_k) = \prod_{k>0}(1 + |\zeta_k|^2)^{-1/2} > 0$. It follows that $\zeta \in l^2$. This implies the following

Theorem 5 *Assume (II.1) in Question 1. Then there exists a unique* $(\zeta_k) \in l^2$ *and a singular inner function* λ *with* $\lambda(0) = 1$ *such that*

$$
\begin{aligned}
k_2(z) &= \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda^{-1}(z) \end{pmatrix} \\
&\quad \times \lim_{n\to\infty} \mathbf{a}(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\bar{\zeta}_n z^n & 1 \end{pmatrix} \ldots \mathbf{a}(\zeta_1) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix}
\end{aligned}
\tag{59}
$$

where the limit is understood in terms of convergence in measure.

Now assume $\zeta \in l^2$ as in (II.2). We will show that this implies (II.1), sans the simultaneous nonvanishing condition, and we will explain why we do not necessarily obtain a factorization as in (II.3). Note we are free to use the unitarity equations for sufficiently regular ζ, e.g. $\zeta \in \mathbf{w}^{1/2}$. In the course of the argument, we will also prove (47), among other formulas.

The following is essentially Lemma 1 of [8].

Proposition 2 *Suppose that* $\zeta = (\zeta_n) \in l^2$. *Let* $k_2^{(N)}$ *be given by*

$$
\begin{pmatrix} d^{(N)*} & -c^{(N)*} \\ c^{(N)} & d^{(N)} \end{pmatrix} := \left(\prod_{n=1}^{N} \mathbf{a}(\zeta_n) \right) \begin{pmatrix} 1 & \zeta_N z^{-N} \\ -\bar{\zeta}_N z^N & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\bar{\zeta}_1 z & 1 \end{pmatrix}. \tag{60}
$$

Then $c^{(N)}$ *and* $d^{(N)}$ *converge uniformly on compact subsets of* Δ *to holomorphic functions* $c = c(\zeta)$ *and* $d = d(\zeta)$*, respectively, as* $N \to \infty$*. The functions* c *and* d *have radial limits at a.e. point of* S^1*,* c *and* d *are uniquely determined by these radial limits,*

$$
k_2(z) = k_2(\zeta)(z) := \begin{pmatrix} d(\zeta)^*(z) & -c(\zeta)^*(z) \\ c(\zeta)(z) & d(\zeta)(z) \end{pmatrix} \in \mathrm{Meas}(S^1, SU(2, \mathbb{C})). \tag{61}
$$

Note that if $\zeta \in l^1$, then the product actually converges absolutely around the circle. So one subtlety here is relaxing summability to square summability. Note also that the proof that (II.1) implies (II.2) shows that there exist convergence in measure limit points. So the second subtlety is showing that there is a unique limit point. We missed one simple point in Lemma 1 of [8]: k_2 actually has values in $SU(2)$. This is a consequence of Lemma 1.

We have now proven the existence of a

$$
k_2(\zeta) = \begin{pmatrix} d_2^* & -c_2^* \\ c_2 & d_2 \end{pmatrix} \tag{62}
$$

as in (II.1), but we have not proven the simultaneous nonvanishing of c_2 and d_2.

We now want to investigate the existence of a triangular factorization

$$k_2 = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 & 0 \\ 0 & \mathbf{a}_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix} \tag{63}$$

where $\mathbf{a}_2 > 0$. Note we have explicit formulas for $\mathbf{a}_2$, γ_2 and δ_2. But we need a formula for x. If we can find x, then we can use (52) to find α_2, β_2. Because of the identity (54) it would only remain to show x is square integrable.

Recall from the appendix to [8] that x^* has the form

$$x^* = \sum_{j=1}^{\infty} x_1^*(\zeta_j, \zeta_{j+1}, \ldots) z^{-j}, \tag{64}$$

where

$$x_1^*(\zeta_1, \ldots) = \sum_{n=1}^{\infty} \zeta_n \left(\prod_{k=n+1}^{\infty} (1 + |\zeta_k|^2) \right) s_n(\zeta_n, \zeta_{n+1}, \bar{\zeta}_{n+1}, ..), \tag{65}$$

$s_1 = 1$ and for $n > 1$,

$$s_n = \sum_{r=1}^{n-1} s_{n,r}, \quad s_{n,r} = \sum c_{i,j} \zeta_{i_1} \bar{\zeta}_{j_1} \zeta_{i_2} \bar{\zeta}_{j_2} .. \zeta_{i_r} \bar{\zeta}_{j_r} \tag{66}$$

where the sum is over multiindices satisfying the constraints

$$\begin{matrix} & j_1 \leq .. \leq j_r \\ & \vee \qquad \quad \vee \\ n \leq & i_1 \leq .. \quad i_r \end{matrix}, \quad \sum_{l=1}^{r} (j_l - i_l) = n - 1, \tag{67}$$

The crucial point is that the $c_{i,j}$ are positive integers, although it is not known how to explicitly compute them. In particular for each n s_n contains the sub-sum $\sum_{m \geq n} \zeta_m \zeta_{m+n-1}^*$.

Now suppose that all of the $\zeta_n \geq 0$. If the sum for x_1^* converges, then the sum

$$\sum_{n=1}^{\infty} \zeta_n \sum_{m \geq n} \zeta_m \zeta_{m+n-1}^* \tag{68}$$

has to converge. But $\zeta \in l^2$ is not a sufficient condition to guarantee the convergence of this sum. Empirically, if $\zeta_n = n^{-p}$ with $p < 5/8$, the sum diverges. From a theoretical point of view, this is the convolution of three functions on $\mathbb{Z}$ evaluated at zero, $\zeta^t * \zeta^t * \zeta$, where $\zeta^t(-m) = \zeta(m)$ is the adjoint; the convolution of two $l^2(\mathbb{Z})$ functions only has the property that it vanishes at infinity, and the convolution of an

$l^2(\mathbb{Z})$ function and a function that vanishes at infinity is not generally defined. This explains why (II.2) in Question 1 does not imply (II.3).

This gap can possibly be (partially) filled by the following hybrid deterministic/probabilistic

Conjecture In reference to Question 1, if $\zeta \in l^2$ as in (II.2) and the phases of the ζ_k are uniform and independent as random variables, then k_2 has a triangular factorization as in (II.3).

To get started on this, we would need to prove the almost sure existence of x_1 above. This has not been done. Instead we will explain the meaning of the operators in the statement of Question 1, which should play an important role in the proof of the conjecture.

Lemma 2 *For sufficiently regular x (which we will clarify in the proof)*

$$\mathbf{a}_2^2 = \frac{\det(1+\dot{B}(x)\dot{B}(x)^*)}{\det(1+\dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*)} \tag{69}$$

$$= 1 + \langle x|(1+\dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*)^{-1}x\rangle_{L^2} = \frac{1}{\langle 1|(1+\dot{B}(x)\dot{B}(x)^*)^{-1}1\rangle_{L^2}} \tag{70}$$

($\langle\cdot\cdot\rangle$ is the L^2 inner product), where $\dot{B}(x)$ denotes the scalar Hankel operator corresponding to the symbol x.

Proof For the first equality see (2.13) of [8]. For the determinants in this formula to make sense, we need $\zeta \in \mathbf{w}^{1/2}$.

As a matrix (relative to the standard Fourier basis)

$$\dot{B}(x)\dot{B}(x)^* - \dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^* = (x_n x_m^*)_{n,m\geq 1} \tag{71}$$

because the n, m entry is

$$\sum_{i\geq 0}(x_{n+i}x_{m+i}^*) - \sum_{i\geq 0}(x_{n+1+i}x_{m+1+i}^*) = x_n x_m^* \tag{72}$$

This is a rank one matrix.

The identity

$$(1+S)(1+T)^{-1} = 1 + (T-S)(1+T)^{-1} \tag{73}$$

implies that $(1+\dot{B}(x)\dot{B}(x)^*)(1+\dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*)^{-1}$ equals

$$1 + \left(\dot{B}(x)\dot{B}(x)^* - \dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*\right)(1+\dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*)^{-1}. \tag{74}$$

This is a rank one perturbation of the identity, and the determinant equals

$$1 + \langle x|(1 + \dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*)^{-1}x\rangle_{L^2}. \tag{75}$$

This proves the second equality. This second formula has a transparent operator-theoretic meaning when the Hankel operator is bounded, and this is the case if $x \in$ BMO.

For the third equality, suppose that $x = \sum_{n\geq 1} x_n z^n \in L^2$. For $i, j \geq 0$, relative to the standard Fourier basis $z^0, z^1, \ldots$ for $\dot{H}^+$, the i, j entry for the matrix representing $\dot{B}(x)\dot{B}(x)^*$ equals

$$\sum_{n=0}^{\infty} x_{i+n}x^*_{j+n}. \tag{76}$$

The matrix representing $\dot{B}(z^{-1}x)\dot{B}(z^{-1}x)^*$ (aside from indexing) is the same as the matrix obtained by deleting the zeroth row and column of the matrix representing $\dot{B}(x)\dot{B}(x)^*$. Thus the third equality is simply Cramer's rule for the inverse. The use of this rule is valid provided $\zeta \in \mathbf{w}^{1/2}$, which guarantees the determinants make sense. However as a formula for $\mathbf{a}_2$, it has a transparent operator-theoretic meaning when $x \in$ BMO. In the next lemma we will see the formula makes sense for $(x_n) \in l^2$.

We will now sharpen this result.

Lemma 3 *Suppose that* $\zeta \in l^2$.

(a) The sequence of positive operators $(1+\dot{B}(x^{(n)})\dot{B}(x^{(n)})^*)^{-1}$ *has a unique norm operator limit, and it is given by the formula*

$$(1 + \dot{B}\dot{B}^*)^{-1}f = c_2(c_2^* f)_{0+} + d_2(d_2^* f)_{0+} \tag{77}$$

[x does not appear in the notation, to emphasize that we are not assuming the existence of x]. Also

$$A(k_2)A(k_2^*)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ (1 + \dot{B}\dot{B}^*)^{-1} f_2 \end{pmatrix}. \tag{78}$$

Similarly the sequence of positive operators $(1 + \dot{B}(x^{(n)})^*\dot{B}(x^{(n)}))^{-1}$ *has a norm operator limit. This limit is unique and (by abuse of notation) denoted* $(1 + \dot{B}^*\dot{B})^{-1}$.

(b) $(1 + \dot{B}\dot{B}^*)^{-1}(z^n)$ *equals*

$$\mathbf{a}_2^{-2}(\gamma_2 \sum_{j=0}^{n-1} \gamma^*_{2,n-j} z^j + \delta_2 \sum_{k=0}^{n} \delta^*_{2,n-j} z^j) = \mathbf{a}_2^{-2}(z^n\gamma_2^{(n-1)*}\gamma_2 + z^n\delta_2^{(n-1)*}\delta_2). \tag{79}$$

For example

$$(1+\dot{B}\dot{B}^*)^{-1}(1) = \mathbf{a}_2^{-2}\delta_2\,, \tag{80}$$

$$(1+\dot{B}\dot{B}^*)^{-1}(z) = \mathbf{a}_2^{-2}(\gamma_{2,1}^*\gamma_2 + (\delta_{2,1}^* + z)\delta_2)\,, \tag{81}$$

$$(1+\dot{B}\dot{B}^*)^{-1}(z^2) = \mathbf{a}_2^{-2}((\gamma_{2,2}^* + \gamma_{2,1}^* z)\gamma_2 + (\delta_{2,2}^* + \delta_{2,1}^* z + z^2)\delta_2) \tag{82}$$

and the diagonal entries are

$$\mathbf{a}_2^{-2} diag\left(1, 1 + |\gamma_{2,1}|^2 + |\delta_{2,1}|^2, ..., 1 + \sum_{k=1}^{n}(|\gamma_{2,k}|^2 + |\delta_{2,k}|^2), ...\right). \tag{83}$$

(c) If x is l^2 and $n \geq -1$, then

$$(1+\dot{B}\dot{B}^*)^{-1}(z^n x) = -\gamma_2 z^n \alpha_2^{(n)*} - \delta_2 z^n \beta_2^{(n)*}\,, \tag{84}$$

in particular

$$(1+\dot{B}\dot{B}^*)^{-1}(z^{-1}x) = -z^{-1}\gamma_2 \tag{85}$$

or equivalently

$$(1+\dot{B}^*\dot{B})^{-1}x^* = -\mathbf{a}_2^{-2}\gamma_2^*. \tag{86}$$

Remark 4 Parts (b) and (c) explicitly determine γ_2 and δ_2 in terms of x. This explains the meaning of the formulas in Theorem 4.

Proof (a) Since $1 + \dot{B}(x^{(n)})^*\dot{B}(x^{(n)}) \geq 1$, it follows that the sequence of operators $(1+\dot{B}(x^{(n)})^*\dot{B}(x^{(n)}))^{-1}$ has strong operator limits. We must prove uniqueness. For this it will suffice to prove the exact formula for $x \in L^2$, because using this formula we can take a limit to obtain the general formula. After discussing the calculations in (c) and (d), we will then explain why this is actually a norm operator limit.

We need several standard facts: (1) If $g = g_- g_0 g_+$, then $Z(g) := C(g)A(g)^{-1} = Z(g_-)$. (2) If g is unitary, then $(1 + Z^*Z)^{-1} = A(g)A(g^{-1})$. And (3) If $g_- = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix}$, then

$$Z(g_-)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} C(x^*)f_2 \\ 0 \end{pmatrix}. \tag{87}$$

It is straightforward to check (1). (2) follows from (1). And (3) is straightforward.

Now suppose that $g = k_2$ and k_2 has a triangular factorization. By (2)

$$\begin{aligned}(1+Z^*Z)^{-1}\begin{pmatrix}f_1\\ f_2\end{pmatrix} &= A(k_2)A(k_2^*)\begin{pmatrix}f_1\\ f_2\end{pmatrix}\\ &= \begin{pmatrix}f_1\\ f_2-(c_2(c_2^*f_2)_-)_{0+}-(d_2(d_2^*f_2)_-)_{0+}\end{pmatrix}.\end{aligned} \tag{88}$$

Now (3) implies

$$\begin{aligned}(1+\dot{B}\dot{B}^*)^{-1} &= 1-(B(c_2)B(c_2)^*+B(d_2)B(d_2)^*)\\ &= A(c_2)A(c_2)^*+A(d_2)A(d_2)^*.\end{aligned} \tag{89}$$

This formula does not depend on the assumption that k_2 has a triangular factorization (hence we can apply the formula to $k_2^{(n)}$ and take a limit). This formula is equivalent to the one in the statement of part (a) of the theorem.

The calculations in (b) are straightforward, given the formula in (a). The calculations in (c) also use the unitarity equation $\mathbf{a}_2^2\alpha_2^* + \gamma_2^* = \delta_2$, multiplied by z^n. Together with the formula in (a) this implies

$$(1+\dot{B}\dot{B}^*)^{-1}(z^n x) = \mathbf{a}_2^{-2}(\gamma_2(z^n\delta_2 - \mathbf{a}_2^2 z^n\alpha_2^{(n)*}) - \delta_2(z^n\gamma_2 + \mathbf{a}_2^2 z^n\beta_2^{(n)*}). \tag{90}$$

This simplifies to the formula in (c).

Finally we explain why the limits in (a) are actually norm limits. Note that $(1+\dot{B}\dot{B}^*)^{-1} \le 1$ as positive operators. The formula for the diagonal in part (b) shows that the diagonal entries monotonely increase to 1 as $n \to \infty$. This implies uniform convergence.

Question 2 If $\zeta \in l^2$, then $0 \le (1+\dot{B}\dot{B}^*)^{-1} \le 1$. Is $(1+\dot{B}\dot{B}^*)^{-1}$ injective? What can we say about the spectrum of $(1+\dot{B}\dot{B}^*)^{-1}$? If $x \in$ VMO, then the spectrum is discrete. Does the spectrum simply become continuous on [0, 1] outside of VMO?

Here is a naive L^2 analogue of Theorem 2.

Question 3 Suppose that $g : S^1 \to SU(2)$ is measurable. Are the following conditions equivalent:

(i) $A(g)$ and $A_1(g)$ are invertible.
(ii) g has a triangular factorization.
(iii) g **and** g^{-1} have (root subgroup) factorizations of the form

$$g = k_1(\eta)^*\begin{pmatrix}e^{\chi} & 0\\ 0 & e^{-\chi}\end{pmatrix}k_2(\zeta), \tag{91}$$

$$g^{-1} = k_1(\eta')^*\begin{pmatrix}e^{\chi'} & 0\\ 0 & e^{-\chi'}\end{pmatrix}k_2(\zeta') \tag{92}$$

where k_1 and k_2 are as in (some form of) Question 1, and $\exp(-\chi_+)$, $\exp(-\chi'_+) \in L^2$.

Remark 5 The conditions (i) and (ii) are invariant with respect interchange of g and g^{-1} (This depends on $A(g^{-1}) = A(g^*) = A(g)^*$ (and similarly for A_1), and $g^{-1} = u(g)^* m(g)^* a(g) l(g)^*$. It is for this reason that we have imposed a condition on both g and its inverse in part (iii). This was not necessary in the $W^{1/2}$ case.

In the remainder of the section, we will explain how these statements have to be modified.

First, it is known that (i) is equivalent to

(ii') g has a triangular factorization, $g = lmau$, **and** the operators

$$R : \mathbb{C}[z] \otimes \mathbb{C}^2 \to \mathbb{C}[z] \otimes \mathbb{C}^2 : \psi_+ \to M_{u^{-1}} \circ P_+ \circ M_{l^{-1}}(\psi_+) \tag{93}$$

(where P_+ is either the projection for the polarization (12) or the shifted polarization) extend to bounded operators.

This is a special case of Theorem 5.1 (page 109) of [7], which establishes a criterion for invertibility of $A(g)$ for more general essentially bounded matrix symbols.

Theorem 6 *If k_1, k'_1, k_2 and k'_2 have triangular factorizations (as in (I.3) and (II.3) of Question 1, then g has a triangular factorization (as in (ii) of Question 3)*

Proof We will recall some more formulas which relate triangular and root subgroup factorization.

Proposition 3 *Suppose that η, χ, ζ are sufficiently regular (e.g. $\mathbf{w}^{1/2}$) Then $g = k_1^* e^{\chi} k_2$ has triangular factorization $g = l(g)m(g)a(g)u(g)$, where*

$$l(g) = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1^* & -(Y^*\alpha_1)_- \\ \beta_1^* & 1-(Y^*\beta_1)_- \end{pmatrix} \begin{pmatrix} e^{-\chi_+^*} & 0 \\ 0 & e^{\chi_+^*} \end{pmatrix} \begin{pmatrix} 1 & M_- \\ 0 & 1 \end{pmatrix}, \tag{94}$$

$$m(g) = \begin{pmatrix} e^{\chi_0} & 0 \\ 0 & e^{-\chi_0} \end{pmatrix}, \quad a(g) = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & (a_1 a_2)^{-1} \end{pmatrix}, \tag{95}$$

$$u(g) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{96}$$

$$= \begin{pmatrix} 1 & M_{0+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\chi_+} & 0 \\ 0 & e^{-\chi_+} \end{pmatrix} \begin{pmatrix} 1-(X^*\gamma_2)_+ & -(X^*\delta_2)_{0+} \\ \gamma_2 & \delta_2 \end{pmatrix}, \tag{97}$$

$Y = \mathbf{a}_1^2 y$, $X = \mathbf{a}_2^{-2} x$, and $M = (a_0 m_0)^{-2} e^{2\chi_+^*} Y + e^{2\chi_+} X^*$.

We claim that the formulas in the Proposition yield a triangular factorization for g. We need to show that the $l(g)$ and $u(g)$ factors are L^2. On S^1

$$|\alpha_1|^2 + |\beta_1|^2 = a_1^{-2} \text{ and } |\gamma_2|^2 + |\delta_2|^2 = a_2^2. \tag{98}$$

Consequently the first column of $l(g)$ and the second row of $u(g)$ are L^2 iff

$$\exp(\mathrm{Re}(\chi_-)) = \exp(-\mathrm{Re}(\chi_+)) \in L^2. \tag{99}$$

We are assuming this in (iii), and hence the first column of $l(g)$ and the second row of $u(g)$ are L^2.

The second column of $l(g)$ and the first row of $u(g)$ appear to be hopeless. But here is the key fact: g has a triangular factorization iff g^{-1} has a triangular factorization (If $g = lmau$, then $g^{-1} = u(g)^* m(g)^* a(g) l(g)^*$). Moreover the problematic second column for $l(g)$ is the adjoint of the second row of $u(g^{-1})$, and similarly the problematic first row of $u(g)$ is the adjoint of the first column of $l(g^{-1})$. It is not a priori clear (and it is undoubtedly not true) that for a general measurable $g : S^1 \to SU(2)$, g has a root subgroup factorization iff g^{-1} has a root subgroup factorization. But we do not have a concrete example to offer. In (iii), we are assuming both g and g^{-1} have root subgroup factorizations. Consequently the second column of $l(g)$ and the first row of $u(g)$ are also L^2 Thus g has a triangular factorization as in (ii). □

Theorem 7 *Assume that g has a triangular factorization. Then g (and g^{-1}) have root subgroup factorizations as in (iii), where we now mean in the sense of (I.1) and (II.1) of Question 1.*

Proof Although somewhat longwinded, it is straightforward to use the formulas in Proposition 3 to find candidates for the factors k_1, χ and k_2, see (3.4)–(3.19) of [8] (when consulting these formulas, note that the χ_+ of this paper is denoted by χ in [8]). We will now list these formulas, explain why they make sense, and note their significance. To begin

$$\mathbf{a}_1 = \exp\left(-\frac{1}{4\pi}\int_{S^1} \log(|l_{11}|^2 + |l_{21}|^2) d\theta\right) \tag{100}$$

and

$$\mathbf{a}_2 = \exp\left(\frac{1}{4\pi}\int_{S^1} \log(|u_{21}|^2 + |u_{22}|^2) d\theta\right). \tag{101}$$

We claim these are finite positive numbers. By assumption l, u are square integrable around S^1, and $0 < \mathbf{a} = \mathbf{a}_1\mathbf{a}_2 < \infty$. This implies $\mathbf{a}_1$ and $\mathbf{a}_2$ are nonzero. Jensen's inequality implies

$$\mathbf{a}_2^2 \leq \int_{S^1} (|u_{21}|^2 + |u_{22}|^2)\frac{d\theta}{2\pi} < \infty. \tag{102}$$

Thus $\mathbf{a}_1$ and $\mathbf{a}_2$ are finite. This proves the claim. On S^1,

$$|l_{11}|^2 + |l_{21}|^2 = \mathbf{a}_1^{-2}\exp(-2Re(\chi_+)) \tag{103}$$

and

$$|u_{21}|^2 + |u_{22}|^2 = \mathbf{a}_2^2\exp(-2Re(\chi_+)). \tag{104}$$

These formulas imply $\exp(-\chi_+) \in L^2$, as in (iii).

$$l_{11} = \alpha_1^*\exp(\chi_-), \quad l_{21} = \beta_1^*\exp(\chi_-), \tag{105}$$

$$u_{21} = \gamma_2\exp(-\chi_+), \quad u_{22} = \delta_2\exp(-\chi_+) \tag{106}$$

and on S^1,

$$|\alpha_1|^2 + |\beta_1|^2 = \mathbf{a}_1^{-2} \text{ and } |\delta_2|^2 + |\gamma_2|^2 = \mathbf{a}_2^2. \tag{107}$$

These formulas enable us to recover measurable loops $k_1, k_2 : S^1 \to SU(2)$,

$$k_1 = \mathbf{a}_1 \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1^* & \alpha_1^* \end{pmatrix} \text{ and } k_2 = \mathbf{a}_2^{-1} \begin{pmatrix} \delta_2^* & -\gamma_2^* \\ \gamma_2 & \delta_2 \end{pmatrix}. \tag{108}$$

Because l^* is invertible at all points of Δ, (105) implies that the entries a_1 and b_1 of k_1 do not simultaneously vanish, and similarly, because u is invertible, the entries c_2 and d_2 do not simultaneously vanish. Using Theorem 1 we can obtain η and ζ from the Taylor series expansions of β_2/α_2 and γ_2/δ_2, and η and ζ are in l^2 because of the finiteness of $\mathbf{a}_1$ and $\mathbf{a}_2$. □

One of several shortcomings of this theorem is that we have assumed that both g and g^{-1} have root subgroup factorizations. This is undesirable because there are (hopelessly) complicated compatibility relations involving the pairs of parameters η, ζ, χ and η', χ', ζ', for g and g^{-1}, respectively.

Example 2 Suppose that $g = k_2(\zeta)$, i.e. η and χ are zero. In this case g^{-1} has the triangular decomposition

$$g^{-1} = g^* = \begin{pmatrix} \alpha_2^*(z) & \gamma_2^*(z) \\ \beta_2^*(z) & \delta_2^*(z) \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 & 0 \\ 0 & \mathbf{a}_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \tag{109}$$

Therefore

$$\frac{\gamma_2(g^{-1})}{\delta_2(g^{-1})}(z) = x_1(\zeta_1, ...)z^1 + ... \tag{110}$$

implying $\xi_n(g^{-1}) = x_n(g)$ and in particular

$$-\zeta_1(g^*) = x_1^*(\zeta_1, ...) \tag{111}$$

The formula for x_1^* is discussed in the appendix in [8]—suffice it to say, it is complicated. □

5 The VMO Theory

In this section we will consider VMO loops and compact operators. Everything we say can be generalized to Besov class $B_p^{1/p}$ loops and Schatten p-class operators. For simplicity of exposition we will focus on the maximal class, VMO.

We begin by recalling basic facts about the abelian case, VMO(S^1, S^1). The notion of degree (or winding number) can be extended from C^0 to VMO(S^1, S^1) (see Sect. 3 of [3] for an amazing variety of formulas, and further references, or pages 98-100 of [7]). Also given $\lambda \in$ VMO(S^1, S^1), we view λ as a multiplication operator on $H = L^2(S^1)$, with the Hardy polarization. We write $\dot{A}(\lambda)$ for the Toeplitz operator, and so on (with the dot), to avoid confusion with the matrix case.

Lemma 4 *There is an exact sequence of topological groups*

$$0 \to 2\pi i\mathbb{Z} \to \text{VMO}(S^1, i\mathbb{R}) \overset{\exp}{\to} \text{VMO}(S^1, S^1) \overset{\text{degree}}{\to} \mathbb{Z} \to 0. \tag{112}$$

Moreover degree$(\lambda) = -$index$(\text{A}(\lambda))$.

This is implicit on pages 100–101 of [7]. The important point is that a VMO function cannot have jump discontinuities. This implies that the kernel of exp is $2\pi i\mathbb{Z}$. Thus the sequence in the statement of the Lemma is continuous and exact.

Remark 6 This should be contrasted with the measurable case. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$ induces a short exact sequence of Polish topological groups

$$0 \to \mathrm{Meas}([0,1],\mathbb{Z}) \to \mathrm{Meas}([0,1],\mathbb{R}) \to \mathrm{Meas}([0,1],\mathbb{T}) \to 0 \tag{113}$$

(see Sect. 2, especially Proposition 9, of [6]). However $\mathrm{Meas}([0,1],\mathbb{Z})$ is not discrete, and (just as the unitary group of an infinite dimensional Hilbert space is contractible—in either the strong operator or norm topology) $\mathrm{Meas}([0,1],\mathbb{T})$ is contractible.

Our aim now is to specialize Theorems 1 and 2 to VMO loops. It seems unlikely that one can characterize the sequences η and ζ that will correspond to VMO loops $k_1, k_2 : S^1 \to SU(2)$, respectively, as in Theorem 1. For this reason we will use $y, x \in \mathrm{VMOA} := \mathrm{VMO}_{0+}$ as parameters.

Proposition 4 *Suppose $k_2 : S^1 \to SU(2)$. The following two conditions are equivalent:*

(II.1) $k_2 \in \mathrm{VMO}$ *is of the form*

$$k_2(z) = \begin{pmatrix} d^*(z) & -c^*(z) \\ c(z) & d(z) \end{pmatrix}, \quad z \in S^1, \tag{114}$$

where $c, d \in H^0(\Delta)$, $c(0) = 0$, $d(0) > 0$, c and d do not simultaneously vanish at a point in Δ.

(II.3) *k_2 has triangular factorization of the form*

$$\begin{pmatrix} 1 & \sum_{j=1}^{\infty} x_j^* z^{-j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a}_2 & 0 \\ 0 & \mathbf{a}_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2(z) & \beta_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix} \tag{115}$$

where $\mathbf{a}_2 > 0$ and $\gamma_2, \delta_2 \in \mathrm{VMO}$.

There is a similar equivalence for k_1.

Proof The equivalence of II.1 and II.3 is proven exactly as in the $W^{1/2}$ case (taking into account the VMO condition of γ_2 and δ_2 in II.3). This uses the invertibility of $A(k_2)$. For the injectivity of $A(k_2)$, see Lemma 6 below (which is more general). The follows since the VMO condition implies that $A(k_2)$ is Fredholm of index zero. □

Theorem 8

(a) For k_2 in the preceding proposition, $x \in rmVMOA$ (i.e. VMO and holomorphic in the disk).
(b) The map $k_2 \to x$ induces a bijection

$$\{k_2 : \textit{II.1 and II.3 hold}\} \leftrightarrow \text{VMOA} : k_2 \leftrightarrow x. \tag{116}$$

(c) In terms of the root subgroup factorization in Theorem 5, the singular inner function $\lambda = 1$.

There is a similar statement for k_1.

Proof (a) The operator $(1 + \dot{B}\dot{B}^*)^{-1}$ is essentially the product $A(k_2)A(k_2^{-1})$, which is of the form 1+compact operator. Thus the inverse $1 + \dot{B}(x)\dot{B}(x)^*$ is also a compact perturbation of the identity. This is equivalent to $x \in$ VMOA. This proves part (a).

To prove part (b), we simply run the argument the opposite direction: if $x \in$ VMOA, then $(1 + \dot{B}\dot{B}^*)^{-1}$, hence also $A(k_2)A(k_2^{-1})$, is a compact perturbation of the identity. This implies k_2 is VMO.

(c) For the Caratheodory function in (58) to be VMO, ν has to be absolutely continuous with respect to Lebesgue measure. Hence $\lambda = 1$.

Theorem 9 *Suppose that $g \in \text{VMO}(S^1, SU(2))$. Assume that Lemma 7 below holds. Then the following are equivalent:*

(a) $A(g)$ and $A_1(g)$ are invertible.
(b) g has a triangular factorization.
(c) g has a (root subgroup) factorization of the form

$$g = k_1(\eta)^* \begin{pmatrix} e^{\chi} & 0 \\ 0 & e^{-\chi} \end{pmatrix} k_2(\zeta) \tag{117}$$

where k_1 and k_2 are as in Theorem 8, $\chi \in \text{VMO}(S^1; i\mathbb{R})$ and $\exp(-\chi_+) \in L^2(S^1)$.

Proof The equivalence of (a) and (b) is true more generally for $g \in QC(S^1, SL(2, \mathbb{C}))$.

To see that (a) and (b) are equivalent to (c), we will need some lemmas. To simplify the notation, let $h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Lemma 5 *With appropriate domains*

$$A(k_1^* e^{\chi h_1} k_2) = A(k_1^* e^{\chi_- h_1}) A(e^{\chi_0 + h_1} k_2). \tag{118}$$

The same is true for A_1 in place of A. Similarly

$$D(k_1^* e^{\chi h_1} k_2) = D(k_1^* e^{\chi_- h_1}) D(e^{\chi_0 + h_1} k_2) \tag{119}$$

and the same is true for D_1 in place of D. □

Proof The first statement is equivalent to showing that $B(k_1^* e^{\chi_- h_1}) C(e^{\chi_0 + h_1} k_2)$ vanishes. Applied to $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H_+$, this equals

$$B\left(\begin{pmatrix} e^{\chi_-} a_1^* & -e^{-\chi_-} b_1 \\ e^{\chi_-} b_1^* & e^{-\chi_-} a_1 \end{pmatrix}\right) C\left(\begin{pmatrix} e^{\chi_+} d_2^* & -e^{-\chi_+} c_2^* \\ e^{-\chi_+} c_2 & e^{-\chi_+} d_2 \end{pmatrix}\right) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{120}$$

$$= \left[\begin{pmatrix} e^{\chi_-} a_1^* & -e^{-\chi_-} b_1 \\ e^{\chi_-} b_1^* & e^{-\chi_-} a_1 \end{pmatrix} \begin{pmatrix} (e^{\chi_+} d_2^* f_1 - e^{-\chi_+} c_2^* f_2)_- \\ 0 \end{pmatrix}\right]_+ \tag{121}$$

$$= \left[\begin{pmatrix} e^{\chi_-} a_1^* (e^{\chi_+} d_2^* f_1 - e^{-\chi_+} c_2^* f_2)_- \\ e^{\chi_-} b_1^* (e^{\chi_+} d_2^* f_1 - e^{-\chi_+} c_2^* f_2)_- \end{pmatrix}\right]_+ = 0. \tag{122}$$

This proves the first statement.

For the second statement involving A_1, we are considering a polarization for H where H_+ now has orthonormal basis $\{\epsilon_i z^j : i = 1, 2, j > 0\} \cup \{\epsilon_1\}$ (see (14)). We let B_1, C_1 denote the Hankel operators relative to this shifted polarization. We must show $B_1(k_1^* e^{\chi_- h_1}) C_1(e^{\chi_0 + h_1} k_2)$ vanishes. The calculation is basically the same, but it depends on our normalizations in a subtle way. Applied to $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H_+$, this equals

$$B_1\left(\begin{pmatrix} e^{\chi_-} a_1^* & -e^{-\chi_-} b_1 \\ e^{\chi_-} b_1^* & e^{-\chi_-} a_1 \end{pmatrix}\right) C_1\left(\begin{pmatrix} e^{\chi_+} d_2^* & -e^{-\chi_+} c_2^* \\ e^{-\chi_+} c_2 & e^{-\chi_+} d_2 \end{pmatrix}\right) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{123}$$

$$= B_1\left(\begin{pmatrix} e^{\chi_-} a_1^* & -e^{-\chi_-} b_1 \\ e^{\chi_-} b_1^* & e^{-\chi_-} a_1 \end{pmatrix}\right) \begin{pmatrix} (e^{\chi_+} d_2^* f_1 - e^{-\chi_+} c_2^* f_2)_- \\ 0 \end{pmatrix} \tag{124}$$

where the vanishing of the second entry uses the fact that $c_2(0) = 0$. This now equals

$$\begin{pmatrix} [e^{\chi_-} a_1^* (e^{\chi_+} d_2^* f_1 - e^{-\chi_+} c_2^* f_2)_-]_{0+} \\ [e^{\chi_-} b_1^* (e^{\chi_+} d_2^* f_1 - e^{-\chi_+} c_2^* f_2)_-]_+ \end{pmatrix} = 0. \tag{125}$$

This proves the second statement.

The third statement is equivalent to $C(k_1^* e^{\chi_- h_1}) B(e^{\chi_0 + h_1} k_2) = 0$. This is a similar calculation. □

Lemma 6 *$A(k_1^* e^{\chi_- h_1})$ and $A(e^{\chi_0 + h_1} k_2)$ are injective on their domains, and similarly for A_1.* □

Proof The four statements are all proved in the same way. We consider the second assertion concerning A. Suppose that

$$A\left(\begin{pmatrix} e^{\chi_+} d_2^* & -e^{-\chi_+} c_2^* \\ e^{-\chi_+} c_2 & e^{-\chi_+} d_2 \end{pmatrix}\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \tag{126}$$

This implies

$$\begin{pmatrix} [e^{\chi_+}(d_2^* f_1 - c_2^* f_2)]_+ \\ e^{-\chi_+}(c_2 f_1 + d_2 f_2) \end{pmatrix}) = 0 \tag{127}$$

The second component implies $c_2 f_1 + d_2 f_2 = 0$, and this implies

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g \begin{pmatrix} d_2 \\ -c_2 \end{pmatrix} \tag{128}$$

where g is holomorphic in the disk. Plug this into the first component to obtain

$$[e^{\chi_+} g(d_2 d_2^* + c_2 c_2^*)]_+ = [e^{\chi_+} g]_+ = 0 \tag{129}$$

which implies $g = 0$. Thus $f = 0$.

Now assume that (c) of Theorem 9 holds. The lemmas imply that the Toeplitz operator $A(g)$ and the shifted Toeplitz operator $A_1(g)$ are injective. Since these operators are Fredholm, they are invertible. Hence (c) implies (a) and (b).

Now assume (a) and (b). We define k_1, k_2 and χ using the explicit formulas in the proof of Theorem 7. Note it is essential that we use these explicit formulas, because (as we saw in the last section) the existence of singular inner functions implies that root subgroup factorization is not unique in general - we have to choose χ wisely! The formula (103) immediately implies that $\exp(-\chi_+) \in L^2$. The crux of the matter is to show that if $g \in$ VMO (or more generally $B_p^{1/p}$), then the factors have the same smoothness property.

Suppose first that $\chi = 0$. In this case

$$A(g)A(g)^* = A(k_1^*)A(k_2)A(k_2)^*A(k_1) \tag{130}$$

$$= 1 - B(k_1)B(k_1)^* - A(k_1^*)B(k_2)B(k_2)^*A(k_1). \tag{131}$$

This implies the following sum is a positive compact operator:

$$B(k_1)B(k_1)^* + A(k_1^*)B(k_2)B(k_2)^*A(k_1). \tag{132}$$

Does this imply that the two summands have to be compact?

Proposition 5 *Assume A and B are positive operators on a Hilbert space H.*

(a) If $A + B$ is finite rank, then A and B are finite rank.
(b) If $A + B$ is compact (or Schatten p-class), the A and B are compact (Schatten p-class, respectively).

□

Proof

(a) For $x \in \ker(A + B)$,

$$\langle Ax, x\rangle + \langle Bx, x\rangle = 0 \tag{133}$$

together with polarization, this implies that $\langle Ax, y\rangle = 0$ for $x, y \in \ker(A + B)$. $ker(A + B)^{\perp}$ is finite dimensional. So the range of A is contained in the finite dimensional subspace

$$\ker(A + B)^{\perp} + A(\ker(A + B)^{\perp}) \tag{134}$$

and similarly for B. This proves A and B are finite rank.

(b) Given n, let K_n (P_n) denote the closed subspace (and the corresponding orthogonal projection) spanned by eigenvectors corresponding to eigenvalues λ for $A + B$ with $\lambda < 1/n$. K_n is $A + B$ invariant and $|A + B|_{K_n} < 1/n$. The orthogonal complement of K_n is finite dimensional. Because $\langle Ax, x\rangle \leq \langle (A + B)x, x\rangle$ for $x \in K_n$ and A is positive, the norm for $|P_n A P_n| < 1/n$. Define $A_n = A - P_n A P_n$. This is a finite rank operator (its range is contained in $K_n^{\perp} + AK_n^{\perp}$) and $|A_n - A| = |P_n A P_n| < 1/n$. This shows that A is a norm limit of finite rank operators. Hence A is compact.

The Schatten p-class claim is done in the same way, using the Schatten p-norm.

Thus if $\chi = 0$, then $g \in \mathrm{VMO}$ implies that $k_1, k_2 \in \mathrm{VMO}$.
Now consider the general case,

$$g = k_1^* e^{\chi h_1} k_2 = \begin{pmatrix} a_1^* e^{\chi} d_2^* - b_1 e^{-\chi} c_2 & -a_1^* e^{\chi} c_2^* - b_1 e^{-\chi} d_2 \\ b_1^* e^{\chi} d_2^* + a_1 e^{-\chi} c_2 & -b_1^* e^{\chi} c_2^* + a_1 e^{-\chi} d_2 \end{pmatrix}. \tag{135}$$

The following is a basic gap in this section, and we will simply assume its truth.

Lemma 7 *(Conjectural) There exists a deformation $\chi_t : S^1 \to i\mathbb{R}$ with $\chi|_{t=0} = \chi$, $\chi_t \in \mathrm{VMO}$ for $t > 0$, and $g_t := k_1^* e^{\chi(t) h_1} k_2 \in \mathrm{VMO}(S^1, SU(2))$.* □

Lemma 5 and some algebraic manipulations imply the following lemma.

Lemma 8 *$A(g_t)$ equals the sum of four terms*

$$A(k_1^*)A(e^{\chi_t h_1})A(k_2) + B(k_1^*)C(e^{\chi_t h_1})A(k_2) \tag{136}$$

$$+ A(k_1^*)B(e^{\chi_t h_1})C(k_2) + B(k_1^*)C(e^{\chi_t h_1})A(e^{-\chi_t h_1})B(e^{\chi_t h_1})C(k_2). \tag{137}$$

The last three terms are compact for $t > 0$. □

This implies that $A(g_t)A(g_t^{-1})$ will be the sum of 16 terms. For $t > 0$ all of the terms, with one exception, are trace class, because e^{χ_t} is smooth. The exceptional term is $A(k_1^*)A(e^{\chi_t h_1})A(k_2)A(k_2^*)A(e^{-\chi_t h_1})A(k_1)$. This can be rewritten as

$$A(k_1^*)A(e^{\chi_t h_1})(1 - B(k_2)B(k_2)^*)A(e^{-\chi_t h_1})A(k_1) = \tag{138}$$

$$\begin{aligned} &A(k_1^*)(1 - B(e^{\chi_t h_1} B(e^{\chi_t h_1})^*)A(k_1) \\ &\qquad - A(k_1^*)A(e^{\chi_t h_1})B(k_2)B(k_2)^* A(e^{-\chi_t h_1})A(k_1). \end{aligned} \tag{139}$$

This equals the identity minus

$$B(k_1^*)B(k_1^*)^* + A(k_1^*)B(e^{\chi_t h_1} B(e^{\chi_t h_1})^*)A(k_1) \tag{140}$$

$$+ A(k_1^*)A(e^{\chi_t h_1})B(k_2)B(k_2)^* A(e^{-\chi_t h_1})A(k_1). \tag{141}$$

This operator is positive because $B(g_t)B(g_t)^*$ is positive. Proposition 5 now implies that $B(k_1)$ and $B(k_2)$ are compact, hence k_1 and k_2 are VMO. This now implies that e^{χ} is VMO. Lemma 4 implies that χ is VMO. This completes the proof of the theorem. □

Theorem 9 implies the following

Corollary 4 VMO$(S^1, SU(2))$ *is a topological manifold, where (y, χ, x) is a topological coordinate system for the open set of loops in* VMO$(S^1, SU(2))$ *with invertible A and A_1.*

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Openness of Regular Regimes of Complex Random Matrix Models

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Dedicated to the memory of Harold Widom

Abstract Consider the general complex polynomial external field

$$V(z) = \frac{z^k}{k} + \sum_{j=1}^{k-1} \frac{t_j z^j}{j}, \qquad t_j \in \mathbb{C}, \quad k \in \mathbb{N}.$$

Fix an equivalence class $\mathcal{T}$ of admissible contours whose members approach ∞ in two different directions and consider the associated max-min energy problem [14]. When $k = 2p$, $p \in \mathbb{N}$, and $\mathcal{T}$ contains the real axis, we show that the set of parameters $t_1, \cdots, t_{2p-1}$ which gives rise to a regular q-cut max-min (equilibrium) measure, $1 \leq q \leq 2p - 1$, is an open set in $\mathbb{C}^{2p-1}$. We use the implicit function theorem to prove that the endpoint equations are solvable in a small enough neighborhood of a regular q-cut point. We also establish the real-analyticity of the

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real and imaginary parts of the end-points for all q-cut regimes, $1 \le q \le 2p-1$, with respect to the real and imaginary parts of the complex parameters in the external field. Our choice of even k and the equivalence class $\mathcal{T} \ni \mathbb{R}$ of admissible contours is only for the simplicity of exposition and our proof extends to all possible choices in an analogous way.

Keywords Equilibrium measure · Orthogonal polynomials · Asymptotic analysis · Phase transition · Random matrices

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1 Introduction and Main Results

The present paper is part of an ongoing project whose main objective is the investigation of the phase diagram and phases of the unitary ensemble of random matrices with a general complex potential

$$V(z;\boldsymbol{t}) = \frac{z^{2p}}{2p} + \sum_{j=1}^{2p-1} \frac{t_j z^j}{j}, \qquad t_j \in \mathbb{C}, \quad p \in \mathbb{N}, \tag{1}$$

in the complex space of the vector of the parameters

$$\boldsymbol{t} = (t_1, \cdots, t_{2p-1}) \in \mathbb{C}^{2p-1}.$$

The unitary ensemble under consideration is defined as the complex measure on the space of $n \times n$ Hermitian random matrices,

$$\frac{1}{\tilde{\mathcal{Z}}_n} e^{-n \mathrm{Tr}\, V(M;\boldsymbol{t})} \mathrm{d}M, \tag{2}$$

where

$$\tilde{\mathcal{Z}}_n(\boldsymbol{t}) = \int_{\mathcal{H}_n} e^{-n \mathrm{Tr}\, V(M;\boldsymbol{t})} \mathrm{d}M \tag{3}$$

is the *partition function*. As well known (see, e.g., [4]), the ensemble of eigenvalues of M,

$$M e_k = z_k e_k, \qquad k = 1, \ldots, n,$$

is given by the probability distribution

$$\frac{1}{\mathcal{Z}_n(\boldsymbol{t})} \prod_{1\le j<k\le n} (z_j - z_k)^2 \prod_{j=1}^{n} \exp\left[-nV(z_j;\boldsymbol{t})\right] \mathrm{d}z_1 \cdots \mathrm{d}z_n, \tag{4}$$

where

$$\mathcal{Z}_n(\boldsymbol{t}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\le j<k\le n} (z_j - z_k)^2 \prod_{j=1}^{n} \exp\left[-nV(z_j;\boldsymbol{t})\right] \mathrm{d}z_1 \cdots \mathrm{d}z_n, \tag{5}$$

is the *eigenvalue partition function*. The partition functions $\mathcal{Z}_n$ and $\tilde{\mathcal{Z}}_n$ are related by the formula,

$$\frac{\mathcal{Z}_n(\boldsymbol{t})}{\tilde{\mathcal{Z}}_n(\boldsymbol{t})} = \frac{1}{\pi^{n(n-1)/2}} \prod_{k=1}^{n} k!. \tag{6}$$

Formulae (5), (6) are well known for real polynomial potentials $V(z)$ of even degree (see, e.g., [4]), and their proof for a complex $V(z)$ goes through without any change.

By Heine's formula (see e.g. [23]) the multiple integral in (5) is, up to a multiplicative constant, the determinant of the Hankel matrix $H_n[w] := \{w_{j+k}\}_{k,j=0,\ldots,n}$, where $w(x;\boldsymbol{t}) \equiv \exp[-nV(x;\boldsymbol{t})]$ and w_ℓ is the ℓ-th moment of the weight $w(x;\boldsymbol{t})$. Correspondingly, one can also consider the system of monic orthogonal polynomials $\{P_n(z;\boldsymbol{t})\}_{n\in\mathbb{Z}_{\ge 0}}$ satisfying

$$\int_\Gamma P_n(z;\boldsymbol{t}) z^k w(z;\boldsymbol{t}) dz = 0\,, \quad \text{for} \quad k = 0, 1, \ldots, n-1, \tag{7}$$

where the infinite contour Γ is in some equivalence class of addmissible contours (see below and Sect. 2.2 for more details). The connection of this system of orthogonal polynomials and the partition function (5) can be seen as follows: the orthogonal polynomial of degree n exists and is unique if the partition function $\mathcal{Z}_n(\boldsymbol{t})$, or the $n\times n$ Hankel determinant $\det H_n[w]$, is nonzero. Indeed, the existence follows from the explicit formula

$$P_n(z;\boldsymbol{t}) \equiv P_n(z) = \frac{1}{\det H_n[w]} \det \begin{pmatrix} w_0 & w_1 & \cdots & w_{n-1} & w_n \\ w_1 & w_2 & \cdots & w_n & w_{n+1} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ w_{n-1} & w_n & \cdots & w_{2n-2} & w_{2n-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}, \tag{8}$$

and the uniqueness follows from the fact that the linear system to find the coefficients of $P_n(z; \boldsymbol{t}) \equiv z^n + \sum_{j=0}^{n-1} a_j(\boldsymbol{t}) z^j$, is of the form $H_n[w]\boldsymbol{a} = \boldsymbol{b}$, and thus can be inverted if the Hankel determinant is nonzero.

It is well known that the normalized counting measure for the zeros of these orthogonal polynomials weakly converges to the associated *equilibrium measure* ν_{eq} (See e.g. [10] and references therein). For a review of the definitions and properties of the equilibrium measure in the cases where the external field is real and complex see Sects. 2.1 and 2.2.

The properties of the equilibrium measure when the external field is real have been studied extensively over the last two decades or so (see e.g. [7, 13, 19] and references therein), and we briefly review these properties in Sect. 2.1. In the real case, the contour of orthogonality for the orthogonal polynomials with respect to $e^{-nV(x;\boldsymbol{t})}$, $\boldsymbol{t} \in \mathbb{R}^{2p-1}$, is the real line and the equilibrium measure is supported on finitely many closed real intervals. One does not need to deal with the problem of *choosing* the contour of integration for orthogonal polynomials in the case where the external field is real, as the solution of the associated extremal problem for the equilibrium measure automatically ensures that the real line is the correct contour of integration.

In this work we are considering polynomials defined by a "complex orthogonality condition", of the form (7). It is easy to see that the polynomials, when they are uniquely determined by the above orthogonality condition, are independent of the choice of contour, within some equivalence class of contours. Moreover, for a given weight function $w(z; \boldsymbol{t}) \equiv e^{-nV(z;\boldsymbol{t})}$, there are multiple possible choices of equivalence classes of contours (see for instance [2, 3, 14]), and each equivalence class yields a different sequence of orthogonal polynomials.

Even though for each choice of the equivalence class of contours, our method would work, for the sake of simplicity of exposition, we will restrict ourselves as follows: We will assume that the external field is a polynomial of degree $2p$ (see (1)), and we will choose the class of contours of integration that are all in the same equivalence class as the real axis (also see Sect. 6).

As opposed to the case of a real measure on the real axis defining more classical polynomials all of whose zeros are real, the case of complex orthogonality produces polynomials whose zeros exhibit more complicated behavior. In fact, as the degree of the polynomials tends to infinity, the zeros accumulate on nontrivial curves in the complex plane.

In order to carry out an asymptotic analysis of the orthogonal polynomials with complex weights, a new problem arises which is the effective selection of a contour of integration for which subsequent analysis is possible. It turns out that the effective selection of the contour of integration determines within it the accumulation set of the zeros of the orthogonal polynomials, which is the support of the equilibrium measure (suitably generalized to the complex case).

The problem of determining this important set in the plane, which is later used as a portion of the contour of integration, is actually connected to a classical energy problem dating back at least to Gauss—the energy of a continuum of particles in the presence of an external field that experiences a repelling force whose potential is logarithmic. The set is determined by considering, for each member Γ of the class of admissible contours $\mathcal{T}$, the energy minimization problem on Γ, and then selecting a contour $\Gamma_0 \in \mathcal{T}$ that maximizes this minimum energy. In other words, Γ_0 solves the following *max-min* problem:

$$\max_{\Gamma \in \mathcal{T}} \left\{ \min_{\substack{\mathrm{supp}\,(\nu) \subset \Gamma \\ \nu(\mathbb{C})=1}} \left\{ \iint_{\Gamma \times \Gamma} \log \frac{1}{|z-s|} \, \mathrm{d}\nu(z) \mathrm{d}\nu(s) + \int_{\Gamma} \Re V(s) \, \mathrm{d}\nu(s) \right\} \right\}. \tag{9}$$

The admissible sectors (in which the admissible equivalence classes of contours could approach ∞) are those in which the requirement

$$\lim_{z \to \infty} \Re V \to +\infty \tag{10}$$

holds, which allows one to associate the *Euler-Lagrange characterization* of the equilibrium measure[19]

$$\begin{aligned} U^{\nu}(z) + \frac{1}{2} \Re V(z) &= \ell, \quad z \in \mathrm{supp}\, \nu, \\ U^{\nu}(z) + \frac{1}{2} \Re V(z) &\geq \ell, \quad z \in \Gamma \setminus \mathrm{supp}\, \nu, \end{aligned} \tag{11}$$

where

$$U^{\nu}(z) = \int_{\Gamma} \log \frac{1}{|z-s|} \, \mathrm{d}\nu(s) \tag{12}$$

is the *logarithmic potential* of the measure ν [19]. There is quite a history of reseach centering on this variational problem in approximation theory and potential theory. See, for example [11, 14–16, 18, 20, 21] and references therein.

In [14] the authors prove the quite general result that for an allowable[1] equivalence class $\mathcal{T}$ of contours, the solution Γ_0 to the above extremal problem exists, the equilibrium measure and, thus, its support J are unique, and the support $J \subset \Gamma_0$ of the equilibrium measure is a finite union of disjoint analytic arcs. Moreover, they show that the support J of the equilibrium measure is part of the *critical graph* of the quadratic differential $Q(z)\mathrm{d}z^2$, that is the totality of solutions to

$$\Re \left(\int_b^z \sqrt{Q(s)} \mathrm{d}s \right) = 0, \tag{13}$$

[1] Characterized by a notion of *non-crossing partitions* of $\{1, \cdots, N\}$, where N is the number of sectors in which (10) holds, see [14].

(see Sect. 2.5 for some background on quadratic differentials and the paragraph that follows Definition 3.1 for details on the connection of this requirement with the Euler-Lagrange characterization of the equilibrium measure). In (13) Q is the *polynomial* (see Proposition 3.7 of [14])

$$Q(z) = \left(-\omega(z) + \frac{V'(z)}{2}\right)^2, \tag{14}$$

in which ω is the *resolvent* of the equilibrium measure

$$\omega(z) = \int_J \frac{d\nu_{eq}(x)}{z-x}, \quad z \in \mathbb{C} \setminus J. \tag{15}$$

Summarizing, we will consider the above max-min variational problem which is associated to the orthogonal polynomials with respect to $e^{-nV(z;\boldsymbol{t})}$, $\boldsymbol{t} \in \mathbb{C}^{2p-1}$, in which the contour $\Gamma_{\boldsymbol{t}}$ in the complex z-plane, being the solution of the max-min problem, is chosen from the members of the equivalence class of contours $\mathcal{T}$ (defined in Sect. 2.2 below - each member being a simply connected curve that tends to ∞ in two different directions, in sectors surrounding the positive and negative real axis). For a "generic" choice of $\boldsymbol{t} \in \mathbb{C}^{2p-1}$, the support $J_{\boldsymbol{t}}$ of the equilibrium measure is a finite union of disjoint analytic arcs (which are also referred to as *cuts*), at each endpoint the density of the equilibrium measure vanishes like a square root $d\nu_V(s;\boldsymbol{t}) = (2\pi i)^{-1}h(s;\boldsymbol{t})\left(\sqrt{R(s;\boldsymbol{t})}\right)_+ ds$, where $h(s;\boldsymbol{t})$ and $R(s;\boldsymbol{t})$ are polynomials in s, and R has the property that its only zeros are simple zeros at the endpoints of the cuts. Moreover, for a generic $\boldsymbol{t}$ the zeros of $h(s;\boldsymbol{t})$ do not lie on $J_{\boldsymbol{t}}$ and one can find a complementary set to $J_{\boldsymbol{t}}$ to build the desired infinite contour $\Gamma_{\boldsymbol{t}}$ so that the requirement outside the support in (11) is satisfied. In fact, for a generic $\boldsymbol{t}$ these complementary contours can all be chosen to satisfy the *strict* inequality in (11), or equivalently chosen so that they all lie in the so-called $\boldsymbol{t}$-stable lands:

$$\left\{z : \Re\eta_q(z;\boldsymbol{t}) < 0\right\}, \tag{16}$$

where

$$\eta_q(z;\boldsymbol{t}) := -\int_{b_q(\boldsymbol{t})}^{z} h(s;\boldsymbol{t})\sqrt{R(s;\boldsymbol{t})}ds, \tag{17}$$

and $b_q(\boldsymbol{t})$ is the rightmost endpoint (for more details see Definition (3.1) and the paragraph that follows it).

However, we may expect that the above *regularity* properties[2] do not hold for certain choices of $\boldsymbol{t}$. For example, for some values of $\boldsymbol{t}$ it could happen that

(a) one or more zeros of $h(s; \boldsymbol{t})$ coincide with the endpoints and thus alter the square root vanishing of the density at one or more endpoints,
(b) one or more zeros of $h(s; \boldsymbol{t})$ may hit the support $J_{\boldsymbol{t}}$ of the equilibrium measure, or
(c) it may not be possible to choose the complementary contours to entirely lie in the $\boldsymbol{t}$-stable lands.

Such values of $\boldsymbol{t}$ at which the aforementioned regularity properties fail, also form *boundaries* in the phase space $\mathbb{C}^{2p-1}$, across which the number of support cuts of the equilibrium measure changes.

Let us highlight these irregularity properties at non-generic parameter values using the complex quartic external field:

$$V(z; \sigma) \equiv \frac{z^4}{4} + \sigma \frac{z^2}{2}, \qquad \sigma \in \mathbb{C},$$

for which one has (regular) one-cut, two-cut, and three-cut regions in the complex σ-plane which are denoted by $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ respectively. In [2] the phase diagrams for a variety of choices of integration contours for this model have been presented. In [6] the particular case of admissible contours that approach ∞ along the real axis was considered and the phase diagram (as shown in Fig. 1[3]) was proven. Using the explicit formulae for the end-points of J_σ and zeros of $h(z; \sigma)$ in the one-cut case, one can easily find that the non-generic parameter values corresponding to case (a) above are only $\sigma = \pm i\sqrt{12}$, for which the points $\mp z_0$ (zeros of $h(z; \sigma)$) coincide with the endpoints $\pm b_1$ [2, 6]. The non-generic points on the boundaries labeled by γ_1 and γ_2 represent the σ values for which the zeros of $h(z; \sigma)$ hit the support of the equilibrium measure (see Fig. 2). Figure 3 corresponds to item (c) above, in which the regions in light blue represent the σ-stable lands. Figures 3a–f show the contour Γ_σ for six choices of parameters $\sigma \in \mathcal{O}_1$, while Fig. 3g corresponds to a *non-generic* value of $\sigma \in \gamma_3$ (see Fig. 1) where the complementary part $\Gamma_\sigma \setminus J_\sigma$ (the orange dashed line in Fig. 3g) can not avoid going through at least one point which does not belong to the σ-stable lands (see item (c) above). Finally Fig. 3h corresponds to $\sigma = -1.35 + 4i$ which is clearly not a one-cut parameter as there is no connection from the endpoint b_1 to ∞ in the sector originally chosen for the orthogonal polynomials, however, it turns out that it is a regular three-cut parameter [6].

It should also be mentioned that transitions through these boundaries correspond qualitatively to phase transitions in the asymptotic behavior of the orthogonal polynomials. For example, in the simpler case of real potentials, if there is one

[2] For a precise definition of regularity see Sects. 2.4, 3 and Definition 3.1.

[3] Figures 1, 2, and 3 are taken from [6].

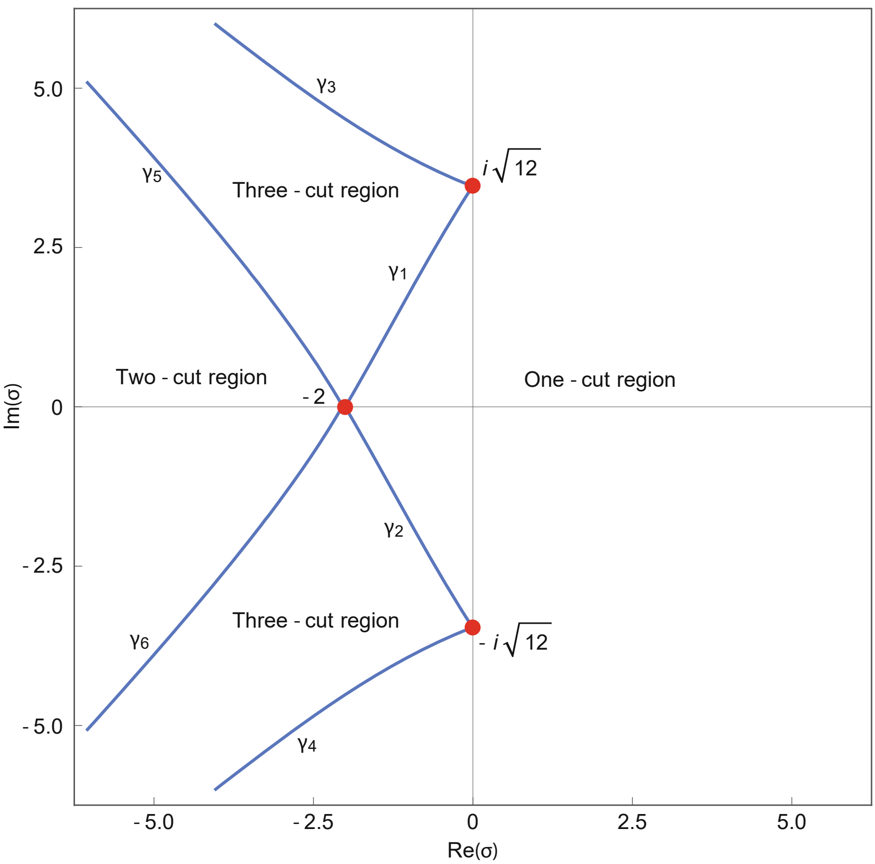

Fig. 1 The phase diagram of the complex quartic random matrix model in the σ-plane.

contour comprising the support, the oscillatory behavior of the polynomails is expressed via trigonometric functions [8], while if there are several intervals, then the oscillatory behavior is described by a Jacobi theta function associated to the Riemann surface of $R(z; \boldsymbol{t})$ [9].

The main purpose of this work is to present a brief self-contained proof of the fact that if for some $\boldsymbol{t}^* \in \mathbb{C}^{2p-1}$ the corresponding equilibrium measure is q-cut regular, then there exists a small enough neighborhood $D_\varepsilon(\boldsymbol{t}^*)$ of $\boldsymbol{t}^*$ so that for all $\boldsymbol{t} \in D_\varepsilon(\boldsymbol{t}^*)$ the associated equilibrium measures are also q-cut regular. Lemma 4.2 of [3] gives another proof of the openness of regular set of parameters using the determinantal form of the function η_q, and uses arguments from [24, 25]. The proof that we present here avoids computations of the Jacobian determinant, but rather has the flavor of a vanishing lemma from the theory of Riemann-Hilbert problems, which permits us to arrive at a contradiction if the Jacobian determinant should vanish at a regular point.

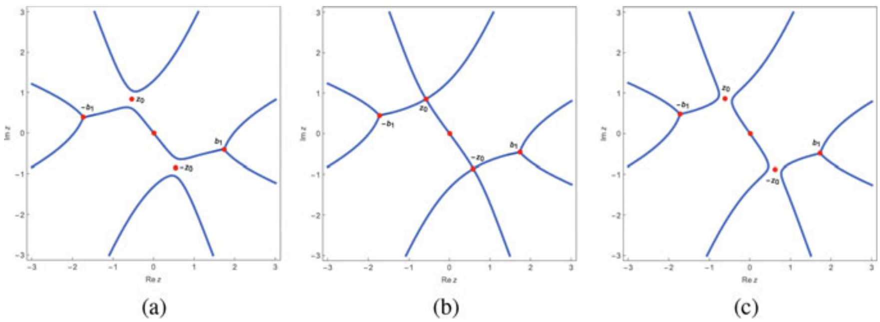

Fig. 2 Snapshots of the continuous deformation (see Theorem 1.3) of the critical graph $\mathscr{J}_\sigma^{(1)}$. (**a**) The critical graph $\mathscr{J}_\sigma^{(1)}$ of the one-cut quadratic differential for the complex quartic model at a $\sigma \in \mathcal{O}_1$. At this value of σ all regularity properties are satisfied. (**b**) The critical graph $\mathscr{J}_\sigma^{(1)}$ at a critical value $\sigma \in \gamma_1$ (see Fig. 1). The zeros of $h(z;\sigma)$ at this value hit J_σ, and thus $\sigma \notin \mathcal{O}_1$. (**c**) The critical graph $\mathscr{J}_\sigma^{(1)}$ at a $\sigma \notin \mathcal{O}_1$. It turns out that this value of σ actually is a regular three-cut value as shown in [6].

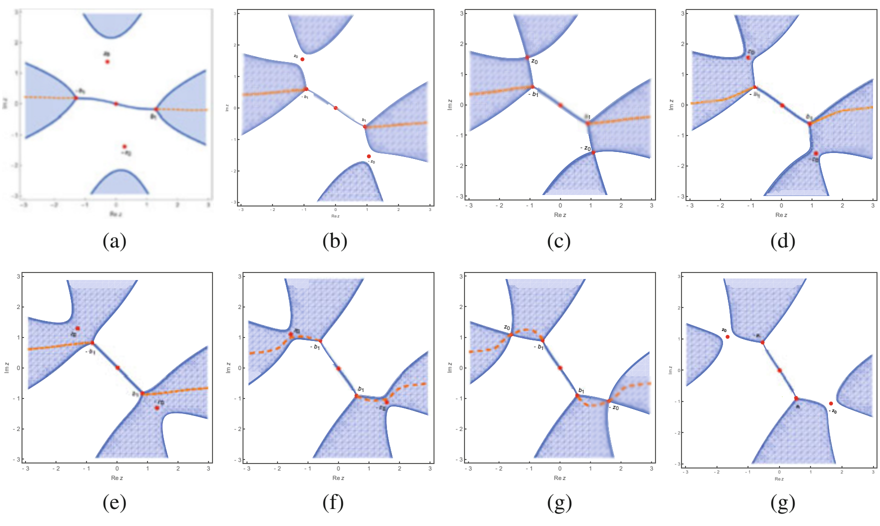

Fig. 3 This sequence of figures shows allowable regions in light blue through which the contour of integration (for the orthogonal polynomials) must pass, for a varying collection of values of σ. Regions in light blue are the σ-stable lands where $\Re[\eta_1(z;\sigma)] < 0$ and the regions in white are the σ-unstable lands where $\Re[\eta_1(z;\sigma)] > 0$ (see (16) and (17)). Notice that the sigma values associated with (g) and (h) do not belong to $\mathcal{O}_1$. $\sigma = 1 + \mathrm{i} \in \mathcal{O}_1$. (**b**) $\sigma = 1 + 3.8\mathrm{i} \in \mathcal{O}_1$. (**c**) $\sigma = 1 + 3.92\mathrm{i} \in \mathcal{O}_1$. (**d**) $\sigma = 1 + 4\mathrm{i} \in \mathcal{O}_1$. (**e**) $\sigma = 4\mathrm{i} \in \mathcal{O}_1$. (**f**) $\sigma = -1 + 4\mathrm{i} \in \mathcal{O}_1$. (**g**) $\sigma_{\mathrm{cr}} \simeq -1.15 + 4\mathrm{i}$. (**h**) $\sigma = -1.35 + 4\mathrm{i}$.

Some of these arguments use ideas based on Riemann surface theory contained in [1] in which Lemma 4.1 provides a proof. Indeed, in Sect. 5 we prove:

Theorem 1.1 *The regular q-cut regime is open.*

The proof of Theorem 1.1 relies upon showing that the underlying equations for finding the endpoints are solvable for every $\boldsymbol{t} \in D_\varepsilon(\boldsymbol{t}^*)$. To this end in § 4 we formulate the end-point equations in the q-cut case and prove the following result:

Theorem 1.2 *The equations which determine the $2q$ endpoints of the regular q-cut regime are solvable for all $\boldsymbol{t}$ in a small enough neighborhood of a regular q-cut point $\boldsymbol{t}^*$, all endpoints $a_j(\boldsymbol{t}), b_j(\boldsymbol{t})$ are distinct, and $\Re a_j(\boldsymbol{t})$, $\Im a_j(\boldsymbol{t})$, $1 \le j \le q$ are real-analytic functions of $\Re t_k$, $\Im t_k$, $1 \le k \le 2p-1$.*

Another important ingredient in the proof of Theorem 1.1, mainly useful for establishing that the *regularity* properties are preserved for every $\boldsymbol{t} \in D_\varepsilon(\boldsymbol{t}^*)$, is the continuity of the critical graph of the associated quadratic differential which, in particular, has within itself the q-cut support of the equilibrium measure. Apart from the continuity of the support $J_{\boldsymbol{t}}$, knowing the continuity of the complementary part of the critical graph, i.e. $\mathscr{J}_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}$, is also very important. This is because the *"closure of a strait"* (recall, for example, the passage from $\sigma = -1 + 4\mathrm{i}$ to $\sigma_{\mathrm{cr}} \simeq -1.15 + 4\mathrm{i}$ depicted in Fig. 3f and g) is directly tied to the behavior of the complementary part $\mathscr{J}_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}$ of the critical graph, which leads to the impossibility of having complementary contours $\Gamma_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}$ to lie entirely in the $\boldsymbol{t}$-stable lands (see the orange dashed line in Fig. 3g). To that end, in § 5, for the entirety of the critical graph $\mathscr{J}_{\boldsymbol{t}}$ we prove:

Theorem 1.3 *The critical graph $\mathscr{J}_{\boldsymbol{t}}$ of the quadratic differential*

$$Q(z;\boldsymbol{t})dz^2 \equiv \left(-\omega(z;\boldsymbol{t}) + \frac{V'(z;\boldsymbol{t})}{2}\right)^2 dz^2,$$

and thus the support $J_{\boldsymbol{t}}$ of the equilibrium measure, deform continuously with respect to $\boldsymbol{t}$.

2 Equilibrium Measure and Quadratic Differentials

2.1 Equilibrium Measure for Orthogonal Polynomials Associated with Real External Fields

Let

$$V(x;\boldsymbol{t}) = \frac{x^{2p}}{2p} + \sum_{j=1}^{2p-1} \frac{t_j z^j}{j},$$

be any polynomial of even degree with real coefficients. Now consider the following energy functional which is defined on the space of probability measures on $\mathbb{R}$:

$$I_V(\nu) := \int_{\mathbb{R}}\int_{\mathbb{R}} \log \frac{1}{|x-y|} d\nu(x) d\nu(y) + \int_{\mathbb{R}} V(x) d\nu(x). \tag{18}$$

The equilibrium measure, ν_{eq}, is a *probability measure* on $\mathbb{R}$ which achieves the infimum of the above functional:

$$\inf_{\mathscr{M}_1(\mathbb{R})} I_V(\nu) = I_V(\nu_{\text{eq}}), \tag{19}$$

where

$$\mathscr{M}_1(\mathbb{R}) := \left\{ \nu : \nu \geq 0, \int_{\mathbb{R}} d\nu = 1 \right\}.$$

For this extremal problem, it is known that (see, e.g., [4, 7, 9])

1. The equilibrium measure *exists* and is *unique*.
2. The equilibrium measure is *absolutely continuous* with respect to the Lebesgue measure,

$$d\nu_{\text{eq}}(z) = \rho_V(z)\, dz.$$

3. The support of ν_{eq} consists of *finitely many closed intervals*,

$$J = \operatorname{supp} \nu_{\text{eq}} = \bigcup_{k=1}^{q} [a_k, b_k],$$

where $q \leq p$. The intervals $\{[a_k, b_k],\ k = 1, \ldots, q\}$ of the support of ν_{eq} are called the *cuts*. We may assume that $a_1 < b_1 < a_2 < b_2 < \ldots < a_q < b_q$.
4. The density of the equilibrium measure on the support J can be written in the form,

$$\rho_V(x) = \frac{1}{2\pi \mathrm{i}} h(x) R_+^{1/2}(x), \quad R(x) = \prod_{k=1}^{q} (x - a_k)(x - b_k), \tag{20}$$

where $h(x)$ is a polynomial, such that $h(x) \geq 0$ for all $x \in J$, and $R^{1/2}(x)$ is the branch on the complex plane of the square root of $R(x)$, with cuts on J, which is positive for large positive x. Respectively, $R_+^{1/2}(x)$ is the value of $R^{1/2}(x)$ on the upper part of the cut.

5. Finally, the polynomial $h(x)$ is the *polynomial part* of the function $\frac{V'(x)}{R^{1/2}(x)}$ at infinity, i.e.,

$$\frac{V'(x)}{R^{1/2}(x)} = h(x) + \mathcal{O}(x^{-1}). \tag{21}$$

   This determines $h(x)$ and hence the equilibrium measure ν_{eq} uniquely, as long as we know the end-points $a_1, b_1, \ldots, a_q, b_q$.

An important property of this minimization problem (19) is that the minimizer ν_{eq} is *uniquely determined* by the Euler–Lagrange variational conditions:

$$2\int_{\mathbb{R}} \log|x-y|\,\mathrm{d}\nu_V(y) - V(x) = l, \quad \text{for} \quad x \in J, \tag{22}$$

$$2\int_{\mathbb{R}} \log|x-y|\,\mathrm{d}\nu_V(y) - V(x) \le l, \quad \text{for} \quad x \in \mathbb{R}\setminus J, \tag{23}$$

for some real constant *Lagrange multiplier* l, which is the *same for all cuts* $[a_k, b_k]$. From this we conclude that

$$\int_{b_k}^{a_{k+1}} h(x)R^{1/2}(x)\,\mathrm{d}x = 0, \quad k = 1, \ldots, q-1. \tag{24}$$

Therefore the polynomial $h(x)$ has a zero on every interval $[b_k, a_{k+1}]$, which means that $\deg h \ge q-1$.

We also consider the *resolvent* of the equilibrium measure defined as

$$\omega(z) = \int_J \frac{\mathrm{d}\nu_V(x)}{z-x}, \quad z \in \mathbb{C}\setminus J. \tag{25}$$

This function, which is very useful to construct the density of the equilibrium measure, has the following analytical and asymptotic properties:

1. $\omega(z)$ is analytic on the set $\mathbb{C}\setminus J$.
2. On J, the equilibrium condition (34) implies that

$$\omega_+(x) + \omega_-(x) = V'(x), \tag{26}$$

   and the Plemelj–Sokhotski formula implies

$$\omega_+(x) - \omega_-(x) = -2\pi i \rho_V(x). \tag{27}$$

   Combining these equations with formula (20) for $\rho_V(x)$, we obtain that

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}. \tag{28}$$

3. As $z \to \infty$,

$$\omega(z) = \frac{1}{z} + \frac{m_1}{z^2} + \dots, \qquad m_k = \int_J x^k \rho_V(x)\, dx. \tag{29}$$

2.2 Equilibrium Measure for Orthogonal Polynomials Associated with Complex External Fields

In this section we follow the work of Kuijlaars and Silva [14] (See also [1, 15, 18]). Let us consider the general complex external field of even degree

$$V(z) = \frac{z^{2p}}{2p} + \sum_{j=1}^{2p-1} \frac{t_j z^j}{j}, \qquad t_j \in \mathbb{C}, \qquad j = 1, \cdots, 2p-1. \tag{30}$$

For $0 < \varepsilon < \pi/4p$, consider the sectors

$$S_\varepsilon^+ = \left\{ z \in \mathbb{C} \;\middle|\; |\arg z| \le \frac{\pi}{4p} - \varepsilon \right\}, \qquad S_\varepsilon^- = \left\{ z \in \mathbb{C} \;\middle|\; |\arg z - \pi| \le \frac{\pi}{4p} - \varepsilon \right\}. \tag{31}$$

Observe that in these sectors we particularly have,

$$\lim_{z\to\infty} \Re V(z) = \infty. \tag{32}$$

By a contour we mean a *continuous curve* $z = z(t)$, $-\infty < t < \infty$, without self-intersections, and we say that a contour Γ is *admissible* if

1. The contour Γ is a finite union of C^1 Jordan arcs.
2. There exists $\varepsilon > 0$ and $r_0 > 0$, such that Γ goes from S_ε^- to S_ε^+ in the sense that $\forall\, r > r_0$, $\exists\, t_0 < t_1$ such that

$$z(t) \in S_\varepsilon^- \setminus D_r \quad \forall\, t < t_0; \quad z(t) \in S_\varepsilon^+ \setminus D_r \quad \forall\, t > t_1,$$

   where D_r is the disk centered at the origin with radius r. We will assume that the contour Γ is oriented from $(-\infty)$ to $(+\infty)$, where $(-\infty)$ lies in the sector S_ε^- and $(+\infty)$ in the sector S_ε^+. The orientation defines an order on the contour Γ.

An example of an admissible contour is the real line. We denote the collection of all admissible contours by $\mathcal{T}$.

For $\Gamma \in \mathcal{T}$, let $\mathcal{P}(\Gamma)$ be the space of probability measures ν on Γ, satisfying

$$\int_{\Gamma} |\Re V(s)| \,\mathrm{d}\nu(s) < \infty. \tag{33}$$

Consider the following real-valued energy functional on $\mathcal{P}(\Gamma)$:

$$I_{V,\Gamma}(\nu) := \iint_{\Gamma\times\Gamma} \log \frac{1}{|z-s|} \,\mathrm{d}\nu(z)\mathrm{d}\nu(s) + \int_{\Gamma} \Re V(s) \,\mathrm{d}\nu(s). \tag{34}$$

Then there exists a unique minimizer $\nu_{V,\Gamma}$ of this functional (see [19]) so that

$$\min_{\nu\in\mathcal{P}(\Gamma)} I_{V,\Gamma}(\nu) = I_{V,\Gamma}(\nu_{V,\Gamma}). \tag{35}$$

The minimizing probability measure $\nu_{V,\Gamma}$ is referred to as the *equilibrium measure* of the functional $I_{V,\Gamma}(\nu)$, and its support is a compact set $J_{V,\Gamma} \subset \Gamma$, and is uniquely determined by the *Euler–Lagrange variational conditions*. Namely, $\nu_{V,\Gamma}$ is the unique probability measure ν on Γ such that there exists a constant l, the Lagrange multiplier, such that

$$\begin{aligned} U^{\nu}(z) + \frac{1}{2}\Re V(z) &= \ell, \quad z \in \operatorname{supp}\nu, \\ U^{\nu}(z) + \frac{1}{2}\Re V(z) &\geq \ell, \quad z \in \Gamma \setminus \operatorname{supp}\nu, \end{aligned} \tag{36}$$

where

$$U^{\nu}(z) = \int_{\Gamma} \log \frac{1}{|z-s|} \,\mathrm{d}\nu(s) \tag{37}$$

is the *logarithmic potential* of the measure ν [19].

Now we maximize the minimized energy functional $I_V(\nu_{V,\Gamma})$ over all admissible contours $\Gamma \in \mathcal{T}$. In [14], the authors prove that the maximizing contour $\Gamma_t \in \mathcal{T}$ exists, and the equilibrium measure

$$\nu_{\mathrm{eq}} \equiv \nu_{V,\Gamma_t}$$

is supported by a set $J_t \subset \Gamma_t$ which is a finite union of *analytic arcs* $\Gamma_t[a_k, b_k] \subset \Gamma_t$,[4] $k = 1, \dots, q$,

$$J_t = \bigcup_{k=1}^{q} \Gamma_t[a_k, b_k],$$

that are *critical trajectories* of a quadratic differential[5] $Q(z)\,\mathrm{d}z^2$, where $Q(z)$ is a polynomial of degree

$$\deg Q(z) = 2 \deg V(z) - 2 = 4p - 2. \tag{38}$$

Moreover, in [14] it is proven that the polynomial $Q(z)$ is equal to

$$Q(z) = \left(-\omega(z) + \frac{V'(z)}{2}\right)^2, \tag{39}$$

where

$$\omega(z) = \int_{J_t} \frac{\mathrm{d}\nu_{\mathrm{eq}}(s)}{z - s} \tag{40}$$

is the resolvent of the measure ν_{eq}. From

$$\frac{1}{z-s} = \frac{1}{z} + \frac{s}{z^2} + \frac{s^2}{z^3} + \dots,$$

we obtain that $\omega(z) = \mathcal{O}(z^{-1})$ as $z \to \infty$:

$$\omega(z) = \frac{1}{z} + \frac{m_1}{z^2} + \dots, \quad \text{with} \quad m_k = \int_{J_t} s^k \mathrm{d}\nu_{\mathrm{eq}}(s). \tag{41}$$

Additionally, the equilibrium measure ν_{eq} is absolutely continuous with respect to the arc length. More precisely we have

$$\mathrm{d}\nu_{\mathrm{eq}}(s) = \frac{1}{\pi \mathrm{i}}\, Q_+(s)^{1/2} \mathrm{d}s, \tag{42}$$

where $Q_+(s)^{1/2}$ is the limiting value of the function

$$Q(z)^{1/2} = -\int_{J_t} \frac{\mathrm{d}\nu_{\mathrm{eq}}(s)}{z - s} + \frac{V'(z)}{2}, \tag{43}$$

[4] Given two points $s_1, s_2 \in \mathbb{C} \cup \{\pm\infty\}$ on Γ_t, by $\Gamma_t(s_1, s_2)$ and $\Gamma_t[s_1, s_2]$ we respectively denote the open and closed "intervals" on Γ_t starting at s_1 and ending at s_2.

[5] See Sect. 2.5 for a review of definitions and basic facts about quadratic differentials.

as $z \to s \in J_{\boldsymbol{t}}$ from the left-hand side of $J_{\boldsymbol{t}}$ with respect to the orientation of the contour $\Gamma_{\boldsymbol{t}}$ from $(-\infty)$ to ∞. A very important result in [14] is that the equilibrium measure ν_{eq} is *unique* as the *max-min measure* which immediately gives the uniqueness of the set $J_{\boldsymbol{t}}$. On the other hand, the infinite contour $\Gamma_{\boldsymbol{t}}$ is *not unique* because it can be deformed outside of the support $J_{\boldsymbol{t}}$ of ν_{eq}, as long as $\Gamma_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}$ lies in the $\boldsymbol{t}$-stable lands. To summarize, one can choose the contour $\Gamma_{\boldsymbol{t}}$ to be the union

$$\Gamma_{\boldsymbol{t}} = J_{\boldsymbol{t}} \cup \widehat{\Gamma}_{\boldsymbol{t}} \cup \check{\Gamma}_{\boldsymbol{t}},$$

where $\widehat{\Gamma}_{\boldsymbol{t}}$ is a (non-unique) set consisting of $q-1$ finite arcs in the $\boldsymbol{t}$-stable lands so that $J_{\boldsymbol{t}} \cup \widehat{\Gamma}_{\boldsymbol{t}}$ is connected and $\check{\Gamma}_{\boldsymbol{t}}$ consists of two (non-unique) infinite arcs in the $\boldsymbol{t}$-stable lands one connecting $-\infty$ to $a_1(\boldsymbol{t})$ and the other connecting $b_q(\boldsymbol{t})$ to $+\infty$.

2.3 The g-Function

As usual we define the "g-function" as

$$g(z) = \int_{J_{\boldsymbol{t}}} \log(z-s)\, \mathrm{d}\nu_{\mathrm{eq}}(s), \tag{44}$$

where for a fixed $s \in J_{\boldsymbol{t}}$, we consider a cut of $\log(z-s)$ to be $\Gamma_{\boldsymbol{t}}(-\infty, s]$. Notice that by (40) we have

$$g'(z) = \int_{J_{\boldsymbol{t}}} \frac{\mathrm{d}\nu_{\mathrm{eq}}(s)}{z-s} = \omega(z). \tag{45}$$

Moreover, from (37), the logarithmic potential $U^{\nu_{\mathrm{eq}}}(z)$ can be written as

$$U^{\nu_{\mathrm{eq}}}(z) = \int_{J_{\boldsymbol{t}}} \log \frac{1}{|z-s|}\, \mathrm{d}\nu_{\mathrm{eq}}(s) = -\Re g(z), \tag{46}$$

and therefore the Euler-Lagrange variational conditions (36) can be expressed as

$$\begin{aligned} &-\Re g(z) + \frac{1}{2}\Re V(z) = \ell, \quad z \in J_{\boldsymbol{t}}, \\ &-\Re g(z) + \frac{1}{2}\Re V(z) \geq \ell, \quad z \in \Gamma_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}. \end{aligned} \tag{47}$$

2.4 Regular and Singular Equilibrium Measures

An equilibrium measure ν_{eq} is called *regular*[6] if the following three conditions hold:

1. The arcs $\Gamma_{\boldsymbol{t}}[a_k, b_k],\ k = 1, \dots, q$, of the support of ν_{eq} are disjoint.
2. The end-points $\{a_k, b_k,\ k = 1, \dots, q\}$ are simple zeros of the polynomial $Q(s)$.
3. There is a contour $\Gamma_{\boldsymbol{t}}$ containing the support $J_{\boldsymbol{t}}$ of ν_{eq} such that

$$U^{\nu_{\mathrm{eq}}}(z) + \frac{1}{2}\Re V(z) > \ell, \quad z \in \Gamma_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}. \tag{48}$$

An equilibrium measure ν_{eq} is called *singular* (or *critical*) if it is not regular.

2.4.1 Regular Equilibrium Measures

Assume that the equilibrium measure ν_{eq} is regular. Because the resolvent

$$\omega(z) = \int_{J_{\boldsymbol{t}}} \frac{\mathrm{d}\nu_{\mathrm{eq}}(s)}{z - s} \tag{49}$$

is analytic on $\mathbb{C} \setminus J_{\boldsymbol{t}}$, one can see from equation (39) that all the zeros of the polynomial $Q(z)$ different from the end-points $\{a_k, b_k,\ \ k = 1, \dots, q\}$ must be of even degree, and thus $Q(z)$ can be expressed as

$$Q(z) = \frac{1}{4} h(z)^2 R(z), \tag{50}$$

where $h(z)$ is some polynomial,

$$h(z) = \prod_{j=1}^{r} (z - z_j), \tag{51}$$

having zeros $z_1, \dots, z_r$ which are distinct from the $2q$ end-points $\{a_k, b_k\}_{k=1}^{q}$, and

$$R(z) = \prod_{k=1}^{q} (z - a_k)(z - b_k). \tag{52}$$

[6] The set of regular q-cut parameters giving rise to regular q-cut equilibrium measures is defined in Definition 3.1.

Therefore,

$$Q(z) = \frac{1}{4} h(z)^2 R(z) = \frac{1}{4} \prod_{j=1}^{r} (z - z_j)^2 \prod_{k=1}^{q} (z - a_k)(z - b_k). \tag{53}$$

In (51), and (53) if $r = 0$, it is understood that $h(z) \equiv 1$. By taking the square root with the plus sign, we obtain that

$$Q(z)^{1/2} = \frac{1}{2} h(z) R(z)^{1/2} = \frac{1}{2} \prod_{j=1}^{r} (z - z_j) \left[\prod_{k=1}^{q} (z - a_k)(z - b_k) \right]^{1/2}, \tag{54}$$

Correspondingly, equation (42) can be rewritten as

$$\mathrm{d}\nu_{\mathrm{eq}}(z) = \frac{1}{2\pi \mathrm{i}} h(z) R_+(z)^{1/2} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \prod_{j=1}^{r} (z - z_j) \left[\prod_{k=1}^{q} (z - a_k)(z - b_k) \right]_+^{1/2} \mathrm{d}z. \tag{55}$$

From (39), (45), and (54) we can write

$$g(z; \boldsymbol{t}) = \frac{V(z; \boldsymbol{t}) + \ell_*^{(q)}(\boldsymbol{t})}{2} + \frac{\eta_q(z; \boldsymbol{t})}{2}, \qquad z \in \mathbb{C} \setminus \Gamma_{\boldsymbol{t}}(-\infty, b_q(\boldsymbol{t})], \tag{56}$$

where

$$\eta_q(z; \boldsymbol{t}) := -\int_{b_q(\boldsymbol{t})}^{z} \prod_{\ell=1}^{r} (s - z_\ell(\boldsymbol{t})) \left[\prod_{j=1}^{q} \big(s - a_j(\boldsymbol{t})\big) \big(s - b_j(\boldsymbol{t})\big) \right]^{1/2} \mathrm{d}s, \tag{57}$$

$z \in \mathbb{C} \setminus \Gamma_{\boldsymbol{t}}(-\infty, b_q(\boldsymbol{t})]$, in which the path of integration does not cross $\Gamma_{\boldsymbol{t}}(-\infty, b_q(\boldsymbol{t})]$,[7] and $\ell_*^{(q)}(\boldsymbol{t})$ is chosen such that $g(z; \boldsymbol{t})$ asymptotically behaves like $\log z$ as $z \to \infty$.[8] Also from (39) and (54) we have

$$g'_+(z; \boldsymbol{t}) + g'_-(z; \boldsymbol{t}) = V'(z; \boldsymbol{t}), \qquad z \in J_{\boldsymbol{t}} = \bigcup_{j=1}^{q} \Gamma_{\boldsymbol{t}}(a_j(\boldsymbol{t}), b_j(\boldsymbol{t})). \tag{58}$$

[7] See the paragraph following (43) regarding the choice of $\Gamma_{\boldsymbol{t}}$, and for the notation $\Gamma_{\boldsymbol{t}}(-\infty, b_q(\boldsymbol{t})]$ see footnote 4.

[8] For the quartic potential discussed in the introduction the explicit formulae for $\ell_*^{(q)}(\boldsymbol{t})$ are derived for $q = 1$ and $q = 2$ in [6].

We use (45) and (50) to rewrite (39) as:

$$g'(z;\boldsymbol{t}) = \frac{1}{2}\left[V'(z;\boldsymbol{t}) - h(z;\boldsymbol{t})R^{1/2}(z;\boldsymbol{t})\right]. \tag{59}$$

2.5 Quadratic Differentials

In this subsection we briefly remind some definitions and basic facts about quadratic differentials from [22]. The zeros and poles of $Q(z)$ are referred to as the *critical points* of the quadratic differential $Q(z)\mathrm{d}z^2$, and all other points are called *regular points* of $Q(z)\mathrm{d}z^2$. For some fixed value $\theta \in [0, 2\pi)$, the smooth curve L_θ along which

$$\arg Q(z)\mathrm{d}z^2 = \theta,$$

is defined as the θ-arc of the quadratic differential $Q(z)\mathrm{d}z^2$, and a maximal θ-arc is called a θ-trajectory. The above equation implies that a θ-arc can only contain regular points of Q, because at the critical points $\arg Q(z)$ is not defined. For a meromorphic quadratic differential, there is only one θ-arc passing through each regular point.

We will refer to a π-trajectory (resp. 0-trajectory) which is incident with a critical point as a *critical trajectory* (resp. *critical orthogonal trajectory*). If b is a critical point of $Q(z)\mathrm{d}z^2$, then the totality of the solutions to

$$\Re\left(\int_b^z \sqrt{Q(s)}\mathrm{d}s\right) = 0,$$

is referred to as the *critical graph* of $\int_b^z \sqrt{Q(s)}\mathrm{d}s$ which is referred to as the *natural parameter* of the quadratic differential $Q(z)\mathrm{d}z^2$ (see §5 of [22]). A Jordan curve Σ composed of open θ-arcs and their endpoints, with respect to some meromorphic quadratic differential $Q(z)\mathrm{d}z^2$, is a simple closed *geodesic polygon* (also referred to as a Q-polygon). The endpoints may be regular or critical points of $Q(z)\mathrm{d}z^2$, which form the vertices of the Q-polygon. Σ is called a *singular geodesic polygon*, if at least one of its end points is a singular point.

Now we can state the *Teichmüller's lemma*: for a meromorphic quadratic differential $Q(z)\mathrm{d}z^2$, assume that Σ is a Q-polygon, and let $\mathtt{V}_\Sigma$ and $\mathtt{Int}\,\Sigma$ respectively denote its set of vertices and interior. Then

$$\#\mathtt{V}_\Sigma - 2 = \sum_{z\in\mathtt{V}_\Sigma}(\mathtt{ord}(z)+2)\frac{\theta(z)}{2\pi} + \sum_{z\in\mathtt{Int}\,\Sigma}\mathtt{ord}(z), \tag{60}$$

where $\theta(z)$ denotes the interior angle of Σ at z, and $\mathtt{ord}(z)$ is the order of the point z with respect to the quadratic differential. That is, $\mathtt{ord}(z) = 0$ for a regular point,

$\mathrm{ord}(z) = n$ if z is a zero of order $n \in \mathbb{N}$, and $\mathrm{ord}(z) = -n$ if z is a pole of order $n \in \mathbb{N}$ of the quadratic differential. We use the Teichmüller's lemma in the proof of Theorem 1.1 in Sect. 5.

3 Endpoint Equations and the Regular q-Cut Regime

Notice that from (58) we have

$$R_+^{-1/2}(z)g_+'(z) = -R_-^{-1/2}(z)\left(V'(z) - g_-'(z)\right) = R_-^{-1/2}(z)g_-'(z) - R_-^{-1/2}(z)V'(z)$$

Therefore by Plemelj-Sokhotskii we have

$$g'(z) = \frac{R^{1/2}(z)}{2\pi \mathrm{i}} \int_J \frac{V'(s)}{R_+^{1/2}(s)} \frac{\mathrm{d}s}{s-z} = -\frac{R^{1/2}(z)}{2\pi \mathrm{i} z} \sum_{\ell=0}^{\infty} \frac{T_\ell}{z^\ell} \tag{61}$$

where

$$T_\ell = \int_J \frac{V'(s)}{R_+^{1/2}(s)} s^\ell \mathrm{d}s\,, \qquad \ell \in \mathbb{N} \cup \{0\}. \tag{62}$$

From (61) and the requirement that $g'(z) = z^{-1} + O(z^{-2})$ as $z \to \infty$, we obtain the following $q+1$ equations:

$$T_\ell = 0\,, \qquad \ell = 0, 1, \ldots, q-1\,, \qquad \text{and} \qquad T_q = -1\,. \tag{63}$$

We have $q-1$ gaps, and thus $q-1$ *gap conditions*:

$$\Re \int_{b_j}^{a_{j+1}} h(s) R^{1/2}(s) \mathrm{d}s = 0\,, \quad j = 1, \ldots, q-1\,. \tag{64}$$

Since the equilibrium measure is positive along the support, we immediately find the following $q-1$ real conditions

$$\Re \int_{a_j}^{b_j} h(s) R_+^{1/2}(s) \mathrm{d}s = 0\,, \quad j = 1, \ldots, q-1\,. \tag{65}$$

Notice that the condition on the last cut

$$\Re \int_{a_q}^{b_q} h(s) R_+^{1/2}(s) \mathrm{d}s = 0\,,$$

is a consequence of the $q-1$ conditions in (65) and should not be considered as an extra requirement.

Being in the q-cut case, we have to determine $2q$ endpoints and thus $4q$ real unknowns $\Re a_1, \Im a_1, \Re a_2, \Im a_2, \cdots \Re b_q, \Im b_q$. These unknowns are determined by the $4q$ real conditions given by (63), (64), and (65).

Let $\mathscr{F}$ be the vector-valued function, whose $4q$ entries are defined as

$$\mathscr{F}_{2\ell} = \Re T_\ell + \delta_{\ell q}, \quad \mathscr{F}_{2\ell+1} = \Im T_\ell, \qquad \ell = 0, \ldots, q\ , \tag{66}$$

$$\mathscr{F}_{2q+1+j} = \Re \int_{a_j}^{b_j} h(s) R_+^{1/2}(s)\mathrm{d}s, \qquad j = 1, \ldots, q-1\ , \tag{67}$$

$$\mathscr{F}_{3q+j} = \Re \int_{b_j}^{a_{j+1}} h(s) R^{1/2}(s)\mathrm{d}s, \qquad j = 1, \ldots, q-1\ . \tag{68}$$

We express the equations (63), (64), and (65) for determining the branch points as

$$\mathscr{F} = 0. \tag{69}$$

From the requirement (41), and equation (59), in particular, we know that

$$\deg V - 1 = \deg h + \frac{\deg R}{2}, \tag{70}$$

therefore, recalling (30), (51), and (52) we obtain

$$r = 2p - 1 - q. \tag{71}$$

Since h is a polynomial, we obtain the following bound on the number of cuts

$$q \leq 2p - 1. \tag{72}$$

Definition 3.1 The regular q-cut regime which is denoted by $\mathbb{O}_q$ is a subset in the phase space $\mathbb{C}^{2p-1}$ which is defined as the collection of all $\boldsymbol{t} \equiv (t_1, \cdots, t_{2p-1}) \in \mathbb{C}^{2p-1}$ such that the points $a_j(\boldsymbol{t}), b_j(\boldsymbol{t})$, with $j = 1, \cdots, q$ as solutions of (69) are all distinct and

1. The set $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ of all points z satisfying

$$\Re\left[\eta_q(z; \boldsymbol{t})\right] = 0,$$

   contains a single Jordan arc connecting $a_j(\boldsymbol{t})$ to $b_j(\boldsymbol{t})$, for each $j = 1, \cdots, q$.
2. The points $z_\ell(\boldsymbol{t})$, $\ell = 1, \cdots, 2p-1-q$, do not lie on $J_{\boldsymbol{t}}^{(q)} := \bigcup_{j=1}^{q} \Gamma_{\boldsymbol{t}}\left[a_j(\boldsymbol{t}), b_j(\boldsymbol{t})\right]$.

3. There exists a complementary arc $\Gamma_{\boldsymbol{t}}(b_q(\boldsymbol{t}), +\infty)$ which lies entirely in the component of the set

$$\left\{z : \Re\left[\eta_q(z; \boldsymbol{t})\right] < 0\right\},$$

which encompasses $(M_1(\boldsymbol{t}), +\infty)$ for some $M_1(\boldsymbol{t}) > 0$.
4. There exists a complementary arc $\Gamma_{\boldsymbol{t}}(-\infty, a_1(\boldsymbol{t}))$ which lies entirely in the component of the set

$$\left\{z : \Re\left[\eta_q(z; \boldsymbol{t})\right] < 0\right\},$$

which encompasses $(-\infty, -M_2(\boldsymbol{t}))$ for some $M_2(\boldsymbol{t}) > 0$.
5. There exists a complementary arc $\Gamma_{\boldsymbol{t}}\left(b_j(\boldsymbol{t}), a_{j+1}(\boldsymbol{t})\right)$, for each $j = 1, \cdots, q-1$ which lies entirely in the component of the set

$$\left\{z : \Re\left[\eta_q(z; \boldsymbol{t})\right] < 0\right\}.$$

Let us now briefly discuss the significance of equation (13) and the requirement (16) in relation to the above definition. Taking real parts from both sides of (56) we obtain

$$-\frac{1}{2}\Re\eta_q(z; \boldsymbol{t}) = -\Re g(z; \boldsymbol{t}) + \frac{1}{2}\Re V(z; \boldsymbol{t}) - \ell, \tag{73}$$

where ℓ denotes $-\frac{1}{2}\Re\ell_*^{(q)}(\boldsymbol{t})$. So for a fixed $\boldsymbol{t}$, comparing with (47), the support of the equilibrium measure, is the collection of q arcs as solutions to $\Re\eta_q(z; \boldsymbol{t}) = 0$ (same as (13)) connecting $a_j(\boldsymbol{t})$ to $b_j(\boldsymbol{t})$, $j = 1, \cdots, q$. For the regular case one also needs to ensure that (48) is also satisfied. In view of (73), this explains why we require that $\Gamma_{\boldsymbol{t}} \setminus J_{\boldsymbol{t}}$ must lie in the so-called $\boldsymbol{t}$-stable lands as defined by (16).

3.1 Structure of the Critical Graph

In this subsection we show basic structural facts about the critical graph $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ for a regular q-cut $\boldsymbol{t}$. Recalling (57) we notice that as $z \to \infty$ we have

$$\eta_q(z; \boldsymbol{t}) = -\frac{z^{r+q+1}}{r+q+1}\left(1 + O(z^{-1})\right) = -\frac{z^{2p}}{2p}\left(1 + O(z^{-1})\right), \tag{74}$$

where we have used (71). Therefore the components of $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ near ∞ must approach the $4p$ distinct angles θ

$$\theta = \frac{\pi}{4p} + \frac{k\pi}{2p}, \qquad k = 0, 1, \cdots, 4p-1, \tag{75}$$

satisfying $\cos(2p\theta) = 0$, where we have parameterized z in the polar form $Re^{i\theta}$. Moreover, at each endpoint there are three critical trajectories of the quadratic differential $Q(z)\mathrm{d}z^2$ making angles of $2\pi/3$ at the critical point. To see this, let α denote either a_j or b_j, $j = 1, \cdots, q$. We have

$$\eta_q(z) = -\int_{b_q}^{\alpha} h(s)\sqrt{R(s)}\mathrm{d}s - \int_{\alpha}^{z} h(s)\sqrt{R(s)}\mathrm{d}s.$$

Notice the first term on the right hand side is an imaginary number, which can be seen if we break it up into integrals over cuts and gaps and using the endpoint conditions (64) and (65). The integrand of the second integral on the right hand side is $O\left((s-\alpha)^{1/2}\right)$, and thus

$$\Re\eta(z) = O\left((z-\alpha)^{3/2}\right), \qquad \text{as} \quad z \to \alpha, \qquad \ell = 1, \cdots, q.$$

This ensures that there are 3 local trajectories emanating from $\alpha \in \{a_j, b_j\}_{j=1}^{q}$ as solutions of $\Re\eta(z) = 0$. Out of these $3 \times 2q$ local critical trajectories, $2q$ of them make the q cuts, and thus we need to determine the destinations of the remaining $4q$ local critical trajectories. Having solutions in the $4p$ directions given in (75) near infinity guides us to investigate if all or some of the $4q$ local critical trajectories can terminate at infinity along one of the $4p$ angles in (75). We define a *hump* to be a part of $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ which a) does not hit any critical points of $Q(z)\mathrm{d}z^2$, and b) starts and ends at ∞ at two of the angles in (75).[9] Notice that there are no singular finite geodesic polygons with one or two vertices associated to the q-cut quadratic differential $Q(z)\mathrm{d}z^2$ given by (53). This is implied by the Teichmüller's lemma (60) and the fact that Q is a polynomial [6].

Let us first assume that that none of $z_1, \cdots, z_r$ lies on $\mathscr{J}_{\boldsymbol{t}}^{(q)} \setminus J_{\boldsymbol{t}}$. By the discussion in the previous paragraph the two local trajectories emanating from one end point (among the $4q$ remaining local trajectories, see the beginning of the previous paragraph) can not connect to one another to form a geodesic polygon with one vertex. Therefore the $4q$ local trajectories have no destiny other than forming some connections among themselves or to terminate at ∞ (see (75)). Now consider the following three cases

1. $p > q$. This means that the remaining $4q$ local trajectories are not enough to exhaust all $4p$ angles given in (75) and thus $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ must also be constituted from $2(p-q)$ humps to correspond to the *unoccupied* $4(p-q)$ directions at infinity.
2. $p = q$. in this case $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ does not have any humps, since the remaining $4q$ local trajectories are enough to exhaust all $4p$ angles given in (75).
3. $p < q$. This means that there are not enough destinations for $4(q-p)$ of the remaining $4q$ local trajectories, and thus the only possibility is that we have $2(q-$

[9] See, e.g. Figs. 2a and 3 except for 3c and 3g.

p) connections among the $4(q-p)$ local trajectories as described in the previous paragraph.

Notice that one can arrive at the above characterization *without* the assumption that none of $z_1, \cdots, z_r$ lies on $\mathscr{J}_{\boldsymbol{t}}^{(q)} \setminus J_{\boldsymbol{t}}$, due to the continuous deformations of $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ (see Theorem 1.3).

Remark 3.2 It is clear that all three cases above are realizable for the quartic potential ($p = 2$) considered in [6], when we can have $q = 1, 2$, and 3.

Remark 3.3 If there are $m \le r$ points $\{z_{k_1}, \cdots, z_{k_m}\} \subset \{z_1, \cdots, z_r\}$ which belong to an unbounded geodesic polygon K with the finite vertex at an endpoint (and the other "vertex" is at infinity), then the separation of the angles between the two edges at ∞ is

$$\theta_\infty = \frac{(2m+1)\pi}{q+r+1} = \frac{(2m+1)\pi}{2p},$$

and therefore K hosts m humps, which is a consequence of the Teichmüller's lemma applied to the polygon K.

4 Solvability of End Point Equations in a Neighborhood of a Regular q-Cut Point: Proof of Theorem 1.2

In this section we want to prove that the equations uniquely determining the endpoints are solvable in a neighborhood of a regular q-cut point. In this section we denote

$$\boldsymbol{t} \equiv \left(\Re t_1, \Im t_1, \cdots, \Re t_{2p-1}, \Im t_{2p-1}\right),$$

and

$$\boldsymbol{x} \equiv \left(\Re a_1, \Im a_1, \ldots, \Re a_q, \Im a_q, \Re b_1, \Im b_1, \ldots, \Re b_q, \Im b_q\right).$$

However when we refer to Definition 3.1, by $\boldsymbol{t}$ we denote the complex vector $(t_1, \cdots, t_{2p-1}) \in \mathbb{C}^{2p-1}$. We can think of $\mathscr{F}$ as a function of $4q$ real variables in the space

$$X := \left\{\left(\Re a_1, \Im a_1, \ldots, \Re a_q, \Im a_q, \Re b_1, \Im b_1, \ldots, \Re b_q, \Im b_q\right) \,:\, a_j, b_j \in \mathbb{C},\ j = 1, \ldots q\right\},$$

and parameters in the space

$$V := \left\{\left(\Re t_1, \Im t_1, \cdots, \Re t_{2p-1}, \Im t_{2p-1}\right) \,:\, t_j \in \mathbb{C},\ j = 1, \cdots 2p-1\right\},$$

recalling (30)[10]. That is

$$\mathscr{F} : X \times V \to \mathbb{R}^{4q} \,, \tag{76}$$

or

$$\mathscr{F} : \mathbb{R}^{4q+4p-2} \to \mathbb{R}^{4q} \,. \tag{77}$$

Notice that the objects T_r, $\int_{b_m}^{a_{m+1}} h(s)R^{1/2}(s)\mathrm{d}s$, and $\int_{a_i}^{b_i} h(s)R_+^{1/2}(s)\mathrm{d}s$ are complex-analytic with respect to a_j, b_j, and t_k, for $0 \le r \le q$, $1 \le m \le q-1$, $1 \le i \le q$, $1 \le j \le q$, and $1 \le k \le 2p-1$. Here we have used the fact that we know the explicit dependence of $h(s;\boldsymbol{t})$ on a_j, b_j, and t_k which can be seen as follows: recall from (59) that

$$h(z;\boldsymbol{t}) = \frac{1}{R^{1/2}(z;\boldsymbol{t})}\left(V'(z;\boldsymbol{t}) - 2g'(z;\boldsymbol{t})\right). \tag{78}$$

Combining this with (61) we obtain

$$h(z;\boldsymbol{t}) = -\frac{1}{\pi \mathrm{i}}\int_J \frac{V'(s;\boldsymbol{t})}{R_+^{1/2}(s;\boldsymbol{t})}\frac{\mathrm{d}s}{s-z} + \frac{V'(z;\boldsymbol{t})}{R^{1/2}(z;\boldsymbol{t})}, \tag{79}$$

and thus,

$$h(z;\boldsymbol{t}) = -\frac{1}{2\pi \mathrm{i}}\oint_{\gamma^*} \frac{V'(s;\boldsymbol{t})}{R^{1/2}(s;\boldsymbol{t})}\frac{\mathrm{d}s}{s-z} \,, \tag{80}$$

where γ^* is a negatively oriented contour which encircles both the support set J and the point z.

This means that the functions $\mathscr{F}_\ell$, $1 \le \ell \le 4q$ are all real-analytic functions of $\Re a_j, \Im a_j, \Re b_j, \Im b_j, \Re t_k, \Im t_k$ for $1 \le j \le q$, and $1 \le k \le 2p-1$. This allows us to use the *real-analytic implicit function theorem.*[11]

We show that if we are in the regular situation, then the Jacobian of the mapping $\mathscr{F}$ with respect to the parameters in X is nonzero. So, if for some $(\boldsymbol{x}^*, \boldsymbol{t}^*) \in X \times V$, we have $\mathscr{F}(\boldsymbol{x}^*, \boldsymbol{t}^*) = 0$ and if

$$\det \left.\begin{pmatrix} \frac{\partial \mathscr{F}_1}{\partial \Re a_1} & \frac{\partial \mathscr{F}_1}{\partial \Im a_1} & \cdots & \frac{\partial \mathscr{F}_1}{\partial \Re b_q} & \frac{\partial \mathscr{F}_1}{\partial \Im b_q} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial \mathscr{F}_{4q}}{\partial \Re a_1} & \frac{\partial \mathscr{F}_{4q}}{\partial \Im a_1} & \cdots & \frac{\partial \mathscr{F}_{4q}}{\partial \Re b_q} & \frac{\partial \mathscr{F}_{4q}}{\partial \Im b_q} \end{pmatrix}\right|_{(\boldsymbol{x}^*,\boldsymbol{t}^*)} \neq 0 \,, \tag{81}$$

[10] Notice that the integrand $h(s)R^{1/2}(s)$ only depends on vectors in $X \times V$ due to (59), (61), and (62).

[11] For the real-analytic version of the implicit function theorem see, e.g. Theorem 2.3.5 of [12], and for the uniqueness of the map ϕ, see e.g. Theorem 9.2 of [17].

then the real-analytic implicit function theorem ensures that there exists a neighborhood Ω_1 of

$$\boldsymbol{t}^* \equiv \left(\Re t_1^*, \Im t_1^*, \cdots, \Re t_{2p-1}^*, \Im t_{2p-1}^*\right) \in \mathbb{R}^{4p-2},$$

a neighborhood Ω_2 of

$$\boldsymbol{x}^* \equiv \left(\Re a_1^*, \Im a_1^*, \ldots, \Re a_q^*, \Im a_q^*, \Re b_1^*, \Im b_1^*, \ldots, \Re b_q^*, \Im b_q^*\right) \in \mathbb{R}^{4q},$$

and a unique real-analytic mapping $\varphi : \Omega_1 \to \Omega_2$ such that $\varphi(\boldsymbol{t}^*) = \boldsymbol{x}^*$, and $\mathscr{F}(\varphi(\boldsymbol{t}), \boldsymbol{t}) = 0$ for all $\boldsymbol{t} \in \Omega_1$.

Due to the continuity of φ, and the fact that $\boldsymbol{t}^*$ is a regular q-cut point (so all $a_j(\boldsymbol{t}^*), b_j(\boldsymbol{t}^*)$ are distinct) we can find a possibly smaller neighborhood $\Omega_0 \subset \Omega_1$ so that for each $\boldsymbol{t} \in \Omega_0$ all end-points $a_j(\boldsymbol{t}), b_j(\boldsymbol{t})$, $j = 1, \cdots, q$, are distinct.

So it only remains to prove that the Jacobian is nonzero at a regular q-cut point. We assume the Jacobian is zero at such a point, and aim for a contradiction. Starting with this assumption, we know that there is $\boldsymbol{0} \neq \tilde{\boldsymbol{x}} \in X$ in the nullspace of the Jacobian matrix. Using $\boldsymbol{x}^*$ and $\tilde{\boldsymbol{x}}$ we define the following 1-parameter family

$$\boldsymbol{x}(\tau) := \boldsymbol{x}^* + \tau \tilde{\boldsymbol{x}} , \qquad \tau \in \mathbb{R} . \tag{82}$$

We obviously have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \boldsymbol{x}(\tau)\bigg|_{\tau=0} = \tilde{\boldsymbol{x}} \neq \boldsymbol{0} . \tag{83}$$

For non-zero values of τ, $\boldsymbol{x}(\tau)$ may not satisfy the end-point equations (69), but we can still think of the entries of $\boldsymbol{x}(\tau)$ as defining "end-points". More precisely, we define the points $a_j(\tau)$ and $b_j(\tau)$, as $\Re a_j(\tau) = \boldsymbol{x}_{2j-1}(\tau)$, $\Im a_j(\tau) = \boldsymbol{x}_{2j}(\tau)$, $\Re b_j(\tau) = \boldsymbol{x}_{2q+2j-1}(\tau)$, $\Im b_j(\tau) = \boldsymbol{x}_{2q+2j}(\tau)$, $j = 1, \cdots, q$. Now, using $a_j(\tau)$ and $b_j(\tau)$ as defined above, we define the τ-dependent objects $R(z;\tau)$, $g'(z;\tau)$ and $T_\ell(\tau)$ using (52), (61) and (62). Now (59) gives an expression for $h(z;\tau)\sqrt{R(z;\tau)}$. We emphasize that for non-zero τ, these objects may not correspond to an equilibrium measure for some potential $V(\tau)$.

Below, we drop the dependence on τ in the notations to simplify our presentation. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_\ell = \sum_{j=1}^{q} \left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau} \frac{\partial}{\partial a_j} + \frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau} \frac{\partial}{\partial b_j} \right) T_\ell , \tag{84}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Re\int_{a_j}^{b_j} h(s)R_+^{1/2}(s)\mathrm{d}s$$
$$= \Re\int_{a_j}^{b_j}\sum_{j=1}^{q}\left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial a_j}+\frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial b_j}\right)\ h(s)R_+^{1/2}(s)\mathrm{d}s\ , \tag{85}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Re\int_{b_j}^{a_{j+1}} h(s)R^{1/2}(s)\mathrm{d}s$$
$$= \Re\int_{b_j}^{a_{j+1}}\sum_{j=1}^{q}\left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial a_j}+\frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau}\frac{\partial}{\partial b_j}\right)\ h(s)R^{1/2}(s)\mathrm{d}s\ . \tag{86}$$

We let α represent an arbitrary branch point a_j or b_j. From (62) we have the identity

$$T_\ell - \alpha T_{\ell-1} = \int_J \frac{V'(s)}{R_+^{1/2}(s)} s^{\ell-1}(s-\alpha)\mathrm{d}s\ , \qquad \ell\in\mathbb{N}\ . \tag{87}$$

Differentiating with respect to α yields

$$\frac{\partial}{\partial\alpha}T_\ell - \alpha\frac{\partial}{\partial\alpha}T_{\ell-1} - T_{\ell-1} = \frac{-1}{2}\int_J \frac{V'(s)}{R_+^{1/2}(s)} s^{\ell-1}\mathrm{d}s = \frac{-1}{2}T_{\ell-1}\ , \tag{88}$$

which implies

$$\frac{\partial}{\partial\alpha}T_\ell - \alpha\frac{\partial}{\partial\alpha}T_{\ell-1} = \frac{1}{2}T_{\ell-1}\ . \tag{89}$$

In view of (63) for $1\le\ell\le q$, when $\tau=0$ we actually have

$$\frac{\partial}{\partial\alpha}T_\ell = \alpha\frac{\partial}{\partial\alpha}T_{\ell-1} = \alpha^\ell\frac{\partial}{\partial\alpha}T_0\ . \tag{90}$$

Lemma 4.1 *We have*

$$\frac{\partial}{\partial\alpha}g'(z) = \frac{-R^{1/2}(z)}{2\pi i(z-\alpha)}\frac{\partial}{\partial\alpha}T_0\ . \tag{91}$$

Proof Let us rewrite (61) as

$$
\begin{aligned}
g'(z) &= \frac{R^{1/2}(z)}{2\pi i}\int_J \frac{V'(s)(s-\alpha)}{R_+^{1/2}(s)}\frac{\mathrm{d}s}{(s-z)(s-\alpha)} \\
&= -\frac{R^{1/2}(z)}{2\pi i(z-\alpha)}\int_J \frac{V'(s)}{R_+^{1/2}(s)}\mathrm{d}s + \frac{R^{1/2}(z)}{2\pi i(z-\alpha)}\int_J \frac{V'(s)(s-\alpha)}{R_+^{1/2}(s)}\frac{\mathrm{d}s}{s-z} \\
&= \frac{R^{1/2}(z)}{2\pi i(z-\alpha)}\left(-T_0 + \int_J \frac{V'(s)(s-\alpha)}{R_+^{1/2}(s)}\frac{\mathrm{d}s}{s-z}\right) .
\end{aligned}
\tag{92}
$$

The advantage of this formula is that the differentiation with respect to α can be pushed through the integral, as the integrand vanishes at α. After taking the derivative with respect to α and straight-forward simplifications we obtain (91)

Returning to (84)–(86), when $\tau = 0$ we have

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}T_\ell = \sum_{j=1}^{q}\left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial a_j}a_j^\ell + \frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial b_j}b_j^\ell\right) , \qquad \ell = 0, \dots q , \tag{93}
$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\Re\int_{a_j}^{b_j} h(s)R_+^{1/2}(s)\mathrm{d}s \\
&\quad = \Re\frac{1}{\pi \mathrm{i}}\int_{a_j}^{b_j}\sum_{j=1}^{q}\left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial a_j}\frac{1}{s-a_j} + \frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial b_j}\frac{1}{s-b_j}\right)\left(R_+^{1/2}(s)\right)\mathrm{d}s,
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
&\frac{d}{d\tau}\Re\int_{b_j}^{a_{j+1}} h(s)R_+^{1/2}(s)\mathrm{d}s \\
&\quad = \Re\frac{1}{\pi \mathrm{i}}\int_{b_j}^{a_{j+1}}\sum_{j=1}^{q}\left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial a_j}\frac{1}{s-a_j} + \frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial b_j}\frac{1}{s-b_j}\right)\left(R^{1/2}(s)\right)\mathrm{d}s,
\end{aligned}
\tag{95}
$$

where in deriving the last two equations we have used (59) and (91). Consider the $2q$-vector

$$
\boldsymbol{W} := \left.\begin{pmatrix} \frac{\partial T_0}{\partial a_1}\frac{\mathrm{d}a_1}{\mathrm{d}\tau} \\ \vdots \\ \frac{\partial T_0}{\partial a_q}\frac{\mathrm{d}a_q}{\mathrm{d}\tau} \\ \frac{\partial T_0}{\partial b_1}\frac{\mathrm{d}b_1}{\mathrm{d}\tau} \\ \vdots \\ \frac{\partial T_0}{\partial b_q}\frac{\mathrm{d}b_q}{\mathrm{d}\tau} \end{pmatrix}\right|_{\tau=0} \tag{96}
$$

that, in view of (93), satisfies the $q+1$ equations

$$
\begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_1 & \cdots & a_q & b_1 & \cdots & b_q \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1^q & \cdots & a_q^q & b_1^q & \cdots & b_q^q \end{pmatrix} \boldsymbol{W} = \boldsymbol{0}\,, \tag{97}
$$

where $a_j,\ b_j,\ j=1,\cdots,q$ are all evaluated at $\tau=0$, in other words they are the actual endpoints corresponding to the solution $(\boldsymbol{x}^*, \boldsymbol{v}^*)$. Furthermore, the integrand in (94) and (95) can be described as follows. We first define

$$
\boldsymbol{\rho}(s) := \begin{pmatrix} \frac{1}{s-a_1} \\ \vdots \\ \frac{1}{s-a_q} \\ \frac{1}{s-b_1} \\ \vdots \\ \frac{1}{s-b_q} \end{pmatrix}\,, \tag{98}
$$

and then

$$
B(s) := \frac{1}{\pi \mathrm{i}}\left(\boldsymbol{W}^T\boldsymbol{\rho}(s)\right)\ R^{1/2}(s)\,, \tag{99}
$$

Where $(\cdot)^T$ denotes the transpose and all objects are evaluated at $\tau=0$. In other words,

$$
B(s) = \frac{1}{\pi \mathrm{i}}\sum_{j=1}^{q}\left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial a_j}\frac{1}{s-a_j} + \frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau}\frac{\partial T_0}{\partial b_j}\frac{1}{s-b_j}\right)\ R^{1/2}(s)\,. \tag{100}
$$

This function, in view of (94) and (95) satisfies

$$\Re \int_{a_j}^{b_j} B_+(s)\mathrm{d}s = 0\,, \qquad j = 1, \ldots, q-1\,, \tag{101}$$

$$\Re \int_{b_j}^{a_{j+1}} B(s)\mathrm{d}s = 0\,, \qquad j = 1, \ldots, q-1\,. \tag{102}$$

Now, using (100) and expanding $(s-\alpha)^{-1}$ for large s and switching the order of summations we obtain

$$\frac{B(s)}{R^{1/2}(s)} = \frac{1}{\pi \mathrm{i} s} \sum_{\ell=0}^{\infty} \frac{1}{s^\ell} \sum_{j=1}^{q} \left(\frac{\mathrm{d}a_j(\tau)}{\mathrm{d}\tau} \frac{\partial T_0}{\partial a_j} a_j^\ell + \frac{\mathrm{d}b_j(\tau)}{\mathrm{d}\tau} \frac{\partial T_0}{\partial b_j} b_j^\ell \right). \tag{103}$$

So, because of (97), and recalling (52) we observe that the behavior of $B(s)$ for s large is given by

$$B(s) = O\left(\frac{1}{s^2}\right)\,. \tag{104}$$

Therefore B can be expressed as

$$B(s) = \frac{Q(s)}{R^{1/2}(s)}\,, \tag{105}$$

where Q is a polynomial of degree at most $q-2$.

Next, we show that B is identically zero. To prove this, we show that the following integral is 0.

$$\iint_{\mathbb{C}} B(z)\overline{B(z)}\mathrm{d}A\,. \tag{106}$$

Lemma 4.2 *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region bounded by C. Let $f(x+iy) \equiv u(x,y) + iv(x,y)$ be analytic in D. We have*

$$\iint_D \frac{\partial}{\partial z} f(z)\, dA = \frac{i}{2} \oint_C f(z)\overline{dz}\,, \tag{107}$$

where Integration with respect to $\overline{dz}$ means: parametrize the contour of integration via $z = z(t)$, then

$$\int f(z)\,\overline{dz} = \int_{t_0}^{t_1} f(z(t))\overline{z'(t)}\,dt\,.$$

Proof We have

$$\iint_D \frac{\partial}{\partial \bar{z}} f(z)\,\mathrm{d}A = \iint_D \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)(u(x,y) + \mathrm{i}v(x,y))\,\mathrm{d}A$$
$$= \frac{1}{2}\iint_D \big((u_x + v_y) + \mathrm{i}(v_x - u_y)\big)\,\mathrm{d}A\,. \tag{108}$$

Now we apply the Green's theorem for the vector fields $\boldsymbol{F}_1(x,y) = (-v(x,y), u(x,y))$, and $\boldsymbol{F}_2(x,y) = (u(x,y), v(x,y))$. So this integral equals

$$\frac{1}{2}\oint_C (-v(x,y), u(x,y))\cdot(x'(t), y'(t))\,\mathrm{d}t + \frac{\mathrm{i}}{2}\oint_C (u(x,y), v(x,y))\cdot(x'(t), y'(t))\,\mathrm{d}t,$$

where $(x(t), y(t))$ is the parametrization of the curve C. We therefore have

$$\iint_D \frac{\partial}{\partial \bar{z}} f(z)\,\mathrm{d}A = \frac{1}{2}\oint_C \Big(x'(t)[-v(x,y) + \mathrm{i}u(x,y)] + y'(t)[u(x,y) + \mathrm{i}v(x,y)]\Big)\,\mathrm{d}t,$$
$$= \frac{\mathrm{i}}{2}\oint_C \Big(u(x,y) + \mathrm{i}v(x,y)\Big)\big(x'(t) - \mathrm{i}y'(t)\big)\,\mathrm{d}t,$$
$$= \frac{\mathrm{i}}{2}\oint_C f(z)\overline{\mathrm{d}z}. \tag{109}$$

Defining

$$u(z) = \int_{+\infty}^{z} B(s)\mathrm{d}s\,, \tag{110}$$

we have

$$B(z)\overline{B(z)} = \frac{\partial}{\partial z}\Big(u(z)\overline{B(z)}\Big)\,, \tag{111}$$

since $\frac{\partial}{\partial z}\overline{B(z)} = 0$. Now, in order to apply Stokes' theorem to the integral (106), we have to apply it in two regions, one *above* the contour of integration Γ, and one *below* the contour of integration Γ.

Let D_r be a disk of radius r centered at the origin. The max-min contour $\Gamma \in \mathcal{T}$ (see Sect. 2.2) divides D_r into two parts: $D_r^{(+)}$ *above* Γ, and $D_r^{(-)}$ *below* Γ. We can write

$$\begin{aligned}\iint_{\mathbb{C}} B(z)\overline{B(z)}\mathrm{d}A &= \iint_{\mathbb{C}} \frac{\partial}{\partial z}\left(u(z)\overline{B(z)}\right)\mathrm{d}A = \lim_{r\to\infty}\iint_{D_r}\frac{\partial}{\partial z}\left(u(z)\overline{B(z)}\right)\mathrm{d}A \\ &= \lim_{r\to\infty}\iint_{D_r^{(+)}}\frac{\partial}{\partial z}\left(u(z)\overline{B(z)}\right)\mathrm{d}A + \lim_{r\to\infty}\iint_{D_r^{(-)}}\frac{\partial}{\partial z}\left(u(z)\overline{B(z)}\right)\mathrm{d}A \\ &= \frac{\mathrm{i}}{2}\lim_{r\to\infty}\oint_{\partial D_r^{(+)}} u(z)\overline{B(z)}\,\overline{\mathrm{d}z} + \frac{\mathrm{i}}{2}\lim_{r\to\infty}\oint_{\partial D_r^{(-)}} u(z)\overline{B(z)}\,\overline{\mathrm{d}z}\,,\end{aligned} \tag{112}$$

where both $\partial D_r^{(+)}$ and $\partial D_r^{(-)}$ are positively oriented. Therefore, due to (104), we find

$$-2i\iint_{\mathbb{C}} B(z)\overline{B(z)}\,\mathrm{d}A = \int_{\Gamma}\left\{\left[u(z)\overline{B(z)}\right]_+ - \left[u(z)\overline{B(z)}\right]_-\right\}\overline{\mathrm{d}z}\,. \tag{113}$$

The contour Γ is comprised of bands $\Gamma[a_j, b_j]$, gaps $\Gamma(b_j, a_{j+1})$, and the two semi-infinite contours ($\Gamma(-\infty, a_1)$ from $-\infty$ to a_1 and $\Gamma(b_q, \infty)$ from b_q to $+\infty$). First observe that

$$\int_{b_q}^{+\infty}\left\{\left[u(z)\overline{B(z)}\right]_+ - \left[u(z)\overline{B(z)}\right]_-\right\}\overline{\mathrm{d}z} = 0\,, \tag{114}$$

since by definition u and B are continuous across the contour from b_q to $+\infty$.

Second, note that for z in the contour from $-\infty$ to a_1, $B(z)$ is continuous across Γ, and so is $u(z)$, since

$$\begin{aligned}u_+(z) - u_-(z) &= \sum_{j=1}^{q}\int_{a_j}^{b_j}\left(B_+(s) - B_-(s)\right)\mathrm{d}s \\ &= \sum_{j=1}^{q}\oint_{\gamma_j} B(s)\mathrm{d}s = \oint_{\gamma^*} B(s)\mathrm{d}s = 0\,,\end{aligned} \tag{115}$$

due to (104), where γ_j is a clockwise contour encircling the cut $\Gamma[a_j, b_j]$, and γ^* is a clockwise contour encircling all cuts. Therefore we also know that

$$\int_{-\infty}^{a_1}\left\{\left[u(z)\overline{B(z)}\right]_+ - \left[u(z)\overline{B(z)}\right]_-\right\}\overline{\mathrm{d}z} = 0\,. \tag{116}$$

So we must consider

$$\int_{a_1}^{b_q} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} \, . \tag{117}$$

There are two types of integrals: those across bands and those across gaps.

For z in a band, say $\Gamma(a_j, b_j)$, the quantity $B(z)$ has a jump discontinuity across the contour: $B_+(z) = -B_-(z)$. So we have

$$\int_{a_j}^{b_j} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \int_{a_j}^{b_j} \overline{B_+(z)} \left(u_+(z) + u_-(z) \right) \overline{\mathrm{d}z} \, , \tag{118}$$

and $u_+ + u_-$ is the following constant:

$$u_+(z) + u_-(z) = -2 \sum_{k=j}^{q-1} \int_{b_k}^{a_{k+1}} B(s)\mathrm{d}s - 2\int_{b_q}^{+\infty} B(s)\mathrm{d}s, \qquad z \in \Gamma(a_j, b_j) \, , \tag{119}$$

and so we find

$$\begin{aligned} &\int_{a_j}^{b_j} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \\ &\qquad \left(-2 \sum_{k=j}^{q-1} \int_{b_k}^{a_{k+1}} B(s)\mathrm{d}s - 2\int_{b_q}^{+\infty} B(s)\mathrm{d}s \right) \int_{a_j}^{b_j} \overline{B_+(z)}\, \overline{\mathrm{d}z} \, . \end{aligned} \tag{120}$$

Note that

$$\int_{a_j}^{b_j} \overline{B_+(z)}\, \overline{\mathrm{d}z} = \overline{\int_{a_j}^{b_j} B_+(z)\mathrm{d}z} = -\int_{a_j}^{b_j} B_+(z)\mathrm{d}z \, , \tag{121}$$

where the last equality follows from (101). Therefore

$$\begin{aligned} &\int_{a_j}^{b_j} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \\ &\qquad \left(2 \sum_{k=j}^{q-1} \int_{b_k}^{a_{k+1}} B(s)\mathrm{d}s + 2\int_{b_q}^{+\infty} B(s)\mathrm{d}s \right) \int_{a_j}^{b_j} B_+(z)\mathrm{d}z \, . \end{aligned} \tag{122}$$

For z in a gap, say $\Gamma(b_j, a_{j+1})$, the quantity $B(z)$ is continuous across the contour Γ, therefore

$$\int_{b_j}^{a_{j+1}} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \int_{b_j}^{a_{j+1}} \overline{B(z)} \left(u_+(z) - u_-(z) \right) \overline{\mathrm{d}z} \, , \tag{123}$$

and $u_+(z) - u_-(z)$ is the constant:

$$u_+(z) - u_-(z) = -2 \sum_{k=j+1}^{q} \int_{a_k}^{b_k} B_+(s)\mathrm{d}s, \qquad \text{for all } z \in \Gamma(b_j, a_{j+1})\,. \tag{124}$$

We have

$$\begin{aligned}\int_{b_j}^{a_{j+1}} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \\ \left(-2 \sum_{k=j+1}^{q} \int_{a_k}^{b_k} B_+(s)\mathrm{d}s \right) \int_{b_j}^{a_{j+1}} \overline{B(z)}\, \overline{\mathrm{d}z}\,.\end{aligned} \tag{125}$$

Note that from (102) we have

$$\int_{b_j}^{a_{j+1}} \overline{B(z)}\, \overline{\mathrm{d}z} = -\int_{b_j}^{a_{j+1}} B(z)\mathrm{d}z\,, \tag{126}$$

therefore

$$\begin{aligned}\int_{b_j}^{a_{j+1}} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \\ \left(2 \sum_{k=j+1}^{q} \int_{a_k}^{b_k} B_+(s)\mathrm{d}s \right) \int_{b_j}^{a_{j+1}} B(z)\mathrm{d}z\,.\end{aligned} \tag{127}$$

Using (122) and (127), we have

$$\begin{aligned}&\int_{a_1}^{b_q} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \\ &\sum_{j=1}^{q} \left(2 \sum_{k=j}^{q} \int_{b_k}^{a_{k+1}} B(s)\mathrm{d}s \right) \int_{a_j}^{b_j} B_+(z)\mathrm{d}z + \sum_{j=1}^{q-1} \left(2 \sum_{k=j+1}^{q} \int_{a_k}^{b_k} B_+(s)\mathrm{d}s \right) \int_{b_j}^{a_{j+1}} B(z)\mathrm{d}z\,.\end{aligned} \tag{128}$$

Note that in (128), $a_{q+1} = +\infty$. Reversing orders of summation in the first term on the r.h.s. of (128), we have

$$\begin{aligned}&\int_{a_1}^{b_q} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} = \\ &2 \sum_{k=1}^{q} \int_{b_k}^{a_{k+1}} B(s)\mathrm{d}s \sum_{j=1}^{k} \int_{a_j}^{b_j} B_+(z)\mathrm{d}z + \sum_{j=1}^{q-1} \left(2 \sum_{k=j+1}^{q} \int_{a_k}^{b_k} B_+(s)\mathrm{d}s \right) \int_{b_j}^{a_{j+1}} B(z)\mathrm{d}z\,.\end{aligned} \tag{129}$$

Exchanging indices of summation in the first term on the r.h.s. of (129), we find

$$\int_{a_1}^{b_q} \left\{ \left[u(z)\overline{B(z)} \right]_+ - \left[u(z)\overline{B(z)} \right]_- \right\} \overline{\mathrm{d}z} =$$

$$2\sum_{j=1}^{q} \int_{b_j}^{a_{j+1}} B(s)\mathrm{d}s \sum_{k=1}^{j} \int_{a_k}^{b_k} B(z)\mathrm{d}z + \sum_{j=1}^{q-1} \left(2 \sum_{k=j+1}^{q} \int_{a_k}^{b_k} B(s)\mathrm{d}s \right) \int_{b_j}^{a_{j+1}} B(z)\mathrm{d}z =$$

$$2\sum_{j=1}^{q-1} \left[\int_{b_j}^{a_{j+1}} B(s)\mathrm{d}s \left(\sum_{k=1}^{q} \int_{a_k}^{b_k} B(z)\mathrm{d}z \right) \right] + \int_{b_q}^{+\infty} B(s)\mathrm{d}s \left(\sum_{k=1}^{q} \int_{a_k}^{b_k} B(z)\mathrm{d}z \right) = 0\,. \tag{130}$$

So we have proven that

$$\iint_{\mathbb{C}} B(z)\overline{B(z)} dA \;=\; 0\,. \tag{131}$$

This of course implies that $B \equiv 0$, and hence by (99) we conclude that

$$\boldsymbol{W} \equiv 0\,. \tag{132}$$

Lemma 4.3 *Let $\alpha \in \{a_j, b_j\ ,\ j = 1, \cdots q\}$, and T_0 given by* (62)*. It holds that*

$$\frac{\partial T_0}{\partial \alpha} \neq 0\,. \tag{133}$$

Proof We have

$$-\,2\pi i \psi(z) = -h(z) R_+^{1/2}(z) = g'(z) - \frac{V'(z)}{2}\,, \tag{134}$$

and our assumptions imply that this quantity vanishes like a square root at each branchpoint α. So we know that

$$\lim_{z\to\alpha} \frac{1}{R^{1/2}(z)} \left(g'(z) - \frac{V'(z)}{2} \right) \neq 0\,. \tag{135}$$

Using (61) we can write

$$\frac{1}{R^{1/2}(z)} \left(g'(z) - \frac{V'(z)}{2} \right) = \frac{1}{4\pi \mathrm{i}} \oint_{\gamma^*} \frac{V'(s)}{R^{1/2}(s)} \frac{\mathrm{d}s}{s-z}\,, \tag{136}$$

where γ^* is a negatively oriented contour which encircles both the support set J and the point z. Taking the limit as $z \to \alpha$, and recalling (135) we find

$$\frac{1}{4\pi \mathrm{i}} \oint_{\gamma^*} \frac{V'(s)}{R^{1/2}(s)} \frac{\mathrm{d}s}{s-\alpha} \neq 0\,. \tag{137}$$

Recalling the definition (62) we can write

$$T_0 = \int_J \frac{V'(s)}{R_+^{1/2}(s)} \mathrm{d}s = \frac{1}{2} \oint_{\gamma^*} \frac{V'(s)}{R^{1/2}(s)} \mathrm{d}s\,. \tag{138}$$

Differentiating with respect to the branchpoint α, we find

$$\frac{\partial T_0}{\partial \alpha} = \frac{1}{4} \oint_{\gamma^*} \frac{V'(s)}{R^{1/2}(s)} \frac{\mathrm{d}s}{s-\alpha}\,, \tag{139}$$

which is nonzero because of (137).

Recalling (96), the above lemma together with (132) imply that

$$\left.\frac{\mathrm{d}a_j}{\mathrm{d}\tau}\right|_{\tau=0} = \left.\frac{\mathrm{d}b_j}{\mathrm{d}\tau}\right|_{\tau=0} = 0\,, \qquad \text{for all } \; j = 1, 2, \ldots, q. \tag{140}$$

We have thus shown that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau} \boldsymbol{x}(\tau)\right|_{\tau=0} = \mathbf{0}\,, \tag{141}$$

which contradicts (83). This proves that the Jacobian (81) is indeed non-zero. We have thus concluded the proof of Theorem 1.2.

5 Openness of the Regular q-Cut Regime

5.1 Proof of Theorem 1.3

The end-points $a_j(\boldsymbol{t}), b_j(\boldsymbol{t}) \in \mathscr{J}_{\boldsymbol{t}}^{(q)}$ deform continuously with respect to $\boldsymbol{t}$ as shown in Theorem 1.2, due to nonsingularity of the Jacobian matrix at a regular q-cut point. Notice that the critical graph of the quadratic differential $Q(z; \boldsymbol{t})\mathrm{d}z^2$ is intrinsic to the polynomial Q and does not depend on the particular branch chosen for its natural parameter $\eta_q(z; \boldsymbol{t})$, for example the one chosen in (57). For the purposes of this proof, for each fixed $\boldsymbol{t}$, unlike our choice in (57), we choose the branch $\tilde{\eta}_q(z; \boldsymbol{t})$ whose branch cut has no intersections with the critical graph $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ and we can characterize the critical graph of $Q(z;\boldsymbol{t})\mathrm{d}z^2$ as the totality of solutions to

$$\Re\tilde{\eta}_q(z;\boldsymbol{t}) = 0. \tag{142}$$

Recalling (57) with the choice of branch discussed above, notice that

$$\frac{\partial\tilde{\eta}_q}{\partial z}(z^*;\boldsymbol{t}^*) = -\prod_{\ell=1}^{r}\left(z^* - z_\ell(\boldsymbol{t}^*)\right)\left[\prod_{j=1}^{q}\left(z^* - a_j(\boldsymbol{t}^*)\right)\left(z^* - b_j(\boldsymbol{t}^*)\right)\right]^{1/2} \neq 0, \tag{143}$$

where $z^* \in \mathscr{J}_{\boldsymbol{t}^*}^{(q)} \setminus \left\{a_j(\boldsymbol{t}^*), b_j(\boldsymbol{t}^*)\right\}_{j=1}^{q}$ does not lie on the branch cut chosen to define $\tilde{\eta}_q$. Since z^* is not on the branch cut, there is a small neighborhood of z^* in which $\tilde{\eta}_q(z,\boldsymbol{t}^*)$ is analytic. By Cauchy-Riemann equations, from (143) we conclude that at least one of the quantities $\frac{\partial\Re\tilde{\eta}_q}{\partial x}(z^*;\boldsymbol{t}^*)$ or $\frac{\partial\Re\tilde{\eta}_q}{\partial y}(z^*;\boldsymbol{t}^*)$ is non zero, $z = x + \mathrm{i}y$. Without loss of generality, let us assume that

$$\frac{\partial\Re\tilde{\eta}_q}{\partial x}(z^*;\boldsymbol{t}^*) \neq 0. \tag{144}$$

Now, think of the left hand side of (142) as a map

$$\Re\tilde{\eta}_q\,(x,\boldsymbol{w}) : \mathbb{R} \times W \to \mathbb{R}, \tag{145}$$

where $z = x + \mathrm{i}y$, and an element $\boldsymbol{w} \in W \simeq \mathbb{R}^{4p-1}$ represents the variable y and the real and imaginary parts of the parameters in the external field:

$$\boldsymbol{w} = \left(y, \Re t_1, \Im, t_1, \cdots, \Re t_{2p-1}, \Im t_{2p-1}\right)^T.$$

Now, by the real-analytic Implicit Function Theorem [12], we know that there exists a neighborhood U of

$$\boldsymbol{w}^* \equiv \left(y^*, \Re t_1^*, \Im, t_1^*, \cdots, \Re t_{2p-1}^*, \Im t_{2p-1}^*\right)$$

and a real-analytic map $\Psi : U \to \mathbb{R}$, with

$$\Psi\left(\boldsymbol{w}^*\right) = x^*$$

and

$$\Re\tilde{\eta}_q(\Psi(\boldsymbol{w}),\boldsymbol{w}) = 0.$$

That is to say that for any y in a small enough neighborhood of y^* and for any $\boldsymbol{t} \equiv \left(\Re t_1, \Im, t_1, \cdots, \Re t_{2p-1}, \Im t_{2p-1}\right)^T$ in a small enough neighborhood of $\boldsymbol{t}^*$, there is an $x \equiv x(y, \boldsymbol{t})$ such that $z = x + iy$ lies on the critical graph $\mathscr{J}_{\boldsymbol{t}}^{(q)}$. The real-analyticity of Ψ, in particular, ensures that $\mathscr{J}_{\boldsymbol{t}}^{(q)}$ deforms continuously with respect to $\boldsymbol{t}$. This finishes the proof of Theorem 1.3.

5.2 *Proof of Theorem 1.1*

Let us start with the following two lemmas.

Lemma 5.1 *The points $z_j(\boldsymbol{t})$, $j = 1, \cdots, r$, depend continuously on $\boldsymbol{t}$.*

Proof The right hand side of (80) clearly depends continuously on $\boldsymbol{t}$ (since the $\boldsymbol{t}$-dependence in R is through the end-points which do depend continuously on $\boldsymbol{t}$). So the zeros of $h(z; \boldsymbol{t})$, being $z_\ell(\boldsymbol{t})$, $\ell = 1, \cdots, r$, depend continuously on $\boldsymbol{t}$. □

Lemma 5.2 *There are no singular finite geodesic polygons with one or two vertices associated with the quadratic differential $Q(z)dz^2$ given by (53).*

Proof The proof follows immediately from the Teichmüller's lemma and the fact that Q is a polynomial. □

Now we prove Theorem 1.1. Let $\boldsymbol{t}^*$ be a regular q-cut point. We show that there exists a small enough neighborhood of $\boldsymbol{t}^*$ in which all the requirements of Definition 3.1 hold simultaneously. We prove this in the following two mutually exclusive cases:

(a) when none of the points $z_\ell(\boldsymbol{t}^*)$ lie on $\mathscr{J}_{\boldsymbol{t}^*}^{(q)} \setminus J_{\boldsymbol{t}^*}^{(q)}$, and
(b) when one or more of the points $z_\ell(\boldsymbol{t}^*)$ lie on $\mathscr{J}_{\boldsymbol{t}^*}^{(q)} \setminus J_{\boldsymbol{t}^*}^{(q)}$.

Let us first consider the case (a) above. So we are at a regular q-cut point $\boldsymbol{t}^*$ where we know that

$$A_\ell(\boldsymbol{t}^*) \neq 0, \qquad \ell = 1, \cdots, r, \tag{146}$$

where

$$A_\ell(\boldsymbol{t}) := \Re \eta_q\left(z_\ell(\boldsymbol{t}); \boldsymbol{t}\right). \tag{147}$$

For $\varepsilon > 0$, let $D_\varepsilon(\boldsymbol{t}^*)$ denote the open set of all points $\boldsymbol{t}$ such that

$$|\Re t_k - \Re t_k^*| < \varepsilon, \quad \text{and} \quad |\Im t_k - \Im t_k^*| < \varepsilon, \qquad \text{for} \qquad k = 1, \cdots, 2p-1.$$

Since the functions $A_\ell(\boldsymbol{t})$ are continuous functions of $\boldsymbol{t}$, for each $\ell = 1, \cdots, r$ there exists $\varepsilon_\ell > 0$ such that for all $\boldsymbol{t} \in D_{\varepsilon_\ell}(\boldsymbol{t}^*)$ the inequalities $A_\ell(\boldsymbol{t}) \neq 0$ hold for each $\ell = 1, \cdots, r$. Let $\varepsilon := \min_{1\le\ell\le r} \varepsilon_\ell$. The claim is that for all $\boldsymbol{t} \in \Omega =: \Omega_0 \cap D_\varepsilon(\boldsymbol{t}^*)$ (see the proof of Theorem 1.2 to recall the open set Ω_0) all requirements of Definition 3.1 hold. It is obvious that the second requirement of Definition 3.1 holds by the choice of ε. Suppose that condition (3) of Definition 3.1 does not hold for some $\tilde{\boldsymbol{t}} \in \Omega$. Let $K(\boldsymbol{t}^*)$ denote the infinite geodesic polygon which hosts the complementary contour $\Gamma_{\boldsymbol{t}}(b_q(\boldsymbol{t}), +\infty)$ as required by condition (3) of Definition 3.1. Due to Theorem 1.3 this is only possible if

(a-i) one or more points on the boundaries $\ell_2^{(b_q)}$ and $\ell_3^{(b_q)}$ of the infinite geodesic polygon $K(\boldsymbol{t}^*)$ continuously deform (as $\boldsymbol{t}^*$ deforms to $\tilde{\boldsymbol{t}}$) to coalesce together and block the access of a complementary contour from b_q to $+\infty$, or

(a-ii) if there are one or more humps in K (see Remark 3.3), one or more points on the boundaries $\ell_2^{(b_q)}$ or $\ell_3^{(b_q)}$ of the infinite geodesic polygon $K(\boldsymbol{t}^*)$ continuously deform (as $\boldsymbol{t}^*$ deforms to $\tilde{\boldsymbol{t}}$) to coalesce with the hump(s) and block the access of a complementary contour from b_q to $+\infty$. This case necessitates $p > q$ which ensures the existence of humps as parts of the critical graph.

Notice that if there are *no humps* in K, in particular when $p = q$ or $p < q$ which means there are no humps at all, then the only possibility to block the access from b_q to $+\infty$ is what mentioned above in case (a-i). We observe that the case (a-i) above is actually impossible by Lemma 5.2 as it necessitates a geodesic polygon with two vertices.

So we just investigate the case (a-ii). Consider a point of coalescence $\tilde{z}$. Notice that $\tilde{z}$ can not be b_q itself because for all $\boldsymbol{t} \in \Omega_0$ there are only three emanating critical trajectories from b_q. At such a point we would have four emanating local trajectories from $\tilde{z}$ (or a higher even number of emanating local trajectories from $\tilde{z}$ if more than just two points come together at $\tilde{z}$) which is an indication that $\tilde{z}$ is a critical point of the quadratic differential. This is a contradiction, since $z_\ell(\tilde{\boldsymbol{t}})$, $\ell = 1, \cdots, r$, do not lie on the critical trajectories by the choice of ε and hence $\tilde{z} \neq z_\ell(\tilde{\boldsymbol{t}})$. Moreover the quadratic differential $Q(z)\mathrm{d}z^2$ given by (53) does not have any critical points other than $a_j(\tilde{\boldsymbol{t}}), b_j(\tilde{\boldsymbol{t}})$ and $z_\ell(\tilde{\boldsymbol{t}})$, $j = 1, \cdots, q$, $\ell = 1, \cdots, r$. This finishes the proof that condition (3) of Definition 3.1 holds for all $\boldsymbol{t} \in \Omega$. Similar arguments show that the conditions (4) and (5) of Definition 3.1 must also hold for all $\boldsymbol{t} \in \Omega$.

Now it only remains to consider the first requirement of Definition 3.1. Assume, for the sake of arriving at a contradiction that for some $\tilde{\boldsymbol{t}} \in \Omega$ there is at least one index $j_1 = 1, \cdots, q$ for which the first requirement fails. Notice that there could not be more than one connection by Lemma 5.2. So the only possibility to consider is that there is *no connection* between $a_{j_1}(\tilde{\boldsymbol{t}})$ and $b_{j_1}(\tilde{\boldsymbol{t}})$. So the three local trajectories emanating from $a_{j_1}(\tilde{\boldsymbol{t}})$ and $b_{j_1}(\tilde{\boldsymbol{t}})$ must end up at ∞ and can not encounter $z_\ell(\tilde{\boldsymbol{t}})$, $\ell = 0, \cdots, r$ by the choice of ε. However this is impossible since there are at least $4(q-1)+6$ rays emanating from the end points and approaching infinity. There are already $4(p-q)$ rays ending up at ∞ from the $2(p-q)$ existing humps. This in total gives at least $4(q-1)+6+4(p-q) = 4p+2$ directions at ∞. Recall that we can have only $4p$ solutions at ∞. This means we have at least 2 more solutions at ∞ than what is allowed. This means that at least 2 rays emanating from the endpoints must connect to one or more humps. But this means we have at least two extra critical points other than $a_j(\tilde{\boldsymbol{t}})$, $b_j(\tilde{\boldsymbol{t}})$ and $z_\ell(\tilde{\boldsymbol{t}})$, $j = 1, \cdots, q$, $\ell = 1, \cdots, r$, which is a contradiction. This finishes the proof that the first requirement of Definition 3.1 holds for all $\boldsymbol{t} \in \Omega$. Therefore, for case (a) we have shown that all the requirements of Definition 3.1 hold simultaneously.

Notice that the proof of case (b) above (when $\boldsymbol{t}^*$ is a regular $q-$cut point and one or more of the points $z_\ell(\boldsymbol{t}^*)$ lie on $\mathscr{J}_{\boldsymbol{t}^*}^{(q)} \setminus J_{\boldsymbol{t}^*}^{(q)}$) is very similar. To that end, let $1 \le m \le r-1$ be such that for all indices

$$\{\ell_1, \cdots, \ell_m\} \subset \{1, \cdots, r\}$$

the points $z_{\ell_k}(\boldsymbol{t}^*)$, $1 \le k \le m$, *do not* lie on $\mathscr{J}_{\boldsymbol{t}^*}^{(q)} \setminus J_{\boldsymbol{t}^*}^{(q)}$. For these indices we define ε_{ℓ_k} as above using the functions A_{ℓ_k} in (147).

Now let us consider the rest of the indices

$$\{\ell_{m+1}, \cdots, \ell_r\} \subset \{1, \cdots, r\}$$

for which the points $z_{\ell_j}(\boldsymbol{t}^*)$, $m+1 \le j \le r$, *do* lie on $\mathscr{J}_{\boldsymbol{t}^*}^{(q)} \setminus J_{\boldsymbol{t}^*}^{(q)}$. We claim that there is an $\varepsilon_{\ell_j} > 0$, for each $m+1 \le j \le r$, such that for all $\boldsymbol{t} \in D_{\varepsilon_{\ell_j}}(\boldsymbol{t}^*)$, the point $z_{\ell_j}(\boldsymbol{t})$ does not lie on $J_{\boldsymbol{t}}^{(q)}$. Indeed, since for each $\boldsymbol{t}$ the set $J_{\boldsymbol{t}}^{(q)}$ is compact, the distance function

$$\mathrm{d}_j(\boldsymbol{t}) := \mathrm{dist}\left(z_{\ell_j}(\boldsymbol{t}), J_{\boldsymbol{t}}^{(q)}\right) \equiv \min_{z \in J_{\boldsymbol{t}}^{(q)}} \{|z_{\ell_j}(\boldsymbol{t}) - z|\}$$

is well-defined and is a continuous function of $\boldsymbol{t}$ due to Theorem 1.3 and Lemma 5.1. For the $\boldsymbol{t}^*$ under consideration we know that $\mathrm{d}_j(\boldsymbol{t}^*) > 0$, and by the continuity of d_j, there is an ε_{ℓ_j} such that for all $\boldsymbol{t}$ in an ε_{ℓ_j}-neighborhood of $\boldsymbol{t}^*$ we have $\mathrm{d}_j(\boldsymbol{t}) > 0$.

Again let $\varepsilon := \min\limits_{1 \le \ell \le r} \varepsilon_\ell$. The claim then is that for all $\boldsymbol{t} \in \Omega$ all requirements of Definition 3.1 hold. It is obvious that the second requirement of Definition 3.1 holds by the choice of ε, and if any other requirement of Definition 3.1 does not hold for some $\boldsymbol{t} \in \Omega$, analogous reasoning as provided in case (a) above shows that one gets a contradiction.

6 Conclusion

In this article we have provided a simple and yet self-contained proof of the openness of the regular q-cut regime when the external field is a complex polynomial of even degree. We have also proven that the solvability of the q-cut end-point equations persists in a small enough neighborhood of a regular q-cut point in the parameter space. In addition, we have shown that the real and imaginary parts of the endpoints are real analytic with respect to the real and imaginary parts of the parameters of the external field, and that the critical graph of the underlying quadratic differential depends continuously on $\boldsymbol{t}$.

As discussed in the introduction, we could have considered other classes of admissible contours different from the one associated with the real axis, for the even degree polynomial (1). Yet, multiple other cases would have arised if one started with an *odd-degree* polynomial external field, then considered its classes of admissible sectors and contours, and finally solved the max-min variational problem for the collection of contours from that class.[12] However, to that end, even though we have made the simplifying assumption on fixing the degree of external field to be even, and our fixed choice of admissible contours, we would like to emphasize that our arguments presented in this paper still work in the other cases as long as one considers a single curve going to infinity inside any two admissible sectors.

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[12] Notice that in the case where the degree of the polynomial external field is odd, the full real axis can not lie in any admissible sector, as the condition (10) is not satisfied as z approaches to ∞ along the negative real axis. See e.g. Figure 1 of [5] for the cubic external field, and Fig. 2 of [14] for a quintic one.

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Focusing Nonlocal Nonlinear Schrödinger Equation with Asymmetric Boundary Conditions: Large-Time Behavior

Anne Boutet de Monvel, Yan Rybalko, and Dmitry Shepelsky

To the memory of Harold Widom

Abstract We consider the focusing integrable nonlocal nonlinear Schrödinger equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0$$

with asymmetric nonzero boundary conditions: $q(x,t) \to \pm A\mathrm{e}^{-2\mathrm{i}A^2t}$ as $x \to \pm\infty$, where $A > 0$ is an arbitrary constant. The goal of this work is to study the asymptotics of the solution of the initial value problem for this equation as $t \to +\infty$. For a class of initial values we show that there exist three qualitatively different asymptotic zones in the (x,t) plane. Namely, there are regions where the parameters are modulated (being dependent on the ratio x/t) and a central region, where the parameters are unmodulated. This asymptotic picture is reminiscent of that for the defocusing classical nonlinear Schrödinger equation, but with some important differences. In particular, the absolute value of the solution in all three regions depends on details of the initial data.

Keywords Nonlocal nonlinear Schrödinger equation · Riemann–Hilbert problem · Large-time asymptotics

Mathematics Subject Classification 35Q53, 37K15, 35Q15, 35B40, 35Q51, 37K40

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1 Introduction

In the present paper we consider the initial value problem for the focusing nonlocal nonlinear Schrödinger (NNLS) equation (we denote the complex conjugate of q by $\bar{q}$)

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \tag{1b}$$

with asymmetric nonzero boundary conditions:

$$q(x,t) \to \pm A\mathrm{e}^{-2\mathrm{i}A^2 t}, \qquad x \to \pm\infty, \quad t \in \mathbb{R}, \tag{1c}$$

for some $A > 0$.

The NNLS Equation

The integrable NNLS equation was obtained by M. Ablowitz and Z. Musslimani as a nonlocal reduction of the Ablowitz-Kaup-Newell-Segur system [2]. This equation satisfies the $\mathscr{P}\mathscr{T}$-symmetric condition [4], i.e., $q(x,t)$ and $\overline{q(-x,-t)}$ are its solutions simultaneously. Thus the NNLS equation is related to the non-Hermitian quantum mechanics [3, 16]. Also this equation has connections with the theory of magnetism, because it is gauge equivalent to the complex Landau-Lifshitz equation [19, 31]. Finally, the NNLS equation is an example of a two-place (Alice-Bob) system [27, 28], which involves the values of the solution at not neighboring points, x and $-x$.

The NNLS equation admits exact solutions with distinctive properties. It has both bright and dark soliton solutions [36], in contrast to its local counterpart, the classical nonlinear Schrödinger (NLS) equation. The simplest one-soliton solution of (1a) on zero background has, in general, periodic (in time) point singularities [2], so the solution becomes unbounded at these points. Different types of exact solutions with various backgrounds can have such isolated blow-up points in the (x,t) plane. For example, solitons with nonzero boundary conditions [1, 17, 21, 22, 26, 33], rogue waves [39] and breathers [35]. Other important exact solutions of the NNLS equation are given in, e.g., [29, 30, 38].

Initial Value Problems

The initial value problem (1a)–(1b) with nonzero background $q(x,t) \to A\mathrm{e}^{\mathrm{i}\theta_\pm(t)}$, as $x \to \pm\infty$ was firstly considered in [1]. It was shown that $\mathrm{e}^{\mathrm{i}\theta_\pm(t)}$ remains bounded as $|t| \to \infty$ only in two cases: $\theta_+(t) - \theta_-(t) = 0$ or $\theta_+(t) - \theta_-(t) = \pi$. Thus bounded (with respect to t) boundary conditions can be either $q(x,t) \to A\mathrm{e}^{2\mathrm{i}A^2 t}$ as $|x| \to \infty$ or $q(x,t) \to \pm A\mathrm{e}^{-2\mathrm{i}A^2 t}$ as $x \to \pm\infty$. The inverse scattering transform method for problems with these two boundary values was developed in [1], where it was shown that the two problems have different continuous spectra. Namely, if $q(x,t) \to A\mathrm{e}^{2\mathrm{i}A^2 t}$ as $|x| \to \infty$, the continuous spectrum consists of the real line and a vertical band $(-\mathrm{i}A, \mathrm{i}A)$, which is reminiscent of the problem for the classical (local) focusing NLS equation on a symmetric [6] or step-like [8] background. For $q(x,t) \to \pm A\mathrm{e}^{-2\mathrm{i}A^2 t}$, $x \to \pm\infty$, the continuous spectrum lies on the real line and has a gap $(-A, A)$, as in the problem for the defocusing NLS equation with symmetric nonzero boundary conditions [14, 24, 40]. Another interesting feature of problem (1) is that the boundary functions $\pm A\mathrm{e}^{-2\mathrm{i}A^2 t}$ are not exact solutions of the NNLS equation. It is in sharp contrast with the local problems, where for the well-posedness it is necessary that the boundary conditions satisfy the equation.

Long-Time Asymptotics

The long-time asymptotics for the defocusing NLS equation with nonzero boundary conditions manifests important nonlinear phenomena, including solitons [9, 37, 40], rarefaction waves, shock waves, and various plane wave type regions [5, 15, 18, 23, 25]. These developments motivate us to study the asymptotics of problem (1) and to highlight its qualitative differences with that for the defocusing NLS equation on a nonzero background, which has a similar spectral picture. We also compare the long-time asymptotic behavior of (1) to that for the Cauchy problem for (1a) with boundary conditions $q(x,t) \to A\mathrm{e}^{2\mathrm{i}A^2 t}$ as $x \to \pm\infty$, which is considered in [32].

Methods

The main technical tool used in this paper is the inverse scattering transform method, which allows us to express the solution of (1) in terms of the solution of an associated Riemann–Hilbert problem. The jump matrix of this problem depends on the parameters (x, t) only via oscillating exponents, so we can apply the Deift and Zhou nonlinear steepest descent method [10, 13] (see also [11, 12] for its extensions) to get the asymptotics of the Riemann–Hilbert problem and, therefore, of the solution $q(x,t)$ of (1).

Organization of the Paper

The article is organized as follows. In Sect. 2 we develop the inverse scattering transform method for (1) and formulate the basic Riemann–Hilbert problem. We also get the one-soliton solution by using the Riemann–Hilbert approach. Section 3 contains our main results, Theorems 3.2 and 3.4, on the long-time asymptotic behavior of $q(x,t)$. More precisely, we present the asymptotics in the "modulated regions" ($|x/4t| > A/2$) in Theorem 3.2, and in the central "unmodulated region" ($0 < |x/4t| < A/2$) in Theorem 3.4. Finally, we discuss the transition inside the unmodulated region as $\xi \to 0$. Theorem 3.9 presents the large time asymptotics with x fixed $\neq 0$, in which case $\xi \to 0$.

2 Inverse Scattering Transform Method

The inverse scattering transform formalism for problem (1) was first developed in [1]. Here we perform the direct and inverse analysis in a different way, in particular we define the inverse transform in terms of an associated Riemann–Hilbert problem formulated in the complex plane of the spectral parameter k entering the standard Lax pair equations for the NNLS equation (1a).

2.1 Direct Scattering

The NNLS equation (1a) is the compatibility condition of the following system of linear equations [2] (the "Lax pair")

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = U\Phi, \tag{2a}$$

$$\Phi_t + 2\mathrm{i}k^2\sigma_3\Phi = V\Phi, \tag{2b}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix, $\Phi(x,t,k)$ is a 2×2 matrix-valued function, $k \in \mathbb{C}$ is the spectral parameter, and $U(x,t)$ and $V(x,t,k)$ are given in terms of $q(x,t)$ as follows:

$$U(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(-x,t) & 0 \end{pmatrix}, \quad V(x,t,k) = \begin{pmatrix} V_{11}(x,t) & V_{12}(x,t,k) \\ V_{21}(x,t,k) & V_{22}(x,t) \end{pmatrix}, \tag{3}$$

where $V_{11} = -V_{22} = \mathrm{i}q(x,t)\bar{q}(-x,t)$, $V_{12} = 2kq(x,t) + \mathrm{i}q_x(x,t)$, and $V_{21} = -2k\bar{q}(-x,t) + \mathrm{i}\bar{q}_x(-x,t)$.

Assuming that

$$\int_{-\infty}^{0} |q(x,t) + A\mathrm{e}^{-2\mathrm{i}A^2 t}|\, \mathrm{d}x < \infty \text{ and } \int_{0}^{\infty} |q(x,t) - A\mathrm{e}^{-2\mathrm{i}A^2 t}|\, \mathrm{d}x < \infty \text{ for all } t \geq 0,$$

we introduce the 2×2 matrix-valued functions $\Psi_j(x,t,k)$, $j = 1, 2$ as the solutions of the following linear Volterra integral equations ($j = 1, 2$) where $k \in \mathbb{R}\setminus[-A, A]$:

$$\Psi_j(x,t,k) = \mathrm{e}^{-\mathrm{i}A^2 t\sigma_3}\mathcal{E}_j(k) + \int_{(-1)^j\infty}^{x} G_j(x,y,t,k)(U(y,t) - U_j(t))\Psi_j(y,t,k)\mathrm{e}^{\mathrm{i}(x-y)f(k)\sigma_3}\,\mathrm{d}y. \tag{4}$$

Here $U_1(t)$ and $U_2(t)$ are the limits of $U(x,t)$ as $x \to \mp\infty$:

$$U(x,t) \to U_j(t), \quad x \to (-1)^j \infty, \tag{5}$$

where

$$U_1(t) = \begin{pmatrix} 0 & -A\mathrm{e}^{-2\mathrm{i}A^2 t} \\ -A\mathrm{e}^{2\mathrm{i}A^2 t} & 0 \end{pmatrix} \quad \text{and} \quad U_2(t) = \begin{pmatrix} 0 & A\mathrm{e}^{-2\mathrm{i}A^2 t} \\ A\mathrm{e}^{2\mathrm{i}A^2 t} & 0 \end{pmatrix}. \tag{6}$$

The kernels $G_j(x,y,t,k)$, $j=1,2$ are defined in terms of functions $\mathcal{E}_j(k)$, $j=1,2$ and $f(k)$ as follows:

$$G_j(x,y,t,k) := \mathrm{e}^{-\mathrm{i}A^2 t\sigma_3} \mathcal{E}_j(k) \mathrm{e}^{-\mathrm{i}(x-y)f(k)\sigma_3} \mathcal{E}_j^{-1}(k) \mathrm{e}^{\mathrm{i}A^2 t\sigma_3}, \tag{7}$$

where

$$\begin{aligned} \mathcal{E}_j(k) &:= \frac{1}{2}\begin{pmatrix} w(k) + \frac{1}{w(k)} & (-1)^j \,\mathrm{i}\left(w(k) - \frac{1}{w(k)}\right) \\ (-1)^{j+1}\,\mathrm{i}\left(w(k) - \frac{1}{w(k)}\right) & w(k) + \frac{1}{w(k)} \end{pmatrix}, \\ w(k) &:= \left(\frac{k-A}{k+A}\right)^{\frac{1}{4}}, \end{aligned} \tag{8}$$

and

$$f(k) := (k^2 - A^2)^{\frac{1}{2}}. \tag{9}$$

Here, the functions $f(k)$ and $w(k)$ are defined for $k \in \mathbb{C} \setminus [-A, A]$ as the branches fixed by the large k asymptotics:

$$f(k) = k + \mathrm{O}(k^{-1}) \quad \text{and} \quad w(k) = 1 + \mathrm{O}(k^{-1}), \quad k \to \infty. \tag{10}$$

We denote by $f_\pm(k)$ and $w_\pm(k)$ the limiting values of the corresponding function as k approaches $(-A, A)$ (oriented from $-A$ to A) from the left/right side (and similarly for $\mathcal{E}_{j\pm}(k)$). In particular, $f_+(k) = \mathrm{i}\sqrt{A^2 - k^2}$ for $k \in (-A, A)$, with $\sqrt{A^2 - k^2} > 0$. Observe that $G(x,y,t,k)$ is entire with respect to k for all x, y, and t.

Since $f(k)$ is real for $k \in \mathbb{R} \setminus [-A, A]$, the integral in (4) converges for such k. Let $Q^{[i]}$ denote the i-th column of a matrix Q, $\mathbb{C}^\pm := \{k \in \mathbb{C} \mid \pm \operatorname{Im} k > 0\}$, and $\overline{\mathbb{C}^\pm} := \{k \in \mathbb{C} \mid \pm \operatorname{Im} k \geq 0\}$. Then we can define $\Psi_j^{[j]}(x,t,k)$, $j=1,2$, and $\Psi_1^{[2]}(x,t,k)$, $\Psi_2^{[1]}(x,t,k)$ on the cut $(-A, A)$ as the limiting values from $\mathbb{C}^+$ and

$\mathbb{C}^-$, respectively (in the following three equations $k \in (-A, A)$):

$$\Psi_{j+}^{[j]}(x,t,k) = \mathrm{e}^{-\mathrm{i}A^2 t\sigma_3}\mathcal{E}_{j+}^{[j]}(k) + \int_{(-1)^j\infty}^{x} G_j(x,y,t,k)(U(y,t)-U_j(t))\Psi_{j+}^{[j]}(y,t,k)\mathrm{e}^{(-1)^{j+1}\mathrm{i}(x-y)f_+(k)}\,\mathrm{d}y, \tag{11}$$

and

$$\Psi_{1-}^{[2]}(x,t,k) = \mathrm{e}^{-\mathrm{i}A^2 t\sigma_3}\mathcal{E}_{1-}^{[2]}(k) + \int_{-\infty}^{x} G_1(x,y,t,k)(U(y,t)-U_1(t))\Psi_{1-}^{[2]}(y,t,k)\mathrm{e}^{-\mathrm{i}(x-y)f_-(k)}\,\mathrm{d}y, \tag{12a}$$

$$\Psi_{2-}^{[1]}(x,t,k) = \mathrm{e}^{-\mathrm{i}A^2 t\sigma_3}\mathcal{E}_{2-}^{[1]}(k) + \int_{+\infty}^{x} G_2(x,y,t,k)(U(y,t)-U_2(t))\Psi_{2-}^{[1]}(y,t,k)\mathrm{e}^{\mathrm{i}(x-y)f_-(k)}\,\mathrm{d}y. \tag{12b}$$

Moreover, when the solution $q(x,t)$ converges exponentially fast to its boundary values, we can define $\Psi_{j-}^{[j]}(x,t,k)$, $j = 1,2$, and $\Psi_{1+}^{[2]}(x,t,k)$, $\Psi_{2+}^{[1]}(x,t,k)$ for $k \in (-A, A)$ by integral equations similar to (11) and (12), respectively.

Proposition 2.1 (Properties of Ψ_j) *$\Psi_1(x,t,k)$ and $\Psi_2(x,t,k)$ have the following properties.*

(i) *The columns $\Psi_1^{[1]}(x,t,k)$ and $\Psi_2^{[2]}(x,t,k)$ are analytic for $k \in \mathbb{C}^+$ and continuous for $k \in \overline{\mathbb{C}^+} \setminus \{\pm A\}$, where $\Psi_j^{[j]}(x,t,k)$ is identified with $\Psi_{j+}^{[j]}(x,t,k)$, $j = 1,2$ for $k \in (-A, A)$.*

$\Psi_1^{[1]}(x,t,k)$ and $\Psi_2^{[2]}(x,t,k)$ have the following behaviors at $k = \infty$ and $k = \pm A$:

$$\Psi_1^{[1]}(x,t,k) = \mathrm{e}^{-\mathrm{i}A^2 t}\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{O}(k^{-1}), \quad k \to \infty, \quad k \in \mathbb{C}^+,$$

$$\Psi_2^{[2]}(x,t,k) = \mathrm{e}^{\mathrm{i}A^2 t}\begin{pmatrix}0\\1\end{pmatrix} + \mathrm{O}(k^{-1}), \quad k \to \infty, \quad k \in \mathbb{C}^+,$$

$$\Psi_1^{[1]}(x,t,k) = \mathrm{O}\big((k \mp A)^{-\frac{1}{4}}\big), \quad k \to \pm A, \quad k \in \mathbb{C}^+,$$

$$\Psi_2^{[2]}(x,t,k) = \mathrm{O}\big((k \mp A)^{-\frac{1}{4}}\big), \quad k \to \pm A, \quad k \in \mathbb{C}^+.$$

(ii) *The columns* $\Psi_1^{[2]}(x,t,k)$ *and* $\Psi_2^{[1]}(x,t,k)$ *are analytic for* $k \in \mathbb{C}^-$ *and continuous for* $k \in \overline{\mathbb{C}^-} \setminus \{\pm A\}$*, where* $\Psi_1^{[2]}(x,t,k)$ *and* $\Psi_2^{[1]}(x,t,k)$ *are identified with* $\Psi_{1-}^{[2]}(x,t,k)$ *and* $\Psi_{2-}^{[1]}(x,t,k)$ *for* $k \in (-A, A)$.

$\Psi_1^{[2]}(x,t,k)$ *and* $\Psi_2^{[1]}(x,t,k)$ *have the following behaviors at* $k = \infty$ *and* $k = \pm A$*:*

$$\begin{aligned}
&\Psi_1^{[2]}(x,t,k) = \mathrm{e}^{\mathrm{i}A^2 t}\begin{pmatrix}0\\1\end{pmatrix} + \mathrm{O}(k^{-1}), && k \to \infty, && k \in \mathbb{C}^-,\\
&\Psi_2^{[1]}(x,t,k) = \mathrm{e}^{-\mathrm{i}A^2 t}\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{O}(k^{-1}), && k \to \infty, && k \in \mathbb{C}^-,\\
&\Psi_1^{[2]}(x,t,k) = \mathrm{O}\big((k \mp A)^{-\frac{1}{4}}\big), && k \to \pm A, && k \in \mathbb{C}^-,\\
&\Psi_2^{[1]}(x,t,k) = \mathrm{O}\big((k \mp A)^{-\frac{1}{4}}\big), && k \to \pm A, && k \in \mathbb{C}^-.
\end{aligned}$$

(iii) *The functions* $\Phi_j(x,t,k)$*,* $j = 1, 2$ *defined by*

$$\Phi_j(x,t,k) := \Psi_j(x,t,k)\mathrm{e}^{-(\mathrm{i}x+2\mathrm{i}tk)f(k)\sigma_3}, \quad k \in \mathbb{R} \setminus [-A, A], \tag{13}$$

are the (Jost) solutions of the Lax pair (2) *satisfying the boundary conditions*

$$\Phi_j(x,t,k) - \Phi_j^{\mathrm{BC}}(x,t,k) \to 0, \quad x \to (-1)^j\infty, \quad k \in \mathbb{R} \setminus [-A, A], \tag{14}$$

where $\Phi_j^{\mathrm{BC}}(x,t,k) := \mathrm{e}^{-\mathrm{i}A^2 t\sigma_3}\mathcal{E}_j(k)\mathrm{e}^{-(\mathrm{i}x+2\mathrm{i}tk)f(k)\sigma_3}$.

(iv) $\det \Psi_j(x,t,k) \equiv 1$ *for* $k \in \mathbb{R} \setminus [-A, A]$.

(v) *The following symmetry relations hold:*

$$\begin{aligned}
&\sigma_1\overline{\Psi_1^{[1]}(-x,t,-\bar{k})} = \Psi_2^{[2]}(x,t,k), && k \in \overline{\mathbb{C}^+} \setminus [-A, A],\\
&\sigma_1\overline{\Psi_{1+}^{[1]}(-x,t,-k)} = \Psi_{2+}^{[2]}(x,t,k), && k \in (-A, A),\\
&\sigma_1\overline{\Psi_1^{[2]}(-x,t,-\bar{k})} = \Psi_2^{[1]}(x,t,k), && k \in \overline{\mathbb{C}^-} \setminus [-A, A],\\
&\sigma_1\overline{\Psi_{1-}^{[2]}(-x,t,-k)} = \Psi_{2-}^{[1]}(x,t,k), && k \in (-A, A),
\end{aligned} \tag{15a}$$

and

$$\begin{aligned}
&\Psi_{1+}^{[1]}(x,t,k) = -\Psi_{1-}^{[2]}(x,t,k), && k \in (-A, A),\\
&\Psi_{2+}^{[2]}(x,t,k) = -\Psi_{2-}^{[1]}(x,t,k), && k \in (-A, A),
\end{aligned} \tag{15b}$$

where $\sigma_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ *is the first Pauli matrix.*

Moreover, when $\Psi_{j-}^{[j]}(x,t,k)$*,* $j=1,2$ *and* $\Psi_{1+}^{[2]}(x,t,k)$*,* $\Psi_{2+}^{[1]}(x,t,k)$ *exist (e.g., when* $q(x,t)$ *converges exponentially fast to its boundary values), they satisfy the following conditions:*

$$\begin{aligned}\Psi_{1-}^{[1]}(x,t,k) &= \Psi_{1+}^{[2]}(x,t,k), \quad k\in(-A,A),\\ \Psi_{2-}^{[2]}(x,t,k) &= \Psi_{2+}^{[1]}(x,t,k), \quad k\in(-A,A).\end{aligned} \tag{16}$$

Proof Items (i)–(iii) follow directly from the integral equations (4). Since the matrix $U(x,t)$ is traceless and $\det\mathcal{E}_j(k)=1$, $j=1,2$, we get item (iv). Finally, (15a) in item (v) follows from the symmetries

$$\begin{aligned}\sigma_1\overline{U}(-x,t)\sigma_1^{-1} &= -U(x,t),\\ \sigma_1\overline{G_1(-x,-y,t,-\bar{k})}\sigma_1^{-1} &= G_2(x,y,t,k), \quad k\in\mathbb{C},\end{aligned} \tag{17}$$

whereas (15b) and (16) follow from the symmetries

$$\mathcal{E}_{j+}(k)=(-1)^{j+1}\,\mathrm{i}\,\mathcal{E}_{j-}(k)\sigma_2, \quad j=1,2, \quad k\in(-A,A), \tag{18}$$

where $\sigma_2=\left(\begin{smallmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{smallmatrix}\right)$ is the second Pauli matrix. □

2.2 Spectral Functions

The Jost solutions $\Phi_1(x,t,k)$ and $\Phi_2(x,t,k)$ of the Lax pair (2) are related by a matrix independent of x and t, which allows us to introduce the scattering matrix $S(k)$ as follows:

$$\Phi_1(x,t,k)=\Phi_2(x,t,k)S(k), \quad k\in\mathbb{R}\setminus[-A,A], \tag{19}$$

or, in terms of $\Psi_j(x,t,k)$, $j=1,2$, and for $k\in\mathbb{R}\setminus[-A,A]$:

$$\Psi_1(x,t,k)=\Psi_2(x,t,k)\mathrm{e}^{-(\mathrm{i}x+2\mathrm{i}tk)f(k)\sigma_3}S(k)\mathrm{e}^{(\mathrm{i}x+2\mathrm{i}tk)f(k)\sigma_3}. \tag{20}$$

From the symmetry relations (15a) it follows that $S(k)$ can be written as

$$S(k)=\begin{pmatrix}a_1(k) & -\overline{b(-k)}\\ b(k) & a_2(k)\end{pmatrix}, \quad k\in\mathbb{R}\setminus[-A,A]. \tag{21}$$

Note that due to the Schwarz symmetry breaking for the solutions $\Psi_j(x,t,k)$, $j=1,2$, see (15a), the values of $a_1(k)$ for $k\in\mathbb{C}^+$ and $a_2(k)$ for $k\in\mathbb{C}^-$ are, in general, *not* related. In particular, this implies that $a_1(k)$ and $a_2(k)$ can have different numbers of zeros in the corresponding complex half-planes.

Relation (20) implies that $a_1(k)$, $a_2(k)$, and $b(k)$ can be found in terms of the initial data alone via the following determinants:

$$a_1(k) = \det\big(\Psi_1^{[1]}(0,0,k), \Psi_2^{[2]}(0,0,k)\big), \qquad k \in \overline{\mathbb{C}^+} \setminus [-A, A], \tag{22a}$$

$$a_2(k) = \det\big(\Psi_2^{[1]}(0,0,k), \Psi_1^{[2]}(0,0,k)\big), \qquad k \in \overline{\mathbb{C}^-} \setminus [-A, A], \tag{22b}$$

$$b(k) = \det\big(\Psi_2^{[1]}(0,0,k), \Psi_1^{[1]}(0,0,k)\big), \qquad k \in \mathbb{R} \setminus [-A, A]. \tag{22c}$$

From (22) and Proposition 2.1 (i) and (ii) we conclude that $a_j(k)$, $j = 1, 2$, and $b(k)$ have the following large k behaviors:

$$\begin{aligned} a_1(k) &= 1 + \mathrm{O}(k^{-1}), \quad k \in \overline{\mathbb{C}^+}, \quad k \to \infty, \\ a_2(k) &= 1 + \mathrm{O}(k^{-1}), \quad k \in \overline{\mathbb{C}^-}, \quad k \to \infty, \\ b(k) &= \mathrm{O}(k^{-1}), \qquad k \in \mathbb{R}, \quad k \to \infty. \end{aligned}$$

Defining $a_{1+}(k)$ and $a_{2-}(k)$ for $k \in (-A, A)$ as the limits of $a_1(k)$ and $a_2(k)$ from $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, we have

$$\begin{aligned} a_{1+}(k) &= \det\big(\Psi_{1+}^{[1]}(0,0,k), \Psi_{2+}^{[2]}(0,0,k)\big), \quad k \in (-A, A), \\ a_{2-}(k) &= \det\big(\Psi_{2-}^{[1]}(0,0,k), \Psi_{1-}^{[2]}(0,0,k)\big), \quad k \in (-A, A). \end{aligned} \tag{23}$$

Moreover, when the initial data $q_0(x)$ converges exponentially fast to its boundary values, we can define $a_{1-}(k)$, $a_{2+}(k)$, and $b_\pm(k)$ for $k \in (-A, A)$ by taking the corresponding limits in (22):

$$a_{1-}(k) = \det\big(\Psi_{1-}^{[1]}(0,0,k), \Psi_{2-}^{[2]}(0,0,k)\big), \quad k \in (-A, A), \tag{24a}$$

$$a_{2+}(k) = \det\big(\Psi_{2+}^{[1]}(0,0,k), \Psi_{1+}^{[2]}(0,0,k)\big), \quad k \in (-A, A), \tag{24b}$$

$$b_\pm(k) = \det\big(\Psi_{2\pm}^{[1]}(0,0,k), \Psi_{1\pm}^{[1]}(0,0,k)\big), \quad k \in (-A, A). \tag{24c}$$

The symmetry relations (15) yield the following symmetries of the spectral functions:

$$\begin{aligned} \overline{a_1(-\bar{k})} &= a_1(k), \quad k \in \overline{\mathbb{C}^+} \setminus [-A, A], \\ \overline{a_2(-\bar{k})} &= a_2(k), \quad k \in \overline{\mathbb{C}^-} \setminus [-A, A], \end{aligned} \tag{25}$$

whereas (16) implies that

$$a_{1\pm}(k) = -a_{2\mp}(k) \quad \text{and} \quad b_\pm(k) = -\overline{b_\mp(-k)}, \qquad k \in (-A, A). \tag{26}$$

From Proposition 2.1 (iv), (13), and (19) it follows that $a_1(k)$, $a_2(k)$, and $b(k)$ satisfy the determinant relations:

$$\begin{aligned} a_1(k)a_2(k) + b(k)\overline{b(-k)} &= 1, \qquad k \in \mathbb{R} \setminus [-A, A], \\ a_{1\pm}(k)a_{2\pm}(k) + b_\pm(k)\overline{b_\pm(-k)} &= 1, \qquad k \in (-A, A). \end{aligned} \tag{27}$$

Finally, we point out that $a_1(k)$, $a_2(k)$, and $b(k)$ are $\mathrm{O}\big((k \mp A)^{-\frac{1}{2}}\big)$ as $k \to \pm A$.

Proposition 2.2 (Pure Step Initial Data) *Consider problem (1) with initial data*

$$q_0(x) = q_{0,R}(x) = \begin{cases} A, & x > R, \\ -A, & x < R, \end{cases} \tag{28}$$

for some $A > 0$ and $R \in \mathbb{R}$. Introduce

$$h(k) := (k^2 + A^2)^{\frac{1}{2}}, \tag{29}$$

which is defined in $\mathbb{C} \setminus [-\mathrm{i}A, \mathrm{i}A]$ and is fixed by the asymptotics $h(k) = k + \mathrm{O}(k^{-1})$ as $k \to \infty$. Define

$$\lambda_j(k) := \mathrm{i}(f(k) + (-1)^{j+1}h(k)), \qquad j = 1, 2. \tag{30}$$

Then the spectral functions associated with this problem have the following form, according to the sign of $R \in \mathbb{R}$:

(i) For $R > 0$,

$$a_1(k) = \frac{1}{2f(k)h(k)}\big(\mathrm{e}^{2\lambda_1(k)R}\big(A^2 + \mathrm{i}k\lambda_2(k)\big) - \mathrm{e}^{2\lambda_2(k)R}\big(A^2 + \mathrm{i}k\lambda_1(k)\big)\big), \tag{31a}$$

$$a_2(k) = \frac{1}{2f(k)h(k)}\big(\mathrm{e}^{-2\lambda_2(k)R}\big(A^2 - \mathrm{i}k\lambda_1(k)\big) - \mathrm{e}^{-2\lambda_1(k)R}\big(A^2 - \mathrm{i}k\lambda_2(k)\big)\big), \tag{31b}$$

$$b(k) = \frac{-\mathrm{i}A}{2f(k)h(k)}\big(\mathrm{e}^{2\mathrm{i}h(k)R}\big(h(k) + k\big) + \mathrm{e}^{-2\mathrm{i}h(k)R}\big(h(k) - k\big)\big). \tag{31c}$$

(ii) For $R = 0$,

$$a_1(k) = a_2(k) = \frac{k}{f(k)}, \qquad b(k) = \frac{-\mathrm{i}A}{f(k)}. \tag{32}$$

(iii) For $R < 0$,

$$a_1(k) = \frac{1}{2f(k)h(k)}\left(\mathrm{e}^{-2\lambda_2(k)R}\left(A^2 - \mathrm{i}k\lambda_1(k)\right) - \mathrm{e}^{-2\lambda_1(k)R}\left(A^2 - \mathrm{i}k\lambda_2(k)\right)\right), \tag{33a}$$

$$a_2(k) = \frac{1}{2f(k)h(k)}\left(\mathrm{e}^{2\lambda_1(k)R}\left(A^2 + \mathrm{i}k\lambda_2(k)\right) - \mathrm{e}^{2\lambda_2(k)R}\left(A^2 + \mathrm{i}k\lambda_1(k)\right)\right), \tag{33b}$$

$$b(k) = \frac{-\mathrm{i}A}{2f(k)h(k)}\left(\mathrm{e}^{2\mathrm{i}h(k)R}\left(h(k) + k\right) + \mathrm{e}^{-2\mathrm{i}h(k)R}\left(h(k) - k\right)\right). \tag{33c}$$

Proof See section "Appendix: Proof of Proposition 2.2" in Appendix. □

Remark 2.3 Note that for any $R \in \mathbb{R}$, $a_1(k)$, $a_2(k)$, and $b(k)$ have no jump across $[-\mathrm{i}A, \mathrm{i}A]$. Also, if we take the limits $R \to \pm 0$ in the expressions of the spectral functions for $R > 0$ and $R < 0$, we arrive at (32).

Remark 2.4 The NNLS equation is not translation invariant. Therefore, shifting the initial data by a constant value can drastically affect the behavior of the solution [34]. Formulas (31)–(33) illustrate this in terms of the spectral functions in the case of pure step initial data (28).

The scattering map associates to $q_0(x)$

(i) the spectral functions $b(k)$ and $a_j(k)$, $j = 1, 2$,
(ii) the discrete data, which are the zeros of $a_j(k)$, $j = 1, 2$ and the associated norming constants.

In studying initial value problems for integrable nonlinear PDEs, the assumptions about these zeros usually rely on properties of the discrete spectrum associated with step-like initial data involving prescribed boundary values, like (1c) (see, e.g., [7, 8, 23, 25, 34]). Alternatively, the discrete spectrum can be added to the formulation of the associated Riemann–Hilbert problem for studying the evolution of more general initial data, which includes solitons [9, 37, 40].

In the present paper we consider initial data which are characterized in spectral terms and which are motivated by the pure step initial data with $R = 0$. Namely, we make the following assumptions.

Assumptions 2.5 (On the Zeros of the Spectral Functions $a_1(k)$ and $a_2(k)$) We assume that

(A1) $a_1(k)$ and $a_2(k)$ do not have zeros in $\overline{\mathbb{C}^+} \setminus (-A, A)$ and $\overline{\mathbb{C}^-} \setminus (-A, A)$, respectively;
(A2) for $k \in (-A, A)$, both $a_{1+}(k)$ and $a_{2-}(k)$ have a simple zero at $k = 0$, i.e.,

$$\begin{aligned} a_{1+}(k) &= a_{10}k + \mathrm{O}(k^2), \quad k \to 0, \quad a_{10} \neq 0, \\ a_{2-}(k) &= a_{20}k + \mathrm{O}(k^2), \quad k \to 0, \quad a_{20} \neq 0. \end{aligned} \tag{34}$$

Then from (26) and (25) it follows that

$$a_{20} = -a_{10} \quad \text{and} \quad \operatorname{Re} a_{10} = 0. \tag{35}$$

2.3 Riemann–Hilbert Problem

Taking into account the analytical properties of the columns of the matrices $\Psi_j(x, t, k)$, $j = 1, 2$ (see Proposition 2.1(i) and (ii)), we define the 2×2 sectionally holomorphic matrix $M(x, t, k)$ as follows:

$$M(x,t,k) = \begin{cases} \mathrm{e}^{\mathrm{i}A^2 t\sigma_3}\left(\frac{\Psi_1^{[1]}(x,t,k)}{a_1(k)}, \Psi_2^{[2]}(x,t,k)\right), & k \in \mathbb{C}^+, \\ \mathrm{e}^{\mathrm{i}A^2 t\sigma_3}\left(\Psi_2^{[1]}(x,t,k), \frac{\Psi_1^{[2]}(x,t,k)}{a_2(k)}\right), & k \in \mathbb{C}^-. \end{cases} \tag{36}$$

By Assumptions 2.5, $a_1(k)$ and $a_2(k)$ have no zeros in the corresponding half-planes and thus the matrix $M(x, t, k)$ does not have poles in $\mathbb{C} \setminus \mathbb{R}$. From the scattering relation (20), the symmetries (15b), and the relations (26) it follows that $M(x, t, k)$ satisfies a multiplicative jump condition:

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \quad k \in \mathbb{R}. \tag{37a}$$

Here and below $M_+(\cdot\,, \cdot\,, k)$ and $M_-(\cdot\,, \cdot\,, k)$ denote the nontangental limits of $M(\cdot\,, \cdot\,, k)$ as k approaches the contour from the left and right sides, respectively (here, the real line $\mathbb{R}$ is oriented from $-\infty$ to $+\infty$). The jump matrix $J(x, t, k)$ has the following form:

$$J(x,t,k) = \begin{cases} \begin{pmatrix} 1 + r_1(k)r_2(k) & r_2(k)\mathrm{e}^{-(2\mathrm{i}x+4\mathrm{i}tk)f(k)} \\ r_1(k)\mathrm{e}^{(2\mathrm{i}x+4\mathrm{i}tk)f(k)} & 1 \end{pmatrix}, & k \in \mathbb{R} \setminus [-A, A], \\ -\mathrm{i}\sigma_2, & k \in (-A, A), \end{cases} \tag{37b}$$

with the reflection coefficients

$$r_1(k) := \frac{b(k)}{a_1(k)} \quad \text{and} \quad r_2(k) := \frac{\overline{b(-k)}}{a_2(k)}, \quad k \in \mathbb{R} \setminus [-A, A]. \tag{37c}$$

Remark 2.6 If $b(k)$ can be analytically continued into a band containing $\mathbb{R}$, we can also define $r_j(k)$, $j = 1, 2$ in this band. Then in view of (26), $r_{1\pm}(k) = r_{2\mp}(k)$ and therefore $1+r_1(k)r_2(k)$ does not have a jump across $(-A, A)$. From the determinant relation (27) it follows that $1 + r_1(k)r_2(k) = a_1^{-1}(k)a_2^{-1}(k)$, so $1 + r_1(k)r_2(k)$ can have simple zeros at $k = \pm A$. This takes place, e.g., for pure step initial data (28) (see [25, Section 3]).

In view of Proposition 2.1 (i) and (ii), and Assumptions 2.5, $M(x, t, k)$ has weak singularities at $k = \pm A$:

$$M(x, t, k) = \mathrm{O}\big((k \pm A)^{-\frac{1}{4}}\big), \quad k \to \mp A. \tag{38}$$

Also it has the normalization condition for large k:

$$M(x, t, k) = I + \mathrm{O}(k^{-1}), \quad k \to \infty, \tag{39}$$

where I is the identity matrix. Finally, $M(x, t, k)$ satisfies the following conditions at $k = 0$:

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^+}} kM^{[1]}(x, t, k) = \frac{\gamma_+}{a_{10}} \mathrm{e}^{-2Ax} M_+^{[2]}(x, t, 0), \tag{40a}$$

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^-}} kM^{[2]}(x, t, k) = \frac{\gamma_-}{a_{20}} \mathrm{e}^{-2Ax} M_-^{[1]}(x, t, 0), \tag{40b}$$

where a_{10} and a_{20} were introduced in (34), and $\gamma_\pm$ are defined as follows:

$$\Phi_{1+}^{[1]}(x, t, 0) = \gamma_+ \Phi_{2+}^{[2]}(x, t, 0) \quad \text{and} \quad \Phi_{1-}^{[2]}(x, t, 0) = \gamma_- \Phi_{2-}^{[1]}(x, t, 0).$$

From (15b) and (15a) one concludes that $\gamma_+ = \gamma_-$ and $|\gamma_+| = 1$.

Remark 2.7 If $b(k)$ can be analytically continued into a band, the norming constants $\gamma_\pm$ can be found in terms of $b(k)$ as follows: $\gamma_+ = b_+(0)$ and $\gamma_- = -\overline{b_-(0)}$.

Thus we arrive at the following basic Riemann–Hilbert (RH) problem:

Basic RH problem Find a sectionally analytic 2×2 matrix $M(x, t, k)$, which

(i) satisfies the jump condition (37) across the real axis,
(ii) has weak singularities (38) at $k = \pm A$,
(iii) converges to the identity matrix as $k \to \infty$,
(iv) and satisfies the singularity conditions (40) at $k = 0$.

Using standard arguments based on Liouville's theorem, it can be shown that the solution of this RH problem is unique, if it exists.

The solution $q(x,t)$ of the initial value problem (1) can be found from the large k expansion of the solution $M(x,t,k)$ of the basic RH problem (follows from (2a)):

$$\begin{aligned} q(x,t) &= 2\mathrm{i}e^{-2\mathrm{i}A^2 t} \lim_{k\to\infty} kM_{12}(x,t,k), \\ q(-x,t) &= -2\mathrm{i}e^{-2\mathrm{i}A^2 t} \lim_{k\to\infty} \overline{kM_{21}(x,t,k)}. \end{aligned} \tag{41}$$

Thus both $q(x,t)$ and $q(-x,t)$ can be found from $M(x,t,k)$ evaluated for $x \geq 0$.

Remark 2.8 Since the jump matrix $J(x,t,k)$ satisfies the condition

$$\sigma_1 \overline{J(-x,t,-k)} \sigma_1^{-1} = \begin{pmatrix} a_2(k) & 0 \\ 0 & \frac{1}{a_2(k)} \end{pmatrix} J(x,t,k) \begin{pmatrix} a_1(k) & 0 \\ 0 & \frac{1}{a_1(k)} \end{pmatrix}, \quad k \in \mathbb{R} \setminus \{\pm A\}, \tag{42}$$

the solution $M(x,t,k)$ of the basic RH problem satisfies the following symmetry conditions (see [33, (2.55)]):

$$M(x,t,k) = \begin{cases} \sigma_1 \overline{M(-x,t,-\bar{k})} \sigma_1^{-1} \begin{pmatrix} \frac{1}{a_1(k)} & 0 \\ 0 & a_1(k) \end{pmatrix}, & k \in \mathbb{C}^+, \\ \sigma_1 \overline{M(-x,t,-\bar{k})} \sigma_1^{-1} \begin{pmatrix} a_2(k) & 0 \\ 0 & \frac{1}{a_2(k)} \end{pmatrix}, & k \in \mathbb{C}^-. \end{cases} \tag{43}$$

2.4 One-Soliton Solution

The one-soliton solution of the focusing NNLS equation satisfying boundary conditions (1c) was obtained in [22, Section 4], by using the Darboux transformation and in [1, Section 3] via the inverse scattering transform method. Here we rederive this soliton solution using the Riemann–Hilbert approach. Consider the basic RH problem in the reflectionless case, i.e., with $r_1(k) \equiv r_2(k) \equiv 0$:

$$M_+^{\mathrm{sol}}(x,t,k) = -\mathrm{i}M_-^{\mathrm{sol}}(x,t,k)\sigma_2, \quad k \in (-A, A), \tag{44a}$$

$$M^{\mathrm{sol}}(x,t,k) = I + \mathrm{O}(k^{-1}), \quad k \to \infty, \tag{44b}$$

$$M^{\mathrm{sol}}(x,t,k) = \mathrm{O}\big((k \mp A)^{-\frac{1}{4}}\big), \quad k \to \pm A, \tag{44c}$$

and with conditions at $k = 0$ of type (40):

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^+}} k(M^{\mathrm{sol}})^{[1]}(x,t,k) = d_0\,\mathrm{e}^{-2Ax}(M^{\mathrm{sol}})^{[2]}_+(x,t,0), \tag{45a}$$

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^-}} k(M^{\mathrm{sol}})^{[2]}(x,t,k) = -d_0\,\mathrm{e}^{-2Ax}(M^{\mathrm{sol}})^{[1]}_-(x,t,0), \tag{45b}$$

for some $d_0 = \frac{\gamma_+}{a_{10}}$, with $|\gamma_+| = 1$.

In the reflectionless case, the spectral functions $a_1(k)$ and $a_2(k)$ are as follows (see the trace formula in [1, Section 3]):

$$a_1(k) = \frac{k + f(k) - \mathrm{i}A}{k + f(k) + \mathrm{i}A} \quad \text{and} \quad a_2(k) = \frac{k + f(k) + \mathrm{i}A}{k + f(k) - \mathrm{i}A}. \tag{46}$$

From (46) we have $a_{10} = -\frac{\mathrm{i}}{2A}$ (see (34)), which implies that

$$d_0 = 2A\mathrm{e}^{\mathrm{i}\phi_0} \quad \text{with some } \phi_0 \in \mathbb{R}. \tag{47}$$

The jump and singularity conditions (44a) and (45) imply that the solution of the RH problem above can be written in the form

$$M^{\mathrm{sol}}(x,t,k) = N(x,t,k)\mathcal{E}_2(k), \quad k \in \mathbb{C}\setminus\{\pm A, 0\}, \tag{48}$$

where $\mathcal{E}_2(k)$ is defined in (8) and $N(x,t,k) = I + \frac{N_1(x,t)}{k}$ with some matrix $N_1(x,t)$. On the other hand, conditions (45) imply that $M_+(x,t,k)$ can be written as follows:

$$M_+(x,t,k) = \begin{pmatrix} \alpha(x,t) & 0 \\ 0 & \beta(x,t) \end{pmatrix}\left(\begin{pmatrix} d_0\mathrm{e}^{-2Ax} & 1 \\ d_0\mathrm{e}^{-2Ax} & 1 \end{pmatrix} + P(x,t)k + \mathrm{O}(k^2)\right)\begin{pmatrix} 1/k & 0 \\ 0 & 1 \end{pmatrix}, \quad k \to 0, \tag{49}$$

with some scalars $\alpha(x,t)$, $\beta(x,t)$, and a matrix-valued function $P(x,t)$. Then, using the relation $N(x,t,k) = M^{\mathrm{sol}}(x,t,k)\mathcal{E}_2^{-1}(k)$ and

$$\mathcal{E}_{2+}^{-1}(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{\mathrm{i}k}{2\sqrt{2}A}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} + \mathrm{O}(k^2), \quad k \to 0, \tag{50}$$

we conclude that α, β, and N_1 are independent of t. Moreover,

$$N_1(x) = d_0\mathrm{e}^{-2Ax}\begin{pmatrix} \alpha(x) & 0 \\ \beta(x) & 0 \end{pmatrix}\mathcal{E}_{2+}^{-1}(0)$$
$$\text{with } \alpha(x) = -\beta(x) = -\frac{\sqrt{2}A}{2A + \mathrm{i}d_0\mathrm{e}^{-2Ax}}. \tag{51}$$

Thus $M^{\mathrm{sol}}(x,t,k)$ is independent of t and has the form

$$M^{\mathrm{sol}}(x,t,k) = \left(I + \frac{\mu(x)}{k}\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\right)\mathcal{E}_2(k) \tag{52}$$

with $\mu(x) = \frac{Ad_0\mathrm{e}^{-2Ax}}{2A+\mathrm{i}d_0\mathrm{e}^{-2Ax}}$. Finally, using (41) and the notation ϕ_0 from (47), we obtain the exact one-soliton solution as follows (see [1, (3.106)] and [22, (17)]):

$$\begin{aligned} q(x,t) &= A\mathrm{e}^{-2\mathrm{i}A^2 t}\left(1 - \frac{2\mathrm{i}\mathrm{e}^{-2Ax+\mathrm{i}\phi_0}}{1+\mathrm{i}\mathrm{e}^{-2Ax+\mathrm{i}\phi_0}}\right)\\ &\equiv A\mathrm{e}^{-2\mathrm{i}A^2 t}\tanh(Ax - \mathrm{i}\phi_0/2 - \mathrm{i}\pi/4). \end{aligned} \tag{53}$$

3 Long-Time Asymptotic Analysis

3.1 Signature Table

Introduce the phase function $\theta(k,\xi)$ as follows:

$$\theta(k,\xi) := 4\xi f(k) + 2kf(k), \quad \xi := \frac{x}{4t}. \tag{54}$$

As noticed above, we can consider $\xi \geq 0$ only. In terms of $\theta(k,\xi)$, the exponentials in (37b) have the form $\mathrm{e}^{2\mathrm{i}t\theta(k,\xi)}$ or $\mathrm{e}^{-2\mathrm{i}t\theta(k,\xi)}$, and the following transformations of the basic RH problem are guided by the signature structure of $\operatorname{Im}\theta(k,\xi)$.

Since $\theta(k,\xi) = 2k^2 + 4\xi k + \mathrm{O}(1)$ as $k \to \infty$, the large k behavior of the signature table for $\operatorname{Im}\theta(k,\xi)$ is the same as for $\operatorname{Im}(4\xi k + 2k^2)$. Though the equation $\frac{\mathrm{d}}{\mathrm{d}k}\theta(k,\xi) = 0$ has two zeros for all $\xi > 0$:

$$k_1(\xi) = -\frac{1}{2}\left(\xi + \sqrt{\xi^2 + 2A^2}\right) \quad \text{and} \quad k_2(\xi) = -\frac{1}{2}\left(\xi - \sqrt{\xi^2 + 2A^2}\right), \tag{55}$$

the signature table of $\operatorname{Im}\theta(k,\xi)$ involves $k_1(\xi)$ only, see Figs. 1 and 2. Namely, one can distinguish two cases:

(1) $\xi \in (A/2, +\infty)$. In this case, the signature table of $\operatorname{Im}\theta(k,\xi)$ is as in Fig. 1. The curves separating the domains where $\operatorname{Im}\theta(k,\xi) > 0$ and $\operatorname{Im}\theta(k,\xi) < 0$ intersect at $k = k_1(\xi)$.

(2) $\xi \in (0, A/2)$. In this case, the signature table of $\operatorname{Im}\theta(k,\xi)$ is as in Fig. 2. The curves separating the domains where $\operatorname{Im}\theta(k,\xi) > 0$ and $\operatorname{Im}\theta(k,\xi) < 0$ intersect at $k = -2\xi$. This is because of

$$\operatorname{Im}\theta_\pm(k,\xi) = \pm 2(2\xi + k)\sqrt{k^2 - A^2}, \quad k \in (-A, A). \tag{56}$$

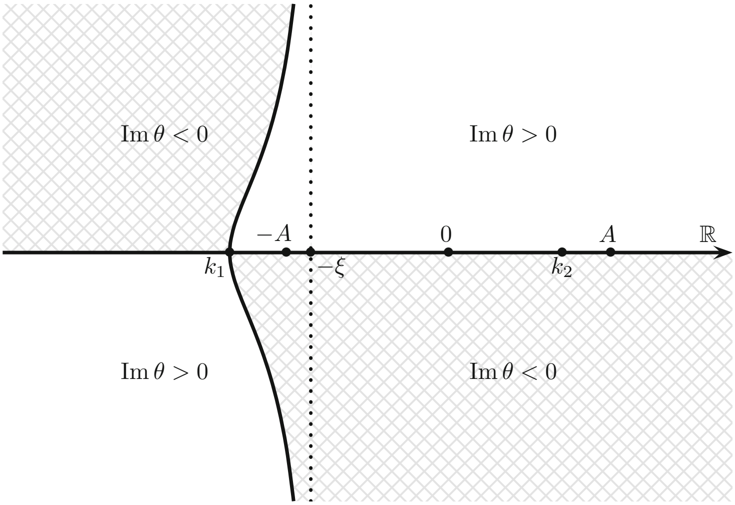

Fig. 1 Signature table of $\mathrm{Im}\,\theta(k,\xi)$ in the modulated wave region $\xi > A/2$.

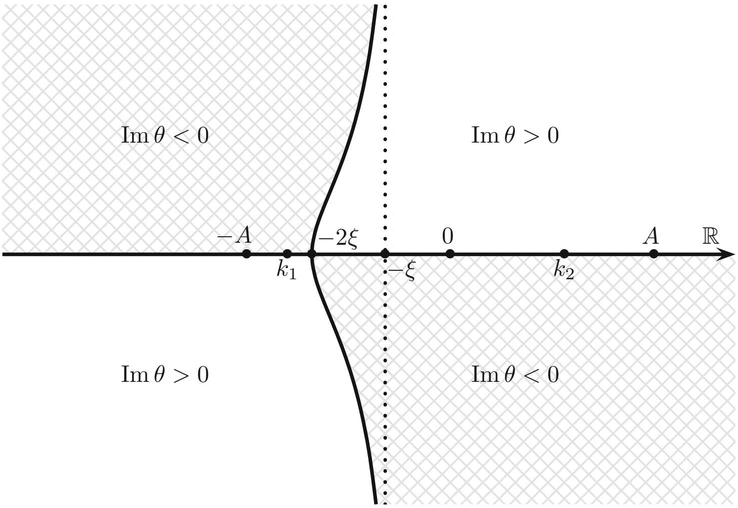

Fig. 2 Signature table of $\mathrm{Im}\,\theta(k,\xi)$ in the central region $0 < \xi < A/2$.

3.2 Modulated Regions $|\xi| \in (A/2, \infty)$

Taking into account the signature structure of $\operatorname{Im}\theta(k,\xi)$ for $-\xi \in (-\infty, -A/2)$ (see Fig. 1), we will use two different triangular factorizations of the jump matrix $J(x,t,k)$ for $k \in \mathbb{R} \setminus [-A, A]$ (cf. [10, 25, 32]):

$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)\mathrm{e}^{2\mathrm{i}t\theta}}{1+r_1(k)r_2(k)} & 1 \end{pmatrix} \begin{pmatrix} 1+r_1(k)r_2(k) & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} \end{pmatrix} \begin{pmatrix} 1 & \frac{r_2(k)\mathrm{e}^{-2\mathrm{i}t\theta}}{1+r_1(k)r_2(k)} \\ 0 & 1 \end{pmatrix}, \quad k \in (-\infty, k_1), \tag{57a}$$

and

$$J(x,t,k) = \begin{pmatrix} 1 & r_2(k)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1(k)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \quad k \in (k_1, -A) \cup (A, \infty). \tag{57b}$$

For getting rid of the diagonal factor in (57a), we introduce the scalar function $\delta(k, k_1)$ as the solution of the following RH problem:

$$\begin{aligned} &\delta_+(k,k_1) = \delta_-(k,k_1)(1 + r_1(k)r_2(k)), && k \in (-\infty, k_1), \\ &\delta(k,k_1) \to 1, && k \to \infty. \end{aligned} \tag{58}$$

The jump function $1 + r_1(k)r_2(k)$ in (58) is, in general, complex-valued for $k \in (-\infty, k_1)$, which is an important difference comparing with the problems for the local equations, where it is real [7, 8, 13, 25]. The nonzero imaginary part of $1 + r_1(k)r_2(k)$ is responsible for the singularity (or zero, depending on the sign) of δ at the endpoint $k = k_1$, which follows from the integral representation for $\delta(k, k_1)$ (cf. [32]):

$$\delta(k,k_1) = \exp\left\{ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{k_1} \frac{\ln(1 + r_1(\zeta)r_2(\zeta))}{\zeta - k} \mathrm{d}\zeta \right\}. \tag{59}$$

Integrating by parts one concludes that

$$\delta(k,k_1) = (k - k_1)^{\mathrm{i}\nu(k_1)} \mathrm{e}^{\chi(k,k_1)}, \tag{60}$$

where

$$\begin{aligned} \chi(k,k_1) &:= -\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{k_1} \ln(k - \zeta)\, \mathrm{d}\ln(1 + r_1(\zeta)r_2(\zeta)), \\ \nu(k_1) &:= -\frac{1}{2\pi} \ln(1 + r_1(k_1)r_2(k_1)) \end{aligned} \tag{61}$$

$$= -\frac{1}{2\pi}\ln|1 + r_1(k_1)r_2(k_1)| - \frac{\mathrm{i}}{2\pi}\Delta(k_1), \tag{62}$$

$$\Delta(k_1) := \int_{-\infty}^{k_1} \mathrm{d}\arg(1 + r_1(\zeta)r_2(\zeta)). \tag{63}$$

To obtain the asymptotics in the modulated regions (see Theorem 3.2 below) we need an additional assumption on the spectral functions (cf. [32]):

Assumption 3.1 (On the Spectral Functions r_1 and r_2)

$$\int_{-\infty}^{k} \mathrm{d}\arg(1 + r_1(\zeta)r_2(\zeta)) \in (-\pi, \pi), \quad \text{for all} \quad k \in (-\infty, -A). \tag{64}$$

This implies that $|\mathrm{Im}\,\nu(k_1)| < \frac{1}{2}$ and, consequently, $\delta^{\sigma_3}(k, k_1)$ has a square integrable singularity at $k = k_1$.

3.2.1 1st Transformation

Using the function $\delta(k, k_1)$ we make the following transformation of $M(x, t, k)$:

$$M^{(1)}(x, t, k) = M(x, t, k)\delta^{-\sigma_3}(k, k_1), \quad k \in \mathbb{C} \setminus \mathbb{R}. \tag{65}$$

Then $M^{(1)}(x, t, k)$ solves the following RH problem:

$$M_+^{(1)}(x, t, k) = M_-^{(1)}(x, t, k)J^{(1)}(x, t, k), \qquad k \in \mathbb{R} \setminus \{\pm A\}, \tag{66a}$$

$$M^{(1)}(x, t, k) = I + \mathrm{O}(k^{-1}), \qquad k \to \infty, \tag{66b}$$

$$M^{(1)}(x, t, k) = \mathrm{O}\left((k \pm A)^{-\frac{1}{4}}\right), \qquad k \to \mp A, \tag{66c}$$

$$M^{(1)}(x, t, k) = \mathrm{O}\begin{pmatrix} (k - k_1)^p & (k - k_1)^{-p} \\ (k - k_1)^p & (k - k_1)^{-p} \end{pmatrix}, \quad k \to k_1, \ p \in (-1/2, 1/2), \tag{66d}$$

where the jump matrix $J^{(1)}(x,t,k)$ has the form

$$J^{(1)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)\delta_-^{-2}(k,k_1)}{1+r_1(k)r_2(k)}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r_2(k)\delta_+^2(k,k_1)}{1+r_1(k)r_2(k)}\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k \in (-\infty, k_1), \\ \begin{pmatrix} 1 & r_2(k)\delta^2(k,k_1)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1(k)\delta^{-2}(k,k_1)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k \in (k_1, -A) \cup (A, \infty), \\ \begin{pmatrix} 0 & -\delta^2(k,k_1) \\ \delta^{-2}(k,k_1) & 0 \end{pmatrix}, & k \in (-A, A). \end{cases} \tag{67}$$

Moreover, $M^{(1)}(x,t,k)$ satisfies singularity conditions at $k = 0$:

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^+}} k \left(M^{(1)}\right)^{[1]}(x,t,k) = \frac{\gamma_+}{a_{10}\,\delta^2(0,k_1)}\mathrm{e}^{-2Ax}\left(M^{(1)}\right)^{[2]}_+(x,t,0), \tag{68a}$$

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^-}} k \left(M^{(1)}\right)^{[2]}(x,t,k) = \frac{\gamma_-\,\delta^2(0,k_1)}{a_{20}}\mathrm{e}^{-2Ax}\left(M^{(1)}\right)^{[1]}_-(x,t,0). \tag{68b}$$

3.2.2 2nd Transformation

Now we are able to get off the real axis and to obtain a RH problem which can be approximated, as $t \to +\infty$, by an exactly solvable problem. We assume that the reflection coefficients $r_j(k)$, $j = 1,2$ can be continued into a band containing the real axis (this takes place, for example, when $q_0(x)$ converges exponentially fast to its boundary values).

Define $M^{(2)}(x,t,k)$ as follows (compare with $M^{(2)}$ in [32] and M in [25]):

$$M^{(2)} = M^{(1)} \times \begin{cases} \begin{pmatrix} 1 & \frac{-r_2(k)\delta^2(k,k_1)}{1+r_1(k)r_2(k)}\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_1; \\ \begin{pmatrix} 1 & 0 \\ -r_1(k)\delta^{-2}(k,k_1)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k \in \hat{\Omega}_2; \\ \begin{pmatrix} 1 & r_2(k)\delta^2(k,k_1)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_3; \\ \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)\delta^{-2}(k,k_1)}{1+r_1(k)r_2(k)}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k \in \hat{\Omega}_4; \\ I, & k \in \hat{\Omega}_0, \end{cases}$$

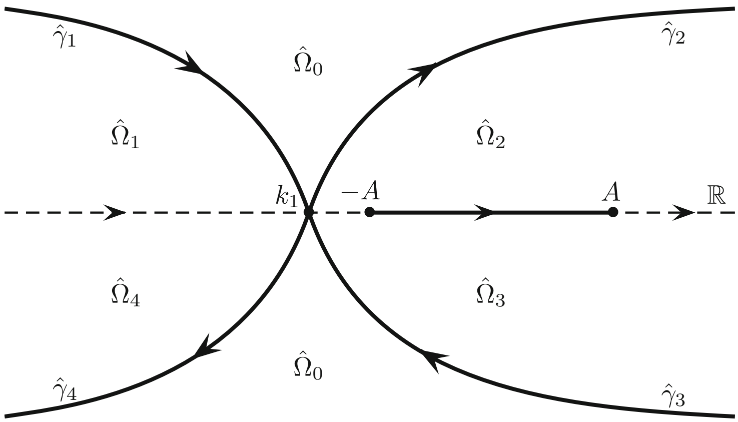

Fig. 3 Modulated wave region: contour $\hat{\Gamma} = \hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_4$ and domains $\hat{\Omega}_j$, $j = 0, \ldots, 4$.

where $\hat{\Omega}_j$, $j = 0, \ldots, 4$ are displayed in Fig. 3. Let $\hat{\Gamma} = \cup_{j=1}^{4} \hat{\gamma}_j$ be the contour also shown in Fig. 3. Then $M^{(2)}(x,t,k)$ solves the following RH problem:

$$M_+^{(2)}(x,t,k) = M_-^{(2)}(x,t,k) J^{(2)}(x,t,k), \qquad k \in \hat{\Gamma} \cup (-A, A), \tag{69a}$$

$$M^{(2)}(x,t,k) = I + \mathrm{O}(k^{-1}), \qquad k \to \infty, \tag{69b}$$

$$M^{(2)}(x,t,k) = \mathrm{O}\left((k \pm A)^{-\frac{1}{4}}\right), \qquad k \to \mp A, \tag{69c}$$

$$M^{(2)}(x,t,k) = \mathrm{O}\begin{pmatrix} (k-k_1)^p & (k-k_1)^{-p} \\ (k-k_1)^p & (k-k_1)^{-p} \end{pmatrix}, \quad k \to k_1, \ \ p \in (-1/2, 1/2), \tag{69d}$$

where, using the relations $r_{1\pm}(k) = r_{2\mp}(k)$ and $\theta_+(k) = -\theta_-(k)$ for $k \in (-A, A)$, one finds that

$$J^{(2)} = \begin{cases} \begin{pmatrix} 0 & -\delta^2(k,k_1) \\ \delta^{-2}(k,k_1) & 0 \end{pmatrix}, & k \in (-A, A); \\ \begin{pmatrix} 1 & \frac{r_2(k)\delta^2(k,k_1)}{1+r_1(k)r_2(k)} \mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\gamma}_1; \\ \begin{pmatrix} 1 & 0 \\ r_1(k)\delta^{-2}(k,k_1)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k \in \hat{\gamma}_2; \\ \begin{pmatrix} 1 & -r_2(k)\delta^2(k,k_1)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\gamma}_3; \\ \begin{pmatrix} 1 & 0 \\ \frac{-r_1(k)\delta^{-2}(k,k_1)}{1+r_1(k)r_2(k)} \mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k \in \hat{\gamma}_4. \end{cases} \tag{70}$$

Using the equalities $r_1(k) = \frac{b_+(0)}{a_{10}k} + \mathrm{O}(1)$ as $k \to 0$ with $k \in \mathbb{C}^+$, $\theta_+(0,\xi) = \mathrm{i}A\frac{x}{t}$, and $\gamma_+ = b_+(0)$ (see Remark 2.7), direct calculations show that $M^{(2)}(x,t,k) = \mathrm{O}(1)$ as $k \to 0$, $k \in \hat{\Omega}_2$. Similarly, it can be shown that $M^{(2)}(x,t,k) = \mathrm{O}(1)$ as $k \to 0$, $k \in \hat{\Omega}_3$. Thus the RH problem for $M^{(2)}$, in contrast to that for $M^{(1)}$, does not involve any singularity conditions at $k = 0$.

In view of the signature table of $\operatorname{Im}\theta(k,\xi)$ (see Fig. 1), the jump matrix $J^{(2)}(x,t,k)$ decays to the identity matrix for $k \in \hat{\Gamma}$, uniformly outside any neighborhood of the stationary phase point $k = k_1$. Arguing as, e.g., in [32, Section 3.2], we eliminate $\delta(k,\xi)$ in the jump for $k \in (-A,A)$ by introducing the scalar function

$$F(k,k_1) := \exp\left\{-\frac{f(k)}{\pi \mathrm{i}}\int_{-A}^{A}\frac{\ln\delta(\zeta,k_1)}{f_-(\zeta)(\zeta-k)}\,\mathrm{d}\zeta\right\}, \quad k \in \mathbb{C}\setminus[-A,A]. \tag{71}$$

This function $F(k,k_1)$ satisfies the jump condition

$$F_+(k,k_1)F_-(k,k_1) = \delta^2(k,k_1), \quad k \in (-A,A), \tag{72}$$

and is bounded at $k = \pm A$. In order to recover $q(x,t)$ from the solution of the RH problem, we need the large k asymptotics of $F(k,k_1)$:

$$\begin{aligned} F(k,k_1) &= \mathrm{e}^{\mathrm{i}F_\infty(k_1)} + \mathrm{O}(k^{-1}), \quad k \to \infty, \\ F_\infty(k_1) &:= -\frac{1}{\pi}\int_{-A}^{A}\frac{\ln\delta(\zeta,k_1)}{f_-(\zeta)}\,\mathrm{d}\zeta. \end{aligned} \tag{73}$$

Substituting (59) into $F_\infty(k_1)$, we have that

$$\operatorname{Re} F_\infty(k_1) = -\frac{1}{2\pi^2}\int_{-A}^{A}\frac{1}{\sqrt{A^2-\zeta^2}}\left(\int_{-\infty}^{k_1}\frac{\ln|1+r_1(s)r_2(s)|}{s-\zeta}\,\mathrm{d}s\right)\mathrm{d}\zeta, \tag{74a}$$

$$\operatorname{Im} F_\infty(k_1) = -\frac{1}{2\pi^2}\int_{-A}^{A}\frac{1}{\sqrt{A^2-\zeta^2}}\left(\int_{-\infty}^{k_1}\frac{\Delta(s)}{s-\zeta}\,\mathrm{d}s\right)\mathrm{d}\zeta, \tag{74b}$$

where $\Delta(s)$ is given by (63) and $\sqrt{A^2-\zeta^2} > 0$.

3.2.3 3rd Transformation

Using $F(k,k_1)$, we define $M^{(3)}(x,t,k)$ as follows:

$$M^{(3)}(x,t,k) = \mathrm{e}^{-\mathrm{i}F_\infty(k_1)\sigma_3}M^{(2)}(x,t,k)F^{\sigma_3}(k,k_1), \quad k \in \mathbb{C}\setminus\{\hat{\Gamma}\cup[-A,A]\}. \tag{75}$$

Then $M^{(3)}$ satisfies the following RH problem with constant jump across $(-A, A)$:

$$M_+^{(3)}(x,t,k) = M_-^{(3)}(x,t,k)J^{(3)}(x,t,k), \qquad k \in \hat{\Gamma} \cup (-A, A), \tag{76a}$$

$$M^{(3)}(x,t,k) = I + \mathrm{O}(k^{-1}), \qquad k \to \infty, \tag{76b}$$

$$M^{(3)}(x,t,k) = \mathrm{O}\left((k \pm A)^{-\frac{1}{4}}\right), \qquad k \to \mp A, \tag{76c}$$

$$M^{(3)}(x,t,k) = \mathrm{O}\begin{pmatrix} (k-k_1)^p & (k-k_1)^{-p} \\ (k-k_1)^p & (k-k_1)^{-p} \end{pmatrix}, \quad k \to k_1, \quad p \in (-1/2, 1/2), \tag{76d}$$

with

$$J^{(3)}(x,t,k) = \begin{cases} -\mathrm{i}\sigma_2, & k \in (-A, A), \\ F^{-\sigma_3}(k,k_1)J^{(2)}(x,t,k)F^{\sigma_3}(k,k_1), & k \in \hat{\Gamma}. \end{cases} \tag{77}$$

Since $F(k, k_1)$ is bounded at $k = 0$, we have $M^{(3)}(x,t,k) = \mathrm{O}(1)$ as $k \to 0$. Thus, similarly to the RH problem for $M^{(2)}$, the RH problem for $M^{(3)}$ does not involve any singularity conditions at $k = 0$.

The solution $q(x,t)$ of the Cauchy problem (1) can be expressed in terms of $M^{(3)}(x,t,k)$ as follows:

$$q(x,t) = 2\mathrm{i}e^{-2\mathrm{i}A^2t+2\mathrm{i}F_\infty(k_1)} \lim_{k\to\infty} kM_{12}^{(3)}(x,t,k), \qquad x > 0, \tag{78a}$$

$$q(x,t) = -2\mathrm{i}e^{-2\mathrm{i}A^2t+2\mathrm{i}\overline{F_\infty(k_1)}} \lim_{k\to\infty} \overline{kM_{21}^{(3)}(-x,t,k)}, \quad x < 0. \tag{78b}$$

3.2.4 Model RH Problem

Arguing as in [32], the RH problem for $M^{(3)}$ can be approximated by a model RH problem whose contour is $(-A, A)$ and whose jump matrix is constant. Using (78), we are able to obtain an asymptotics of $q(x,t)$ including at least the first decaying term [32]. For the sake of brevity, we present here, in Theorem 3.2 below, the leading (non-decaying) terms only.

Theorem 3.2 (Modulated Regions $|\xi| > A/2$) *Assume that the initial data $q_0(x)$ approaches its boundary values (1c) exponentially fast and that the associated spectral functions $a_j(k)$ and $r_j(k)$, $j = 1, 2$ satisfy Assumptions 2.5 and 3.1, respectively.*

Then the solution $q(x,t)$ *of problem* (1) *has the following long-time asymptotics along the rays* $\xi \equiv \frac{x}{4t} = \mathrm{const}$, *uniformly in any compact subset of* $\{\xi \in \mathbb{R} : |\xi| \in (A/2, +\infty)\}$*:*

$$q(x,t) = \begin{cases} A\mathrm{e}^{-2\,\mathrm{Im}\,F_\infty(k_1(|\xi|))}\mathrm{e}^{-2\mathrm{i}(A^2 t - \mathrm{Re}\,F_\infty(k_1(|\xi|)))} + E(x,t), & \xi > A/2, \\ -A\mathrm{e}^{2\,\mathrm{Im}\,F_\infty(k_1(|\xi|))}\mathrm{e}^{-2\mathrm{i}(A^2 t - \mathrm{Re}\,F_\infty(k_1(|\xi|)))} + E(x,t), & \xi < -A/2, \end{cases} \tag{79}$$

where k_1 *and* $F_\infty(k_1)$ *are defined by* (55) *and* (74)*, respectively, and with error terms* $E(x,t) = \mathrm{O}(t^{-\frac{1}{2}-\mathrm{Im}\,\nu(k_1(|\xi|))} + t^{-\frac{1}{2}+\mathrm{Im}\,\nu(k_1(|\xi|))})$.

Remark 3.3 In contrast to the plane wave regions for problems for the defocusing NLS equation [5, 15, 23, 25], the modulus of the main term in (79) depends on the direction ξ. Notice that the absolute value of the main term of the asymptotics in the plane wave regions [32] and the so-called "modulated constant" regions [33, 34] in problems for the NNLS equation with nonzero symmetric and step-like boundary conditions also depends on the direction ξ.

3.3 *Central Region* ($|\xi| \in (0, A/2)$)

For this region, in contrast to the modulated regions (see Sect. 3.2), the sign-changing critical point $k = -2\xi$ lies on the cut $(-A, A)$ (see Fig. 2). Since $\mathrm{Im}\,\theta(k,\xi)$ does not vanish on the cut ($\pm\,\mathrm{Im}\,\theta_\pm(k,\xi) < 0$ for $k \in (-A, -2\xi)$ and $\pm\,\mathrm{Im}\,\theta_\pm(k,\xi) > 0$ for $k \in (-2\xi, A)$), we are able to obtain the asymptotics with exponential precision (see [23] and [25, Section 5.5]). Moreover, no additional conditions on the winding of the argument are needed, because in the central region there is no need to deal with a model problem on the cross.

3.3.1 1st Transformation

The first transformation is similar to that in the modulated region, but with $\delta(k, -A)$ instead of $\delta(k, k_1)$ (cf. (65)):

$$M^{(1)}(x,t,k) = M(x,t,k)\delta^{-\sigma_3}(k,-A), \quad k \in \mathbb{C} \setminus \mathbb{R}. \tag{80}$$

Then $M^{(1)}(x,t,k)$ solves a RH problem similar to that in the modulated regions, but with, in general, a *strong* singularity at $k = -A$. The form of this singularity depends on whether the quantity $1 + r_1(-A)r_2(-A)$ is equal to zero or not (see Remark 2.6). Here we only consider the most complicated case, when $1 + r_1(-A)r_2(-A) = 0$.

Using the results of [20, Sections 8.1 and 8.5] about the behavior of Cauchy-type integrals at the end points and the relation $\ln(-A) = \ln A + \mathrm{i}\pi$, we have that

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{-A} \frac{\ln \frac{\zeta + A}{\zeta}}{\zeta - k} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \ln A \cdot \ln(k+A) + \frac{1}{4\pi \mathrm{i}} \ln^2(k+A) + \Phi_{-A}(k), \tag{81}$$

where $\Phi_{-A}(k)$ is analytic in a neighborhood of $k = -A$. Since

$$\int_{-\infty}^{-A} \mathrm{d} \arg(1 + r_1(\zeta) r_2(\zeta)) = \int_{-\infty}^{-A} \mathrm{d} \arg \frac{\zeta + A}{\zeta} (1 + r_1(\zeta) r_2(\zeta))$$

and $\ln^2(k+A) = \ln^2 |k+A| + \arg^2(k+A) + 2\mathrm{i} \arg(k+A) \cdot \ln(k+A)$, we obtain the following behavior of $\delta(k, -A)$ at $k = -A$:

$$\delta(k, -A) = (k+A)^{\frac{1}{2\pi}(\Delta(-A) + \arg(k+A))} \delta_{-A}(k), \tag{82}$$

where $\Delta(-A)$ is given by (63) and $\delta_{-A}(k)$ is bounded at $k = -A$. Then $M^{(1)}$ has the following behavior at $k = -A$:

$$M^{(1)}(x, t, k) = \mathrm{O} \begin{pmatrix} (k+A)^{-\frac{1}{2\pi}(\Delta(-A) + \arg(k+A)) - \frac{1}{4}} & (k+A)^{\frac{1}{2\pi}(\Delta(-A) + \arg(k+A)) - \frac{1}{4}} \\ (k+A)^{-\frac{1}{2\pi}(\Delta(-A) + \arg(k+A)) - \frac{1}{4}} & (k+A)^{\frac{1}{2\pi}(\Delta(-A) + \arg(k+A)) - \frac{1}{4}} \end{pmatrix}, \quad k \to -A. \tag{83}$$

3.3.2 2nd Transformation

Further, we define $M^{(2)}(x, t, k)$ as in Sect. 3.2.2 for the modulated wave case, but with domains $\hat{\Omega}_j$, $j = 0, \ldots, 4$ displayed in Fig. 4. In that case (see Fig. 4) the points of intersection $\hat{k}_1$ and $\hat{k}_2$ of the real axis with $\hat{\gamma}_1$ and $\hat{\gamma}_4$, then with $\hat{\gamma}_2$ and $\hat{\gamma}_3$ are simply chosen such that $-A < \hat{k}_1 < -2\xi < \hat{k}_2 < 0$. Since $1 + r_1(k) r_2(k)$ has a simple zero at $k = -A$, choosing $\arg(k+A) \in (2\pi, 3\pi)$ for $k \in \mathbb{C}^+$ in the second column of $M^{(1)}$ as $k \to -A$ and $\arg(k+A) \in (-3\pi, -2\pi)$ for $k \in \mathbb{C}^-$ in the first column of $M^{(1)}$ as $k \to -A$ (see (83)) we obtain the behavior (83) for $M^{(2)}$ with $\arg(k+A) \in (-\pi, \pi)$. Moreover, similarly to Sect. 3.2, $k = 0$ lies on the boundary of the domains $\hat{\Omega}_2$ and $\hat{\Omega}_3$ and thus $M^{(2)}(x, t, k)$ turns to be bounded at $k = 0$ as well.

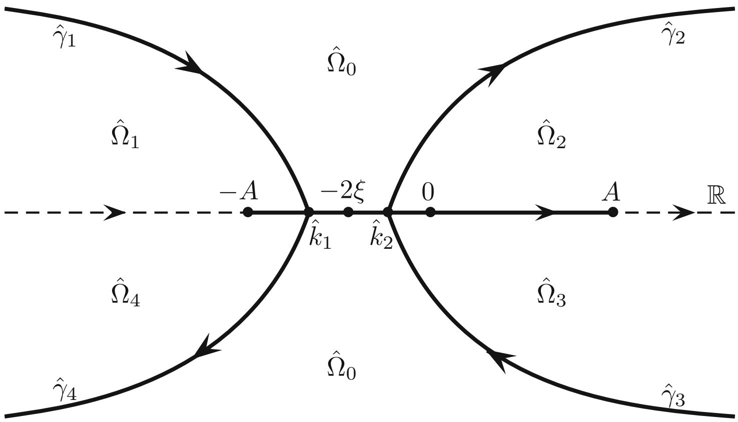

Fig. 4 Central region: contour $\hat\Gamma = \hat\gamma_1 \cup \cdots \cup \hat\gamma_4$ and domains $\hat\Omega_j$, $j = 0, \ldots, 4$.

3.3.3 3rd Transformation

We define $M^{(3)}(x,t,k)$ as in Sect. 3.2.3, but with $F(k,-A)$ instead of $F(k,k_1)$. From (82) and [20, Section 8.6] we conclude that $F(k,-A)$ behaves at $k=-A$ as follows:

$$F(k,-A) = (k+A)^{\frac{1}{2\pi}(\Delta(-A)+\arg(k+A))} F_{-A}(k), \tag{84}$$

where $F_{-A}(k)$ is bounded at $k=-A$. Therefore, $M^{(3)}(x,t,k) = \mathrm{O}\big((k+A)^{-\frac14}\big)$ as $k \to -A$. The jump matrix $J^{(3)}$ associated with $M^{(3)}$ is defined similarly to (77), with $F(k,k_1)$ replaced by $F(k,-A)$ and with the contour $\hat\Gamma$ displayed in Fig. 4.

3.3.4 Model RH Problem

Taking into account that $J^{(3)}(x,t,k)$, $k \in \hat\Gamma$ (see Fig. 4) approaches exponentially fast the identity matrix (as $t \to +\infty$), uniformly with respect to $k \in \hat\Gamma$, we arrive at the following asymptotics for $q(\pm x, t)$:

$$q(x,t) = 2\mathrm{i}\mathrm{e}^{-2\mathrm{i}A^2t+2\mathrm{i}F_\infty(-A)} \lim_{k\to\infty} kM_{12}^{\mathrm{mod}}(k) + \mathrm{O}(\mathrm{e}^{-ct}), \qquad x>0,\ t\to+\infty, \tag{85a}$$

$$q(-x,t) = -2\mathrm{i}\mathrm{e}^{-2\mathrm{i}A^2t+2\mathrm{i}\overline{F_\infty(-A)}} \lim_{k\to\infty} \overline{kM_{21}^{\mathrm{mod}}(k)} + \mathrm{O}(\mathrm{e}^{-ct}), \quad x>0,\ t\to+\infty, \tag{85b}$$

with some $c > 0$, and where $M^{\mathrm{mod}}(k)$ is analytic in $\mathbb{C} \setminus [-A, A]$ and solves the following RH problem with constant jump matrix across the contour $(-A, A)$:

$$M^{\mathrm{mod}}_{+}(k) = -\mathrm{i}M^{\mathrm{mod}}_{-}(k)\sigma_2, \qquad k \in (-A, A), \tag{86a}$$

$$M^{\mathrm{mod}}(k) = I + \mathrm{O}(k^{-1}), \qquad k \to \infty, \tag{86b}$$

$$M^{\mathrm{mod}}(k) = \mathrm{O}\left((k \pm A)^{-\frac{1}{4}}\right), \quad k \to \mp A. \tag{86c}$$

From (18) it follows that $M^{\mathrm{mod}}(k) = \mathcal{E}_2(k)$. Combining this with (85), we arrive at

Theorem 3.4 (Unmodulated Regions $0 < |\xi| < A/2$**)** *Assume that the initial data* $q_0(x)$ *approaches exponentially fast its boundary values* (1c) *and that the associated spectral functions* $a_j(k)$*,* $j = 1, 2$ *satisfy Assumptions* 2.5.

Then the solution $q(x,t)$ *of problem* (1) *has the following long-time asymptotics along the rays* $\xi = \frac{x}{4t} = \text{const}$*, uniformly in any compact subset of* $\{\xi \in \mathbb{R} : |\xi| \in (0, A/2)\}$*:*

$$q(x,t) = \begin{cases} A\mathrm{e}^{-2\,\mathrm{Im}\,F_\infty(-A)}\mathrm{e}^{-2\mathrm{i}(A^2t - \mathrm{Re}\,F_\infty(-A))} + \mathrm{O}(\mathrm{e}^{-ct}), & 0 < \xi < A/2, \\ -A\mathrm{e}^{2\,\mathrm{Im}\,F_\infty(-A)}\mathrm{e}^{-2\mathrm{i}(A^2t - \mathrm{Re}\,F_\infty(-A))} + \mathrm{O}(\mathrm{e}^{-ct}), & -A/2 < \xi < 0, \end{cases} \tag{87}$$

with some $c > 0$ *independent of* ξ*. Here* $F_\infty(-A)$ *is given by* (74) *with* $k_1 = -A$*.*

Remark 3.5 The asymptotics in the central (unmodulated) regions is established without additional restrictions on the winding of the argument of the spectral data (cf. Theorem 3.2 and, e.g., [32, 34]). To the best of our knowledge, it is the first discovered zone for nonlocal integrable equations where the asymptotics of the solution does not depend on the behavior of the argument of a dedicated spectral function.

Remark 3.6 The asymptotics of $q(x,t)$ for $\xi \in (-A/2, 0)$ and $\xi \in (0, A/2)$ does not depend on the direction ξ. However, both $|q(x,t)|$ and $\arg q(x,t)$ depend on the initial data through $F_\infty(-A)$.

The central region can be compared with the central plateau zone for the defocusing NLS equation, where the asymptotics is also obtained with exponential precision, but the modulus of the solution does not depend on the initial data [5, 15, 23, 25].

Remark 3.7 Since $k_1(\frac{A}{2}) = -A$, the main terms in the unmodulated regions, see (87), match those in the modulated regions (see (79)) at $\xi = \pm\frac{A}{2}$.

Remark 3.8 The asymptotic formulas (87) do not match as $\xi \to \pm 0$. However, in the central region, the solution $q(x,t)$ can approach a tanh-like function as $t \to +\infty$ (see Theorem 3.9 below).

3.4 Transition at $\xi = 0$

In this section we analyse the asymptotics of the solution as $\xi \to \pm 0$. For this, we consider (x,t) with $x = x_0 > 0$ *fixed* and $t \to +\infty$.

3.4.1 First Transformations

We perform three transformations of the basic RH problem similar to those made in Sect. 3.3. However, since $\xi \to +0$, we choose the contour $\hat{\Gamma}$ (see Fig. 5) such that its points of intersection $\hat{k}_1$ and $\hat{k}_2$ with the real axis satisfy $-A < \hat{k}_1 < 0 < \hat{k}_2 < A$.

In contrast to the cases presented in Sects. 3.2 and 3.3, now the point $k = 0$ lies on the boundary of $\hat{\Omega}_0$. It follows that the RH problems for both $M^{(2)}(x,t,k)$ and $M^{(3)}(x,t,k)$ involve singularity conditions at $k = 0$; particularly, these conditions for $M^{(3)}$ read as follows:

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^+}} k\big(M^{(3)}\big)^{[1]}(x,t,k) = \frac{\gamma_+ F_+^2(0,-A)}{a_{10}\,\delta^2(0,-A)} \mathrm{e}^{-2Ax}\big(M^{(3)}\big)_+^{[2]}(x,t,0), \tag{88a}$$

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^-}} k\big(M^{(3)}\big)^{[2]}(x,t,k) = \frac{\gamma_- \delta^2(0,-A)}{a_{20}\,F_-^2(0,-A)} \mathrm{e}^{-2Ax}\big(M^{(3)}\big)_-^{[1]}(x,t,0). \tag{88b}$$

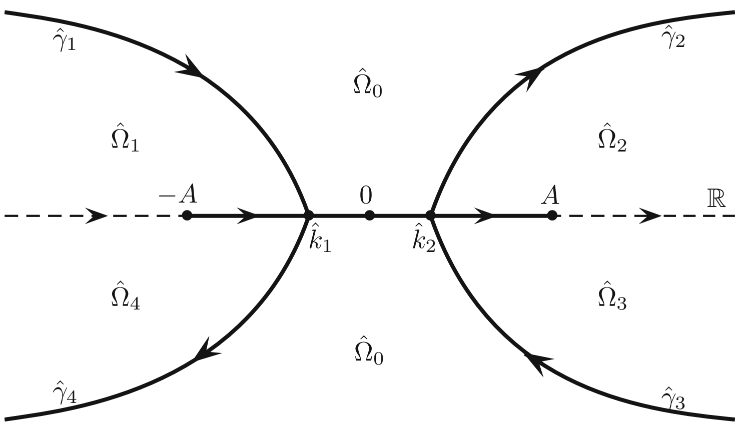

Fig. 5 Transition region: contour $\hat{\Gamma} = \hat{\gamma}_1 \cup \cdots \cup \hat{\gamma}_4$ and domains $\hat{\Omega}_0, \ldots, \hat{\Omega}_4$.

3.4.2 Model RH Problem

The solution $M^{(3)}(x,t,k)$ of the RH problem relative to the contour $\hat{\Gamma} \cup (-A, A)$ (see Fig. 5) can be approximated by the solution $M^{\mathrm{mod}}(x,k)$ of a model problem, which is as follows (cf. (44) and (45)):

$$M_+^{\mathrm{mod}}(x,k) = -\mathrm{i} M_-^{\mathrm{mod}}(x,k)\sigma_2, \quad k \in (-A, A), \tag{89a}$$

$$M^{\mathrm{mod}}(x,k) = I + \mathrm{O}(k^{-1}), \quad k \to \infty, \tag{89b}$$

$$M^{\mathrm{mod}}(x,k) = \mathrm{O}\big((k \pm A)^{-\frac{1}{4}}\big), \quad k \to \mp A, \tag{89c}$$

with singularity conditions at $k = 0$:

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^+}} kM^{\mathrm{mod}[1]}(x,k) = \frac{\gamma_+ F_+^2(0,-A)}{a_{10}\,\delta^2(0,-A)} \mathrm{e}^{-2Ax} M_+^{\mathrm{mod}[2]}(x,0), \tag{90a}$$

$$\lim_{\substack{k\to 0,\\ k\in\mathbb{C}^-}} kM^{\mathrm{mod}[2]}(x,k) = \frac{\gamma_- \delta^2(0,-A)}{a_{20}\,F_-^2(0,-A)} \mathrm{e}^{-2Ax} M_-^{\mathrm{mod}[1]}(x,0). \tag{90b}$$

Indeed, writing

$$M^{(3)}(x,t,k) = M^{\mathrm{err}}(x,t,k)M^{\mathrm{mod}}(x,k), \tag{91}$$

M^{err} satisfies the following RH problem on the contour $\hat{\Gamma}$:

$$M_+^{\mathrm{err}}(x,t,k) = M_-^{\mathrm{err}}(x,t,k)J^{\mathrm{err}}(x,t,k), \quad k \in \hat{\Gamma}, \tag{92a}$$

$$M^{\mathrm{err}}(x,k) = I + \mathrm{O}(k^{-1}), \quad k \to \infty, \tag{92b}$$

$$M^{\mathrm{err}}(x,k) = \mathrm{O}\left((k \pm A)^{-\frac{1}{2}}\right), \quad k \to \mp A, \tag{92c}$$

where $J^{\mathrm{err}}(x,t,k)$, $k \in \hat{\Gamma}$ can be uniformly estimated with exponentially small error for large t:

$$J^{\mathrm{err}}(x,t,k) = M^{\mathrm{mod}}(x,k)(I + \mathrm{O}(\mathrm{e}^{-ct}))(M^{\mathrm{mod}})^{-1}(x,k), \quad t \to +\infty, \tag{93}$$

with some $c > 0$ which does not depend on x. It follows that for all x such that $2A + \mathrm{i}d(A)\mathrm{e}^{-2Ax} \neq 0$ (see (52)),

$$M_1^{\mathrm{err}}(x,t) := \lim_{k\to\infty} k\big(M^{\mathrm{err}}(x,t,k) - I\big) = \frac{\mathrm{O}(\mathrm{e}^{-ct})}{2A + \mathrm{i}d(A)\mathrm{e}^{-2Ax}}, \quad t \to +\infty, \tag{94}$$

where $\mathrm{O}(\mathrm{e}^{-ct})$ is independent of x and

$$d(A) := \frac{\gamma_+ F_+^2(0,-A)}{a_{10}\,\delta^2(0,-A)}, \tag{95}$$

with $\delta(k,-A)$ and $F(k,-A)$ given by (59) and (71), respectively. From (89) and (94) we conclude that $q(x,t)$ and $q(-x,t)$ can be found in terms of the solution $M^{\mathrm{mod}}(x,k)$ as follows:

$$q(x,t) = 2\mathrm{i}\mathrm{e}^{-2\mathrm{i}A^2t+2\mathrm{i}F_\infty(-A)} \lim_{k\to\infty} k\tilde{M}_{12}(x,k) + \mathrm{O}(\mathrm{e}^{-ct}), \qquad x>0,\ t\to+\infty, \tag{96a}$$

$$q(-x,t) = -2\mathrm{i}\mathrm{e}^{-2\mathrm{i}A^2t+2\mathrm{i}\overline{F_\infty(-A)}} \lim_{k\to\infty} \overline{k\tilde{M}_{21}(x,k)} + \mathrm{O}(\mathrm{e}^{-ct}), \quad x>0,\ t\to+\infty. \tag{96b}$$

Then, arguing as in Sect. 2.4, we can explicitly solve the RH problem for $M^{\mathrm{mod}}(x,k)$ and thus arrive at

Theorem 3.9 (Transition at $\xi=0$) *Assume that the initial data $q_0(x)$ approaches exponentially fast its boundary values (1c) and that the associated spectral functions $a_j(k)$, $j=1,2$ satisfy Assumptions 2.5.*

Then the solution $q(x,t)$ of problem (1) has the following asymptotics as $t\to+\infty$ along the rays $x=\mathrm{const}$, excluding $x=0$ and also $x=x' := \frac{1}{2A}\ln\frac{-\mathrm{i}d(A)}{2A}$ if x' is real and positive, and $x=x'' := -\frac{1}{2A}\ln\frac{\mathrm{i}\overline{d(A)}}{2A}$ if x'' is real and negative:

$$q(x,t) = \begin{cases} A\mathrm{e}^{-2\,\mathrm{Im}\,F_\infty(-A)}\mathrm{e}^{-2\mathrm{i}(A^2t-\mathrm{Re}\,F_\infty(-A))}\cdot\frac{2A-\mathrm{i}d(A)\mathrm{e}^{-2Ax}}{2A+\mathrm{i}d(A)\mathrm{e}^{-2Ax}} + \mathrm{O}(\mathrm{e}^{-ct}), & x>0,\\ -A\mathrm{e}^{2\,\mathrm{Im}\,F_\infty(-A)}\mathrm{e}^{-2\mathrm{i}(A^2t-\mathrm{Re}\,F_\infty(-A))}\cdot\frac{2A\mathrm{e}^{-2Ax}+\mathrm{i}\overline{d(A)}}{2A\mathrm{e}^{-2Ax}-\mathrm{i}\overline{d(A)}} + \mathrm{O}(\mathrm{e}^{-ct}), & x<0, \end{cases} \tag{97}$$

with some $c>0$ independent of x. Here $F_\infty(-A)$ and $d(A)$ are given by (74) and (95), respectively.

Remark 3.10 As $x\to\pm\infty$, the main terms in (97) match those in (87).

Remark 3.11 The main term of the asymptotics in (97) is continuous at $x=0$ only if $d(A)$ and $\mathrm{Im}\,F_\infty(-A)$ satisfies one of the two conditions:

- $\mathrm{Im}\,F_\infty(-A)=0$ and $|d(A)|=2A$ with $d(A)\neq 2\mathrm{i}A$,
- $d(A)=-2\mathrm{i}A$ (without condition on $\mathrm{Im}\,F_\infty(-A)$).

Appendix: Proof of Proposition 2.2

Proof of item (ii) Substituting $q_{0,R}(x)$ with $R = 0$ (see (28)) to (4), we obtain that $\Psi_j(0,0,k) = \mathcal{E}_j(k)$, $j = 1, 2$. Using (20), we have $S(k) = \mathcal{E}_2^{-1}(k)\mathcal{E}_1(k)$, which implies (32) in view of (21).

Proof of item (i) For the initial data $q_{0,R}(x)$ with $R > 0$, from the integral representations (4) we have that

$$\Psi_2(R,0,k) = \mathcal{E}_2(k) \tag{A.1}$$

and that the (11) and (12) entries of $\Psi_1(x,0,k)$ satisfy the following integral equations for $x \in [-R, R]$:

$$(\Psi_1)_{11}(x,0,k) = \\ e_1(k) + 2Ae_1(k)e_2(k)\int_{-R}^{x}\left(1 - \mathrm{e}^{2\mathrm{i}f(k)(x-y)}\right)(\Psi_1)_{11}(y,0,k)\,\mathrm{d}y, \tag{A.2a}$$

$$(\Psi_1)_{12}(x,0,k) = \\ -e_2(k) - 2Ae_1(k)e_2(k)\int_{-R}^{x}\left(1 - \mathrm{e}^{-2\mathrm{i}f(k)(x-y)}\right)(\Psi_1)_{12}(y,0,k)\,\mathrm{d}y, \tag{A.2b}$$

where

$$e_1(k) := \frac{1}{2}\left(w(k) + \frac{1}{w(k)}\right), \quad e_2(k) := \frac{\mathrm{i}}{2}\left(w(k) - \frac{1}{w(k)}\right), \tag{A.3}$$

with $w(k)$ given in (8). The entries $(\Psi_1)_{21}(x,0,k)$ and $(\Psi_1)_{22}(x,0,k)$ can be expressed in terms of $(\Psi_1)_{11}(x,0,k)$ and $(\Psi_1)_{12}(x,0,k)$ as follows (for $x \in [-R, R]$):

$$(\Psi_1)_{21}(x,0,k) = e_2(k) + 2A\int_{-R}^{x}\left(e_2^2(k) + e_1^2(k)\mathrm{e}^{2\mathrm{i}f(k)(x-y)}\right)(\Psi_1)_{11}(y,0,k)\,\mathrm{d}y, \tag{A.4a}$$

$$(\Psi_1)_{22}(x,0,k) = e_1(k) + 2A\int_{-R}^{x}\left(e_1^2(k) + e_2^2(k)\mathrm{e}^{-2\mathrm{i}f(k)(x-y)}\right)(\Psi_1)_{12}(y,0,k)\,\mathrm{d}y. \tag{A.4b}$$

In order to find $\Psi_1(R,0,k)$, we first solve the integral equations (A.2) and then substitute the solutions into (A.4) with $x = R$. Using the equality $e_1(k)e_2(k) = -\frac{\mathrm{i}A}{2f(k)}$, equation (A.2a) can be reduced to the following Cauchy problem for a linear ordinary differential equation (where $x \in [-R, R]$):

$$\begin{cases} \frac{d^2}{dx^2}(\Psi_1)_{11}(x,0,k) - 2\mathrm{i}f(k)\frac{d}{dx}(\Psi_1)_{11}(x,0,k) + 2A^2(\Psi_1)_{11}(x,0,k) = 0, \\ (\Psi_1)_{11}(-R,0,k) = e_1(k), \qquad \frac{d}{dx}(\Psi_1)_{11}(-R,0,k) = 0. \end{cases} \tag{A.5}$$

The solution of (A.5) has the form (for $x \in [-R, R]$):

$$(\Psi_1)_{11}(x,0,k) = \frac{\mathrm{i}e_1(k)\lambda_2(k)}{2h(k)}\mathrm{e}^{\lambda_1(k)(x+R)} - \frac{\mathrm{i}e_1(k)\lambda_1(k)}{2h(k)}\mathrm{e}^{\lambda_2(k)(x+R)}, \tag{A.6}$$

where $h(k)$ and $\lambda_j(k)$, $j = 1, 2$ are given by (29) and (30), respectively. Then, substituting (A.6) into (A.4a) and using the relations $\frac{\lambda_1(k)}{\lambda_2(k)} = -\frac{f(k)h(k)+k^2}{A^2}$, $\frac{\lambda_2(k)}{\lambda_1(k)} = \frac{f(k)h(k)-k^2}{A^2}$, and $\frac{e_1^2(k)}{e_2^2(k)} = -\frac{(k+f(k))^2}{A^2}$, we obtain:

$$\begin{aligned} &(\Psi_1)_{21}(R,0,k) \\ &\quad = e_2(k) + \mathrm{i}A\frac{e_1(k)e_2^2(k)}{h(k)}\left(\frac{\lambda_2(k)}{\lambda_1(k)}\left(\mathrm{e}^{2\lambda_1(k)R}-1\right) - \frac{\lambda_1(k)}{\lambda_2(k)}\left(\mathrm{e}^{2\lambda_2(k)R}-1\right)\right) \\ &\qquad + \mathrm{i}A\frac{e_1^3(k)}{h(k)}\left(\mathrm{e}^{2\lambda_2(k)R} - \mathrm{e}^{2\lambda_1(k)R}\right) \\ &\quad = \frac{A^2e_2(k)}{2f(k)h(k)}\left(\mathrm{e}^{2\lambda_1(k)R}\left(\frac{\lambda_2(k)}{\lambda_1(k)} - \frac{e_1^2(k)}{e_2^2(k)}\right) - \mathrm{e}^{2\lambda_2(k)R}\left(\frac{\lambda_1(k)}{\lambda_2(k)} - \frac{e_1^2(k)}{e_2^2(k)}\right)\right) \\ &\quad = \frac{e_2(k)}{2h(k)}\left(\mathrm{e}^{2\lambda_1(k)R}(2k - \mathrm{i}\lambda_1(k)) - \mathrm{e}^{2\lambda_2(k)R}(2k - \mathrm{i}\lambda_2(k))\right). \end{aligned} \tag{A.7}$$

Similarly, from the integral equation (A.2b) we deduce that, for $x \in [-R, R]$,

$$(\Psi_1)_{12}(x,0,k) = \frac{\mathrm{i}e_2(k)\lambda_1(k)}{2h(k)}\mathrm{e}^{-\lambda_2(k)(x+R)} - \frac{\mathrm{i}e_2(k)\lambda_2(k)}{2h(k)}\mathrm{e}^{-\lambda_1(k)(x+R)}, \tag{A.8}$$

and, therefore, from (A.4b) we get, using that $\frac{e_2^2(k)}{e_1^2(k)} = -\frac{(f(k)-k)^2}{A^2}$:

$$\begin{aligned} &(\Psi_1)_{22}(R,0,k) \\ &\quad = e_1(k) + \mathrm{i}A\frac{e_1^2(k)e_2(k)}{h(k)}\left(\frac{\lambda_2(k)}{\lambda_1(k)}\left(\mathrm{e}^{-2\lambda_1(k)R}-1\right) - \frac{\lambda_1(k)}{\lambda_2(k)}\left(\mathrm{e}^{-2\lambda_2(k)R}-1\right)\right) \\ &\qquad + \mathrm{i}A\frac{e_2^3(k)}{h(k)}\left(\mathrm{e}^{-2\lambda_2(k)R} - \mathrm{e}^{-2\lambda_1(k)R}\right) \end{aligned}$$

$$= \frac{A^2 e_1(k)}{2f(k)h(k)} \left(\mathrm{e}^{-2\lambda_1(k)R} \left(\frac{\lambda_2(k)}{\lambda_1(k)} - \frac{e_2^2(k)}{e_1^2(k)} \right) - \mathrm{e}^{-2\lambda_2(k)R} \left(\frac{\lambda_1(k)}{\lambda_2(k)} - \frac{e_2^2(k)}{e_1^2(k)} \right) \right)$$

$$= \frac{e_1(k)}{2h(k)} \left(\mathrm{e}^{-2\lambda_2(k)R}(2k + \mathrm{i}\lambda_2(k)) - \mathrm{e}^{-2\lambda_1(k)R}(2k + \mathrm{i}\lambda_1(k)) \right). \tag{A.9}$$

Finally, substituting (A.1) and (A.6)–(A.9) into

$$S(k) = \mathrm{e}^{\mathrm{i}Rf(k)\sigma_3} \Psi_2^{-1}(R, 0, k) \Psi_1(R, 0, k) \mathrm{e}^{-\mathrm{i}Rf(k)\sigma_3} \tag{A.10}$$

and using the relations $e_1^2(k) = \frac{f(k)+k}{2f(k)}$ and $e_2^2(k) = \frac{f(k)-k}{2f(k)}$, we arrive at (31).

Proof of item (iii) Let the entries of the 2×2 matrix $\hat{\Psi}_1(x, k)$ satisfy (A.2) and (A.4) for $x \in [R, -R]$ (recall that here $R < 0$). Then from the integral representation for $\Psi_2(x, 0, k)$, see (4), we conclude that the entries of $\Psi_2(x, 0, k)$ can be found via $\hat{\Psi}_1(x, k)$ as follows:

$$\begin{aligned} (\Psi_2)_{11}(x, 0, k) &= (\hat{\Psi}_1)_{11}(x, k), \qquad (\Psi_2)_{12}(x, 0, k) = -(\hat{\Psi}_1)_{12}(x, k), \\ (\Psi_2)_{21}(x, 0, k) &= -(\hat{\Psi}_1)_{21}(x, k), \quad (\Psi_2)_{22}(x, 0, k) = (\hat{\Psi}_1)_{22}(x, k). \end{aligned} \tag{A.11}$$

Therefore, using the expressions for the entries of the matrix $\hat{\Psi}_1(R, k)$ obtained in the proof of item (i), we obtain $\Psi_2(R, 0, k)$. Since $\Psi_1(R, 0, k) = \mathcal{E}_1(k)$, from (A.10) and (A.11) we have (33).

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Algebras of Commuting Differential Operators for Kernels of Airy Type

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To the memory of Harold Widom, with admiration

Abstract Instances of commuting differential and integral operators were discovered by C. Tracy and H. Widom and used to derive asymptotic expansions of Fredholm determinants. Recently, we proved that all rational, symmetric Darboux transformations of the Bessel, Airy, and exponential bispectral functions vastly generalize these examples. In this paper, we give a classification of the Airy family using a differential Galois group action on the Lagrangian locus of the Airy adelic Grassmannian and initiate the study of the full algebra of differential operators commuting with an integral operator. We obtain explicit formulas for the two differential operators of lowest orders that commute with the level one and two integral operators obtained in the Darboux process. Both pairs commute with each other and, in the level one case, are shown to satisfy an algebraic relation defining an elliptic curve.

Keywords Commuting integral and differential operators · Bispectral functions · Fourier algebras · Adelic Grassmannian · Differential Galois groups

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1 Introduction

Our contribution to this volume bears a connection with a phenomenon uncovered by Craig Tracy and Harold Widom [38] in their work on level spacing in Random Matrix Theory. For a double scaling limit at the "edge of the spectrum" they observed that the resulting integral operator with the Airy kernel acting on an appropriate interval admits a commuting second order differential operator. This highly exceptional fact is put to good use in section IV of their paper where a number of asymptotic results for several quantities of interest are given.

In the context of Random Matrix Theory the existence of such a commuting pair of operators had been exploited earlier, for instance in work by M. Mehta [29] and W. Fuchs [14]. In this case one is interested in the "bulk of the spectrum" and the role of the Airy kernel is taken up by the more familiar sinc kernel. Both of these situations deal with the Gaussian Unitary Ensemble.

The consideration of either the Laguerre or the Jacobi ensembles at the "edge of the spectrum" gives rise to the Bessel kernel. This case, as well as the corresponding commuting pair of integral-differential operators is considered by C. Tracy and H. Widom in [37]. There, once again, this exceptional fact is exploited in section III to derive a number of important asymptotic results.

In this paper we concentrate on the "exceptional fact" mentioned above in three different situations relevant to Random Matrix theory. This fact had emerged in other areas of mathematics. In a ground-breaking collection of papers by D. Slepian, H. Landau and H. Pollak done at Bell labs in the 1960's [27, 28, 32–36] instances of this phenomenon were discovered and used in a key way in communication-signal processing theory. In fact, some precedents can be traced further back, see [4, 22]. For an up-to-date treatment of the numerical issues involving the prolate spheroidal function, one can see [30].

Incidentally in the case of the Bessel kernel the existence of a commuting operator was already proved by D. Slepian, while the situation of the Airy kernel appears for the first time in C. Tracy and H. Widom's paper mentioned above. The so called "prolate spheroidal wave functions," which arise in the case of the sinc kernel and their corresponding integral-differential pair of operators, have played an important role in areas far removed from signal processing that motivated the research of Slepian and collaborators. We give only two instances of this, but we are sure that other people can provide other examples: the paper by J. Kiukas and R. Werner [24] in connection with Bell's inequalities, and the program by A. Connes in connection with the Riemann hypothesis with C. Consani, M. Marcolli and H. Moscovici [10–12].

One should also mention that the Airy function itself and variants of it have played an important role in other very active areas of current research, such as quantum gravity and intersection theory on moduli space of curves, see [25, 41].

In all the three instances discussed above (namely the sinc, Bessel and Airy kernels), the commuting differential operator has been found by a direct computation that relies heavily on integration by parts. The interest in understanding and extending this exceptional phenomenon in a variety of other situations has produced some few more examples, see [5, 9, 15–17, 19, 20].

The bispectral problem formulated 1986 in [13] aimed at a conceptual understanding of the phenomenon of integral operators admitting a commuting differential operator. The idea is that all known kernels with this property are built from bispectral functions, that is functions in two complex variables that are eigenfunctions of differential operators in each of them. There has been a substantial amount of research on this problem [18, 21], which started with the classification of all bispectral differential operators of second order [13] and culminated in the classification of bipectral functions of rank 1 in [39] and the construction of bispectral functions of arbitrary rank via Darboux transformations [2, 23] and automorphisms of the first Weyl algebra [1, 3].

Since the mid 80s, the belief that the two problems, bispectrality and the existence of a commuting pair made up of a differential and an integral operator were closely connected has been driving research on both fronts. However, for a long time there no general argument proving that bispectral functions give rise to kernels of integral operators with the commutativity property. This was finally settled in [8] where it was proved to be the case for self-adjoint bispectral functions of rank 1 and 2.

More recently we proved in [6, 7] that all bispectral functions of rank 1 give rise to integral operators that reflect a differential operator rather than plain commute with it.

All of the previous results on integral operators address the construction of a single differential operator commuting with it. The purpose of this paper is to initiate the systematic study of the *algebras* of differential operators that commute with a given integral operator. We start with the Airy example considered by C. Tracy and H. Widom and consider all self-adjoint bispectral Darboux transformations. This is an infinite dimensional manifold which sits canonically in the infinite dimensional Grassmannian of all Darboux transformation from the Airy function, obtained from factorizations of polynomials of the Airy operator

$$L(x, \partial_x) = \partial^2 - x. \tag{1}$$

We give a conceptual classification of the former manifold as the fixed point set of a Lagrangian Grassmannian with respect to the canonical action of the associated differential Galois group. The Lagrangian Grassmannian in question is the sub-Grassmannian with respect to a canonical symplectic form. We consider the first two instances of self-adjoint bispectral Darboux transformations coming from factorizations of

$$(L - t_1)^2 \quad \text{and} \quad (L - t_2)^4$$

of the form P^*P for a differential operator $P(x, \partial_x)$ with rational coefficients. The corresponding bispectral functions, referred to here as *level one* and *level two* bispectral functions, are significantly more complicated than the bispectral Airy function $\mathrm{Ai}(x+z)$. The integral operators that they give rise to depend on parameters classifying different factorizations. For each integral operator, we compute explicitly the differential operators of the lowest two orders and prove that they are algebraically dependent. In the level one situation, the commuting operators have order 4 and 6. They generate the algebra of all differential operators commuting with the integral operator and satisfy an algebraic relation which happens to be an elliptic curve. In the level two situation, the lowest two commuting operators have order 10 and 12. However, we are also able to find commuting operators of order 14, 16, and 18 and to prove that these differential operators commute with each other. In a future publication, we will return to the problem of studying algebras of differential operators commuting with a fixed integral operator and will present general structural results for the algebra of differential operator commuting with all integral operators which are built from bispectral functions, and which are motivated by the examples in this paper.

This paper is written as a small token of admiration and gratitude to the amazing mathematical work of Harold Widom. Widom started mathematical life as an algebraist working with Irving Kaplansky at Chicago, before becoming mainly an analyst through the influence of Mark Kac at Cornell. This paper uses tools from both analysis and algebra, uniting Widom's dual mathematical history. His influence will be a lasting one, and we will miss him badly.

2 Bispectral Functions, Fourier Algebras and Prolate Spheroidal Type Commutativity

2.1 Bispectrality and Fourier Algebras

For an open subset $U \subseteq \mathbb{C}$, denote by $\mathfrak{D}(U)$ the algebra of differential operators on U with meromorphic coefficients.

Definition 1 ([13]) Let U and V be two domains in $\mathbb{C}$. A nonconstant meromorphic function $\Psi(x, z)$ defined on $U \times V \subseteq \mathbb{C}^2$ is called *bispectral* if there exist differential operators $B(x, \partial_x) \in \mathfrak{D}(U)$ and $D(z, \partial_z) \in \mathfrak{D}(V)$ such that

$$B(x, \partial_x)\Psi(x, z) = g(z)\Psi(x, z),$$

$$D(z, \partial_z)\Psi(x, z) = f(x)\Psi(x, z)$$

for some nonconstant functions $f(x)$ and $g(z)$ meromorphic on U and V, respectively.

Denote by $\mathrm{Ai}(x)$ the classical Airy function. The function

$$\Psi_{\mathrm{Ai}}(x,z) := \mathrm{Ai}(x+z)$$

is bispectral because

$$L(x,\partial_x)\Psi_{\mathrm{Ai}}(x,z) = z\Psi_{\mathrm{Ai}}(x,z) \quad \text{and} \quad L(z,\partial_z)\Psi_{\mathrm{Ai}}(x,z) = x\Psi_{\mathrm{Ai}}(x,z), \tag{2}$$

where $L(x,\partial_x)$ is the Airy operator (1). The differential equations satisfied by a bispectral function are captured by the following definition.

Definition 2 ([1]) Let $\Psi(x,z)$ be a bispectral meromorphic function defined on $U \times V \subseteq \mathbb{C}^2$. Define the *left* and *right Fourier algebras* of differential operators for Ψ by

$$\mathfrak{F}_x(\Psi) = \{R(x,\partial_x) \in \mathfrak{D}(U) : \text{there exists a differential operator } S(z,\partial_z) \in \mathfrak{D}(V) \text{ satisfying } R(x,\partial_x)\Psi(x,z) = S(z,\partial_z)\Psi(x,z)\}$$

and

$$\mathfrak{F}_z(\Psi) = \{S(z,\partial_z) \in \mathfrak{D}(V) : \text{there exists a differential operator } R(x,\partial_x) \in \mathfrak{D}(U) \text{ satisfying } R(x,\partial_x)\Psi(x,z) = S(z,\partial_z)\Psi(x,z)\}.$$

By [8, Proposition 2.4], for every bispectral meromorphic function $\Psi : U \times V \to \mathbb{C}$, there exists a canonical anti-isomorphism

$$b_\Psi : \mathfrak{F}_x(\Psi) \to \mathfrak{F}_z(\Psi),$$

given by $b_\Psi(R(x,\partial_x)) = S(z,\partial_z)$ if

$$R(x,\partial_x)\Psi(x,z) = S(z,\partial_z)\Psi(x,z).$$

We call this the *generalized Fourier map* associated to $\Psi(x,z)$. Define the *co-order* of an element $R(x,\partial_x) \in \mathfrak{F}_x(\Psi)$ by

$$\mathrm{cord} R := \mathrm{ord} b_\Psi(R).$$

Analogously, we define the co-order of $S(z,\partial_z) \in \mathfrak{F}_z(\Psi)$ by $\mathrm{cord} S := \mathrm{ord} b_\Psi^{-1}(S)$. The Fourier algebras of $\Psi(x,z)$ have natural $\mathbb{N} \times \mathbb{N}$-filtrations:

$$\mathfrak{F}_x(\Psi)^{\ell,m} = \{R(x,\partial_x) \in \mathfrak{F}_x(\Psi) : \ \mathrm{ord} R \le \ell, \mathrm{cord} R \le m\},$$
$$\mathfrak{F}_z(\Psi)^{m,\ell} = \{S(z,\partial_z) \in \mathfrak{F}_z(\Psi) : \ \mathrm{ord} S \le m, \mathrm{cord} S \le \ell\},$$

where $\mathbb{N} = \{0, 1, \ldots\}$ and $b_\Psi(\mathfrak{F}_x(\Psi)^{\ell,m}) = \mathfrak{F}_z(\Psi)^{m,\ell}$. The commutative algebras

$$\mathfrak{B}_x(\Psi) := \bigcup_{\ell \geq 0} \mathfrak{F}_x(\Psi)^{\ell,0} \quad \text{and} \quad \mathfrak{B}_z(\Psi) := \bigcup_{m \geq 0} \mathfrak{F}_z(\Psi)^{0,m}$$

are precisely the algebras of differential operators in x and z, respectively, for which $\Psi(x, z)$ is a eigenfunction.

Example 1 The Airy bispectral function $\Psi_{\text{Ai}}(x, z)$ satisfies

$$\begin{aligned} L(x, \partial_x)\Psi_{\text{Ai}}(x, z) &= z\Psi_{\text{Ai}}(x, z), \\ \partial_x \Psi_{\text{Ai}}(x, z) &= \partial_z \Psi_{\text{Ai}}(x, z), \\ x\Psi_{\text{Ai}}(x, z) &= L(z, \partial_z)\Psi_{\text{Ai}}(x, z). \end{aligned}$$

The Fourier algebras $\mathfrak{F}_x(\Psi_{\text{Ai}})$ and $\mathfrak{F}_z(\Psi_{\text{Ai}})$ coincide with the first Weyl algebra in the variables x and z, respectively, and the generalized Fourier map $b_{\Psi_{\text{Ai}}}$ is the anti-isomorphism from the first Weyl algebra in x to the first Weyl algebra in z given by

$$b_{\Psi_{\text{Ai}}}(x) = \partial_z^2 - z, \quad b_{\Psi_{\text{Ai}}}(\partial_x) = \partial_z.$$

Furthermore,

$$\dim \mathfrak{F}_x(\Psi_{\text{Ai}})^{2\ell,2m} = \ell m + \ell + m + 1,$$

see [8, Sect. 3.1 and Lemma 5.5]. On the level of Wilson's adelic Grassmannian, the anti-isomorphism b_ψ is equivalent to Wilson's bispectral involution [39]. More generally, every anti-automorphism of the first Weyl algebra determines a bispectral function as proved in [3]. □

Definition 3 A rational Darboux transformation from the bispectral Airy function $\Psi_{\text{Ai}}(x, z)$ is a function of the form

$$\Psi(x, z) := \frac{P(x, \partial_x)\Psi_{\text{Ai}}(x, z)}{q(z)p(x)} \tag{3}$$

such that

$$\Psi_{\text{Ai}}(x, z) = Q(x, \partial_x)\frac{\Psi(x, z)}{\widetilde{q}(z)\widetilde{p}(x)} \tag{4}$$

for some differential operators P and Q with polynomial coefficients and polynomials $p(x)$, $\widetilde{p}(x)$, $q(z)$ and $\widetilde{q}(z)$ with coefficients in $\mathbb{C}$. We define the *bidegree* of such a transformation to be the pair $(\text{ord} P, \text{cord} P)$.

In this setting we have $Q, P \in \mathfrak{F}_x(\Psi_{\mathrm{Ai}})$, $\widetilde{p}(x), p(x) \in \mathfrak{F}_x(\Psi_{\mathrm{Ai}})^{0,m}$ and $\widetilde{q}(x), q(x) \in \mathfrak{F}_z(\Psi_{\mathrm{Ai}})^{0,\ell}$ for some $\ell, m \in \mathbb{N}$. Furthermore, Eqs. (3)–(4) imply that

$$Q(x, \partial_x)\frac{1}{\widetilde{p}(x)p(x)}P(x, \partial_x)\Psi_{\mathrm{Ai}}(x, z) = \widetilde{q}(z)q(z)\Psi_{\mathrm{Ai}}(x, z),$$

and thus by Example 1,

$$Q(x, \partial_x)\frac{1}{\widetilde{p}(x)p(x)}P(x, \partial_x) = \widetilde{q}(L(x, \partial_x))q(L(x, \partial_x)). \tag{5}$$

Theorem 1 *[1, 3, 23] All rational Darboux transformations of the bispectral Airy function $\Psi_{\mathrm{Ai}}(x, z)$ are bispectral functions. More precisely, if $\Psi(x, z)$ is as in Definition 3, then it satisfies the spectral equations*

$$\frac{1}{p(x)}P(x, \partial_x)Q(x, \partial_x)\frac{1}{\widetilde{p}(x)}\Psi(x, z) = q(z)\widetilde{q}(z)\Psi(x, z),$$

$$\frac{1}{q(z)}b_{\Psi_{\mathrm{Ai}}}(P)(z, \partial_z)b_{\Psi_{\mathrm{Ai}}}(S)(z, \partial_z)\frac{1}{\widetilde{q}(z)}\Psi(x, z) = p(x)\widetilde{p}(x)\Psi(x, z).$$

2.2 Prolate Spheroidal Type Commutativity

A rational Darboux transformation $\Psi(x, z)$ of the bispectral Airy function of bidegree (d_1, d_2) is called *self-adjoint* if it has a presentation as in Definition 3 such that

$$Q(x, \partial_x) = P^*(x, \partial_x)$$

and $\widetilde{p}(x) = p(x), \widetilde{q}(z) = q(z)$. Here $P \mapsto P^*$ denotes the formal adjoint. It follows from (5) that P has even order. A rational Darboux transformation $\Psi(x, z)$ of the Airy bispectral function $\Psi_{\mathrm{Ai}}(x, z)$ is self-adjoint if and only if the spectral algebras $\mathfrak{B}_x(\Psi)$ and $\mathfrak{B}_z(\Psi)$ are preserved under the formal adjoint, and this condition is satisfied if and only if $\Psi(x, z)$ is an eigenfunction of nonconstant, formally symmetric differential operators in x and z (i.e., operators that are fixed by the formal adjoint), see [8, Remark 3.17 and Proposition 3.18].

For self-adjoint rational Darboux transformations $\Psi(x, z)$ of $\Psi_{\mathrm{Ai}}(x, z)$, both Fourier algebras $\mathfrak{F}_x(\Psi)$ and $\mathfrak{F}_z(\Psi)$ are preserved under the formal adjoint and

$$(b_\Psi(R))^* = b_\Psi(R^*) \quad \text{for all} \quad R \in \mathfrak{F}_x(\Psi), \tag{6}$$

see [8, Proposition 3.24 and 3.25]. Define

$$\mathfrak{F}_{x,\mathrm{sym}}(\Psi) := \{R \in \mathfrak{F}_x(\Psi) : R^* = R\}.$$

By (6) for all $R \in \mathfrak{F}_{x,\mathrm{sym}}(\Psi)$,

$$(b_\Psi(R))^* = b_\Psi R.$$

Example 2 ([8, Lemma 5.5]) For all $\ell, m \in \mathbb{N}$, $\mathfrak{F}^{2\ell,2m}_{x,\mathrm{sym}}(\Psi_{\mathrm{Ai}})$ has a basis given by

$$\{L(x,\partial_x)^j x^k + x^k L(x,\partial_x)^j : 0 \le j \le l,\ 0 \le k \le m\},$$

and in particular, $\mathfrak{F}^{2\ell,2m}_{x,\mathrm{sym}}(\Psi_{\mathrm{Ai}}) = (\ell+1)(m+1)$. □

For $\epsilon > 0$ consider the sector

$$\Sigma_\epsilon = \{re^{i\theta} \in \mathbb{C} : r > 0,\ |\theta| < \pi/6 - \epsilon\}.$$

The Airy function $\mathrm{Ai}(x)$ of the first kind is holomorphic on this domain and has the asymptotic expansion

$$\mathrm{Ai}(x) = e^{-\frac{2}{3}x^{3/2}}\left(\sum_{j=1}^{\infty} c_j x^{-j/4}\right)$$

for some $c_j \in \mathbb{R}$ where $x^{1/4}$ is interpreted as the principal 4th root of x. Furthermore, any rational Darboux transformation of $\Psi_{\mathrm{Ai}}(x,z)$ equals $\Psi(x,z) = \frac{1}{p(x)q(z)}P(x,\partial_x)\Psi_{\mathrm{Ai}}(x,z)$ for some polynomials $p(x), q(z)$ and a differential operator $P(x,\partial_x)$ with polynomial coefficients. Thus, for any bispectral Darboux transformation of $\Psi_{\mathrm{Ai}}(x,z)$ we have the asymptotic estimate

$$\|\partial_x^j \partial_z^k \Psi(x,z)\| = e^{-\frac{2}{3}(x+z)^{3/2}}\mathcal{O}((|x|+|z|)^{(j+k)/2+m})$$

on Σ_ϵ for some integer m. The transformation $z \mapsto (2/3)z^{3/2}$ sends Σ_ϵ to the sector $\{re^{i\theta} \in \mathbb{C} : r > 0,\ |\theta| < \pi/4 - 3\epsilon/2\}$. Therefore if $\Gamma_1, \Gamma_2 \subseteq \Sigma_\epsilon$ are smooth, semi-infinite curves inside this domain with parametrizations $\gamma_i(t) : [0,\infty) \to \mathbb{C}$ then the real part of $-2(\gamma_1(t)+\gamma_2(s))^{3/2}/3$ will go to $-\infty$ as $t \to \infty$ or $s \to \infty$. The above asymptotic estimate now shows that $\Psi(x,z)$ satisfies

$$\int_{\Gamma_1} |x^m z^n \partial_x^j \partial_z^k \Psi(x,z)|dx \in L^\infty(\Gamma_2) \text{ and } \int_{\Gamma_2} |x^m z^n \partial_x^j \partial_z^k \Psi(x,z)|dz \in L^\infty(\Gamma_1),$$

for every pair of smooth, semi-infinite curves $\Gamma_1, \Gamma_2 \subseteq \Sigma_\epsilon$.

Recall that the *bilinear concomitant* of a differential operator

$$R(x, \partial_x) = \sum_{j=0}^{m} d_j(x)\partial_x^j.$$

is the bilinear form $\mathcal{C}_R(-, -; p)$ defined on pairs of functions $f(x), g(x)$, which are analytic at $p \in \mathbb{C}$ by

$$\begin{aligned}\mathcal{C}_R(f, g; p) &= \sum_{j=1}^{m}\sum_{k=0}^{j-1}(-1)^k f^{(j-1-k)}(x)(d_j(x)g(x))^{(k)}|_{x=p}\\ &= \sum_{j=1}^{m}\sum_{k=0}^{j-1}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^k f^{(j-1-k)}(x)d_j(x)^{(k-\ell)}g(x)^{(\ell)}|_{x=p}.\end{aligned}$$

See for example [22, Chapter 5, Section 3].

Theorem 2 ([8]) *Let $\Psi(x, z)$ be a self-adjoint bispectral Darboux transformation of the Airy bispectral function $\Psi_{\mathrm{Ai}}(x, z)$ of bidegree (d_1, d_2) and let Γ_1 and Γ_2 be two semi-infinite, smooth curves in Σ_ϵ for some $\epsilon > 0$, whose finite endpoints are t_1 and t_2, respectively. Assume moreover that $\Psi(x, z)$ is holomorphic in a neighborhood of $\Gamma_1 \times \Gamma_2$ and that the operators in $\mathcal{F}_x(\Psi)$ and $\mathcal{F}_z(\Psi)$ have holomorphic coefficients in a neighborhood of Γ_1 and Γ_2, respectively. Then the following hold:*

1. $\dim \mathfrak{F}_{x,\mathrm{sym}}^{2\ell,2m}(\Psi) \geq (\ell + 1)(m + 1) + 1 - d_1 d_2.$
2. *If a differential operator $S(z, \partial_z) \in \mathfrak{F}_{z,\mathrm{sym}}(\Psi)$ satisfies*

$$\mathcal{C}_S(-, -; t_1) \equiv 0 \quad \textit{and} \quad \mathcal{C}_{b_\Psi^{-1}(S)}(-, -; t_2) \equiv 0,$$

then it commutes with the integral operator

$$T : f(z) \mapsto \int_{\Gamma_1} K(z, w) f(w) dw \quad \textit{with} \quad K(z, w) = \int_{\Gamma_2} \Psi(x, z)\Psi(x, w) dx.$$

3. *If* $\dim \mathfrak{F}_{z,\mathrm{sym}}^{2\ell,2\ell} \geq \ell(\ell + 1) + 2$*, in particular if $\ell = d_1 d_2$, then there exists a differential operator $S(z, \partial_z) \in \mathfrak{F}_{z,\mathrm{sym}}^{\ell,\ell}(\Psi)$ of positive order satisfying the assumption and conclusion in part (2).*

As a special case of this theorem, we are able to recover the commuting integral and differential operators studied by Tracy and Widom in [38]. In particular, if we take $\Psi = \Psi_{\mathrm{Ai}}$ in the theorem, then it guarantees the existence of a differential operator of order 2 commuting with the integral operator

$$T_{\mathrm{Ai}} f(z) \mapsto \int_{t_1}^{\infty} K_{\mathrm{Ai}}(z, w) f(w) dw,$$

with kernel

$$K_{\mathrm{Ai}}(z, w) = \int_{t_2}^{\infty} \mathrm{Ai}(x+z)\mathrm{Ai}(x+w)dx$$
$$= \frac{\mathrm{Ai}'(t_2+z)\mathrm{Ai}(t_2+w) - \mathrm{Ai}(t_2+z)\mathrm{Ai}'(t_2+w)}{z-w}.$$

Solving the associated system of linear equations for the vanishing concomitant, we discover that the differential operator

$$S_{\mathrm{Ai}}(z, \partial_z) := \partial_z(t_1 - z)\partial_z + (t_2 - t_1)z + z^2$$

satisfies the condition that $\mathcal{C}_{S_{\mathrm{Ai}}}(f, g; t_1) = 0$ for all functions f, g analytic at t_1. Its preimage under the generalized Fourier map

$$b_{\Psi}^{-1}(S_{\mathrm{Ai}}(z, \partial_z)) = \partial_x(t_2 - x)\partial_x + (t_1 - t_2)x + x^2$$

also satisfies the condition $\mathcal{C}_{b_{\Psi}^{-1}S_{\mathrm{Ai}}}(f, g; t_2) = 0$ for all functions f, g analytic at t_2. Therefore the differential operator $S_{\mathrm{Ai}}(z, \partial_z)$ commutes with T_{Ai}. This is precisely the differential operator discovered by Tracy and Widom in [38].

3 Classification of Self-adjoint Rational Darboux Transformations of the Bispectral Airy Function

In this section, we will classify the self-adjoint rational Darboux transformations of the bispectral Airy function by leveraging two tools: (1) the technology of differential Galois theory, and (2) the classification of self-adjoint Darboux transformations in terms of Lagrangian subspaces of symplectic vector spaces found in [8]. A similar classification is performed in [2] using the entirely different technique of performing an explicit asymptotic analysis of Wronskians associated to subspaces of the kernel. More explicity, in this section we wish to classify factorizations of the form

$$P(x, \partial_x)^* \frac{1}{p(x)^2} P(x, \partial_x) = q(L(x, \partial_x))^2 \tag{7}$$

where here p and q are polynomials and $P(x, \partial_x)$ is a differential operator with polynomial coefficients. Without loss of generality, we take $q(z)$ to be monic so that $p(x)$ is the leading coefficient of the operator $P(x, \partial_x)$. The associated self-adjoint rational Darboux transformation of the bispectral function $\mathrm{Ai}(x+z)$ is then defined by

$$\Psi(x, z) = \frac{1}{p(x)q(z)} P(x, \partial_x) \cdot \mathrm{Ai}(x+z).$$

3.1 Lagrangian Subspaces and Concomitant

We begin by recalling the classification of self-adjoint factorizations of self-adjoint differential operators found in [8]. To begin, let $A(x, \partial_x)$ be a differential operator and recall the standard fact that the concomitant $\mathcal{C}_A(f, g; x)$ is independent of x for all $f \in \ker(A)$ and $g \in \ker(A^*)$.

Lemma 1 ([40], Section 3) *Let $A(x, \partial_x)$ be a linear differential operator. Then the concomitant of A defines a canonical nondegenerate pairing*

$$\ker(A) \times \ker(A^*) \to \mathbb{C}, \ \ (f, g) \mapsto \mathcal{C}_A(f, g).$$

Combining this with the identity $\mathcal{C}_A(f, g) = -\mathcal{C}_{A^*}(g, f)$, we see that the concomitant restricts to a symplectic bilinear form on $\ker(A)$ when $A(x, \partial_x)$ is formally symmetric.

We will also rely on the following formula for concomitants of differential operator products.

Lemma 2 ([40], Lemma 3.6) *Let $A(x, \partial_x) = A_1(x, \partial_x)A_2(x, \partial_x)$. Then*

$$\mathcal{C}_A(f, g; x) = \mathcal{C}_{A_1}(A_2 f, g; x) + \mathcal{C}_{A_2}(f, A_1^* g; x).$$

From this, we see that if $A = A^*$, then $\ker(A_2) \subseteq \ker(A)$ and $\ker(A_1^*) \subseteq \ker(A)$ are orthogonal under the pairing defined by the concomitant of $A(x, \partial_x)$.

As is well-known in the theory of factorizations of linear differential operators, a factorization of a differential operator

$$A(x, \partial_x) = A_1(x, \partial_x)A_2(x, \partial_x)$$

corresponds to a choice of a subspace $V \subseteq \ker(A)$. The subspace V corresponds to the kernel of $A_2(x, \partial_x)$ and determines the value of the operator $A_2(x, \partial_x)$ up to a left multiple by a function of x. As is readily seen from the previous lemma, the kernel of $A_1(x, \partial_x)^*$ is completely determined by V and given by the orthogonal complement

$$V^\perp = \{g \in \ker(A^*) : \mathcal{C}_A(f, g) = 0 \ \forall f \in V\}.$$

Thus to obtain factorizations of the form (7), we search in particular for subspaces $V \subseteq \ker(q(L)^2)$ satisfying $V^\perp = V$. In other words, we search for Lagrangian subspaces of the symplectic vector space $\ker(q(L)^2)$. To summarize, we have the following proposition.

Proposition 1 *Factorizations of the form (7) with $p(x)$ and the coefficients of $P(x, \partial_x)$ not necessarily rational functions, correspond precisely to Lagrangian subspaces of the symplectic vector space $\ker(q(L)^2)$ whose symplectic form is defined by the concomitant of $q(L)^2$.*

3.2 Differential Galois Theory

Our next task is to determine the symmetric factorizations obtained in the previous section which are rational. For the convenience of the reader, we briefly outline the requisite basics of Picard-Vessiot extensions and the Fudamental Theorem of Differential Galois Theory. We direct the interested reader to [31] for a more thorough treatment.

Definition 4 Let (K, ∂) be a differential field and let $A \in K[\partial]$ be a linear differential operator with coefficients in K. The *Picard-Vessiot* extension of K associated with $A(x, \partial_x)$ is a differential field extension (F, ∂) of K whose constants all belong to K and which is generated by the solutions of the homogeneous equation $Ag = 0$.

Picard-Vessiot extensions of a differential field play precisely the role of Galois extensions in field theory. Likewise, the usual Galois group is replaced by a similar object consisting of field automorphisms respecting differentiation.

Definition 5 The *differential Galois group* $\mathrm{DGal}(F/K)$ consists of all K-linear field automorphisms $\sigma : F \to F$ of F satisfying $\sigma(\partial \cdot a) = \partial \cdot \sigma(a)$ for all $a \in F$.

Analogous to the case of Galois extensions of fields, we have the following theorem relating differential subextensions and Zariski-closed subgroups of the differential Galois group (see [31, Proposition 1.34]).

Theorem 3 (Fundamental Theorem of Differential Galois Theory) *Let (K, ∂) be a differential field whose subfield of constants is algebraically closed and let (F, ∂) be a Picard-Vessiot extension of K. Then there is a bijective correspondence between differential subfields $K \subseteq F' \subseteq F$ and Zariski-closed subgroups $G' \subseteq$ $\mathrm{DGal}(F/K)$ given by*

$$G' \subseteq \mathrm{DGal}(K/F) \mapsto K^{G'} = \{a \in K : \sigma(a) = a,\ \forall \sigma \in G'\},$$

$$K \subseteq F' \subseteq F \mapsto \mathrm{DGal}(F'/K) = \{\sigma \in \mathrm{DGal}(F/K) : \sigma(a) = a,\ \forall a \in F'\}.$$

Furthermore, this correspondence restricts to a correspondence between Picard-Vessiot subextensions of F/K and normal subgroups of $\mathrm{DGal}(F/K)$.

We will not rely on the full force of this correspondence, and therefore will not have to recall the precise nature of the topological structure of $\mathrm{DGal}(F/K)$ as a group subscheme of a general linear group. Instead, we will use only the immediate fact that

$$K = \{a \in F : \sigma(a) = a,\ \forall \sigma \in \mathrm{DGal}(F/K)\}. \tag{8}$$

Since differential operators are determined (up to a multiple) by their kernels, rationality of a differential operator may be characterized by differential Galois invariance of the associated kernel.

Theorem 4 *Let $A(x, \partial_x)$ be a differential operator with rational coefficients and let F be the Picard-Vessiot extension of $\mathbb{C}(x)$ for A. Consider a factorization $A(x, \partial_x) = A_1(x, \partial_x)A_2(x, \partial_x)$ with A_2 monic. Then $A_1(x, \partial_x)$ and $A_2(x, \partial_x)$ have rational coefficients if and only if* $\ker(A_2) \subseteq \ker(A)$ *is invariant under the action of* $\mathrm{DGal}(F/\mathbb{C}(x))$.

Proof For $\sigma \in \mathrm{DGal}(F/\mathbb{C}(x))$, let $\sigma(A_j) := \sigma(A_j)(x, \partial_x)$ denote the operator obtained by applying the automorphism to the coefficients. Since the automorphism preserves differentiation, we know that

$$\sigma(A_j)(x, \partial_x) \cdot \sigma(a) = \sigma(A_j(x, \partial_x) \cdot a), \ \forall a \in F.$$

If $A_1(x, \partial_x)$ and $A_2(x, \partial_x)$ have rational coefficients, then clearly $\sigma(A_j) = A_j$ and therefore $\ker(\sigma(A_j)) = \ker(A_j)$. Thus the kernel of $A_j(x, \partial_x)$ is invariant under the action of $\mathrm{DGal}(F/\mathbb{C}(x))$.

Conversely, suppose that $\ker(A_2) \subseteq \ker(A)$ is invariant under the action of the differential Galois group, ie. $\sigma(\ker(A_2)) = \ker(A_2)$ Then $\sigma(A_2) \cdot \sigma(a) = \sigma(A_2 \cdot a) = \sigma(0) = 0$ for all $a \in \ker(A)$ and therefore $\ker(A_2) \subseteq \ker(\sigma(A_2))$. Since the order of A_2 and the order of $\sigma(A_2)$ are the same, their kernels will have the same dimension. Therefore $\ker(\sigma(A_2)) = \ker(A_2)$ and consequently $\sigma(A_2) = bA_2$ for some $b \in F$. Since A_2 has leading coefficient 1, it follows that $b = 1$. Hence $\sigma(A_2) = A_2$ and from the Fundamental Theorem of Differential Galois Theory, the coefficients of A_2 must all be rational functions. Lastly, since A and A_2 have rational coefficients, it follows that A_1 has rational coefficients.

Corollary 1 *Let $A(x, \partial_x)$ be a self-adjoint differential operator with rational coefficients and let F be the Picard-Vessiot extension for A. Then the self-adjoint, rational factorization of $A(x, \partial_x)$ correspond precisely with the* $\mathrm{DGal}(F/\mathbb{C}(x))$*-invariant Lagrangian subspaces of* $\ker(A)$.

Proof This follows immediately from the theorem and the results of the previous subsection.

3.3 The Classification

Now let $a_1, \ldots, a_r \in \mathbb{C}$ be the distinct roots of $q(z)$ and write

$$q(z) = (z - a_1)^{d_1} \ldots (z - a_r)^{d_r}$$

for some positive integers $d_1, \dots, d_r$ and distinct $a_1, \dots, a_r \in \mathbb{C}$. The kernel of $q(L)^2$ for $L(x, \partial_x) = \partial_x^2 - x$ the Airy operator is given by the following lemma.

Lemma 3 *The kernel of $q(L)^2$ has basis given by*

$$\{\mathrm{Ai}^{(j)}(x+a_i), \mathrm{Bi}^{(j)}(x+a_i) : 1 \le k \le r,\ 0 \le j \le 2d_k\}$$

where here $\mathrm{Ai}(x)$ *and* $\mathrm{Bi}(x)$ *are the Airy functions of the first and second kind, respectively.*

Proof To prove this, we will rely on the fundamental relation

$$L(x, \partial_x)\partial_x = \partial_x L(x, \partial_x) + 1,$$

which implies that

$$(L(x, \partial_x) - a_k)^m \partial_x^n = \sum_{j=0}^{m \wedge n} \binom{m}{j} \frac{n!}{(n-j)!} \partial_x^{n-j} (L(x, \partial_x) - a_k)^{m-j}.$$

Thus for all $0 \le n < 2d_k - 1$, we have

$$\begin{aligned}(L(x, \partial_x) - a_k)^{2d_k} &\mathrm{Ai}^{(n)}(x+a_k) \\ &= \sum_{j=0}^{n} \binom{2d_k}{j} \frac{n!}{(n-j)!} \partial_x^{n-j} (L(x, \partial_x) - a_k)^{2d_k - j} \mathrm{Ai}(x+a_k) = 0.\end{aligned}$$

Hence $\mathrm{Ai}^{(n)}(x+a_k) \in \ker((L(x, \partial_x) - a_k)^{2d_k}) \subseteq \ker(q(L)^2)$ for all $0 \le n < 2d_k$. The same calculation shows that $\mathrm{Bi}^{(n)}(x+a_k) \in \ker(q(L)^2)$ for all $0 \le n < 2d_k$.

Thus the Picard-Vessiot extension of the differential field $(\mathbb{C}(x), \partial_x)$ corresponding to the linear differential operator $q(L(x, \partial_x))^2$ is finitely generated by $4r$ elements

$$F^q = \mathbb{C}(x)(\mathrm{Ai}(x+a_k), \mathrm{Ai}'(x+a_k), \mathrm{Bi}(x+a_k), \mathrm{Bi}'(x+a_k) : 1 \le k \le r).$$

Using this, we see that the differential Galois group of F is isomorphic to r copies of $\mathrm{SL}_2(\mathbb{C})$.

Lemma 4 *The differential Galois group consists of all differential $\mathbb{C}(x)$-linear morphisms*

$$\sigma : F^q \to F^q, \quad \begin{cases} \mathrm{Ai}(x+a_k) \mapsto \alpha_k \mathrm{Ai}(x+a_k) + \beta_k \mathrm{Bi}(x+a_k) \\ \mathrm{Bi}(x+a_k) \mapsto \gamma_k \mathrm{Ai}(x+a_k) + \delta_k \mathrm{Bi}(x+a_k) \end{cases} \forall 1 \le k \le r,$$

where here $\alpha_k, \beta_k, \gamma_k, \delta_k \in \mathbb{C}$ with $\alpha_k\delta_k - \beta_k\gamma_k = 1$.

Proof The fact that

$$\begin{array}{l} \mathrm{Ai}(x+a_k) \mapsto \alpha \mathrm{Ai}(x+a_k) + \beta \mathrm{Bi}(x+a_k) \\ \mathrm{Bi}(x+a_k) \mapsto \gamma \mathrm{Ai}(x+a_k) + \delta \mathrm{Bi}(x+a_k) \end{array}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C}),$$

is a differential automorphism is standard. See for example [31, Example 8.15]. Therefore, we need only show that this accounts for all differential automorphisms.

If $\sigma : F^q \to F^q$ is a differential automorphism fixing $\mathbb{C}(x)$, then

$$\frac{\sigma(\mathrm{Ai}(x+a_k))''}{\sigma(\mathrm{Ai}(x+a_k))} = \sigma\left(\frac{\mathrm{Ai}''(x+a_k)}{\mathrm{Ai}(x+a_k)}\right) = \sigma(x+a_k) = x+a_k.$$

Thus $\sigma(\mathrm{Ai}(x+a_k))$ must be a solution of the differential equation $y'' = (x+a_k)y$, and therefore a linear combination of $\mathrm{Ai}(x+a_k)$ and $\mathrm{Bi}(x+a_k)$ for all k. A similar statement holds for $\sigma(\mathrm{Bi}(x+a_k))$ so that

$$\sigma : \begin{cases} \mathrm{Ai}(x+a_k) \mapsto \alpha_k \mathrm{Ai}(x+a_k) + \beta_k \mathrm{Bi}(x+a_k) \\ \mathrm{Bi}(x+a_k) \mapsto \gamma_k \mathrm{Ai}(x+a_k) + \delta_k \mathrm{Bi}(x+a_k) \end{cases} \forall 1 \le k \le r$$

for some $\alpha_k, \beta_k, \gamma_k, \delta_k \in \mathbb{C}$. Lastly, the Wronskian identity implies

$$W(\mathrm{Ai}(x+a_k), \mathrm{Bi}(x+a_k)) = \mathrm{Ai}'(x+a_k)\mathrm{Bi}(x+a_k) - \mathrm{Ai}(x+a_k)\mathrm{Bi}'(x+a_k) = \frac{1}{\pi}\cdot$$

Since the Wronskian is skew-symmetric, we can conclude that

$$\begin{aligned} \frac{1}{\pi} &= \sigma(W(\mathrm{Ai}(x+a_k), \mathrm{Bi}(x+a_k))) \\ &= W(\sigma(\mathrm{Ai}(x+a_k)), \sigma(\mathrm{Bi}(x+a_k))) \\ &= (\alpha\delta - \beta\gamma) W(\mathrm{Ai}(x+a_k), \mathrm{Bi}(x+a_k)) = (\alpha\delta - \beta\gamma)/\pi. \end{aligned}$$

Hence $\alpha\delta - \beta\gamma = 1$.

Using this, we can obtain the following characterization of the Galois-invariant subspaces of $\ker(q(L)^2)$.

Lemma 5 *Suppose that $V \subseteq \ker(q(L)^2)$ is a subspace. Then V is invariant under the action of the differential Galois group if and only if V is spanned by* pairs *of elements of the form*

$$\sum_{j=0}^{2d_k-1} \alpha_{kj} \mathrm{Ai}^{(j)}(x+a_k), \quad \sum_{j=0}^{2d_k-1} \alpha_{kj} \mathrm{Bi}^{(j)}(x+a_k).$$

Proof Clearly any subspace spanned by pairs of elements of this form is invariant under the action of the Galois group, since the the action restricts to an action sending each of the functions in the pair to a linear combination of the functions in the pair. Thus it suffices to show the converse.

Let $f(x)$ be a nonzero element of V. Then

$$f(x) = \sum_{k=1}^{r} \sum_{j=0}^{2d_k-1} \alpha_{kj} \mathrm{Ai}^{(j)}(x+a_k) + \beta_{kj} \mathrm{Bi}^{(j)}(x+a_k).$$

Consider the differential automorphisms σ_k and τ_k which fix $\mathrm{Ai}(x+a_j)$ and $\mathrm{Bi}(x+a_j)$ and satisfy

$$\sigma_k : \mathrm{Ai}(x+a_k) \mapsto -\mathrm{Bi}(x+a_k),\ \mathrm{Bi}(x+a_k) \mapsto \mathrm{Ai}(x+a_k),$$

$$\tau_k : \mathrm{Ai}(x+a_k) \mapsto \mathrm{Ai}(x+a_k) + \mathrm{Bi}(x+a_k),\ \mathrm{Bi}(x+a_k) \mapsto \mathrm{Bi}(x+a_k).$$

We see that

$$\tau_k(f(x)) - f(x) = \sum_{j=0}^{2d_k-1} \alpha_{kj} \mathrm{Bi}^{(j)}(x+a_k) \in V.$$

Following up by applying σ_k, we see that

$$\sigma_k(\tau_k(f(x)) - f(x)) = \sum_{j=0}^{2d_k-1} \alpha_{kj} \mathrm{Ai}^{(j)}(x+a_k) \in V.$$

Likewise, one may show $\sum_{j=0}^{2d_k-1} \beta_{kj} \mathrm{Ai}^{(j)}(x+a_k),\ \sum_{j=0}^{2d_k-1} \beta_{kj} \mathrm{Bi}^{(j)}(x+a_k) \in V$ and since k was arbitrary, the statement of the Lemma follows immediately.

Our explicit description of the kernel of $q(L)^2$ allows us to give a concrete formula for the symplectic form on $\ker(q(L)^2)$ defined by the bilinear concomitant. We start with a combinatorial Lemma.

Lemma 6 *Let a, b, m be integers. Then*

$$\sum_{k=0}^{m} (-1)^k \binom{k+a}{k} \binom{b}{m-k} = \binom{b-1-a}{m}.$$

Proof We use the binomial series expansion on the identity

$$(1-z)^{-a-1}(1-z)^b = (1-z)^{b-a-1}$$

to find

$$\sum_{j,k=0}^{\infty} \binom{-a-1}{k} \binom{b}{j} (-1)^{j+k} z^{j+k} = \sum_{m=0}^{\infty} \binom{b-1-a}{m} (-1)^m z^m.$$

Now comparing coefficients of z^m:

$$\sum_{k=0}^{m} \binom{-a-1}{k} \binom{b}{m-k} (-1)^{j+k} = \binom{b-1-a}{m}.$$

Noting that $\binom{-a-1}{k} = (-1)^k \binom{k+a}{a}$, the statement of the lemma follows immediately.

Proposition 2 *Let* $f(x), g(x) \in \{\mathrm{Ai}(x), \mathrm{Bi}(x)\}$ *and choose* $0 \le m < 2d_j$ *and* $0 \le n < 2d_k$. *Then*

$$\begin{aligned} &\mathcal{C}_{q(L)^2}(f^{(m)}(x+a_j), g^{(n)}(x+a_k)) \\ &\qquad = \delta_{jk} \frac{m!n!W(f,g)}{(m+n-2d_k+1)!}\, \partial_z^{m+n-2d_k+1}\Big|_{z=a_k} \cdot \left(\frac{q(z)^2}{(z-a_k)^{2d_k}} \right) \end{aligned}$$

for all nonnegative integers m, n *with* $m+n \ge 2d_k - 1$ *and is zero otherwise.*

Proof For simplicity of notation, we will let $h(z) = q(z)^2$ and write f and g in place of $f(x+a_k)$ and $g(x+b_k)$, respectively. First note that if $j \neq k$ then $f^{(m)} \in \ker((L-a_j)^{2d_j})$ and $g^{(n)} \in \ker(h(L)(L-a_j)^{-2d_j})$, which is the orthogonal complement of the subspace $\ker((L-a_j)^{2d_j})$ of $\ker(h(L))$. Hence $\mathcal{C}_{h(L)}(f^{(m)}, g^{(n)}) = 0$. Thus it suffices to consider the case when $j = k$.

Let $\widetilde{h}(z) = h(z)/(z-a_k)^{2d_k}$. Applying Lemma 2 and the fundamental relation $L\partial_x = \partial_x L + 1$ we see that

$$\begin{aligned} \mathcal{C}_{h(L)}(f^{(m)}, g^{(n)}) &= \mathcal{C}_{h(L)\partial_x^m}(f, g^{(n)}) \\ &= \mathcal{C}_{\widetilde{h}(L)(L-a_k)^{2d_k}\partial_x^m}(f, g^{(n)}) \\ &= \sum_{s=0}^{m} \binom{m}{s} \frac{(2d_k)!}{(2d_k-s)!} \mathcal{C}_{\widetilde{h}(L)\partial_x^{m-s}(L-a_k)^{2d_k-s}}(f, g^{(n)}). \end{aligned}$$

Now using the fact that the concomitant of L is the Wronskian and again applying Lemma 2 and the more general relation

$$\widetilde{h}(L(x,\partial_x))\partial_x^m = \sum_{s=0}^{m} \binom{m}{s} \partial_x^{m-s} \widetilde{h}^{(s)}(L(x,\partial_x))$$

we see that

$$
\begin{aligned}
&\mathcal{C}_{h(L)}(f^{(m)}, g^{(n)})\\
&\quad = \sum_{s=0}^{m} \binom{m}{s} \frac{(2d_k)!}{(2d_k - s)!} (-1)^{m-s} W(f, (L - a_k)^{2d_k - s - 1} \partial_x^{m-s} \widetilde{h}(L) \partial_x^n \cdot g)\\
&\quad = \sum_{t=0}^{n} \binom{n}{t} \widetilde{h}^{(t)}(a_k) \sum_{s=0}^{m} \binom{m}{s} \frac{(2d_k)!}{(2d_k - s)!} (-1)^{m-s}\\
&\qquad\qquad W(f, (L - a_k)^{2d_k - s - 1} \partial_x^{n+m-s-t} \cdot g).
\end{aligned}
$$

From this it is clear that if $n + m < 2d_k - 1$ then the concomitant is zero. Thus without loss of generality we take $m + n \geq 2d_k - 1$. Then for $\ell = n + m - 2d_k + 1$

$$
\begin{aligned}
&\mathcal{C}_{h(L)}(f^{(m)}, g^{(n)})\\
&\quad = \sum_{t=0}^{n} \binom{n}{t} \widetilde{h}^{(t)}(a_k) \sum_{s=0}^{m} \binom{m}{s} \frac{(2d_k)!}{(2d_k - s)!} (-1)^{m-s}\\
&\qquad\qquad W(f, (L - a_k)^{2d_k - s - 1} \partial_x^{n+m-s-t} \cdot g)\\
&\quad = \sum_{t=0}^{n\wedge\ell} \binom{n}{t} \widetilde{h}^{(t)}(a_k) \sum_{s=0}^{m} \binom{m}{s} \frac{(2d_k)!}{(2d_k - s)!} (-1)^{m-s} \frac{(m+n-s-t)!}{(\ell - t)!}\\
&\qquad\qquad W(f, \partial_x^{\ell - t} \cdot g)\\
&\quad = \sum_{t=0}^{n\wedge\ell} \binom{n}{t} \widetilde{h}^{(t)}(a_k) \frac{m!(n-t)!}{(\ell - t)!} \sum_{s=0}^{m} \binom{2d_k}{s} \binom{m+n-s-t}{m-s} (-1)^{m-s}\\
&\qquad\qquad W(f, \partial_x^{\ell - t} \cdot g).
\end{aligned}
$$

Now reindexing the sum and applying the previous lemma, we obtain

$$
\begin{aligned}
&\mathcal{C}_{h(L)}(f^{(m)}, g^{(n)})\\
&= \sum_{t=0}^{n\wedge\ell} \binom{n}{t} \widetilde{h}^{(t)}(a_k) \frac{m!(n-t)!}{(\ell - t)!} \sum_{s=0}^{m} \binom{2d_k}{m-s} \binom{s+n-t}{s} (-1)^s W(f, \partial_x^{\ell - t} \cdot g)\\
&= \sum_{t=0}^{n\wedge\ell} \binom{n}{t} \widetilde{h}^{(t)}(a_k) \frac{m!(n-t)!}{(\ell - t)!} \binom{2d_k - 1 - n + t}{m} W(f, \partial_x^{\ell - t} \cdot g).
\end{aligned}
$$

The binomial coefficient in the last sum is nonzero if and only if $\ell \leq t$. Since the sum is taken between $t = 0$ and $t = \ell$, the only nonzero term comes from when $t = \ell$. Thus

$$\mathcal{C}_{h(L)}(f^{(m)}, g^{(n)}) = \frac{m!n!}{(n+m-2d_k+1)!}\widetilde{h}^{(n+m-2d_k+1)}(a_k)W(f,g).$$

The rational Darboux transformations of $\mathrm{Ai}(x+z)$ come directly from factorizations of the form (7) with $P(x, \partial_x)$ having rational coefficients. As we have outlined above, these correspond precisely to the Galois-invariant Lagrangian subspaces of $\ker(q(L)^2)$. This characterization is made explicit in the next theorem.

Theorem 5 (Classification Theorem) *Let $f_m, g_m \in \ker(q(L)^2)$ for $1 \leq m \leq d$ be $2d$ linearly independent functions of the form*

$$f_i(x) = \sum_{m=0}^{2d_{\ell_i}-1} \alpha_{im}\mathrm{Ai}^{(m)}(x + a_{\ell_i}), \quad g_i(x) = \sum_{n=0}^{2d_{\ell_i}-1} \alpha_{in}\mathrm{Bi}^{(n)}(x + a_{\ell_i})$$

satisfying the condition that

$$\sum_{m+n\geq 2d_k-1}^{2d_k-1} \alpha_{im}\alpha_{jn}\frac{m!n!}{(m+n-2d_k+1)!}\,\partial_z^{m+n-2d_k+1}\Big|_{z=a_k}\cdot\left(\frac{q(z)^2}{(z-a_k)^{2d_k}}\right) = 0$$

for all k and for all i, j with $\ell_i = \ell_j = k$. Then the differential operator $P(x, \partial_x)$ of order $2d$ defined in terms of a Wronskian by

$$P(x, \partial_x)\cdot f := W(f_1, f_2, \ldots, f_d, g_1, g_2, \ldots, g_d, f)$$

has rational coefficients and satisfies

$$P(x, \partial_x)^*\frac{1}{p(x)^2}P(x, \partial_x) = q(L(x, \partial_x))^2$$

for some rational function $p(x)$. Furthermore every self-adjoint rational factorization of $q(L(x, \partial_x))^2$ is of this form.

Proof This follows directly from our direct calculation of the concomitant along with our characterization of the Galois-invariant subspaces of the kernel.

This result is particularly nice in the situation that $q(z) = (z - s_1)^d$, so that the concomitant has the simple form

$$\mathcal{C}_{q(L)^2}(f^{(m)}(x+s_1), g^{(n)}(x+s_1)) = \begin{cases} \frac{m!n!}{\pi}, & f = \mathrm{Ai},\ g = \mathrm{Bi},\ m+n = 2d-1 \\ \frac{-m!n!}{\pi}, & f = \mathrm{Bi},\ g = \mathrm{Ai},\ m+n = 2d-1 \\ 0, & \text{otherwise.} \end{cases}$$

The payout of our dive through all the differential Galois theory and symplectic geometry above is that we immediately provide *explicit factorizations* of $(L(x,\partial_x)-s_1)^2$ and $(L(x,\partial_x)-s_1)^4$.

Corollary 2 *Let $s_1 \in \mathbb{C}$. Then up to a function multiple, the only self-adjoint rational factorizations of $(L(x,\partial_x)-s_1)^2$ are the trivial one and*

$$P_1(x,\partial_x)^* \frac{1}{(x+s_1)^2} P_1(x,\partial_x) = (L(x,\partial_x)-s_1)^2$$

for

$$P_1(x,\partial_x) = (x+s_1)\partial_x^2 - \partial_x - (x+s_1)^2.$$

Proof From the previous Theorem, we know we must choose functions

$$f_1(x) = \alpha_{11}\mathrm{Ai}(x+s_1)+\alpha_{12}\mathrm{Ai}'(x+s_1), \quad g_1(x) = \alpha_{11}\mathrm{Bi}(x+s_1)+\alpha_{12}\mathrm{Bi}'(x+s_1)$$

satisfying $2\alpha_{11}\alpha_{12}1!1!/\pi = 0$. Thus either $\alpha_{11} = 0$ or $\alpha_{12} = 0$ and without loss of generality we may take the remaining coefficient to be π. In the first case, the operator $P(x,\partial_x)$ is

$$P(x,\partial_x)\cdot f = W(\mathrm{Ai}'(x+s_1),\mathrm{Bi}'(x+s_1), f) = (x+s_1)f''(x)-f'(x)-(x+s_1)^2 f.$$

In the second case, the operator $P(x,\partial_x)$ is

$$P(x,\partial_x)\cdot f = W(\mathrm{Ai}(x+s_1),\mathrm{Bi}(x+s_1), f) = f''(x)-(x+s_1)f.$$

Thus in this second case $P(x,\partial_x) = L(x,\partial_x)-s_1$, giving us the trivial factorization of $(L(x,\partial_x)-s_1)^2$.

Corollary 3 *Let $s_1 \in \mathbb{C}$. Then up to a function multiple, the self-adjoint rational factorizations of $(L(x,\partial_x)-s_1)^4$ are of the form*

$$P_2(x,\partial_x)^* \frac{1}{(x+s_1)^2} P_2(x,\partial_x) = (L(x,\partial_x)-s_1)^4$$

for

$$P_2(x,\partial_x)\cdot f = W(f_1, f_2, g_1, g_2, f),$$

where here

$$f_k(x) = \sum_{j=0}^{3}\alpha_{kj}\mathrm{Ai}^{(j)}(x+s_1), \quad g_k(x) = \sum_{j=0}^{3}\alpha_{kj}\mathrm{Bi}^{(j)}(x+s_1)$$

for some constants α_{kj} satisfying the three relations

$$6\alpha_{m3}\alpha_{n0} + 2\alpha_{m1}\alpha_{n2} + 2\alpha_{m2}\alpha_{n1} + 6\alpha_{m3}\alpha_{n0} = 0, \quad 1 \leq m \leq n \leq 2.$$

Proof This follows immediately from the Classification Theorem.

The operator $P_2(x, \partial_x)$ in this latter situation is more complicated. First of all, it features the factorizations from the previous corollary, as may be obtained from taking $\alpha_{13} = \alpha_{23} = 0$. Thus to get new factorizations, we can without loss of generality take $\alpha_{13} = \alpha_{23} = 1$. Then the three relations simplify to $\alpha_{m0} = -\alpha_{m1}\alpha_{m2}/3$ for $m = 1, 2$ plus a choice of either $\alpha_{11} = \alpha_{21}$ or $\alpha_{12} = \alpha_{22}$. For sake of concreteness, we choose $\alpha_{11} = \alpha_{21}$ and take $\alpha_{22} = 1$, $\alpha_{12} = 0$, and $s_1 = 0$. This determines all parameters, except for α_{11} and the associated operator $P(x, \partial_x)$ is explicitly computed to be

$$\begin{aligned} P_2(x, \partial_x) = &\left(x^4 - 4x^3\alpha_{11} + \frac{10}{3}x^2\alpha_{11}^2 + \left(\frac{4}{3}\alpha_{11}^3 + 4\right)x + \frac{1}{9}\alpha_{11}^4 - 8\alpha_{11}\right)\partial_x^4 \\ &+ \left(-4x^3 + 12x^2\alpha_{11} - \frac{20}{3}\alpha_{11}^2 x - \frac{4}{3}\alpha_{11}^3 - 4\right)\partial_x^3 \\ &+ \left(-2x^5 + 8x^4\alpha_{11} - \frac{20}{3}\alpha_{11}^2 x^3 - \left(\frac{8}{3}\alpha_{11}^3 + 2\right)x^2 - \left(\frac{2}{9}\alpha_{11}^4 - 4\alpha_{11}\right)x + \frac{10}{3}\alpha_{11}^2\right)\partial_x^2 \\ &+ \left(2x^4 - 4x^3\alpha_{11} - \frac{4}{3}x\alpha_{11}^3 - 16 - \frac{2}{9}\alpha_{11}^4 + 36\alpha_{11}\right)\partial_x \\ &+ x^6 - 4x^5\alpha_{11} + \frac{10}{3}x^4\alpha_{11}^2 + \left(\frac{4}{3}\alpha_{11}^3 + 8\right)x^3 + \left(\frac{1}{9}\alpha_{11}^4 - 22\alpha_{11}\right)x^2 + \frac{16}{3}x\alpha_{11}^2 \\ &+ 2\alpha_{11}^3 + 16. \end{aligned} \tag{9}$$

4 Commuting Differential Operators for the Level One Kernels

In this section, we explore the commuting differential operators for integral operators with *level one Airy kernels*, ie. those defined by bispectral functions Ψ obtained from self-adjoint rational Darboux transformations of $(L(x, \partial_x) - s_1)^2$. There is only one such bispectral function, determined by the factorization of $(L(x, \partial_x) - s_1)^2$ in Corollary 2. Using the operator $P_1(x, \partial_x)$ described in this Corollary, the associated bispectral function is

$$\Psi_1(x, z) = \frac{1}{(x + s_1)(z - s_1)} P_1(x, \partial_x) \cdot \Psi_{\text{Ai}}(x, z) = \text{Ai}(x + z) - \frac{\text{Ai}'(x + z)}{(x + s_1)(z - s_1)}.$$

Let $\widetilde{P}_1(z, \partial_z) = b_{\Psi_{\rm Ai}}(P_1(x, \partial_x))$, $p_1(x) = x + s_1$ and $q_1(z) = z - s_1$. For every $R(x, \partial_x) \in \mathfrak{F}_x(\Psi_{\rm Ai})$ and $S(z, \partial_z) = b_{\Psi_{\rm Ai}}(R(x, \partial_x))$, we have the identities

$$\frac{1}{p_1(x)} P_1(x, \partial_x)^* R(x, \partial_x) P_1(x, \partial_x) \frac{\Psi_1(x, z)}{p_1(x)} = (q_1(z)) S(z, \partial_z)(q_1(z)) \cdot \Psi_1(x, z),$$

$$\frac{1}{q_1(z)} \widetilde{P}_1(z, \partial_z)^* S(z, \partial_z) \widetilde{P}_1(z, \partial_z) \frac{\Psi_1(x, z)}{q_1(z)} = (p_1(x)) R(x, \partial_x)(p_1(x)) \cdot \Psi_1(x, z),$$

and the more complicated identity

$$\left(\frac{1}{p_1(x)} P_1(x, \partial_x) R(x, \partial_x) p(x) + p(x) R(x, \partial_x)^* P_1(x, \partial_x)^* \frac{1}{p_1(x)}\right) \cdot \Psi_1(x, z)$$
$$= \left(\frac{1}{q_1(z)} \widetilde{P}_1(z, \partial_z) S(z, \partial_z) q(z) + q(z) S(z, \partial_z)^* \widetilde{P}_1(z, \partial_z)^* \frac{1}{q_1(z)}\right) \cdot \Psi_1(x, z). \tag{10}$$

Comparing the orders of these operators, we see that $\mathfrak{F}^{2\ell,2m}_{x,\rm sym}(\Psi_1)$ contains the direct sum

$$\mathfrak{F}^{2\ell,2m}_{x,\rm sym}(\Psi_1) \supseteq \frac{1}{p_1(x)} P_1(x, \partial_x)^* \mathfrak{F}^{2\ell-4,2m}_{x,\rm sym}(\Psi_{\rm Ai}) P_1(x, \partial_x) \frac{1}{p_1(x)}$$
$$\oplus\, p_1(x) \mathfrak{F}^{2,2m-4}_{x,\rm sym}(\Psi_{\rm Ai}) p_1(x) \oplus \mathfrak{E} \oplus \mathbb{C}$$

for all $\ell, m \geq 2$, where here $\mathfrak{E}$ is a set of additional operators stemming from Eq. (10)

$$\mathfrak{E} = \left\{\frac{1}{p_1(x)} P_1(x, \partial_x) R(x, \partial_x) p(x) + p(x) R(x, \partial_x)^* P_1(x, \partial_x)^* \frac{1}{p_1(x)} : R(x, \partial_x) \in \mathfrak{F}^{1,1}_x(\Psi_{\rm Ai})\right\}.$$

Explicit calculation shows that $\mathfrak{E}$ is two dimensional. Consequently the dimension of $\mathfrak{F}^{2\ell,2m}_{x,\rm sym}(\Psi_1)$ is at least $(\ell - 1)(m + 1) + 2(m - 1) + 2 + 1 = (\ell + 1)(m + 1) - 1$. One can show that this is precisely the dimension for all $m, n > 1$ and that both $\mathfrak{F}_x(\Psi)$ and $\mathfrak{F}_z(\psi)$ are equal to algebras of differential operators on a rational curve with a cuspidal singularity of degree 2 at the origin.

Let T_1 be the integral operator

$$T_1 : f(z) \mapsto \int_{t_1}^{\infty} K_1(z, w) f(w) dw, \quad K_1(z, w) = \int_{t_2}^{\infty} \Psi_1(x, z) \Psi_1(x, w) dx.$$

The specific value of the kernel $K_1(z,w)$ is determined via integration by parts to be

$$K_1(z,w) = \frac{q_1(w)}{q_1(z)} K_{\mathrm{Ai}}(z,w) + \mathcal{C}_{P_1}(\psi_{\mathrm{Ai}}(x,z), \psi_1(x,w)/p_1(x); t_2).$$

From the previous estimate of the dimension of $\mathfrak{F}^{2\ell,2m}_{x,\mathrm{sym}}(\Psi_1)$, we see that T_1 will commute with a differential operator $S_1(z,\partial_z)$ in $\mathfrak{F}^{4,4}_{z,\mathrm{sym}}(\Psi_1)$.

The values of the commuting integral and differential operators will in general depend on s_1, albeit predictably. If we make the s_1-dependence of $\Psi(x,z) = \Psi(x,z;s_1)$ explicit, we see $\Psi(x,z;s_1) = \Psi(x+s_1, z-s_1; 0)$ and consequently the differential operator $S_1(z,\partial_z)$ commuting with T_1 for arbitrary s_1 is the same as in the case $s_1 = 0$, but with z replaced by $z - s_1$ and t_2 replaced by $t_2 + s_1$. Thus without loss of generality we will take $s_1 = 0$.

Explicitly computing the condition of the vanishing of the concomitant and solving the resulting linear system of equations yields the operator of order 4

$$S_1(z,\partial_z) = \frac{1}{z}\left(\sum_{k=0}^{2} \partial_z^k a_k(z)(z-t_1)^k \partial_z^k\right)\frac{1}{z}$$

where here

$$\begin{aligned}
a_2(z) &= z^2,\\
a_1(z) &= -2(z^4 + (t_2 - t_1)z^3 - 3t_1),\\
a_0(z) &= z^3(z^3 + 2(t_2-t_1)z^2 + (t_2-t_1)^2 z - 8) + (t_1+t_2)z^2/3.
\end{aligned}$$

The dimension estimates also imply the existence of a commuting differential operator of order 6, which we find to be

$$\widetilde{S}_1(z,\partial_z) = \frac{1}{z}\left(\sum_{k=0}^{3} \partial_z^k \widetilde{a}_k(z)(z-t_1)^k \partial_z^k\right)\frac{1}{z}$$

where here

$$\begin{aligned}
\widetilde{a}_3(z) &= z^2,\\
\widetilde{a}_2(z) &= -3(z^4 + (t_2-t_1)z^3 - 4t_1),\\
\widetilde{a}_1(z) &= 3(z^6 + 2(t_2-t_1)z^5 + (t_2-t_1)^2 z^4 - 10z^3 + (5t_1 - 4t_2)z^2 - 3t_1(t_2-t_1)z),\\
\widetilde{a}_0(z) &= -z^8 - 3(t_2-t_1)z^7 - 3(t_2-t_1)^2 z^6 - ((t_2-t_1)^3 - 32)z^5\\
&\quad + (42t_2 - 63t_1)z^4 + (36t_1^2 - 48t_1t_2 + 12t_2^2)z^3 + t_1t_2(t_1+t_2)z^2 + 12t_1^2 - 6t_1t_2.
\end{aligned}$$

The operators $S_1(z, \partial_z)$ and $\widetilde{S}_1(z, \partial_z)$ commute and thus satisfy an algebraic relation. The relation is

$$\widetilde{S}_1^2 = S_1^3 - \frac{t_1^2 - t_1 t_2 + t_2^2}{3} S_1 + \frac{(t_1 - 2t_2)(2t_1 - t_2)(t_1 + t_2)}{3^3}.$$

The discriminant of the polynomial on the right hand side is

$$\Delta = -\frac{16}{27}(260t_1^6 - 780t_1^5 t_2 - 627t_1^4 t_2^2 + 2554t_1^3 t_2^3 - 627t_1^2 t_2^4 - 780t_1 t_2^5 + 260t_2^6),$$

so for generic values of t_1 and t_2, the associated algebraic variety is an elliptic curve.

5 Commuting Differential Operators for the Level Two Kernels

In this section, we explore the commuting differential operators for integral operators with *level two Airy kernels*, ie. those defined by bispectral functions Ψ obtained from self-adjoint rational Darboux transformations of $(L(x, \partial_x) - s_1)^2(L(x, \partial_x) - s_2)^2$. We will focus on the particular case when $s_1 = s_2$, leaving the other situation to a future publication. Note also that due to the nice translation behavior of $\psi_{\mathrm{Ai}}(x, z)$, we can easily rederive the formula for general values of s_1 from the case when $s_1 = 0$. So for sake of simplicity, we will take $s_1 = 0$.

There are many bispectral functions in the level two case, all of which are determined by the factorizations of $L(x, \partial_x)^4$ in Corollary 3, which in turn are determined by a choice of α_{jk} for $j = 1, 2$ and $0 \leq k \leq 3$ satisfying the constraints of the Corollary. The precise value $P_2(x, \partial_x)$ and the commuting operator is very complicated in general. To facilitate our computations, and the inclusion of exact formulas in our paper, we will take $\alpha_{31} = \alpha_{32} = 1$, $\alpha_{11} = \alpha_{21}$ and take $\alpha_{22} = 1$, $\alpha_{12} = 0$, so that $P_2(x, \partial_x)$ is given by (9). Additionally we will take $\alpha_{11} = 0$ so that $P_2(x, \partial_x)$ has the simplified formula

$$P_2(x, \partial_x) = x(x^3+4)\partial_x^4 - 4(x^3+1)\partial_x^3 - 2x^2(x^3+1)\partial_x^2 + 2x(x^3-8)\partial_x + x^6 + 8x^3 + 16.$$

Let $q_2(z) = z^2$ and $p_2(x) = x(x^3 + 4)$. The corresponding bispectral function is defined by

$$\begin{aligned}\Psi_2(x, z) &= \frac{1}{p_2(x)q_2(z)} P_2(x, \partial_x) \cdot \Psi_{\mathrm{Ai}}(x, z) \\ &= \mathrm{Ai}(x + z) + \frac{6(x^3 + x^2 z + 2)}{p_2(x)q_2(z)} \mathrm{Ai}(x + z) - \frac{4(x^3 w + 3x + w)}{p_2(x)q_2(z)} \mathrm{Ai}'(x + z).\end{aligned}$$

The Fourier algebras for $\Psi_2(x, z)$ are given by algebras of differential operators on some rank 1, torsion-free modules over certain rational curves with cuspidal singularities. Specifically, let $\mathscr{A}_x = \{f(x) \in \mathbb{C}[x] : p(x) | f'(x)\}$ be the coordinate ring of a singular rational curve X with cusps of order 2 at the roots of $p(x)$. Then

$$\mathfrak{F}_x(\Psi_2) = \{D(x, \partial_x) : D(x, \partial_x) \cdot \mathscr{A}_x \subseteq \mathscr{A}_x\}$$

is the algebra of differential operators on X. Likewise, let $\mathscr{A}_z = \mathbb{C}[z^4, z^5]$ be the affine coordinate ring of a rational curve Z with a higher-order cusp at 0 and consider the torsion-free rank 1 $\mathscr{A}_z$-module $\mathscr{M}_z = \mathrm{Span}_{\mathbb{C}}\{z^{-2}, z^{-1}\} \oplus z^2\mathbb{C}[z]$. Then

$$\mathfrak{F}_z(\Psi_2) = \{D(z, \partial_z) : D(z, \partial_z) \cdot \mathscr{M}_z \subseteq \mathscr{M}_z\}$$

is the algebra of differential operators on the line bundle $\mathcal{L}$ over Z associated to $\mathscr{M}_z$.

The generalized Fourier map b_Ψ may be described in terms of $b_{\Psi_{\mathrm{Ai}}}$ by

$$b_\Psi(A(x, \partial_x)) = \frac{1}{q_2(z)} b_{\Psi_{\mathrm{Ai}}} \left[P_2(x, \partial_x)^* \frac{1}{p_2(x)} A(x, \partial_x) \frac{1}{p_2(x)} P_2(x, \partial_x) \right] \frac{1}{q_2(z)}.$$

Let T_2 be the integral operator

$$T_2 : f(z) \mapsto \int_{t_1}^{\infty} K_2(z, w) f(w) dw, \quad K_2(z, w) = \int_{t_2}^{\infty} \Psi_2(x, z)\Psi_2(x, w) dx.$$

The specific value of the kernel $K_2(z, w)$ is determined via integration by parts to be

$$K_2(z, w) = \frac{q_2(w)}{q_2(z)} K_{\mathrm{Ai}}(z, w) + \mathcal{C}_{P_2}(\psi_{\mathrm{Ai}}(x, z), \psi_2(x, w)/p_2(x); t_2).$$

Computer calculation finds $\dim \mathfrak{F}^{10,10}_{x,\mathrm{sym}}(\Psi_2) = 32$, and therefore T_2 will commute with a differential operator $S_2(z, \partial_z)$ in $\mathfrak{F}^{10,10}_{z,\mathrm{sym}}(\Psi_2)$.

Taking $t_1 = t_2 = 1$, and solving the linear system describing the vanishing of the concomitants, we find differential operators of order 10, 12, 14, 16, and 18 commuting with T_2. The operators $S_2(z, \partial_z)$ and $\widetilde{S}_2(z, \partial_z)$ of order 10 and 12 are given by

$$S_2(z, \partial_z) = \frac{1}{z^2} \left(\sum_{k=0}^{5} \partial_z^k (1 - z)^k a_k(z) \partial_z^k \right) \frac{1}{z^2},$$

$$a_0(z) = z^{14} - 200z^{11} + 170z^{10} + 5640z^8 - 7360z^7 + 2160z^6 - 11520z^5 - 2880z + 4320,$$

$$
\begin{aligned}
a_1(z) &= 5z^{12} - 580z^9 + 380z^8 + 6240z^6 - 3700z^5 - 960z^2 - 9600z + 4800,\\
a_2(z) &= 10z^{10} - 560z^7 + 180z^6 + 960z^4 + 1800z^3 + 300z^2,\\
a_3(z) &= 10z^8 - 180z^5 - 100z^4 - 420z + 1260,\\
a_4(z) &= 5z^6 - 70z^2,\\
a_5(z) &= z^4;
\end{aligned}
$$

$$
\widetilde{S}_2(z, \partial_z) = \frac{1}{z^2}\left(\sum_{k=0}^{6} \partial_z^k (1-z)^k \widetilde{a}_k(z) \partial_z^k\right)\frac{1}{z^2},
$$

$$
\begin{aligned}
\widetilde{a}_0(z) &= z^{16} - 340z^{13} + 504z^{12} + 21040z^{10} - 52200z^9 + 28812z^8\\
&\quad - 192000z^7 + 490464z^6 - 328320z^5 - 201600z + 130464,\\
\widetilde{a}_1(z) &= 6(z^{14} - 220z^{11} + 300z^{10} + 7000z^8 - 14212z^7 + 5148z^6\\
&\quad - 16800z^5 + 13568z^4 + 13568z^3 + 2368z^2 - 6240z + 12480,\\
\widetilde{a}_2(z) &= 3(5z^{12} - 640z^9 + 760z^8 + 7800z^6 - 8792z^5 - 2996z^4 - 3120z^2\\
&\quad - 36000z + 50400,\\
\widetilde{a}_3(z) &= 4z^2(5z^8 - 310z^5 + 270z^4 + 600z^2 + 1566z - 2268),\\
\widetilde{a}_4(z) &= 3(5z^8 - 100z^5 - 224z + 784),\\
\widetilde{a}_5(z) &= 6(z-2)z^2(z+2)(z^2+4),\\
\widetilde{a}_6(z) &= z^4.
\end{aligned}
$$

From Burchnall-Chaundy Theory and its extensions (see for example [26]), we know that each pair of operators must satisfy a polynomial relation. Together, the algebra they generate is the coordinate ring of an affine curve. However, the precise relations that are satisfied are sufficiently complicated so as to be omitted from the paper.

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A Random Walk on the Rado Graph

Sourav Chatterjee, Persi Diaconis, and Laurent Miclo

Dedicated to our friend and coauthor Harold Widom.

Abstract The Rado graph, also known as the random graph $G(\infty, p)$, is a classical limit object for finite graphs. We study natural ball walks as a way of understanding the geometry of this graph. For the walk started at i, we show that order $\log_2^* i$ steps are sufficient, and for infinitely many i, necessary for convergence to stationarity. The proof involves an application of Hardy's inequality for trees.

Keywords Rado graph · Random graph · Random walk · Mixing speed · Cheeger's inequality · Hardy's inequality

Mathematics Subject Classification (2020) 05C81, 05C80, 05C63, 60J10, 60J27, 37A25, 46E39

1 Introduction

The Rado graph R is a natural limit of the set of all finite graphs (Fraissé limit, see Sect. 2.1). In Rado's construction, the vertex set is $\mathbb{N} = \{0, 1, 2, \ldots\}$. There is an undirected edge from i to j if $i < j$ and the i^{th} binary digit of j is a one (where the 0^{th} digit is the first digit from the right). Thus, 0 is connected to all odd numbers, 1 is connected to 0 and all j which are 2 or 3 (mod 4) and so on. There are many

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alternative constructions. For $p \in (0, 1)$, connecting i and j with probability p gives the Erdős–Rényi graph $G(\infty, p)$, which is (almost surely) isomorphic to R. Further constructions are in Sect. 2.1.

Let $(Q(j))_{0 \leqslant j < \infty}$ be a positive probability on $\mathbb{N}$ (so, $Q(j) > 0$ for all j, and $\sum_{j=0}^{\infty} Q(j) = 1$). We study a 'ball walk' on R generated by Q: from $i \in \mathbb{N}$, pick $j \in N(i)$ with probability proportional to $Q(j)$, where $N(i) = \{j : j \sim i\}$ is the set of neighbors of i in R. Thus, the probability of moving from i to j in one step is

$$K(i, j) = \begin{cases} Q(j)/Q(N(i)) & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

As explained below, this walk is connected, aperiodic and reversible, with stationary distribution

$$\pi(i) = \frac{Q(i)Q(N(i))}{Z}, \tag{2}$$

where Z is the normalizing constant.

It is natural to study the mixing time—the rate of convergence to stationarity. The following result shows that convergence is extremely rapid. Starting at $i \in \mathbb{N}$, order $\log_2^* i$ steps suffice, and for infinitely many i, are needed.

Theorem 1 *Let $Q(j) = 2^{-(j+1)}$, $0 \leqslant j < \infty$. For $K(i, j)$ and π defined at (1) and (2) on the Rado graph R,*

1. *For universal $A, B > 0$, we have for all $i \in \mathbb{N}$, $\ell \geqslant 1$,*

$$\|K_i^\ell - \pi\| \leqslant A e^{\log_2^* i} e^{-B\ell}.$$

2. *For universal $C > 0$, if $2^{(k)} = 2^{2^{\cdot^{\cdot^{2}}}}$ is the tower of 2's of height k,*

$$\|K_{2^{(k)}}^\ell - \pi\| \geqslant C$$

for all $\ell \leqslant k$. Here $\|K_i^\ell - \pi\| = \frac{1}{2}\sum_{j=0}^{\infty} |K^\ell(i, j) - \pi(j)|$ is the total variation distance and $\log_2^* i$ is the number of times $\log_2$ needs to be applied, starting from i, to get a result $\leqslant 1$.

The proofs allow for some variation in the measure Q. They also work for the $G(\infty, p)$ model of R, though some modification is needed since then K and π are random.

Theorem 1 answers a question in Diaconis and Malliaris [8], who proved the lower bound. Most Markov chains on countable graphs restrict attention to locally finite graphs [25]. For Cayley graphs, Bendikov and Saloff-Coste [1] begin the study of more general transitions and point out how few tools are available. See also [12, 20]. Studying the geometry of a space (here R) by studying the properties of the

Laplacian (here $I - K$) is a classical pursuit ("Can you hear the shape of a drum?")—see [16].

Section 2 gives background on the Rado graph, Markov chains, ball walks, and Hardy's inequalities. Section 3 gives preliminaries on the behavior of the neighborhoods of the $G(\infty, p)$ model. The lower bound in Theorem 1 is proved in Sect. 4. Both Sects. 3 and 4 give insight into the geometry of R. The upper bound in Theorem 1 is proved by proving that the Markov chain K has a spectral gap. Usually, a spectral gap alone does not give sharp rates of convergence. Here, for any start i, we show the chain is in a neighborhood of 0 after order $\log_2^* i$ steps. Then the spectral gap shows convergence in a bounded number of further steps. This argument works for both models of R. It is given in Sect. 5.

The spectral gap for the $G(\infty, p)$ model is proved in Sect. 6 using a version of Cheeger's inequality for trees. For Rado's binary model, the spectral gap is proved by a novel version of Hardy's inequality for trees in Sect. 7. This is the first probabilistic application of this technique, which we hope will be useful more generally. There are two appendices containing technical details for the needed versions of Cheeger's and Hardy's inequalities.

2 Background on R, Markov Chains, and Hardy's Inequalities

2.1 The Rado Graph

A definitive survey on the Rado graph (with full proofs) is in Peter Cameron's fine article [6]. We have also found the Wikipedia entry on the Rado graph and Cameron's follow-up paper [7] useful.

In Rado's model, the graph R has vertex set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and an undirected edge from i to j if $i < j$ and the i^{th} digit of j is a one. There are many other constructions. The vertex set can be taken as the prime numbers that are 1 (mod 4) with an edge from p to q if the Legendre symbol $(\frac{p}{q}) = 1$. In [8], the graph appears as an induced subgraph of the commuting graph of the group $U(\infty, q)$—infinite upper-triangular matrices with ones on the diagonal and entries in $\mathbb{F}_q$. The vertices are points of $U(\infty, q)$. There is an edge from x to y if and only if the commutator $x^{-1}y^{-1}xy$ is zero. The infinite Erdős–Rényi graphs $G(\infty, p)$ are almost surely isomorphic to R for all p, $0 < p < 1$.

The graph R has a host of fascinating properties:

- It is stable in the sense that deleting any finite number of vertices or edges yields an isomorphic graph. So does taking the complement.
- It contains all finite or countable graphs as induced subgraphs. Thus, the (countable) empty graph and complete graphs both appear as induced subgraphs.
- The diameter of R is two—consider any $i \neq j \in \mathbb{N}$ and let k be a binary number with ones in positions i and j and zero elsewhere. Then $i \sim k \sim j$.

- Each vertex is connected to “half” of the other vertices: 0 is connected to all the odd vertices, 1 to 0 and all numbers congruent to 2 or 3 (mod 4), and so on.
- R is highly symmetric: Any automorphism between two induced subgraphs can be extended to all of R (this is called homogeneity). The automorphism group has the cardinality of the continuum.
- R is the “limit” if the collection of all finite graphs (Fraissé limit). Let us spell this out. A *relational structure* is a set with a finite collection of relations (we are working in first order logic without constants or functions). For example, $\mathbb{Q}$ with $x < y$ is a relational structure. A graph is a set with one symmetric relation. The idea of a “relational sub-structure” clearly makes sense. A class $\mathcal{C}$ of structures has the *amalgamation property* if for any $A, B_1, B_2 \in \mathcal{C}$ with embeddings $A \overset{f_1}{\to} B_1$ and $A \overset{f_2}{\to} B_2$, there exists $C \in \mathcal{C}$ and embeddings $B_1 \overset{g_1}{\to} C$ and $B_2 \overset{g_2}{\to} C$ such that $g_1 f_1 = g_2 f_2$. A countable relational structure M is *homogeneous* if any isomorphism between finite substructures can be extended to an automorphism of M. Graphs and $\mathbb{Q}$ are homogeneous relational structures. A class $\mathcal{C}$ has the joint embedding property if for any $A, B \in \mathcal{C}$ there is a $C \in \mathcal{C}$ so that A and B are embeddable in C.

  Theorem 2 (Fraissé) *Let $\mathcal{C}$ be a countable class of finite structures with the joint embedding property and closed under ‘induced’ isomorphism with amalgamation. Then there exists a unique countable homogeneous $\mathcal{M}$ with $\mathcal{C}$ as induced substructures.*

  The rationals $\mathbb{Q}$ are the Fraissé limit of finite ordered sets. The Rado graph R is the Fraissé limit of finite graphs. We have (several times!) been told “for a model theorist, the Rado graph is just as interesting as the rationals”.

There are many further, fascinating properties of R; see [6].

2.2 Markov Chains

A *transition matrix* $K(i, j)$, $0 \leqslant i, j < \infty$, $K(i, j) \geqslant 0$, $\sum_{j=0}^{\infty} K(i, j) = 1$ for all i, $0 \leqslant i < \infty$, generates a Markov chain through its powers

$$K^{\ell}(i, j) = \sum_{k=0}^{\infty} K(i, k) K^{\ell-1}(k, j).$$

A probability distribution $\pi(i)$, $0 \leqslant i < \infty$, is *reversible* for K if

$$\pi(i)K(i, j) = \pi(j)K(j, i) \quad \text{for all } 0 \leqslant i, j < \infty. \tag{3}$$

Example With definitions (1), (2) on the Rado graph, if $i \sim j$,

$$\pi(i)K(i,j) = \frac{Q(i)Q(N(i))}{Z}\frac{Q(j)}{Q(N(i))} = \frac{Q(i)Q(j)}{Z} = \pi(j)K(j,i).$$

(Both sides are zero if $i \not\sim j$.)

In the above example, we think of $K(i,j)$ as a 'ball walk': From i, pick a neighbor j with probability proportional to $Q(j)$ and move to j. We initially found the neat reversible measure surprising. Indeed, we and a generation of others thought that ball walks would have Q as a stationary distribution. Yuval Peres points out that, given a probability $Q(j)$ on the vertices, assigning symmetric weight $Q(i)Q(j)$ to $i \sim j$ gives this K for the weighted local walk. A double ball walk—"from i, choose a neighbor j with probability proportional to $Q(j)$, and from j, choose a neighbor k with probability proportional to $Q(k)/Q(N(k))$"—results in a reversible Markov chain with Q as reversing measure. Note that these double ball walks don't require knowledge of normalizing constants. All of this suggests ball walks as reasonable objects to study.

Reversibility (3) shows that π is a stationary distribution for K:

$$\sum_{i=0}^{\infty}\pi(i)K(i,j) = \sum_{i=0}^{\infty}\pi(j)K(j,i) = \pi(j)\sum_{i=0}^{\infty}K(j,i) = \pi(j).$$

In our setting, since the Rado graph has diameter 2, the walk is connected. It is easy to see that it is aperiodic. Thus, the π in (2) is the unique stationary distribution. Now, the fundamental theorem of Markov chain theory shows, for every starting state i, $K^{\ell}(i,j) \to \pi(j)$ as $\ell \to \infty$, and indeed,

$$\lim_{\ell\to\infty}\|K_i^{\ell} - \pi\| = 0.$$

Reversible Markov chains have real spectrum. Say that (K,π) has a *spectral gap* if there is $A > 0$ such that for every $f \in \ell^2(\pi)$,

$$\sum_i (f(i) - \overline{f})^2\pi(i) \leqslant A\sum_{i,j}(f(i) - f(j))^2\pi(i)K(i,j), \tag{4}$$

where $\overline{f} = \sum_{i=0}^{\infty} f(i)\pi(i)$. (Then the gap is at least $1/A$.) For chains with a spectral gap, for any i,

$$4\|K_i^{\ell} - \pi\|^2 \leqslant \frac{1}{\pi(i)}\left(1 - \frac{1}{A}\right)^{2\ell}. \tag{5}$$

Background on Markov chains, particularly rates of convergence, can be found in the readable book of Levin and Peres [19]. For the analytic part of the theory, particularly (4) and (5), and many refinements, we recommend [23].

There has been a healthy development in Markov chain circles around the theme ‘How does a Markov chain on a random graph behave?’. One motivation being, ‘What does a typical convergence rate look like?’. The graphs can be restricted in various natural ways (Cayley graphs, regular graphs of fixed degree or fixed average degree, etc.). A survey of by now classical work is Hildebrand’s survey of ‘random-random walks’ [14]. Recent work by Bordenave and coauthors can be found from [4, 5]. For sparse Erdős–Rényi graphs, there is remarkable work on the walk restricted to the giant component. See [22], [11] and [3].

It is worth contrasting these works with the present efforts. The above results pick a neighbor uniformly at random. In the present paper, the ball walk drives the walk back towards zero. The papers above are all on finite graphs. The Markov chain of Theorem 1 makes perfect sense on finite graphs. The statements and proofs go through (with small changes) to show that order $\log_2^* i$ steps are necessary and sufficient. (For the uniform walk on $G(n, 1/2)$, a bounded number of steps suffice from most initial states, but there are states from which $\log_2^* n$ steps are needed.)

2.3 Hardy’s Inequalities

A key part of the proof of Theorem 1 applies Hardy’s inequalities for trees to prove a Poincaré inequality (Cf. (4)) and hence a bound on the spectral gap. Despite a large expository literature, Hardy’s inequalities remain little known among probabilists. Our application can be read without this expository section but we hope that some readers find it useful. Extensive further references, trying to bridge the gap between probabilists and analysts, is in [17].

Start with a discrete form of Hardy’s original inequality [13, pp. 239–243]. This says that if $a_n \geqslant 0$, $A_n = a_1 + \cdots + a_n$, then

$$\sum_{n=1}^{\infty} \frac{A_n^2}{n^2} \leqslant 4 \sum_{n=1}^{\infty} a_n^2,$$

and the constant 4 is sharp. Analysts say that “the Hardy operator taking $\{a_n\}$ to $\{A_n/n\}$ is bounded from ℓ^2 to ℓ^2”. Later writers showed how to put weights in. If $\mu(n)$ and $\nu(n)$ are positive functions, one aims for

$$\sum_{n=1}^{\infty} A_n^2 \mu(n) \leqslant A \sum_{n=1}^{\infty} a_n^2 \nu(n),$$

for an explicit A depending on $\mu(n)$ and $\nu(n)$. If $\mu(n) = 1/n^2$ and $\nu(n) = 1$, this gives the original Hardy inequality. To make the transition to a probabilistic application, take $a(n) = g(n) - g(n-1)$ for g in ℓ^2. The inequality becomes

$$\sum_{n=1}^{\infty} g(n)^2 \mu(n) \leqslant A \sum_{n=1}^{\infty} (g(n) - g(n-1))^2 \nu(n). \tag{6}$$

Consider a 'birth and death chain' which transits from j to $j+1$ with probability $b(j)$ and from j to $j-1$ with probability $d(j)$. Suppose that this has stationary distribution $\mu(j)$ and that $\sum_j g(j)\mu(j) = 0$. Set $\nu(j) = \mu(j)d(j)$. Then (6) becomes (following simple manipulations)

$$\mathrm{Var}(g) \leqslant A \sum_{j,k} (g(j) - g(k))^2 \mu(j) K(j,k) \tag{7}$$

with $K(j,k)$ the transition matrix of the birth and death chain. This gives a Poincaré inequality and spectral gap estimate. A crucial ingredient for applying this program is that the constant A must be explicit and manageable. For birth-death chains, this is indeed the case. See [21] or the applications in [9]. The transition from (6) to (7) leans on the one-dimensional setup of birth-death chains. While there is work on Hardy's inequalities in higher dimensions, it is much more complex; in particular, useful forms of good constants A seem out of reach. In [21], Miclo has shown that for a general Markov transition matrix $K(i,j)$, a spanning tree in the graph underlying K can be found. There is a useful version of Hardy's inequality for trees due to Evans, Harris and Pick [10]. This is the approach developed in Sect. 7 below which gives further background and details.

Is approximation by trees good enough? There is some hope that the best tree is good enough (see [2]). In the present application, the tree chosen gives the needed result.

2.4 *The* log* *Function*

Take any $a > 1$. The following is a careful definition of $\log_a^* x$ for $x \geqslant 0$. First, an easy verification shows that the map $x \mapsto (\log x)/x$ on $(0, \infty)$ is unimodal, with a unique maximum at $x = e$ (where its value is $1/e$), and decaying to $-\infty$ as $x \to 0$ and to 0 as $x \to \infty$. Thus, if $a > e^{1/e}$, then for any $x > 0$,

$$\log_a x = \frac{\log x}{\log a} \leqslant \frac{x}{e \log a} < x.$$

Since $\log_a$ is a continuous map, this shows that if we start with any $x > 0$, iterative applications of $\log_a$ will eventually lead to a point in $(0, a)$ (because there are no fixed points of $\log_a$ above that, by the above inequality), and then another application of $\log_a$ will yield a negative number. This allows us to define $\log_a^* x$ as the minimum number of applications of $\log_a$, starting from x, that gets us a nonpositive result.

If $a \leqslant e^{1/e}$, the situation is a bit more complicated. Here, $\log a \leqslant 1/e$, which is the maximum value of the unimodal map $x \mapsto (\log x)/x$. This implies that there exist exactly two points $0 < y_a \leqslant x_a$ that are fixed points of $\log_a$ (with $y_a = x_a$ if $a = e^{1/e}$). Moreover, $\log_a x < x$ if $x \notin [y_a, x_a]$, and $\log_a x \geqslant x$ if $x \in [y_a, x_a]$. Thus, the previous definition does not work. Instead, we define $\log_a^* x$ to be the minimum number of applications of $\log_a$, starting from x, that leads us to a result $\leqslant x_a$. In both cases, defining $\log_a^* 0 = 0$ is consistent with the conventions. Note that $\log_a^* x \geqslant 0$ for all $x \geqslant 0$.

3 The Geometry of the Random Model

Throughout this section the graph is $G(\infty, 1/2)$ — an Erdős–Rényi graph on $\mathbb{N} = \{0, 1, 2, \ldots\}$ with probability $1/2$ for each possible edge. From here on, we will use the notation $\mathbb{N}_+$ to denote the set $\{1, 2, \ldots\}$ of strictly positive integers. Let $Q(x) = 2^{-(x+1)}$ for $x \in \mathbb{N}$. The transition matrix

$$K(x, y) = \frac{Q(y)}{Q(N(x))} \mathbb{1}_{\{y \in N(x)\}}$$

and its stationary distribution $\pi(x) = Z^{-1} Q(x) Q(N(x))$ are thus random variables. Note that $N(x)$, the neighborhood of x, is random. The main result of this section shows that this graph, with vertices weighted by $Q(x)$, has its geometry controlled by a tree rooted at 0. This tree will appear in both lower and upper bounds on the mixing time for the random model.

To describe things, let $p(x) = \min N(x)$ (p is for 'parent', not to be confused with the edge probability p in $G(\infty, p)$). We need some preliminaries about $p(x)$.

Lemma 1 *Let $\mathcal{B}$ be the event that for all $x \in \mathbb{N}_+$, $p(x) < x$. Then we have that $\mathbb{P}(\mathcal{B}) \geqslant 1/4$.*

Proof Denote $\overline{E} := \{\{x, y\} : x \neq y \in \mathbb{N}\}$, and for any $e \in \overline{E}$, consider $B_e = \mathbb{1}_E(e)$, where E is the set of edges in $G(\infty, 1/2)$, so that $(B_e)_{e \in \overline{E}}$ is a family of independent Bernoulli variables of parameter $1/2$.

For $x \in \mathbb{N}_+$, define A_x the event that x is not linked in $\mathcal{G}$ to a smaller vertex. Namely, we have formally

$$A_x := \bigcap_{y \in [\![0, x-1]\!]} \{B_{\{y,x\}} = 0\},$$

where $[\![0, x-1]\!] := \{0, 1, \ldots, x-1\}$. Note that the family $(A_x)_{x\in\mathbb{N}_+}$ is independent, and in particular, its events are pairwise independent. We are thus in position to apply Kounias–Hunter–Worsley bounds [15, 18, 26] (see also the survey [24]), to see that for any $n \in \mathbb{N}_+$,

$$\mathbb{P}\left(\bigcup_{x\in[\![1,n]\!]} A_x\right) \leqslant \min\left\{\sum_{x\in[\![1,n]\!]} \mathbb{P}(A_x) - \mathbb{P}(A_1)\sum_{y\in[\![2,n]\!]} \mathbb{P}(A_y), 1\right\},$$

where we used that $\mathbb{P}(A_1) \geqslant \mathbb{P}(A_2) \geqslant \cdots \geqslant \mathbb{P}(A_n)$, which holds because

$$\forall\, x \in \mathbb{N}_+, \qquad \mathbb{P}(A_x) = \prod_{y\in[\![0,x-1]\!]} \mathbb{P}(B_{\{y,x\}}) = \frac{1}{2^x}.$$

We deduce that

$$\mathbb{P}\left(\bigcup_{x\in[\![1,n]\!]} A_x\right) \leqslant \min\left\{\sum_{x\in[\![1,n]\!]} \frac{1}{2^x} - \frac{1}{2}\sum_{y\in[\![2,n]\!]} \frac{1}{2^y}, 1\right\} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2^{n+1}}.$$

Letting n tends to infinity, we get $\mathbb{P}\left(\bigcup_{x\in\mathbb{N}_+} A_x\right) \leqslant \frac{3}{4}$. To conclude, note that $\mathcal{B}^{\mathrm{c}} = \bigcup_{x\in\mathbb{N}_+} A_x$. □

Remark 1 Assume that instead of $1/2$, the edges of $\overline{E}$ belong to E with probability $p \in (0, 1)$ (still independently), the corresponding notions receive p in index. The above computations show $\mathbb{P}_p(\mathcal{B}) \geqslant 1 - (2 - 3p + p^2) \wedge 1$, so that $\mathbb{P}_p(\mathcal{B})$ goes to 1 as p goes to 1, but this bounds provides no information for $p \in (0, (3 - \sqrt{5})/2]$.

In fact the above observation shows that the Kounias–Hunter–Worsley bound is not optimal, at least for small $p > 0$. So let us give another computation of $\mathbb{P}_p(\mathcal{B})$:

Lemma 2 *Consider the situation described in Remark 1, with* $p \in (0, 1)$. *We have*

$$\mathbb{P}_p(\mathcal{B}) = \left(\sum_{n\in\mathbb{N}} p(n)(1-p)^n\right)^{-1}$$

where $p(n)$ *is the number of partitions of* n. *In particular* $\mathbb{P}(\mathcal{B}) > 0$ *for all* $p \in (0, 1)$.

Proof Indeed, we have $\mathcal{B} = \bigcap_{x\in\mathbb{N}_+} A_x^{\mathrm{c}}$, so that by independence of the A_x, for $x \in \mathbb{N}_+$,

$$\mathbb{P}_p(\mathcal{B}) = \prod_{x\in\mathbb{N}_+} \mathbb{P}(A_x^{\mathrm{c}}) = \left(\prod_{x\in\mathbb{N}_+} \frac{1}{1-(1-p)^x}\right)^{-1} = \left(\prod_{x\in\mathbb{N}_+}\sum_{n\in\mathbb{N}} (1-p)^{xn}\right)^{-1}$$

Let $\mathcal{N}$ be the set of sequences of integers $(n_l)_{l\in\mathbb{N}_+}$ with all but finitely many elements equal to zero. Applying the distributive law to the above expression, we have

$$\mathbb{P}_p(\mathcal{B}) = \Bigg(\sum_{(n_l)_{l\in\mathbb{N}_+}\in\mathcal{N}} \prod_{x\in\mathbb{N}_+} (1-p)^{x n_x} \Bigg)^{-1} = \Bigg(\sum_{n\in\mathbb{N}} p(n)(1-p)^n \Bigg)^{-1}$$

where $p(n)$ is the number of ways to write n as $\sum_{x\in\mathbb{N}_+} x n_x$, with $(n_l)_{l\in\mathbb{N}_+} \in \mathcal{N}$. □

Consider the set of edges

$$F := \{\{x, p(x)\} : x \in \mathbb{N}_+\}$$

and the corresponding graph $\mathcal{T} := (\mathbb{N}, F)$. Under $\mathcal{B}$, it is clear that $\mathcal{T}$ is a tree. But this is always true:

Lemma 3 *The graph $\mathcal{T}$ is a tree.*

Proof The argument is by contradiction. Assume that $\mathcal{T}$ contains a cycle, say $(x_l)_{l\in\mathbb{Z}_n}$ with $n \geqslant 3$. Let us direct the a priori unoriented edges $\{x_l, x_{l+1}\}$, for $l \in \mathbb{Z}_n$, by putting an arrow from x_l to x_{l+1} (respectively from x_{l+1} to x_l) if $p(x_l) = x_{l+1}$ (resp. $p(x_{l+1}) = x_l$). Note that we either have

$$\forall\, l \in \mathbb{Z}_n,\ x_l \to x_{l+1}, \ \text{ or } \ \forall\, l \in \mathbb{Z}_n,\ x_{l+1} \to x_l, \tag{8}$$

because otherwise there would exist $l \in \mathbb{Z}_n$ with two arrows exiting from x_l, a contradiction. Up to reindexing $(x_l)_{l\in\mathbb{Z}_n}$ as $(x_{-l})_{l\in\mathbb{Z}_n}$, we can assume that (8) holds.

Fix some $l \in \mathbb{Z}_n$. Since $p(x_l) = x_{l+1}$, we have $x_l \in N(x_{l+1})$, so $x_{l+2} = p(x_{l+1}) \leqslant x_l$. Due to the fact that $x_l \neq x_{l+2}$ (recall that $n \geqslant 3$), we get $x_{l+2} < x_l$. Starting from x_0 and iterating this relation (in a minimal way, $n/2$ times if n is even, or n times if n is odd), we obtain a contradiction: $x_0 < x_0$. Thus, $\mathcal{T}$ must be a tree. □

Let us come back to the case where $p = 1/2$. The following result gives an idea of how far $p(x)$ is from x, for $x \in \mathbb{N}_+$.

Lemma 4 *Almost surely, there exist only finitely many $x \in \mathbb{N}_+$ such that $p(x) > 2\log_2(1+x)$. In particular, a.s. there exists a (random) finite $C \geqslant 2$ such that*

$$\forall\, x \in \mathbb{N}_+, \qquad p(x) \leqslant C \log_2(1+x).$$

Proof The first assertion follows from the Borel–Cantelli lemma, as follows. For any $x \in \mathbb{N}_+$, consider the event

$$A_x := \{p(x) > 2\log_2(1+x)\}.$$

Denoting $\lfloor\cdot\rfloor$ the the integer part, we compute

$$\sum_{x\in\mathbb{N}_+}\mathbb{P}(A_x)=\sum_{x\in\mathbb{N}_+}\mathbb{P}(B_{\{0,x\}}=0,B_{\{1,x\}}=0,\ldots,B_{\{\lfloor 2\log_2(1+x)\rfloor,x\}}=0)$$
$$=\sum_{x\in\mathbb{N}_+}\frac{1}{2^{1+\lfloor 2\log_2(1+x)\rfloor}}\leqslant\sum_{x\in\mathbb{N}_+}\frac{1}{(1+x)^2}<+\infty.$$

Having shown that a.s. there exists only a finite number of integers $x\in\mathbb{N}_+$ satisfying $p(x)>2\log_2(1+x)$, denote these points as $x_1,\ldots,x_N$, with $N\in\mathbb{N}$. To get the second assertion, it is sufficient to take $C:=\max\{p(x_l)/\log(1+x_l)\;:\;l\in[\![1,N]\!]\}$, with the convention that $C:=2$ if $N=0$. □

4 The Lower Bound

The lower bound in Theorem 1, showing that order $\log_2^* i$ steps are necessary for infinitely many i is proved in [8] for the binary model of the Rado graph and we refer there for the proof. A different argument is needed for the $G(\infty,1/2)$ model. This section gives the details (see Theorem 3 below).

Let μ be the stationary distribution of our random walk on $G(\infty,1/2)$ (with $Q(j)=2^{-(j+1)}$, as in Theorem 1), given a realization of the graph. Note that μ is random. For each $x\in\mathbb{N}$, let τ_x be the mixing time of the walk starting from x, that is, the smallest n such that the law of the walk at time n, starting from x, has total variation distance $\leqslant 1/4$ from μ. Note that the τ_x's are also random.

Theorem 3 *Let τ_x be as above. Then with probability one,*

$$\limsup_{x\to\infty}\frac{\tau_x}{\log_{16}^* x}\geqslant 1.$$

Equivalently, with probability one, given any $\varepsilon>0$, $\tau_x\geqslant(1-\varepsilon)\log_{16}^* x$ for infinitely many x.

We need the following lemma.

Lemma 5 *With probability one, there is an infinite sequence $x_0<x_1<x_2<\cdots\in\mathbb{N}$ such that:*

1. *For each i, x_{i+1} is connected to x_i by an edge, but not connected by an edge to any other number in $\{0,1,\ldots,2x_i-1\}$.*
2. *For each i, $2^{3x_i}\leqslant x_{i+1}\leqslant 2^{3x_i+1}-1$.*

Proof Define a sequence $y_0, y_1, y_2, \ldots$ inductively as follows. Let y_0 be an arbitrary element of $\mathbb{N}$. For each i, let y_{i+1} be the smallest element in $\{2^{3y_i}, 2^{3y_i}+1, \ldots, 2^{3y_i+1}-1\}$ that has an edge to y_i, but to no other number in $\{0, 1, \ldots, 2y_i-1\}$. If there exists no such number, then the process stops. Let A_i be the event that y_i exists. Note that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$.

Let $F(x) := 2^{3x}$, $G(x) := 2^{3x+1}-1$, $a_0 = b_0 = y_0$, and for each $i \geqslant 1$, let

$$a_i := \underbrace{F \circ F \circ \cdots \circ F}_{i \text{ times}}(y_0), \quad b_i := \underbrace{G \circ G \circ \cdots \circ G}_{i \text{ times}}(y_0).$$

Since $2^{3y_i} \leqslant y_{i+1} \leqslant 2^{3y_i+1}-1$ for each i, it follows by induction that $a_i \leqslant y_i \leqslant b_i$ for each i (if y_i exists). Now fix some $i \geqslant 1$. Since the event A_{i-1} is determined by $y_1, \ldots, y_{i-1}$, and these random variables can take only finitely many values (by the above paragraph), we can write A_{i-1} as a finite union of events of the form $\{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\}$, where $c_1 < c_2 < \cdots < c_{i-1} \in \mathbb{N}$.

Now note that for any $c_1 < \cdots < c_{i-1}$, the event $A_i \cap \{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\}$ happens if and only if $\{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\}$ happens and there is some $y \in \{2^{3c_{i-1}}, 2^{3c_{i-1}}+1, \ldots, 2^{3c_{i-1}+1}-1\}$ that has an edge to c_{i-1}, but to no other number in $\{0, \ldots, 2c_{i-1}-1\}$. The event $\{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\}$ is in $\mathcal{F}_{c_{i-1}}$, where $\mathcal{F}_x$ denotes the σ-algebra generated by the edges between all numbers in $\{0, \ldots, x\}$. On the other hand, on the event $\{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\}$, it is not hard to see that

$$\mathbb{P}(A_i | \mathcal{F}_{c_{i-1}}) = 1 - (1 - 2^{-2c_{i-1}})^{2^{3c_{i-1}}}.$$

Thus,

$$\begin{aligned}
&\mathbb{P}(A_i \cap \{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\}) \\
&= \mathbb{P}(y_1 = c_1, \ldots, y_{i-1} = c_{i-1})(1 - (1 - 2^{-2c_{i-1}})^{2^{3c_{i-1}}}) \\
&\geqslant \mathbb{P}(y_1 = c_1, \ldots, y_{i-1} = c_{i-1})(1 - e^{-2^{c_{i-1}}}),
\end{aligned}$$

where in the last step we used the inequality $0 \leqslant 1 - x \leqslant e^{-x}$ (which holds for all $x \in [0, 1]$). Note that the term inside the parentheses on the right side is an increasing function of c_{i-1}, and the maximum possible value of y_{i-1} is b_{i-1}. Thus, summing both sides over all values of $c_1, \ldots, c_{i-1}$ such that $\{y_1 = c_1, \ldots, y_{i-1} = c_{i-1}\} \subseteq A_{i-1}$, we get $\mathbb{P}(A_i) = \mathbb{P}(A_i \cap A_{i-1}) \geqslant \mathbb{P}(A_{i-1})(1 - e^{-2^{b_{i-1}}})$. Proceeding inductively, this gives

$$\mathbb{P}(A_1 \cap \cdots \cap A_i) \geqslant \prod_{k=0}^{i-1}(1 - e^{-2^{b_k}}).$$

Taking $i \to \infty$, we get $\mathbb{P}(B) \geqslant \prod_{k=0}^{\infty}(1-e^{-2^{b_k}})$, where $B := \bigcap_{k=1}^{\infty} A_k$. Now recall that the event B, as well as the numbers $b_0, b_1, \ldots$, are dependent on our choice of y_0. To emphasize this dependence, let us write them as $B(y_0)$ and $b_k(y_0)$. Then by the above inequality,

$$\sum_{y_0 \in \mathbb{N}} \mathbb{P}(B(y_0)^c) \leqslant \sum_{y_0 \in \mathbb{N}} \left(1 - \prod_{k=0}^{\infty}(1 - e^{-2^{b_k(y_0)}})\right),$$

where $B(y_0)^c$ denotes the complement of $B(y_0)$. Due to the extremely rapid growth of $b_k(y_0)$ as $k \to \infty$, and the fact that $b_0(y_0) = y_0$, it is not hard to see that the right side is finite. Therefore, by the Borel–Cantelli lemma, $B(y_0)^c$ happens for only finitely many y_0 with probability one. In particular, with probability one, $B(y_0)$ happens for some y_0. This completes the proof. □

Proof (Of Theorem 3) Fix a realization of $G(\infty, 1/2)$. Let x be so large that $\mu([x, \infty)) < 1/10$, and $\prod_{k=1}^{\infty}(1 - 2^{-a_k(x)+1}) \geqslant 9/10$.

Let $x_0, x_1, x_2, \ldots$ be a sequence having the properties listed in Lemma 5 (which exists with probability one, by the lemma). Discarding some initial values if necessary, let us assume that $x_0 > x$. By the listed properties, it is obvious that $x_i \to \infty$ as $i \to \infty$. Thus, to prove Theorem 3, it suffices to prove that

$$\liminf_{i \to \infty} \frac{\tau_{x_i}}{\log_{16}^* x_i} \geqslant 1. \tag{9}$$

We will now deduce this from the properties of the sequence.

Suppose that our random walk starts from x_i for some $i \geqslant 1$. Since x_i connects to x_{i-1} by an edge, but not to any other number in $\{0, \ldots, 2x_{i-1} - 1\}$, we see that the probability of the walk landing up at x_{i-1} in the next step is at least

$$1 - \frac{1}{2^{-x_i}} \sum_{k=2x_i}^{\infty} 2^{-k} = 1 - 2^{-x_i+1}.$$

Proceeding by induction, this shows that the chance that the walk lands up at x_0 at step i is at least $\prod_{k=1}^{i}(1 - 2^{-x_k+1})$. Let μ_i be the law of walk at step i (starting from x_i, and conditional on the fixed realization of our random graph). Then by the above deduction and the facts that $x_0 > x$ and $x_k \geqslant a_k(x_0) \geqslant a_k(x)$, we have

$$\mu_i([x, \infty)) \geqslant \prod_{k=1}^{i}(1 - 2^{-x_k+1}) \geqslant \prod_{k=1}^{i}(1 - 2^{-a_k(x)+1}).$$

By our choice of x, the last expression is bounded below by $9/10$. But $\mu([x, \infty)) < 1/10$. Thus, the total variation distance between μ_i and μ is at least $8/10$. In particular, $\tau_{x_i} > i$. Now, $x_i \leqslant 2^{3x_{i-1}+1} - 1 \leqslant 16^{x_{i-1}}$, which shows that

$\log^*_{16} x_i \leqslant \log^*_{16} x_{i-1} + 1$. Proceeding inductively, we get $\log^*_{16} x_i \leqslant i + \log^*_{16} x_0$. Thus, $\tau_{x_i} > \log^*_{16} x_i - \log^*_{16} x_0$. This proves (9). □

5 The Upper Bound (Assuming a Spectral Gap)

This section gives the upper bound for both the binary and random model of the Rado graph. Indeed, the proof works for a somewhat general class of graphs and more general base measures Q. The argument assumes that we have a spectral gap estimate. These are proved below in Sects. 6 and 7. We give this part of the argument first because, as with earlier sections, it gives a useful picture of the random graph.

Take any undirected graph on the nonnegative integers, with the property:

$$\begin{cases} \text{There exists } C > 0 \text{ such that for any } j \geqslant 2, \\ j \text{ is connected to some } k \leqslant C \log j. \end{cases} \tag{10}$$

Let $\{X_n\}_{n \geqslant 0}$ be the Markov chain on this graph, which, starting at state i, jumps to a neighbor j with probability proportional to $Q(j) = 2^{-(j+1)}$. The following is the main result of this section.

Theorem 4 *Let K be the transition kernel of the Markov chain defined above. Suppose that K has a spectral gap. Let μ be the stationary distribution of the chain, and let $a := e^{1/C}$. Then for any $i \in \mathbb{N}$ and any $\ell \geqslant 1$,*

$$\|K_i^\ell - \mu\| \leqslant C_1 e^{\log^*_a i} e^{-C_2 \ell},$$

where C_1 and C_2 are positive constants that depend only the properties of the chain (and not on i or ℓ).

By Lemma 4, $G(\infty, 1/2)$ satisfies the property (10) with probability one, for some C that may depend on the realization of the graph. The Rado graph also satisfies property (10), with $K = 1/\log 2$. Thus, the random walk starting from j mixes in time $\log^*_2 j$ on the Rado graph, provided that it has a spectral gap. For $G(\infty, 1/2)$, assuming that the walk has a spectral gap, the mixing time starting from j is $\log^*_a j$, where a depends on the realization of the graph. The spectral gap for $G(\infty, 1/2)$ will be proved in Sect. 6, and the spectral gap for the Rado graph will be established in Sect. 7. Therefore, this proves Theorem 1 and also the analogous result for $G(\infty, 1/2)$.

Proof (Of Theorem 4) Note that $a > 1$. Let $Z_n := \log^*_a X_n$. We claim that there is some j_0 sufficiently large, and some positive constant c, such that

$$\mathbb{E}(e^{Z_{n+1}} | \mathcal{F}_n) \leqslant e^{Z_n - c} \quad \text{if } Z_n > j_0, \tag{11}$$

where $\mathcal{F}_n$ is the σ-algebra generated by $X_0, \ldots, X_n$. (The proof is given below.) This implies that if we define the stopping time $S := \min\{n \geqslant 0 : X_n \leqslant j_0\}$, then $\{e^{Z_{S\wedge n}+c(S\wedge n)}\}_{n\geqslant 0}$ is a supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geqslant 0}$ (see details below). Moreover, it is nonnegative. Thus, if we start from the deterministic initial condition $X_0 = j$, then for any n,

$$\mathbb{E}(e^{Z_{S\wedge n}+c(S\wedge n)}|X_0 = j) \leqslant e^{Z_{S\wedge 0}+c(S\wedge 0)} = e^{\log_a^* j}.$$

But $Z_{S\wedge n} \geqslant 0$. Thus, $\mathbb{E}(e^{c(S\wedge n)}|X_0 = j) \leqslant e^{\log_a^* j}$. Taking $n \to \infty$ and applying the monotone convergence theorem, we get

$$\mathbb{E}(e^{cS}|X_0 = j) \leqslant e^{\log_a^* j}. \tag{12}$$

Now take any $j \geqslant 1$ and $n \geqslant 1$. Let μ be the stationary distribution, and let $\mu_{j,n}$ be the law of X_n when $X_0 = j$. Take any $A \subseteq \{0, 1, \ldots\}$. Then for any $m \leqslant n$,

$$\begin{aligned}\mu_{j,n}(A) &= \mathbb{P}(X_n \in A|X_0 = j)\\ &= \sum_{i=0}^{m}\sum_{l=0}^{j_0} \mathbb{P}(X_n \in A|S = i,\ X_i = l,\ X_0 = j)\mathbb{P}(S = i,\ X_i = l|X_0 = j)\\ &\quad + \mathbb{P}(X_n \in A|S > m,\ X_0 = j)\mathbb{P}(S > m|X_0 = j).\end{aligned}$$

But

$$\mathbb{P}(X_n \in A|S = i,\ X_i = l,\ X_0 = j) = \mathbb{P}(X_n \in A|X_i = l) = \mu_{l,n-i}(A),$$

and

$$\mu(A) = \sum_{i=0}^{m}\sum_{l=0}^{j_0} \mu(A)\mathbb{P}(S = i,\ X_i = l|X_0 = j) + \mu(A)\mathbb{P}(S > m|X_0 = j).$$

Thus, $|\mu_{j,n}(A) - \mu(A)|$ can be bounded above by

$$\sum_{i=0}^{m}\sum_{l=0}^{j_0} |\mu_{l,n-i}(A) - \mu(A)|\mathbb{P}(S = i,\ X_i = l|X_0 = j) + \mathbb{P}(S > m|X_0 = j).$$

Now, if our Markov chain has a spectral gap, there exist constants C_1 and C_2 depending only on j_0 and the spectral gap, such that

$$|\mu_{l,n-i}(A) - \mu(A)| \leqslant C_1 e^{-C_2(n-i)} \leqslant C_1 e^{-C_2(n-m)}$$

for all $0 \leqslant i \leqslant m$ and $0 \leqslant l \leqslant j_0$. Using this bound and the bound (12) on $\mathbb{E}(e^{cS}|X_0 = j)$ obtained above, we get

$$|\mu_{j,n}(A) - \mu(A)| \leqslant C_1 e^{-C_2(n-m)} + e^{\log_a^* j - cm}.$$

Taking $m = \lceil n/2 \rceil$, we get the desired result. □

Proof (Of inequality (11)*)* It suffices to take $n = 0$. Suppose that $X_0 = j$ for some $j \geqslant 1$. By assumption, there is a neighbor k of j such that $k \leqslant K \log j = \log_a j$. Assuming that j is sufficiently large (depending on K), we have that for any $l \leqslant k$,

$$\log_a^* l \leqslant \log_a^* k \leqslant \log_a^*(\log_a j) = \log_a^* j - 1.$$

Also, $\log_a^* l \leqslant \log_a^* j$ for any $l \leqslant j$. Thus,

$$\mathbb{E}(e^{Z_1 - Z_0}|X_0 = j) \leqslant e^{-1}\mathbb{P}(X_1 \leqslant k|X_0 = j) + \mathbb{P}(k < X_1 \leqslant j|X_0 = j) + \sum_{l>j} e^{\log_a^* l - \log_a^* j}\mathbb{P}(X_1 = l|X_0 = j).$$

Now for any $l \geqslant k$,

$$\mathbb{P}(X_1 = l|X_0 = j) \leqslant \frac{\mathbb{P}(X_1 = l|X_0 = j)}{\mathbb{P}(X_1 = k|X_0 = j)} = \frac{Q(l)}{Q(k)} = 2^{-(l-k)}.$$

Thus,

$$\sum_{l>j} e^{\log_a^* l - \log_a^* j}\mathbb{P}(X_1 = l|X_0 = j) \leqslant \sum_{l>j} e^{\log_a^* l - \log_a^* j} 2^{-(l-k)},$$

which is less than $1/4$ if j is sufficiently large (since $k \leqslant \log_a^* j$). Next, let L be the set of all $l > k$ that are connected to j. Then

$$\mathbb{P}(X_1 > k|X_0 = j) \leqslant \frac{\mathbb{P}(X_1 > k|X_0 = j)}{\mathbb{P}(X_1 \geqslant k|X_0 = j)} = \frac{\sum_{l \in L} 2^{-l}}{2^{-k} + \sum_{l \in L} 2^{-l}}.$$

Since the map $x \mapsto x/(2^{-k} + x)$ is increasing, this shows that

$$\mathbb{P}(X_1 > k|X_0 = j) \leqslant \frac{\sum_{l>k} 2^{-l}}{2^{-k} + \sum_{l>k} 2^{-l}} = \frac{1}{2}.$$

Combining, we get that for sufficiently large j,

$$\begin{aligned}\mathbb{E}(e^{Z_1-Z_0}|X_0=j) &\leqslant e^{-1}\mathbb{P}(X_1\leqslant k|X_0=j)+\mathbb{P}(X_1>k|X_0=j)+\frac{1}{4}\\ &= e^{-1}+(1-e^{-1})\mathbb{P}(X_1>k|X_0=j)+\frac{1}{4}\\ &\leqslant e^{-1}+\frac{1-e^{-1}}{2}+\frac{1}{4}=\frac{3+2e^{-1}}{4}<1.\end{aligned}$$

□

Proof (Of the Supermartingale Property) Note that

$$\begin{aligned}&\mathbb{E}(e^{Z_{S\wedge(n+1)}+c(S\wedge(n+1))}|\mathcal{F}_n)\\ &=\sum_{i=0}^{n}\mathbb{E}(e^{Z_{S\wedge(n+1)}+c(S\wedge(n+1))}1_{\{S=i\}}|\mathcal{F}_n)+\mathbb{E}(e^{Z_{S\wedge(n+1)}+c(S\wedge(n+1))}1_{\{S>n\}}|\mathcal{F}_n)\\ &=\sum_{i=0}^{n}\mathbb{E}(e^{Z_i+ci}1_{\{S=i\}}|\mathcal{F}_n)+\mathbb{E}(e^{Z_{n+1}+c(n+1)}1_{\{S>n\}}|\mathcal{F}_n).\end{aligned}$$

The events $\{S=i\}$ are $\mathcal{F}_n$-measurable for all $0\leqslant i\leqslant n$, and so is the event $\{S>n\}$. Moreover, $Z_0,\ldots,Z_n$ are also $\mathcal{F}_n$-measurable. Thus, the above expression shows that

$$\mathbb{E}(e^{Z_{S\wedge(n+1)}+c(S\wedge(n+1))}|\mathcal{F}_n)=1_{\{S\leqslant n\}}e^{Z_{S\wedge n}+c(S\wedge n)}+1_{\{S>n\}}\mathbb{E}(e^{Z_{n+1}+c(n+1)}|\mathcal{F}_n).$$

But if $S>n$, then $Z_n>j_0$, and therefore by (11),

$$\mathbb{E}(e^{Z_{n+1}+c(n+1)}|\mathcal{F}_n)\leqslant e^{Z_n-c+c(n+1)}=e^{Z_n+cn}.$$

Thus,

$$\begin{aligned}\mathbb{E}(e^{Z_{S\wedge(n+1)}+c(S\wedge(n+1))}|\mathcal{F}_n) &\leqslant 1_{\{S\leqslant n\}}e^{Z_{S\wedge n}+c(S\wedge n)}+1_{\{S>n\}}e^{Z_n+cn}\\ &= e^{Z_{S\wedge n}+c(S\wedge n)}.\end{aligned}$$

□

6 Spectral Gap for the Random Model

Our next goal is to show that the random reversible couple (K,π) admits a spectral gap. The arguments make use of the ideas and notation of Sect. 3. In particular, recall the event $\mathcal{B}=\{p(x)<x\ \ \forall\, x\in\mathbb{N}_+\}$ from Lemma 1 and the random tree $\mathcal{T}$ with

edge set F from Lemma 3. The argument uses a version of Cheeger's inequality for trees which is further developed in Appendix 1.

Proposition 1 *On $\mathcal{B}$, there exists a random constant $\Lambda > 0$ such that*

$$\forall f \in L^2(\pi), \qquad \Lambda\pi[(f - \pi[f])^2] \leqslant \mathcal{E}(f)$$

where in the r.h.s. $\mathcal{E}$ is the Dirichlet form defined by

$$\forall f \in L^2(\pi), \qquad \mathcal{E}(f) := \frac{1}{2} \sum_{x,y \in \mathbb{N}} (f(y) - f(x))^2 \pi(x) K(x, y).$$

Taking into account that for any $f \in L^2(\pi)$, the variance $\pi[(f - \pi[f])^2]$ of f with respect to π is bounded above by $\pi[(f - f(0))^2]$, the previous result is an immediate consequence of the following existence of positive first Dirichlet eigenvalue under $\mathcal{B}$.

Proposition 2 *On $\mathcal{B}$, there exists a random constant $\Lambda > 0$ such that*

$$\forall f \in L^2(\pi), \qquad \Lambda\pi[(f - f(0))^2] \leqslant \mathcal{E}(f). \tag{13}$$

The proof of Proposition 2 is based on the pruning of $\mathcal{G}$ into $\mathcal{T}$ and then resorting to Cheeger's inequalities for trees. More precisely, let us introduce the following notations. Define the Markov kernel $K_{\mathcal{T}}$ as

$$\forall x, y \in \mathbb{N}, \qquad K_{\mathcal{T}}(x, y) := \begin{cases} K(x, y) & \text{if } \{x, y\} \in F, \\ 1 - \sum_{z \in \mathbb{N} \setminus \{x\}} K_{\mathcal{T}}(x, z) & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this kernel is reversible with respect to π. The corresponding Dirichlet form is given, for any $f \in L^2(\pi)$, by

$$\mathcal{E}_{\mathcal{T}}(f) := \sum_{x,y \in \mathbb{N}} (f(y) - f(x))^2 \frac{\pi(x) K_{\mathcal{T}}(x, y)}{2} = \sum_{\{x,y\} \in F} (f(y) - f(x))^2 \pi(x) K(x, y)$$

It will be convenient to work with $\widetilde{\mathcal{E}} := Z\mathcal{E}_{\mathcal{T}}$, where Z is the normalizing constant of π, as in equation (2). Define a nonnegative measure μ on $\mathbb{N}_+$ as

$$\forall x \in \mathbb{N}_+, \qquad \mu(x) := Q(x)Q(p(x)). \tag{14}$$

Proposition 3 *On $\mathcal{B}$, there exists $\lambda > 0$ such that*

$$\forall f \in L^2(\mu), \qquad \lambda\mu[(f - f(0))^2] \leqslant \widetilde{\mathcal{E}}(f). \tag{15}$$

This result immediately implies Proposition 2. Indeed, due on one hand to the inclusion $N(x) \subset [\![p(x), \infty[\![$ and on the other hand to the nature of Q, we have

$$\forall\, x \in \mathbb{N}_+, \qquad Q(p(x)) \leqslant Q(N(x)) \leqslant 2Q(p(x)). \tag{16}$$

Thus for any $f \in L^2(\mu)$,

$$\begin{aligned}
\lambda\pi[(f - f(0))^2] &= \frac{\lambda}{Z} \sum_{x\in\mathbb{N}_+} (f(x) - f(0))^2 Q(x)Q(N(x)) \\
&\leqslant \frac{2\lambda}{Z} \sum_{x\in\mathbb{N}_+} (f(x) - f(0))^2 Q(x)Q(p(x)) \\
&= \frac{2\lambda}{Z}\mu[(f - f(0))^2] \leqslant \frac{2}{Z}\widetilde{\mathcal{E}}(f) = 2\mathcal{E}_{\mathcal{T}}(f) \leqslant 2\mathcal{E}(f),
\end{aligned}$$

and thus, Proposition 2 holds with $\Lambda := \lambda/2$.

The proof of Proposition 3 is based on a Dirichlet-variant of the Cheeger inequality (which is in fact slightly simpler than the classical one, see Appendix 1). For any $A \subset \mathbb{N}_+$, define $\partial A := \{\{x, y\} : x \in A, y \notin A\} \subset \overline{E}$. Endow $\overline{E}$ with the measure ν induced by, for any $\{x, y\} \in \overline{E}$,

$$\nu(\{x, y\}) := Z\pi(x)K_{\mathcal{T}}(x, y) = \begin{cases} Q(x)Q(y) & \text{if } \{x, y\} \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Define the Dirichlet–Cheeger constant

$$\iota := \inf_{A\in\mathcal{A}} \frac{\nu(\partial A)}{\mu(A)} \geqslant 0$$

where $\mathcal{A} := \{A \subset \mathbb{N}_+ : A \neq \emptyset\}$. The proof of the traditional Markovian Cheeger's inequality given in the lectures by Saloff-Coste [23] implies directly that the best constant λ in Proposition 3 satisfies $\lambda \geqslant \iota^2/2$. Thus it remains to check:

Proposition 4 *On $\mathcal{B}$, we have $\iota \geqslant 1/2$ and in particular $\iota > 0$.*

Proof Take any nonempty $A \in \mathcal{A}$ and decompose it into its connected components with respect to $\mathcal{T}$: $A = \bigsqcup_{i\in\mathcal{I}} A_i$, where the index set $\mathcal{I}$ is at most denumerable. Note that

$$\mu(A) = \sum_{i\in\mathcal{I}} \mu(A_i), \quad \nu(A) = \sum_{i\in\mathcal{I}} \nu(A_i),$$

where the second identity holds because there are no edges in F connecting two different A_i's. Thus, it follows that $\iota = \inf_{A\in\widetilde{\mathcal{A}}} \nu(\partial A)/\mu(A)$, where $\widetilde{\mathcal{A}}$ is the set of subsets of $\mathcal{A}$ which are $\mathcal{T}$-connected.

Consider $A \in \widetilde{\mathcal{A}}$, it has a smallest element $a \in \mathbb{N}_+$ (since $0 \notin A$). Let T_a be the subtree of descendants of A in $\mathcal{T}$ (i.e., the set of vertices from $\mathbb{N}_+$ whose non-self-intersecting path to 0 passes through a). We have $A \subset T_a$, and $\partial A \supset \{a, p(a)\} = \partial T_a$, and it follows that $\nu(\partial A)/\mu(A) \geqslant \nu(\partial T_a)/\mu(T_a)$. We deduce that

$$\iota \geqslant \inf_{a \in \mathbb{N}_+} \frac{\nu(\partial T_a)}{\mu(T_a)} = \inf_{a \in \mathbb{N}_+} \frac{Q(a)Q(p(a))}{\mu(T_a)}.$$

On $\mathcal{B}$, we have for any $a \in \mathbb{N}_+$, on the one hand

$$\forall\, x \in T_a, \qquad p(x) \geqslant p(a), \tag{17}$$

and on the other hand

$$T_a \subset [\![a, \infty[\![. \tag{18}$$

We get $\mu(T_a)$ equals

$$\sum_{x \in T_a} Q(x)Q(p(x)) \geqslant Q(p(a)) \sum_{x \in T_a} Q(x) \geqslant Q(p(a)) \sum_{x \in [\![a,\infty[\![} Q(x) = 2Q(p(a))Q(a).$$

It follows that $\iota \geqslant 1/2$. $\square$

Lemma 4 can now be used to see that the ball Markov chain on the random graph has a.s. a spectral gap. Indeed, we deduce from Lemma 4 that there exists a (random) vertex $x_0 \in \mathbb{N}$ such that for any $x > x_0$, $p(x) < x$. Consider

$$x_1 := \max\{p(x) \;:\; x \in [\![1, x_0]\!]\}.$$

It follows that for any $a > x_1$, we have, for all $\forall\, x \in T_a$, $p(x) < x$. (To see this, take any path $a_0, a_1, \ldots$ in T_a, starting at $a_0 = a$, so that $p(a_i) = a_{i-1}$ for each i. Let k be the first index such that $a_k \geqslant a_{k+1}$, assuming that there exists such a k. Then $a_{k+1} \leqslant x_0$, and so $a_k = p(a_{k+1}) \leqslant x_1$. But this is impossible, since $a_0 \leqslant a_k$ and $a_0 > x_1$.) In particular, we see that (17) and (18) hold for $a > x_1$. As a consequence,

$$\inf_{a > x_1} \frac{\nu(\partial T_a)}{\mu(T_a)} \geqslant \frac{1}{2}.$$

By the finiteness of $[\![1, x_1]\!]$, we also have $\inf_{a \in [\![1,x_1]\!]} \nu(\partial T_a)/\mu(T_a) > 0$. So, finally,

$$\iota = \inf_{a \in \mathbb{N}_+} \frac{\nu(\partial T_a)}{\mu(T_a)} \;>\; 0,$$

which shows that $G(\infty, 1/2)$ has a spectral gap a.s.

7 Spectral Gap for the Rado Graph

This section proves the needed spectral gap for the Rado graph. Here the graph has vertex set $\mathbb{N}$ and an edge from i to j if i is less than j and the ith bit of j is a one. We treat carefully the case of a more general base measure, $Q(x) = (1-\delta)\delta^x$. As delta tends to 1, sampling from this Q is a better surrogate for "pick a neighboring vertex uniformly". Since the normalization doesn't enter, throughout take $Q(x) = \delta^x$. The heart of the argument is a discrete version of Hardy's inequality for trees. This is developed below with full details in Appendix 2.

Consider the transition kernel K reversible with respect to π and associated to the measure Q given by $Q(x) := \delta^x$ for all $x \in \mathbb{N}$, where $\delta \in (0, 1)$ (instead of $\delta = 1/2$ as in the introduction, up to the normalization). Recall that

$$\forall\, x, y \in \mathbb{N}, \qquad K(x, y) := \frac{Q(y)}{Q(N(x))}\mathbb{1}_{N(x)}(y),$$

$$\forall\, x \in \mathbb{N}, \qquad \pi(x) = Z^{-1}Q(x)Q(N(x)),$$

where $N(x)$ is the set of neighbors of x induced by K and where $Z > 0$ is the normalizing constant. Here is the equivalent of Proposition 3:

Proposition 5 *We have*

$$\lambda \geqslant \frac{1-\delta}{16(2 \vee \lceil \log_2 \log_2(2/\log_2(1/\delta))\rceil)}.$$

This bound will be proved via Hardy's inequalities. If we resort to Dirichlet–Cheeger, we rather get

$$\lambda \geqslant \frac{(1-\delta)^2}{2}. \tag{19}$$

To see the advantage of Proposition 5, let δ come closer and closer to 1, namely, approach the problematic case of "pick a neighbor uniformly at random". In this situation, the r.h.s. of the bound of Proposition 5 is of order

$$\frac{1-\delta}{16\lceil \log_2 \log_2(1/(1-\delta))\rceil}$$

which is better than (19) as δ goes to $1-$.

Here we present the Hardy's inequalities method to get Proposition 5 announced above. Our goal is to show that K admits a positive first Dirichlet eigenvalue:

Proposition 6 *There exists* $\Lambda > 0$ *depending on* $\delta \in (0, 1)$ *such that*

$$\forall\, f \in L^2(\pi), \qquad \Lambda\pi[(f - f(0))^2] \leqslant \frac{1}{2}\sum_{x,y\in\mathbb{N}} (f(y) - f(x))^2\,\pi(x)K(x, y).$$

It follows that the reversible couple (K, π) admits a spectral gap bounded below by Λ given above. Indeed, it is an immediate consequence of the fact that for any $f \in L^2(\pi)$, the variance of f with respect to π is bounded above by $\pi[(f - f(0))^2]$.

The proof of Proposition 6 is based on a pruning of K and Hardy's inequalities for trees. Consider the set of unoriented edges induced by K: $E := \{\{x, y\} \in \mathbb{N}\times\mathbb{N} : K(x, y) > 0\}$ (in particular, E does not contain the self-edges or singletons). For any $x \in \mathbb{N}_+$, let $p(x)$ the smallest bit equal to 1 in the binary expansion of x, i.e.,

$$p(x) := \min\{y \in \mathbb{N} : K(x, y) > 0\}.$$

Define the subset F of E by

$$F := \{\{x, p(x)\} \in E : x \in \mathbb{N}_+\}$$

and the function ν on F via

$$\forall \{x, p(x)\} \in F, \qquad \nu(\{x, p(x)\}) := Z\pi(x)K(x, p(x)) = Q(x)Q(p(x)).$$

To any $f \in L^2(\pi)$, associate the function $(df)^2$ on F given by

$$\forall \{x, p(x)\} \in F, \qquad (df)^2(\{x, p(x)\}) := (f(x) - f(p(x)))^2.$$

Finally, consider the (non-negative) measure μ defined on $\mathbb{N}_+$ via

$$\forall x \in \mathbb{N}_+, \qquad \mu(x) := Q(x)Q(p(x)). \tag{20}$$

Then we have:

Proposition 7 *There exists* $\lambda > 0$ *depending on* $\delta \in (0, 1)$ *such that*

$$\forall f \in L^2(\mu), \qquad \lambda\mu[(f - f(0))^2] \leqslant \sum_{e\in F}(df)^2(e)\nu(e).$$

This result implies Proposition 6. Indeed, note that by the definition of Q,

$$\forall x \in \mathbb{N}_+, \qquad Q(p(x)) \leqslant Q(N(x)) \leqslant \frac{1}{1-\delta}Q(p(x)). \tag{21}$$

Thus, for any $f \in L^2(\mu)$,

$$\begin{aligned}\lambda\pi[(f - f(0))^2] &= \frac{\lambda}{Z}\sum_{x\in\mathbb{N}_+}(f(x) - f(0))^2 Q(x)Q(N(x)) \\ &\leqslant \frac{\lambda}{(1-\delta)Z}\sum_{x\in\mathbb{N}_+}(f(x) - f(0))^2 Q(x)Q(p(x))\end{aligned}$$

$$= \frac{\lambda}{(1-\delta)Z}\mu[(f - f(0))^2] \leqslant \frac{1}{(1-\delta)Z}\sum_{e\in F}(df)^2(e)\nu(e)$$

$$\leqslant \frac{1}{2(1-\delta)}\sum_{x,y\in\mathbb{N}}(f(y)-f(x))^2\pi(x)K(x,y)$$

namely Proposition 6 holds with $\Lambda := \lambda(1-\delta)$.

Note that $\mathbb{N}$ endowed with the set of non-oriented edges F has the structure of a tree. We interpret 0 as its root, so that for any $x \in \mathbb{N}_+$, $p(x)$ is the parent of x. Note that for any $x \in \mathbb{N}$, the children of x are exactly the numbers $y2^x$, where y is an odd number. We will denote $h(x)$ the height of x with respect to the root 0 (thus, the odd numbers are exactly the elements of $\mathbb{N}$ whose height is equal to 1).

According to [21] (see also Evans, Harris and Pick [10]), the best constant λ in Proposition 7, say λ_0, can be estimated up to a factor 16 via Hardy's inequalities for trees, see (23) below. To describe them we need several notations.

Let $\mathcal{T}$ the set of subsets T of $\mathbb{N}_+$ satisfying the following conditions

- T is non-empty and connected (with respect to F),
- T does not contain 0,
- there exists $M \geqslant 1$ such that $h(x) \leqslant M$ for all $x \in T$,
- if $x \in T$ has a child in T, then all children of x belong to T.

Note that any $T \in \mathcal{T}$ admits a closest element to 0, call it $m(T)$. Note that $m(T) \neq 0$. When T is not reduced to the singleton $\{m(T)\}$, then $T \setminus \{m(T)\}$ has a denumerable infinity of connected components which are indexed by the children of $m(T)$. Since these children are exactly the $y2^{m(T)}$, where $y \in \mathcal{I}$, the set of odd numbers, call $T_{y2^{m(T)}}$ the connected component of $T \setminus \{m(T)\}$ associated to $y2^{m(T)}$. Note that $T_{y2^{m(T)}} \in \mathcal{T}$. We extend ν as a functional on $\mathcal{T}$, via the iteration

- when T is the singleton $\{m(T)\}$, we take $\nu(T) := \nu(\{m(T), p(m(T))\})$,
- when T is not a singleton, decompose T as $\{m(T)\} \sqcup \bigsqcup_{y\in\mathcal{I}} T_{y2^{m(T)}}$, then ν is defined as

$$\frac{1}{\nu(T)} = \frac{1}{\nu(\{m(T)\})} + \frac{1}{\sum_{y\in\mathcal{I}}\nu(T_{y2^{m(T)}})}. \tag{22}$$

For $x \in \mathbb{N}_+$, let S_x be the set of vertices $y \in \mathbb{N}_+$ whose path to 0 passes through x. For any $T \in \mathcal{T}$ we associate the subset

$$T^* := (S_{m(T)} \setminus T) \sqcup L(T)$$

where $L(T)$ is the set of leaves of T, namely the $x \in T$ having no children in T. Equivalently, T^* is the set of all descendants of the leaves of T, themselves included.

Consider $\mathcal{S} \subset \mathcal{T}$ the set of $T \in \mathcal{T}$ which are such that $m(T)$ is an odd number. Finally, define

$$A := \sup_{T\in\mathcal{S}} \frac{\mu(T^*)}{\nu(T)}.$$

We are interested in this quantity because of the Hardy inequalities:

$$A \leqslant \frac{1}{\lambda_0} \leq 16A, \tag{23}$$

where recall that λ_0 is the best constant in Proposition 7. (In [21], only finite trees were considered, the extension to infinite trees is given in Appendix 2). So, to prove Proposition 7, it is sufficient to show that A is finite. To investigate A, we need some further definitions. For any $x \in \mathbb{N}_+$, let

$$b(x) := \frac{Q(2^x)}{Q(p(x))}.$$

A finite path from 0 in the direction to infinity is a finite sequence $z := (z_n)_{n\in[\![0,N]\!]}$ of elements of $\mathbb{N}_+$ such that $z_0 = 0$ and $p(z_n) = z_{n-1}$ for any $n \in [\![1, N]\!]$. On such a path z, we define the quantity

$$B(z) := \sum_{n\in[\![1,N]\!]} b(z_n).$$

The following technical result is crucial for our purpose of showing that A is finite.

Lemma 6 *For any finite path from* 0 *in the direction to infinity* $z := (z_n)_{n\in[\![0,N]\!]}$, *we have* $B(z) \leqslant C$, *where* $C := \sum_{l\in\mathbb{N}} \delta^{2^{2^l}-l} < +\infty$.

Proof Note that for any $n \in [\![1, N]\!]$, $h(z_n) = n$. Furthermore, for any $x \in \mathbb{N}_+$, we have $h(x) \leqslant x$ and we get $h(p(z_n)) = h(z_n) - 1 = n - 1$, so that $p(z_n) \geqslant n - 1$. Writing $z_n = y_n 2^{p(z_n)}$, for some odd number y_n, it follows that

$$b(z_n) = \frac{Q(2^{y_n 2^{p(z_n)}})}{Q(p(z_n))} = \delta^{2^{y_n 2^{p(z_n)}}-p(z_n)} \leqslant \delta^{2^{2^{p(z_n)}}-p(z_n)} \leqslant \delta^{2^{2^{n-1}}-n-1}.$$

The desired result follows at once. □

We need two ingredients about ratios $\mu(T^*)/\nu(T)$. Here is the first one.

Lemma 7 *For any* $T \in \mathcal{T}$ *which is a singleton, we have* $\frac{\mu(T^*)}{\nu(T)} \leqslant \frac{1}{1-\delta}$.

Proof When T is the singleton $\{m(T)\}$, on the one hand we have

$$\nu(T) = \nu(\{p(m(T)), m(T)\}) = \mu(m(T)).$$

On the other hand, T^* is the subtree growing from $m(T)$, namely the subtree containing all the descendants of $m(T)$. Note two properties of T^*:

$$T^* \subset \{y \in \mathbb{N}_+ \,:\, y \geqslant m(T)\} \text{ and } \forall\, y \in T^*,\ p(y) \geqslant p(m(T)), \tag{24}$$

and we further have $p(y) \geqslant m(T)$ for any $y \in T^* \setminus \{m(T)\}$. It follows that

$$\begin{aligned}\mu(T^*) =& \sum_{y \in T^*} Q(y)Q(p(y)) \leqslant Q(p(m(T))) \sum_{y \geqslant m(T)} Q(y) = Q(p(m(T))) \sum_{y \geqslant m(T)} \delta^y \\ =& \ Q(p(m(T)))\frac{Q(m(T))}{1-\delta} = \frac{1}{1-\delta}\mu(m(T)).\end{aligned} \tag{25}$$

Thus, we get $\mu(T^*)/\nu(T) \leqslant \frac{1}{1-\delta}$. □

For the second ingredient, we need some further definitions. The length $\ell(T)$ of $T \in \mathcal{T}$ is given by $\ell(T) \coloneqq \max_{x \in T} h(x) - \min_{x \in T} h(x)$, and for any $l \in \mathbb{N}$, we define

$$\mathcal{T}_l \coloneqq \{T \in \mathcal{T} \,:\, \ell(T) \leqslant l\}$$

Lemma 8 *For any $l \in \mathbb{N}$, we have* $\sup_{T \in \mathcal{T}_l} \frac{\mu(T^*)}{\nu(T)} < +\infty$.

Proof We will prove the finiteness by induction over $l \in \mathbb{N}$. First, note that $\mathcal{T}_0$ is the set of singletons, and so Lemma 7 implies that $\sup_{T \in \mathcal{T}_0} \frac{\mu(T^*)}{\nu(T)} \leqslant \frac{1}{1-\delta}$. Next, assume that the supremum is finite for some $l \in \mathbb{N}$ and let us show that it is also finite for $l+1$.

Consider $T \in \mathcal{T}_{l+1}$, with $\ell(T) = l+1$; in particular, T is not a singleton. Decompose T as $\{m(T)\} \sqcup \bigsqcup_{y \in \mathcal{I}} T_{y2^{m(T)}}$ and recall the relation (22). Since $T^* = \bigsqcup_{y \in \mathcal{I}} T^*_{y2^{m(T)}}$, it follows that

$$\begin{aligned}\frac{\mu(T^*)}{\nu(T)} &= \sum_{y \in \mathcal{I}} \mu(T^*_{y2^{m(T)}}) \left(\frac{1}{\nu(\{m(T)\})} + \frac{1}{\sum_{y \in \mathcal{I}} \nu(T_{y2^{m(T)}})} \right) \\ &= \frac{\sum_{y \in \mathcal{I}} \mu(T^*_{y2^{m(T)}})}{\nu(\{m(T)\})} + \frac{\sum_{y \in \mathcal{I}} \mu(T^*_{y2^{m(T)}})}{\sum_{y \in \mathcal{I}} \nu(T_{y2^{m(T)}})} \\ &\leqslant \frac{\mu(\sqcup_{y \in \mathcal{I}} T^*_{y2^{m(T)}})}{\mu(m(T))} + \sup\left\{ \frac{\mu(T^*_{y2^{m(T)}})}{\nu(T_{y2^{m(T)}})} \,:\, y \in \mathcal{I} \right\}.\end{aligned} \tag{26}$$

Consider the first term on the right. Given $y \in \mathcal{I}$, the smallest possible element of $T^*_{y2^{m(T)}}$ is $y2^{m(T)}$, and we have for any $x \in T^*_{y2^{m(T)}}$,

$$p(x) \geqslant p(y2^{m(T)}) = m(T).$$

Thus we have the equivalent of (24):

$$\bigsqcup_{y\in\mathcal{I}} T^*_{y2^{m(T)}} \subset \{y \in \mathbb{N}_+ \,:\, y \geqslant 2^{m(T)}\},\ \forall\, x \in \bigsqcup_{y\in\mathcal{I}} T^*_{y2^{m(T)}},\ p(x) \geqslant m(T). \tag{27}$$

Following the computation (25), we get

$$\mu\left(\bigsqcup_{y\in\mathcal{I}} T^*_{y2^{m(T)}}\right) < \frac{1}{1-\delta} Q(m(T))Q(2^{m(T)}),$$

where the inequality is strict, because in (27) we cannot have equality for all $x \in \bigsqcup_{y\in\mathcal{I}} T^*_{y2^{m(T)}}$. It follows that

$$\frac{\sum_{y\in\mathcal{I}} \mu(T^*_{y2^{m(T)}})}{\mu(m(T))} < \frac{1}{1-\delta}\frac{Q(m(T))Q(2^{m(T)})}{Q(m(T))Q(p(m(T)))} = \frac{b(m(T))}{1-\delta} \leqslant \frac{C}{1-\delta} \tag{28}$$

where C is the constant introduced in Lemma 6. Since for any $y \in \mathcal{I}$, we have $T_{y2^{m(T)}} \in \mathcal{T}_l$, we deduce the desired result from the induction hypothesis. □

We are now ready to prove Proposition 7.

Proof (Of Proposition 7) Fix some $T \in \mathcal{S}$, we are going to show that $\mu(T^*)/\nu(T) \leqslant 1 + C/(1-\delta)$, where C is the constant introduced in Lemma 6. Due to Lemma 7, this bound is clear if T is a singleton. When T is not the singleton $\{m(T)\}$, decompose T as $\{m(T)\} \sqcup \bigsqcup_{y\in\mathcal{I}} T_{y2^{m(T)}}$ and let us come back to (26). Denote $z_1 := m(T)$ and

$$\epsilon := \frac{b(z_1)}{1-\delta} - \frac{\sum_{y\in\mathcal{I}} \mu(T^*_{y2^{m(T)}})}{\mu(m(T))}$$

which is positive according to (28). Coming back to (26), we have shown

$$\frac{\mu(T^*)}{\nu(T)} \leqslant \frac{b(z_1)}{1-\delta} + \frac{\mu(T^*_{z_2})}{\nu(T_{z_2})}$$

where $z_2 \in \{y2^{m(T)} \,:\, y \in \mathcal{I}\}$ is such that

$$\sup\left\{\frac{\mu(T^*_{y2^{m(T)}})}{\nu(T_{y2^{m(T)}})} \,:\, y \in \mathcal{I}\right\} \leqslant \frac{\mu(T^*_{z_2})}{\nu(T_{z_2})} + \epsilon.$$

To get the existence of z_2, we used that the supremum is finite, as ensured by Lemma 8.

By iterating this procedure, define a finite path from 0 in the direction to infinity $z := (z_n)_{n\in[\![0,N]\!]}$, such that for any $n \in [\![1, N-1]\!]$,

$$\frac{\mu(T^*_{z_n})}{\nu(T_{z_n})} \leqslant \frac{b(z_n)}{1-\delta} + \frac{\mu(T^*_{z_{n+1}})}{\nu(T_{z_{n+1}})}$$

and T_{z_N} is a singleton. We have $N \leqslant \max\{h(x) \,:\, x \in T\}$. We deduce that

$$\frac{\mu(T^*)}{\nu(T)} \leqslant \frac{B(z)}{1-\delta} + \frac{\mu(T^*_{z_N})}{\nu(T_{z_N})} \leqslant \frac{C+1}{1-\delta},$$

as desired. □

To get an explicit bound in terms of δ, it remains to investigate the quantity C.

Lemma 9 *We have*

$$C \leqslant \begin{cases} 2 & \textit{if } \delta \in (0, 1/\sqrt{2}], \\ 1 + \left\lceil \log_2 \log_2 \left(\frac{2}{\log_2(1/\delta)}\right)\right\rceil & \textit{if } \delta \in (1/\sqrt{2}, 1). \end{cases}$$

Proof Consider $l_0 := \min(l \in \mathbb{N}_+ \,:\, \delta^{2^{2^l}-l} \leqslant 1/2)$. Elementary computations show that

$$\forall\, l \geqslant 1, \qquad 2^{2^{l+1}} - l - 1 \geqslant 2(2^{2^l} - l),$$

so we get

$$\sum_{l\geqslant l_0} \delta^{2^{2^l}-l} \leqslant \sum_{n\geqslant 0} \frac{1}{2^{2^n}} \leqslant \sum_{n\geqslant 1} \frac{1}{2^n} = 1.$$

Since we have for any $l \in \mathbb{N}$, $2^{2^l} - l \geqslant 0$, we deduce

$$C \leqslant 1 + \sum_{l\in[\![0,l_0-1]\!]} \delta^{2^{2^l}-l} \leqslant 1 + l_0.$$

It is not difficult to check that for any $l \geqslant 1$, $2^{2^l} - l \geqslant \frac{1}{2}2^{2^l}$, so that

$$\begin{aligned} l_0 &= \min\{l \in \mathbb{N}_+ \,:\, 2^{2^l} - l \geqslant 1/\log_2(1/\delta)\} \leqslant \min\{l \in \mathbb{N}_+ \,:\, 2^{2^l} \geqslant 2/\log_2(1/\delta)\} \\ &= 1 \vee \lceil \log_2 \log_2(2/\log_2(1/\delta))\rceil. \end{aligned}$$

The announced result follows from the fact

$$\log_2 \log_2(2/\log_2(1/\delta)) \geqslant 1 \Leftrightarrow \delta \geqslant \frac{1}{\sqrt{2}}.$$

□

The following observations show that Q needs to be at least decaying exponentially for the Hardy inequality approach to work.

Remark 2

(a) In view of the expression of π, it is natural to try to replace (20) by

$$\forall\, x \in \mathbb{N}_+, \qquad \mu(x) := Z\pi(x) = Q(x)Q(N(x)).$$

But then in Lemma 7, where we want the ratios $\mu(T^*)/\nu(T)$ to be bounded above for singletons T, we end up with the fact that

$$\frac{Q(N(m(T)))}{Q(p(m(T)))} = \frac{\mu(T)}{\nu(T)} \leqslant \frac{\mu(T^*)}{\nu(T)}$$

must be bounded above for singletons T. Namely an extension of (21) must hold: there exists a constant $c > 0$ such that

$$\forall\, x \in \mathbb{N}_+, \qquad Q(N(x)) \leqslant cQ(p(x)). \tag{29}$$

Writing $x = y2^p$, with $y \in \mathcal{I}$ and $p \in \mathbb{N}$, we must have $Q(N(y2^p)) \leqslant cQ(p)$. Take $y = 1 + 2 + 4 + \cdots + 2^l$, then we get that $p, p+1, \ldots, p+l$ all belong to $Q(N(y2^p))$, so that $Q(\{p, p+1, \ldots, p+l\}) \leqslant cQ(p)$, and letting l go to infinity, it follows that $Q([\![p, \infty[\![) \leqslant cQ(p)$, namely, Q has exponential tails.

(b) Other subtrees of the graph generated by K could have been considered. It amounts to choose the parent of any $x \in \mathbb{N}_+$. But among all possible choices of such a neighbor, the one with most weight is $p(x)$, at least if Q is decreasing. In view of the requirement (29), it looks like the best possible choice.

(c) If one is only interested in Proposition 7 with μ defined by (20), then many more probability measures Q can be considered, in particular any polynomial probability of the form $Q(x) := \frac{1}{\zeta(l)(x+1)^l}$, for any $x \in \mathbb{N}$, where ζ is the Riemann function and $l > 1$.

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Appendix 1: Dirichlet–Cheeger Inequalities

We begin by showing the Dirichlet–Cheeger inequality that we have been using in the previous sections. It is a direct extension (even simplification) of the proof of the Cheeger inequality given in Saloff-Coste [23]. We end this appendix by proving that it is in general not possible to compare linearly the Dirichlet–Cheeger constant of an absorbed Markov chain with the largest Dirichlet–Cheeger constant induced on a spanning subtree.

Let us work in continuous time. Consider L a sub-Markovian generator on a finite set V. Namely, $L := (L(x, y))_{x,y\in V}$, whose off-diagonal entries are non-negative and whose row sums are non-positive. Assume that L is irreducible and reversible with respect to a probability π on V.

Let $\lambda(L)$ be the smallest eigenvalue of $-L$ (often called the Dirichlet eigenvalue). The variational formula for eigenvalues shows that

$$\lambda(L) = \min_{f\in\mathbb{R}^V\setminus\{0\}} \frac{-\pi[fL[f]]}{\pi[f^2]}. \tag{30}$$

The Dirichlet–Cheeger constant $\iota(L)$ is defined similarly, except that only indicator functions are considered in the minimum:

$$\iota(L) = \min_{A\subset V,\, A\neq\emptyset} \frac{-\pi[\mathbb{1}_A L[\mathbb{1}_A]]}{\pi[A]}. \tag{31}$$

Here is the Dirichlet–Cheeger inequality:

Theorem 5 *Assuming* $L \neq 0$, *we have*

$$\frac{\iota(L)^2}{2\ell(L)} \leqslant \lambda(L) \leqslant \iota(L)$$

where $\ell(L) := \max\{|L(x,x)| : x \in V\} > 0$.

When L is Markovian, the above inequalities are trivial and reduce to $\iota(L) = \lambda(L) = 0$. Indeed, it is sufficient to consider $f = \mathbb{1}$ and $A = V$ respectively in the r.h.s. of (30) and (31). Thus there is no harm in supposing furthermore that L is strictly sub-Markovian: at least one of the row sums is negative. To bring this situation back to a Markovian setting, it is usual to extend V into $\overline{V} := V \sqcup \{0\}$ where $0 \notin V$ is a new point. Then one introduces the extended Markov generator $\overline{L}$ on $\overline{V}$ via

$$\forall\, x, y \in \overline{V}, \qquad \overline{L}(x, y) := \begin{cases} L(x, y) & \text{if } x, y \in V, \\ -\sum_{z\in V} L(x, z) & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the point 0 is absorbing for the Markov processes associated to $\overline{L}$.

It is convenient to give another expression for $\iota(L)$. Consider the set of edges $\overline{E} := \{\{x, y\} : x \neq y \in \overline{V}\}$. We define a measure μ on $\overline{E}$:

$$\forall\, e := \{x, y\} \in \overline{E}, \qquad \mu(e) := \begin{cases} \pi(x)L(x, y) \text{ if } x, y \in V, \\ \pi(x)\overline{L}(x, 0) \text{ if } y = 0, \\ \pi(y)\overline{L}(y, 0) \text{ if } x = 0. \end{cases}$$

(Note that the reversibility assumption was used to ensure that the first line is well-defined.) Extend any $f \in \mathbb{R}^V$ into the function $\overline{f}$ on $\overline{V}$ by making it vanish at 0 and define

$$\forall\, e := \{x, y\} \in \overline{E}, \qquad |d\overline{f}|(e) := |\overline{f}(y) - \overline{f}(x)|.$$

With these definitions we can check that

$$\forall\, f \in \mathbb{R}^V, \qquad -\pi[fL[f]] = \sum_{e \in \overline{E}} |d\overline{f}|^2(e)\mu(e).$$

These notations enable to see (31) as a L^1 version of (30):

Proposition 8 *We have*

$$\iota(L) = \min_{f \in \mathbb{R}^V \setminus \{0\}} \frac{\sum_{e \in \overline{E}} |d\overline{f}|(e)\mu(e)}{\pi[|f|]}.$$

Proof Restricting the minimum in the r.h.s. to indicator functions, we recover the r.h.s. of (31). It is thus sufficient to show that for any given $f \in \mathbb{R}^V \setminus \{0\}$,

$$\frac{\sum_{e \in \overline{E}} |d\overline{f}|(e)\mu(e)}{\pi[|f|]} \geqslant \iota(L). \tag{32}$$

Note that $|d\overline{f}|(e) \geqslant |d|\overline{f}||(e)$ for any $e \in \overline{E}$, so without lost of generality, we can assume $f \geqslant 0$. For any $t \geqslant 0$, consider the set F_t and its indicator function given by

$$F_t := \{\overline{f} > t\} = \{f > t\} \text{ and } f_t := \mathbb{1}_{F_t}.$$

Note that

$$\forall\, x \in V, \qquad f(x) = \int_0^{+\infty} f_t(x)\, dt,$$

so that by integration,

$$\pi[f] = \int_0^{+\infty} \pi[F_t]\, dt.$$

Furthermore, we have

$$\sum_{e\in\overline{E}} |d\overline{f}|(e)\mu(e) = \sum_{e=:\{x,y\}\,:\,\overline{f}(y)>\overline{f}(x)} (\overline{f}(y)-\overline{f}(x))\mu(e) = \sum_{e=:\{x,y\}\,:\,\overline{f}(y)>\overline{f}(x)} \int_{\overline{f}(x)}^{\overline{f}(y)} \mu(e)\, dt$$
$$= \int_0^{+\infty} \sum_{e=:\{x,y\}\,:\,\overline{f}(y)>t\geqslant\overline{f}(x)} \mu(e)\, dt = \int_0^{+\infty} \mu(\partial F_t)\, dt,$$

where for any $A \subset V$, we define

$$\partial A := \{\{x, y\} \in \overline{E} \;:\; x \in A \text{ and } y \notin A\}.$$

Note that for any such A, we have $\mu(\partial A) = -\pi[\mathbb{1}_A L[\mathbb{1}_A]]$, so that

$$\sum_{e\in\overline{E}} |d\overline{f}|(e)\mu(e) = -\int_0^{+\infty} \pi[f_t L[f_t]]\, dt \geqslant \iota(L) \int_0^{+\infty} \pi[F_t]\, dt = \iota(L)\pi[f],$$

showing (32). □

Proof (Of Theorem 5) Given $g \in \mathbb{R}^V$, let $f = g^2$. By Proposition 8, we compute

$$\iota(L)\pi[f] \leqslant \sum_{e\in\overline{E}} |d\overline{f}|(e)\mu(e) = \sum_{e=:\{x,y\}\in\overline{E}} |\overline{g}^2(y) - \overline{g}^2(x)|\mu(e)$$
$$= \sum_{e=:\{x,y\}\in\overline{E}} |\overline{g}(y) - \overline{g}(x)||\overline{g}(y) + \overline{g}(x)|\mu(e)$$
$$\leqslant \sqrt{\sum_{e=:\{x,y\}\in\overline{E}} (\overline{g}(y) - \overline{g}(x))^2\mu(e)}\sqrt{\sum_{e=:\{x,y\}\in\overline{E}} (\overline{g}(y) + \overline{g}(x))^2\mu(e)}$$
$$\leqslant \sqrt{-\pi[gL[g]]}\sqrt{2\sum_{e=:\{x,y\}\in\overline{E}} (\overline{g}^2(y) + \overline{g}^2(x))\mu(e)}$$
$$= \sqrt{-\pi[gL[g]]}\sqrt{4\sum_{e=:\{x,y\}\in\overline{E}} \overline{g}^2(x)\mu(e)}$$
$$= \sqrt{-\pi[gL[g]]}\sqrt{2\sum_{x\in V} g^2(x)\pi(x) \sum_{y\in\overline{V}\setminus\{x\}} \overline{L}(x, y)}$$

$$= \sqrt{-\pi[gL[g]]}\sqrt{2\sum_{x\in V} g^2(x)\pi(x)|L(x,x)|}$$

$$\leqslant \sqrt{2\ell(L)}\sqrt{-\pi[gL[g]]}\sqrt{\pi[g^2]} = \sqrt{2\ell(L)}\sqrt{-\pi[gL[g]]}\sqrt{\pi[f]}.$$

Thus, we have

$$\frac{\iota(L)^2}{2\ell(L)}\pi[g^2] \leqslant -\pi[gL[g]],$$

which gives the desired lower bound for $\lambda(L)$. The upper bound is immediate. □

The unoriented graph associated to L is $\overline{G} := (\overline{V}, \overline{E}_L)$ where $\overline{E}_L := \{e \in \overline{E} : \mu(e) > 0\}$. Consider $\mathbb{T}$, the set of all subtrees of $\overline{G}$, and for any $T \in \mathbb{T}$, consider the sub-Markovian generator L_T on V associated to T via

$$L_T(x,y) := \begin{cases} L(x,y) & \text{if } \{x,y\} \in \overline{E}(T), \\ -\sum_{z\in V\setminus\{x\}} L_T(x,z) & \text{if } x = y \text{ and } \{x,0\} \notin \overline{E}(T), \\ -\sum_{z\in V\setminus\{x\}} L_T(x,z) - \overline{L}(x,0) & \text{if } x = y \text{ and } \{x,0\} \in \overline{E}(T), \\ 0 & \text{otherwise,} \end{cases}$$

where $x, y \in V$ and $\overline{E}(T)$ is the set of (unoriented) edges of T.

Note that L_T is also reversible with respect to π (it is irreducible if and only if 0 belongs to a unique edge of $\overline{E}(T)$). Denote μ_T the corresponding measure on $\overline{E}$. It is clear that $\mu_T \leqslant \mu$, so we get $\iota(L_T) \leqslant \iota(L)$. In the spirit of Benjamini and Schramm [2], we may wonder if conversely, $\iota(L)$ could be bounded above in terms of $\max_{T\in\mathbb{T}} \iota(L_T)$. A linear comparison is not possible:

Proposition 9 *It does not exist a universal constant $\chi > 0$ such that for any L as above, $\chi\iota(L) \leqslant \max_{T\in\mathbb{T}} \iota(L_T)$.*

Proof Let us construct a family $(L^{(n)})_{n\in\mathbb{N}_+}$ of sub-Markovian generators such that

$$\lim_{n\to\infty} \frac{\max_{T\in\mathbb{T}} \iota(L_T^{(n)})}{\iota(L^{(n)})} = 0 \tag{33}$$

For any $n \in \mathbb{N}_+$, the state space $V^{(n)}$ of $L^{(n)}$ is $[\![n]\!] \times \{0, 1\}$ (more generally, all notions associated to $L^{(n)}$ will marked by the exponent (n)). Denote $V_0^{(n)} := [\![n]\!] \times \{0\}$ and $V_1^{(n)} := [\![n]\!] \times \{1\}$. We take

$$L^{(n)}(x, y) := \begin{cases} \epsilon & \text{if } x \in V_i^{(n)},\ y \in V_{1-i}^{(n)} \text{ with } i \in \{0, 1\}, \\ n\epsilon + 1 & \text{if } x = y \in V_0^{(n)}, \\ n\epsilon & \text{if } x = y \in V_1^{(n)}, \\ 0 & \text{otherwise,} \end{cases}$$

where $x, y \in V^{(n)}$, and $\epsilon > 0$, that will depend on n, is such that $n\epsilon < 1/2$.

Recall that 0 is the cemetery point added to $V^{(n)}$, we have

$$\forall\, x \in V^{(n)}, \qquad \overline{L}^{(n)}(x, 0) = \begin{cases} 1 \text{ if } x \in V_0^{(n)}, \\ 0 \text{ if } x \in V_1^{(n)}. \end{cases}$$

Note that $\pi^{(n)}$ is the uniform probability on $V^{(n)}$. Let us show that

$$\iota(L^{(n)}) = n\epsilon. \tag{34}$$

Consider any $\emptyset \neq A \subset V^{(n)}$, and decompose $A = A_0 \sqcup A_1$, with $A_0 := A \cap V_0^{(n)}$ and $A_1 := A \cap V_1^{(n)}$. Denote $a_0 := |A_0|$ and $a_1 := |A_1|$. We have that ∂A is given by

$$\{\{x, y\} : x \in A_0,\ y \in V_1^{(n)} \setminus A_1\}\{\{x, y\} : x \in V_0^{(n)} \setminus A_0,\ y \in A_1\} \sqcup \{\{x, 0\} : x \in A_0\},$$

and thus $\mu^{(n)}(\partial A) = \frac{1}{2n}(\epsilon(a_0(n - a_1) + a_1(n - a_0)) + a_0)$. It follows that

$$\frac{\mu^{(n)}(\partial A)}{\pi^{(n)}(A)} = n\epsilon + \frac{a_0(1 - 2\epsilon a_1)}{a_0 + a_1}.$$

Taking into account that $1 - 2\epsilon a_1 > 0$, the r.h.s. is minimized with respect to $a_0 \in [\![0, n]\!]$ when $a_0 = 0$ and we then get (independently of a_1), $\mu^{(n)}(\partial A)/\pi^{(n)}(A) = n\epsilon$. We deduce (34).

Consider any $T \in \mathbb{T}^{(n)}$ and let us check that

$$\iota(L_T^{(n)}) \leqslant \epsilon. \tag{35}$$

Observe there exists $x \in V_1^{(n)}$ such that there is a unique $y \in V_0^{(n)}$ with $\{x, y\}$ being an edge of T. Indeed, put on the edges of T the orientation toward the root 0. Thus from any vertex $x \in V_1^{(n)}$ there is a unique exiting edge (but it is possible there are several incoming edges). Necessarily, there is a vertex in $V_0^{(n)}$ whose edge exits to 0. So there are at most $n - 1$ vertices from $V_0^{(n)}$ whose exit edge points toward $V_1^{(n)}$. In particular, there is at least one vertex from $V_1^{(n)}$ which is not pointed out by a vertex from $V_0^{(n)}$. We can take x to be this vertex from $V_1^{(n)}$ and $y \in V_0^{(n)}$ is the vertex pointed out by the oriented edge exiting from x.

Considering the singleton $\{x\}$, we get

$$\mu_T^{(n)}(\partial\{x\}) = \mu_T(\{x, y\}) = \frac{\epsilon}{2n} \text{ and } \pi^{(n)}(x) = \frac{1}{2n}.$$

implying (35) (a little more work would prove that an equality holds there). As a consequence, we see that $\max_{T\in\mathbb{T}^{(n)}} \iota(L_T^{(n)}) \leqslant \epsilon$. Taking for instance $\epsilon := 1/(4n)$ to fulfill the condition $n\epsilon < 1/2$, we obtain $\frac{\max_{T\in\mathbb{T}^{(n)}} \iota(L_T^{(n)})}{\iota(L^{(n)})} \leqslant \frac{1}{n}$, and (33) follows. □

Appendix 2: Hardy's Inequalities

Our goal here is to extend the validity of Hardy's inequalities on finite trees to general denumerable trees, without assumption of local finiteness. We begin by recalling the Hardy's inequalities on finite trees. Consider $\mathcal{T} = (\overline{V}, \overline{E}, 0)$ a finite tree rooted in 0, whose vertex and (undirected) edge sets are $\overline{V}$ and $\overline{E}$. Denote $V := \overline{V} \setminus \{0\}$, for each $x \in V$, the parent $p(x)$ of x is the neighbor of x in the direction of 0. The other neighbors of x are called the children of x and their set is written $C(x)$. For $x = 0$, by convention $C(0)$ is the set of neighbors of 0. Let be given two positive measures μ, ν defined on V. Consider $c(\mu, \nu)$ the best constant $c \geqslant 0$ in the inequality

$$\forall f \in \mathbb{R}^V, \qquad \mu[f^2] \leqslant c \sum_{x\in V}(f(p(x)) - f(x))^2\nu(x) \tag{36}$$

where f was extended to 0 via $f(0) := 0$.

According to [21] (see also Evans, Harris and Pick [10]), $c(\mu, \nu)$ can be estimated up to a factor 16 via Hardy's inequalities for trees, see (39) below. To describe them we need several notations.

Let $\mathbb{T}$ the set of subsets T of V satisfying the following conditions

- T is non-empty and connected (in $\mathcal{T}$),
- T does not contain 0,
- if $x \in T$ has a child in T, then all children of x belong to T.

Note that any $T \in \mathbb{T}$ admits a closest element to 0, call it $m(T)$, we have $m(T) \neq 0$. When T is not reduced to the singleton $\{m(T)\}$, the connected components of $T \setminus \{m(T)\}$ are indexed by the set of the children of $m(T)$, namely $C(m(T))$. For $y \in C(m(T))$, denote by T_y the connected component of $T \setminus \{m(T)\}$ containing y. Note that $T_y \in \mathbb{T}$.

We extend ν as a functional on $\mathbb{T}$, via the iteration

- when T is the singleton $\{m(T)\}$, we take $\nu(T) := \nu(m(T))$,
- when T is not a singleton, decompose T as $\{m(T)\} \sqcup \bigsqcup_{y\in C(m(T))} T_y$, then ν satisfies

$$\frac{1}{\nu(T)} = \frac{1}{\nu(m(T))} + \frac{1}{\sum_{y\in C(m(T))} \nu(T_y)}. \tag{37}$$

For $x \in V$, let S_x be the set of vertices $y \in V$ whose path to 0 pass through x. For any $T \in \mathbb{T}$ we associate the subset

$$T^* := (S_{m(T)} \setminus T) \sqcup L(T)$$

where $L(T)$ is the set of leaves of T, namely the $x \in T$ having no children in T. Equivalently, T^* is the set of all descendants of the leaves of T, themselves included.

Consider $\mathbb{S} \subset \mathbb{T}$, the set of $T \in \mathbb{T}$ which are such that $m(T)$ is a child of 0. Finally, define

$$b(\mu, \nu) := \max_{T\in\mathbb{S}} \frac{\mu(T^*)}{\nu(T)}. \tag{38}$$

We are interested in this quantity because of the Hardy inequality:

$$b(\mu, \nu) \leqslant c(\mu, \nu) \leq 16\, b(\mu, \nu). \tag{39}$$

Our goal here is to extend this inequality to the situation where V is denumerable and where μ and ν are two positive measures on V, with $\sum_{x\in V} \mu(x) < +\infty$.

Remark 3 Without lost of generality, we can assume 0 has only one child, because what happens on different S_x and S_y, where both x and y are children of 0, can be treated separately.

More precisely, while V is now (denumerable) infinite, we first assume that the height of $\mathcal{T} := (\overline{V}, \overline{E}, 0)$ is finite (implying that $\mathcal{T}$ cannot be locally finite). Recall that the height $h(x)$ of a vertex $x \in \overline{V}$ is the smallest number of edges linking x to 0. The assumption that $\sup_{x\in\overline{V}} h(x) < +\infty$ has the advantage that the iteration (37) enables us to compute ν on $\mathbb{T}$, starting from the highest vertices from an element of $\mathbb{T}$. Then $b(\mu, \nu)$ is defined exactly as in (38), except the maximum has to be replaced by a supremum. Extend $c(\mu, \nu)$ as the minimal constant $c \geqslant 0$ such that (36) is satisfied, with the possibility that $c(\mu, \nu) = +\infty$ when there is no such c. Note that in (36), the space $\mathbb{R}^V$ can be reduced and replaced by $\mathcal{B}(V)$, the space of bounded mappings on V:

Lemma 10 *We have*

$$c(\mu, \nu) = \sup_{f \in \mathcal{B}(V)\setminus\{0\}} \frac{\mu[f^2]}{\sum_{x\in V}(f(p(x)) - f(x))^2 \nu(x)}.$$

Proof Denote $\widetilde{c}(\mu, \nu)$ the above r.h.s. A priori we have $c(\mu, \nu) \geqslant \widetilde{c}(\mu, \nu)$. To prove the reverse bound, consider any $f \in \mathbb{R}^V$ and consider for $M > 0$, $f_M := (f \wedge M) \vee (-M)$. Note that

$$\sum_{x\in V}(f_M(p(x)) - f_M(x))^2 \nu(x) \leqslant \sum_{x\in V}(f(p(x)) - f(x))^2 \nu(x).$$

(This a general property of Dirichlet forms and comes from the 1-Lipschitzianity of the mapping $\mathbb{R} \ni r \mapsto (r \wedge M) \vee (-M)$.) Since $f_M \in \mathcal{B}(V)$, we have

$$\begin{aligned}\mu[f_M^2] &\leqslant \widetilde{c}(\mu, \nu) \sum_{x\in V}(f_M(p(x)) - f_M(x))^2 \nu(x) \\ &\leqslant \widetilde{c}(\mu, \nu) \sum_{x\in V}(f(p(x)) - f(x))^2 \nu(x).\end{aligned}$$

Letting M go to infinity, we get at the limit by monotonous convergence

$$\mu[f^2] \leqslant \widetilde{c}(\mu, \nu) \sum_{x\in V}(f(p(x)) - f(x))^2 \nu(x).$$

Since this is true for all $f \in \mathbb{R}^V$, we deduce that $c(\mu, \nu) \leqslant \widetilde{c}(\mu, \nu)$. □

Consider $(x_n)_{n\in\mathbb{N}_+}$ an exhaustive sequence of $\overline{V}$, with $x_0 = 0$ and such that for any $n \in \mathbb{N}_+$, $\overline{V}_n := \{x_0, x_1, \ldots, x_n\}$ is connected. We denote $\mathcal{T}_n$ the tree rooted on 0 induced by $\mathcal{T}$ on $\overline{V}_n$ and as above, $V_n := \overline{V}_n \setminus \{0\} = \{x_1, \ldots, x_n\}$. For any $n \in \mathbb{N}_+$ and $x \in V_n$, introduce the set

$$R_n(x) := \{x\} \bigsqcup_{y\in C(x)\setminus V_n} S_y.$$

In words, this is the set of elements of V whose path to 0 first enters V_n at x.

From now on, we assume that 0 has only one child, taking into account Remark 3. It follows that

$$V = \bigsqcup_{x\in V_n} R_n(x). \tag{40}$$

Let μ_n and ν_n be the measures defined on V_n via

$$\forall\, x \in V_n, \qquad \begin{cases} \mu_n(x) := \mu(R_n(x)), \\ \nu_n(x) := \nu(x). \end{cases}$$

The advantage of the μ_n and ν_n is that they brought us back to the finite situation while enabling to approximate $c(\mu, \nu)$:

Proposition 10 *We have* $\lim_{n\to\infty} c(\mu_n, \nu_n) = c(\mu, \nu)$.

Proof We first check that the limit exists. For $n \in \mathbb{N}_+$, consider the sigma-field $\mathcal{F}_n$ generated by the partition (40). To each $\mathcal{F}_n$-measurable function f, associate the function f_n defined on V_n by

$$\forall\, x \in V_n, \qquad f_n(x) := f(x).$$

This function determines f, since for any $x \in V_n$ and any $y \in R_n(x)$, $f(y) = f_n(x)$. Furthermore, we have:

$$\mu[f^2] = \mu_n[f_n^2]$$
$$\sum_{x\in V}(f(p(x)) - f(x))^2\nu(x) = \sum_{x\in V_n}(f_n(p(x)) - f_n(x))^2\nu_n(x).$$

It follows that

$$c(\mu_n, \nu_n) = \sup_{f\in\mathcal{B}(\mathcal{F}_n)\setminus\{0\}} \frac{\mu[f^2]}{\sum_{x\in V}(f(p(x)) - f(x))^2\nu(x)},$$

where $\mathcal{B}(\mathcal{F}_n)$ is the set of $\mathcal{F}_n$-measurable functions, which are necessarily bounded, i.e., belong to $\mathcal{B}(V)$. Since for any $n \in \mathbb{N}_+$ we have $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, we get that the sequence $(c(\mu_n, \nu_n))_{n\in\mathbb{N}_+}$ is non-decreasing and, taking into account Lemma 10, that

$$\lim_{n\to\infty} c(\mu_n, \nu_n) \leqslant c(\mu, \nu).$$

To get the reverse bound, first assume that $c(\mu, \nu) < +\infty$. For given $\epsilon > 0$, find a function $f \in \mathcal{B}(V)$ with

$$\frac{\mu[f^2]}{\sum_{x\in V}(f(p(x)) - f(x))^2\nu(x)} \geqslant c(\mu, \nu) - \epsilon.$$

Consider π the normalization of μ into a probability measure and let f_n be the conditional expectation of f with respect to π and to the sigma-field $\mathcal{F}_n$. Note that the f_n are uniformly bounded by $\|f\|_\infty$. Thus by the bounded martingale convergence theorem and since π gives a positive weight to any point of V, we have

$$\forall\, x \in V, \qquad \lim_{n\to\infty} f_n(x) = f(x).$$

From Fatou's lemma, we deduce

$$\liminf_{n\to\infty} \sum_{x\in V_n} (f_n(p(x)) - f_n(x))^2 \nu_n(x) = \liminf_{n\to\infty} \sum_{x\in V_n} (f_n(p(x)) - f_n(x))^2 \nu(x)$$
$$\geqslant \sum_{x\in V} \liminf_{n\to\infty} [(f_n(p(x)) - f_n(x))^2 \mathbb{1}_{V_n}(x)]\, \nu(x) = \sum_{x\in V} (f(p(x)) - f(x))^2 \nu(x).$$

By another application of the bounded martingale convergence theorem, we get

$$\lim_{n\to\infty} \mu_n[f_n^2] = \lim_{n\to\infty} \mu[f_n^2] = \mu[f^2],$$

so that

$$\limsup_{n\to\infty} \frac{\mu_n[f_n^2]}{\sum_{x\in V}(f_n(p(x)) - f_n(x))^2\nu(x)} \geqslant \frac{\mu[f^2]}{\sum_{x\in V}(f(p(x)) - f(x))^2\nu(x)}.$$

It follows that $\lim_{n\to\infty} c(\mu_n, \nu_n) \geqslant c(\mu, \nu) - \epsilon$, and since $\epsilon > 0$ can be chosen arbitrary small,

$$\lim_{n\to\infty} c(\mu_n, \nu_n) \geqslant c(\mu, \nu).$$

It remains to deal with the case where $c(\mu, \nu) = +\infty$. Then for any $M > 0$, we can find a function $f \in \mathcal{B}(V)$ with

$$\frac{\mu[f^2]}{\sum_{x\in V}(f(p(x)) - f(x))^2\nu(x)} \geqslant M.$$

By the above arguments, we end up with $\lim_{n\to\infty} c(\mu_n, \nu_n) \geqslant M$, and since M can be arbitrary large, $\lim_{n\to\infty} c(\mu_n, \nu_n) = +\infty = c(\mu, \nu)$. □

Our next goal is to show the same result holds for $b(\mu, \nu)$. We need some additional notations. The integer $n \in \mathbb{N}_+$ being fixed, denote $\mathbb{T}_n$ and $\mathbb{S}_n$ the sets $\mathbb{T}$ and $\mathbb{S}$ associated to $\mathcal{T}_n$. The functional ν_n is extended to $\mathbb{T}_n$ via the iteration (37) understood in $\mathcal{T}_n$. To any $T \in \mathbb{T}_n$, associate T_n the minimal element of $\mathbb{T}$ containing T. It is obtained in the following way: to any $x \in T$, if x has a child in T, then add all the children of x in V, and otherwise do not add any other points.

Lemma 11 *We have the comparisons*

$$\nu_n(T) \geqslant \nu(T_n) \;\; and \;\; \mu_n(T^*) \leqslant \mu(T_n^*),$$

where T^* is understood in $\mathcal{T}_n$ (and T_n^* in $\mathcal{T}$).

Proof The first bound is proven by iteration on the height of $T \in \mathbb{T}_n$.

- If this height is zero, then T is a singleton and T_n is the same singleton, so that $\nu_n(T) = \nu(T_n)$.
- If the height $h(T)$ of T is at least equal to 1, decompose

$$T = \{m_n(T)\} \sqcup \bigsqcup_{y \in C_n(m_n(T))} T_{n,y}$$

where $m_n(\cdot)$, $C_n(\cdot)$ and $T_{n,\cdot}$ are the notions corresponding to $m(\cdot)$, $C(\cdot)$ and $T_{\cdot}$ in $\mathcal{T}_n$.

Note that T and T_n have the same height and decompose

$$T_n = \{m(T_n)\} \sqcup \bigsqcup_{z \in C(m(T_n))} T_{n,z}.$$

On the one hand, we have $m(T_n) = m_n(T)$ and $C_n(m_n(T)) \subset C(m_n(T))$ and on the other hand, we have for any $y \in C_n(m_n(T))$, $\nu_n(T_y) \geqslant \nu((T_y)_n) = \nu(T_{n,y})$, due to the iteration assumption and to the fact that the common height of T_y and $(T_y)_n$ is at most equal to $h(T) - 1$. The equality $(T_y)_n = T_{n,y}$ is due to the fact that $T_{n,y}$ is obtained by the same completion of T_y as the one presented for T just above the statement of Lemma 11, and thus coincides with $(T_y)_n$. It follows that

$$\begin{aligned}
\frac{1}{\nu_n(T)} &= \frac{1}{\nu_n(m_n(T))} + \frac{1}{\sum_{y \in C_n(m_n(T))} \nu_n(T_y)} \\
&= \frac{1}{\nu(m(T_n))} + \frac{1}{\sum_{y \in C_n(m_n(T))} \nu_n(T_y)} \leq \frac{1}{\nu(m(T_n))} + \frac{1}{\sum_{y \in C_n(m_n(T))} \nu(T_{n,y})} \\
&\leqslant \frac{1}{\nu(m(T_n))} + \frac{1}{\sum_{y \in C(m(T_n))} \nu(T_{n,y})} = \frac{1}{\nu(T_n)},
\end{aligned}$$

establishing the wanted bound $\nu_n(T) \geqslant \nu(T_n)$. We now come to the second bound of the above lemma. By definition, we have

$$T^* = \sqcup_{x \in L_n(T)} S_{n,y},$$

where $L_n(T)$ is the set of leaves of T in $\mathcal{T}_n$ and $S_{n,y}$ is the subtree rooted in y in $\mathcal{T}_n$.

Note that $L_n(T) \subset L(T_n)$ and by definition of μ_n, we have

$$\forall\, y \in L_n(T), \qquad \mu_n(S_{n,y}) = \mu(S_y).$$

It follows that

$$\mu_n(T^*) = \sum_{x \in L_n(T)} \mu_n(S_{n,y}) = \sum_{x \in L_n(T)} \mu(S_y) \leqslant \sum_{x \in L(T_n)} \mu(S_y) = \mu(T_n^*).$$

□

Let $\widetilde{\mathbb{S}}_n$ be the image of $\mathbb{S}_n$ under the mapping $\mathbb{S}_n \ni T \mapsto T_n \in \mathbb{S}$. Since $\mathbb{S}_n \ni T \mapsto T_n \in \widetilde{\mathbb{S}}_n$ is a bijection, we get from Lemma 11,

$$b(\mu_n, \nu_n) := \max_{T \in \mathbb{S}_n} \frac{\mu_n(T^*)}{\nu_n(T)} \leqslant \max_{T_n \in \mathbb{S}_n} \frac{\mu(T_n^*)}{\nu(T_n)} \leqslant b(\mu, \nu),$$

so that

$$\limsup_{n\to\infty} b(\mu_n, \nu_n) \leqslant b(\mu, \nu). \tag{41}$$

Let us show more precisely:

Proposition 11 *We have* $\lim_{n\to\infty} b(\mu_n, \nu_n) = b(\mu, \nu)$.

Proof According to (41), it remains to show that

$$\liminf_{n\to\infty} b(\mu_n, \nu_n) \geqslant b(\mu, \nu). \tag{42}$$

Consider $T \in \mathbb{S}$ such that the ration $\mu(T^*)/\nu(T)$ serves to approximate $b(\mu, \nu)$, namely up to an arbitrary small $\epsilon > 0$ if $b(\mu, \nu) < +\infty$ or is an arbitrary large quantity if $b(\mu, \nu) = +\infty$. Define

$$\forall\, n \in \mathbb{N}_+, \qquad T^{(n)} := T \cap V_n.$$

Arguing as at the end of the proof of Proposition 10, we will deduce (42) from

$$\lim_{n\to\infty} \frac{\mu_n((T^{(n)})^*)}{\nu_n(T^{(n)})} = \frac{\mu(T^*)}{\nu(T)},$$

where $(T^{(n)})^*$ is understood in $\mathcal{T}_n$. This convergence will be the consequence of

$$\lim_{n\to\infty} \mu_n((T^{(n)})^*) = \mu(T^*), \tag{43}$$

$$\lim_{n\to\infty} \nu_n(T^{(n)}) = \nu(T). \tag{44}$$

For (43), note that

$$\mu(T^*) = \sum_{x \in L(T)} \mu(S_y),$$

and as we have seen at the end of the proof of Lemma 11,

$$\mu(T^*) = \sum_{x \in L_n(T^{(n)})} \mu(S_y).$$

Thus (43) follows by dominated convergence (since $\mu(V) < +\infty$), from

$$\forall\, x \in T, \qquad \lim_{n\to\infty} \mathbb{1}_{L_n(T^{(n)})}(x) = \mathbb{1}_{L(T)}(x).$$

To show the latter convergences, consider two cases:

- If $x \in L(T)$, then we will have $x \in L_n(T^{(n)})$ as soon as $x \in V_n$.
- If $x \in T \setminus L(T)$, then we will have $x \notin L_n(T^{(n)})$ as soon as V_n contains one of the children of x in T.

We now come to (44), and more generally let us prove by iteration over their height, that for any $\widetilde{T} \in \mathbb{T}$ and $\widetilde{T} \subset T$, we have

$$\lim_{n\to\infty} \uparrow \nu_n(\widetilde{T} \cap V_n) = \nu(\widetilde{T}), \tag{45}$$

i.e., the limit is non-decreasing. Indeed, if $\widetilde{T}$ has height 0, it is a singleton $\{x\}$, we have $\nu_n(\widetilde{T} \cap V_n) = \nu(\widetilde{T})$ as soon as x belongs to V_n, insuring (45).

Assume that $\widetilde{T}$ has height a $h \geqslant 1$ and that (45) holds for any $\widetilde{T}$ whose height is at most equal to $h-1$. Write as usual

$$\frac{1}{\nu(\widetilde{T})} = \frac{1}{\nu(m(\widetilde{T}))} + \frac{1}{\sum_{y \in C(m(\widetilde{T}))} \nu(\widetilde{T}_y)}. \tag{46}$$

Assume that n is large enough so that $C(m(\widetilde{T})) \cap V_n \neq \emptyset$ and in particular $m(\widetilde{T}) \in V_n$ and $m_n(\widetilde{T} \cap V_n) = m(\widetilde{T})$. Thus we also have

$$\begin{aligned} \frac{1}{\nu_n(\widetilde{T} \cap V_n)} &= \frac{1}{\nu_n(m_n(\widetilde{T} \cap V_n))} + \frac{1}{\sum_{y \in C_n(m_n(\widetilde{T} \cap V_n))} \nu_n((\widetilde{T} \cap V_n)_y)} \\ &= \frac{1}{\nu(m(\widetilde{T}))} + \frac{1}{\sum_{y \in C_n(m(\widetilde{T}))} \nu_n(\widetilde{T}_y \cap V_n)}. \end{aligned} \tag{47}$$

On the one hand, the set $C_n(m(\widetilde{T}))$ is non-decreasing and its limit is $C(m(\widetilde{T}))$, and on the other hand, due to the induction hypothesis, we have for any $y \in C(m(\widetilde{T}))$,

$$\lim_{n\to\infty} \uparrow \nu_n(\widetilde{T}_y \cap V_n) = \nu(\widetilde{T}_y).$$

By monotone convergence, we get

$$\lim_{n\to\infty} \uparrow \sum_{y\in C_n(m(\widetilde{T}))} \nu_n(\widetilde{T}_y \cap V_n) = \sum_{y\in C(m(\widetilde{T}))} \nu(\widetilde{T}_y),$$

which leads to (45), via (46) and (47). This ends the proof of (42). □

The conjunction of Propositions 10 and 11 leads to the validity of (39), when V is denumerable with $\mathcal{T}$ of finite height.

Let us now remove the assumption of finite height. The arguments are very similar to the previous one, except that the definition of $b(\mu, \nu)$ has to be modified (μ and ν are still positive measures on V, with μ of finite total mass). More precisely, for any $M \in \mathbb{N}_+$, consider $V_M := \{x \in V : h(x) \leqslant M\}$. Define on V_M the measure ν_M as the restriction to V_M of ν and μ_M via

$$\forall\, x \in V_M, \qquad \mu_M(x) := \begin{cases} \mu(x) & \text{if } h(x) < M, \\ \mu(S_x) & \text{if } h(x) = M. \end{cases}$$

By definition, we take

$$b(\mu, \nu) := \lim_{M\to\infty} b(\mu_M, \nu_M).$$

This limit exists and the convergence is monotone, since he have for any $M \in \mathbb{N}_+$, $b(\mu_M, \nu_M) = \max_{T\in\mathbb{S}_M} \frac{\mu(T^*)}{\nu(T)}$, where $\mathbb{S}_M := \{T \in \mathbb{S} : T \subset V_M\}$. Note that a direct definition of $b(\mu, \nu)$ via the iteration (37) is not possible: we could not start from leaves that are singletons.

By definition, $c(\mu, \nu)$ is the best constant in (36). It also satisfies $c(\mu, \nu) := \lim_{M\to\infty} c(\mu_M, \nu_M)$, as can be seen by adapting the proof of Proposition 10. We conclude that (39) holds by passing at the limit in

$$\forall\, M \in \mathbb{N}_+, \qquad b(\mu_M, \nu_M) \leqslant c(\mu_M, \nu_M) \le 16\, b(\mu_M, \nu_M).$$

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Widom Factors and Szegő–Widom Asymptotics, a Review

Jacob S. Christiansen, Barry Simon, and Maxim Zinchenko

Dedicated with great respect to the memory of Harold Widom, 1932–2021.

Abstract We survey results on Chebyshev polynomials centered around the work of H. Widom. In particular, we discuss asymptotics of the polynomials and their norms and general upper and lower bounds for the norms. Several open problems are also presented.

Keywords Chebyshev polynomials · Widom factors · Szegő–Widom asymptotics · Totik–Widom upper bound

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1 Introduction

Let $\mathfrak{e} \subset \mathbb{C}$ be a compact, not finite set and denote by

$$\|f\|_{\mathfrak{e}} := \sup_{z \in \mathfrak{e}} |f(z)|$$

the supremum norm of a continuous, complex-valued function f on $\mathfrak{e}$. A classical problem in approximation theory is, for every $n \geq 1$, to find the unique monic degree n polynomial, T_n, which minimizes $\|P\|_{\mathfrak{e}}$ among all monic degree n polynomials, P. The resulting sequence is called the *Chebyshev polynomials* of $\mathfrak{e}$.

By the maximum principle, we may assume that $\mathfrak{e}$ is polynomially convex. This means that $\Omega := (\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$ is connected so that $\mathfrak{e}$ has no inner boundary.

It is only in the case of $\mathfrak{e}$ being a (possibly elliptical) disk or a line segment that explicit formulas for all T_n's are available. The Chebyshev polynomials of the unit disk are simply $T_n(z) = z^n$, while the ones for the interval $[-1, 1]$ (or any ellipse with foci at ± 1) are given by

$$T_n(x) = 2^{-n+1} \cos(n\theta),$$

where $x = \cos\theta$.

In addition to this, there are certain sets generated by polynomials (such as lemniscates and Julia sets) for which a subsequence of T_n can be written in closed form. For general $\mathfrak{e}$, however, the best one can hope for is to determine the asymptotic behavior of T_n. In this article we seek to present what is known about the asymptotics of Chebyshev polynomials. Had it not been for Widom's landmark paper [49], there probably wouldn't be much to say.

To get started, we briefly introduce some notions from potential theory (see, e.g., [8, 27–29, 33] for more details). Let $\mathrm{C}(\mathfrak{e})$ denote the logarithmic capacity of $\mathfrak{e}$. When $\mathfrak{e}$ is non-polar (i.e., $\mathrm{C}(\mathfrak{e}) > 0$), we denote by $d\rho_{\mathfrak{e}}$ the equilibrium measure of $\mathfrak{e}$ and by $G := G_{\mathfrak{e}}$ the Green's function of $\mathfrak{e}$. These are closely linked by the relation

$$G(z) = -\log\bigl[\mathrm{C}(\mathfrak{e})\bigr] + \int \log|z - x|\, d\rho_{\mathfrak{e}}(x). \tag{1}$$

For subsets $\mathfrak{f} \subset \mathfrak{e}$, we shall also refer to $\rho_{\mathfrak{e}}(\mathfrak{f})$ as the harmonic measure of $\mathfrak{f}$. The set $\mathfrak{e}$ is called regular if G vanishes at all points of $\mathfrak{e}$ (equivalently, G is continuous on all of $\mathbb{C}$).

The general results for Chebyshev polynomials are few, but important. Szegő [40] showed that

$$\|T_n\|_{\mathfrak{e}} \geq \mathrm{C}(\mathfrak{e})^n. \tag{2}$$

This applies to all compact sets $\mathfrak{e} \subset \mathbb{C}$ and is optimal since equality occurs for all n when $\mathfrak{e}$ is a disk. When $\mathfrak{e} \subset \mathbb{R}$, Schiefermayr [37] improved upon (2) by showing that

$$\|T_n\|_{\mathfrak{e}} \geq 2\,\mathrm{C}(\mathfrak{e})^n, \quad n \geq 1, \tag{3}$$

which is again optimal (take $\mathfrak{e}$ to be an interval). Szegő [40], using prior results of Faber [21] and Fekete [22], also proved the following asymptotic result:

$$\lim_{n\to\infty} \|T_n\|_{\mathfrak{e}}^{1/n} = \mathrm{C}(\mathfrak{e}). \tag{4}$$

This certainly puts a growth restriction on $\|T_n\|_{\mathfrak{e}}$ but is not strong enough to force $\|T_n\|_{\mathfrak{e}}/\,\mathrm{C}(\mathfrak{e})^n$ to be bounded. We shall discuss which extra assumptions on $\mathfrak{e}$ may imply this in Sects. 2 and 3.

The polynomials themselves also obey nth root asymptotics. For a non-polar compact set $\mathfrak{e} \subset \mathbb{C}$, we have that

$$|T_n(z)|^{1/n} \to \mathrm{C}(\mathfrak{e})\exp\bigl[G(z)\bigr] \tag{5}$$

uniformly on any closed set disjoint from $\mathrm{cvh}(\mathfrak{e})$, the convex hull of $\mathfrak{e}$. This result is implicitly in Widom [48], where he shows that all zeros of T_n must lie in $\mathrm{cvh}(\mathfrak{e})$ before proceeding to the asymptotics. See also Ullman [46] and Saff–Totik [36, Chap. III].

"All asymptotic formulas have refinements," quoting the introduction of [49]. And this is precisely what we aim at, just as Widom did. As (2)–(5) suggest, it is natural to scale T_n by a factor of $\mathrm{C}(\mathfrak{e})^n$. We shall study the limiting behavior of the so-called *Widom factors*

$$W_n(\mathfrak{e}) := \|T_n\|_{\mathfrak{e}}/\,\mathrm{C}(\mathfrak{e})^n. \tag{6}$$

If this scaled version of the norms does not have a limit, can we then at least single out the possible limit points? Regarding the polynomials T_n, we aim at strong asymptotics or what we shall refer to as *Szegő–Widom asymptotics*.

The first result in this direction goes back to Faber [21]. When $\mathfrak{e}$ is a closed Jordan region, there is a Riemann map of Ω onto the unit disk, $\mathbb{D}$. We uniquely fix this map, B, by requiring that

$$B(z) = \frac{\mathrm{C}(\mathfrak{e})}{z} + O(1/z^2) \tag{7}$$

near ∞. Assuming that $\partial\mathfrak{e}$ is analytic, Faber showed that $W_n(\mathfrak{e}) \to 1$ and, more importantly, that

$$\frac{T_n(z)B(z)^n}{\mathrm{C}(\mathfrak{e})^n} \to 1 \tag{8}$$

uniformly for z in a neighborhood of $\overline{\Omega}$.

The picture changes completely when $\mathfrak{e}$ consists of more than one component. In his work on Chebyshev polynomials of two intervals, Akhiezer [1, 2] proved that either $W_n(\mathfrak{e})$ is asymptotically periodic or else the set of limit points of $W_n(\mathfrak{e})$ fills up an entire interval. But it was only Widom [49] who lifted the theory to $\mathfrak{e}$ being a union of disjoint compact subsets of $\mathbb{C}$ and developed a framework to distinguish between periodicity and almost periodicity.

In replacement of the Riemann map, we introduce (on Ω) a multivalued analytic function $B := B_{\mathfrak{e}}$ which is determined by

$$|B(z)| = \exp\bigl[-G(z)\bigr] \tag{9}$$

and (7) near ∞. One can construct this B using the fact that $-G$ is locally the real part of an analytic function whose exponential ($=B$) can be continued along any curve in Ω. By the monodromy theorem, the continuation is the same for homotopic curves and, due to (9), going around a closed curve γ can only change B by a phase factor. Hence there is a character $\chi_{\mathfrak{e}}$ of the fundamental group $\pi_1(\Omega)$ so that going around γ changes B by $\chi_{\mathfrak{e}}([\gamma])$. More explicitly, if γ winds around a subset $\mathfrak{f} \subset \mathfrak{e}$ and around no other points of $\mathfrak{e}$, then the multiplicative change of phase of B around γ is given by

$$\exp\bigl[-2\pi i \rho_{\mathfrak{e}}(\mathfrak{f})\bigr]. \tag{10}$$

In line with Faber and (8), Widom looked at $T_n(z)B(z)^n/\mathrm{C}(\mathfrak{e})^n$ for the "new" B and noted that its character $\chi_{\mathfrak{e}}^n$ only has a limit when $\chi_{\mathfrak{e}}$ is trivial (i.e., Ω is simply connected). So there is no hope of finding a pointwise limit except when $\mathfrak{e}$ just has one component. Widom's stroke of genius was to find a good candidate for the asymptotics when $\mathfrak{e}$ has several components. For every character χ in $\pi_1(\Omega)^*$ there exists a so-called *Widom minimizer* which we shall denote by F_χ. This is the unique element of $H^\infty(\Omega, \chi)$ (i.e., the set of bounded analytic χ-automorphic functions on Ω) with $F_\chi(\infty) = 1$ and for which

$$\|F_\chi\|_\infty = \inf\bigl\{\|h\|_\infty : h \in H^\infty(\Omega, \chi),\ h(\infty) = 1\bigr\}. \tag{11}$$

Writing F_n as shorthand notation for F_{χ^n}, the *Widom surmise* is the notion that

$$\frac{T_n(z)B(z)^n}{\mathrm{C}(\mathfrak{e})^n} - F_n(z) \to 0. \tag{12}$$

When it holds uniformly on compact subsets of the universal cover of Ω, we say that $\mathfrak{e}$ has Szegő–Widom asymptotics.

Widom [49] proved that one has this type of asymptotics when $\mathfrak{e}$ is a finite union of disjoint Jordan regions with smooth boundaries and conjectured that this should also hold for finite gap sets (in $\mathbb{R}$). A main result of [13] was to settle this conjecture. By streamlining the method of proof, this was then extended to a large class of infinite gap sets in [14] (see Sect. 2 for further details).

The framework of characters is also useful when describing the fluctuation of $W_n(\mathfrak{e})$. In [49], Widom proved that

$$W_n(\mathfrak{e})/\|F_n\|_\infty \to 1 \tag{13}$$

for finite unions of disjoint Jordan regions and established the counterpart (with 1 replaced by 2 on the right-hand side) for finite gap sets. The behavior of $\|F_n\|_\infty$ very much depends on the character $\chi_\mathfrak{e}$. If $\chi_\mathfrak{e}^n = 1$ for some n, then the sequence is periodic (with period at most n) and otherwise it is merely *almost periodic*. This is precisely the pattern that Akhiezer discovered for two intervals. We shall discuss the possible limit points in more detail in Sect. 2.

The paper is organized as follows. In Sect. 2 we discuss bounds and asymptotics for Chebyshev polynomials of compact subsets of the real line. Then in Sect. 3 we survey similar results for Chebyshev and weighted Chebyshev polynomials of subsets of the complex plane, including results on the asymptotic distribution of zeros. Open problems are formulated along the way.

We would be remiss if not mentioning related problems, such as the Ahlfors problem [19], and similar classes of polynomials or functions, for instance, residual polynomials [17, 54] and rational Chebyshev functions [20]. But to consider the subject in more depth, we decided to merely focus on the Chebyshev problem.

2 Real Chebyshev Polynomials

As we shall see, there is a rather complete theory for Chebyshev polynomials of compact sets $\mathfrak{e} \subset \mathbb{R}$. This is in part due to what is called *Chebyshev alternation*. We say that P_n, a real degree n polynomial, has an alternating set in $\mathfrak{e}$ if there exists $n+1$ points in $\mathfrak{e}$, say $x_0 < x_1 < \ldots < x_n$, so that

$$P_n(x_j) = (-1)^{n-j}\|P_n\|_\mathfrak{e}. \tag{14}$$

The alternation theorem gives the following characterization of the nth Chebyshev polynomial of $\mathfrak{e}$: *T_n always has an alternating set in $\mathfrak{e}$ and, conversely, any monic degree n polynomial with an alternating set in $\mathfrak{e}$ must be equal to T_n.*

This result, in turn, has consequences for the zeros of T_n. Not only do all of them lie in $\text{cvh}(\mathfrak{e})$, but any gap of $\mathfrak{e}$ (i.e., a bounded component of $\mathbb{R} \setminus \mathfrak{e}$) contains at most one zero of T_n. The alternating set need not be unique and usually isn't. However, it always contains the endpoints of $\text{cvh}(\mathfrak{e})$. See, e.g., [13] for proofs and more details.

We now turn the attention to the Widom factors which were introduced in (6). By [37] we always have $W_n(\mathfrak{e}) \geq 2$ and, as proven in [15], equality occurs for $n = km$ (with $m \geq 1$) precisely when

$$\mathfrak{e} = P^{-1}\big([-2, 2]\big) \tag{15}$$

for some degree k polynomial, $P(z) = cz^k +$ lower order terms. In that case, T_{km} is nothing but the mth Chebyshev polynomial of $[-2, 2]$ composed with P and divided by c^m. It also follows that equality holds for all n if and only if $\mathfrak{e}$ is an interval. A stronger and related result of Totik [42] states that if $\lim_{n\to\infty} W_n(\mathfrak{e}) = 2$, then $\mathfrak{e}$ must be an interval.

Interestingly, the sets that appear in (15) are not only of interest for the lower bound; they play a key role in the theory. For $\mathfrak{e} \subset \mathbb{R}$, we introduce the so-called *period-n sets*, $\mathfrak{e}_n$, (aka *n-regular sets* [39]) by

$$\mathfrak{e}_n := T_n^{-1}\big([-\|T_n\|_{\mathfrak{e}}, \|T_n\|_{\mathfrak{e}}]\big). \tag{16}$$

Clearly, T_n is also the Chebyshev polynomial of $\mathfrak{e}_n \supset \mathfrak{e}$ and furthermore we have that

$$\|T_n\|_{\mathfrak{e}} = 2\,\mathrm{C}(\mathfrak{e}_n)^n. \tag{17}$$

Due to alternation we can write any period-n set as

$$\mathfrak{e}_n = \bigcup_{j=1}^{n} [\alpha_j, \beta_j], \tag{18}$$

where $\alpha_1 < \beta_1 \le \ldots \le \alpha_n < \beta_n$ are the solutions of $T_n(x) = \pm\|T_n\|_{\mathfrak{e}}$. So T_n is strictly monotone on each of the bands $[\alpha_j, \beta_j]$ and $\mathfrak{e}_n \subset \mathrm{cvh}(\mathfrak{e})$. Note that α_1 and β_n always belong to $\mathfrak{e}$ while for $j = 1, \ldots, n-1$, at least one of β_j and α_{j+1} must lie in $\mathfrak{e}$. Therefore, any gap of $\mathfrak{e}$ can at most overlap with one of the bands of $\mathfrak{e}_n$.

The period-n sets are well suited for potential theory. For instance, the Green's function and equilibrium measure of $\mathfrak{e}_n$ are explicitly given by

$$G_n(z) = \frac{1}{n}\log\left|\frac{\Delta_n(z)}{2} + \sqrt{\left(\frac{\Delta_n(z)}{2}\right)^2 - 1}\,\right| \tag{19}$$

and

$$d\rho_n(x) = \frac{1}{\pi n}\frac{|\Delta_n'(x)|}{\sqrt{4-\Delta_n(x)^2}}\,dx, \quad x \in \mathfrak{e}_n, \tag{20}$$

where Δ_n is defined by

$$\Delta_n(z) := 2T_n(z)/\|T_n\|_{\mathfrak{e}}. \tag{21}$$

In particular, each band of $\mathfrak{e}_n$ has ρ_n-measure $1/n$. See, e.g., [13] for proofs and further details.

Comparing the Green's functions for $\mathfrak{e}$ and $\mathfrak{e}_n$ at ∞, and letting $\{K_j\}$ account for the gaps of $\mathfrak{e}$, we see that

$$\log\bigl[\mathrm{C}(\mathfrak{e}_n)/\,\mathrm{C}(\mathfrak{e})\bigr] = \int_{\mathfrak{e}_n} \bigl[G(x) - G_n(x)\bigr]d\rho_n(x) \le \frac{1}{n}\sum_j \max_{x\in K_j} G(x) \tag{22}$$

which combined with (17) then yields

$$\|T_n\|_{\mathfrak{e}}/\,\mathrm{C}(\mathfrak{e})^n \le 2\sum_j \max_{x\in K_j} G(x). \tag{23}$$

This observation leads to an upper bound on $W_n(\mathfrak{e})$ for a large class of compacts sets $\mathfrak{e} \subset \mathbb{R}$. When $\mathfrak{e}$ is regular (for potential theory), the Green's function vanishes at all endpoints of the K_j's and since G is also concave on the gaps, it attains its maximum on K_j at the unique critical point, c_j, in that gap. A regular compact set $\mathfrak{e} \subset \mathbb{R}$ (or $\mathbb{C}$) is called a *Parreau–Widom set* (in short, PW) if

$$PW(\mathfrak{e}) := \sum_j G(c_j) < \infty, \tag{24}$$

where the sum is over all points $c_j \in \mathbb{R} \setminus \mathfrak{e}$ for which $\nabla G(c_j) = 0$. Such sets are known to have positive Lebesgue measure (see, e.g., [12] for details). One of the main results of [13] that we have now deduced is the following:

Theorem 2.1 *If $\mathfrak{e} \subset \mathbb{R}$ is a PW set, then the Widom factors are bounded. Explicitly, we have that*

$$\|T_n\|_{\mathfrak{e}} \le 2\exp\bigl[PW(\mathfrak{e})\bigr]\,\mathrm{C}(\mathfrak{e})^n. \tag{25}$$

Remarks

- (i) Sets which obey (24) were introduced by Parreau [32] in the context of Riemann surfaces. They later appeared in Widom's work on multi-valued analytic functions [50, 51] and the name was coined by Hasumi in his monograph [25].
- (ii) Examples of PW sets include finite gap sets but also sets that are homogeneous in the sense of Carleson [11], e.g., fat Cantor sets.
- (iii) Upper bounds of the form $\|T_n\|_{\mathfrak{e}} \le K \cdot \mathrm{C}(\mathfrak{e})^n$ are also referred to as *Totik–Widom* bounds. Here $K > 0$ is a constant that does not depend of n.

As alluded to in the introduction, the Widom factors are not always bounded. It was proven in [9] that they are unbounded when $\mathfrak{e}$ is the Julia set of $(z-\lambda)^2$ and $\lambda > 2$. Interestingly, $W_{2^n}(\mathfrak{e})$ is bounded (in fact, constant) in that case, while $W_{2^n-1}(\mathfrak{e}) \to \infty$. There are more elaborate examples of very thin Cantor-type sets for which $W_n(\mathfrak{e})$ grows subexponentially of any order, see Goncharov–Hatinoğlu [24] for details. But it is not known if the Widom factors of, e.g., the middle third

Cantor set are bounded. The best result in this direction, due to Andrievskii [6], states that when $\mathfrak{e} \subset \mathbb{R}$ is uniformly perfect there exists a constant $c > 0$ such that $W_n(\mathfrak{e}) = O(n^c)$. We pose the following question:

Open Problem 2.2 Does there exist a Lebesgue measure zero set or merely a non-PW set $\mathfrak{e} \subset \mathbb{R}$ for which the Widom factors are bounded?

It remains to consider the fluctuation and possible limit points of $W_n(\mathfrak{e})$. We shall do so in conjunction with the asymptotics of the polynomials.

Let us start by explaining, following [13, 14], how one can establish Szegő–Widom asymptotics. Since every band of $\mathfrak{e}_n$ has ρ_n-measure $1/n$, the nth power of $B_n := B_{\mathfrak{e}_n}$ is single-valued. In fact,

$$B_n(z)^{\pm n} = \frac{\Delta_n(z)}{2} \mp \sqrt{\left(\frac{\Delta_n(z)}{2}\right)^2 - 1} \tag{26}$$

with Δ_n as in (21). It follows that

$$\frac{2T_n(z)}{\|T_n\|_{\mathfrak{e}}} = B_n(z)^n + B_n(z)^{-n} \tag{27}$$

and this is the key formula we need. As a side remark we note that when $\mathfrak{e} = [-1, 1]$, (27) corresponds to the familiar formula

$$T_n(z) = \frac{1}{2^n}\left(\left(z - \sqrt{z^2 - 1}\right)^n + \left(z + \sqrt{z^2 - 1}\right)^n\right). \tag{28}$$

The idea is now to recast (27) in the form

$$\frac{T_n(z)B(z)^n}{\mathrm{C}(\mathfrak{e})^n} = \left(1 + B_n(z)^{2n}\right)\frac{M_n(z)}{M_n(\infty)}, \tag{29}$$

where

$$M_n(z) = B(z)^n / B_n(z)^n. \tag{30}$$

Since $\sup_{n,\, z \in K} |B_n(z)| < 1$ on any compact subset K of the universal cover of Ω, the task is reduced to proving that

$$F_n(z) - M_n(z)/M_n(\infty) \to 0 \tag{31}$$

and this can be done by controlling the limit points of M_n.

In order to go beyond finite gap sets, some issues have to be sorted out. First of all, for which infinite gap sets do the Widom minimizers at all exist and are they unique? Secondly, how are the limit points of M_n related to the Widom minimizers and do these minimizers depend continuously on the character so that one can pass to the limit along convergent subsequences?

The answer to both of the above questions are rooted in Widom's work. In [51], he proved that (24) holds if and only if there is a nonzero element in $H^\infty(\Omega, \chi)$ for every $\chi \in \pi_1(\Omega)^*$. Hence, by compactness, Widom minimizers exist for all PW sets. Uniqueness requires a separate argument for which we refer the reader to [14] and [47]. Note also that Theorem 2.1 implies $|M_n| \leq 1$ in the PW regime. So limit points do exist in that setting by Montel's theorem.

To proceed with the analysis, it is instructive to also consider the problem dual to (11). The function $Q_\chi \in H^\infty(\Omega, \chi)$ which satisfies

$$Q_\chi(\infty) = \sup\big\{g(\infty) : g \in H^\infty(\Omega, \chi),\ \|g\|_\infty = 1,\ g(\infty) > 0\big\} \tag{32}$$

is called the *dual Widom maximizer*. Clearly, we have

$$Q_\chi = F_\chi/\|F_\chi\|_\infty, \quad F_\chi = Q_\chi/Q_\chi(\infty), \quad Q_\chi(\infty) = 1/\|F_\chi\|_\infty \tag{33}$$

and therefore the two problems either both or neither have unique solutions. By controlling the zeros of T_n in gaps of $\mathfrak{e}$ and using the fact that

$$|M_n(z)| = \exp\left[-n \int_{\cup_j K_j} G(x, z)\, d\rho_n(x)\right], \tag{34}$$

one can prove that the limit points of M_n are dual Widom maximizers. More precisely, the approach of [14] reveals that limit points of M_n are Blaschke products with at most one zero per gap of $\mathfrak{e}$ and such character automorphic products are indeed dual Widom maximizers.

As for the final issue, Widom [50] noted that "It is natural to ask (and important to know) whether Q_χ is continuous as a function of χ on the compact group $\pi_1(\Omega)^*$." He pointed out that this can easily fail to hold (e.g., if $\mathfrak{e}$ has isolated points) but was not able to characterize those sets for which we have continuity. Years later, this was settled by Hayashi and Hasumi (see [25, 26]). Continuity in χ is equivalent to having a so-called *direct Cauchy theorem* (DCT) on Ω. There seems to be no obvious geometric interpretation of this DCT property; while it may fail for a general PW set, it always holds when $\mathfrak{e}$ is homogeneous (see, e.g., [55] for further details).

We should point out that DCT is responsible for the almost periodic behavior of the Widom minimizers. That is,

$$n \mapsto \|F_n\|_\infty \text{ is an almost periodic function} \tag{35}$$

and

$$n \mapsto F_n(z) \text{ is almost periodic uniformly for } z \text{ in compact subsets of the universal cover of } \Omega. \tag{36}$$

Recall namely that $n \mapsto x_n$ is almost periodic precisely when $\{x_n\}$ is the orbit of a continuous function on a torus (possibly of infinite dimension). Since the character group $\pi_1(\Omega)^*$ is topologically a torus, we are led to (35) and (36).

After this extended discussion, we are now ready to formulate the main result of [14]:

Theorem 2.3 *If $\mathfrak{e} \subset \mathbb{R}$ is a PW set and obeys the DCT condition, then the Chebyshev polynomials of $\mathfrak{e}$ have strong Szegő–Widom asymptotics. That is, the Widom surmise* (12) *and also* (35) *and* (36) *hold. Moreover,*

$$\lim_{n\to\infty} \frac{\|T_n\|_{\mathfrak{e}}}{\mathrm{C}(\mathfrak{e})^n \|F_n\|_\infty} = 2. \tag{37}$$

Remarks

(i) The additional word "strong" is used here to include the almost periodicity of (35) and (36).

(ii) The last statement also follows from (29) by noting that $\sup_{z\in\Omega} |1+B_n(z)^{2n}| = 2$ since there are points $x \in \mathfrak{e}_n$ with $|B_n(x)| = 1$.

The above theorem enables us to shed more light on the fluctuation of the Widom factors. When $\mathfrak{e} \subset \mathbb{R}$ is a PW set with DCT, the function $n \mapsto W_n(\mathfrak{e}) := \|T_n\|_{\mathfrak{e}} / \mathrm{C}(\mathfrak{e})^n$ is asymptotically almost periodic. The set of limit points may or may not fill up the entire interval between the lower bound (=2) and the upper bound from Theorem 2.1. Generically, it will (as explained in [15]; see also below) but this is not the case when, for instance, $\mathfrak{e}$ is a period-n set. For in that case we have $\chi_{\mathfrak{e}}^n = 1$ and the function in (35) becomes periodic.

Following [14], we say that $\mathfrak{e}$ has a *canonical generator* if the orbit $\{\chi_{\mathfrak{e}}^n\}_{n\in\mathbb{Z}}$ is dense in $\pi_1(\Omega)^*$. This holds if and only if for all decompositions $\mathfrak{e} = \mathfrak{e}_1 \cup \ldots \cup \mathfrak{e}_l$ into disjoint closed sets and rational numbers $\{q_j\}_{j=1}^{l-1}$ (not all zero), we have that

$$\sum_{j=1}^{l-1} q_j \rho_{\mathfrak{e}}(\mathfrak{e}_j) \neq 0 \pmod 1. \tag{38}$$

In particular, a finite gap set has a canonical generator precisely when the harmonic measures of the bands are rationally independent (except that they sum to 1). One can show that the property of having a canonical generator is generic (see [14] for details) and it implies that any number ≥ 2 and $\leq 2\exp\big[PW(\mathfrak{e})\big]$ is a limit point of $W_n(\mathfrak{e})$.

To end this section, we return to the open problem formulated a few pages ago. It was proven in [14] that if $\mathfrak{e}$ has a canonical generator and obeys a Totik–Widom bound (as in Theorem 2.1), then it must be a PW set. This provides some evidence that the answer could be in the negative. However, there are also results pulling in the opposite direction. While $\liminf W_n(\mathfrak{e}) = 2$ for any PW set with DCT (as proven in [17]), it is not always the case that lim sup is equal to $2\exp\big[PW(\mathfrak{e})\big]$ when $\mathfrak{e}$ is a period-n set and $n \geq 2$. For instance, one can prove that strict inequality applies when $\mathfrak{e}$ is the degenerate period-3 set

$$\big[-\sqrt{3}, 0\big] \cup \big[\sqrt{3}, 2\big].$$

We thus wonder if some cleverly arranged limit of period-n sets could provide an example of a non-PW set with bounded Widom factors.

3 Complex Chebyshev Polynomials

In this section we consider Chebyshev and weighted Chebyshev polynomials for compact subsets of the complex plane. In particular, we discuss Widom's contribution to the subject as well as several recent refinements.

Throughout the section we will assume that $\mathrm{C}(\mathfrak{e}) > 0$ and let w be a nonnegative upper semi-continuous weight function on $\mathfrak{e}$ (this ensures that w is bounded) which is nonzero at infinitely many points of $\mathfrak{e}$. Under these assumptions, there exists for each $n \geq 1$ a unique weighted Chebyshev polynomial $T_{n,w} := T_{n,w}^{(\mathfrak{e})}$ that minimizes $\|wT_{n,w}\|_{\mathfrak{e}}$ among monic polynomials of degree n.

In [48], Widom proved that one has root asymptotics analogous to (4) for a fairly general class of monic extremal polynomials which, in particular, includes the $L^p(w\, d\rho_{\mathfrak{e}})$-extremal polynomials for $0 < p < \infty$ and weights w satisfying $w > 0\ d\rho_{\mathfrak{e}}$-a.e. This type of asymptotics for the $L^1(w\, d\rho_{\mathfrak{e}})$-extremal polynomials, P_n, combined with (4) and the two-sided estimate

$$\|P_n\|_{L^1(wd\rho_{\mathfrak{e}})} \leq \|T_{n,w}\|_{L^1(wd\rho_{\mathfrak{e}})} \leq \|wT_{n,w}\|_{\mathfrak{e}} \leq \|wT_n\|_{\mathfrak{e}} \leq \|w\|_{\mathfrak{e}}\|T_n\|_{\mathfrak{e}} \tag{39}$$

yields that the weighted Chebyshev polynomials obey the root asymptotics

$$\lim_{n\to\infty} \|wT_{n,w}\|_{\mathfrak{e}}^{1/n} = \mathrm{C}(\mathfrak{e}) \tag{40}$$

whenever $w > 0\ d\rho_{\mathfrak{e}}$-a.e.

As explained below, there is also a lower bound, an asymptotic upper bound, and strong asymptotics for the weighted Chebyshev polynomials under an additional assumption on the weight function w, namely the so-called Szegő condition

$$S(w) := \exp\left[\int \log w(z)\, d\rho_{\mathfrak{e}}(z)\right] > 0. \tag{41}$$

A generalization of Szegő's lower bound (2) to the weighted case was observed for finite unions of Jordan regions by Widom [49, Sect. 8] and extended to general non-polar compact sets $\mathfrak{e} \subset \mathbb{C}$ in [31]. It relies on (41) and states that

$$\|wT_{n,w}\|_{\mathfrak{e}} \geq S(w)\, \mathrm{C}(\mathfrak{e})^n. \tag{42}$$

In addition, it was shown in [31] that unlike the unweighted case, this lower bound is sharp even for real sets $\mathfrak{e}$ (cf. (3)). Moreover, equality in (42) occurs for some n if and only if there exists a monic polynomial, P_n, of degree n such that $P_n(z) = 0$ implies $G(z) = 0$ and $w(z)|P_n(z)| = \|wP_n\|_{\mathfrak{e}}$ for $d\rho_{\mathfrak{e}}$-a.e. $z \in \mathfrak{e}$, in which case $T_{n,w} = P_n$.

Next, we turn to upper bounds. A collection of very general bounds for unweighted Chebyshev polynomials were obtained by Andrievskii [5, 6] and Andrievskii–Nazarov [7]. See also Totik–Varga [44]. In particular, it was shown that if $\mathfrak{e} \subset \mathbb{C}$ is a finite union of quasiconformal arcs and/or Jordan regions bounded by quasiconformal curves (aka quasidisks), then a Totik–Widom upper bound

$$\|T_n\|_{\mathfrak{e}} \leq K \cdot \mathrm{C}(E)^n$$

holds for some constant K. This result includes a large class of regions with pathological boundaries, for example, the Koch snowflake. In addition, in the absence of any smoothness it was shown that for compact sets $\mathfrak{e}$ with finitely many components, the Widom factors $W_n(\mathfrak{e})$ can grow at most logarithmically in n. Yet, in this setting no example of unboundedness is known. Numerical evidence points in the direction of bounded Widom factors, at least in the case of Jordan regions. But no proof is currently available.

Open Problem 3.1 Does there exist a compact set $\mathfrak{e} \subset \mathbb{C}$ with finitely many components for which the Widom factors are unbounded?

The above mentioned results are qualitative in nature as the involved constants are large and their dependence on the set $\mathfrak{e}$ is rather implicit. Going in the other direction and assuming smoothness of the components of $\mathfrak{e}$ typically yields more explicit constants and even precise asymptotics for the Widom factors and the Chebyshev polynomials.

Suppose now that $\mathfrak{e} \subset \mathbb{C}$ is a finite disjoint union of C^{2+} arcs and/or Jordan regions with C^{2+} boundaries. Assume also that the weight function w is supported on the boundary of $\mathfrak{e}$. Under these assumptions, Widom [49, Sect. 11] obtained the asymptotic upper bound

$$\limsup_{n\to\infty} \|wT_{n,w}\|_{\mathfrak{e}} / \mathrm{C}(\mathfrak{e})^n \leq 2S(w) \exp\big[PW(\mathfrak{e})\big], \tag{43}$$

compare with (25). This asymptotic bound is sharp within the class of real sets (i.e., $\mathfrak{e}$ consisting only of arcs lying on the real line). However, in the case of $\mathfrak{e}$ consisting only of regions, Widom [49, Sect. 8] established the improved asymptotic upper bound

$$\limsup_{n\to\infty} \|wT_{n,w}\|_{\mathfrak{e}} / \mathrm{C}(\mathfrak{e})^n \leq S(w) \exp\big[PW(\mathfrak{e})\big]. \tag{44}$$

More remarkably, in that case Widom showed that we have Szegő–Widom asymptotics for the weighted Chebyshev polynomials $T_{n,w}$ and their norms $\|wT_{n,w}\|_{\mathfrak{e}}$ (i.e., the weighted analogs of (12) and (13)). The improved asymptotic bound (44) is also sharp; in fact, by (42), equality is attained when $\mathfrak{e}$ consists of a single region since in that case the Green's function has no critical points and thus $PW(\mathfrak{e}) = 0$.

For special subsets of the complex plane, we also have non-asymptotic upper bounds that parallel the real case. The following two results are taken from [16].

Theorem 3.2 *If* $\mathfrak{e} \subset \mathbb{C}$ *is a solid lemniscate, that is,*

$$\mathfrak{e} = \{z \in \mathbb{C} : |P(z)| \leq \alpha\} \tag{45}$$

for some polynomial P *of degree* $k \geq 1$ *and* $\alpha > 0$*, then*

$$\|T_n\|_{\mathfrak{e}} \leq K \cdot \mathrm{C}(\mathfrak{e})^n, \tag{46}$$

where the constant K *is given by*

$$K = \max_{j=1,\ldots,k} W_j(\mathfrak{e}). \tag{47}$$

The other special case is motivated by an old result of Faber [21] stating that the Chebyshev polynomials of an ellipse are the same as the ones for the interval between the two foci. This in particular leads to explicit values of the Widom factors for ellipses. By further developing the results of Fischer [23] for two intervals, one can produce general results for level sets of the Green's function.

Theorem 3.3 *If* $\mathfrak{e}_0 \subset \mathbb{R}$ *is a PW set and*

$$\mathfrak{e} = \{z \in \mathbb{C} : G_{\mathfrak{e}_0}(z) \leq \alpha\} \tag{48}$$

for some $\alpha > 0$*, then*

$$\left\|T_n^{(\mathfrak{e})}\right\|_{\mathfrak{e}} \leq \left(1 + e^{-n\alpha}\right) \exp\left[PW(\mathfrak{e}_0)\right] \mathrm{C}(\mathfrak{e})^n. \tag{49}$$

In addition, if $\mathfrak{e}_0$ *is a period-n set then the Chebyshev polynomials of degree nk for the sets* $\mathfrak{e}$ *and* $\mathfrak{e}_0$ *coincide and*

$$\left\|T_{nk}^{(\mathfrak{e})}\right\|_{\mathfrak{e}} = \cosh(nk\alpha) \left\|T_{nk}^{(\mathfrak{e}_0)}\right\|_{\mathfrak{e}_0}. \tag{50}$$

As mentioned above, in the case where $\mathfrak{e}$ consists of finitely many C^{2+} Jordan regions, Widom obtained both Szegő–Widom asymptotics and asymptotics of the Widom factors for the weighted Chebyshev polynomials. In the case of arcs, however, very little is known. For weighted Chebyshev polynomials on finitely many interval (i.e., in the special case of arcs lying on the real line), Widom [49, Sect. 11] merely established asymptotics of the Widom factors and conjectured the corresponding Szegő–Widom asymptotics for the polynomials. This conjecture was proven in the unweighted case in [13], but remains open for the weighted case.

In [45], Totik–Yuditskii extended the asymptotics of the Widom factors for weighted Chebyshev polynomials to the case of $\mathfrak{e}$ consisting of finitely many intervals and C^{2+} Jordan regions symmetric with respect to the real line. Yet, the case of sets consisting of finitely many smooth components some or all of which are arcs in general position in the complex plane has proven to be much more difficult. Widom made conjectures regarding that case, but subsequent works [18, 41, 45] have shown that these conjectures are incorrect. In particular, Widom expected that generically the asymptotic upper bound (43) is attained for sets $\mathfrak{e}$ with finitely many smooth components when at least one of them is an arc. While this was shown to be false in [45], the same work [45] (and [41] in the unweighted case) also showed that Widom was qualitatively correct in expecting larger asymptotics when an arc component is present. In addition, for unweighted Chebyshev polynomials it was shown in [42] that sets $\mathfrak{e}$ containing an arc lead to an increased lower bound

$$\|T_n\|_{\mathfrak{e}} \geq (1+\beta)\,\mathrm{C}(\mathfrak{e})^n, \quad n \geq 1, \tag{51}$$

for some $\beta > 0$ that depends only on $\mathfrak{e}$ (cf. (2)).

So far, the only nontrivial example of an arc for which the asymptotics is known is a single arc on the unit circle. In that case, Widom expected the asymptotics to be the same as for an interval. However, it was observed in [41] that for the circular arc $\mathfrak{e} = \left\{e^{i\theta} : \theta \in [-\alpha, \alpha]\right\}$ (with $0 < \alpha < \pi$), the unweighted Widom factors obey the asymptotics

$$\lim_{n\to\infty} W_n(\mathfrak{e}) = 1 + \cos\left(\frac{\alpha}{2}\right). \tag{52}$$

This shows that the case of a circular arc continuously interpolates between the case of a region (e.g., $W_n(D) \equiv 1$ for a closed disk D) and the case of a flat arc (e.g., $W_n(I) \equiv 2$ for an interval I). In addition, it was shown in [38] that the Widom factors for a circular arc are strictly monotone increasing. The Szegő–Widom asymptotics for the unweighted Chebyshev polynomials of a circular arc was derived by Eichinger [18] and the behavior is indeed different from the case of an interval.

At this point, we also mention a curious observation made in [4, Thm. 5.1]. For the polynomials orthogonal with respect to the equilibrium measure $d\rho_{\mathfrak{e}}$ on a circular arc $\mathfrak{e}$, the square of the associated L^2-Widom factors have the same asymptotics as $W_n(\mathfrak{e})$ in (52). This suggests that the two quantities might also coincide for other smooth arcs in the complex plane. Since the asymptotics of the L^2-Widom factors for a C^{2+} arc is known (see [3, 49]), we are led to the following conjecture:

Conjecture 3.4 If $\mathfrak{e}$ is a smooth arc in the complex plane, then

$$\lim_{n\to\infty} W_n(\mathfrak{e}) = 2\pi S(w_{\mathfrak{e}})\,\mathrm{C}(\mathfrak{e}), \tag{53}$$

where $w_{\mathfrak{e}} = \frac{1}{2\pi}\left(\frac{\partial G}{\partial n_+} + \frac{\partial G}{\partial n_-}\right)$ is the density of the equilibrium measure $d\rho_{\mathfrak{e}}$ with respect to arc-length.

For C^{2+} arcs, Alpan [3] showed that the conjectured asymptotic value satisfies

$$1 < 2\pi S(w_{\mathfrak{e}})\,\mathrm{C}(\mathfrak{e}) \leq 2 \tag{54}$$

with the upper bound being strict if and only if $\frac{\partial G}{\partial n_+}(z) \neq \frac{\partial G}{\partial n_-}(z)$ for some non-endpoint $z \in \mathfrak{e}$. The latter holds, for example, for non-analytic arcs. Partial progress towards the above conjecture is also reported in [3, Thm. 1.3] where the asymptotic upper bound (43) is improved by replacing the constant 2 with the smaller constant $2\sqrt{\pi S(w_{\mathfrak{e}})\,\mathrm{C}(\mathfrak{e})}$.

The study of Chebyshev polynomials for subsets of the complex plane has another interesting and challenging direction which concerns the asymptotic behavior of their zeros. Let $w_1, \ldots, w_n$ be the zeros of T_n counting multiplicity and denote by

$$d\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{w_j} \tag{55}$$

the normalized zero-counting measure for T_n. The limit points of $\{d\mu_n\}_{n=1}^{\infty}$ as $n \to \infty$ are called *density of Chebyshev zeros* for $\mathfrak{e}$.

In [48], Widom proved that for any closed subset S of Ω, the unbounded component of $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$, there is an upper bound on the number of zeros of T_n in S which depends only on S and not on n. This implies the following general result on the density of Chebyshev zeros as stated in [16].

Theorem 3.5 *Any limit point $d\mu_\infty$ of the zero-counting measures $d\mu_n$ is supported in the polynomial convex hull of $\mathfrak{e}$. Moreover, for all $z \in \Omega$ we have that*

$$\int \log|z - w|\, d\mu_\infty(w) = \int \log|z - w|\, d\rho_{\mathfrak{e}}(w). \tag{56}$$

This theorem says that $d\rho_{\mathfrak{e}}$ is the balayage (see, e.g., [36, Sect. II.4]) of $d\mu_\infty$ onto $\partial\mathfrak{e}$, equivalently, the balayage of $d\mu_n$ converges to $d\rho_{\mathfrak{e}}$; ideas that go back at least to Mhaskar–Saff [30]. It is an intriguing question to understand whether or not the zero-counting measures $d\mu_n$ (or some subsequence thereof) converge to the equilibrium measure $d\rho_{\mathfrak{e}}$. In [35], Saff–Totik proved the following result.

Theorem 3.6 *Let $\mathfrak{e} \subset \mathbb{C}$ be a compact set with connected interior and complement. Then:*

(*a*) *If $\mathfrak{e}$ is an analytic Jordan region (i.e., $\partial\mathfrak{e}$ is an analytic simple curve), then there is a neighborhood U of $\partial\mathfrak{e}$ so that for all large n, T_n has no zeros in U.*

(*b*) *If $\partial\mathfrak{e}$ has a neighborhood U and there is a sequence $n_j \to \infty$ so that $\mu_{n_j}(U) \to 0$, then $\mathfrak{e}$ is an analytic Jordan region.*

Accordingly, for analytic Jordan regions the equilibrium measure is never a density of Chebyshev zeros and one may start wondering where these densities are supported. Interestingly, and around the same time, Widom [52] had a similar result for nonselfadjoint Toeplitz matrices and Faber polynomials of the second kind.

On the other hand, Blatt–Saff–Simkani [10] proved the following result.

Theorem 3.7 *Let $\mathfrak{e} \subset \mathbb{C}$ be a polynomially convex set with empty interior. Then, as $n \to \infty$, the Chebyshev zero-counting measures $d\mu_n$ converge weakly to $d\rho_{\mathfrak{e}}$.*

As explained below, there are also local versions of the above two theorems (see [16] for proofs).

Theorem 3.8 *Let $\mathfrak{e} \subset \mathbb{C}$ be a polynomially convex set and suppose $U \subset \mathbb{C}$ is an open connected set with connected complement so that $U \cap \partial\mathfrak{e}$ is a continuous arc that divides U into two pieces, $\mathfrak{e}^{int} \cap U$ and $(\mathbb{C} \setminus \mathfrak{e}) \cap U$. If $M_n(U)$ denotes the number of zeros of T_n in U and*

$$\liminf_{n\to\infty} \frac{M_n(U)}{n} = 0 \tag{57}$$

then

$$U \cap \partial\mathfrak{e} \text{ is an analytic arc.} \tag{58}$$

It readily follows that if $\mathfrak{e}$ is a Jordan region whose boundary curve is piecewise analytic but not analytic at some corner points, then at least these corner points are points of density for the zeros of T_n. Moreover, if $\partial\mathfrak{e}$ is nowhere analytic then all of the boundary points are points of density for the zeros. In that light, it might be tempting to expect that the zero-counting measures $d\mu_n$ converge to the equilibrium measure $d\rho_{\mathfrak{e}}$ whenever $\partial\mathfrak{e}$ is nowhere analytic—and this was conjectured in [16]. However, Totik [43] recently disproved such a statement (which was also considered by Widom [53] in the context of nonselfadjoint Toeplitz matrices).

Nevertheless, local convergence to the equilibrium measure can be proved in some cases.

Theorem 3.9 *Let $\mathfrak{e} \subset \mathbb{C}$ be a polynomially convex set and suppose $U \subset \mathbb{C}$ is an open connected set whose complement is also connected. Assume that $\mathrm{C}(U \cap \mathfrak{e}) > 0$ but that $U \cap \mathfrak{e}$ has two-dimensional Lebesgue measure zero. Then, as $n \to \infty$, the zero-counting measures $d\mu_n$ restricted to U converge weakly to the equilibrium measure $d\rho_{\mathfrak{e}}$ restricted to U.*

Another interesting result on convergence to the equilibrium measure is given by Saff–Stylianopoulos [34]. They prove that if $\partial\mathfrak{e}$ has an inward pointing corner (more generally, a non-convex type singularity), then the zero-counting measures $d\mu_n$ always converge weakly to $d\rho_{\mathfrak{e}}$. For example, if $\mathfrak{e}$ is a non-convex polygon then their hypothesis holds. The case of convex polygons, on the other hand, leads to an interesting open problem.

Open Problem 3.10 What are the density of Chebyshev zeros when $\mathfrak{e}$ is a convex polygon?

This is not even known for the equilateral triangle, although numerical computations present some evidence for convergence to the equilibrium measure. The other natural candidate for the limit points of zeros is the skeleton consisting of the line segments from the centroid of the triangle to the vertices.

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Some Recent Progress on the Stationary Measure for the Open KPZ Equation

Ivan Corwin

Dedicated to the Memory of Harold Widom

Abstract This note is an expanded version of a lecture I gave in fall 2021 at the MSRI program "Universality and Integrability in Random Matrices and Interacting Particle Systems". I will focus on the behavior of the stationary measure for the open KPZ equation, a paradigmatic model for interface growth in contact with boundaries. Much of this will review elements of my joint work with A. Knizel as well as with H. Shen, as well as subsequent works of W. Bryc, A. Kuznetsov, Y. Wang, and J. Wesołowski and of G. Barraquand and P. Le Doussal. The basis for this advance is fundamental work of B. Derrida, M. Evans, V. Hakim and V. Pasquier from 1993, of T. Sasamoto, M. Uchiyama and M. Wadati from 2003, and of W. Bryc and J. Wesołowski from 2010 and 2017. I will try to explain how all of this fits together, without laboring details for the sake of exposition. Though this work does not directly follow from Harold Widom's own work, it (and a great deal of my research) is very much inspired by his and Craig Tracy's work on ASEP.

Keywords Asymmetric simple exclusion process · Askey-Wilson processes · Matrix product ansatz · Kardar-Parisi-Zhang stochastic partial differential equation

1 Preface

The purpose of this note is to describe the stationary measure for the open KPZ (Kardar-Parisi-Zhang) equation and some of the ideas related to its construction. The KPZ equation has been a subject of intense study due to its value as a model for

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random growth and as a singular stochastic PDE, its remarkable connection with integrable systems, its ubiquitous occurrence in seemingly unrelated problems in mathematics and physics, e.g., see the reviews [13, 14, 29, 36, 46].

The open KPZ equation is supposed to model stochastic growth in contact with boundaries, or equivalently (through differentiation) stochastic transport between reservoirs. As is often the case in the study of Markov processes, a fundamental question here is to understand the structure of its stationary state—for instance, whether it is unique and how it can be characterized.

Remarkably, as I will explain below, this stationary measure can be described in terms of the Brownian motion measure reweighted in terms of certain exponential functionals—a subject which has attracted great interest within other realms of probability theory in its own right, e.g. [20, 37, 42].

2 An Aside on q-Pochhammer Symbol Asymptotics

I start with something that may seem far from the advertised subject. A devoted reader who gets to the very end of this note (see Sect. 8.5) will see that, indeed, this is quite key in the asymptotic analysis that leads to the main result presented in this note. Indeed, it is very much in the spirit of Harold's work that in the end, things boil down to involved asymptotic analysis.

The q-Pochhammer symbol is defined for $a \in \mathbb{C}$ and $|q| < 1$ by

$$(a; q)_\infty := (1 - a)(1 - qa)(1 - q^2 a) \cdots .$$

This convergent infinite product defines an analytic function (in a) which arises in many contexts. In combinatorics it encodes certain generating functions. Two of the simplest examples are

$$(-q; q)_\infty = \sum_{\lambda \text{ strict}} q^{|\lambda|} \qquad \text{and} \qquad (q; q)_\infty^{-1} = \sum_{\lambda} q^{|\lambda|}$$

where λ denotes a partition (i.e., a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$), strict means that $\lambda_1 > \lambda_2 > \cdots$, and $|\lambda| := \lambda_1 + \lambda_2 + \cdots$.

The q-Pochhammer symbol is key to defining various q-deformed variants of classical special functions. These come up, for example, in the Askey-Wilson scheme of orthogonal polynomials, see, for example, Sect. 8.3 below or [1].

The q-gamma function is a deformed special function, defined for $z \in \mathbb{C}$ and $|q| < 1$ as

$$\Gamma_q(z) := (1 - q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty}.$$

It is not too hard to see that

$$\lim_{q\to 1} \Gamma_q(z) = \Gamma(z),$$

for instance by observing convergence to the Euler product formula for the gamma function.

It is an interesting and (as will be clear much later) valuable question to determine the nature of this convergence. For instance, the gamma function has various asymptotic behaviors as $|z| \to \infty$, in a manner depending on the direction to which z goes to complex infinity. Does the q-gamma function enjoy similar asymptotic behavior and is this uniform, in any sense, as q tends to 1?

In 1984, Moak [44] first took up this question, proving a Stirling's type expansion for $\Gamma_q(z)$ with $q \in (0, 1)$ fixed and z tending to infinity with $\arg(z) \in (-\pi/2 + \delta, \pi/2 - \delta)$ for any fixed ϵ. Soon after, McIntosh [43] addressed the case where z is fixed but q tends to 1 from below, see also work of Daalhuis [22] and more recently Katsurada [38] and Zhang [55].

The most complicated element in $\Gamma_q(z)$ is the denominator $(q^z; q)_\infty$. Consider $z \in \mathbb{C}$ and

$$q = e^{-\epsilon}$$

as ϵ tends to 0. In that case, q^z approaches 1 (for z fixed) and hence many terms in the product defining $(q^z; q)_\infty$ will approach 0. In fact, a quick calculation shows that there are order ϵ^{-1} such terms. This suggests that $\log(q^z; q)_\infty$ will decay like a negative constant times ϵ^{-1}. This is the case, and the constant is remarkably given by $\zeta(2)$. The expansion continues like

$$\log(q^z; q)_\infty = -\tfrac{\pi^2}{6}\epsilon^{-1} - \left(z - \tfrac{1}{2}\right)\log(\epsilon) - \log\left(\tfrac{\Gamma(z)}{2\pi}\right) + \cdots .$$

The $\Gamma(z)$ term in this expansion is what ultimately leads to the limit $\Gamma_q(z) \to \Gamma(z)$. The $\cdots$ lower order terms hide a lot here. For instance, it is possible to go out to arbitrary order $m \in \mathbb{Z}_{\geq 1}$ in ϵ so that the lower order terms take the form

$$\cdots = -\sum_{n=1}^{m} \frac{B_{n+1}(z)B_n}{n(n+1)!}\epsilon^n + \mathrm{Error}_m(\epsilon, z)$$

where $B_k(z)$ and B_k are the Bernoulli polynomials and Bernoulli numbers, respectively, and the remaining error in this approximation is denoted as $\mathrm{Error}_m(\epsilon, z)$.

Of course, everything is now moved into studying the behavior of the residual error term $\mathrm{Error}_m(\epsilon, z)$. Namely, how does it behave as ϵ and z vary? For ϵ fixed and z varying in certain regions of the complex plane, or for ϵ tending to 1 and z in fixed compact regions of the complex plane, this was understood in the earlier mentioned works. However, what if z varies in a region that is not bounded as ϵ tends to 0? It turns out that exactly this type of control is pivotal in my derivation of the open KPZ equation stationary measure with Knizel in [17] since things eventually boil down to studying asymptotics of integrals involving factors like $(q^z; q)_\infty$ for z varying over regions that grow as q goes to 1. In order to apply the dominated convergence theorem, some type of uniform control is needed. In particular, in my work with Knizel we make an estimate [17, Proposition 2.2] that shows that for any $\delta \in (0, 1/2)$ and $b \in (m-1, m)$, there exists $C, \epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ and all $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < 5/\epsilon$,

$$|\mathrm{Error}_m(\epsilon, z)| \le C\left(\epsilon(1+|z|)^2 + \epsilon^b(1+|z|)^{1+2b+\delta}\right).$$

While we made no claim as to whether this is an optimal bound, it does suffice for our purposes in applying dominated convergence. There is a similar sort of expansion for $(-q^z; q)_\infty$, though I will not record it here.

The proof of this asymptotic expansion and error bound in [17, Proposition 2.2] builds on work of Zhang [55], correcting some mistakes therein and extending from compact domains for z to unbounded ones. The starting point is the following Mellin-Barnes integral representation

$$\log(q^z; q)_\infty = -\frac{1}{2\pi}\int_{c-\iota\infty}^{c+\iota\infty} \Gamma(s)\zeta(s+1)\zeta(s, z)\epsilon^{-s} ds$$

where $q = e^{-\epsilon}$, and where $c > 1$, $\mathrm{Re}(z) > 0$, the integral is over a vertical contour, and $\zeta(s, z)$ is the Hurwitz zeta function. Asymptotic of this integral formula requires a well-controlled understanding of asymptotics of the terms in the integrand. For $\Gamma(s)$ and $\zeta(s+1)$ such control is well-known, while for $\zeta(s, z)$ it needs to be derived, again based on Mellin-Barnes integral formulas which express $\zeta(s, z)$ in terms of simpler functions (namely, the gamma and zeta function). Finally, observe that the above Mellin-Barnes formula for $\log(q^z; q)_\infty$ was restricted to $\mathrm{Re}(z) > 0$. The move into the other half of the complex plane is facilitated by certain identities involving the Jacobi theta functions. I will not give any further details here. An interested reader can find more in the final section of [17].

3 What Is the Open KPZ Equation?

Since this is not a survey on the KPZ equation as a stochastic PDE, I will try to stay brief on this side of the story and instead refer an interested reader to my earlier survey [13] and more recent expository piece with Shen [19].

The open KPZ equation is a singular stochastic PDE describing the evolution of a *height function* $h(t, x)$ taking real values, with time $t \in \mathbb{R}_{\ge 0}$, and space $x \in [0, 1]$. The evolution is a function of a time-space white noise, denoted by $\xi(t, x)$, and described by the equation

$$\partial_t h = \tfrac{1}{2}\partial_{xx}h + \tfrac{1}{2}(\partial_x h)^2 + \xi$$

with the boundary conditions for all $t \in \mathbb{R}_{>0}$ depending on two parameters $u, v \in \mathbb{R}$ given by

$$\partial_x h = \begin{cases} u, & x = 0 \\ -v & x = 1. \end{cases}$$

Since the noise driving this equation is uncorrelated in time, it follows that this defines a Markov process. At a physical level, this equation is supposed to model the evolution of an interface that is subject to smoothing (the Laplacian), growth normal to the interface (the non-linearity) and a stochastic driving force (the white noise). The boundary condition is suppose to indicate that local to the end of the interval, the slope is maintained at specific values. From a physical perspective, this could be caused by an interaction between the boundary and growing media. See Fig. 1 for an illustration of this growth process over time.

Another physical interpretation of this equation is in terms of the dynamics on $\partial_x h$. The resulting stochastic PDE is known as the stochastic Burger's equation and it is supposed to model stochastic conservative transport along the interval [0, 1]. The boundary conditions then correspond to maintaining specific densities or potentials for reservoirs at the two endpoints of the interval.

When I introduce the open ASEP height and particle process later and describe its limit to the open KPZ equation, both of these interpretations may come into further focus.

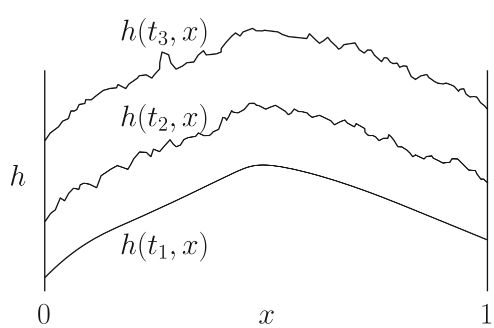

Fig. 1 An illustration of the open KPZ equation evolution at three times t_1, t_2 and t_3. The height is plotted as a function of space x. The height functions, in general, can overlap since they may both grow and shrink. At the left boundary, the slope is maintained as u and at the right it is maintained as $-v$. The initially smooth height profile is roughened with time, as it approaches the stationary measure.

3.1 Long-Time and Stationary Behavior

The driving question behind studying the open KPZ stationary measure is to understand the long-time behavior of the open KPZ equation. Does the height function have a limit (in distribution) as time goes to infinity? The easy answer to this question is NO. There is nothing pinning the height function down and hence over time the height will drift away to infinity. Of course, the relevant question is not about the absolute height, but rather the relative height—that is the height function centered by its value at $x = 0$. Now there is good reason to believe that the law of this profile should converge as time grows to a limit distribution on height functions which take value 0 at $x = 0$, and that the law should not depend on the initial distribution of the profile. I will return to this, as of yet, open problem a bit later.

The law on a height function $h_{u,v} : [0, 1] \to \mathbb{R}$ with $h_{u,v}(0) = 0$ will be said to be a *stationary measure* for the open KPZ equation if when started with $h(0, \cdot) \overset{(d)}{=} h_{u,v}(\cdot)$, it follows that $h(t, \cdot) - h(t, 0) \overset{(d)}{=} h_{u,v}(\cdot)$ for all subsequent times $t > 0$. Of course, it is reasonable to expect that the long-time limit profile will converge to such a stationary measure, and in the case of the open KPZ equation that for each choice of u, v there is just one such stationary measure. Though these are great questions that should be addressed, here I will focus on the question of how to construct and describe a stationary measure for the open KPZ equation.

3.2 Defining the Open KPZ Equation

Before proceeding in this direction, let me briefly address the highly non-trivial task of making mathematical sense of what it means to solve the open KPZ equation. Even a näive understanding of time-space white noise suggests that the spatial trajectory of $h(t, x)$ should have the regularity of Brownian motion and hence be Hölder $1/2-$. Thus, making sense of the KPZ equation non-linearity becomes challenging in this case, as does making sense of the meaning of the boundary condition. This can all be done by appealing to Hairer's regularity structure framework, see [32], by smoothing ξ and then performing suitable renormalization (in fact if the smoothing is only on space, Bertini and Cancrini's earlier approach from [4] should suffice).

The simplest way, however, to define the open KPZ equation is through the open stochastic heat equation (SHE). For the sake of completeness I will recall this definition from [18] here. In fact, most of what I discuss in this note can be appreciated without a deep understanding of the stochastic PDE side of this story.

The *Hopf-Cole solution* to the open KPZ equation is defined as $h(t, x) := \log z(t, x)$ where $z(t, x)$ solves the SHE

$$\partial_t z = \tfrac{1}{2}\partial_{xx} z + \xi z$$

with boundary conditions

$$\partial_x z = \begin{cases} (u - \frac{1}{2})z, & x = 0, \\ -(v - \frac{1}{2})z & x = 1. \end{cases}$$

The inclusion of this $1/2$ factor is just a convention that ensures that the point $u, v = 0$ is special (in terms of the phase diagram for the open KPZ equation). The above form of the SHE can be defined and solved via classical methods of semi-linear stochastic PDEs. In particular, a process $z(t, x)$ is a (mild) solution to the SHE if $z(t, \cdot)$ is adapted to the filtration generated by the initial data $z_0(\cdot)$ and the time-space white noise ξ up to time t, and if z satisfies for all $t \in \mathbb{R}_{>0}$ and $x \in [0, 1]$ the equation

$$z(t, x) = \int_0^1 p_{u,v}(t; x, y) z_0(y) dy + \int_0^1 \int_0^t p_{u,v}(t - s; x, y) z(s, y) \xi(ds, dy).$$

The last integral is in the sense of Itô and $p_{u,v}(t; x, y)$ is the heat kernel on $[0, 1]$ satisfying

$$\partial_t p_{u,v}(t; x, y) = \partial_{xx} p_{u,v}(t; x, y), \qquad p_{u,v}(0; x, y) = \delta_{x=y}$$

with boundary conditions for all $y \in [0, 1]$ and $t \in \mathbb{R}_{>0}$

$$\partial_x p_{u,v} = \begin{cases} (u - \frac{1}{2}) p_{u,v}, & x = 0, \\ -(v - \frac{1}{2}) p_{u,v} & x = 1. \end{cases}$$

The open SHE admits a chaos series expansion and can (though it has not been precisely given in the literature) also be interpreted as a partition function for a continuum directed random polymer model in which the underlying path measure is that of Brownian motion which either dies or splits at the boundaries 0 and 1, depending on the signs of u and v.

4 Constructing the Stationary Measure

I will relate here the main result of my joint work with A. Knizel, given there as [17, Theorem 1.2]. Essentially, we show that for each pair $(u, v) \in \mathbb{R}$ of boundary parameters there exists a stationary measure for the corresponding version of the open KPZ equation; for general (u, v) we provide some properties of our measure while for (u, v) such that $u + v \geq 0$ we are able to completely characterize the measure in terms of an exact formula for its multi-point Laplace transform. In phrasing our results, there is a bit of subtlety stemming from the fact that we did not show that these are the unique stationary measures (though we conjecture this to be the case).

As suggested above, our first result is that for each pair $(u, v) \in \mathbb{R}$ there exists a stationary measure for the corresponding open KPZ equation with those boundary parameters. We denote a random function distributed according to this measure by $h_{u,v}(\cdot)$. As I will explain later in this note, these measures arise as subsequential limits of a stationary measure for a discrete approximation to the open KPZ equation, namely the open ASEP.

In the special case that $u + v = 0$, the random function $h_{u,v}(\cdot)$ has the law of a standard Brownian motion of drift $u = -v$. In fact, in that case the ASEP approximation scheme is not just tight but has a unique limit point. That is not to say that this implies that there is a unique stationary measure in this case, only that the ASEP stationary measures has a unique limit, denoted by $h_{u,-u}(\cdot)$.

For general $(u, v) \in \mathbb{R}$, the increments of $h_{u,v}(\cdot)$ satisfy a property that we call *stochastic sandwiching*. The simplest and most useful implication of this sandwiching is the following. There exists a coupling (i.e., a common probability space which supports these random processes) of $h_{u,v}(\cdot)$, $h_{u,-u}(\cdot)$ and $h_{-v,v}(\cdot)$ such that for all $0 \le x \le y \le 1$, almost surely

$$h_{-v,v}(y) - h_{-v,v}(x) \le h_{u,v}(y) - h_{u,v}(x) \le h_{u,-u}(y) - h_{u,-u}(x)$$

when $u + v > 0$; when $u + v < 0$, the above holds with the inequalities swapped.

For $u + v > 0$, we completely characterize the law of $h_{u,v}(\cdot)$ in terms of its multi-point Laplace transform. Before stating that, let me relate the simplest version of this formula which characterizes $h_{u,v}(1)$. Physically, this represents the total displacement of the open KPZ height interface on the interval. Due to its Brownianity, in the case where $u + v = 0$ this increment is clearly a Gaussian random variable of mean $u = -v$ and variance 1. When $u, v > 0$ the law of $h_{u,v}(1)$ is determined by the following formula

$$\mathbb{E}\left[e^{-sh_{u,v}(1)}\right] = e^{s^2/4} \frac{\int_0^\infty e^{-r^2} \mu_s(r)dr}{\int_0^\infty e^{-r^2} \mu_0(r)dr}, \quad \text{for } \mu_s(r) = \frac{\left|\Gamma(\frac{s}{2} + u + \iota r)\Gamma(-\frac{s}{2} + v + \iota r)\right|^2}{|\Gamma(2\iota r)|^2} \tag{1}$$

and where the Laplace variable s is allowed to vary in $(0, 2v)$. When $u + v > 0$ yet one of the variables is negative, there is a similar albeit slightly more complicated formula involving a measure with and atomic part as well as an absolutely continuous part.

Turning to the general formula, let $C_{u,v} = 2\mathbf{1}_{u\notin(0,1)} + 2u\mathbf{1}_{u\in(0,1)}$ and consider any $d \in \mathbb{Z}_{\ge 1}$, any $0 = x_0 < x_1 < \cdots < x_d \le x_{d+1} = 1$ and any $c_1, \ldots, c_d$ such that $c_1 + \cdots + c_d < C_{u,v}$. Then, letting $s_k = c_k + \cdots + c_d$ for $k = 1, \ldots, d$ and

$s_{d+1} = 0$, the multi-point Laplace transform is given by the formula

$$\mathbb{E}\left[e^{-\sum_{k=1}^{d} c_k h_{u,v}(x_k)}\right] = \frac{\mathbb{E}\left[e^{\frac{1}{4}\sum_{k=1}^{d+1}(s_k^2 - \mathbb{T}_{s_k})(x_k - x_{k-1})}\right]}{\mathbb{E}\left[e^{-\frac{1}{4}\mathbb{T}_0}\right]}. \tag{2}$$

The expectation on the left side above is over the open KPZ stationary measure on $h_{u,v}(\cdot)$ while on the right it is over the measure on a stochastic process $(\mathbb{T}_s)_{s\in[0,C_{u,v})}$ that we termed the *continuous dual Hahn process*. The transition probabilities for this process are given in terms of the orthogonality probability measure for the continuous dual Hahn polynomials, and the measure according to which we initialize this process in the above identity takes a similar form in terms of a ratio of products of gamma functions. Since this involves a number of formulas that probably will not be illuminating, I will not write anything further on this definition here. An interested reader is referred to [17] or the subsequent work of Bryc [7] for more on this process and the formulas used to define it.

The above characterization of the stationary measure $h_{u,v}(\cdot)$ in the case where $u + v > 0$ is, in my own opinion, a bit less than fulfilling. While it uniquely characterizes the law it does not provide a direct description of $h_{u,v}(\cdot)$ in terms of a "nice" stochastic process, such as is the case when $u + v = 0$ (when $h_{u,-u}(\cdot)$ is Brownian). In fact, it is challenging (though certainly possible) to take the limit of $u + v \searrow 0$ in our Laplace transform formula in order to recover the Laplace transform of the Brownian motion $h_{u,-u}(\cdot)$.

Thankfully (for us), this lack of a nice stochastic process description was fairly quickly resolved, as I explain now.

5 Inverting the Multipoint Laplace Transform Formula

It took a few years for Knizel and me to complete our work, proving the above results. Pretty early on, though, we had a good sense of what the final formula should look like and we communicated this to Bryc when he visited us as Columbia for a seminal. He had worked with Wang in [9] on inverting some similar (albeit less complicated) Laplace transform formulas arising for the limit of the ASEP stationary measure, so it seems reasonable to see if he had ideas on how to proceed. The seed was planted and as luck would have it, around the time that Knizel and I were finishing our paper, Bryc and his collaborators (Kuznetsov, Wang, and Wesołowski) had figured out how to do the desired inversion. In fact, this was extremely helpful since it provided a check for our formulas and helped reveal a few missing factors that we hunted down and repaired.

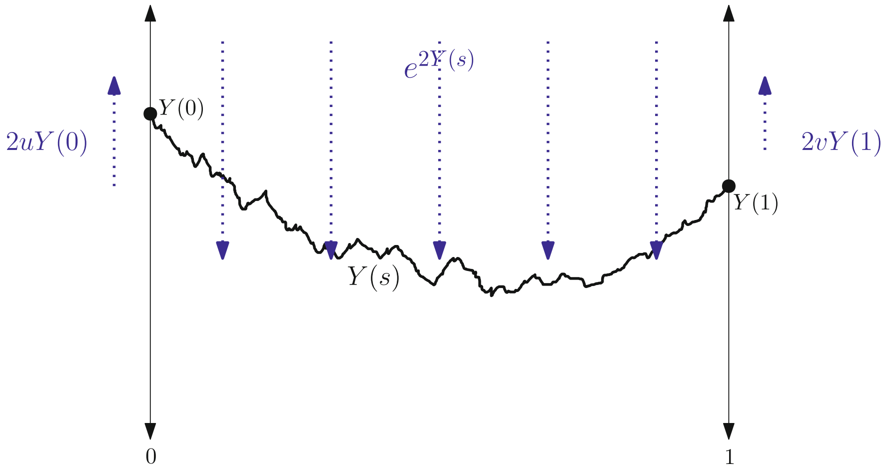

Fig. 2 A depiction of $Y(\cdot)$. In this illustration we assume $u, v > 0$ so the starting and ending points are probabilistically rewarded for going up with a linear energy contribution. However, the exponential energy term $e^{Y(s)}$ pushes the curves down. It is this balance of energetic constraints (depicted in dark blue) that results in the reweighted measure being normalizable despite the fact that the free starting point Brownian motion is an infinite measure.

The description that they arrived at in [12] can be written as follows. The process $x \mapsto h_{u,v}(x)$ for $x \in [0, 1]$ has the same law as the process

$$x \mapsto B(x) + Y(0) - Y(x). \tag{3}$$

Here B denotes a Brownian motion started at zero and with variance $1/2$ and mean 0 at time 1. The process Y is independent of B and has a Radon-Nikodym derivative against a free starting point Brownian motion with variance $1/2$ and no drift which is proportional to (see also Fig. 2)

$$\exp\left(2uY(0) - \int_0^1 e^{2Y(s)} ds + 2vY(1)\right). \tag{4}$$

By a free starting point Brownian motion, I mean the infinite measure on paths where the starting point is distributed according to Lebesgue measure on $\mathbb{R}$ and the trajectory from there is distributed according to a Brownian motion with that starting point. The time interval for this Brownian motion is $[0, 1]$ and the variance at time 1 is $1/2$. It is a non-trivial calculation to show that despite the free Brownian motion being an infinite measure, after multiplying by the above Radon-Nikodyn derivative, the resulting measure has finite mass and can be normalized to define a probability measure, i.e., the measure on Y. The work in [12, Proposition 1.6] is valid for $u + v > 0$ and $\min(u, v) > -1$, though the description seems to make sense for general $u + v > 0$ and that extension should be possible with what is already known.

Soon after the work of [12] was posted, Barraquand and Le Doussal [3] used a different set of tools from physics (coming from the study of 1d Liouville quantum gravity and not addressed there with mathematical rigor) to independently invert the Laplace transform. They arrived at a slightly different description that, in fact, has some advantages. They show that the process $x \mapsto h_{u,v}(x)$ for $x \in [0, 1]$ has the same law as the process

$$x \mapsto B(x) + \tilde{Y}(x). \tag{5}$$

As before, B denotes a Brownian motion started at zero and with variance $1/2$ and mean 0 at time 1. The process $\tilde{Y}$ is independent of B and has a Radon-Nikodym derivative against Brownian motion started at zero and with variance $1/2$ and mean 0 at time 1 which is propositional to

$$\left(\int_0^1 e^{-2\tilde{Y}(s)}ds\right)^{-u} \left(\int_0^1 e^{2\tilde{Y}(1)-2\tilde{Y}(s)}ds\right)^{-v}. \tag{6}$$

The factor $e^{-2v\tilde{Y}(1)}$ which appears above can be thought of as introducing a drift so we can also characterize $\tilde{Y}$ as having a Radon-Nikodym derivative against Brownian motion started at zero and with variance $1/2$ and mean $-v$ at time 1 which is propositional to

$$\left(\int_0^1 e^{-2\tilde{Y}(s)}ds\right)^{-u-v}. \tag{7}$$

It may not seem so immediate how to move between the two descriptions above for $h_{u,v}$. Such a matching is shown in [8]. Here is a brief sketch. Starting with (3), write $\hat{Y}(x) = Y(0) - Y(x)$. The aim is to show that $\hat{Y}(\cdot)$ and $\tilde{Y}(\cdot)$ have the same law. Notice that the Radon-Nikodym derivative from (4) can be rewritten in terms of $\hat{Y}$ and $Y(0)$ as

$$\exp\left(2(u+v)Y(0) - e^{2Y(0)}\int_0^1 e^{-2\hat{Y}(s)}ds\right)e^{-2v\hat{Y}(1)}.$$

The law of $\hat{Y}(\cdot)$ and $Y(0)$ are independent, and hence it is possible to integrate out $Y(0)$ since this does not figure into the description of $h_{u,v}$. Recall the integral identity

$$\int_{-\infty}^{\infty} e^{2ax-be^{2x}} dx = \tfrac{1}{2} b^{-a}\Gamma(a)$$

which holds provided the real part of a and b are strictly positive. Choose $a = u+v$ and $b = \int_0^1 e^{-2\hat{Y}(s)}ds$. Since both a and b are strictly positive, after integrating this yields the Radon-Nikodym derivative expression in (6) (written there in terms of $\tilde{Y}$, not $\hat{Y}$) and hence implies that $\hat{Y}$ and $\tilde{Y}$ have the same law.

As far as how to link either of these descriptions to the Laplace transform formula, I will just say a bit since this is a substantial calculation. As is generally the case, given the description of the process $h_{u,v}$ as above, it is much easier to verify that it has the correct Laplace transform than it is to invert the Laplace transform from the start without any inspiration. Taking the Laplace transform relies on the following ideas (much more detail can be found in the papers [12] and [3]). First, recall that for Brownian motions subject to energetic penalization by an exponential potential, the subMarkov generator $\mathcal{L}$ that is relevant acts on a suitable space of functions f as $\mathcal{L}f := f''(x) - e^{2x}f(x)$. Define $f_u(x) := K_{\iota u}(e^x)$ where K is the modified Bessel function (or Macdonald function) given, for $u \in \mathbb{R}$ and $x > 0$, by $K_{\iota u}(x) = \int_0^\infty e^{-x\cosh(w)}\cos(uw)dw$. Then the heat kernel from x to y in time t for this operator takes the form

$$\int_0^\infty e^{-tu^2} f_u(x) f_u(y) \frac{2du}{\pi |\Gamma(\iota u)|^2}$$

as can be shown by appealing to the eigenrelations $\big(\mathcal{L}f_u\big)(x) = -u^2 f_u(x)$. This leads to formulas for the finite dimensional distributions of $h_{u,v}$ in terms of integrals of these functions. Finally, the Laplace transform can be computed by appealing repeatedly to the two identities:

$$\int_{-\infty}^{\infty} e^{tx} f_u(x) f_v(x) dx = \frac{2^{t-3}}{\Gamma(t)} \left| \Gamma\left(\frac{t+\iota(u+v)}{2}\right) \Gamma\left(\frac{t+\iota(u-v)}{2}\right)\right|^2,$$

$$\int_{-\infty}^{\infty} e^{tx} f_u(x) dx = 2^{t-2} \left| \Gamma\left(\frac{t+\iota u}{2}\right)\right|^2,$$

where $u, v \in \mathbb{R}$ and $\mathrm{Re}(t) > 0$. The gamma function products arising here ultimately account for those in the Laplace transform formula.

Let me close this section with a few comments and discussions on directions related to the results described in this and the previous section.

A keen reader will have noticed that the Laplace transform formula in [17] is only given for $u+v > 0$. I will discuss later about the origins of this restriction. The natural question here is to identify the Laplace transform or better yet the law of $h_{u,v}$ when $u+v < 0$. In fact, it is even non-trivial to take the limit where $u+v \searrow 0$ in the Laplace transform formula (2) or the formulas in (3) and (4) for $h_{u,v}$ from [12]. At least in the case of (2), this limit can be performed with some care and does recover the Brownian motion Laplace transform as one would hope. By comparison, the formula for $h_{u,v}$ in (5) and (6) (or even better (7)) immediately admits this limit—when $u + v = 0$ in (7) the Radon-Nikodym derivative is identically equal to 1 and hence $\tilde{Y}$ is a Brownian motion of standard deviation $\sqrt{2}$ and drift $-4v$. Plugging this into (5) shows that $h_{u,v}$ is a standard Brownian motion with drift $u = -v$, as it should be.

The description for $h_{u,v}$ from [3] in (5), in fact, readily admits and extension to all u and v, not just $u+v > 0$. In that case, the process $\tilde{Y}(\cdot)$ is still well-defined. What is not clear (though is conjectured to be true in [3]) is whether the process $h_{u,v}$ defined through (5) in this case when $u + v < 0$ is, in fact, stationary for the KPZ equation. Proving (or disproving) this seems like a great question. A natural approach might be to try to use some variant of uniqueness of analytic continuation—to show that the stationary measure depends in some sort of analytic manner on the boundary parameters u and v, and likewise $h_{u,v}$ defined above by (5). This is all rather vague since the correct notion of analyticity as well as the means to prove it is currently unclear to me.

Another extension discussed in [3] is to consider the open KPZ equation stationary measure on a general interval $[0, L]$, or even on a half-infinite interval $[0, \infty)$. In both cases they write down candidates for the stationary measure. It should be possible to modify the approach from [17] to address these cases. One nice observation in [3] is that the stationary measure process $h_{u,v}$ actually arose (in a very different context) in 2004 work of Hariya and Yor [37]. In that case, their motivation was to study properties of exponential transforms of Brownian motion.

On the subject of the form of the Radon-Nikodym derivatives above, I just want to mention that they bare remarkable similarities to the type of reweighting that arose in my study with Hammond of the KPZ line ensemble [16].

My final remark returns to the question of uniqueness (for each pair u, v) of the open KPZ stationary measures $h_{u,v}(\cdot)$. In [17] we conjectured that this is true, and moreover that the open KPZ equation is ergodic and satisfies a one-force one-solution principle which essentially says that if you start at time $-T$ with two different choices of initial data, and then look at the solution around time 0, as $T \to \infty$, the height increments of the two processes will converge to be the same, and hence independent of the initial data. There are some similar results to this proved in the literature for the periodic boundary condition KPZ equation or for some other similar models, see for example [2, 27, 34, 35, 47, 52].

6 Open ASEP to KPZ

How does one even start to show that there exists a stationary measure for the open KPZ equation, let alone check that it is given in the form that we claim? Usually the idea is to construct the generator or semi-group for the open KPZ equation and then check that the generator acts on a given stationary measure to give zero or that the semi-group preserves the stationary measure. Mixing properties can then be studied by finding the spectral gap or probing clever couplings.

This approach has been successfully implemented for the KPZ equation on a torus or to some extent on the full line where the stationary measure is purely Brownian, see for example [28, 34, 35]. For the open KPZ equation this approach has not been implemented, though I would be quite interested to see it done.

Instead, in [17] we proceed through a discretization of the open KPZ equation—the interacting particle system called *open ASEP*. In a nutshell, the idea is to first show that under special scaling, the height function for open ASEP converges to the open KPZ equation, provided that the initial data has a limit and satisfies some reasonable hypotheses (i.e., its has Brownian-like Hölder behavior). Open ASEP is a finite state space Markov process and for each system size N it has a unique stationary measure. The challenge then becomes to show that these stationary measures converge to a limit as N goes to infinity and satisfy the desired hypotheses. In fact, we only show the existence of a limit in the case where $u + v > 0$ (I will explain what these mean in the open ASEP context below). For general u, v we are able to show tightness of the N-indexed sequence of open ASEP stationary measures which translates into the existence of subsequential limits that are stationary measures for the open KPZ equation. Of course, if we knew uniqueness of the open KPZ stationary measures, that would imply there is only one limit point.

In this section, I am going to try to explain how open ASEP approximates the open KPZ equation. This is based on work of mine with Shen in [18] and a subsequent extension by Parekh [45]. Open ASEP is quite an interesting and well-studied object in its own right, and the rest of this note will almost entirely focus on it. As such, I will start out here with a bit of background, in particular its phase diagram. Then I will return to the connection to open KPZ.

6.1 Introducing Open ASEP

Open ASEP was first introduced by MacDonald et al. in 1968 [41] as a model for the dynamics of ribosomes on an mRNA chain during the synthesis of proteins. Already in that work, their main interest was in studying its stationary measure. Within probability, Spitzer initiated study of a general class of exclusion processes in his 1970 work [50] and Liggett introduced open ASEP to the community in his

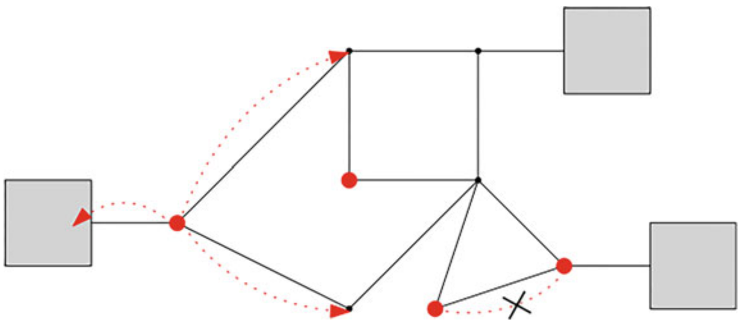

Fig. 3 An exclusion process on a network with reservoirs. Dotted red lines show the possible jumps of the left-most particle and the dotted red line with an $\times$ through it, shows an excluded jump. Other possible jumps and jump rates are not shown.

1975 work [39] as a tool to study the nature of stationary measures for ASEP on the full line and half-line.

Let me now introduce open ASEP. Consider the exclusion process in Fig. 3. Particles (red dots) occupy vertices of a graph and jump along edges according to exponential rates subject to the exclusion rule (jumps to occupied sites are suppressed). There are *reservoirs* (grey squares) which (according to exponential rates) insert particles into unoccupied neighboring sites, or remove particles from occupied neighboring sites. This models transport through a network with sources and sinks.

The presence of reservoirs leads to remarkable physical behavior. Since the total number of particles is not conserved, even when in its steady state (i.e., started its unique stationary measure) there will typically be a net flow of particles through this system—like how a flowing stream may have a stationary density profile while still moving water from source to sink. In statistical physics such models are said to have a *non-equilibrium steady state* (e.g. see the review [6]). Because the Markovian dynamics which produce these steady states are not reversible, non-equilibrium steady states do not take the Boltzmann weight form common in equilibrium statistical mechanics. This renders the study of thermodynamic (i.e. large system size) limits of non-equilibrium steady states quite challenging.

The presence of reservoirs can also induce phase transitions as the number of nodes (N in our case) in the network grows. I am unaware of a precise statements to this effect when dealing with general networks but for the one-dimensional asymmetric simple exclusion process with reservoirs (i.e., open ASEP) such a phase transition is understood. Besides serving as a transport model, I will explain below how the height function for open ASEP is also connected with stochastic interface growth and, through exponentiation, to a discrete stochastic heat equation.

Let me start by defining the open ASEP in one-dimension. Figure 4 gives an illustration of it along with its height function. We consider an N site nearest neighbor graph $\{1, \ldots, N\}$ with edges between consecutive numbers. Particles jump left according to exponential clocks of rate $q < 1$ and right at rate 1. These jumps

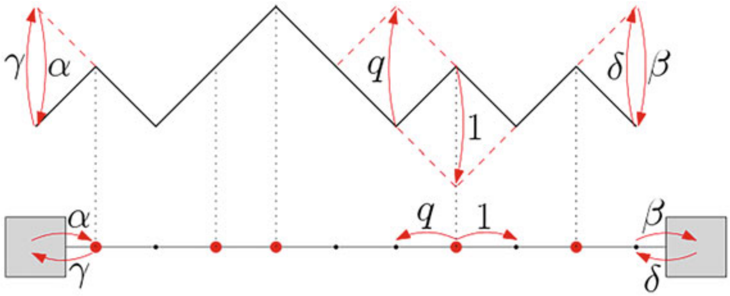

Fig. 4 A given instance of open ASEP with $N = 10$ is illustrated both in terms of occupation variables (the red dots) and its height function h_N (the piecewise linear function drawn above the particle configuration). The red arrows indicate some possible moves with the associated exponential rates labeled.

(and all others) are independent and only taken if the exclusion rule is observed—particles cannot move to occupied sites. In addition to this bulk evolution, there are two reservoirs which interact with site 1 and site N. At rate α a particle enters site 1 (provided it is empty) and at rate γ a particle is removed from site 1 (provided it is occupied); similarly at rate δ a particle enters site N (provided it is empty) and at rate β a particle is removed from site N (provided it is occupied). This process can be encode as $\tau(t) = \big(\tau_1(t), \ldots, \tau_N(t)\big)$ where $\tau_x(t) = 1$ if site x is occupied at time t and otherwise $\tau_x(t) = 0$ if it is unoccupied. The (backward) generator of the process $\tau(t)$ is discussed later around (16). It is important to note that since there is no conservation of particle number (particles are created and removed) there is a single invariant measure (or steady state) for this process. The physics literature often denotes this measure as $\pi_N(\tau)$, with the dependence on the other parameters implicit, and denotes the expectation of a function $f : \{0, 1\}^{[\![1,N]\!]} \to \mathbb{R}$ under π_N by

$$\langle f \rangle_N := \sum_{\tau \in \{0,1\}^{[\![1,N]\!]}} f(\tau) \cdot \pi_N(\tau). \tag{8}$$

While the existence of this measure π_N is clear, it is not obvious how it behaves as N tend to infinity. I will return to this important point a bit later.

There is one piece of information that is lost in the open ASEP occupation process—the count of how many particles have entered or exited from the boundaries. Going to the height function process remedies this. I will define the height function process $h_N(t, x)$ as follows. Of course, the subscript indicates the lattice size N, and the time t and spatial location x are now both written as arguments. The dependence of $h_N(t, x)$ on the other parameters $q, \alpha, \beta, \gamma, \delta$ will be generally suppressed from the notation, though they will eventually all depend on N non-trivially. The height function is defined for $t \geq 0$ and $x \in \{0, \dots N\}$ as

$$h_N(t,x) := h_N(t,0) + \sum_{i=1}^{x} \big(2\tau_i(t) - 1\big), \qquad \text{where} \quad h_N(t,0) := -2\mathcal{N}_N(t) \tag{9}$$

and where the net current $\mathcal{N}_N(t)$ records the total number of particles that have entered into site 1 from the left reservoir up to time t minus the number of particles that have exited from site 1 into the left reservoir up to time t. It is convenient to linearly interpolate to define a continuous height function in space.

The open ASEP height function process does not have an invariant probability measure. Indeed, due to the net current $\mathcal{N}_N(t)$, the only invariant measure takes the form of infinite counting measure on the height at the origin and then the induced measure (coming from π_N on the occupation variables τ) on the height function increments from there. Another way of saying this is that if $h_N(x)$ is the random height function defined by $h_N(x) := \sum_{i=1}^{x} \big(2\tau_i(t) - 1\big)$ where τ is distributed according to the invariant measure $\pi_N(\tau)$, then starting the open ASEP height process with initial data $h_N(0,\cdot) = h_N(\cdot)$ implies that for all later times $t > 0$, $h_N(t,x) - h_N(t,0)$ will still have the law of $h_N(x)$ as a process in x. Thus, I call $h_N(\cdot)$ the stationary measure. Here I use stationary instead of invariant to emphasize that it is the increment process that is invariant.

6.2 *Phase Diagram*

The boundary reservoirs play a key role in determining the limiting behavior of open ASEP as N goes to infinity. Define the current in stationarity to be

$$J_N := \frac{\big\langle \alpha(1-\tau_1) - \gamma\tau_1 \big\rangle_N}{1-q}.$$

The term $\alpha(1-\tau_1)$ accounts for the rate α at which particles enter the system provided $\tau_1 = 0$ and the term $\gamma\tau_1$ for the rate γ at which they depart provided $\tau_1 = 1$. The different measures the instantaneous rate of signed movement across the bond between the reservoir and site 1.

If open ASEP starts according to its invariant measure π_N, then it follows that for all $t > 0$

$$J_N = \frac{\big\langle \mathcal{N}_N(t) \big\rangle_N}{t(1-q)}.$$

I am abusing notation here since now $\langle \cdot \rangle_N$ represent the expectation of the open ASEP occupation process $\tau(t)$ started from in its invariant measure $\pi_N(\cdot)$ at time $t = 0$, not just the expectation of a function of the state space. The quantity J_N

represents the average (normalized by the bulk drift $1-q$) current of particles moving through the system at stationarity. In fact, since particles are conserved by the bulk dynamics, this average current will be the same everywhere in the system.

There is a remarkable phase diagram for the large N limit of J_N which highlights the role of the boundary rates. The existence of such a limit is non-trivial, let alone computing its value as a function of $q, \alpha, \beta, \gamma, \delta$. I will give a simple (and non-rigorous) heuristic to derive it, though a more complete derivation in the physics literature follows from the matrix product ansatz (which will be introduced near the end of this note) in [25] for $q=0$ and [51] for general q. In fact, a version of this phase diagram arose in a closely related context in early work of [40].

The limit $J := \lim_{N\to\infty} J_N$ exists and depends on two parameters ρ_ℓ and ρ_r which have nice physical interpretations as effective boundary densities at the left and right boundaries.

Consider N very large and focus on the invariant measure near site 1. Without any justification, imagine that locally and asymptotically in N, the invariant measure looks like a product Bernoulli measure there with density ρ_ℓ for some $\rho_\ell \in (0,1)$. Provided this, it is possible to determine what value ρ_ℓ must take through a simple consideration. Since particles are conserved within the bulk of open ASEP, in stationarity the net number of particles moving from the left reservoir into site 1 must equal the net number moving from site 1 to 2. Taking expectations, this implies a simple conservation equation

$$\alpha(1-\rho_\ell) - \gamma\rho_\ell = (1-q)\rho_\ell(1-\rho_\ell).$$

The first term on the left accounts for the rate α at which particles enter from the boundary to site 1 provided it is unoccupied (which happens in stationarity with probability $1-\rho_\ell$) and the second term accounts for the rate γ at which particles exit to the boundary from site 1 provided it is occupied (which happens in stationarity with probability ρ_ℓ). On the right, the factor $\rho_\ell(1-\rho_\ell)$ is the probability (under the Bernoulli product measure assumption) of having a particle and a hole next to each other in any prescribed order. The factor $(1-q)$ comes from the rate of particles jumping right minus the rate jumping left. Solving this quadratic equation yields ρ_ℓ. A similar consideration around N yields ρ_r. In both cases, there is just one positive solution which yields the effective density at the boundaries.

It is convenient (for later purposes) to write the solutions to the above quadratic equations explicitly in the following form:

$$\rho_\ell = \frac{1}{1+C}, \qquad \rho_r = \frac{A}{1+A} \tag{10}$$

where $A, C > 0$ are given by

$$A = \kappa^+(q,\beta,\delta), \quad B = \kappa^-(q,\beta,\delta), \quad C = \kappa^+(q,\alpha,\gamma), \quad D = \kappa^-(q,\alpha,\gamma) \tag{11}$$

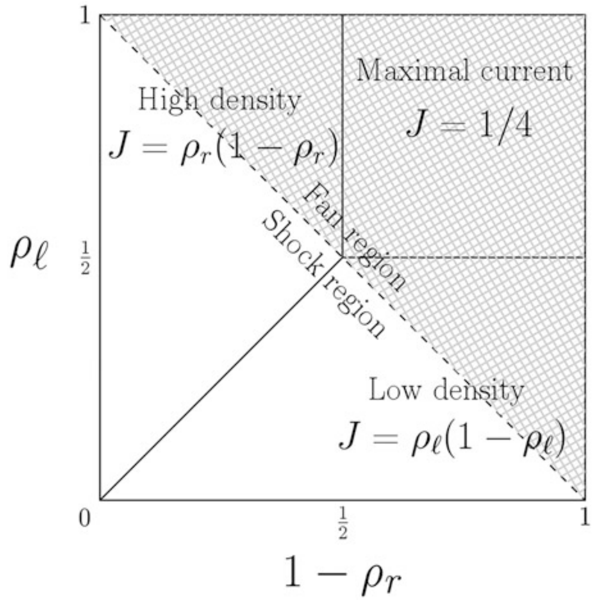

Fig. 5 The phase diagram for the current J of open ASEP as a function of the effective densities ρ_ℓ and ρ_r.

with

$$\kappa^{\pm}(q, x, y) := \frac{1}{2x}\left(1 - q - x + y \pm \sqrt{(1 - q - x + y)^2 + 4xy}\right). \tag{12}$$

The κ^- terms are the negative roots of the quadratic and are introduced for later purposes. Notice that for q fixed, (11) actually gives a bijection between $\{(\alpha, \beta, \gamma, \delta) : \alpha, \beta > 0, \gamma, \delta \geq 0\}$ and $\{(A, B, C, D) : A, C > 0, B, D \in (-1, 0]\}$.

The phase diagram for the value of J is determined entirely by the values of ρ_ℓ and ρ_r (or equivalently A and C) as follows (see also Fig. 5):

- *Maximal current phase ($\rho_\ell > 1/2$ and $1 - \rho_r > 1/2$, or equivalently $C < 1$ and $A < 1$)*: The left boundary creates particles at a fast enough rate (and does not remove them too quickly) and the right boundary removes particles at a fast enough rate (and does not create them too quickly) so that the system is able to transport particles from left to right in the bulk at its level of maximal efficiency. The maximal rate of transport for particles is $1/4$ and is achieved when the bulk density is $1/2$. To see this, note that if there is local product measure of density ρ in the bulk of ASEP then the rate at which particles move (normalized by $1 - q$) will be $\rho(1 - \rho)$. This is maximized at $\rho = 1/2$ and takes value $1/4$ in that case. Thus, $J = 1/4$ in this phase and the density should look locally in the bulk like product Bernoulli with density $1/2$.
- *Low density phase ($\rho_\ell + \rho_r < 1$, or equivalently $C < A$ and $A > 1$)*: The left boundary creates particles relatively slowly and the right boundary removes them

fast enough so that they do not build up there. As such, the density in the bulk is determined by the effective density on the left, ρ_ℓ and the current $J = \rho_\ell(1-\rho_\ell)$.

- *High density phase ($\rho_\ell + \rho_r > 1$, or equivalently $C > A$ and $C > 1$)*: By applying a particle-hole transform this is equivalent to the low density phase for the holes. In particular, now the right boundary removes particles so slowly and there is enough input from the left boundary that the density builds up and becomes ρ_r. Thus, the current $J = \rho_r(1 - \rho_r)$.

The line $\rho_\ell + \rho_r = 1$ involves coexistence of the high and low density phases and displays interesting behavior that I will not touch on here. Another point of interest is the *triple point* when $\rho_\ell = \rho_r = 1/2$. This point will play an key role since it is around there that the KPZ equation arises.

Besides the phases addressed above, there is one other important division of the phase diagram:

- *Fan region ($\rho_\ell > \rho_r$, or equivalently $AC < 1$)*: Very close (in a scaling going to zero relative to N) to the left boundary, the density exceeds that of the bulk. This is called the fan region since going from high to low density in the Burgers equation produces a rarefaction fan.
- *Shock region ($\rho_\ell < \rho_r$, or equivalently $AC > 1$)*: Very close (in a scaling going to zero relative to N) to the left boundary, the density is lower than that of the bulk. Thus, in the same spirit of the Burgers equation, one sees shock-type behavior here.

The boundary between the fan and shock regions, when $\rho_\ell = \rho_r$ or equivalently $AC = 1$, is special since it is the only part of the phase diagram where the invariant measure is simple. Along this line, in the finite N open ASEP, the invariant measure is Bernoulli product with density $\rho = \rho_\ell = \rho_r$. I will recall later how this important fact is shown.

Beyond determining the current J, the above phase diagram also dictates the nature of the fluctuations and large deviations for the open ASEP height function stationary measure and process. For instance, the fluctuations as a spatial process for the stationary measure has been determined for TASEP ($q = 0$) by Derrida, Lebowitz and Enaud [26] and for ASEP (general $q \in (0, 1)$) more recently by Bryc and Wang [9]. In fact, the work of [9] is what prompted my interest in the open KPZ stationary measure and what indicated to me that it should be possible using, in part, methods in that work.

6.3 Microscopic Hopf-Cole Transform

I want to now address the question of how open ASEP and open KPZ are related. The study of stochastic PDE limits of interacting particle systems has been a hot topic in the past decade, and has a significantly longer history. Typically such limits arise when the system size and time scale are taken to infinity appropriately while various parameters are tuned in a critical manner as well. In the case of the KPZ equation on the entire real line, the earliest example of such a convergence result is the 1995 work of Bertini and Giacomin [5]. That work relied on the starting observation which came from earlier work of Gärtner [31] that

$$Z_t(x) := e^{-\lambda h_N(t,x)+\nu t}, \qquad \text{with} \quad \lambda = \tfrac{1}{2}\log q, \quad \nu = 1 + q + -2\sqrt{q} \tag{13}$$

satisfies a microscopic version of the SHE. This transform can be thought of as a microscopic version of the Hopf-Cole transform. The analysis from there on was challenging, but possible since a well-developed solution theory for the SHE was already developed.

I have a few papers from the past decade devoted to generalizing the Bertini-Giacomin approach in different directions and to different types of interacting particle systems. Always, the starting point is the microscopic Hopf-Cole transform. Often this arises as a result of a hidden Markov duality, though I will not say more about this here. The need for such a transform severely limits the applicability of this technique. However, work of Dembo and Tsai [23], and then subsequent developments by Yang have demonstrated that even in instances where the microscopic Hopf-Cole transform does not exactly satisfy a version of the SHE, it is still possible to show convergence to the SHE, and hence of the height process to the KPZ equation. In fact, Yang's recent work [54] shows that the open KPZ equation arises as a limit of a wide variety of open exclusion processes. Other techniques such as energy solutions, paracontrolled distributions and regularity structures also provide routes to derive the KPZ equation as a limit of interacting particle systems.

In the case of the open ASEP, the first proof of its convergence to the open KPZ equation came in my work with Shen [18]. As indicated above, the starting point was our observation that open ASEP satisfies an exact microscopic Hopf-Cole transform. In fact, a special case of this observation showed up around the same time in work of Goncalves et al. [33].

In order for the microscopic Hopf-Cole transform to work for open ASEP, it is necessary to impose a restriction on the parameters. With Shen, we called this the *Liggett condition* since it arose in his much earlier work [39, 40]. The condition is that

$$\frac{\alpha}{1} + \frac{\gamma}{q} = 1 = \frac{\beta}{1} + \frac{\delta}{q}. \tag{14}$$

Under this condition the effective reservoir densities simplify so that $\rho_\ell = \alpha$ and $\rho_r = 1 - \beta$. Moreover, this condition implies that the role of the reservoirs can be replaced by a simpler boundary interaction—if there is no particle at site 1, then with rate 1 the system attempts to pull a particle out of the reservoir to fill site 1, and that attempt is successful with probability α; if there is a particle at site 1, then with rate q the system attempts to push that particle out into the reservoir, and that attempt is successful with probability $1 - \alpha$. There is a similar dynamic with the reservoir near N. Thus, under Liggett's condition, the left reservoir can be thought of as a standard ASEP site which is occupied or vacant with probability α or $1-\alpha$ at any given moment of time, independently of all other times. This implication of the condition was explained initially in [39].

Under Liggett's condition, $Z(t,x)$ from (13) satisfies the following discrete stochastic differential equation version of the SHE:

$$dZ_t(x) = \tfrac{1}{2}\Delta Z_t(x) + dM_t(x) \tag{15}$$

for all $x \in \{0,\ldots,N\}$ subject to the boundary conditions that for all $t \geq 0$

$$Z_t(-1) = \mu_\ell Z_t(0) \qquad Z_t(N+1) = \mu_r Z_t(N)$$

where μ_ℓ and μ_r take values in $[q^{1/2}, q^{-1/2}]$ and are given by

$$\mu_\ell = q^{-1/2} - \alpha(q^{-1/2} - q^{1/2}) \qquad \text{and} \qquad \mu_r = q^{-1/2} - \beta(q^{-1/2} - q^{1/2}).$$

The term $M_t(x)$ represents a martingale with an explicit bracket process that I will not record here. The meaning of the boundary condition (which is an inhomogeneous discrete Robin boundary condition) is that when considering $\Delta Z_t(x)$ for $x = 0$ or $x = N$, the term which involves $x = -1$ or $x = N+1$ is replaced by use of the boundary condition above.

The convergence result that Shen and I proved, and that will be described in the next section, uses this transformation as the starting point and thus requires that the parameters satisfy Liggett's condition. This reduces the number of boundary parameters from four to two, though these remaining two parameters can be thought of as the reservoir potentials ρ_ℓ and ρ_r and by tuning them we are able to access the full two-parameter family of open SHE/KPZ equation boundary parameters. It would be nice to see a proof of the open ASEP to KPZ scaling limit which does not require Liggett's condition (note that this condition seems to be necessary in Yang's work [54] as well).

6.4 Convergence to Open KPZ

In order to prove convergence of the discrete SHE (15) to the continuum SHE, it is necessary to introduce two more scalings of the remaining parameters (which we can take to be q, ρ_ℓ and ρ_r). The first is known as weak asymmetry scaling. In the full-line setting of Bertini and Giacomon, they introduce a scaling parameter ϵ and take q scaled close to 1 on the order of $\epsilon^{1/2}$, space scale down by ϵ and time scaled down by ϵ^2. It takes a bit of work to see why this is a natural choice of scaling and, in my opinion, is best seen by studying scalings under which the KPZ equation remains fixed (e.g., see [15]).

In the context of the open ASEP, space is always fixed to involve N sites. This suggests to think of N and ϵ^{-1} as being the same (or at least proportional). Informed by this and earlier scaling of Bertini and Giacomin, I will assume below that

$$q = \exp\left(-\frac{2}{\sqrt{N}}\right)$$

and call this weak asymmetry scaling. Notice that $q \approx 1 - 2N^{-1/2}$, which matches with the $\epsilon^{1/2}$ notion of weak asymmetry coming from Bertini and Giacomin. Likewise, further define

$$h^{(N)}(t,x) := N^{-1/2} h_N(\tfrac{1}{2}e^{N^{-1/2}} N^2 t, Nx) + (\tfrac{1}{2}N^{-1} + \tfrac{1}{24})t, \quad z^{(N)}(t,x) := e^{h^{(N)}(t,x)}.$$

The scaling of space like Nx means that in $h^{(N)}(t,x)$ and $z^{(N)}(t,x)$, x can vary over $[0,1]$. The scaling of time like $\frac{1}{2}e^{N^{-1/2}}N^2 t$ captures the N^2 scaling from Bertini and Giacomin, as well as some additional corrections that simplify coefficients elsewhere. The rest of the definition is fixed by the microscopic Hopf-Cole transform—namely the fact that $z^{(N)}(t,x)$ should satisfy a scaled version of the discrete SHE. Indeed, the factor $(\frac{1}{2}N^{-1} + \frac{1}{24})t$ comes from the expansion of $\nu\frac{1}{2}e^{N^{-1/2}}N^2 t$ down to order 1 terms in N.

So far, none of this scaling has involved the boundary parameters or conditions. Since $h^{(N)}(t,x)$ involves diffusive scaling between space and the scale of the height function it is natural to imagine that the density of particles in open ASEP should be close to $1/2$ in order for this scaling to make sense. In fact, it should be within order $N^{-1/2}$ of density $1/2$ to respect this scaling. This implies that ρ_ℓ and ρ_r should be scaled in this manner around density $1/2$, i.e., in an $N^{-1/2}$ window around the triple point of the phase diagram. Specifically, assume now that for some $u, v \in \mathbb{R}$,

$$\rho_\ell = \frac{1}{2} + \frac{u}{2}N^{-1/2} + o(N^{-1/2}), \qquad \rho_r = \frac{1}{2} - \frac{v}{2}N^{-1/2} + o(N^{-1/2}).$$

The above assumptions have all been on parameters, but it is also necessary to assume something about the initial data. Since N is varying, for each N there will be a different choice of initial data—denote the height function for that initial data by $h_N(\cdot)$. This is an abuse of notation from earlier where I let this denote the stationary height function. For the moment, I will just take this to be any choice of initial data. Introduce its scaled version $h^{(N)}(x) := N^{-1/2}h_N(Nx)$ and exponential $Z^{(N)}(x) := e^{h^{(N)}(x)}$. These definitions match with the $t = 0$ height function scaling introduced above.

The following *Hölder bounds* assumption will be sufficient to state the KPZ convergence result: For all $\theta \in (0, 1/2)$ and every $n \in \mathbb{Z}_{\geq 1}$ there exists $C(n), C(\theta, n) > 0$ such that for every $x, x' \in [0,1]$ and $N \in \mathbb{Z}_{\geq 1}$

$$||Z^{(N)}(x)||_n \leq C(n), \qquad \text{and} \qquad ||Z^{(N)}(x) - Z^{(N)}(x')||_n \leq C(\theta, n)|x - x'|^\theta$$

where $||\cdot||_n := \mathbb{E}[|\cdot|^n]^{1/n}$ and $\mathbb{E}$ is the expectation over the initial data.

Notice that assuming initial data satisfying these Hölder bounds, the initial data $Z^{(N)}(\cdot)$ will form a tight sequence of random functions. With some work (as done in [18]) it can be shown that the discrete SHE preserves this class of initial data. This is the first step to proving convergence to the continuum SHE/KPZ equation since it implies tightness of the entire process. The second step is to show that all subsequential limits satisfy the desired SHE/KPZ equation.

I will informally state the main convergence result from [18] and [45]. This was proved in [18] under the assumption that $u, v \geq 1/2$ and extended by Parekh [45] to general $u, v \in \mathbb{R}$ case. The reason why the general case was harder is that the boundary condition for $u, v \geq 1/2$ can be thought of as repulsive or killing in terms of the Feynman-Kac representation, and thus the heat kernel for the associated Laplacian tends to decay to zero, simplifying various estimates. In the general case, the heat kernel may grow exponentially, thus complicating matters.

The combination of these two works showed that, provided a sequence of N-indexed open ASEP processes with parameters satisfying Liggett's condition, weak asymmetry and triple point scaling, and initial data satisfying Hölder bounds, then the following holds. For any fixed time horizon $T > 0$, the law of $\{Z^{(N)}(\cdot, \cdot)\}_N$ is tight in the Skorokhod space $D([0, T], C([0, 1]))$ of time-space processes that are CADLAG in time and continuous in space. Moreover, any limit point is in $C([0, T], C([0, 1]))$, i.e., continuous in both time and space. If there exists a (possibly random) non-negative-valued function $z_0 \in C([0, 1])$ such that $Z^{(N)}(\cdot)$ converges to $z_0(\cdot)$ along a subsequence as N goes to infinity in the space of continuous processes on $[0, 1]$, then along that same subsequence $Z^{(N)}(\cdot, \cdot)$ converges to $z(\cdot, \cdot)$ in $D([0, T], C([0, 1]))$ where $z(\cdot, \cdot)$ is the unique (mild) solution to the SHE with boundary parameters u and v and initial data $z_0(\cdot)$, see Sect. 3.2. In other words, this implies that the corresponding height function $h^{(N)}(\cdot, \cdot)$ converges to $h(\cdot, \cdot)$, the Hopf-Cole solution to the KPZ equation.

Before ending this section, I want to just remark on how the above scaling limit result relates to the mixing time conjecture for open ASEP and to the uniqueness conjecture for the stationary measure of open KPZ. Besides controlling the current and stationary measure density, the phase diagram for the open ASEP is supposed to control the mixing time, i.e., the time that it takes for a general initial state to converge (e.g. in total variation distance) close to the stationary measure. Recently, Gantert et al. [30] have made progress on characterizing the mixing time. In the high and low density states, and assuming a strict asymmetry (i.e. $q < 1$) they show that the mixing time grows linearly in the system size N. In the case of the triple point between all three phases, they are able to give an upper bound of order N^3. However, the expectation is that the true mixing time at this point and in the maximal current phase is of order $N^{3/2}$ and Schmid [49] has since proved this in the case of TASEP ($q = 0$).

How does this ASEP mixing time behavior relate to the scaling limit of open ASEP to open KPZ? The mixing time of $N^{3/2}$ should have a prefactor that depends on the strength of the asymmetry. In particular, I expect it to behave like $(1 - q)^{-1}N^{3/2}$. Under our weak asymmetry, this behaves like N^2 which is exactly the time scaling in order to arrive at the open KPZ equation. Thus, if someone can prove mixing for open ASEP in a manner sufficiently uniform over q, this could yield a route to study the mixing of the open KPZ equation as well (and hence imply the uniqueness of the stationary measure). Of course, this is probably not the easiest (or most direct) route to prove the uniqueness result about open KPZ.

7 Taking Asymptotics of the Open ASEP Stationary Measure

The convergence result described above shows that under special scaling and assuming Hölder bounds on the open ASEP initial data, there is tightness of the scaled open ASEP height function and all subsequential limits of the initial data yield subsequential limits of the process which solve the open KPZ equation (in fact, things are phrased in terms of the SHE). Since my goal (i.e., the results claimed in Sect. 4) is to construct a stationary measure for the open KPZ equation, I will consider now what happens when the convergence result is applied to the open ASEP stationary measure.

There are two types of results in Sect. 4—those that deal with the existence and general properties of the stationary measure and those that address the exact formulas in the case where $u + v > 0$. Let me start by addressing the first type of result.

There are two inputs that we appeal to about the open ASEP stationary measure. The first is that when $\rho_\ell = \rho_r = \rho$, the corresponding stationary measure is Bernoulli with density parameter ρ. Liggett [39] provided a proof of this (under the Liggett condition) and a more general result from the matrix product ansatz [25], as explained further in Sect. 8.2. It is straight-forward to see that in this case the open ASEP stationary measure height function satisfies the necessary Hölder bounds since it is just a simple random walk trajectory.

The second input is a microscopic version of the stochastic sandwiching result that I described earlier for the open KPZ stationary measure. Before stating it, let me motivate it. Imagine you have two versions of open ASEP, one with boundary parameters $\alpha, \beta, \gamma, \delta$ and the other with $\alpha', \beta', \gamma', \delta'$. If $\alpha \le \alpha'$, $\beta \ge \beta'$, $\gamma \ge \gamma'$, and $\delta \le \delta'$, then in the primed system there is an increased rate at which particles enter and a decreased rate at which they exit the system. It would reason then that the stationary measure for the primed system should have more particles than in the original system. This is true, as is the stronger statement that π_N is stochastically dominated (see the definition below) by π'_N.

Consider two measures π and π' on $\{0, 1\}^{\{1,\dots,N\}}$. The measure π is said to be stochastically dominated by π' (written $\pi \preceq \pi'$) if there exists a coupling of π and π' on which all sites occupied under π are likewise occupied under π'. More explicitly, this means that there exists a probability measure μ on $\{0, 1\}^{\{1,\dots,N\}} \times \{0, 1\}^{\{1,\dots,N\}}$ such that if let (τ, τ') be sampled according to μ (here τ and τ' take values in $\{0, 1\}^{\{1,\dots,N\}}$) then marginally τ has law π, τ' has law π' and almost surely $\tau \le \tau'$ in the sense that $\tau_i \le \tau'_i$ for all $i \in \{1, \dots, N\}$.

This result can be generalized to consider three sets of boundary parameters, and it is this consideration that leads to stochastic sandwiching for the open ASEP. After all, the height function increments are just sums of occupation variables, and thus stochastic domination of the individual occupation variables certainly implies that of their sums.

I will briefly explain how to demonstrate the above stochastic domination of the stationary measures. However, before that let me note how it implies the general u, v Hölder bounds. When $u + v = 0$ it is possible to choose $\rho_\ell = \rho_r$ and thus deal with product Bernoulli measure. When $u + v > 0$ or $u + v < 0$, the parameters $\alpha, \beta, \gamma, \delta$ can be adjusted in the spirit explained above to show that the corresponding stationary measure is stochastically sandwiched between two different product Bernoulli measures. The density parameter for the upper and lower bounding Bernoulli measures only differ by order $N^{-1/2}$ which is compatible with the diffusive scaling that is applied to the height function. Consequently, the Hölder bounds follow by applying the sandwiching in concert with the analogous bounds for simple random walks that are converging to drifted Brownian motions.

As far as proving the stochastic domination, the main idea is to use second class particles. This is the same mechanism that is used to show what is termed attractiveness of the usual ASEP. Consider starting both the original and the primed version of open ASEP in state $\tau(0)$ and $\tau'(0)$ in such a way that $\tau(0) \leq \tau'(0)$ (e.g. they could both start out entirely empty). Let $\alpha(0) = \tau'(0) - \tau(0)$ denote the occupation variables for what are called *second class* particles. For ASEP on the full line, the basic coupling provides a dynamic on the pair (τ, α) such that at any later time $\tau'(t)$ and $\tau(t) + \alpha(t)$ have the same distribution.

In order to demonstrate such a coupling on the interval, the basic coupling needs to be augmented at the boundary. This coupling at the boundary is best explained with the help of Fig. 6. The red particles are those of τ and the blue are those of α. The arrows and labels represent the transitions and rates associated with the boundary coupling dynamics. For instance, if there is neither a τ or α particle at site 1, then at rate α a τ-particle can enter, and at rate $\alpha' - \alpha$ an α-particle can enter. Thus, if I am only keeping track of the τ-particles, the transition from empty to

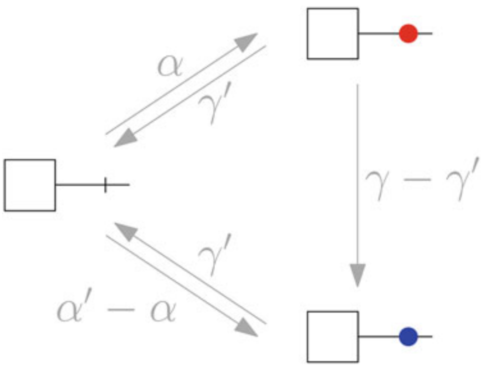

Fig. 6 Coupling τ and τ' at the boundary. Particles in τ are red dots, and those in τ' are blue dots. The transition rates and transitions are labeled in grey.

occupied occurs at rate α and if I am tracking the $\tau + \alpha$ particles (i.e. the occurrence of their, which should match the τ' dynamic), this occurs at rate $\alpha + (\alpha' - \alpha) = \alpha'$ as desired. Similar considerations imply that these dynamics project onto the τ and τ' dynamics marginally, and hence shows that this constitutes the desired coupling. This type of coupling is also used in the open ASEP mixing time work of Gantert, Nestoridi and Schmid [30]. In fact, in the case when Liggett's condition is assumed, this monotonicity seems to first have been shown in Liggett's 1975 paper [39] as Corollary 3.8, based on a calculation involving the open ASEP generator.

Given the two inputs I have discussed above along with the convergence result from Sect. 6.4, let me complete the construction of the open KPZ stationary measure. As indicated above, the stochastic sandwiching and Bernoulli cases provide a way to demonstrate the Hölder bounds for $h^{(N)}(x) := N^{-1/2} h_N(Nx)$ (where h_N is the open ASEP stationary measure in question under the Liggett condition, weak asymmetry and triple-point scaling). The calculation for the Bernoulli case is fairly simple, and is transferred to the general $u + v \neq 0$ case through the sandwiching and some simply inequalities.

Once the Hölder bounds are in place, the argument proceeds by observing that these bounds imply tightness of the initial data, and hence (through the results discussed in Sect. 6.4) also tightness of the process $h^{(N)}(\cdot, \cdot)$. Any subsequential limit will solve the open KPZ equation and any subsequential limit of the initial data $h^{(N)}(\cdot)$ will be a stationary measure (since the same held true before taking the limit). Of course, if someone proves that there is only one open KPZ stationary measure (for each pair of $u, v \in \mathbb{R}$) then this would imply that $h^{(N)}(\cdot)$ converges to it.

The above argument shows everything claimed in Sect. 4 except for the Laplace transform formula that characterizes the $u + v > 0$ stationary measure. The next section explains the origin and derivation of such formulas.

8 Matrix Product Ansatz and Askey-Wilson Processes

It is at this point in the talk version of this note that my time usually is close to cutting off (of course, much of what has been said above is severely curtailed in the hour long talk as well). So, I would like to use this space to give a sketch of the progression of ideas that form the starting point to derive the Laplace transform formula for the open KPZ stationary measure.

The first major breakthrough in studying the stationary measure for open ASEP was in Liggett's 1975 work [39] where he discovered that the stationary measure satisfies a recursion relation with respect to the system size N. Namely, the size N stationary measure is expressible in terms of the size $N - 1$ case (see [39, Theorem 3.2]). In that same work, Liggett proved (based on a fairly simple generator calculation) that assuming the Liggett condition (14), if $\rho_\ell = \rho_r$ then the stationary measure is Bernoulli with that density parameter. He also proved certain useful monotonicity results about the stationary measure, including with respect to changes in N and in the boundary parameters. Liggett's main motivation in that work was to use open ASEP to approximate half-line and then full-line ASEP and study the convergence of different choices of initial data in that final setting to Bernoulli product measures.

A decade and a half later, there was significant renewed interest in the open ASEP stationary measure, this time from the perspective of statistical physics. The phase diagram and computation of thermodynamic quantities became centrally important. Though initially Liggett's recursion was used to make such calculations (see, e.g. [24]), an alternative algebraic approach soon presented itself as quite useful in extracting asymptotics.

8.1 Deriving the Matrix Product Ansatz

Derrida et al. introduced the *matrix product ansatz* (MPA) in their seminar 1993 work [25]. They primarily focused on the case of open TASEP ($q = 0$) for which they could find useful representations of the matrices. However, they also explained how to formulate the ansatz for the general open ASEP case. The MPA has found many uses since then in the study of boundary driven integrable spin chains and particle systems, and I will not try to survey this literature. Instead, let me briefly introduce the MPA for open ASEP and describe how it is derived. After that I will explain how to go from there to our open KPZ result.

As earlier in the text, let $\tau = (\tau_1, \ldots, \tau_N) \in \{0, 1\}^N$ denote the state-space for open ASEP, and $\tau(t)$ denote the occupation process at time t. Then, for any $\eta, \tau \in \{0, 1\}^N$, and $P_\eta(t, \tau) := \mathbb{P}(\tau(t) = \tau | \tau(0) = \eta)$, the Kolmogorov backward equation (or master equation) says that $\partial_t P_\eta(t, \tau) = L^* P_\eta(t, \tau)$ where L^* acts on the τ variable and represents the backwards generator of open ASEP. In particular, the stationary measure must have a zero time derivative and hence, recalling the notation $\pi_N(\tau)$ introduced earlier for the stationary measure, it must satisfy the defining relation $L^*\pi_N(\tau) = 0$. Since there is no particle conservation and all states communicate in open ASEP, the Perron-Frobenius theorem implies that (up to constant scaling) there is a unique eigenfunction for L^* with zero eigenvalue. This means that for any function $f_N(\tau)$ that is not identically zero and which satisfies $L^* f_N(\tau) = 0$, $\pi_N(\tau)$ must equal $f_N(\tau)/Z_N$ where Z_N is the sum of $f_N(\tau)$ over all $\tau \in \{0, 1\}^N$.

Let me now write down explicitly the relation $L^* f_N(\tau) = 0$, separating things out in terms of particle movement between the left reservoir and site 1, sites i and $i + 1$ in the bulk for $i = 1, \ldots N - 1$, and site N and the right reservoir:

$$L^* f_N(\tau) := L^*_\ell f_N(\tau) + \sum_{i=1}^{N-1} L^*_{i,i+1} f_N(\tau) + L^*_r f_N(\tau) = 0 \tag{16}$$

where

$$L_\ell^* f_N(\tau) := \sum_{\sigma_1} (h_\ell)_{\tau_1;\sigma_1} f_N(\sigma_1, \tau_2, \ldots \tau_N),$$

$$L_r^* f_N(\tau) := \sum_{\sigma_N} (h_r)_{\tau_N;\sigma_N} f_N(\tau_1, \ldots, \tau_{N-1}, \sigma_N),$$

$$L_{i,i+1}^* f_N(\tau) := \sum_{\sigma_i,\sigma_{i+1}} (h)_{\tau_i,\tau_{i+1};\sigma_i,\sigma_{i+1}} f_N(\tau_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \tau_N),$$

for each $i \in \{1, \ldots, N-1\}$. In the above expressions, the sums over the σ's are take over values in $\{0, 1\}$, h_ℓ and h_r are 2×2 matrices and h is a 4×4 matrix. These matrices are given as

$$h_\ell = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h_r = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix} \tag{17}$$

where for h_ℓ (and similarly h_r) the term $(h_\ell)_{i;j}$ corresponds to row $i+1$ and column $j+1$ with $i, j, \in \{0, 1\}$ and for h, the term $(h)_{i,i';j,j'}$ corresponds to row $i' + 2i + 1$ and column $j' + 2j + 1$ with $i, i', j, j' \in \{0, 1\}$.

In order for (16) to hold, it would be sufficient (though certainly not necessary) that there exist two constants x_0 and x_1 such that the following relations hold for all choices of $\tau_1, \ldots, \tau_N$:

$$L_\ell^* f_N(\tau) = x_{\tau_1} f_{N-1}(\hat{\tau}_1), \tag{18}$$

$$L_{i,i+1}^* f_N(\tau) = -x_{\tau_i} f_{N-1}(\hat{\tau}_i) + x_{\tau_{i+1}} f_{N-1}(\hat{\tau}_{i+1}), \tag{19}$$

$$L_r^* f_N(\tau) = -x_{\tau_N} f_{N-1}(\hat{\tau}_N), \tag{20}$$

where $\hat{\tau}_i$ denotes the vector τ with the ith coordinate (i.e., τ_i) removed (e.g. $\hat{\tau}_1 = (\tau_2, \ldots, \tau_N)$). Clearly, if these relations hold, then summing the left-hand sides yields the left-hand side of (16) while summing the right-hand side yields 0 by telescoping.

Of course, the question is whether f_N actually satisfies this relation and if so, whether there exists a manageable representation for the solution to this recursion. Inspired by earlier work in integrable systems, Derrida et al. [25] proposed an ansatz for the form that f_N could take. Consider the class of functions $\tilde{f}_N(\tau)$ that can be written in the form

$$\tilde{f}_N(\tau_1, \ldots, \tau_N) = \langle W|(\tau_1 D + (1-\tau_1)E) \cdots (\tau_N D + (1-\tau_N)E)|V\rangle$$

where D and E are matrices (possibly infinite dimensional), $\langle W|$ is a row vector and $|V\rangle$ is a column vector. Here the dimension for all of these matrices and vectors are assumed to match and the multiplication is well-defined (i.e. everything is convergent if things are infinite dimensional). For instance, if everything is one-dimensional, then the class of measures that can be defined by $\tilde{f}_N$ above (after normalizing) is exactly that of product Bernoulli measure, all with the same parameter.

Assuming the general form of $\tilde{f}_N$ above, it is possible to deduce conditions on the matrices and vectors that are necessary in order that $\tilde{f}_N$ satisfies (18)–(20). Consider (18). In terms of the matrix product, this asks that

$$\langle W|(-\alpha E+\gamma D)\prod_{i=2}^{N}(\tau_i D+(1-\tau_i)E)|V\rangle = x_0\langle W|\prod_{i=2}^{N}(\tau_i D+(1-\tau_i)E)|V\rangle, \tag{21}$$

$$\langle W|(\alpha E-\gamma D)\prod_{i=2}^{N}(\tau_i D+(1-\tau_i)E)|V\rangle = x_1\langle W|\prod_{i=2}^{N}(\tau_i D+(1-\tau_i)E)|V\rangle \tag{22}$$

where the product should be understood as ordered from left-to-right in terms of increasing index. Clearly this implies that $x_0 = -x_1$ and since everything scales homogeneously, it is fine to take $x_1 = 1$. In order for the above relations to hold, it is sufficient then, that

$$\langle W|(\alpha E-\gamma D) = \langle W|.$$

Similar reasoning show that in order for $\tilde{f}_N$ to satisfy (20) it is sufficient that

$$(\beta D-\delta E)|V\rangle = |V\rangle.$$

Likewise, the relation in (19) will be satisfied by $\tilde{f}_N$ as long as

$$DE - qED = D + E.$$

Notice that the quadratic term on the right-hand side above arises since the bulk relation (19) involves summing over two particles.

To summarize, provided non-trivial matrices and vectors (potentially infinite dimensional, though necessarily such that products are well-defined) that satisfy the three relations

$$DE - qED = D + E, \qquad \langle W|(\alpha E-\gamma D) = \langle W|, \qquad (\beta D-\delta E)|V\rangle = |V\rangle \tag{23}$$

then necessarily the corresponding $\tilde{f}_N(\tau)$ will satisfy the defining conditions asked for $f_N(\tau)$, and hence yield the stationary measure (after normalizing). This quadratic algebra is sometimes called the DEHP algebra. Of course, there is a priori no reason to expect that there exist matrix representations to the DEHP algebra.

8.2 Representations for the DEHP Algebra

As an immediate application, it is now possible to use the MPA to give a sufficient condition under which the stationary measure for open ASEP is homogeneous product Bernoulli. If the matrices and vectors are all one-dimensional (which implies the Bernoulli product measure) then the DEPH algebra relations reduce to

$$de(1-q) = d + e, \qquad \alpha e - \gamma d = 1, \qquad \beta d - \delta e = 1$$

for scalars d, e. In order for there to exist such d and e is suffices that the open ASEP parameters satisfies

$$(1-q)(\alpha+\delta)(\beta+\gamma) = (\alpha+\beta+\gamma+\delta)(\alpha\beta-\gamma\delta).$$

If we further assume that Liggett's condition holds (thus expressing γ in terms of α, and δ in terms of β) then the above equation is satisfied when $\alpha+\beta=1$, which was precisely the condition mentioned in Sect. 7.

Putting aside the one-dimensional case, Derrida et al. [25] showed that any other matrix representation to (23) must be infinite dimensional. Finding such representations is a highly non-trivial task. When $q = 0$, Derrida et al. [25] provided a few such representations. Different matrix representations have proven useful in making various types of calculations involving the large N limit of the stationary measure, for example including computing the stationary current, and other correlation functions.

For the general $q \neq 0$ and general $\alpha, \beta, \gamma, \delta$ parameter case, it took about a decade until Uchiyama et al. [51] provided the first representations to the DEHP algebra. Their infinite dimensional matrix representations (that I will call the USW representation) were written in terms of the Askey-Wilson orthogonal polynomials (in particular, the associated Jacobi matrix). This remarkable link between orthogonal polynomials and the open ASEP stationary measure was already present in a simpler case (when $\gamma = \delta = 0$) in earlier work of just Sasamoto [48] in which the Al-Salam-Chihara polynomials replaced the Askey-Wilson polynomials. Another notable development in this area came in work of Corteel and Williams [21] who, using a different matrix representation, found a combinatorial description for the open ASEP stationary measure in terms of certain types of tableaux combinatorics. A nice exposition on this direction and further developments around it can be found in Williams' recent expository piece [53].

Since my aim here is explain how the work of [51] leads to the Askey-Wilson process formulas of Bryc and Wesołowski [11], I will recall a variant of the USW matrix representation for the DEHP algebra in the notation of [11].

It will be helpful to work with the parameterizations of $\alpha, \beta, \gamma, \delta$ in terms of A, B, C, D as facilitated by the bijection in (11). The USW matrix representation is infinite dimensional and written in terms of one-sided infinite matrices and vectors (i.e., indexed by natural numbers). I will use bold face variables to distinguish this representation as well as associated matrices used in describing it. The row vector $\langle \mathbf{W}| = [1, 0, 0, \ldots]$ and column vector $|\mathbf{V}\rangle = [1, 0, 0, \ldots]^T$ are both simple. The matrices $\mathbf{E}$ and $\mathbf{D}$ are more complicated and can be written in terms of the identity matrix $\mathbf{I}$ and two other matrices $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{D} = \frac{1}{1-q}\mathbf{I} + \frac{1}{\sqrt{1-q}}\mathbf{x}, \qquad \mathbf{E} = \frac{1}{1-q}\mathbf{I} + \frac{1}{\sqrt{1-q}}\mathbf{y}. \tag{24}$$

The $\mathbf{x}$ and $\mathbf{y}$ matrices are tridiagonal and admit explicit formulas for the coefficients. Since I do not want to get lost in the (very important) details here, I will abstain from recording them precisely, but rather just focus on their key properties. First and foremost, they are such that $\mathbf{D}$ and $\mathbf{E}$, along with the simple $\langle \mathbf{W}|$ and $|\mathbf{V}\rangle$, satisfy the DEHP algebra. That they satisfy the quadratic relation in the DEHP algebra is equivalent to the q-commutation relation $\mathbf{xy} - q\mathbf{yx} = \mathbf{I}$. This relation, in conjunction with the knowledge of the first entry of the three non-trivial diagonals in both $\mathbf{x}$ and $\mathbf{y}$ uniquely determine their values. The values of these non-trivial entries can be determined from the two boundary relations in the DEHP algebra.

8.3 Askey-Wilson Polynomials and Processes

The other key property of the $\mathbf{x}$ and $\mathbf{y}$ matrices is that they appear in the Jacobi matric that describes the three step recurrence for Askey-Wilson orthogonal polynomials. Let me briefly introduce these polynomials since they will be needed in order to relate the MPA to the Askey-Wilson process (which is not defined in terms of these polynomials, but rather the orthogonality measure associated to them).

I will use the notation of [11] whereby they write the Askey-Wilson polynomials as $\bar{w}_n(x; a, b, c, d, q)$. The parameters should be assumed to satisfy $q \in (-1, 1)$, $a, b, c, d \in \mathbb{C}$ with $abcd, abcdq, ab, abq \notin [1, \infty)$. These polynomials are typically described in terms of their three term recurrence, though I will not write this down here explicitly. Under some additional conditions on parameters (that I will also suppress here) these polynomials will be orthogonal with respect to a probability distribution known as the *Askey-Wilson measure*. This measure has the form

$$\nu(dx; a, b, c, d, q) := f(x; a, b, c, d, q)\mathbf{1}_{|x|<1} + \sum_{y \in F(a,b,c,d,q)} p(y; a, b, c, d, q)\delta_y(dx). \tag{25}$$

The first part above is an absolutely continuous measure on $x \in (-1, 1)$ with density

$$f(x; a, b, c, d, q) = \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{2\pi(abcd; q)_\infty\sqrt{1-x^2}} \left|\frac{(e^{2\iota\theta}; q)_\infty}{(ae^{\iota\theta}, be^{\iota\theta}, ce^{\iota\theta}, de^{\iota\theta}; q)_\infty}\right|^2 \tag{26}$$

where we recall $(a; q)_\infty = (1-a)(1-qa)(1-q^2a)\cdots$, $(a_1, \ldots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty$, and θ is defined by the relation $\cos(\theta) = x$. The second part of (25) is an atomic measure supported at locations y in the finite set $F(a, b, c, d, q)$ with masses $p(y; a, b, c, d, q)$. The points and their masses are explicit as well, though I will not record them here. Though I will not explain it, these atoms only become important when the open KPZ parameters u, v are not both positive. The formula (2) that I gave earlier for the Laplace transform of the stationary open KPZ height increment from 0 to 1 was exactly in the case of $u, v > 0$ in which case it is only the absolutely continuous part above in (25) that is present.

In a spirit similar (albeit considerably more involved) to how one uses the Gaussian distribution (which is the orthogonality measure for Hermite polynomials) to define Brownian motion, Bryc and Wesołowski [10] used the measure ν to define a process that they called the *Askey-Wilson process*. The existence and path properties (e.g. continuity versus jumps) of this process are non-trivial. It is defined only for an interval I of time that is dependent upon the five parameters A, B, C, D, q that determine it. These parameters are also subject to certain conditions for existence of the process. Following [10] and [11], let $Y = (Y_t)_{t\in I}$ denote this process. It is a Markov process with marginal distribution for each $t \in I$ given by the measure

$$\pi_t(dx) := \nu(dx; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}, q)$$

and with transition probabilities for $s < t$ both in I of the form

$$P_{s,t}(x, dy) := \nu(dy; A\sqrt{t}, B\sqrt{t}, \sqrt{s/t}(x + \sqrt{x^2-1}), \sqrt{s/t}(x - \sqrt{x^2-1}), q).$$

It takes some work to show that this is well-defined, i.e., satisfying Chapman-Kolmogorov.

Bryc and Wesołowski's original motivation for introducing the Askey-Wilson process came from the study of *quadratic harness* which are stochastic processes X_t such that the expectation of X_t and of X_t^2, conditioned on the process outside an interval containing t, are given as linear and quadratic functions (respectively) of the values of the process X at the boundary of said interval. Brownian motion or a Poisson jump process are arguably the simplest examples of such harnesses. The question addressed in [10] was how to characterize the space of all standard quadratic harnesses (standard means that X_t has the same mean and covariance as standard Brownian motion), and it turned out that this was achieved by the Askey-Wilson processes.

The first time I heard about Askey-Wilson processes was at a conference on "Orthogonal polynomials, applications in statistics and stochastic processes" in 2010 at Warwick. This was quite a notable event for me since it was among the first times I gave a talk at a conference, especially one overseas. I have a vivid memory of sitting in the lecture theater and listening to Wesołowski give a talk entitled "Quadratic harnesses and Askey-Wilson polynomials". One of the reasons it was so vivid is that I was completely confused as to why anyone would study these seemingly very complicated processes. I suppose the moral of this brief anecdote is that when someone gives a talk about something that seems very complicated, it is often worth paying attention even if you do not immediately understand why.

A property of the Askey-Wilson polynomials and process is that when combined, they form a family of orthogonal martingales. Specifically, let

$$Z_t := \frac{2\sqrt{t}}{\sqrt{1-q}} Y_t$$

and define the (infinite) row vector-valued function $(x,t) \mapsto \langle \mathbf{r}_t(x)|$ where

$$\langle \mathbf{r}_t(x)| := [r_0(x;t), r_1(x;t), \ldots]$$

and where $r_n(x;t)$ are polynomials of degree n in the variable x and given in terms of the Askey-Wilson polynomial $\bar{w}_n$ by

$$r_n(x;t) = t^{n/2}\bar{w}_n\left(\frac{\sqrt{1-q}}{2\sqrt{2}}; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}, q\right).$$

The process $\langle \mathbf{r}_t(Z_t)|$ satisfies the following properties (proved in [10]):

1. $r_0(x;t) = 1$ for all x,
2. $\mathbb{E}[r_n(Z_t;t)r_m(Z_t;t)] = 0$ if $m \neq n$,
3. $\mathbb{E}[r_n(Z_t;t)|Z_s] = \mathbb{E}[r_n(Z_t;t)|\mathcal{F}_s] = r_n(Z_s;s)$ for $s \leq t$ and $\mathcal{F}_s = \sigma(Z_v : v \leq s)$ the sigma-algebra generated by Z_v for all $v \leq s$.

The three term recurrence relation for Askey-Wilson polynomials can be rewritten in terms of the r_n version of these polynomials in the following succinct manner:

$$x\langle \mathbf{r}_t(x)| = \langle \mathbf{r}_t(x)|(t\mathbf{x} + \mathbf{y}) \tag{27}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the matrices discussed earlier around (24). Since these are tridiagonal matrices, they imply that $xr_n(x;t)$ can be expressed as a sum of $r_{n-1}(x;t)$, $r_n(x;t)$ and $r_{n+1}(x;t)$ with coefficients that depend on t in a linear manner but not on the variable x.

8.4 Connecting Matrix Product Ansatz and Askey-Wilson Processes

With all of the notation and properties introduced above, I am now in a position to relate the beautiful proof that appears in [11] and relates the open ASEP stationary measure, via the matrix product ansatz and the Uchiyama-Sasamoto-Wadati representation, to the Askey-Wilson process.

The matrix product ansatz gives a formula for the stationary measure. However, by summing over all states it also give a compact representation for the following generating function for $\pi_N(\tau)$ (recall the notation on the left-hand side means averaging a function of τ against this measure)

$$\left\langle \prod_{j=1}^{N} t_j^{\tau_j} \right\rangle_N = \frac{\langle \mathbf{W}| \prod_{j=1}^{N}(\mathbf{E}+t_j\mathbf{D})|\mathbf{V}\rangle}{\langle \mathbf{W}| \prod_{j=1}^{N}(\mathbf{E}+\mathbf{D})|\mathbf{V}\rangle}.$$

As before, the product is ordered from left to right in increasing order of indices. Focusing on the numerator and using the USW representation (24) of the DEHP algebra yields

$$\langle \mathbf{W}| \prod_{j=1}^{N}(\mathbf{E}+t_j\mathbf{D})|\mathbf{V}\rangle = (1-q)^{-N}\langle \mathbf{W}| \prod_{j=1}^{N} \big((1+t_j)\mathbf{I}+\sqrt{1-q}(t_j\mathbf{x}+\mathbf{y})\big)|\mathbf{V}\rangle$$
$$:= (1-q)^{-N}\Pi.$$

Since $r_0(x;t) \equiv 1$ and since $\mathbb{E}[r_n(Z_t;t)] = \mathbb{E}[r_n(Z_t;t)r_0(Z_t;t)] = 0$ if $n \neq 0$, the vector $\langle \mathbf{W}|$ can be rewritten in terms of the vector $\mathbb{E}[\langle \mathbf{r}_t(Z_t)|]$ for any choice of t. By linearity of the expectation,

$$\Pi = \mathbb{E}\left[\langle \mathbf{r}_{t_1}(Z_{t_1})| \prod_{j=1}^{N} \big((1+t_j)\mathbf{I}+\sqrt{1-q}(t_j\mathbf{x}+\mathbf{y})\big)|\mathbf{V}\rangle \right].$$

The three term recurrence relation (27) can now be applied to peal off the first term in the product above. In particular, observe that for any t,

$$\langle \mathbf{r}_t(Z_t)|\big((1+t)\mathbf{I}+\sqrt{1-q}(t\mathbf{x}+\mathbf{y})\big) = (1+t+\sqrt{1-q}Z_t)\langle \mathbf{r}_t(Z_t)|.$$

The term involving $\mathbf{I}$ follows since that is just the identity matrix, and the other follows from (27). This means that Π can be rewritten as

$$\Pi = \mathbb{E}\left[(1+t+\sqrt{1-q}Z_{t_1})\langle \mathbf{r}_{t_1}(Z_{t_1})| \prod_{j=2}^{N} \big((1+t_j)\mathbf{I}+\sqrt{1-q}(t_j\mathbf{x}+\mathbf{y})\big)|\mathbf{V}\rangle \right].$$

Provided that $t_2 \geq t_1$, the martingale property for each $r_n(Z_t; t)$ implies that

$$\langle \mathbf{r}_{t_1}(Z_{t_1})| \prod_{j=2}^{N} \big((1+t_j)\mathbf{I} + \sqrt{1-q}(t_j\mathbf{x}+\mathbf{y})\big)|\mathbf{V}\rangle$$
$$= \mathbb{E}\left[\langle \mathbf{r}_{t_2}(Z_{t_2})| \prod_{j=2}^{N} \big((1+t_j)\mathbf{I} + \sqrt{1-q}(t_j\mathbf{x}+\mathbf{y})\big)|\mathbf{V}\rangle|\mathscr{F}_{t_1} \right].$$

Plugging this into the above expression for Π and using the tower property for martingales to remove the conditional expectation yields

$$\Pi = \mathbb{E}\left[(1+t+\sqrt{1-q}Z_{t_1})\langle \mathbf{r}_{t_2}(Z_{t_2})| \prod_{j=2}^{N} \big((1+t_j)\mathbf{I} + \sqrt{1-q}(t_j\mathbf{x}+\mathbf{y})\big)|\mathbf{V}\rangle| \right].$$

This procedure can be repeated providing $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_N$, yielding

$$\Pi = \mathbb{E}\left[\prod_{j=1}^{N} (1+t_j+\sqrt{1-q}Z_{t_j})\langle \mathbf{r}_{t_N}(Z_{t_N})|\mathbf{V}\rangle \right].$$

However, since $|\mathbf{V}\rangle = [1,0,\ldots]^T$ and $\langle \mathbf{r}_{t_N}(Z_{t_N})|\mathbf{V}\rangle = r_0(Z_{t_N}; t_N) = 1$. Thus,

$$\Pi = \mathbb{E}\left[\prod_{j=1}^{N} (1+t_j+\sqrt{1-q}Z_{t_j}) \right].$$

In the above calculation it was important that the Askey-Wilson process be defined for the values of t_j used, and for the parameters A, B, C and D that encode the boundary parameters for open ASEP. Without going into the details, provided that $AC < 1$, the Askey-Wilson process is well-defined for $t \in (0,\infty)$ and thus the above argument proves the following result [10, Theorem 1]: For $0 < t_1 \leq \cdots \leq t_N$,

$$\left\langle \prod_{j=1}^{N} t_j^{\tau_j} \right\rangle_N = \frac{\mathbb{E}\left[\prod_{j=1}^{N}(1+t_j+\sqrt{1-q}Z_{t_j})\right]}{\mathbb{E}\left[(2+\sqrt{1-q}Z_1)^N\right]} \tag{28}$$

where the $\langle\cdot\rangle_N$ on the left-hand side is the stationary state expectation for open ASEP and the $\mathbb{E}[\cdot]$ on the right-hand side is the expectation with respect to the Askey-Wilson process (with the two processes related in terms of their parameters A, B, C, D and q). The condition $AC < 1$ corresponds exactly to the fan region discussed in Sect. 6.2 and ultimately is the reason why the open KPZ stationary measure Laplace transform is only computed for $u+v > 0$ (the limiting fan region).

8.5 Coming Full Circle to Asymptotics

To close, I would like to briefly explain why (28) is quite useful for asymptotics. Recall from earlier the formula (1) for the Laplace transform for the total change in height for the open KPZ equation across the interval [0, 1]. The microscopic analog for this is $h_N(N) = 2(\tau_1 + \cdots + \tau_N) - N$.

If $t_1 = \cdots = t_N = e^{2s}$ for $s \in \mathbb{R}$ then (28) implies the Laplace transform formula

$$\langle e^{sh_N(N)} \rangle = e^{-sN} \frac{\mathbb{E}\left[(1 + e^{2s} + \sqrt{1-q}Z_{e^{2s}})^N\right]}{\mathbb{E}\left[(2 + \sqrt{1-q}Z_1)^N\right]}. \tag{29}$$

The right-hand side numerator (and similarly denominator) can be written as an integral against the law of $Z_{e^{2s}}$, i.e., against the (scaled) Askey-Wilson orthogonality measure. As N grows, the complexity of the right-hand side does not. In fact, as is often the case in asymptotic analysis, owing to the power of N, the right-hand side will actually simplify in the $N \to \infty$ limit, even when the special scalings from Sect. 6 are applied. This should be compared to the original form of the matrix product ansatz which requires multiplication of N matrices, the complexity of which grows considerably, even if they are tridiagonal.

I should be clear that the asymptotics of a formula like (29) is not simple. However, it can be done. In [11], they apply their Askey-Wilson process formula (28) to prove a large deviation principle for the stationary measure of open ASEP with fixed parameters as N goes to infinity. Soon after, Bryc and Wang [9] used (28) to study the fluctuation scaling limit for the stationary measure, again for all parameters fixed and $N \to \infty$.

My results with Knizel, in particular, the open KPZ Laplace transform formula discussed in Sect. 4 also proceed through (28). In our case, all of the parameters are being scaled in an $N^{-1/2}$ window around their limiting values: q approaching 1, $\alpha, \beta, \gamma, \delta$ approaching $1/2$ and (in order to accommodate the scaling of $h_N(N)$), s approaching 0. Inputting these scalings into a formula like (29), it eventually becomes clear that the main contribution to the integral against the Askey-Wilson process marginal distribution comes when Z is within order N^{-1} of 1. Zooming into this scale eventually (in the $N \to \infty$ limit) leads to formulas involving a *tangent process* to the Askey-Wilson process—the continuous dual Hahn process mentioned earlier in Sect. 4. In terms of (29), all of this scaling also introduces a diverging Jacobian factor which is compensated by the decay of the integrand. The point-wise limit of the integrand and measure exists based on convergence of q-gamma functions to gamma functions (recall that q is tending to 1 and there are lots of q-Pochhammer symbols). However, in order to conclude that the integral itself converges (and hence deduce formulas like (1)) requires an application of the dominated convergence theorem. This, in turn, relies on uniform control over the behavior of q-gamma functions with its variable varying in vertical strips in the complex plane of height of order N (though Z has bounded support, after scaling by order N^{-1} around $Z = 1$, the support becomes of order N). So, to bring things full circle, it is exactly in proving this type of dominated convergence bounds that the results of Sect. 2 become necessary.

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Probability of Two Large Gaps in the Bulk and at the Edge of the Spectrum of Random Matrices

B. Fahs, I. Krasovsky, and T.-H. Maroudas

In memory of Harold Widom, 1932–2021

Abstract We present the probability of two large gaps (intervals without eigenvalues) in the bulk and also in the edge scaling limit of the Gaussian Unitary Ensemble of random matrices.

Keywords Random matrices · Fredholm determinants · Gap probability

Let $K^{\sin}$, K^{Ai} be the (trace class) operators on $L^2(sA)$, where $sA = \{sx : x \in A\}$ for $s > 0$ and $A \subset \mathbb{R}$ a finite union of intervals, with kernels

$$K^{\sin}(x, y) = \frac{\sin(x - y)}{\pi(x - y)},$$

$$K^{\mathrm{Ai}}(z, z') = \frac{\mathrm{Ai}(z)\mathrm{Ai}'(z') - \mathrm{Ai}'(z)\mathrm{Ai}(z')}{z - z'} = \int_0^\infty \mathrm{Ai}(z + \zeta)\mathrm{Ai}(z' + \zeta)d\zeta,$$

respectively.

Consider the Fredholm determinants

$$P^{\sin}(sA) = \det(I - K^{\sin})_{sA}, \qquad P^{\mathrm{Ai}}(sA) = \det(I - K^{\mathrm{Ai}})_{sA}. \tag{1}$$

The determinants (1), called the sine- and Airy-kernel determinants, respectively, are the probability of gaps (intervals without eigenvalues) sA in the bulk and in the

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edge scaling limit, respectively, of the Gaussian Unitary Ensemble (GUE) of random matrices. These determinants may be described in terms of solutions to integrable systems of partial differential equations (see [27, 31, 32], and [33] for an overview). A fascinating problem is to find their asymptotic expansion for large s.

First consider the case when A is a single interval. For the sine-kernel determinant, it is sufficient, because of the translation invariance and rescaling, to consider $A = (-1, 1)$, and for the Airy-kernel determinant, the most interesting case is $A = (-1, +\infty)$. In both models the asymptotics of the determinant are by now well established.

In the case of the sine-kernel determinant, asymptotics of the logarithm of $P^{\sin}$ in (1) have the form:

$$\log P^{\sin}(-s, s) = -\frac{s^2}{2} - \frac{1}{4}\log s + c_0 + O(s^{-1}), \qquad s \to \infty, \tag{2}$$

where

$$c_0 = \frac{1}{12}\log 2 + 3\zeta'(-1). \tag{3}$$

Here $\zeta'(z)$ is the derivative of Riemann's zeta function.

The leading term $-\frac{s^2}{2}$ was found by Dyson in 1962 in [19]. Dyson used Coulomb gas arguments. The terms $-\frac{s^2}{2} - \frac{1}{4}\log s$ were computed by des Cloizeaux and Mehta [18] in 1973 who used the fact that the eigenfunctions of K_s are spheroidal functions. The constant c_0 appearing in (3), known as the Widom-Dyson constant, was identified by Dyson [20] in 1976 who used the inverse scattering techniques and the earlier work of Widom [35]. In that work, Widom computed asymptotics of Toeplitz determinants with symbols supported on an arc of the unit circle. The arguments in [19], [18], and [20] are not fully rigorous. The first proof of the main term, i.e. the fact that $\log P^{\sin}(-s, s) = -\frac{s^2}{2}(1 + o(1))$, was given by Widom [36] in 1994. The full asymptotic expansion (2), apart from the expression (3) for c_0, was proved by Deift Its and Zhou [14] in 1997. The authors of [14] used Riemann-Hilbert techniques to determine asymptotics of the logarithmic derivative $\frac{d}{ds}\log P^{\sin}(sA)$, where A is one (or a union of several) interval(s). The asymptotics for $P_s(A)$ were then obtained in [14] by integrating the logarithmic derivative with respect to s. The reason the expression for c_0 was not established in [14] is that there is no initial integration point $s = s_0$ where $P^{\sin}(sA)$ would be known explicitly. In [28], the author was able to justify the value of c_0 in (3) by using a different differential identity for associated Toeplitz determinants and again the result of Widom [35]. An alternative proof of (3) was given in [15], which was based on another differential identity for Toeplitz determinants. In [15], the result of Widom on Toeplitz determinants in [35] was also rederived this way. Both [28] and [15] relied on Riemann-Hilbert techniques. Yet another proof of (3) was given by Ehrhardt [22] who used a very different approach of operator theory.

In the case of the Airy-kernel determinant, the asymptotics of the logarithm of P^{Ai} in (1) have the form:

$$\log P^{\mathrm{Ai}}(-s,+\infty) = -\frac{1}{12}s^3 - \frac{1}{8}\log s + \chi_{\mathrm{Airy}} + O\left(\frac{1}{s^{3/2}}\right), \quad s \to \infty, \tag{4}$$

where

$$\chi_{\mathrm{Airy}} = \frac{1}{24}\log 2 + \zeta'(-1). \tag{5}$$

The determinant $P^{\mathrm{Ai}}(-s,+\infty)$ is the Tracy-Widom distribution [31]—the distribution of the largest eigenvalue of the GUE. The same determinant also describes the distribution of the longest increasing subsequence in a random permutation [1]. Its large s asymptotics were first considered by Tracy and Widom [31] in 1994, who observed that

$$P^{\mathrm{Ai}}(-s,+\infty) = \exp\left\{-\int_{-s}^{\infty}(x+s)u^2(x)dx\right\}, \tag{6}$$

where $u(x)$ is the Hastings-McLeod solution of the Painlevé II equation

$$u''(x) = xu(x) + 2u^3(x), \tag{7}$$

specified by the following asymptotic condition:

$$u(x) \sim \mathrm{Ai}(x) \qquad \text{as} \quad x \to +\infty. \tag{8}$$

The asymptotics of the logarithmic derivative $(d/ds)\log P^{\mathrm{Ai}}(-s,+\infty)$ follow, up to a constant (which is in fact zero), from (8) and the known asymptotics of the Hastings-McLeod solution at $-\infty$. Integrating in s, Tracy and Widom obtained (4) up to an undetermined constant χ_{Airy}, whose value they conjectured to be (5). Two independent and complete proofs of (4), confirming the value of χ_{Airy}, were given in [16] and [2].

Note that analogous results on the probability of a large gap were obtained for the Bessel-kernel determinant in [17, 23], see [29] for an overview.

Return now to the case of the sine-kernel determinant. If A is a union of several intervals, it was shown by Widom in [37] that

$$\frac{d}{ds}\log P^{\sin}(sA) = -C_1 s + C_2(s) + o(1), \qquad s \to \infty, \tag{9}$$

where $C_1 > 0$ and $C_2(s)$ is a bounded oscillatory function. The constant C_1 can be computed explicitly, but $C_2(s)$ is an implicit solution of a Jacobi inversion problem. This result was extended and made more explicit by Deift Its and Zhou in [14].

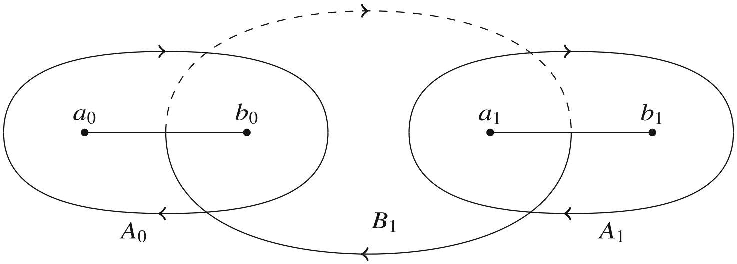

Fig. 1 Cycles on the Riemann surface Σ.

From now on we will concentrate on the case of two gaps, $A = (a_0, b_0) \cup (a_1, b_1)$. We proceed to introduce the necessary notation, following [14], and will subsequently present a complete solution.

Let $p(z) = (z - a_0)(z - b_0)(z - a_1)(z - b_1)$, and consider the two-sheeted Riemann surface Σ of the function $p(z)^{1/2}$. On the first sheet $p(z)^{1/2}/z^2 \to 1$ as $z \to \infty$, while on the second, $p(z)^{1/2}/z^2 \to -1$ as $z \to \infty$. The sheets are glued at the cuts (a_0, b_0) and (a_1, b_1). Each point $z \in \overline{\mathbb{C}} \setminus A$ has two images on Σ. The Riemann surface Σ is topologically a torus.

Let the elliptic integrals L_j, M_j (which depend on the end-points a_k, b_k) be given by

$$L_j = \int_{a_1}^{b_1} \frac{x^j dx}{\sqrt{|p(x)|}} = \frac{i}{2}\int_{A_1} \frac{x^j dx}{p(x)^{1/2}}, \quad M_j = \int_{b_0}^{a_1} \frac{x^j dx}{\sqrt{|p(x)|}} = \frac{1}{2}\int_{B_1} \frac{x^j dx}{p(x)^{1/2}}, \tag{10}$$

$j = 0, 1, 2$, where the loops (cycles) A_j, B_j are shown in Fig. 1. The loops A_0, A_1 lie on the first sheet, and the loop B_1 passes from one to the other: the part of it denoted by a solid line is on the first sheet, the other is on the second.

Let

$$\psi(z) = \frac{q(z)}{p(z)^{1/2}}, \qquad q(z) = (z - x_1)(z - x_2) = z^2 + q_1 z + q_0, \\ q_1 = -(x_1 + x_2), \qquad q_0 = x_1 x_2, \tag{11}$$

where the constants $x_1 \in (a_0, b_0)$ and $x_2 \in (a_1, b_1)$ are defined by the conditions

$$\int_{A_j} \psi(z)dz = 0, \qquad j = 0, 1. \tag{12}$$

It follows that

$$q_1 = -\frac{a_0 + b_0 + a_1 + b_1}{2}, \qquad q_0 = -\left(L_2 + q_1 L_1\right)\frac{1}{L_0}. \tag{13}$$

Note that (12) implies that $\psi(z)$ has no residue at infinity. More precisely, we obtain as $z \to \infty$ on the first sheet

$$\psi(z) = 1 + \frac{G_0}{z^2} + O(z^{-3}), \qquad G_0 = q_0 - \frac{1}{2}q_1^2 + \frac{1}{4}(a_0^2 + a_1^2 + b_0^2 + b_1^2). \tag{14}$$

As shown in [14], $G_0 > 0$.

Let

$$\tau = i\frac{M_0}{L_0}, \qquad \Omega = -\frac{1}{2\pi}\int_{B_1}\psi(\zeta)d\zeta = \frac{1}{\pi}\int_{b_0}^{a_1}\psi(\zeta)d\zeta = \frac{1}{L_0}, \tag{15}$$

where the integration $\int_{b_0}^{a_1}\psi(x)dx$ is taken on the first sheet, and where the last equation for Ω follows by Riemann's period relations (Lemma 3.45 in [14] for $n = 1$). Note that $i\tau < 0$, $\Omega \in \mathbb{R}$. Recall the definition of the third Jacobian θ-function $\theta_3(z;\tau)$:

$$\theta_3(z) = \theta_3(z;\tau) = \sum_{m\in\mathbb{Z}} e^{2\pi i m z + \pi i \tau m^2}. \tag{16}$$

The θ-function satisfies the following periodicity relations, see e.g. [34],

$$\theta_3(z+1) = \theta_3(z) \quad \text{and} \quad \theta_3(z+\tau) = e^{-2\pi i z - \pi i \tau}\theta_3(z). \tag{17}$$

We are now ready to present the result.

Theorem 1

$$\log P^{\sin}((sa_0, sb_0) \cup (sa_1, sb_1)) = -s^2 G_0 - \frac{1}{2}\log s + \log\frac{\theta_3(s\Omega;\tau)}{\theta_3(0;\tau)}$$
$$+\frac{1}{4}\log(a_1 - a_0)(b_1 - b_0) - \frac{1}{8}\sum_{j=0}^{1}\log|q(a_j)q(b_j)| + 2c_0 + O(s^{-1}), \qquad s \to \infty, \tag{18}$$

with G_0 as in (14), c_0 as in (3), and τ, Ω as in (15).

Deift et al. found in [14] the asymptotics of the derivative w.r.t. s, $\frac{d}{ds}P^{\sin}(s \cup_{j=0}^{m} (a_j, b_j))$. In particular, for the present case $m = 1$, this allowed them to obtain the $-s^2G_0$ term, the $G_1 \log s$ term (with G_1 written in terms of a limit of an integral of a combination of θ-functions), and the $\log\theta_3(s\Omega;\tau)$ term. The constant in s term remained undetermined (for the same reason as given above in the case of one interval). The proof of Theorem 1 was concluded in [25] where the constant term was found and proved, and the fact that $G_1 = -1/2$ was established. In [25] we also make a conjecture of the constant term in the general case of $m + 1$

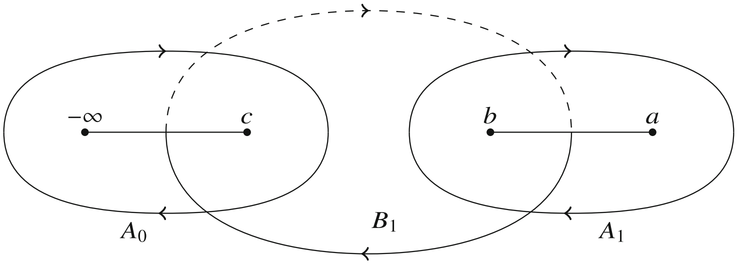

Fig. 2 Cycles on the Riemann surface $\widetilde{\Sigma}$.

gaps. Combined with our previous work [24], Fahs and Krasovsky [25] also gives a full description of all the transition asymptotics between one and two gaps (that is when the edges a_k, b_k are allowed to depend on s).

The method of [25] is based on differential identities for $\frac{d}{da_k}P^{\sin}$, $\frac{d}{db_k}P^{\sin}$, and on the fact that when the distance between the gaps is large relative to their length, the probability of two gaps may be approximated by the product of two single-gap probabilities. The latter fact provides a starting point for the integration. The main technical challenge in [25] was the integration over a_k, b_k of the asymptotic expressions for the differential identities.

We now discuss the corresponding result for the Airy-kernel determinant on two intervals $A = (c, b) \cup (a, +\infty)$. The most interesting case and the one we consider is when $a, b, c < 0$. This corresponds for large s to the asymptotics of the Tracy-Widom distribution for the gap $(-|a|s, +\infty)$ in the presence of one additional gap $(-s|c|, -s|b|)$. Similarly to the sine case above, if the distance between gaps is large compared to $b - c$ and $|a|$, the logarithm of the probability is approximately the sum of the (rescaled) Tracy-Widom distribution asymptotics (4) and the one-gap asymptotics (2).

Let $\widetilde{p}(z) = (z-a)(z-b)(z-c)$, and consider the two-sheeted Riemann surface $\widetilde{\Sigma}$ of the function $\widetilde{p}(z)^{1/2}$. On the first sheet $\widetilde{p}(z)^{1/2} > 0$ for $z > a$. The cuts are shown on Fig. 2.

Consider the function

$$\widetilde{\psi}(z) = \frac{\widetilde{q}(z)}{\widetilde{p}(z)^{1/2}}, \qquad \widetilde{q}(z) = z^2 + \widetilde{q}_1 z + \widetilde{q}_0, \tag{19}$$

where the polynomial $\widetilde{q}(z)$ is determined by the conditions

$$\int_{B_1} \widetilde{\psi}(\zeta)d\zeta = 0, \qquad \int_a^z \widetilde{\psi}(\zeta)d\zeta = \frac{2}{3}z^{3/2} + O\left(\frac{1}{z^{1/2}}\right), \quad z \to \infty. \tag{20}$$

It follows that

$$\widetilde{q}_1 = -\frac{a+b+c}{2}, \qquad \widetilde{q}_0 = -\frac{\widetilde{M}_2 + \widetilde{q}_1 \widetilde{M}_1}{\widetilde{M}_0} = \frac{1}{3}(ab+ac+bc) + \frac{1}{3}\widetilde{q}_1 \frac{\widetilde{M}_1}{\widetilde{M}_0}. \tag{21}$$

Here we define (cf. (10))

$$\widetilde{L}_j = \frac{i}{2}\int_{A_1} \frac{x^j dx}{\widetilde{p}(x)^{1/2}}, \qquad \widetilde{M}_j = \frac{1}{2}\int_{B_1} \frac{x^j dx}{\widetilde{p}(x)^{1/2}}, \qquad j = 0, 1, 2. \tag{22}$$

Furthermore, the function $\widetilde{\psi}(z)$ admits a large-z asymptotic expansion of the form

$$\widetilde{\psi}(z) = z^{1/2} - \frac{1}{2}\frac{\alpha_1}{z^{3/2}} - \frac{3}{2}\frac{\alpha_2}{z^{5/2}} + O\left(z^{-7/2}\right), \qquad z \to \infty, \tag{23}$$

where, in particular,

$$\alpha_2 = -\frac{1}{12}\left(a^3 + b^3 + c^3 - (a+b)(a+c)(b+c) - 8\widetilde{q}_0\widetilde{q}_1\right). \tag{24}$$

Let

$$\widetilde{\tau} = i\frac{\widetilde{L}_0}{\widetilde{M}_0}, \qquad \widetilde{\Omega} = -\frac{1}{2\pi i}\int_{A_1} \widetilde{\psi}(\zeta)d\zeta. \tag{25}$$

Similarly to (15), we have that $i\widetilde{\tau} < 0$, $\widetilde{\Omega} \in \mathbb{R}$.

We now state our result.

Theorem 2 *The following asymptotics hold*

$$\log P^{\mathrm{Ai}}((sc, sb) \cup (sa, +\infty)) = -s^3\alpha_2 - \frac{1}{2}\log s + \log\frac{\theta_3(s^{3/2}\widetilde{\Omega}; \widetilde{\tau})}{\theta_3(0; \widetilde{\tau})}$$
$$+ \frac{1}{4}\log(a-c) - \frac{1}{8}\log|2\widetilde{q}(a)\widetilde{q}(b)\widetilde{q}(c)| + c_0 + \chi_{\mathrm{Airy}} + o(1), \qquad s \to +\infty, \tag{26}$$

where the constants c_0, χ_{Airy} are given by (3) *and* (5)*, respectively.*

This theorem was established in [30] where we followed the method of [25].

In the recent work [3], Blackstone Charlier and Lenells have simultaneously and independently analyzed the large-s asymptotics of $\log P^{\mathrm{Ai}}(sA)$. They found the expansion $-\alpha_2 s^3 - \frac{1}{2}\log s + \log\theta_3(s^{3/2}\widetilde{\Omega}) + \chi' + O(1/s)$ with an undetermined constant term $\chi' = \chi'(a, b, c)$. (This analysis was then extended by the authors to the case of n gaps in the bulk of the Airy process in [4], and in the Bessel process in [5].) The authors of [3–5] followed the approach of [14], and used Riemann-Hilbert analysis to obtain the asymptotics of the derivative $\frac{d}{ds}\log P^{\mathrm{Ai}}(sA)$. By contrast, our approach of [25, 30] uses the differential identity w.r.t. a_k, b_k, which allows to determine the constant.

A related problem where θ-functions also appear is the computation of the asymptotics of the determinant $\det(I - \gamma K^{\sin})_{(-s,s)}$, where $\gamma \in [0, 1]$. This determinant is the gap probability of a thinned version of the sine process. Namely, if we independently remove each particle from the sine process with probability $1 - \gamma$, then the probability of the resulting process having no particles in $(-s, s)$ is given by $\det(I - \gamma K^{\sin})_{(-s,s)}$. When most of the particles are removed, the resulting process is close to a Poisson process, and thus, as γ changes between 1 and 0, the determinant interpolates between a random matrix gap probability and the gap probability of a Poisson process. It is of interest to compute the behaviour of $\det(I - \gamma K^{\sin})_{(-s,s)}$ as $s \to \infty$, and in a certain double scaling limit where $\gamma \to 1$, fluctuations involving θ-functions appear in the asymptotic description of the determinant. This problem was first studied by Dyson [21], and recently addressed in [7–9] (see also [6] and references therein for work on an analogous Airy kernel determinant).

For further related results on gap probabilities see [10–13, 26] and references therein.

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Multiplicative Properties of Infinite Block Toeplitz and Hankel Matrices

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Dedicated to the memory of Harold Widom who made fundamental contributions to the theory of (finite and infinite) Toeplitz and Hankel matrices

Abstract We give a simple characterization of when the product of two infinite block Toeplitz matrices is another infinite block Toeplitz matrix. We then use this characterization to answer the question of when two infinite block Toeplitz matrices commute. This approach gives a unified treatment for two related problems which are often studied separately for infinite scalar Toeplitz-like and Hankel-like matrices. Related results for products of two infinite block Hankel matrices are also obtained.

Keywords Toeplitz matrix · Hankel matrix · Block Toeplitz operator · Block Hankel operator

1 Introduction

A Toeplitz matrix and a Hankel matrix are defined by

$$T = \left[a_{i-j}\right]_{i,j=1}^{n} \quad \text{and} \quad H = \left[b_{i+j}\right]_{i,j=1}^{n},$$

respectively, where a_i and b_i are complex numbers.

Toeplitz and Hankel matrices, more generally structured matrices as in [28], have been studied intensively in the last few decades. They have wide applications in signal and imaging processing, control theory and other branches of numerical and computational sciences [21]. Fast algorithms about these matrices have been

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developed and used successfully in various applications [26] and [27]. Spectral properties of banded Toeplitz matrices have been documented in the book [7]. Their algebraic properties have also been discussed in book form and in a number of papers, for example, see book [19] and papers [11, 17, 20, 30] and references therein.

An infinite Toeplitz matrix and an infinite Hankel matrix are defined by

$$T = \left[a_{i-j}\right]_{i,j=1}^{\infty} \quad \text{and} \quad H = \left[b_{i+j}\right]_{i,j=1}^{\infty}, \tag{1}$$

respectively, where a_i and b_i are complex numbers subject to some boundedness conditions, which will be made precise in next section. The infinite Toeplitz matrices and infinite Hankel matrices (called Toeplitz operators and Hankel operators) have also been studied extensively in operator theory and robust control literature [8, 9, 12] and [29]. The seminal paper [10] by Brown and Halmos laid down some basic algebraic proprieties of Toeplitz operators. Their work has been generalized by many authors, as seen by several hundreds of citations of their paper; for example, see papers [2, 3] and [13]. Many authors studied analogous Toeplitz operators and Hankel operators on various function spaces and the matrix representations of these operators are in general similar and more complicated than the form (1).

In this paper we study some basic algebraic properties of infinite block Toeplitz matrices and infinite block Hankel matrices (called block Toeplitz operators and block Hankel operators), where a_i and b_i are matrices of any fixed size instead of scalars. Some algebraic properties of block Toeplitz operators and block Hankel operators were discussed in [14] and [18], but the results were sometimes complicated or incomplete. However, recently satisfactory characterizations of normal block Toeplitz operators and normal block Hankel operators are given in [16] even when a_i and b_i are operators on a Hilbert space. See recent papers [22] and [23] for some algebraic properties of finite block Toeplitz matrices which are notably different from the algebraic properties of infinite block Toeplitz matrices discussed here.

Next we recall a couple of results from [10] in a slightly different way and state our generalizations using matrix terminology.

Theorem A *Let* $T_1 = \left[a_{i-j}\right]_{i,j=1}^{\infty}$ *and* $T_2 = \left[b_{i-j}\right]_{i,j=1}^{\infty}$ *be two infinite Toeplitz matrices. Then* T_1T_2 *is another infinite Toeplitz matrix if and only if either* T_1 *is lower triangular or* T_2 *is upper triangular. In other words,* T_1T_2 *is another infinite Toeplitz matrix if and only if one of the following holds.*

(1) $a_i = 0$ *for all* $i \geq 1$.
(2) $b_i = 0$ *for all* $i \leq -1$.

Theorem A is Theorem 8 in [10].

Theorem 1.1 *Let* $T_1 = \left[A_{i-j}\right]_{i,j=1}^{\infty}$ *and* $T_2 = \left[B_{i-j}\right]_{i,j=1}^{\infty}$ *be two infinite Toeplitz matrices, where* A_i *are matrices of size* $m \times n$ *and* B_i *are matrices of size* $n \times k$. *Then* $T_1T_2 = \left[C_{i-j}\right]_{i,j=1}^{\infty}$, *where* C_i *are matrices of size* $m \times k$, *if and only if one of the followings holds.*

(1) $A_i = 0$ *for all* $i \geq 1$.
(2) $B_i = 0$ *for all* $i \leq -1$.
(3) *There exist an* $n \times n$ *invertible matrix* U *and a positive integer* l *(*$1 \leq l < n$*) such that the last* $n - l$ *columns of* $A_i U$ *are zero for all* $i \geq 1$ *and the first* l *rows of* $U^{-1} B_i$ *are zero for all* $i \leq -1$.

The theorem above is just the matrix formulation of Theorem 4.12 below.

Remark 1.2 We can view Condition (1) and Condition (2) in Theorem 1.1 as corresponding to $l = 0$ and $l = n$ in Condition (3) respectively. The proof of Theorem 4.12 will also give a meaning of l and thus tells exactly when each condition is satisfied.

We now state a related result about the product of two infinite block Hankel matrices. The result for the scalar case is implicitly in [10] by the connection between Hankel and Toeplitz operators (see Lemma 2.1). However, Brown and Halmos [10] did not use Hankel operators.

Theorem B *Let* $H_1 = \left[a_{i+j}\right]_{i,j=1}^{\infty}$ *and* $H_2 = \left[b_{i+j}\right]_{i,j=1}^{\infty}$ *be two scalar infinite Hankel matrices. Then* $H_1 H_2 = 0$ *if and only if either* $H_1 = 0$ *or* $H_2 = 0$.

Theorem 1.3 *Let* $H_1 = \left[A_{i+j}\right]_{i,j=1}^{\infty}$, $H_2 = \left[B_{i+j}\right]_{i,j=1}^{\infty}$ *be two block infinite Hankel matrices, where* A_i *are matrices of size* $m \times n$ *and* B_i *are matrices of size* $n \times k$. *Then* $H_1 H_2 = 0$ *if and only if one of the followings holds*

(1) $H_1 = 0$.
(2) $H_2 = 0$.
(3) *There exist a positive integer* l *(*$1 \leq l < n$*) and two matrices* A *and* B *of sizes* $l \times n$ *and* $n \times (n - l)$ *such that* $AB = 0$ *and* $H_1 = \left[F_{i+j} A\right]_{i,j=1}^{\infty}$, $H_2 = \left[B G_{i+j}\right]_{i,j=1}^{\infty}$, *where* F_i *and* G_i *are matrices of sizes* $m \times l$ *and* $(n-l) \times k$ *for all* $i \geq 2$.

Proof One merit of such a result is that the proof of one direction is almost trivial, as we demonstrate now. Assume Condition (3) holds. Then

$$H_1 H_2 = \left[F_{i+j} A\right]_{i,j=1}^{\infty} \left[B G_{i+j}\right]_{i,j=1}^{\infty} = \left[\sum_{r=1}^{\infty} F_{i+r} A B G_{r+j}\right] = 0,$$

since $AB = 0$. □

The theorem above is the matrix formulation of Theorem 4.1. Again we can view Condition (1) and Condition (2) as corresponding to $l = 0$ and $l = n$ in case (3) respectively.

Theorem C *Let* $H_1 = \left[a_{i+j}\right]_{i,j=1}^{\infty}$, $H_2 = \left[b_{i+j}\right]_{i,j=1}^{\infty}$, $H_3 = \left[c_{i+j}\right]_{i,j=1}^{\infty}$, $H_4 = \left[d_{i+j}\right]_{i,j=1}^{\infty}$ *be four nonzero scalar infinite Hankel matrices. Then* $H_1 H_2 = H_3 H_4$ *if and only if there exists a constant* λ *such that* $H_1 = \lambda H_3$ *and* $H_4 = \lambda H_2$.

Theorem C is a special case of Theorem 5 in [18]. It can also be derived from Theorem 9 in [10].

Theorem 1.4 *Let* $H_1 = \left[A_{i+j}\right]_{i,j=1}^{\infty}$, $H_2 = \left[B_{i+j}\right]_{i,j=1}^{\infty}$, $H_3 = \left[C_{i+j}\right]_{i,j=1}^{\infty}$, $H_4 = \left[D_{i+j}\right]_{i,j=1}^{\infty}$ *be four nonzero block infinite Hankel matrices, where* A_i, B_i, C_i, D_i *are matrices of compatible sizes. Then* $H_1H_2 = H_3H_4$ *if and only if there exist four matrices* A, B, C *and* D *such that* $AB = CD$ *and* $H_1 = \left[F_{i+j}A\right]_{i,j=1}^{\infty}$ $H_2 = \left[BG_{i+j}\right]_{i,j=1}^{\infty}$, $H_3 = \left[F_{i+j}C\right]_{i,j=1}^{\infty}$, $H_4 = \left[DG_{i+j}\right]_{i,j=1}^{\infty}$, *where* F_i *and* G_i *are matrices of appropriate sizes for all* $i \geq 2$.

Proof We present the almost trivial proof of one direction. Assume

$$H_1 = \left[F_{i+j}A\right]_{i,j=1}^{\infty}, \qquad H_2 = \left[BG_{i+j}\right]_{i,j=1}^{\infty},$$

$$H_3 = \left[F_{i+j}C\right]_{i,j=1}^{\infty}, \qquad H_4 = \left[DG_{i+j}\right]_{i,j=1}^{\infty}.$$

Then

$$H_1H_2 = \left[F_{i+j}A\right]\left[BG_{i+j}\right] = \left[\sum_{r=1}^{\infty} F_{i+r}ABG_{r+j}\right],$$

$$H_3H_4 = \left[F_{i+j}C\right]\left[DG_{i+j}\right] = \left[\sum_{r=1}^{\infty} F_{i+r}CDG_{r+j}\right].$$

Therefore $H_1H_2 = H_3H_4$ follows from $AB = CD$. □

The above theorem is the matrix formulation of Theorem 5.2.

Other generalizations, such as when two infinite block Toeplitz matrices commute, are slightly more complicated to state even though we feel we have found the simplest answers possible. So for more precise statements and proofs, we will introduce our function spaces, block Toeplitz operators and block Hankel operators in the next section.

The proofs of our results are elementary and sometimes straightforward; however, the formulations of the results are modeled after the elegant and inspirational results of Brown and Halmos [10]. It is our hope that this paper will play a similar role to motivate the study of algebraic properties of block Toeplitz and Hankel operators on various function spaces as the paper [10] did for algebraic properties of scalar Toeplitz and Hankel operators.

We would like to mention two-sided infinite block Laurent matrices. A two-sided infinite block Laurent matrix is defined by

$$\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & A_0 & A_{-1} & A_{-2} & \ddots \\ \ddots & A_1 & [A_0] & A_{-1} & \ddots \\ \ddots & A_2 & A_1 & A_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2}$$

where A_i are matrices of an arbitrary but fixed size for $-\infty < i < \infty$ and $[A_0]$ denotes the $(0, 0)$-entry. Then the product of such two matrices is still a two-sided infinite block Laurent matrix. Two such matrices of compatible sizes commute if and only if their symbols (to be defined in next section) commute. We observe that an infinite block Toeplitz matrix is just the low right corner of a two-sided infinite block Laurent matrix. Similarly one can show that an infinite block Hankel matrix is unitarily equivalent to the upper right corner of a two-sided infinite block Laurent matrix.

The author is honored to dedicate this paper in memory of Professor Harold Widom who made many fundamental contributions to the theory and application of Hankel matrices, Toeplitz matrices, Hankel operators, and Toeplitz operators; see, for example, references [4–6, 32–34], and [31].

2 Block Toeplitz and Hankel Operators

Let $\mathbb{T}$ be the unit circle in the complex plane. Let $L^2 = L^2(\mathbb{T})$ be the set of all square-integrable functions on $\mathbb{T}$. Each function $f \in L^2$ has a Fourier series expansion

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \quad \text{for } \theta \in [0, 2\pi]$$

and

$$\|f(z)\|^2 = \int_{\mathbb{T}} |f(z)|^2 \, dm(z) = \sum_{n=-\infty}^{\infty} |f_n|^2$$

where $m(z)$ is the normalized Lebesgue measure on $\mathbb{T}$. The function $f(e^{i\theta})$ has a unique harmonic extension into the unit disk D as follows

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} f_n r^{|n|} e^{in\theta}, \quad 0 \le r \le 1.$$

Let L^∞ be the set of essentially bounded functions on $\mathbb{T}$. Let $H^\infty = L^\infty \cap H^2$; that is, H^∞ is the set of all bounded analytic functions on D. Given $\varphi \in L^\infty$, the Laurent operator M_φ is the multiplication operator defined by

$$M_\varphi g = \varphi g, \quad g \in L^2.$$

Then with respect to the bases $\{e^{in\theta}\}_{n=-\infty}^{\infty}$ of L^2, the matrix representation of M_φ is a two-sided infinite Laurent matrix (with scalar entries) as in (2). The function φ is called the symbol of M_φ. Furthermore $\|M_\varphi\| = \|\varphi\|_\infty$. If $\varphi_1, \varphi_2 \in L^\infty$, then $M_{\varphi_1} M_{\varphi_2} = M_{\varphi_1 \varphi_2} = M_{\varphi_2} M_{\varphi_1}$.

The Hardy space H^2 is the closed subspace of L^2 spanned by analytic polynomials. In other words, each $f \in H^2$ has a Fourier series expansion

$$f(e^{i\theta}) = \sum_{n=0}^{\infty} f_n e^{in\theta} \quad \text{for } \theta \in [0, 2\pi]$$

and we can view f as an analytic function inside the unit disk D with power series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad |z| < 1.$$

Given $\varphi \in L^\infty$, the Toeplitz operator T_φ and the Hankel operator H_φ are defined by

$$T_\varphi(g) = P(\varphi g) \quad \text{and} \quad H_\varphi(g) = JP^\perp(\varphi g), \quad (g \in H^2)$$

where P and $P^\perp$ denote the orthogonal projections that map L^2 onto H^2 and $H^{2\perp}$ respectively, and J denotes the unitary operator on L^2 defined by $Jf(z) = \overline{z} f(\overline{z})$. The function φ is called the symbol of T_φ and H_φ. It is an elementary fact that

$$\|T_\varphi\| = \|\varphi\|_\infty := ess \sup_{z \in \mathbb{T}} |\varphi(z)|$$

and it is an important theorem (Nehari's Theorem [25]) that

$$\|H_\varphi\| = \inf_{f \in H^\infty} \|\varphi - f\|_\infty .$$

To define block Toeplitz and Hankel operators with matrix-valued symbols, we first define the vector-valued L^2 space and H^2 space. Let $\mathbb{C}^n$ be the standard n-dimensional complex Hilbert space with the inner product defined by

$$\langle u, v\rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} u_i \overline{v_i} \quad \text{where } u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \ v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Let $M_{m\times n}$ be the set of all constant $m \times n$ matrices. Let L_n^2 be the set of all $\mathbb{C}^n$-valued square-integrable functions on the unit circle $\mathbb{T}$ and H_n^2 be the corresponding Hardy space. For $f, g \in L_n^2$, the inner product $\langle f, g\rangle$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{T}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} \, dm(z).$$

Let $L_{m\times n}^\infty$ be the set of matrix-valued functions Φ on the unit circle $\mathbb{T}$ with finite operator supremum norm. The operator supremum norm of Φ is defined by

$$\|\Phi\|_\infty := \operatorname*{ess\,sup}_{z\in\mathbb{T}} \|\Phi(z)\|,$$

where $\|\Phi(z)\|$ denotes the matrix norm of $\Phi(z)$ (as an operator from $\mathbb{C}^n$ into $\mathbb{C}^m$). Let $H_{m\times n}^\infty$ denote the set of analytic elements in $L_{m\times n}^\infty$. Given $\Phi \in L_{m\times n}^\infty$, the Laurent operator M_Φ is the multiplication operator from L_n^2 into L_m^2 defined by

$$M_\Phi g = \Phi g, \quad g \in L_n^2.$$

Similar to the definition of Toeplitz and Hankel operators with scalar-valued symbols, if Φ is a matrix-valued function in $L_{m\times n}^\infty$, then the Toeplitz operator T_Φ and the Hankel operator H_Φ from H_n^2 into H_m^2 are defined by

$$T_\Phi f = P(\Phi f) \quad \text{and} \quad H_\Phi f = JP^\perp(\Phi f) \quad (f \in H_n^2),$$

where P and $P^\perp$ denote the orthogonal projections that map L_m^2 onto H_m^2 and $\left(H_m^2\right)^\perp$, respectively, and J denotes the unitary operator on L_m^2 given by $J(g)(z) = \overline{z}g(\overline{z})$ for $g \in L_m^2$. Note that $JP^\perp = PJ$. Similarly,

$$\|T_\Phi\| = \|\Phi\|_\infty,$$

and by matrix-valued Nehari's Theorem in [1]

$$\|H_\Phi\| = \inf_{F\in H_{m\times n}^\infty} \|\Phi - F\|_\infty .$$

For $\Phi \in L^\infty_{m\times n}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

One can easily see that $\widetilde{\widetilde{\Phi}}(z) = \Phi(z)$.

It is instructive to write down two different representations of Toeplitz operators and Hankel operators with matrix-valued symbols.

In the first representation, note that H^2_n and H^2_m can be viewed as the direct sum of n and m copies of H^2, respectively,

$$H^2_n = H^2 \oplus H^2 \oplus \cdots \oplus H^2 \ (n \text{ copies of } H^2) \quad \text{and}$$
$$H^2_m = H^2 \oplus H^2 \oplus \cdots \oplus H^2 \ (m \text{ copies of } H^2).$$

With respect to the decompositions above, we can write

$$\Phi(z) = \left[\varphi_{ij}(z)\right]_{m\times n} \in L^\infty_{m\times n}, \quad T_\Phi = \left[T_{\varphi_{ij}(z)}\right]_{m\times n}, \quad H_\Phi = \left[H_{\varphi_{ij}(z)}\right]_{m\times n},$$

where $T_{\varphi_{ij}(z)}$ and $H_{\varphi_{ij}(z)}$ are Toeplitz and Hankel operators with scalar-valued symbols, respectively.

In the second representation, write

$$\Phi(z) = \sum_{j=-\infty}^{\infty} \Phi_j z^j \in L^\infty_{m\times n} \quad (z = e^{i\theta}) \tag{3}$$

where $\Phi_j \in M_{m\times n}$. Then with respect to the decomposition

$$H^2_n = \mathbb{C}^n \oplus z\mathbb{C}^n \oplus z^2\mathbb{C}^n \oplus \cdots \quad \text{and} \quad H^2_m = \mathbb{C}^m \oplus z\mathbb{C}^m \oplus z^2\mathbb{C}^m \oplus \cdots$$

we have

$$T_\Phi = \begin{pmatrix} \Phi_0 & \Phi_{-1} & \Phi_{-2} & \cdots \\ \Phi_1 & \Phi_0 & \Phi_{-1} & \ddots \\ \Phi_2 & \Phi_1 & \Phi_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad H_\Phi = \begin{pmatrix} \Phi_{-1} & \Phi_{-2} & \Phi_{-3} & \cdots \\ \Phi_{-2} & \Phi_{-3} & \Phi_{-4} & \iddots \\ \Phi_{-3} & \Phi_{-4} & \iddots & \iddots \\ \vdots & \iddots & \iddots & \end{pmatrix}.$$

So they are the infinite block Toeplitz and Hankel matrices. Note that H_Φ only depends on Φ_j for $j < 0$.

If $\Phi(z) \in L^\infty_{m\times n}$ has the Fourier series expansion as in (3), we can write

$$\Phi(z) = \Phi_+(z) + \Phi_0 + \Phi^*_-(z),$$

where $\Phi_+(z) \in zH^2_{m\times n}$, $\Phi_-(z) \in zH^2_{n\times m}$ and $\Phi_0 \in M_{m\times n}$. Similarly, if $\Psi(z) \in L^\infty_{n\times k}$ we can write

$$\Psi(z) = \Psi_+(z) + \Psi_0 + \Psi_-^*(z).$$

As in the scalar case, operators such as T_{Φ_+}, $T_{\Phi_+\Psi_-^*}$ and $T_{\Phi_+}T_{\Psi_-^*}$ are densely defined on all vector-valued analytic polynomials. Let I_n be the identity operator on $\mathbb{C}^n$. Let $S_n = T_{zI_n}$ be the unilateral shift operator on H^2_n, so S_n is the shift with multiplicity n. Then the Toeplitz operator T_Φ is characterized as a bounded operator satisfying the equation $S_m^* T_\Phi S_n = T_\Phi$. The Hankel operator is characterized as a bounded operator satisfying the equation $H_\Phi S_n = S_m^* H_\Phi$. The following lemma on several basic relations between block Toeplitz and block Hankel operators is well known.

Lemma 2.1 *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then*

$$T_\Phi^* = T_{\Phi^*},\ H_\Phi^* = H_{\widetilde{\Phi}},\ T_{\Phi\Psi} - T_\Phi T_\Psi = H_{\Phi^*}^* H_\Psi. \tag{4}$$

Since $H_\Psi = 0$ if $\Psi(z) \in H^\infty_{n\times k}$, the following result is clear from (4).

Corollary 2.2 *If either $\Phi(z) \in \overline{H^\infty_{m\times n}}$ or $\Psi(z) \in H^\infty_{n\times k}$, then $T_\Phi T_\Psi = T_{\Phi\Psi}$. In particular $T_\Phi T_\Psi$ is another block Toeplitz operator.*

When $\Phi(z)$ and $\Psi(z)$ are scalar-valued functions, Brown and Halmos [10] proved the converse of the above corollary and thus characterized when the product of two scalar Toeplitz operators is another Toeplitz operator. This converse does not hold for matrix-valued Toeplitz operators. We now study the question when $T_\Phi T_\Psi$ is another Toeplitz operator. Our result (Theorem 4.12) says that if $T_\Phi T_\Psi$ is another block Toeplitz operator, then part of $\Phi(z)$ is in $\overline{H^\infty_{m\times n}}$ and part of $\Psi(z)$ is in $H^\infty_{n\times k}$. The following proposition follows from a general result in Proposition 4.1 in [9].

Proposition 2.3 *If $T_\Phi T_\Psi = T_\Omega$ for some $\Omega \in L^\infty_{m\times k}$, then $\Omega = \Phi\Psi$.*

Now by Lemma 2.1, $T_{\Phi\Psi} - T_\Phi T_\Psi = H_{\Phi^*}^* H_\Psi$. Therefore $T_{\Phi\Psi} - T_\Phi T_\Psi = 0$ if and only if $H_{\Phi^*}^* H_\Psi = 0$. To study this equation, we first establish a lemma. Let $e_1, e_2, \cdots, e_n$ be the standard basis of $\mathbb{C}^n$. Then

$$(I - S_n S_n^*) = \sum_{i=1}^n e_i \otimes e_i.$$

In other words, $I - S_n S_n^*$ is the projection onto the constant vectors in H^2_n.

The following simple lemma is crucial in our approach because it provides a direct link between Hankel operators and their symbols.

Lemma 2.4 *Let* $\Phi(z) = L^\infty_{m\times n}$ *and* $\Psi(z) \in L^\infty_{n\times k}$. *Then*

$$H^*_{\Phi^*} H_\Psi - S^*_m H^*_{\Phi^*} H_\Psi S_k = \sum_{i=1}^{n} H^*_{\Phi^*} e_i \otimes H^*_\Psi e_i.$$

Proof By Lemma 2.1,

$$\begin{aligned} H^*_{\Phi^*} H_\Psi - S^*_m H^*_{\Phi^*} H_\Psi S_k &= H^*_{\Phi^*} H_\Psi - H^*_{\Phi^*} S_n S^*_n H_\Psi \\ &= H^*_{\Phi^*}(I - S_n S^*_n) H_\Psi = \sum_{i=1}^{n} H^*_{\Phi^*} e_i \otimes H^*_\Psi e_i. \end{aligned}$$

The proof is complete. □

3 Some Lemmas on Finite Rank Operators

We also need the following lemma, for which it is best to use more abstract notations. This lemma has three versions and each version seems to give an interesting different perspective. Let H and K be two complex Hilbert spaces. Let $h \in H$ and $k \in K$. The rank one operator $k \otimes h$ from H into K is defined by

$$(k \otimes h)\, x = \langle x, h\rangle\, k, \quad x \in H.$$

For a constant matrix A, we write $A_{n\times l}$ to indicate that $A \in M_{n\times l}$.

Lemma 3.1 (First Version) *Let* $x_i \in H$ *and* $y_i \in K$ *for* $1 \le i \le n$. *Then*

$$\sum_{i=1}^{n} x_i \otimes y_i = 0 \tag{5}$$

if and only one of the followings holds.

(i) All $x_i = 0$ *for* $1 \le i \le n$.
(ii) All $y_i = 0$ *for* $1 \le i \le n$.
(iii) For some $0 < l < n$, *there exist matrices* $A_{l\times n}$ *and* $B_{(n-l)\times n}$ *such that* $AB^* = 0$ *and*

$$\begin{bmatrix} x_1 \cdots x_n \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_l \end{bmatrix} A, \qquad \begin{bmatrix} y_1 \cdots y_n \end{bmatrix} = \begin{bmatrix} w_{l+1} \cdots w_n \end{bmatrix} B.$$

for some $u_i \in H$ *for* $1 \le i \le l$ *and* $w_i \in K$ *for* $l+1 \le i \le n$.

Proof The sufficiency of (iii) is by a direct verification. Now we prove the necessity of (iii). Let the dimension of the subspace spanned by $\{x_i, 1 \le i \le n\}$ be l. If $l = 0$, we have (i). If $l = n$, we have (ii). So we assume $0 < l < n$. We first assume $\{x_i, 1 \le i \le l\}$ are linearly independent. The general case will be reduced to this case by using a permutation matrix. Write

$$x_i = \sum_{j=1}^{l} c_{ji} x_j, \quad l+1 \le i \le n. \tag{6}$$

Plugging the above equation into (5), we have

$$\begin{aligned}
&\sum_{i=1}^{n} x_i \otimes y_i \\
&= \sum_{i=1}^{l} x_i \otimes y_i + \sum_{j=l+1}^{n} x_j \otimes y_j = \sum_{i=1}^{l} x_i \otimes y_i + \sum_{j=l+1}^{n} \left(\sum_{i=1}^{l} c_{ij} x_i \right) \otimes y_j \\
&= \sum_{i=1}^{l} x_i \otimes y_i + \sum_{i=1}^{l} x_i \otimes \left(\sum_{j=l+1}^{n} \overline{c_{ij}} y_j \right) = \sum_{i=1}^{l} x_i \otimes (y_i + \sum_{j=l+1}^{n} \overline{c_{ij}} y_j) = 0.
\end{aligned}$$

Therefore

$$y_i = - \sum_{j=l+1}^{n} \overline{c_{ij}} y_j, \quad 1 \le i \le l.$$

Set

$$A = \left[I_l \ C \right]_{l \times n}, \qquad B = \left[-C^* \ I_{n-l} \right]_{(n-l) \times n}, \tag{7}$$

where $C = \left[c_{ji} \right]$ is the matrix of size $l \times (n-l)$ defined in (6). Then

$$\left[x_1 \cdots x_n \right] = \left[x_1 \cdots x_l \right] A, \tag{8}$$

$$\left[y_1 \cdots y_n \right] = \left[y_{l+1} \cdots y_n \right] B, \tag{9}$$

and most importantly, $AB^* = -I_l C + C I_{n-l} = 0$.

In the general case, assume $\left\{ x_{n_i}, 1 \le i \le l \right\}$ are linearly independent. There exists a permutation matrix $P_{n \times n}$ such that

$$\left[x_1 \cdots x_n \right] = \left[x_{n_1} \cdots x_{n_l} \ x_{n_{l+1}} \cdots x_{n_n} \right] P,$$

$$\left[y_1 \cdots y_n \right] = \left[y_{n_1} \cdots y_{n_l} \ y_{n_{l+1}} \cdots y_{n_n} \right] P.$$

Since

$$\sum_{i=1}^{n} x_{n_i} \otimes y_{n_i} = \sum_{i=1}^{n} x_i \otimes y_i = 0,$$

by the previous case, there exist A and B as in (7) such that $AB^* = 0$ and

$$\begin{aligned} \left[x_{n_1} \cdots x_{n_l} \; x_{n_{l+1}} \cdots x_{n_n} \right] &= \left[x_{n_1} \cdots x_{n_l} \right] A \\ \left[x_{n_1} \cdots x_{n_l} \; x_{n_{l+1}} \cdots x_{n_n} \right] &= \left[y_{n_{l+1}} \cdots y_{n_n} \right] B. \end{aligned}$$

Then AP and BP satisfy the equation $AP(BP)^* = AP^2B^* = AB^* = 0$. Furthermore

$$\begin{aligned} \left[x_1 \cdots x_n \right] &= \left[x_{n_1} \cdots x_{n_l} \; x_{n_{l+1}} \cdots x_{n_n} \right] P \\ &= \left[x_{n_1} \cdots x_{n_l} \right] AP, \\ \left[y_1 \cdots y_n \right] &= \left[y_{n_1} \cdots y_{n_l} \; y_{n_{l+1}} \cdots y_{n_n} \right] P \\ &= \left[y_{n_{l+1}} \cdots y_{n_n} \right] BP. \end{aligned}$$

The proof is complete. □

The following lemma restates the above result more succinctly.

Lemma 3.2 (Second Version) *Notations as above. Then*

$$\sum_{i=1}^{n} x_i \otimes y_i = 0$$

if and only if there exist matrices $C_{n\times n}$ and $D_{n\times n}$ such that $CD^* = 0$ and

$$\left[x_1 \cdots x_n \right] = \left[x_1 \cdots x_n \right] C, \qquad \left[y_1 \cdots y_n \right] = \left[y_1 \cdots y_n \right] D$$

Proof In the case all $x_i = 0$ for $1 \le i \le n$, we just set $C = 0$ and $D = I_n$. In the case all $y_i = 0$ for $1 \le i \le n$, we just set $C = I_n$ and $D = 0$. Otherwise, as in proof of the previous lemma, if $\{x_i, 1 \le i \le l\}$ are linearly independent, set

$$C = \begin{bmatrix} A \\ 0 \end{bmatrix}_{n\times n}, \quad D = \begin{bmatrix} 0 \\ B \end{bmatrix}_{n\times n},$$

where A and B are as in (7). Then

$$\left[x_1 \cdots x_n \right] = \left[x_1 \cdots x_n \right] C, \quad \left[y_1 \cdots y_n \right] = \left[y_1 \cdots y_n \right] D.$$

Furthermore,

$$CD^* = \begin{bmatrix} A \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ B \end{bmatrix}^* = \begin{bmatrix} A \\ 0 \end{bmatrix} \begin{bmatrix} 0 & B^* \end{bmatrix} = 0.$$

In the general case where $\{x_{n_i}, 1 \le i \le l\}$ are linearly independent, the proof can be done by using a permutation matrix. □

The next lemma is based on the following observation: the simplest situation for $\sum_{i=1}^{n} x_i \otimes y_i = 0$ is that for some $0 \le l \le n$,

$$x_i = 0, 1 \le i \le l, \quad \text{and} \quad y_i = 0, l+1 \le i \le n$$

with the understanding that $l = n$ corresponds to all $x_i = 0$ for $1 \le i \le n$ and $l = n$ corresponds to all $y_i = 0$ for $1 \le i \le n$.

Lemma 3.3 (Third Version) *Notations as above. Then*

$$\sum_{i=1}^{n} x_i \otimes y_i = 0$$

if and only if there exist an invertible matrix $U_{n\times n}$ *and some* $0 \le l \le n$ *such that*

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_l & 0 & \cdots & 0 \end{bmatrix} U, \tag{10}$$

$$\begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & w_{l+1} & \cdots & w_n \end{bmatrix} U^{*-1} \tag{11}$$

for some $u_i \in H$ *for* $1 \le i \le l$ *and* $w_i \in K$ *for* $l+1 \le i \le n$.

Proof The sufficiency is by a direct verification. We now prove the necessity. In the case all $x_i = 0$ for $1 \le i \le n$, we have $U = I_n$ and $l = n$. In the case all $y_i = 0$ for $1 \le i \le n$, we have $U = I_n$ and $l = 0$. Otherwise, as in proof of the previous lemma, if $\{x_i, 1 \le i \le l\}$ are linearly independent for $1 \le l < n$, set

$$U = \begin{bmatrix} I_l & C \\ 0 & I_{n-l} \end{bmatrix}$$

where $C = \begin{bmatrix} c_{ji} \end{bmatrix}$ is the matrix of size $(n-l) \times l$ defined in (6). Note that

$$U^{*-1} = \begin{bmatrix} I_l & C^* \\ 0 & I_{n-l} \end{bmatrix}^{-1} = \begin{bmatrix} I_l & -C^* \\ 0 & I_{n-l} \end{bmatrix}.$$

It follows from (8) and (9) that

$$\begin{bmatrix} x_1 \cdots x_n \end{bmatrix} = \begin{bmatrix} x_1 \cdots x_l \; 0 \cdots 0 \end{bmatrix} \begin{bmatrix} I_l & C \\ 0 & I_{n-l} \end{bmatrix},$$

$$\begin{bmatrix} y_1 \cdots y_n \end{bmatrix} = \begin{bmatrix} 0 \cdots 0 \; y_{l+1} \cdots y_n \end{bmatrix} \begin{bmatrix} I_l & 0 \\ -C^* & I_{n-l} \end{bmatrix}.$$

In the general case where $\{x_{n_i}, 1 \le i \le l\}$ are linearly independent, the proof can be done by using a permutation matrix. □

4 Products of Block Toeplitz and Hankel Operators

The following is one of our main results and also has three versions.

Theorem 4.1 (First Version) *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then $H^*_{\Phi^*} H_\Psi = 0$ if and only if there exist $\widehat{\Phi}(z) \in L^\infty_{m\times l}$ and $\widehat{\Psi}(z) \in L^\infty_{(n-l)\times k}$ for some $0 \le l \le n$ and two constant matrices $A \in M_{l\times n}$ and $B \in M_{n\times(n-l)}$ such that*

$$\Phi(z) - \widehat{\Phi}(z)A \in \overline{H^\infty_{m\times n}}, \quad \Psi(z) - B\widehat{\Psi}(z) \in H^\infty_{n\times k} \quad \textit{and} \quad AB = 0.$$

Proof The proof of sufficiency is straightforward. Write $\widehat{\Phi}(z) = \left[\widehat{\varphi}_{ij}(z)\right]_{m\times l} \in L^\infty_{m\times l}$ and $\widehat{\Psi}(z) = \left[\widehat{\psi}_{ij}(z)\right]_{l\times k} \in L^\infty_{m\times l}$. Then $\Phi(z) - \widehat{\Phi}(z)A \in \overline{H^\infty_{m\times n}}$ and $\Psi(z) - B\widehat{\Psi}(z) \in H^\infty_{n\times k}$ imply that

$$H^*_{\Phi^*} = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right]_{m\times l} A \quad \text{and} \quad H_\Psi = B\left[H_{\widehat{\psi}_{ij}(z)}\right]_{l\times k}.$$

Therefore

$$H^*_{\Phi^*} H_\Psi = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right] AB \left[H_{\widehat{\psi}_{ij}(z)}\right] = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right] \cdot 0 \cdot \left[H_{\widehat{\psi}_{ij}(z)}\right] = 0.$$

The proof of the necessity is more involved, but the main part has already been done in Lemmas 2.4 and 3.1. By Lemma 2.4, $H^*_{\Phi^*} H_\Psi = 0$ implies that

$$\sum_{i=1}^n H^*_{\Phi^*} e_i \otimes H^*_\Psi e_i = \sum_{i=1}^n H_{\widetilde{\Phi}^*} e_i \otimes H_{\widetilde{\Psi}} e_i = 0. \tag{12}$$

By Lemma 3.1, there exist matrices $C_{l\times n}$ and $D_{(n-l)\times n}$ such that $CD^* = 0$ and

$$\begin{bmatrix} H_{\widetilde{\Phi}^*} e_1 \cdots H_{\widetilde{\Phi}^*} e_n \end{bmatrix} = \begin{bmatrix} u_1(z) \cdots u_l(z) \end{bmatrix} C \quad = U(z)C, \tag{13}$$

$$\begin{bmatrix} H_{\widetilde{\Psi}} e_i \cdots H_{\widetilde{\Psi}} e_i \end{bmatrix} = \begin{bmatrix} w_{l+1}(z) \cdots w_n(z) \end{bmatrix} D = W(z)D \tag{14}$$

for some $U(z) \in \overline{H^2_{m\times l}}$, $W(z) \in \overline{H^2_{(n-l)\times k}}$. Since H_Ψ only depends on $\Psi_-^*(z)$, without loss of generality, assume $\Phi^* = \Phi_+^*$ and $\Psi = \Psi_-^*(z)$. Write

$$\widetilde{\Phi^*} = \widetilde{\Phi_+^*} = \left[\varphi_{ij}\right]_{m\times n}, \quad \widetilde{\Psi(z)} = \widetilde{\Psi_-^*(z)} = \left[\psi_{ij}\right]_{k\times n}.$$

Then

$$\left[\, H_{\widetilde{\Phi^*}} e_1 \cdots H_{\widetilde{\Phi^*}} e_n \,\right] = \widetilde{\Phi_+^*} \quad \text{and}$$
$$\left[\, H_{\widetilde{\Psi}} e_1 \cdots H_{\widetilde{\Psi}} e_n \,\right] = \widetilde{\Psi_-^*(z)}.$$

Now (13) and (14) becomes

$$\widetilde{\Phi_+^*} = U(z)C \quad \text{or} \quad \Phi_+ = U(\overline{z})C$$
$$\widetilde{\Psi_-^*(z)} = W(z)D \quad \text{or} \quad \Psi_-^*(z) = D^*\widetilde{W(z)}$$

for some $U(\overline{z}) \in zH^2_{m\times l}$ and $\widetilde{W(z)} \in \overline{zH^2_{(n-l)\times k}}$. In fact it follows from the proof of Lemma 3.1 that $U(\overline{z})$ are some columns of Φ_+ and $\widetilde{W(z)}$ are some columns of $\Psi_-^*(z)$. So $U(\overline{z})$ is the analytic part of some function in $L^\infty_{m\times l}$ and $\widetilde{W(z)}$ is the co-analytic part of some function in $L^\infty_{n\times k}$. Equivalently, we have $\Phi(z) - \widehat{\Phi}(z)A \in \overline{H^\infty_{m\times n}}$, $\Psi(z) - B\widehat{\Psi}(z) \in H^\infty_{n\times k}$ for some $\widehat{\Phi}(z) \in L^\infty_{m\times l}$ and $\widehat{\Psi}(z) \in L^\infty_{(n-l)\times k}$, where $A = C$, $B = D^*$, and $AB = CD^* = 0$. The proof is complete. □

Next we emphasize the cases $l = 0$ and $l = n$ in the above theorem. We need the following definition.

Definition 4.2 Let $\Phi(z) \in L^2_{m\times n}$. The column rank of $\Phi(z)$ is the dimension of the space spanned by the columns of $\Phi(z)$ as a subspace of L^2_m. The row rank of $\Phi(z)$ is the dimension of the space spanned by the rows of $\Phi(z)$ as a subspace of L^2_n.

In general, there is no relation between the column rank of $\Phi(z)$ and the row rank of $\Phi(z)$. It follows from the proof Lemma 3.1 that l in Theorem 4.1 can be taken as the column rank of Φ_+ or the row rank of $\Psi_-^*(z)$.

Corollary 4.3 *Let $\Phi(z) = \Phi_+(z) + \Phi_0 + \Phi_-^*(z) \in L^\infty_{m\times n}$ and $\Psi(z) = \Psi_+(z) + \Psi_0 + \Psi_-^*(z) \in L^\infty_{n\times k}$. If the column rank of Φ_+ is n, then $H^*_{\Phi^*}H_\Psi = 0$ implies that $H_\Psi = 0$. Similarly if the row rank of $\Psi_-^*(z)$ is n, then $H^*_{\Phi^*}H_\Psi = 0$ implies that $H_{\Phi^*} = 0$.*

Proof We observe that the column rank of Φ_+ is the same as the column rank of $\widetilde{\Phi_+^*}$. If the column rank of $\widetilde{\Phi_+^*}$ is n, then $\left\{ H_{\widetilde{\Phi^*}} e_1 \cdots H_{\widetilde{\Phi^*}} e_n \right\}$ are linearly independent, so Eq. (12) implies that $\left[\, H_{\widetilde{\Psi}} e_1 \cdots H_{\widetilde{\Psi}} e_n \,\right] = \widetilde{\Psi_-^*(z)} = 0$. Thus $H_\Psi = 0$. The proof of the other assertion in this corollary is similar. □

Theorem 4.4 (Second Version) *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then $H^*_{\Phi^*}H_\Psi = 0$ if and only if there exist two constant matrices $A_{n\times n}$ and $B_{n\times n}$ such that*

$$\Phi(z) - \Phi(z)A \in \overline{H^\infty_{m\times n}}, \quad \Psi(z) - B\Psi(z) \in H^\infty_{n\times k} \quad \textit{and} \quad AB = 0.$$

Proof The proof is similar to the proof Theorem 4.1 by using Lemma 3.2. □

Theorem 4.5 (Third Version) *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then $H^*_{\Phi^*}H_\Psi = 0$ if and only if there exist some $0 \le l \le n$ and an invertible constant matrix $U_{n\times n}$ such that*

$$\Phi(z)U^{-1} = \begin{bmatrix} \widehat{\Phi}(z) & F^* \end{bmatrix} \quad \textit{and} \quad U\Psi(z) = \begin{bmatrix} G \\ \widehat{\Psi}(z) \end{bmatrix},$$

where $\widehat{\Phi}(z) \in L^\infty_{m\times l}$, $F^* \in \overline{H^\infty_{m\times(n-l)}}$, $\widehat{\Psi}(z) \in L^\infty_{(n-l)\times k}$ and $G \in H^\infty_{l\times k}$.

Proof The proof of necessity is similar to the proof Theorem 4.1 by using Lemma 3.3. The almost trivial proof of the sufficiency is really the strength of this result. Note that since U is a constant matrix and $H_F = H_G = 0$,

$$H^*_{\Phi^*}H_\Psi = H^*_{(\Phi U^{-1})^*}H_{U\Psi} = \begin{bmatrix} H^*_{(\widehat{\Phi}(z))^*} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ H_{\widehat{\Psi}(z)} \end{bmatrix} = 0$$

The proof is complete. □

Remark 4.6 It follows from the proofs of Lemmas 3.1 and 3.3 that $\widehat{\Phi}(z)$ and $\widehat{\Psi}(z)$ can be taken to be some columns of $\Phi(z)$ and some rows of $\Psi(z)$, respectively.

When $m = k = 1$, we have the following result for a finite sum of product of Hankel operators. The characterizations in Theorem 5 in [18] for the case $k = 1$ are more complicated.

Corollary 4.7 *Let $\varphi_i, \psi_i \in L^\infty$ for $i = 1, \cdots, n$. Then $\sum_{i=1}^n H_{\varphi_i}H_{\psi_i} = 0$ if and only if there exist some $0 \le l \le n$ and an invertible constant matrix $U_{n\times n}$ such that*

$$\begin{bmatrix} H_{\varphi_1} & \cdots & H_{\varphi_n} \end{bmatrix} = \begin{bmatrix} H_{f_1} & \cdots & H_{f_l} & 0 & \cdots & 0 \end{bmatrix} U \quad \textit{and}$$

$$\begin{bmatrix} H_{\psi_1} & \cdots & H_{\psi_n} \end{bmatrix}^T = U^{-1} \begin{bmatrix} 0 & \cdots & 0 & H_{f_{l+1}} & \cdots & H_{f_n} \end{bmatrix}$$

where $f_i \in L^\infty$ for $i = 1, \cdots, n$.

By Proposition 2.3 and the results above, we have the three versions for characterizing when the product of two block Toeplitz operator is another block Toeplitz operator. See also Theorem 6 in [18] for special cases with different characterizations.

Theorem 4.8 (First Version) *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then $T_\Phi T_\Psi$ is another block Toeplitz operator if and only if there exist $\widehat{\Phi}(z) \in L^\infty_{m\times l}$ and $\widehat{\Psi}(z) \in L^\infty_{(n-l)\times k}$ for some $0 \le l \le n$, and two constant matrices $A \in M_{l\times n}$ and $B \in M_{n\times(n-l)}$ such that*

$$\Phi(z) - \widehat{\Phi}(z)A \in \overline{H^\infty_{m\times n}}, \quad \Psi(z) - B\widehat{\Psi}(z) \in H^\infty_{n\times k} \quad \textit{and} \quad AB = 0.$$

Furthermore in this case $T_\Phi T_\Psi = T_{\Phi\Psi}$.

Again we emphasize the cases $l = 0$ and $l = n$ in the above theorem.

Corollary 4.9 *Let $\Phi(z) = \Phi_+(z) + \Phi_0 + \Phi_-^*(z) \in L^\infty_{m\times n}$ and $\Psi(z) = \Psi_+(z) + \Psi_0 + \Psi_-^*(z) \in L^\infty_{n\times k}$. If the column rank of Φ_+ is n, then $T_\Phi T_\Psi$ is another block Toeplitz operator if and only if $\Psi(z) \in H^\infty_{n\times k}$. Similarly if the row rank of $\Psi_-^*(z)$ is n, $T_\Phi T_\Psi$ is another block Toeplitz operator if and only if $\Phi(z) \in \overline{H^\infty_{m\times n}}$.*

The following corollary extends Corollary 2 in [10], which characterizes when T_Φ^{-1} is also a Toeplitz operator.

Corollary 4.10 *Let $\Phi(z) = \Phi_+(z) + \Phi_0 + \Phi_-^*(z) \in L^\infty_{n\times n}$.*

(a) Assume the column rank of Φ_+ is n. If T_Φ is invertible, then a necessary and sufficient condition that T_Φ^{-1} be a Toeplitz operator is that Φ is analytic.
(b) Assume the row rank of Φ_-^* is n. If T_Φ is invertible, then a necessary and sufficient condition that T_Φ^{-1} be a Toeplitz operator is that Φ is co-analytic.

Proof If T_Φ is invertible and $T_\Phi^{-1} = T_{\Psi(z)}$ for some $\Psi(z) \in L^\infty_{n\times n}$. By Corollary 4.9, $\Psi(z) \in H^\infty_{n\times n}$. But $\Phi(z)\Psi(z) = I_n$ (the $n \times n$ identity matrix). Thus $\Phi(z) = \Psi(z)^{-1}$ is also analytic. □

Theorem 4.11 (Second Version) *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then $T_\Phi T_\Psi$ is another Toeplitz operator if and only if there exist two constant matrices $A_{n\times n}$ and $B_{n\times n}$ such that*

$$\Phi(z) - \Phi(z)A \in \overline{H^\infty_{m\times n}}, \quad \Psi(z) - B\Psi(z) \in H^\infty_{n\times k} \quad \textit{and} \quad AB = 0.$$

Proof We would like to give the almost trivial proof of the sufficiency. Write

$$\Phi(z) = \Phi(z)A + F^*, \quad \Psi(z) = B\Psi(z) + G$$

where $F \in H^{\infty}_{m\times n}$ and $G \in H^{\infty}_{n\times k}$. Then

$$\begin{aligned} T_\Phi T_\Psi &= \left(T_{\Phi(z)A} + T_{F^*}\right)\left(T_{B\Psi(z)} + T_G\right) \\ &= T_\Phi T_A T_B T_\Psi + T_{\Phi(z)AG} + T_{F^*B\Psi(z)} + T_{F^*G} \\ &= T_{\Phi(z)AG} + T_{F^*B\Psi(z)} + T_{F^*G}. \end{aligned}$$

The proof is complete. □

Theorem 4.12 (Third Version) *Let $\Phi(z) \in L^{\infty}_{m\times n}$ and $\Psi(z) \in L^{\infty}_{n\times k}$. Then $T_\Phi T_\Psi$ is another Toeplitz operator if and only if there exist some $0 \le l \le n$ and an invertible constant matrix $U_{n\times n}$ such that*

$$\Phi(z)U^{-1} = \begin{bmatrix} \widehat{\Phi}(z) & F^* \end{bmatrix} \quad \text{and} \quad U\Psi(z) = \begin{bmatrix} G \\ \widehat{\Psi}(z) \end{bmatrix},$$

where $\widehat{\Phi}(z) \in L^{\infty}_{m\times l}$, $F^* \in \overline{H^{\infty}_{m\times(n-l)}}$, $\widehat{\Psi}(z) \in L^{\infty}_{(n-l)\times k}$ and $G \in H^{\infty}_{l\times k}$.

Proof We would like to give the almost trivial proof of the sufficiency. Since $F^* \in \overline{H^{\infty}_{m\times(n-l)}}$ and $G \in H^{\infty}_{l\times k}$,

$$\begin{aligned} T_\Phi T_\Psi = T_{\Phi U^{-1}} T_{U\Psi} &= \begin{bmatrix} T_{\widehat{\Phi}(z)} & T_{F^*} \end{bmatrix} \begin{bmatrix} T_G \\ T_{\widehat{\Psi}(z)} \end{bmatrix} \\ &= T_{\widehat{\Phi}(z)} T_G + T_{F^*} T_{\widehat{\Psi}(z)} = T_{\widehat{\Phi}(z)G} + T_{F^*\widehat{\Psi}(z)} = T_{\widehat{\Phi}(z)G + F^*\widehat{\Psi}(z)}. \end{aligned}$$

The proof is complete. □

When $m = k = 1$, we have the following result for a finite sum of products of Toeplitz operators.

Corollary 4.13 *Let $\varphi_i, \psi_i \in L^\infty$ for $i = 1, \cdots, n$. Then $\sum_{i=1}^n T_{\varphi_i} T_{\psi_i}$ is a Toeplitz operator if and only if there exist some $0 \le l \le n$ and an invertible constant matrix $U_{n\times n}$ such that*

$$\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix} = \begin{bmatrix} f_1 & \cdots & f_l & \overline{h_{l+1}} & \cdots & \overline{h_n} \end{bmatrix} U \quad \text{and}$$

$$\begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix}^T = U^{-1} \begin{bmatrix} h_1 & \cdots & h_l & f_{l+1} & \cdots & f_n \end{bmatrix},$$

where $f_i \in L^\infty$ and $h_i \in H^\infty$ for $i = 1, \cdots, n$. In this case $\sum_{i=1}^n T_{\varphi_i} T_{\psi_i} = T_f$, where

$$f = \sum_{i=1}^{l} f_i h_i + \sum_{j=l+1}^{n} \overline{h_j} f_j.$$

Let $\Phi(z) \in L^{\infty}_{m\times n}$ and $\Psi(z) \in L^{\infty}_{n\times k}$. The semi-commutator of T_Φ and T_Ψ is defined to be

$$[T_\Phi, T_\Psi) = T_{\Phi\Psi} - T_\Phi T_\Psi.$$

For a fixed $\Phi(z)$, let

$$S(\Phi, k) = \left\{\Psi \in L^{\infty}_{n\times k} : [T_\Phi, T_\Psi) = 0\right\}.$$

Theorem 4.12 suggests a way of describing the set $S(\Phi, k)$. Let U be an invertible constant matrix $U_{n\times n}$ such that $\Phi(z)U^{-1} = \begin{bmatrix} \Omega(z) & F^*(z) \end{bmatrix}$, where $\Omega(z) \in L^{\infty}_{m\times l}$ and $F^* \in \overline{H^{\infty}_{m\times(n-l)}}$. Furthermore we can assume the column rank of $\Omega_+(z)$ is l. Then

$$S(\Phi, k) = \left\{\Psi = U^{-1}\begin{bmatrix} G(z) \\ \Delta(z) \end{bmatrix} \in L^{\infty}_{n\times k} : G \in H^{\infty}_{l\times k},\ \Delta(z) \in L^{\infty}_{(n-l)\times k}\right\}.$$

To see this, let $\Psi \in S(\Phi, k)$. We write

$$\Psi = U^{-1}\begin{bmatrix} G(z) \\ \Delta(z) \end{bmatrix} \in L^{\infty}_{n\times k}.$$

Then

$$\begin{aligned}
T_{\Phi\Psi} - T_\Phi T_\Psi &= T_{\Phi U U^{-1}\Psi} - T_{\Phi U} T_{U^{-1}\Psi} \\
&= T_{\Omega(z)G(z)} + T_{F^*(z)\Delta(z)} - \begin{bmatrix} T_{\Omega(z)} & T_{F^*(z)} \end{bmatrix}\begin{bmatrix} T_{G(z)} \\ T_{\Delta(z)} \end{bmatrix} \\
&= T_{\Omega(z)G(z)} - T_{\Omega(z)}T_{G(z)} + T_{F^*(z)\Delta(z)} + T_{F^*(z)}T_{\Delta(z)} \\
&= T_{\Omega(z)G(z)} - T_{\Omega(z)}T_{G(z)}.
\end{aligned}$$

Therefore $T_{\Phi\Psi} - T_\Phi T_\Psi = 0$ if and only if $T_{\Omega(z)G(z)} - T_{\Omega(z)}T_{G(z)}$. Since the column rank of $\Omega_+(z)$ is l, by Corollary 4.9, $G \in H^{\infty}_{l\times k}$.

When $\Phi(z)$ and $\Psi(z)$ are scalar functions, by Corollary 1 in [10], $T_\Phi T_\Psi = 0$ if and only if and only either $T_\Phi = 0$ or $T_\Psi = 0$ (there are no zero divisors). In the matrix-valued case, this is not true even when the column rank of Φ_+ is n. The following simple example illustrates this phenomena. Let $f, g \in H^\infty$, then

$$\begin{bmatrix} T_f, T_g \end{bmatrix}\begin{bmatrix} T_g \\ -T_f \end{bmatrix} = T_f T_g - T_g T_f = T_{fg} - T_{gf} = 0.$$

The following result follows from Proposition 2.3 and Theorem 4.12.

Corollary 4.14 *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. Then $T_\Phi T_\Psi = 0$ if and only the following two conditions hold.*

(1) $\Phi\Psi = 0$.
(2) There exist some $0 \le l \le n$ and an invertible constant matrix $U_{n\times n}$ such that

$$\Phi(z)U^{-1} = \begin{bmatrix} \widehat{\Phi}(z) & F^* \end{bmatrix} \quad \text{and} \quad U\Psi(z) = \begin{bmatrix} G \\ \widehat{\Psi}(z) \end{bmatrix},$$

where $\widehat{\Phi}(z) \in L^\infty_{m\times l}$, $F^* \in \overline{H^\infty_{m\times(n-l)}}$, $\widehat{\Psi}(z) \in L^\infty_{(n-l)\times k}$ and $G \in H^\infty_{l\times k}$.

We emphasize the cases $l = 0$ and $l = n$ as a separate result.

Corollary 4.15 *Let $\Phi(z) \in L^\infty_{m\times n}$ and $\Psi(z) \in L^\infty_{n\times k}$. If the column rank of Φ_+ is n, then $T_\Phi T_\Psi = 0$ if and only if $\Psi(z) \in H^\infty_{n\times k}$ and $\Phi\Psi = 0$. Similarly if the row rank of $\Psi^*_-(z)$ is n, then $T_\Phi T_\Psi = 0$ if and only if $\Phi(z) \in \overline{H^\infty_{m\times n}}$ and $\Phi\Psi = 0$.*

5 Commuting Block Toeplitz Operators

In this section we study when two block Toeplitz operators commute (Theorem 5.4). In fact, the commuting problem of two block Toeplitz operators reduces to the commuting problem of four block Hankel operators (Theorem 5.2). We first characterize when four block Toeplitz operators satisfy a commuting relation by a simple and direct application of Theorem 4.8.

It is quite surprising and amazing that we can derive the solution to the commuting problem from the solution of the zero product problem, as in Corollary 4.14, since in literature, these two problems (for the scalar case) are often studied separately. The commuting problem receives more attentions and is usually more difficult. We also studied these two questions separately at the beginning. In fact we studied square matrix-valued functions at first as traditionally done and changed to general matrix-valued functions later on. This is yet another strong reason to study the block Toeplitz and Hankel operators with non square matrix-valued symbols because it could give a unified approach to several related problems. Without much extra work we can state our results for four block Toeplitz operators.

Theorem 5.1 (First Version) *Let $\Phi_1(z) \in L^\infty_{m\times n}$, $\Psi_1(z) \in L^\infty_{n\times k}$, $\Phi_2(z) \in L^\infty_{m\times j}$, and $\Psi_2(z) \in L^\infty_{j\times k}$. Then $T_{\Phi_1}T_{\Psi_1} = T_{\Phi_2}T_{\Psi_2}$ if and only if the following two conditions hold.*

(1) $\Phi_1(z)\Psi_1(z) = \Phi_2(z)\Psi_2(z)$.

(2) There exist $\widehat{\Phi}(z) \in L^\infty_{m\times l}$ *and* $\widehat{\Psi}(z) \in L^\infty_{(n+j-l)\times k}$ *for some* $0 \le l \le n+j$ *and four constant matrices* A, B, C *and* D *such that* $AB = CD$ *and*

$$\Phi_1(z) - \widehat{\Phi}(z)A \in \overline{H^\infty_{m\times n}}, \qquad \Psi_1(z) - B\widehat{\Psi}(z) \in H^\infty_{n\times k} \tag{15}$$

$$\Phi_2(z) - \widehat{\Phi}(z)C \in \overline{H^\infty_{m\times j}}, \qquad \Psi_2(z) - D\widehat{\Psi}(z) \in H^\infty_{j\times k}. \tag{16}$$

Proof Note that $T_{\Phi_1}T_{\Psi_1} = T_{\Phi_2}T_{\Psi_2}$ if and only if

$$T_\Omega T_\Delta = 0 \quad \text{where} \quad \Omega = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}_{m\times(n+j)}, \quad \Delta = \begin{bmatrix} -\Psi_1 \\ \Psi_2 \end{bmatrix}_{(n+j)\times k}.$$

By Theorem 4.8, there exist $\widehat{\Phi}(z) \in L^\infty_{m\times l}$ and $\widehat{\Psi}(z) \in L^\infty_{(n+j-l)\times k}$ for some $0 \le l \le n+j$, and two matrices $U_{m\times l}$ and $V_{(n+j-l)\times k}$ such that $UV = 0$ and

$$\Omega(z) - \widehat{\Phi}(z)U \in \overline{H^\infty_{m\times(n+j)}}, \qquad \Delta(z) - D\widehat{\Psi}(z) \in H^\infty_{(n+j)\times k}. \tag{17}$$

Write

$$U = \begin{bmatrix} A & B \end{bmatrix}, \qquad V = \begin{bmatrix} -C \\ D \end{bmatrix}$$

where $A \in M_{m\times n}$, $B \in M_{m\times j}$, $C \in M_{n\times k}$, $D \in M_{j\times k}$. Then $UV = 0$ is the same as $AB = CD$ and Condition (17) is equivalent to Condition (15) and Condition (16). □

The following is the result for four block Hankel operators.

Theorem 5.2 (First Version) *Let* $\Phi_1(z) \in L^\infty_{m\times n}$, $\Psi_1(z) \in L^\infty_{n\times k}$, $\Phi_2(z) \in L^\infty_{m\times j}$, $\Psi_2(z) \in L^\infty_{j\times k}$. *Then* $H^*_{\Phi_1^*}H_{\Psi_1} = H^*_{\Phi_2^*}H_{\Psi_2}$ *if and only if there exist* $\widehat{\Phi}(z) \in L^\infty_{m\times l}$ *and* $\widehat{\Psi}(z) \in L^\infty_{(n+j-l)\times k}$ *for some* $0 \le l \le n+j$ *and four constant matrices* A, B, C *and* D *such that* $AB = CD$ *and*

$$\Phi_1(z) - \widehat{\Phi}(z)A \in \overline{H^\infty_{m\times n}}, \qquad \Psi_1(z) - B\widehat{\Psi}(z) \in H^\infty_{n\times k} \tag{18}$$

$$\Phi_2(z) - \widehat{\Phi}(z)C \in \overline{H^\infty_{m\times j}}, \qquad \Psi_2(z) - D\widehat{\Psi}(z) \in H^\infty_{j\times k}. \tag{19}$$

Proof We prove the sufficiency. Write

$$\widehat{\Phi}(z) = \left[\widehat{\varphi}_{ij}(z)\right]_{m\times l}, \qquad \widehat{\Psi}(z) = \left[\widehat{\psi}_{ij}(z)\right]_{(n+j-l)\times k},$$

where $\alpha_{ij}(z)$ and $\beta_{ij}(z)$ are scalar L^∞ functions. Conditions (18) and (19) imply that

$$H^*_{\Phi_1^*} = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right]_{m\times l} A \quad \text{and} \quad H_{\Psi_1} = B\left[H_{\widehat{\psi}_{ij}(z)}\right]_{l\times k},$$
$$H^*_{\Phi_2^*} = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right]_{m\times l} C \quad \text{and} \quad H_{\Psi_2} = D\left[H_{\widehat{\psi}_{ij}(z)}\right]_{l\times k}.$$

Therefore

$$H^*_{\Phi_1^*}H_{\Psi_1} = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right] AB \left[H_{\widehat{\psi}_{ij}(z)}\right] = \left[H^*_{\overline{\widehat{\varphi}_{ij}(z)}}\right] CD \left[H_{\widehat{\psi}_{ij}(z)}\right] = H^*_{\Phi_2^*}H_{\Psi_2}.$$

The proof is complete. □

In fact Theorem 5.1 follows from Theorem 5.2 by the following proposition, which in turn follows from Lemma 2.1 and Proposition 2.3. However, we consider Theorem 5.2 to be satisfactory since four constant matrices $A, B, C,$ and D are arbitrary except the condition $AB = CD$, while Theorem 5.1 can still be improved because besides $AB = CD$, Condition (1) in Theorem 5.1 also imposes some restrictions on A, B, C and D. The l in the theorem above can be taken as the column rank of $\left[\, \Phi_{1+} \ \Phi_{2+} \,\right]$.

Proposition 5.3 *Let $\Phi_1(z) \in L^\infty_{m\times n}, \Psi_1(z) \in L^\infty_{n\times k}, \Phi_2(z) \in L^\infty_{m\times j}, \Psi_2(z) \in L^\infty_{j\times k}$. Then $T_{\Phi_1}T_{\Psi_1} = T_{\Phi_2}T_{\Psi_2}$ if and only if $\Phi_1(z)\Psi_1(z) = \Phi_2(z)\Psi_2(z)$ and $H^*_{\Phi_1^*}H_{\Psi_1} = H^*_{\Phi_2^*}H_{\Psi_2}$.*

We now study when two block Toeplitz operators commute. We first recall terminology, which is one of the motivations behind the study of commuting two block Toeplitz operators.

Let $\Phi(z) \in L^\infty_{n\times n}$ and $\Psi(z) \in L^\infty_{n\times n}$ be two square matrix-valued functions. Recall that the commutator of two Toeplitz operators of T_Φ and T_Ψ is defined to be

$$[T_\Phi, T_\Psi] = T_\Phi T_\Psi - T_\Psi T_\Phi.$$

Note that

$$[T_\Phi, T_\Psi] = T_\Phi T_\Psi - T_\Psi T_\Phi = T_\Omega T_\Delta \quad \text{where} \ \ \Omega = \left[\, \Phi \ \Psi \,\right], \ \ \Delta = \begin{bmatrix} -\Psi \\ \Phi \end{bmatrix}.$$

The commutator $[T_\Phi, T_\Psi]$ is a useful tool for studying Toeplitz operators [9]. Write

$$\Phi(z) = \Phi_+(z) + \Phi_0 + \Phi_-^*(z), \qquad \Psi(z) = \Psi_+(z) + \Psi_0 + \Psi_-^*(z).$$

Theorem 5.4 *Let $\Phi(z) \in L^\infty_{n\times n}$ and $\Psi(z) \in L^\infty_{n\times n}$. Then $T_\Phi T_\Psi = T_\Psi T_\Phi$ if and only the following two conditions hold.*

(1) $\Phi\Psi = \Psi\Phi$.
(2) *There exist* $\widehat{\Phi}(z) \in zH^2_{n\times l}$ *and* $\widehat{\Psi}(z) \in \overline{zH^2_{(2n-l)\times n}}$ *for some* $0 \le l \le 2n$ *and four constant matrices* A, B, C *and* D *such that* $AB = CD$

$$\Phi_+(z) = \widehat{\Phi}(z)A, \qquad \Phi^*_-(z) = D\widehat{\Psi}(z),$$
$$\Psi_+(z) = \widehat{\Phi}(z)C, \qquad \Psi^*_-(z) = B\widehat{\Psi}(z).$$

Proof The result follows from Theorem 5.1 or Theorem 5.2 by noting that

$$\begin{aligned}[T_\Phi, T_\Psi] &= T_\Phi T_\Psi - T_\Psi T_\Phi \\ &= T_\Phi T_\Psi - T_{\Phi\Psi} + T_{\Psi\Phi} - T_\Psi T_\Phi + T_{\Phi\Psi-\Psi\Phi} \\ &= -H^*_{\Phi^*} H_\Psi + H^*_{\Psi^*} H_\Phi + T_{\Phi\Psi-\Psi\Phi},\end{aligned} \tag{20}$$

and $H^*_{\Phi^*} H_\Psi = H^*_{\Phi^*} H_\Psi$ is equivalent to $H^*_{\Phi^*_+} H_{\Psi^*_-} = H^*_{\Psi^*_+} H_{\Phi^*_-}$. □

We remark different (and more complicated) characterizations of $T_\Phi T_\Psi = T_\Psi T_\Phi$ were also given in [18].

The cases $l = 0$ and $l = 2n$ lead to the following result.

Corollary 5.5 *Let* $\Phi(z) \in L^\infty_{n\times n}$ *and* $\Psi(z) \in L^\infty_{n\times n}$.

(a) If the column rank of $\begin{bmatrix} \Phi_+ & \Psi_+ \end{bmatrix}$ *is* $2n$, *then* $T_\Phi T_\Psi = T_\Psi T_\Phi$ *if and only if* $\Phi\Psi = \Psi\Phi$ *and* $\Phi(z) \in H^\infty_{n\times n}$ *and* $\Psi(z) \in H^\infty_{n\times n}$. *In this case* $T_\Phi T_\Psi = T_{\Phi\Psi} = T_{\Psi\Phi} = T_\Psi T_\Phi$.
(b) If the row rank of $\begin{bmatrix} -\Psi^*_- \\ \Phi^*_- \end{bmatrix}$ *is* $2n$, *then* $T_\Phi T_\Psi = T_\Psi T_\Phi$ *if and only if* $\Phi\Psi = \Psi\Phi$ *and* $\Phi(z) \in \overline{H^\infty_{n\times n}}$ *and* $\Psi(z) \in \overline{H^\infty_{n\times n}}$. *In this case* $T_\Phi T_\Psi = T_{\Phi\Psi} = T_{\Psi\Phi} = T_\Psi T_\Phi$.

Recall in the scalar case, Theorem 9 in [10] says except the two scenarios for $T_\Phi T_\Psi = T_\Psi T_\Phi$ described in the corollary above, the only other option is that $\Psi = \lambda\Phi$. To capture this case for block Toeplitz operators, we need a lemma for finite rank operators. This lemma is used implicitly in the proof of Theorem 3.2 in [14] and can be proved by a similar method as Lemma 3.1.

Lemma 5.6 *Let* $x_i \in H, y_i \in K$ *for* $1 \le i \le n$ *and* $w_j, z_j \in K$ *for* $1 \le j \le k$. *If* $\{x_i, 1 \le i \le n\}$ *are linearly independent and* $\{z_j, 1 \le j \le k\}$ *are linearly independent, then*

$$\sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{k} w_j \otimes z_j \tag{21}$$

if and only if there exists a matrix $A_{k\times n}$ such that

$$\begin{bmatrix} y_1 \cdots y_n \end{bmatrix} = \begin{bmatrix} z_1 \cdots z_k \end{bmatrix} A, \qquad \begin{bmatrix} w_1 \cdots w_k \end{bmatrix} = \begin{bmatrix} x_1 \cdots x_n \end{bmatrix} A^*.$$

The following theorem is a refinement of Theorem 5.2. In the square matrix case ($m = n = j = k$), this result was proved in Theorem 7 [14].

Theorem 5.7 *Let $\Phi_1(z) \in L^\infty_{m\times n}$, $\Psi_1(z) \in L^\infty_{n\times k}$, $\Phi_2(z) \in L^\infty_{m\times j}$, $\Psi_2(z) \in L^\infty_{j\times k}$. If the column rank of Φ_{1+} is n and the row rank of Ψ^*_{2-} is j, then $H^*_{\Phi_1^*} H_{\Psi_1} = H^*_{\Phi_2^*} H_{\Psi_2}$ if and only if there exists a constant matrix $A_{j\times n}$ such that $\Phi_2 - \Phi_1 A^* \in \overline{H^\infty_{m\times j}}$ and $\Psi_1 - A^*\Psi_2 \in H^\infty_{m\times j}$.*

Proof The proof is similar to the proof of Theorem 4.1. We include the short proof for clarity. Again the proof of sufficiency is meant to be straightforward. Write $\Phi_{1+}(z) = [\varphi_{il}(z)]_{m\times n}$ and $\Psi^*_{2-}(z) = [\psi_{il}(z)]_{j\times k}$, then $\Phi_{2+} = \Phi_{1+}A^*$ and $\Psi^*_{1-}(z) = A^*\Psi^*_{2-}$. Thus

$$H^*_{\Phi_1^*} H_{\Psi_1} = H^*_{\Phi^*_{1+}} H_{\Psi^*_{1-}} = \left[H^*_{\overline{\varphi_{il}(z)}} \right] A^* \left[H_{\psi_{il}(z)} \right],$$

$$H^*_{\Phi_2^*} H_{\Psi_2} = H^*_{\Phi^*_{2+}} H_{\Psi^*_{2-}} = \left[H^*_{\overline{\varphi_{il}(z)}} \right] A^* \left[H_{\psi_{il}(z)} \right].$$

Now we prove the necessity. Without loss of generality, assume $\Phi_1^* = \Phi^*_{1+}$, $\Phi_2^* = \Phi^*_{2+}$, $\Psi_1 = \Psi^*_{1-}(z)$ and $\Psi_2 = \Psi^*_{2-}(z)$. Let $\{e_i\}_{i=1}^n$ and $\{f_l\}_{l=1}^j$ be the standard bases of $\mathbb{C}^n$ and $\mathbb{C}^j$ respectively. By Lemma 2.4, $H^*_{\Phi_1^*} H_{\Psi_1^*} = H^*_{\Phi_2} H_{\Psi_2^*}$ implies that

$$\sum_{i=1}^n H^*_{\Phi_1^*} e_i \otimes H^*_{\Psi_1} e_i = \sum_{l=1}^j H^*_{\Phi_2^*} f_l \otimes H^*_{\Psi_2} f_l.$$

Equivalently

$$\sum_{i=1}^n H_{\widetilde{\Phi_1^*}} e_i \otimes H_{\widetilde{\Psi_1}} e_i = \sum_{l=1}^j H_{\widetilde{\Phi_2^*}} f_l \otimes H_{\widetilde{\Psi_2}} f_l.$$

Since the column rank of Φ_1 is n and the row rank of Ψ_2 is j, $\left\{ H_{\widetilde{\Phi_1^*}} e_i, 1 \le i \le n \right\}$ are linearly independent and $\left\{ H_{\widetilde{\Psi_2}} f_l, 1 \le l \le j \right\}$ are linearly independent. By Lemma 5.6, there exist a constant matrix $A_{j\times n}$ such that

$$\left[H_{\widetilde{\Psi_1^*}} e_1 \cdots H_{\widetilde{\Psi_1^*}} e_n \right] = \left[H_{\widetilde{\Psi_2}} f_1 \cdots H_{\widetilde{\Psi_2}} f_l \right] A,$$

$$\left[H_{\widetilde{\Phi_2^*}} f_1 \cdots H_{\widetilde{\Phi_2^*}} f_n \right] = \left[H_{\widetilde{\Phi_1^*}} e_1 \cdots H_{\widetilde{\Phi_1^*}} e_n \right] A^*.$$

Equivalently

$$\widetilde{\Psi_1} = \widetilde{\Psi_2} A \qquad \text{or} \qquad \Psi_1 = A^* \Psi_2$$

$$\widetilde{\Phi_2^*(z)} = \widetilde{\Phi_1^*(z)} A^* \quad \text{or} \quad \Phi_2(z) = \Phi_1 A^*.$$

The proof is complete. □

Remark 5.8 Without the assumption on the column rank of Φ_{1+} and the row rank of Ψ_{2-}^*, the theorem above does not hold, and this can be seen from Theorem 5.2. It seems four matrices A, B, C and D are needed in Theorem 5.2.

Corollary 5.9 *Let $\varphi_i, \psi_i \in L^\infty$ for $i = 1, \cdots, n$ and $f_j, g_j \in L^\infty$ for $j = 1, \cdots, k$. Assume $\{\varphi_{i+}, i = 1, \cdots, n\}$ are linearly independent and $\{g_{j-}, j = 1, \cdots, k\}$ are linearly independent. Then $\sum_{i=1}^n H_{\overline{\varphi_i}}^* H_{\psi_i} = \sum_{j=1}^k H_{\overline{f_j}}^* H_{g_j}$ if and only if there exist a matrix $A_{k\times n}$ such that*

$$\left[\, H_{\psi_1} \cdots H_{\psi_n} \,\right] = A^* \left[\, H_{g_1} \cdots H_{g_k} \,\right],$$

$$\left[\, H_{\overline{f_1}} \cdots H_{\overline{f_k}} \,\right] = \left[\, H_{\overline{\varphi_1}} \cdots H_{\overline{\varphi_n}} \,\right] A^T,$$

where A^T is the transpose of A.

Corollary 5.10 *Let $\varphi_i, \psi_i \in L^\infty$ for $i = 1, \cdots, n$ and $f_j, g_j \in L^\infty$ for $j = 1, \cdots, k$. Assume $\{\varphi_{i+}, i = 1, \cdots, n\}$ are linearly independent and $\{g_{j-}, j = 1, \cdots, k\}$ are linearly independent. Then $\sum_{i=1}^n T_{\varphi_i} T_{\psi_i} = \sum_{j=1}^k T_{f_i} T_{g_i}$ if and only if $\sum_{i=1}^n \varphi_i \psi_i = \sum_{j=1}^k f_j g_j$ and there exist a matrix $A_{k\times n}$ such that*

$$\left[\, \psi_1 \cdots \psi_n \,\right] - A^* \left[\, g_1 \cdots g_k \,\right] \in H^\infty_{n\times 1},$$

$$\left[\, \overline{f_1} \cdots \overline{f_k} \,\right] - \left[\, \overline{\varphi_1} \cdots \overline{\varphi_n} \,\right] A^T \in H^\infty_{k\times 1}.$$

The following result for commuting block Toeplitz operators in the scalar case reduces to Theorem 9 in [10]. It follows immediately from Theorem 5.7 and Proposition 5.3.

Theorem 5.11 *Let $\Phi(z) \in L^\infty_{n\times n}$ and $\Psi(z) \in L^\infty_{n\times n}$. If the column rank of Φ_+ is n and the row rank of Φ_- is n, then $T_\Phi T_\Psi = T_\Psi T_\Phi$ if and only if $\Phi\Psi = \Psi\Phi$ and there exists a constant matrix $A_{n\times n}$ such that $\Psi_-^* = A\Phi_-^*$ and $\Psi_+ = \Phi_+ A$.*

6 Unbounded Block Toeplitz and Block Hankel Operators

In this section we show that results of previous sections are also valid for unbounded block Toeplitz and block Hankel operators. Let P_n be the set of analytic polynomial in H_n^2, that is,

$$P_n = \left\{ p(z) \in H_n^2 : p(z) = \sum_{i=0}^{m} c_i z^i,\ c_i \in \mathbb{C}^n,\ m \geq 0 \right\}.$$

Then P_n is dense in H_n^2. Let $\Phi(z) \in L^2_{m\times n}$, $\Psi(z) \in L^2_{n\times k}$ and $\Omega(z) \in L^2_{m\times k}$. Then T_Φ and T_Ψ are defined on P_n and P_k respectively, however $T_\Phi T_\Psi$ is not necessary defined on P_k. As in [15] (where unbounded Hankel operators with scalar symbols were discussed), we can introduce bilinear forms $B_{\Phi,\Psi}$ and D_Ω defined on $P_k \times P_m$ by

$$B_{\Phi,\Psi}(f, g) = \langle T_\Psi f, T_{\Psi^*} g\rangle_{H_m^2}, \quad f \in P_k,\ g \in P_m, \quad \text{and}$$
$$D_\Omega(f, g) = \langle T_\Omega, g\rangle_{H_m^2}, \qquad f \in P_k,\ g \in P_m.$$

To be rigorous, we will carefully state and prove a couple of results. We first prove the analogue of Proposition 2.3.

Proposition 6.1 *Let $\Phi(z) \in L^2_{m\times n}$ and $\Psi(z) \in L^2_{n\times k}$. If*

$$\langle T_\Phi f, T_{\Psi^*} g\rangle_{H_m^2} = \langle T_\Omega f, g\rangle_{H_m^2}, \quad f \in P_k,\ g \in P_m,$$

for some $\Omega \in L^2_{m\times k}$, then $\Omega = \Phi\Psi$.

Proof Write $z = e^{i\theta}$,

$$\Phi(e^{i\theta}) = \sum_{j=-\infty}^{\infty} \Phi_j z^j, \quad \Psi(e^{i\theta}) = \sum_{j=-\infty}^{\infty} \Psi_j z^j, \quad \Omega(e^{i\theta}) = \sum_{j=-\infty}^{\infty} \Omega_j z^j.$$

By definition, for $f = z^{u+l}c \in P_k$, $g = z^{v+l}d \in P_m$ where $c \in \mathbb{C}^k$, $d \in \mathbb{C}^m$ and u and v are nonnegative integers,

$$0 = \langle T_\Omega f, g\rangle_{H_m^2} - \langle T_\Phi f, T_{\Psi^*} g\rangle_{H_m^2} = \langle P[\Omega f], g\rangle_{H_m^2} - \left\langle P[\Psi f], \Phi^* g\right\rangle_{L_m^2}$$
$$= \left\langle \sum_{j=-u-l}^{\infty} \Omega_j c z^{j+u+l}, z^{v+l} d \right\rangle_{H_m^2} - \left\langle \sum_{j=-u-l}^{\infty} \Psi_j c z^{j+u+l}, \sum_{j=\infty}^{\infty} \Phi_j^* z^{-j+v+l} d \right\rangle_{H_m^2}$$

$$= \langle \Omega_{v-u} c, d \rangle_{\mathbb{C}^m} - \sum_{j=-u-l}^{\infty} \left\langle \Psi_j c, \Phi_{v-u-j}^* d \right\rangle_{\mathbb{C}^m}$$

$$= \langle \Omega_{v-u} c, d \rangle_{\mathbb{C}^m} - \sum_{j=-u-l}^{\infty} \left\langle \Phi_{v-u-j} \Psi_j c, d \right\rangle_{\mathbb{C}^m} .$$

Let $l \to \infty$, we have

$$\langle \Omega_{v-u} c, d \rangle_{\mathbb{C}^m} - \sum_{j=-\infty}^{\infty} \left\langle \Phi_{v-u-j} \Psi_j c, d \right\rangle_{C^m} = 0.$$

Since $c \in \mathbb{C}^k$ and $d \in \mathbb{C}^m$ are arbitrary, therefore

$$\Omega_{v-u} = \sum_{j=-\infty}^{\infty} \Phi_{v-u-j} \Psi_j \quad \text{and} \quad \Omega = \Phi \Psi. \tag{22}$$

The proof is complete. □

Remark 6.2 We note that $\Phi\Psi$ does not necessarily belong to $L^2_{m\times k}$ if $\Phi(z) \in L^2_{m\times n}$ and $\Psi(z) \in L^2_{n\times k}$. Thus in the above proposition we actually proved $\Phi\Psi \in L^2_{m\times k}$. We also remark that the infinite series in (22) is convergent by the assumption $\Phi(z) \in L^2_{m\times n}$ and $\Psi(z) \in L^2_{n\times k}$.

Even though, H_Ψ only depends on the co-analytic part $\Psi^*_-(z)$, it is cumbersome that we have to assume $\Psi \in L^\infty_{n\times k}$ before. Now we can just assume $\Psi(z) \in \overline{zH^2_{n\times k}}$. The analogue of Lemma 2.4 can be proved similarly.

Lemma 6.3 *Let $\Phi(z) \in zH^2_{m\times n}$ and $\Psi(z) \in \overline{zH^2_{n\times k}}$. Then for $f \in P_k$, $g \in P_m$,*

$$\langle H_\Psi f, H_{\Phi^*} g \rangle_{H^2_n} - \langle H_\Psi z f, H_{\Phi^*} z g \rangle_{H^2_n} = \sum_{i=1}^{n} \langle H_\Psi f, e_i \rangle_{H^2_n} \langle e_i, H_{\Phi^*} g \rangle_{H^2_n}$$
$$= \left\langle \left(\sum_{i=1}^{n} H^*_{\Phi^*} e_i \otimes H^*_\Psi e_i \right) f, g \right\rangle.$$

We now state a couple of sample results for unbounded block Toeplitz and Hankel operators.

Theorem 6.4 *Let $\Phi(z) \in zH^2_{m\times n}$ and $\Psi(z) \in \overline{zH^2_{n\times k}}$. Then*

$$\langle H_\Psi f, H_{\Phi^*} g \rangle_{H^2_n} = 0$$

for all $f \in P_k, g \in P_m$ *if and only if there exist* $\widehat{\Phi}(z) \in zH^2_{m\times l}$ *and* $\widehat{\Psi}(z) \in zH^2_{(n-l)\times k}$ *for some* $0 \le l \le n$ *and two constant matrices* $A \in M_{l\times n}$ *and* $B \in M_{n\times(n-l)}$ *such that* $\Phi(z) = \widehat{\Phi}(z)A$, $\Psi(z) = B\widehat{\Psi}(z)$ *and* $AB = 0$.

Proof By Lemma 6.3, if $\langle H_\Psi f, H_{\Phi^*} g\rangle_{H^2_n} = 0$ for all $f \in P_k, g \in P_m$, then

$$\left\langle \left(\sum_{i=1}^{n} H^*_{\Phi^*} e_i \otimes H^*_{\Psi} e_i \right) f, g \right\rangle = 0.$$

Note that $\sum_{i=1}^{n} H^*_{\Phi^*} e_i \otimes H^*_{\Psi} e_i$ is a bounded operator, and P_k and P_m are dense in H^2_k and H^2_m. Hence

$$\sum_{i=1}^{n} H^*_{\Phi^*} e_i \otimes H^*_{\Psi} e_i = 0.$$

The rest of proof goes exactly as the proof of Theorem 4.1 by using Lemma 3.1. □

Theorem 6.5 *Let* $\Phi(z) \in L^2_{m\times n}$ *and* $\Psi(z) \in L^2_{n\times k}$. *Then*

$$\langle T_\Phi f, T_{\Psi^*} g\rangle_{H^2_m} = \langle T_\Omega f, g\rangle_{H^2_m}, \quad f \in P_k, \ g \in P_m,$$

for some $\Omega \in L^2_{m\times k}$ *if and only if there exist* $\widehat{\Phi}(z) \in L^2_{m\times l}$ *and* $\widehat{\Psi}(z) \in L^2_{(n-l)\times k}$ *for some* $0 \le l \le n$, *and two constant matrices* $A \in M_{l\times n}$ *and* $B \in M_{n\times(n-l)}$ *such that* $\Phi(z) - \widehat{\Phi}(z)A \in \overline{H^2_{m\times n}}$, $\Psi(z) - B\widehat{\Psi}(z) \in H^2_{n\times k}$, *and* $AB = 0$. *Furthermore, in this case* $\Omega = \Phi\Psi$.

It is natural to ask when the product of two Hankel operators is still a Hankel operator or when a Toeplitz operator and a Hankel operator commute. Indeed, these questions have been studied in literature, for example, see [24] and [35] for complete answers to these two questions in the scalar case on Hardy space and [14] for some partial answers in the matrix-valued case on Hardy space.

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Toeplitz and Related Operators on Polyanalytic Fock Spaces

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In Memory of Harold Widom

Abstract We give a characterization of compact and Fredholm operators on polyanalytic Fock spaces in terms of limit operators. As an application we obtain a generalization of the Bauer–Isralowitz theorem using a matrix valued Berezin type transform. We then apply this theorem to Toeplitz and Hankel operators to obtain necessary and sufficient conditions for compactness. As it turns out, whether or not a Toeplitz or Hankel operator is compact does not depend on the polyanalytic order. For Hankel operators this even holds on the true polyanalytic Fock spaces.

Keywords Polyanalytic Fock space · Toeplitz operators · Hankel operators · Compactness · Essential spectrum

1 Introduction

Polyanalytic functions on $\mathbb{C}$ (also called polyentire functions) are smooth functions $f : \mathbb{C} \to \mathbb{C}$ in the variables z and $\bar{z}$ that satisfy

$$\frac{\partial^n f}{(\partial \bar{z})^n} = 0$$

for some $n \in \mathbb{N}$. They can be represented in the form

$$f(z) = \sum_{j=0}^{n-1} h_j(z)\bar{z}^j,$$

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where the h_j are entire functions. We then say that f is of polyanalytic order at most n. In terms of regularity, polyanalytic functions are somewhere in between (complex) analytic and real analytic functions. They still satisfy a Cauchy type integral equation and are subject to Liouville's theorem, but the maximum principle fails as well as the strong form of the identity theorem. For example, $f(z) = 1 - |z|^2$ defines a polyanalytic function that vanishes on the unit circle and has a maximum at 0; two features a non-zero analytic function cannot have. But of course we still have a weaker form of the identity theorem, which also holds for real analytic functions: A polyanalytic function that vanishes on an open set is equal to 0 everywhere. We refer to [3] for an overview of results on polyanalytic functions.

Polyanalytic functions have been studied for over a century as they naturally appear in the theory of elasticity [15, 20]. However, it was only recently discovered that certain polyanalytic function spaces possess an interesting creation-annihilation structure similar to the quantum harmonic oscillator [23]. Subsequently, several connections to time-frequency analysis, signal processing and quantum mechanics have been found. We refer the interested reader to [1, 22] for now and return to related work after some introductory material.

Let μ denote the Gaussian measure on $\mathbb{C}$ defined by

$$\mathrm{d}\mu(z) = \frac{1}{\pi} e^{-|z|^2} \, \mathrm{d}z. \tag{1}$$

The polyanalytic Fock space $\mathcal{F}_n^2$ is the closed subset of $L^2(\mathbb{C}, \mu)$ consisting of polyanalytic functions of order at most n. For $n = 1$ we of course get the classical Fock space $\mathcal{F}^2 = \mathcal{F}_1^2$ of analytic functions. Vasilevski [23] now observed that the polyanalytic Fock spaces can be decomposed into an orthogonal sum of so-called true polyanalytic Fock spaces

$$\mathcal{F}_n^2 = \bigoplus_{k=1}^{n} \mathcal{F}_{(k)}^2,$$

where $\mathcal{F}_{(k)}^2$ consists of those $f \in L^2(\mathbb{C}, \mu)$ that can be written as

$$f(z) = \frac{1}{(k-1)!} e^{|z|^2} \frac{\partial^{k-1}}{\partial z^{k-1}} \left(e^{-|z|^2} g(z) \right)$$

for an entire function g. Now consider the following operators defined on $\mathcal{F}_n^2$:

$$\mathfrak{a}^\dagger := \left(-\frac{\partial}{\partial z} + \bar{z} \right) \quad \text{and} \quad \mathfrak{a} := \frac{\partial}{\partial \bar{z}}.$$

The operator $\frac{1}{\sqrt{k}}\mathfrak{a}^\dagger$ is an isometric isomorphism between $\mathcal{F}^2_{(k)}$ and $\mathcal{F}^2_{(k+1)}$ with inverse $\frac{1}{\sqrt{k}}\mathfrak{a}$. In particular, $N := \mathfrak{a}^\dagger\mathfrak{a}$ is the counting operator, that is,

$$Nf = kf \quad \text{for } f \in \mathcal{F}^2_{(k+1)},$$

and it holds $[\mathfrak{a}, \mathfrak{a}^\dagger] = I$. Summing up all the true polyanalytic Fock spaces, we obtain $L^2(\mathbb{C}, \mu)$:

$$L^2(\mathbb{C}, \mu) = \bigoplus_{k=1}^{\infty} \mathcal{F}^2_{(k)}; \tag{2}$$

see [23, Corollary 2.4].

Just like $\mathcal{F}^2$, the true polyanalytic Fock spaces are reproducing kernel Hilbert spaces. Their reproducing kernels are given by

$$\begin{aligned} K_{(k)}(z, w) &= \frac{1}{(k-1)!}\left(-\frac{\partial}{\partial \bar{w}} + w\right)^{k-1}\left(-\frac{\partial}{\partial z} + \bar{z}\right)^{k-1} e^{z\bar{w}} \\ &= L^0_{k-1}(|z-w|^2)e^{z\bar{w}}, \end{aligned}$$

where for $\alpha \in \mathbb{N}_0$ the

$$L^\alpha_k(x) := \sum_{j=0}^{k}(-1)^j\binom{k+\alpha}{k-j}\frac{x^j}{j!}$$

are the generalized Laguerre polynomials. Consequently, the orthogonal projection onto $\mathcal{F}^2_{(k)}$ is given by

$$P_{(k)}f(z) = \int_{\mathbb{C}} f(w)L^0_{k-1}(|z-w|^2)e^{z\bar{w}}\,\mathrm{d}\mu(w)$$

for $z \in \mathbb{C}$, $f \in L^2(\mathbb{C}, \mu)$. As $\mathcal{F}^2_n$ is equal to the orthogonal sum of true polyanalytic Fock spaces, the orthogonal projection P_n onto $\mathcal{F}^2_n$ is just the sum of the $P_{(k)}$. Using an identity for Laguerre polynomials, we get

$$P_n f(z) = \int_{\mathbb{C}} f(w)L^1_{n-1}(|z-w|^2)e^{z\bar{w}}\,\mathrm{d}\mu(w)$$

for $z \in \mathbb{C}$, $f \in L^2(\mathbb{C}, \mu)$.

Via the decomposition (2), we can define the isometry

$$\mathfrak{A}^\dagger : L^2(\mathbb{C}, \mu) \to L^2(\mathbb{C}, \mu), \quad \mathfrak{A}^\dagger f = \mathfrak{A}^\dagger \sum_{k=1}^{\infty} f_k := \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\mathfrak{a}^\dagger f_k,$$

where $f_k := P_{(k)} f \in \mathcal{F}^2_{(k)}$ is the k-th component of $f \in L^2(\mathbb{C}, \mu)$. Its adjoint is of course given by

$$\mathfrak{A}\colon L^2(\mathbb{C}, \mu) \to L^2(\mathbb{C}, \mu), \quad \mathfrak{A} f = \mathfrak{A} \sum_{k=1}^{\infty} f_k = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k-1}} \mathfrak{a} f_k.$$

By definition, we have $\mathfrak{A}^\dagger(\mathcal{F}^2_{(k)}) = \mathcal{F}^2_{(k+1)}$, $\mathfrak{A}(\mathcal{F}^2_{(k+1)}) = \mathcal{F}^2_{(k)}$ and $\mathfrak{A}(\mathcal{F}^2_{(1)}) = \{0\}$. In particular, $\mathfrak{A}^\dagger$ and $\mathfrak{A}$ can be seen as the forward and backward shift on $\bigoplus\limits_{k=1}^{\infty} \mathcal{F}^2_{(k)} \cong \ell^2(\mathbb{N}, \mathcal{F}^2)$. It is also clear that $\mathfrak{A}\mathfrak{A}^\dagger = I$ and $\mathfrak{A}^\dagger\mathfrak{A} = (I - P_{(1)})$.

For bounded functions f we can now define polyanalytic Toeplitz and Hankel operators in the usual way:

$$\begin{aligned}
T_{f,(k)}\colon \mathcal{F}^2_{(k)} \to \mathcal{F}^2_{(k)}, &\quad T_{f,(k)} g = P_{(k)}(fg),\\
T_{f,n}\colon \mathcal{F}^2_n \to \mathcal{F}^2_n, &\quad T_{f,n} g = P_n(fg),\\
H_{f,(k)}\colon \mathcal{F}^2_{(k)} \to L^2(\mathbb{C}, \mu), &\quad H_{f,(k)} g = (I - P_{(k)})(fg),\\
H_{f,n}\colon \mathcal{F}^2_n \to L^2(\mathbb{C}, \mu), &\quad H_{f,n} g = (I - P_n)(fg).
\end{aligned}$$

For $k = 1$ (or equivalently $n = 1$) we just get the usual Toeplitz and Hankel operators on the standard analytical Fock space $\mathcal{F}^2 = \mathcal{F}^2_1 = \mathcal{F}^2_{(1)}$.

Shortly after Axler and Zheng [2] proved a similar result for the Bergman space over the unit disk, Engliš [8] showed that a Toeplitz operator on $\mathcal{F}^2$ is compact if and only if its Berezin transform vanishes at infinity. In fact, in both cases this is true not only for Toeplitz operators but for any finite sum of finite products of Toeplitz operators. This was later generalized to what we shall call the Bauer–Isralowitz theorem: A bounded linear operator on $\mathcal{F}^2$ is compact if and only if it is in the C^*-algebra generated by all Toeplitz operators and its Berezin transform vanishes at infinity. Motivated by this result, many allegedly larger C^*-algebras of operators, such as the sufficiently and weakly localized operators [14, 26], have been introduced where the same result would hold. However, in 2015 Xia [25] proved the surprising result that all these algebras actually coincide with the closure of the set of all Toeplitz operators. All the different approaches notably had some limit operator type arguments in common, reminiscent of the Fredholm theory of sequence spaces. Consequently, in analogy to the sequence space case, another C^*-algebra, called the band-dominated operators, was introduced in [9], which formalized the limit operator idea. It was later shown in [4] that this would be again the same algebra, but the band-dominated approach of [9] also provided a characterization of Fredholm operators. Moreover, it allowed to study Hankel operators, providing quick proofs for well-known compactness results as well as some new insights [13].

Apart from different algebras, several authors also started considering different domains such as bounded symmetric domains [11] and Bergman-type function spaces [19] just to name a few. The structure of these results is always very similar. In order to be compact, an operator must be contained in a certain C^*-algebra and the Berezin transform must vanish at the boundary of the domain. In this paper we now present an example where the Berezin transform is not strong enough to characterize compactness, even within the reasonable algebra of band-dominated operators, which, as mentioned above, at least in case of the analytic Fock space is just the closure of the set of Toeplitz operators.

In [22], Rozenblum and Vasilevski showed that a Toeplitz operator on a true polyanalytic Fock space $\mathcal{F}^2_{(k)}$ is unitarily equivalent to a Toeplitz operator on $\mathcal{F}^2$, but with possibly very irregular symbol. They then offer the options of either considering Toeplitz operators on 'bad' spaces with 'nice' symbols or Toeplitz operators on 'nice' spaces with 'bad' symbols. Rozenblum and Vasilevski conclude that the second option is more promising. Our (operator algebraic) point of view here is somewhat different as the inner structure of the polyanalytic function spaces does not matter very much in our analysis. Indeed, the band-dominated operators look exactly the same on each true polyanalytic Fock space (in the sense that the algebras are isomorphic) and hence many results can be reduced directly to the analytic case via $\mathfrak{A}$ and $\mathfrak{A}^\dagger$. This also lets us circumvent some of the problems in [18], which prevented a generalization of the Bauer–Isralowitz theorem to polyanalytic Fock spaces. However, the downside is that our construction of the generalized Berezin transform is rather ad-hoc and therefore appears to be less natural. Nevertheless, we manage to obtain a generalization of the Bauer–Isralowitz theorem for polyanalytic Fock spaces using a matrix valued Berezin type transform. It is then not very surprising that our limit operator approach also provides generalizations of other typical results in the area such as the characterization of compact Hankel operators in terms of VMO-functions and the corresponding formula for the essential spectrum of Toeplitz operators. We note that the essential spectrum of polyanalytic Toeplitz operators did not get much attention in the literature so far. The only related (but much weaker) result known to the author is for the polyanalytic Bergman space over the unit disk and due to Wolf [24]. Maybe the most unexpected result of this paper is that the compactness of Hankel operators $H_{f,(k)}$ on $\mathcal{F}^2_{(k)}$ does not depend on the order k. It is somewhat expected from the work of Rozenblum and Vasilevski [22] that if $H_{f,(1)}$ is compact, then all other Hankel operators $H_{f,(k)}$ are compact as well, but the other direction appears to be rather surprising in this context.

2 Properties of $\mathfrak{A}$ and $\mathfrak{A}^\dagger$

We first introduce some more notation that will be needed later on. The algebra of bounded linear operators between two Hilbert spaces H_1 and H_2 will be denoted by $\mathcal{L}(H_1, H_2)$. If $H_1 = H_2$, we will just write $\mathcal{L}(H_1)$. Similarly, we will use $\mathcal{K}(H_1, H_2)$ and $\mathcal{K}(H_1)$ for the compact operators between the respective Hilbert spaces. The identity operator on any Hilbert space will be denoted by I. Furthermore, the characteristic function of a set $K \subseteq \mathbb{C}$ will be denoted by $\mathbb{1}_K$. An open ball in $\mathbb{C}$ with midpoint z and radius r will be denoted by $B(z, r)$.

The first property we are going to state is rather obvious, but still worth noting for later.

Proposition 1 *For every $k \in \mathbb{N}$ we have $\mathfrak{A}^\dagger P_{(k)} \mathfrak{A} = P_{(k+1)}$.*

Proof As $\mathfrak{A}\mathfrak{A}^\dagger = I$, $\mathfrak{A}^\dagger P_{(k)} \mathfrak{A}$ is an orthogonal projection onto $\mathcal{F}^2_{(k+1)}$, hence equal to $P_{(k+1)}$. □

Next, we need the concept of band-dominated operators. The notion originates in the theory of sequence spaces (see e.g. [21]), but has also been introduced to Bergman and Fock spaces a few years ago in order to study compactness and Fredholm problems [9, 10]. A slightly more systematic introduction of band-dominated operators to non-discrete spaces is given in [12].

Definition 2 An operator $T \in \mathcal{L}(L^2(\mathbb{C}, \mu))$ is called a band operator if

$$\sup \left\{ \operatorname{dist}(K, K') : K, K' \subseteq \mathbb{C}, M_{\mathbb{1}_{K'}} T M_{\mathbb{1}_K} \neq 0 \right\} < \infty,$$

where $\operatorname{dist}(K, K') := \inf\limits_{w \in K, z \in K'} |w - z|$ is the distance between the sets K and K'. T is called band-dominated if it is the norm limit of a sequence of band operators. The set of band-dominated operators is denoted by BDO^2. An operator T defined on $\mathcal{F}^2_{(k)}$ or $\mathcal{F}^2_n$ is called band-dominated if $TP_{(k)} \in \mathrm{BDO}^2$ or $TP_n \in \mathrm{BDO}^2$, respectively. The sets of band-dominated operators in $\mathcal{L}(\mathcal{F}^2_{(k)})$ and $\mathcal{L}(\mathcal{F}^2_n)$ will be denoted by $\mathcal{A}^2_{(k)}$ and $\mathcal{A}^2_n$, respectively.

It turns out that the sets BDO^2, $\mathcal{A}^2_{(k)}$ and $\mathcal{A}^2_n$ are actually C^*-algebras that contain all compact operators; see [12, Theorems 3.7 and 3.10]. Luckily, for most integral operators it is relatively straightforward to prove their membership in BDO^2. The following lemma is a special case that essentially follows from Young's inequality.

Lemma 3 *Let ν be the Gaussian measure defined by $\mathrm{d}\nu(z) := \frac{1}{2\pi} e^{-\frac{1}{2}|z|^2} \mathrm{d}z$, $g \in L^1(\mathbb{C}, \nu)$ and*

$$(Tf)(z) := \int_{\mathbb{C}} f(w) g(z - w) e^{z\bar{w}} \, \mathrm{d}\mu(w)$$

for $z \in \mathbb{C}$ and $f \in L^2(\mathbb{C}, \mu)$. Then T defines a bounded linear operator on $L^2(\mathbb{C}, \mu)$, $T \in \mathrm{BDO}^2$ and $\|T\| \le 2 \|g\|_{L^1(\mathbb{C}, \nu)}$.

Proof For $m \ge 0$ we define

$$(T_m f)(z) := \int_{\mathbb{C}} f(w) g(z - w) \mathbb{1}_{B(0,m)}(z - w) e^{z\bar{w}} \, \mathrm{d}\mu(w)$$

for $z \in \mathbb{C}$ and $f \in L^2(\mathbb{C}, \mu)$. This implies

$$\begin{aligned} &|(T - T_m)f(z)|\, e^{-\frac{1}{2}|z|^2} \\ &\qquad \le \frac{1}{\pi} \int_{\mathbb{C}} |f(w)|\, e^{-\frac{1}{2}|w|^2} \mathbb{1}_{\mathbb{C}\setminus B(0,m)}(z-w)\, |g(z-w)|\, e^{-\frac{1}{2}|z-w|^2} \, \mathrm{d}w. \end{aligned}$$

Using Young's inequality, we get

$$\|(T - T_m)f\| \le 2\,\|f\|\, \left\| g \mathbb{1}_{\mathbb{C}\setminus B(0,m)} \right\|_{L^1(\mathbb{C},\nu)}.$$

For $m = 0$ we obtain the boundedness of T and the norm estimate. As the operators T_m are obviously band operators, we also get $T \in \mathrm{BDO}^2$. □

With this lemma we can now prove the following important proposition. We actually do not know if $\mathfrak{A}$ and $\mathfrak{A}^\dagger$ are in BDO^2 themselves, but for our purposes it is sufficient to know $\mathfrak{A}P_{(k)}, \mathfrak{A}^\dagger P_{(k)} \in \mathrm{BDO}^2$.

Proposition 4 *$\mathfrak{A}P_{(k)}$ and $\mathfrak{A}^\dagger P_{(k)}$ are contained in BDO^2 for all $k \in \mathbb{N}$.*

Proof As $\mathfrak{A}^\dagger P_{(k)}$ is the adjoint of $\mathfrak{A}P_{(k+1)}$ and BDO^2 is a C^*-algebra, it suffices to show that $\mathfrak{A}P_{(k)} \in \mathrm{BDO}^2$ for all $k \in \mathbb{N}$. $\mathfrak{A}P_{(1)}$ vanishes, so assume $k \ge 2$. For $k \ge 2$ the operator $\mathfrak{A}P_{(k)}$ can be written as an integral operator:

$$\begin{aligned} \mathfrak{A}P_{(k)}f(z) &= \frac{1}{\sqrt{k-1}} \mathfrak{a} P_{(k)} f(z) \\ &= \frac{1}{\sqrt{k-1}} \frac{\partial}{\partial \bar{z}} \int_{\mathbb{C}} f(w) L_{k-1}^0(|z-w|^2) e^{z\bar{w}} \, \mathrm{d}\mu(w) \\ &= -\frac{1}{\sqrt{k-1}} \int_{\mathbb{C}} f(w)(z-w) L_{k-2}^1(|z-w|^2) e^{z\bar{w}} \, \mathrm{d}\mu(w), \end{aligned}$$

where $f \in L^2(\mathbb{C}, \mu)$, $z \in \mathbb{C}$ and we used $(L_{k-1}^0)' = -L_{k-2}^1$. Choosing $g(z) := -\frac{1}{\sqrt{k-1}} z L_{k-2}^1(|z|^2)$, we obtain $\mathfrak{A}P_{(k)} \in \mathrm{BDO}^2$ by Lemma 3. □

As BDO^2 is a C^*-algebra, we also have $P_{(k)} = \mathfrak{A}P_{(k+1)}\mathfrak{A}^\dagger P_{(k)} \in \mathrm{BDO}^2$. Obviously, one could also check this directly by the same argument as in Proposition 4.

Corollary 5 *$P_{(k)}, P_n \in \mathrm{BDO}^2$ for all $k, n \in \mathbb{N}$.*

As the multiplication operators M_f for $f \in L^\infty(\mathbb{C}, \mu)$ are obviously contained in BDO^2, all the Toeplitz and Hankel operators defined in the introduction are band-dominated.

Corollary 6 *Let $k, n \in \mathbb{N}$ and $f \in L^\infty(\mathbb{C}, \mu)$. Then the operators $T_{f,(k)}$, $T_{f,n}$, $H_{f,(k)}$ and $H_{f,n}$ are band-dominated.*

We also get the following corollary of Proposition 4. It shows that the algebras of band-dominated operators on each true polyanalytic Fock space are isomorphic. This will allow us to jump back and forth between the spaces and, in particular, obtain a generalization of the Bauer–Isralowitz theorem.

Corollary 7 *The C^*-algebras $\mathcal{A}^2_{(k)}$ are isomorphic for $k \in \mathbb{N}$. The isomorphism $\mathcal{A}^2_{(1)} \to \mathcal{A}^2_{(k)}$ is given by $T \mapsto (\mathfrak{A}^\dagger)^{k-1} T \mathfrak{A}^{k-1}$.*

For $z \in \mathbb{C}$ the Weyl operators $W_z \colon L^2(\mathbb{C}, \mu) \to L^2(\mathbb{C}, \mu)$ are defined by

$$(W_z f)(w) = f(w - z) k_z(w),$$

where the k_z denote the normalized reproducing kernels on $\mathcal{F}^2$, that is,

$$k_z(w) := e^{w\bar{z} - \frac{1}{2}|z|^2}.$$

The following properties of Weyl operators are well-known and easy to check: W_z is unitary with $W_z^* = W_{-z}$ and

$$W_z W_w = e^{-i \operatorname{Im}(z\bar{w})} W_{z+w}. \tag{3}$$

Moreover, W_z obviously leaves $\mathcal{F}^2_{(1)}$ invariant. Our next proposition shows that all the true polyanalytic Fock spaces are actually left invariant.

Proposition 8 *We have $\mathfrak{A} W_z = W_z \mathfrak{A}$ and $\mathfrak{A}^\dagger W_z = W_z \mathfrak{A}^\dagger$ for all $z \in \mathbb{C}$. In particular, $\mathcal{F}^2_{(k)}$ is invariant under W_z and we have $P_{(k)} W_z = W_z P_{(k)}$ for all $k \in \mathbb{N}$.*

Proof Let $g \in \mathcal{F}^2_{(k)}$. Then

$$(\mathfrak{a} W_z g)(w) = \frac{\partial}{\partial \bar{w}}\big(g(w - z) k_z(w)\big) = \frac{\partial g}{\partial \bar{w}}(w - z) k_z(w) = (W_z \mathfrak{a} g)(w),$$

hence $\mathfrak{A} W_z = W_z \mathfrak{A}$. Taking adjoints yields the other equality since $W_z^* = W_{-z}$. The second claim follows via Proposition 1 as $\mathcal{F}^2_{(1)}$ is invariant under W_z. □

Another useful property of the Weyl operators is

$$W_{-z} M_f W_z = M_{f(\cdot + z)} \tag{4}$$

for all $f \in L^\infty(\mathbb{C}, \mu)$ and $z \in \mathbb{C}$; see e.g. [9, Lemma 17].

3 Limit Operator Methods

In this section we will briefly recall the limit operator methods developed in [12] and show how they can be applied to operators on polyanalytic Fock spaces. The main idea of [12] was to formulate a collection of very general assumptions that are needed to characterize compact and Fredholm operators in terms of limit operators. The assumptions are as follows.

Assumption 9 (Space) Let (X, d) be a proper metric space of bounded geometry that satisfies property A'. Assume that d is unbounded and let μ be a Radon measure on X.

In our case we have $X = \mathbb{C}$, d is the usual Euclidean metric and μ is the Gaussian measure defined in (1). We will skip the definitions of the more technical terms and just mention that they are more or less trivially satisfied here. For details, we refer to [12, Example 6.5].

Assumption 10 (Subspaces and Projection) Let $p \in (1, \infty)$ and let $M^p \subseteq L^p(X, \mu)$ be a closed subspace with bounded projection $P \in \mathrm{BDO}^p$. Moreover, assume that $M_{\mathbb{1}_K} P$ and $PM_{\mathbb{1}_K}$ are compact for all compact subsets $K \subset X$.

For simplicity we only consider the case $p = 2$ here. M^2 will be any of the polyanalytic Fock spaces $\mathcal{F}^2_{(k)}$ or $\mathcal{F}^2_n$. In each case we choose the orthogonal projection discussed in the introduction for P. For these we have $P \in \mathrm{BDO}^2$ as shown in Corollary 5. That $M_{\mathbb{1}_K} P$ and $PM_{\mathbb{1}_K}$ are compact follows from a Hilbert–Schmidt type argument and will be proven in Theorem 14 below.

Assumption 11 (Shifts) Fix $x_0 \in X$. For $x \in X$ let $\phi_x \colon X \to X$ be a bijective isometry with $\phi_x(x_0) = x$. Assume that $\mu \circ \phi_x \ll \mu \ll \mu \circ \phi_x$ and let h_x be a measurable function such that $|h_x|^p = \frac{\mathrm{d}(\mu \circ \phi_x)}{\mathrm{d}\mu}$ μ-almost everywhere. Assume that the maps $x \mapsto \phi_x(y)$ and $x \mapsto h_x(y)$ are continuous for μ-almost every $y \in X$. For $p \in (1, \infty)$ and $x \in X$ let $U_x^p \colon L^p(X, \mu) \to L^p(X, \mu)$ be defined by $U_x^p f := (f \circ \phi_x) \cdot h_x$ and assume that $x \mapsto M_{\mathbb{1}_K} U_x^p P (U_x^p)^{-1} M_{\mathbb{1}_{K'}}$ extends continuously to the Stone-Čech compactification βX of X for all compact sets $K, K' \subset X$.

We choose $x_0 = 0$ and $\phi_x(z) := z + x$ for each $x \in \mathbb{C}$. For $\mu \circ \phi_x$ we get

$$\mathrm{d}(\mu \circ \phi_x)(z) = \frac{1}{\pi} e^{-|z+x|^2} \,\mathrm{d}z = e^{-2\,\mathrm{Re}(z\bar{x}) - |x|^2} \mathrm{d}\mu(z)$$

and hence $\mu \circ \phi_x \ll \mu \ll \mu \circ \phi_x$ for all $x \in \mathbb{C}$. For h_x we choose $h_x(z) := e^{-z\bar{x} - \frac{1}{2}|x|^2} = k_{-x}(z)$. For U_x^2 we then have

$$(U_x^2 f)(z) = f(z + x) k_{-x}(z) = (W_{-x} f)(z).$$

The last assumption trivially holds as $U_x^2 = W_{-x}$ and P commute in our case by Proposition 8.

We have to admit that in the following the notation is a bit unfortunate as the sign convention is conflicting between [12] and earlier works such as [5] and [9]. We will use the former convention because it makes it compatible with the boundary values of the Berezin transform introduced later and consequently lets some results appear much cleaner (cf. Lemma 18 or Lemma 19 below). Anyway, let $x \in \beta\mathbb{C} \setminus \mathbb{C}$ and choose a net (z_γ) in $\mathbb{C}$ that converges to $x \in \beta\mathbb{C} \setminus \mathbb{C}$. For $H := \mathcal{F}^2_{(k)}$ or $H := \mathcal{F}^2_n$ and $T \in \mathcal{L}(H)$ band-dominated we define

$$T_x g := \lim_{z_\gamma \to x} W_{-z_\gamma} T W_{z_\gamma} g$$

for $g \in H$. This strong limit is guaranteed to exist and independent of the chosen net as shown in [12, Theorem 4.11]. T_x is called a limit operator of $T \in \mathcal{L}(H)$. The main results of [12] are now as follows:

Theorem 12 (Corollary 4.24 of [12]) *Assume that the Assumptions 9–11 are satisfied. Then $K \in \mathcal{L}(M^p)$ is compact if and only if K is band-dominated and $K_x = 0$ for all $x \in \beta X \setminus X$.*

Theorem 13 (Theorem 4.38 of [12]) *Assume that the Assumptions 9–11 are satisfied. Further assume that $T \in \mathcal{L}(M^p)$ is band-dominated. Then T is Fredholm if and only if T_x is invertible for all $x \in \beta X \setminus X$.*

Translated to the case at hand we obtain the following theorem as a special case.

Theorem 14 *Let $H := \mathcal{F}^2_{(k)}$ or $H := \mathcal{F}^2_n$ for some $k, n \geq 1$.*

(a) $K \in \mathcal{L}(H)$ is compact if and only if K is band-dominated and $K_x = 0$ for all $x \in \beta\mathbb{C} \setminus \mathbb{C}$.
(b) Let $T \in \mathcal{L}(H)$ be band-dominated. Then T is Fredholm if and only if T_x is invertible for all $x \in \beta\mathbb{C} \setminus \mathbb{C}$.

Proof We only need to show that the Assumptions 9–11 are satisfied for the true polyanalytic Fock spaces $\mathcal{F}^2_{(k)}$. For $\mathcal{F}^2_n = \bigoplus_{k=1}^{n} \mathcal{F}^2_{(k)}$ the theorem then follows as well because the theory is stable under orthogonal sums (compare with the remark at the end of [12, Example 6.8]). Alternatively, the proof below also works for $\mathcal{F}^2_n$.

So let $H := \mathcal{F}^2_{(k)}$ for some $k \in \mathbb{N}$. After the discussion above, the only condition left to prove is the compactness of $M_{\mathbb{1}_K} P_{(k)}$ and $P_{(k)} M_{\mathbb{1}_K}$ for compact sets $K \subset \mathbb{C}$.

We have

$$P_{(k)} M_{\mathbb{1}_K} f(z) = \int_K f(w) L^0_{k-1}(|z-w|^2) e^{z\bar{w}} \, \mathrm{d}\mu(w)$$

for $f \in L^2(\mathbb{C}, \mu)$, $z \in \mathbb{C}$. This integral operator is Hilbert–Schmidt because

$$\begin{aligned}\int_{\mathbb{C}} \int_K \left| L_{k-1}^0(|z-w|^2) e^{z\bar{w}} \right|^2 \mathrm{d}\mu(w)\, \mathrm{d}\mu(z) \\ &= \frac{1}{\pi^2} \int_{\mathbb{C}} \int_K \left| L_{k-1}^0(|z-w|^2) \right|^2 e^{-|z-w|^2} \,\mathrm{d}w\, \mathrm{d}z \\ &= \frac{1}{\pi^2} \int_{\mathbb{C}} \int_K \left| L_{k-1}^0(|z|^2) \right|^2 e^{-|z|^2} \,\mathrm{d}w\, \mathrm{d}z \\ &< \infty.\end{aligned}$$

A similar argument shows that $M_{\mathbb{1}_K} P_{(k)}$ is also Hilbert–Schmidt. □

4 Generalized Berezin Transforms

On $\mathcal{F}_1^2 = \mathcal{F}^2$ one defines the Berezin transform $\mathcal{B}(T)$ of an operator $T \in \mathcal{L}(\mathcal{F}_1^2)$ as

$$[\mathcal{B}(T)](z) := \langle T k_z, k_z \rangle .$$

From the Bauer–Isralowitz theorem we know that the compact operators on $\mathcal{F}_1^2$ can be characterized via the Berezin transform. Namely, $T \in \mathcal{L}(\mathcal{F}_1^2)$ is compact if and only if T is in the Toeplitz algebra and $\mathcal{B}(T) \in C_0(\mathbb{C})$ [5, Theorem 1.1]. Note that in this case the Toeplitz algebra coincides with the algebra of band-dominated operators $\mathcal{A}_1^2$ [4, Theorem 4.20].

As $\mathcal{F}_n^2$ is also a reproducing kernel Hilbert space, one can of course ask the same question: Can the compact operators be characterized via the Berezin transform? Unfortunately, the answer is no for $n \geq 2$ (at least within $\mathcal{A}_n^2$). Define the normalized reproducing kernels $k_{z,n}$ as usual:

$$k_{z,n}(w) := \frac{K(w,z)}{\|K(\cdot,z)\|} = \frac{1}{\sqrt{n}} L_{n-1}^1(|z-w|^2) e^{w\bar{z} - \frac{1}{2}|z|^2}.$$

Then

$$\left\langle (P_{(1)} - P_{(2)}) k_{z,n}, k_{z,n} \right\rangle = \left\langle P_{(1)} k_{z,n}, k_{z,n} \right\rangle - \left\langle P_{(2)} k_{z,n}, k_{z,n} \right\rangle = \frac{1}{n} - \frac{1}{n} = 0$$

for all $z \in \mathbb{C}$. However, $P_{(1)} - P_{(2)}$ is obviously not compact.

Nevertheless, one can still characterize compactness on $\mathcal{F}_n^2$ with a real analytic function, but it turns out to be matrix-valued. The set of complex $n \times n$ matrices will be denoted by $\mathbb{C}^{n \times n}$.

Definition 15 Let $k, n \in \mathbb{N}$. For $T \in \mathcal{L}(\mathcal{F}^2_{(k)})$ we define

$$\mathcal{B}_{(k)}(T) \colon \mathbb{C} \to \mathbb{C}, \quad [\mathcal{B}_{(k)}(T)](z) := \langle T l_{z,k}, l_{z,k} \rangle$$

and for $T \in \mathcal{L}(\mathcal{F}^2_n)$ we define

$$\mathcal{B}_n(T) \colon \mathbb{C} \to \mathbb{C}^{n \times n}, \quad [\mathcal{B}_n(T)](z) := \begin{pmatrix} \langle T l_{z,1}, l_{z,1} \rangle & \dots & \langle T l_{z,n}, l_{z,1} \rangle \\ \vdots & & \vdots \\ \langle T l_{z,1}, l_{z,n} \rangle & \dots & \langle T l_{z,n}, l_{z,n} \rangle \end{pmatrix},$$

where $l_{z,k} \in \mathcal{F}^2_{(k)}$ is given by $l_{z,k}(w) = \frac{1}{\sqrt{(k-1)!}}(\bar{w} - \bar{z})^{k-1} e^{w\bar{z} - \frac{1}{2}|z|^2}$. We will also use the notation $\widetilde{T}$ for the Berezin transform of $P_1 T|_{\mathcal{F}^2_1} \colon \mathcal{F}^2_1 \to \mathcal{F}^2_1$. For $f \in L^\infty(\mathbb{C}, \mu)$ we will use the abbreviations $\mathcal{B}_n(f) := \mathcal{B}_n(T_{f,n})$, $\mathcal{B}_{(k)}(f) := \mathcal{B}_{(k)}(T_{f,(k)})$ and $\tilde{f} := \mathcal{B}_1(f)$.

Note that $l_{z,k}$ is indeed in $\mathcal{F}^2_{(k)}$ as

$$l_{z,k}(w) = (\mathfrak{A}^\dagger)^{k-1} k_z(w), \tag{5}$$

which is easily seen by induction. These generalized Berezin transforms thus inherit all the basic properties of the usual Berezin transform $\mathcal{B} = \mathcal{B}_1$. In particular, $\mathcal{B}_{(k)} \colon \mathcal{L}(\mathcal{F}^2_{(k)}) \to C(\mathbb{C})$ and $\mathcal{B}_n \colon \mathcal{L}(\mathcal{F}^2_n) \to C(\mathbb{C} \to \mathbb{C}^{n \times n})$ are injective bounded linear operators and their images consist of real analytic, Lipschitz continuous functions.

This also makes it evident why this is suitable to characterize compactness. Namely, $T \in \mathcal{L}(\mathcal{F}^2_n)$ is compact if and only if $P_{(k)} T|_{\mathcal{F}^2_{(j)}} \in \mathcal{L}(\mathcal{F}^2_{(j)}, \mathcal{F}^2_{(k)})$ is compact for all $j, k = 1, \dots, n$, which is equivalent to $P_{(1)} \mathfrak{A}^{k-1} T (\mathfrak{A}^\dagger)^{j-1}|_{\mathcal{F}^2_{(1)}} \in \mathcal{L}(\mathcal{F}^2_{(1)})$ being compact. As the Berezin transform of $P_{(1)} \mathfrak{A}^{k-1} T (\mathfrak{A}^\dagger)^{j-1}|_{\mathcal{F}^2_{(1)}}$ is given by

$$\left\langle P_{(1)} \mathfrak{A}^{k-1} T (\mathfrak{A}^\dagger)^{j-1} k_z, k_z \right\rangle = \langle T l_{z,j}, l_{z,k} \rangle,$$

the following theorem now follows directly from the Bauer–Isralowitz theorem on $\mathcal{F}^2$ [5, Theorem 1.1], [4, Theorem 4.20] mentioned above and Corollary 7.

Theorem 16 *Let $j, k, n \in \mathbb{N}$.*

(a) $T \in \mathcal{L}(\mathcal{F}^2_{(j)}, \mathcal{F}^2_{(k)})$ is compact if and only if $T P_{(j)} \in \mathrm{BDO}^2$ and $z \mapsto \langle T l_{z,j}, l_{z,k} \rangle$ is in $C_0(\mathbb{C})$.

(b) $T \in \mathcal{L}(\mathcal{F}^2_n)$ is compact if and only if $T \in \mathcal{A}^2_n$ and $\mathcal{B}_n(T) \in C_0(\mathbb{C} \to \mathbb{C}^{n \times n})$.

As $\mathfrak{A}^\dagger$ commutes with the Weyl operators (see Proposition 8), $\mathcal{B}_n$ also preserves the shift action.

Lemma 17 *Let $T \in \mathcal{L}(\mathcal{F}_n^2)$. We have*

$$[\mathcal{B}_n(W_{-\zeta} T W_\zeta)](z) = [\mathcal{B}_n(T)](z + \zeta)$$

for all $\zeta, z \in \mathbb{C}$.

Proof Let $j, k \in \{1, \dots, n\}$. By (3) and (5) we have

$$W_\zeta l_{z,j} = W_\zeta (\mathfrak{A}^\dagger)^{j-1} k_z = e^{-i \operatorname{Im}(\zeta \bar{z})} W_{z+\zeta} (\mathfrak{A}^\dagger)^{j-1} \mathbb{1},$$

hence

$$\begin{aligned} \langle W_{-\zeta} T W_\zeta l_{z,j}, l_{z,k} \rangle &= \left\langle T W_{z+\zeta} (\mathfrak{A}^\dagger)^{j-1} \mathbb{1}, W_{z+\zeta} (\mathfrak{A}^\dagger)^{k-1} \mathbb{1} \right\rangle \\ &= \left\langle T (\mathfrak{A}^\dagger)^{j-1} k_{z+\zeta}, (\mathfrak{A}^\dagger)^{k-1} k_{z+\zeta} \right\rangle \\ &= \langle T l_{z+\zeta,j}, l_{z+\zeta,k} \rangle . \end{aligned}$$

□

Let us define the oscillation of a bounded continuous function $f : \mathbb{C} \to \mathbb{C}^{n \times n}$ at z as

$$\operatorname{Osc}_z(f) := \sup \{ \| f(z) - f(w) \| : |z - w| \le 1 \},$$

where $\|\cdot\|$ is of course the usual matrix norm on $\mathbb{C}^{n \times n}$ induced by the Euclidean norm on $\mathbb{C}^n$. As $z \mapsto \operatorname{Osc}_z(f)$ is continuous and bounded, we can extend it to the Stone-Čech compactification of $\mathbb{C}$. We will use the notation $\operatorname{Osc}_x(f)$ for the extension evaluated at some point $x \in \beta\mathbb{C}$.

We say that a bounded continuous function $f : \mathbb{C} \to \mathbb{C}^{n \times n}$ has vanishing oscillation and write $f \in \mathrm{VO}(\mathbb{C} \to \mathbb{C}^{n \times n})$ if

$$\lim_{|z| \to \infty} \operatorname{Osc}_z(f) = 0.$$

For $n = 1$ we just write $f \in \mathrm{VO}(\mathbb{C})$ as usual. Note that $f \in \mathrm{VO}(\mathbb{C} \to \mathbb{C}^{n \times n})$ if and only if all of its matrix entries are in $\mathrm{VO}(\mathbb{C})$.

The following is an adaptation of [11, Theorem 36]. Here, I_n denotes he $n \times n$ identity matrix.

Lemma 18 *Let $\lambda \in \mathbb{C}$ and $x \in \beta\mathbb{C} \setminus \mathbb{C}$.*

(a) For $T \in \mathcal{A}^2_{(k)}$ we have $T_x = \lambda I$ if and only if $[\mathcal{B}_{(k)}(T)](x) = \lambda$ and $\operatorname{Osc}_x(\mathcal{B}_{(k)}(T)) = 0$.
(b) For $T \in \mathcal{A}^2_n$ we have $T_x = \lambda I$ if and only if $[\mathcal{B}_n(T)](x) = \lambda I_n$ and $\operatorname{Osc}_x(\mathcal{B}_n(T)) = 0$.

Proof The proofs of (a) and (b) are identical, so we only show (b).

Assume that $T_x = \lambda I$. Choose a net (z_γ) in $\mathbb{C}$ that converges to x. By Lemma 17, it follows

$$\lim_{z_\gamma \to x} [\mathcal{B}_n(T)](z + z_\gamma) = \lim_{z_\gamma \to x} [\mathcal{B}_n(W_{-z_\gamma} T W_{z_\gamma})](z) = [\mathcal{B}_n(T_x)](z) = \lambda I_n \tag{6}$$

for all $z \in \mathbb{C}$. In particular, choosing $z = 0$, we obtain $[\mathcal{B}_n(T)](x) = \lambda I_n$.

Now assume that $\operatorname{Osc}_x(\mathcal{B}_n(T)) \neq 0$. Then there is an $\varepsilon > 0$ and a net (z_γ) converging to x such that $\operatorname{Osc}_{z_\gamma}(\mathcal{B}_n(T)) > 2\varepsilon$ for all γ. For every γ we may choose a $w_\gamma \in \mathbb{C}$ with $|z_\gamma - w_\gamma| \leq 1$ such that

$$\left\| [\mathcal{B}_n(T)](z_\gamma) - [\mathcal{B}_n(T)](w_\gamma) \right\| > \varepsilon.$$

Without loss of generality we may assume that the net $(z_\gamma - w_\gamma)$ converges to some $z \in \overline{B(0,1)}$. The Lipschitz continuity of $\mathcal{B}_n(T)$ then implies that there exists a $C \geq 0$ such that

$$\begin{aligned}
&\left\| [\mathcal{B}_n(T)](z_\gamma) - [\mathcal{B}_n(T)](w_\gamma) \right\| \\
&\quad \leq \left\| [\mathcal{B}_n(T)](z_\gamma) - [\mathcal{B}_n(T)](z_\gamma - z) \right\| + \left\| [\mathcal{B}_n(T)](z_\gamma - z) - [\mathcal{B}_n(T)](w_\gamma) \right\| \\
&\quad \leq \left\| [\mathcal{B}_n(T)](z_\gamma) - [\mathcal{B}_n(T)](z_\gamma - z) \right\| + C \left| z_\gamma - z - w_\gamma \right|.
\end{aligned}$$

Using (6), this tends to 0, which is a contradiction. Thus $\operatorname{Osc}_x(\mathcal{B}_n(T)) = 0$.

Now assume that $\operatorname{Osc}_x(\mathcal{B}_n(T)) = 0$ and $[\mathcal{B}_n(T)](x) = \lambda I_n$. Choose a net (z_γ) in $\mathbb{C}$ that converges to x. Then, by Lemma 17 again,

$$\begin{aligned}
[\mathcal{B}_n(T_x)](0) &= \lim_{z_\gamma \to x} [\mathcal{B}_n(W_{-z_\gamma} T W_{z_\gamma})](0) = \lim_{z_\gamma \to x} [\mathcal{B}_n(T)](z_\gamma) = [\mathcal{B}_n(T)](x) \\
&= \lambda I_n.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\| [\mathcal{B}_n(T_x)](w) - [\mathcal{B}_n(T_x)](0) \| \\
&\qquad = \lim_{z_\gamma \to x} \left\| [\mathcal{B}_n(W_{-z_\gamma} T W_{z_\gamma})](w) - [\mathcal{B}_n(W_{-z_\gamma} T W_{z_\gamma})](0) \right\| \\
&\qquad = \lim_{z_\gamma \to x} \left\| [\mathcal{B}_n(T)](w + z_\gamma) - [\mathcal{B}_n(T)](z_\gamma) \right\| \\
&\qquad \leq \lim_{z_\gamma \to x} \operatorname{Osc}_{z_\gamma}(\mathcal{B}_n(T)) \\
&\qquad = 0
\end{aligned}$$

for $|w| \leq 1$. As $\mathcal{B}_n(T_x)$ is real analytic, the identity theorem implies $[\mathcal{B}_n(T_x)](w) = \lambda I_n$ for all $w \in \mathbb{C}$. The injectivity of $\mathcal{B}_n$ thus shows $T_x = \lambda I$ as expected. □

Lemma 19 *Let $x \in \beta\mathbb{C} \setminus \mathbb{C}$ and $f \in \mathrm{VO}(\mathbb{C})$. Then $(T_{f,n})_x = f(x)I$ for all $n \in \mathbb{N}$.*

Proof Choose a net (z_γ) in $\mathbb{C}$ that converges to x. By Combining (4) and Proposition 8, we get

$$W_{-z_\gamma} T_{f,n} W_{z_\gamma} = T_{f(\cdot+z_\gamma),n}.$$

For $|w| \leq 1$ we have

$$\begin{aligned}\left|f(w+z_\gamma) - f(x)\right| &\leq \left|f(w+z_\gamma) - f(z_\gamma)\right| + \left|f(z_\gamma) - f(x)\right| \\ &\leq \mathrm{Osc}_{z_\gamma}(f) + \left|f(z_\gamma) - f(x)\right|.\end{aligned}$$

Since $f \in \mathrm{VO}(\mathbb{C})$, this converges to 0 uniformly in w. A straightforward induction argument now shows that the net $(f(\cdot + z_\gamma))$ converges to the constant function $f(x)$ uniformly on compact sets. This implies $T_{f(\cdot+z_\gamma),n} \to f(x)I$ in the strong operator topology. □

We now combine the previous two lemmas to our next theorem, which can be viewed as a generalization of Theorem 14 (a). The formula for the essential spectrum is well-known in case $k = 1$ (see [7, Theorem 19]).

Theorem 20 *Let $T \in \mathcal{A}^2_{(k)}$. The following are equivalent:*

(i) Every limit operator of T is a multiple of the identity.
(ii) $T = T_{f,(k)} + K$, where $f \in \mathrm{VO}(\mathbb{C})$ and K is compact.

In that case f can be chosen as $\mathcal{B}_{(k)}(T)$ and we have

$$\mathrm{sp}_{\mathrm{ess}}(T) = [\mathcal{B}_{(k)}(T)](\beta\mathbb{C} \setminus \mathbb{C}).$$

Proof Assume that every limit operator of T is a multiple of the identity. Then Lemma 18 implies $f := \mathcal{B}_{(k)}(T) \in \mathrm{VO}(\mathbb{C})$. It remains to show that $T - T_{f,(k)}$ is compact. Let $x \in \beta\mathbb{C} \setminus \mathbb{C}$. We know from Lemma 19 that $(T_{f,n})_x = f(x)I$ for every $n \in \mathbb{N}$. Since $P_{(k)}$ commutes with the Weyl operators (see Proposition 8) and $T_{f,(k)} = P_{(k)} T_{f,n}|_{\mathcal{F}^2_{(k)}}$ for any $n \geq k$, this means that we also have $(T_{f,(k)})_x = f(x)I$. On the other hand, we also have $T_x = f(x)I$ by Lemma 18. It follows $(T - T_{f,(k)})_x = 0$. As $x \in \beta\mathbb{C} \setminus \mathbb{C}$ was arbitrary, Theorem 14 (a) implies that $T - T_{f,(k)}$ is indeed compact. Moreover, we get the formula $\mathrm{sp}_{\mathrm{ess}}(T) = [\mathcal{B}_{(k)}(T)](\beta\mathbb{C} \setminus \mathbb{C})$ via Theorem 14 (b).

Now assume that $T = T_{f,(k)} + K$ for some $f \in \mathrm{VO}(\mathbb{C})$ and $K \in \mathcal{K}(\mathcal{F}^2_{(k)})$. Using Theorem 14 (a) and Lemma 19 again, we obtain

$$T_x = (T_{f,(k)})_x + K_x = f(x)I$$

for every $x \in \beta\mathbb{C} \setminus \mathbb{C}$. □

Next, we want to prove a version of this result for operators $T \in \mathcal{A}_n^2$. However, a priori it is not clear what function f one should take for the decomposition $T = T_{f,n} + K$ because $\mathcal{B}_n(T)$ is matrix valued. It turns out that any of the functions $\mathcal{B}_{(k)}(T) := \mathcal{B}_{(k)}(P_{(k)}T|_{\mathcal{F}^2_{(k)}})$, $k = 1, \ldots, n$ works because they only differ by a C_0-function in that case.

Lemma 21 *Let $T \in \mathcal{A}_n^2$ and assume that every limit operator of T is a multiple of the identity. Then*

$$g := \widetilde{T} - \mathcal{B}_{(k)}(T) \in C_0(\mathbb{C})$$

for all $k = 1, \ldots, n$. In particular, $T_{g,(k)}$ is compact.

Proof Let $k \in \{1, \ldots, n\}$, $x \in \beta\mathbb{C} \setminus \mathbb{C}$ and $T_x = \lambda I$ for some $\lambda \in \mathbb{C}$. This implies that $(P_{(k)}T|_{\mathcal{F}^2_{(k)}})_x = \lambda I$ as well because $P_{(k)}$ commutes with W_z by Proposition 8. Thus $[\mathcal{B}_{(k)}(T)](x) = \lambda$ for all $k = 1, \ldots, n$ by Lemma 18. As $x \in \beta\mathbb{C} \setminus \mathbb{C}$ was arbitrary, $\widetilde{T} = \mathcal{B}_{(1)}(T)$ and $\mathcal{B}_{(k)}(T)$ agree on $\beta\mathbb{C} \setminus \mathbb{C}$, which proves the first part of the lemma. That C_0-functions produce compact Toeplitz operators is easy to show directly, but also follows from Theorem 14 (a) and Lemma 19.

We can now formulate a version of Theorem 20 for $T \in \mathcal{A}_n^2$.

Theorem 22 *Let $T \in \mathcal{A}_n^2$. The following are equivalent:*

(i) *Every limit operator of T is a multiple of the identity.*
(ii) *$T = T_{f,n} + K$, where $f \in \mathrm{VO}(\mathbb{C})$ and K is compact.*

In that case f can be chosen as $\widetilde{T}$ and we have

$$\mathrm{sp}_{\mathrm{ess}}(T) = \widetilde{T}(\beta\mathbb{C} \setminus \mathbb{C}).$$

Proof Assume (i) and write $T = \sum\limits_{j,k=1}^{n} P_{(k)}TP_{(j)}$. Combining Theorem 20 with Lemma 21 we get that $P_{(k)}TP_{(k)} - P_{(k)}T_{\widetilde{T},n}P_{(k)}$ is compact and $\widetilde{T} \in \mathrm{VO}(\mathbb{C})$. $P_{(k)}TP_{(j)}$ and $P_{(k)}T_{\widetilde{T},n}P_{(j)}$ are also compact for $j \neq k$. In both cases this follows from Theorem 14 as all limit operators are zero, respectively. We infer that $T - T_{\widetilde{T},n}$ is compact, which implies (ii).

The other direction and the formula for the essential spectrum follow from Theorem 14 and Lemma 19 just like in the proof of Theorem 20. ◻

5 Compact Toeplitz and Hankel Operators

In [22] it was observed that a Toeplitz operator on a true polyanalytic Fock space $\mathcal{F}^2_{(k)}$ is unitarily equivalent to a Toeplitz operator on the analytic Fock space $\mathcal{F}^2$ with a much more irregular, possibly distributional symbol. After this observation, Rozenblum and Vasilevski offer a choice of considering "operators with nice symbols in 'bad' spaces or operators in nice spaces with 'bad' symbols". Conversely, one would therefore expect that if we take a very good symbol, e.g. a symbol that induces a compact Toeplitz operator on $\mathcal{F}^2$, then this symbol would also induce an in this sense very good (or even better) Toeplitz operator on the polyanalytic spaces. The next result thus may not be very surprising and in fact follows directly from our considerations above. A related result was proven by different means in [18, Proposition 4.6].

Theorem 23 *Let $k, n \in \mathbb{N}$ and $f \in L^\infty(\mathbb{C}, \mu)$. $T_{f,1}$ is compact if and only if $T_{f,n}$ is compact. In particular, if $T_{f,(1)}$ is compact, then every $T_{f,(k)}$ is compact as well.*

Proof As $T_{f,1}$ is a compression of $T_{f,n}$, it is clear that if $T_{f,n}$ is compact, then $T_{f,1}$ is necessarily compact as well.

So assume that $T_{f,1}$ is compact. By Theorem 16, we need to show that $\mathcal{B}_n(T_{f,n})$ is in $C_0(\mathbb{C} \to \mathbb{C}^{n\times n})$, that is, $\lim\limits_{|z|\to\infty} \langle T_{f,n} l_{z,j}, l_{z,k}\rangle = 0$ for all $j, k = 1, \ldots, n$. We have

$$\begin{aligned}
&\langle T_{f,n} l_{z,j}, l_{z,k}\rangle \\
&= \frac{1}{\sqrt{(j-1)!}\sqrt{(k-1)!}} \int_{\mathbb{C}} f(w)(\bar{w} - \bar{z})^{j-1}(w-z)^{k-1} e^{w\bar{z}+\bar{w}z-|z|^2}\, \mathrm{d}\mu(w) \\
&= \langle T_{f,1} \hat{l}_{z,k}, \hat{l}_{z,j}\rangle,
\end{aligned}$$

where $\hat{l}_{z,k}(w) = \frac{1}{\sqrt{(k-1)!}}(w-z)^{k-1} e^{w\bar{z}-\frac{1}{2}|z|^2}$. Note that $\hat{l}_{z,k} \in \mathcal{F}^2_1$ and

$$\hat{l}_{z,k} = W_z m_k,$$

where m_k is the monomial given by $m_k(w) = \frac{1}{\sqrt{(k-1)!}} w^{k-1}$. We thus have

$$\langle T_{f,n} l_{z,j}, l_{z,k}\rangle = \langle T_{f,1} W_z m_k, W_z m_j\rangle = \langle W_{-z} T_{f,1} W_z m_k, m_j\rangle,$$

which converges to 0 as $|z| \to \infty$ by Theorem 14 (a). □

Combining this result with Theorem 22, we obtain the following generalization. It particularly applies to Toeplitz operators of the form $\lambda I + K$ with $K \in \mathcal{K}(\mathcal{F}^2_n)$, which is the case considered in [18, Proposition 4.6].

Corollary 24 *Let $n \in \mathbb{N}$ and $f \in L^\infty(\mathbb{C}, \mu)$. Every limit operator of $T_{f,1}$ is a multiple of the identity if and only if every limit operator of $T_{f,n}$ is a multiple of the identity.*

Proof The "if" direction is again obvious. So assume that every limit operator of $T_{f,1}$ is a multiple of the identity. Theorem 22 implies that $\tilde{f} \in \mathrm{VO}$ and $T_{f-\tilde{f},1}$ is compact. This means that $T_{f-\tilde{f},n}$ is also compact by Theorem 23. Using

Theorem 22 again, we see that every limit operator of $T_{f,n}$ is a multiple of the identity as well.

Next, we turn our attention to Hankel operators. The theory developed in Sect. 3 cannot be applied directly to them because they do not map into a polyanalytic Fock space. This can of course be circumvented by considering $H_{f,n}^* H_{f,n} \in \mathcal{L}(\mathcal{F}_n^2)$ instead, which we will do later on. However, we will take a slightly more general approach first which provides a compactness characterization for all operators acting on $L^2(\mathbb{C}, \mu)$. For this we use the decomposition

$$L^2(\mathbb{C}, \mu) = \bigoplus_{k=1}^{\infty} \mathcal{F}_{(k)}^2 \cong \ell^2(\mathbb{N}, \mathcal{F}^2)$$

again. The (non-commutative) algebra $\ell^\infty(\mathbb{N}, \mathcal{L}(\mathcal{F}^2))$ acts on $\ell^2(\mathbb{N}, \mathcal{F}^2)$ via multiplication, that is, if $g \in \ell^\infty(\mathbb{N}, \mathcal{L}(\mathcal{F}^2))$, then

$$m_g \colon \ell^2(\mathbb{N}, \mathcal{F}^2) \to \ell^2(\mathbb{N}, \mathcal{F}^2), \quad (m_g f)(k) = g(k)f(k)$$

for $f \in \ell^2(\mathbb{N}, \mathcal{F}^2)$, $k \in \mathbb{N}$. Moreover, we have the usual shift operators V acting on $\ell^2(\mathbb{N}, \mathcal{F}^2)$ defined by

$$(Vf)(1) := 0, \quad (Vf)(k+1) := f(k)$$

for $f \in \ell^2(\mathbb{N}, \mathcal{F}^2)$, $k \in \mathbb{N}$. One can now define band-dominated operators on $\ell^2(\mathbb{N}, \mathcal{F}^2)$ with respect to their matrix structure as follows.

Definition 25 (Definition 2.1.5 in [21]) Let $\omega \in \mathbb{N}$. Operators of the form

$$T = \sum_{j=0}^{\omega} m_{g_j} V^j + \sum_{j=1}^{\omega} m_{g_{-j}} (V^*)^j$$

with $g_j \in \ell^\infty(\mathbb{N}, \mathcal{L}(\mathcal{F}^2))$ are called band operators on $\ell^2(\mathbb{N}, \mathcal{F}^2)$. Operators obtained as the norm limit of a sequence of band operators are then called band-dominated on $\ell^2(\mathbb{N}, \mathcal{F}^2)$.

One could of course also define band-dominated operators on $\ell^2(\mathbb{N}, \mathcal{F}^2)$ just like we did for $L^2(\mathbb{C}, \mu)$ in Definition 2 and it is not difficult to show that this would be an equivalent definition (see e.g. [21, Theorem 2.1.6]). However, using this version makes it a little more obvious why these operators are called band-dominated and it will also be more straightforward to use in the proof of the next theorem. We emphasize that it is important not to confuse the two notions of band-dominated operators, though, which is why we added the suffix "on $\ell^2(\mathbb{N}, \mathcal{F}^2)$". Indeed, they act on different spaces ($L^2(\mathbb{C}, \mu)$ vs. $\ell^2(\mathbb{N}, \mathcal{F}^2)$) and are not equivalent via the isomorphism

$$U: L^2(\mathbb{C}, \mu) \to \ell^2(\mathbb{N}, \mathcal{F}^2).$$

Example 26

(a) Consider the Hankel operator $H_{f,1}$ for $f(z) = e^{i|z|^2}$. Then $H_{f,1}$ is band-dominated by Corollary 6. We claim that $UH_{f,1}U^{-1}$ is not band-dominated on $\ell^2(\mathbb{N}, \mathcal{F}^2)$. So assume by contradiction that $UH_{f,1}U^{-1}$ is band-dominated on $\ell^2(\mathbb{N}, \mathcal{F}^2)$. Then necessarily $\|(I - P_n)H_{f,1}P_1\| \to 0$ as $n \to \infty$ by [17, Propositions 1.20 and 1.48]. The Toeplitz operator $T_{f,1}$ is compact, which can be deduced directly from its eigenvalues (see [6, Example (A)]). By Theorem 23, this means that for every $n \in \mathbb{N}$ the operator $P_n M_f P_1 = T_{f,n} P_1$ is compact as well. Combining this with $\|(I - P_n)H_{f,1}P_1\| \to 0$ shows that $H_{f,1}P_1$ is also compact. But this would mean that

$$P_1 = P_1 M_{\bar{f}} M_f P_1 = P_1 M_{\bar{f}} H_{f,1} P_1 + P_1 M_{\bar{f}} T_{f,1} P_1,$$

is compact, which is obviously a contradiction. This shows that $UH_{f,1}U^{-1}$ is not band-dominated on $\ell^2(\mathbb{N}, \mathcal{F}^2)$.

(b) To give an operator $T \notin \mathrm{BDO}^2$ such that UTU^{-1} is band-dominated on $\ell^2(\mathbb{N}, \mathcal{F}^2)$ is much simpler; many such examples can be found in the literature. As a specific example we mention the operator

$$T: L^2(\mathbb{C}, \mu) \to L^2(\mathbb{C}, \mu), \quad (Tf)(z) = f(-z).$$

T leaves every $\mathcal{F}^2_{(k)}$ invariant, which implies that UTU^{-1} is a band operator (with just one diagonal) on $\ell^2(\mathbb{N}, \mathcal{F}^2)$. On the other hand,

$$\widetilde{T}(z) = \langle Tk_z, k_z \rangle = \langle k_{-z}, k_z \rangle = e^{-2|z|^2} \to 0$$

as $|z| \to \infty$. So if T was band-dominated, then T restricted to $\mathcal{F}^2_1$ would be compact by Theorem 16, which it obviously is not. Further, maybe more interesting examples are provided in [4, Example 2], for instance.

The next theorem now provides a limit operator type characterization of compact operators on $L^2(\mathbb{C}, \mu)$.

Theorem 27 *Let $T \in \mathcal{L}(L^2(\mathbb{C}, \mu))$. Then T is compact if and only if $T \in \mathrm{BDO}^2$, UTU^{-1} is band-dominated on $\ell^2(\mathbb{N}, \mathcal{F}^2)$, $W_{-z}TW_z \to 0$ strongly as $|z| \to \infty$ and*

$$\lim_{k\to\infty} \left\| P_n \mathfrak{A}^k T (\mathfrak{A}^\dagger)^k P_n \right\| = 0 \tag{7}$$

for every $n \in \mathbb{N}$.

Proof Define

$$\mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P}) := \left\{K \in \mathcal{L}(\ell^2(\mathbb{N},\mathcal{F}^2)) : \|K - \hat{P}_n K\| + \|K - K\hat{P}_n\| \to 0 \text{ as } n \to \infty\right\},$$

where $\hat{P}_n := UP_nU^{-1}$ is the orthogonal projection onto $\ell^2(\{1,\dots,n\},\mathcal{F}^2)$. We will first show that $K \in \mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$ if and only if K is band-dominated on $\ell^2(\mathbb{N},\mathcal{F}^2)$ and

$$\lim_{k\to\infty} \left\| \hat{P}_n (V^*)^k K V^k \hat{P}_n \right\| = 0. \tag{8}$$

This follows from a standard limit operator argument; nevertheless, we provide some details for the convenience of the reader. First observe that if K belongs to $\mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$, then

$$\|K - \hat{P}_n K \hat{P}_n\| \le \|K - \hat{P}_n K\| + \|\hat{P}_n K - \hat{P}_n K \hat{P}_n\| \to 0 \tag{9}$$

as $n \to \infty$. $\hat{P}_n K \hat{P}_n$ is obviously a band operator on $\ell^2(\mathbb{N},\mathcal{F}^2)$ and so all $K \in \mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$ are band-dominated on $\ell^2(\mathbb{N},\mathcal{F}^2)$. Moreover, since $\hat{P}_n V^k = 0$ for $k \ge n$, we have

$$\lim_{k\to\infty} \left\|(V^*)^k K V^k\right\| = \lim_{k\to\infty} \left\|(V^*)^k (K - K\hat{P}_n) V^k\right\| = 0.$$

In particular, $\lim\limits_{k\to\infty} \left\| \hat{P}_n (V^*)^k K V^k \hat{P}_n \right\| = 0$.

Conversely, assume that K is a band operator on $\ell^2(\mathbb{N},\mathcal{F}^2)$ and satisfies

$$\left\| \hat{P}_n (V^*)^k K V^k \hat{P}_n \right\| \to 0 \text{ as } k \to \infty.$$

Write K as $\sum\limits_{j=0}^{\omega} m_{g_j} V^j + \sum\limits_{j=1}^{\omega} m_{g_{-j}} (V^*)^j$ for some $g_j \in \ell^\infty(\mathbb{N},\mathcal{L}(\mathcal{F}^2))$. The condition $\left\| \hat{P}_n (V^*)^k K V^k \hat{P}_n \right\| \to 0$ implies that $\left\|m_{g_j}(k)\right\| \to 0$ for $k \to \infty$, $j \in \{-\omega,\dots,\omega\}$. It follows

$$\left\|m_{g_j} - \hat{P}_n m_{g_j}\right\| = \left\|m_{g_j} - m_{g_j}\hat{P}_n\right\| \le \sup_{k \ge n+1} \left\|m_{g_j}(k)\right\| \to 0$$

as $k \to \infty$ and thus $m_{g_j} \in \mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$ for each $j \in \{-\omega,\dots,\omega\}$. As $\mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$ is an algebra and invariant under multiplying by V or V^*, this shows $K \in \mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$. $\mathcal{K}(\ell^2(\mathbb{N},\mathcal{F}^2),\mathcal{P})$ is also closed and therefore the same conclusion holds if K is assumed to be a band-dominated instead of a band operator.

We already observed $V = U\mathfrak{A}^\dagger U^{-1}$ in the introduction and so (8) for $K = UTU^{-1}$ is the same as (7). Hence, it suffices to prove that T is compact if and only if $T \in \mathrm{BDO}^2$, $UTU^{-1} \in \mathcal{K}(\ell^2(\mathbb{N}, \mathcal{F}^2), \mathcal{P})$ and $W_{-z}TW_z \to 0$ strongly as $|z| \to \infty$. So assume that T is compact. Then $T \in \mathrm{BDO}^2$ is [12, Theorem 3.7 (d)] and $W_{-z}TW_z \to 0$ as $|z| \to \infty$ follows from [12, Proposition 4.20]. Moreover, it is clear that $\mathcal{K}(\ell^2(\mathbb{N}, \mathcal{F}^2), \mathcal{P})$ contains all compact operators as $\hat{P}_n \to I$ in the strong operator topology. Conversely, assume that $T \in \mathrm{BDO}^2$, $UTU^{-1} \in \mathcal{K}(\ell^2(\mathbb{N}, \mathcal{F}^2), \mathcal{P})$ and $W_{-z}TW_z \to 0$ strongly as $|z| \to \infty$. By (9), $\|T - P_nTP_n\| \to 0$ as $n \to \infty$. As P_n and W_z commute (see Proposition 8), we have $W_{-z}P_nTP_nW_z \to 0$ strongly as $|z| \to \infty$. By Theorem 14 (a), P_nTP_n is compact for every $n \in \mathbb{N}$. This shows that T is compact as well, which completes the proof. □

For operators T that satisfy either $\|T(I - P_n)\| \to 0$ or $\|(I - P_n)T\| \to 0$ as $n \to \infty$, such as Hankel operators, Theorem 27 simplifies significantly.

Corollary 28 *Assume that* $T \in \mathcal{L}(L^2(\mathbb{C}, \mu))$ *satisfies either* $\|T(I - P_n)\| \to 0$ *or* $\|(I - P_n)T\| \to 0$ *as* $n \to \infty$. *Then* T *is compact if and only if* $T \in \mathrm{BDO}^2$ *and* $W_{-z}TW_z \to 0$ *strongly as* $|z| \to \infty$.

Proof As T is compact if and only if T^*T is compact, is suffices to consider T^*T. We only prove the case when $\|T(I - P_n)\| \to 0$; the other case is similar. The condition $\|T(I - P_n)\| \to 0$ implies

$$\left\|(I - P_n)T^*T\right\| = \left\|T^*T(I - P_n)\right\| \to 0$$

and therefore $UT^*TU^{-1} \in \mathcal{K}(\ell^2(\mathbb{N}, \mathcal{F}^2), \mathcal{P})$. The result now follows by the same arguments as in the last paragraph of the proof of Theorem 27. □

The next corollary is now immediate. It can be viewed as a partial generalization of [13, Theorem 3.1].

Corollary 29 *Let* $f \in L^\infty(\mathbb{C})$ *and* $k, n \in \mathbb{N}$.

(a) $H_{f,(k)}$ *is compact if and only if* $W_{-z}H_{f,(k)}W_z \to 0$ *strongly as* $|z| \to \infty$.
(b) $H_{f,n}$ *is compact if and only if* $W_{-z}H_{f,n}W_z \to 0$ *strongly as* $|z| \to \infty$.

Of course, we also have the classical characterization of compact Hankel operators in terms of the Berezin transform. Quite surprisingly, the compactness of $H_{f,(k)}$ does not depend on k. For $f \in L^\infty(\mathbb{C}, \mu)$ we say that $f \in \mathrm{VMO}(\mathbb{C})$ if

$$\widetilde{|f|^2} - |\tilde{f}|^2 \in C_0(\mathbb{C}).$$

Theorem 30 *Let* $k, n \in \mathbb{N}$ *and* $f \in L^\infty(\mathbb{C})$. *The following are equivalent:*

(a) $H_{f,(k)}$ *is compact,*
(b) $H_{f,n}$ *is compact,*
(c) $f \in \mathrm{VMO}(\mathbb{C})$.

In particular, $H_{f,(k)}$ is compact if and only if $H_{\bar{f},(k)}$ is compact and $H_{f,n}$ is compact if and only if $H_{\bar{f},n}$ is compact.

Proof Assume that $H_{f,(k)}$ is compact. Then, by Corollary 29, $W_{-z}H_{f,(k)}W_z \to 0$ strongly as $|z| \to \infty$. Let $x \in \beta\mathbb{C} \setminus \mathbb{C}$ and choose a net (z_γ) that converges to x. Then $W_{-z_\gamma}T_{f,(k)}W_{z_\gamma}$ converges strongly to $(T_{f,(k)})_x$. It follows that

$$\begin{aligned} f(\cdot + z_\gamma)g = W_{-z_\gamma}M_f W_{z_\gamma} g &= W_{-z_\gamma}T_{f,(k)}W_{z_\gamma} g + W_{-z_\gamma}H_{f,(k)}W_{z_\gamma} g \\ &\to (T_{f,(k)})_x g \end{aligned}$$

for all $g \in \mathcal{F}^2_{(k)}$ as $z_\gamma \to x$. In particular, for $g(w) = \bar{w}^{k-1}$, we get

$$\lim_{z_\gamma \to x} \int_{\mathbb{C}} \left|f(w + z_\gamma) - \psi(w)\right|^2 |w|^{2k-2} \,\mathrm{d}\mu(w) = 0, \tag{10}$$

where $\psi(w) := \frac{((T_{f,(k)})_x g)(w)}{\bar{w}^{k-1}}$ for $w \neq 0$. As $\left\|f(\cdot + z_\gamma)\right\|_\infty \leq \|f\|_\infty$ for all γ, we also have $\|\psi\|_\infty \leq \|f\|_\infty$. Moreover, $g\psi \in \mathcal{F}^2_{(k)}$. By Liouville's theorem for polyanalytic functions, this implies that $g\psi$ is a polynomial in w and $\bar{w}$ of degree at most $k-1$ (see [3, Theorem 2.2] or [16, Corollary 1]). But the only true polyanalytic functions of order k that are also a polynomial of degree $k-1$ are multiples of g. It follows that ψ is constant. Furthermore, (10) implies

$$\lim_{z_\gamma \to x} \int_{\mathbb{C}} \left|f(w + z_\gamma) - \psi\right|^2 |w|^{2k-2} |\phi(w)|^2 \,\mathrm{d}\mu(w) = 0$$

for all bounded functions ϕ. This shows that the net of multiplication operators $M_{f(\cdot+z_\gamma)}$ converges strongly to ψI on the set $\left\{g\phi \in L^2(\mathbb{C}, \mu) : \phi \in L^\infty(\mathbb{C}, \mu)\right\}$. This set is dense in $L^2(\mathbb{C}, \mu)$ and therefore $M_{f(\cdot+z_\gamma)} \to \psi I$ strongly on $L^2(\mathbb{C}, \mu)$ as $z_\gamma \to x$. This of course implies $(T_{f,(k)})_x = \psi I$ and $(T_{|f|^2,(k)})_x = |\psi|^2 I$, but also $W_{-z_\gamma}H_{f,(k')}W_{z_\gamma} \to 0$ and $W_{-z_\gamma}H_{f,n}W_{z_\gamma} \to 0$ in the strong operator topology for any $k', n \in \mathbb{N}$. Corollary 29 therefore reveals that $H_{f,(k')}$ and $H_{f,n}$ are compact. The same argument as above, for $k = 1$, also shows that if $H_{f,n}$ is compact, then $W_{-z_\gamma}H_{f,(k')}W_{z_\gamma} \to 0$ for any $k' \in \mathbb{N}$. This proves the equivalence of (a) and (b), and that the compactness is independent of k and n. For $n = 1$, the equivalence of (b) and (c) is well-known (see e.g. [27, Theorems 8.5 and 8.13]), but also follows directly from our previous results. Indeed, $M_{f(\cdot+z_\gamma)} \to \psi I$ implies

$$\widetilde{|f|^2}(x) - |\tilde{f}(x)|^2 = |\psi|^2 - |\psi|^2 = 0$$

for all $x \in \beta\mathbb{C}^n \setminus \mathbb{C}^n$ via Lemma 18, hence $f \in \mathrm{VMO}(\mathbb{C})$.

Conversely, assume that $f \in \mathrm{VMO}(\mathbb{C})$. Then the formula $H_{f,1}^* H_{f,1} = T_{|f|^2,1} - T_{\bar{f},1} T_{f,1}$ implies $\mathcal{B}_1(H_{f,1}^* H_{f,1}) = \widetilde{|f|^2} - \|T_{f,1} k_z\|^2$. Clearly, $|\tilde{f}| \le \|T_{f,1} k_z\|$ and so

$$0 \le \mathcal{B}_1(H_{f,1}^* H_{f,1}) \le \widetilde{|f|^2} - |\tilde{f}|^2.$$

It follows $\mathcal{B}_1(H_{f,1}^* H_{f,1}) \in C_0(\mathbb{C})$. Theorem 16 now implies that $H_{f,1}^* H_{f,1}$ is compact. □

6 Remarks and Open Problems

It was quite surprising to the author that the compactness of the Hankel operators $H_{f,(k)}$ is independent of k. Indeed, one is tempted to define $\mathrm{VMO}_{(k)}$-spaces consisting of bounded functions f satisfying

$$\mathcal{B}_{(k)}(|f|^2) - |\mathcal{B}_{(k)}(f)|^2 \in C_0(\mathbb{C}).$$

The same argument as in the proof of Theorem 30 then shows that $H_{f,(k)}$ is compact if and only if $f \in \mathrm{VMO}_{(k)}$. But as it turns out, these $\mathrm{VMO}_{(k)}$-spaces are all the same. This naturally leads to the following question.

Question 31 Is the compactness of Toeplitz operators $T_{f,(k)}$ also independent of k? The argument used in the proof of Theorem 23 does not quite work for $k \ge 2$, unfortunately.

In this paper we quite heavily used the functions $\mathcal{B}_n$, which we introduced to generalize the Berezin transform. However, as the polyanalytic Fock spaces are reproducing kernel Hilbert spaces, it seems more natural to use the Berezin transform induced by the normalized reproducing kernels. This did not turn out to be very fruitful for our approach. Nevertheless, we pose the following question.

Question 32 Does the more standard Berezin transform $z \mapsto \langle Tk_{z,n}, k_{z,n} \rangle$ have any useful properties in connection with compactness or Fredholmness problems on $\mathcal{F}_n^2$? It is fairly obvious that if $T \in \mathcal{L}(\mathcal{F}_n^2)$ is compact, then $\langle Tk_{z,n}, k_{z,n} \rangle \to 0$ as $|z| \to \infty$, but we also know that this Berezin transform is not strong enough to characterize compactness in $\mathcal{A}_n^2$. Nevertheless, if restricted to Toeplitz operators (with certain symbols) maybe something can still be said.

We have seen in Corollary 6 that every Toeplitz operator with bounded symbol is band-dominated. Bauer and Fulsche [4] showed that the algebra of band-dominated operators on $\mathcal{F}^2$ is generated by Toeplitz operators. A natural question is therefore:

Question 33 Are $\mathcal{A}_{(k)}^2$ and $\mathcal{A}_n^2$ also generated by Toeplitz operators for $k, n \ge 2$? If not, can the band-dominated operators in Theorem 16 and related results at least be replaced by an algebra of Toeplitz operators?

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Strong Szegő Theorem on a Jordan Curve

Kurt Johansson

Dedicated to the memory of Harold Widom 1932–2021

Abstract We consider certain determinants with respect to a sufficiently regular Jordan curve γ in the complex plane that generalize Toeplitz determinants which are obtained when the curve is the circle. This also corresponds to studying a planar Coulomb gas on the curve at inverse temperature $\beta = 2$. Under suitable assumptions on the curve we prove a strong Szegő type asymptotic formula as the size of the determinant grows. The resulting formula involves the Grunsky operator built from the Grunsky coefficients of the exterior mapping function for γ. As a consequence of our formula we obtain the asymptotics of the partition function for the Coulomb gas on the curve. This formula involves the Fredholm determinant of the absolute value squared of the Grunsky operator which equals, up to a multiplicative constant, the Loewner energy of the curve. Based on this we obtain a new characterization of curves with finite Loewner energy called Weil-Petersson quasicircles.

Keywords Strong Szegő theorem · Toeplitz determinant · Weil-Petersson quasicircle · Loewner energy

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1 Introduction and Results

1.1 Definitions and Results

Let γ be a Jordan curve in the complex plane and $g : \gamma \mapsto \mathbb{C}$ a given function on the curve. Define the determinant

$$D_n[e^g] = \det\left(\int_\gamma \zeta^j \bar{\zeta}^k e^{g(\zeta)}\,|d\zeta|\right)_{0\le j,k<n} \tag{1}$$

assuming that all integrals exist. In the case $\gamma = \mathbb{T}$, the unit circle, this is $(2\pi)^n$ times the Toeplitz determinant with symbol e^g. These determinants are related to orthogonal polynomials on the curve γ when the weight function $w = e^g$ is positive, see [17, Sec. 16.2]. Note that our definition is different from the one in [17]; the D_n there is our D_{n+1}/L^{n+1} where L is the length of the curve. In the case when $g = 0$ these orthogonal polynomials were introduced by Szegő in [16], and some properties of the polynomials and the determinants, like (15) below, were investigated. By Andrieff's identity, we have the integral formula

$$D_n[e^g] = \frac{1}{n!}\int_{\gamma^n} \prod_{1\le\mu\ne\nu\le n} |\zeta_\mu - \zeta_\nu| \prod_{\mu=1}^{n} e^{g(\zeta_\mu)} \prod_{\mu=1}^{n} |d\zeta_\mu|. \tag{2}$$

Note that $D_n[1]$ is the *partition function* for a planar Coulomb gas on the curve γ,

$$Z_n(\gamma) = D_n[1] = \frac{1}{n!}\int_{\gamma^n} \exp\left[-\sum_{1\le\mu\ne\nu\le n} \log|\zeta_\mu - \zeta_\nu|^{-1}\right] \prod_{\mu=1}^{n} |d\zeta_\mu|. \tag{3}$$

In the case of Toeplitz determinants the strong Szegő limit theorem gives a precise asymptotic formula for $D_n[e^g]/D_n[1]$ where $D_n[1] = Z_n(\mathbb{T}) = (2\pi)^n$, see [2, 15] for background on, and proofs of, this theorem. In this case the partition function is easy to compute which is not the case for other curves. We want to generalize the strong Szegő theorem to the case of a more general Jordan curve and also understand the asymptotics of the partition function. Asymptotic properties of the determinant (1) were studied in [16] and [6, Sec. 6.2], where the asymptotics for a quotient of consecutive determinants was given, see (15). A strong Szegő theorem for $D_n[e^g]/D_n[1]$ was proved in [9, Sec. III]. In this paper we prove a precise asymptotic formula for $D_n[e^g]$ under a somewhat weaker, but certainly not optimal, condition on the curve γ, and our assumption on the function g is optimal. See Sect. 1.2 below for further comments and background. The asymptotics of the partition function $Z_n(\gamma)$ turns out to be interesting and we will discuss its asymptotics under optimal conditions in Sect. 1.3. We will not prove any results for the β-ensemble corresponding to (2), but we give a heuristic discussion in Sects. 1.4 and 5.

We can expect that the leading order asymptotics of $D_n[e^g]$ should be given by $\exp(-n^2 V(\gamma))$, where $V(\gamma)$ is the *logarithmic energy* of γ. The logarithmic energy is defined by

$$V(\gamma) = \inf_{\mu} \int_\gamma \int_\gamma \log |\zeta_1 - \zeta_2|^{-1} \, d\mu(\zeta_1) d\mu(\zeta_2),$$

where the infimum is over all probability measures μ on γ. The *logarithmic capacity* of γ can be defined by $\mathrm{cap}(\gamma) = \exp(-V(\gamma))$. Let Ω be the unbounded component of the complement of γ and let $\mathrm{cap}(\gamma)\phi : \{|z| > 1\} \mapsto \Omega$ be the exterior mapping function with the expansion,

$$\phi(z) = z + \phi_0 + \phi_{-1} z^{-1} + \dots \tag{4}$$

around infinity. If $|z| > 1$, $|\zeta| > 1$, we have the expansion

$$\log \frac{\phi(\zeta) - \phi(z)}{\zeta - z} = - \sum_{k,\ell=1}^{\infty} a_{k\ell} \zeta^{-k} z^{-\ell}, \tag{5}$$

where $a_{k\ell} = a_{\ell k} \in \mathbb{C}$ are the *Grunsky coefficients*, see e.g. [13, Sec. 3.1]. If γ is a quasicircle, i.e. it is the image of the unit circle under a quasiconformal mapping of the plane, there is a constant $\kappa < 1$ such that

$$\sum_{k=1}^{\infty} \left| \sum_{\ell=1}^{\infty} \sqrt{k\ell}\, a_{k\ell} w_\ell \right|^2 \le \kappa^2 \sum_{k=1}^{\infty} |w_k|^2, \tag{6}$$

and

$$\left| \sum_{k,\ell=1}^{\infty} \sqrt{k\ell}\, a_{k\ell} w_k w_\ell \right| \le \kappa \sum_{k=1}^{\infty} |w_k|^2, \tag{7}$$

called the *Grunsky inequalities*, see [13, Sec. 9.4]. Write

$$b_{k\ell} = \sqrt{k\ell}\, a_{k\ell} = b_{k\ell}^{(1)} + \mathrm{i} b_{k\ell}^{(2)}, \tag{8}$$

where $b_{k\ell}^{(j)} \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$. Consider the *Grunsky operator* B and its real and imaginary parts,

$$B = (b_{k\ell})_{k,\ell \ge 1}, \quad B^{(j)} = (b_{k\ell}^{(j)})_{k,\ell \ge 1},\ j = 1, 2, \tag{9}$$

which are bounded operators on $\ell^2(\mathbb{C})$ by (6) with norm $\le \kappa < 1$. Define the operator K on $\ell^2(\mathbb{C}) \oplus \ell^2(\mathbb{C})$, by

$$K = \begin{pmatrix} B^{(1)} & B^{(2)} \\ B^{(2)} & -B^{(1)} \end{pmatrix}.$$

Note that K is real and symmetric.

Expand the function $g(\mathrm{cap}(\gamma)\phi(e^{\mathrm{i}\theta}))$ in a Fourier series,

$$g(\mathrm{cap}(\gamma)\phi(e^{\mathrm{i}\theta})) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta, \tag{10}$$

where $a_k, b_k \in \mathbb{C}$ and we assume that $g(\mathrm{cap}(\gamma)\phi(e^{\mathrm{i}\theta}))$ is integrable. Define the infinite column vector in $\ell^2(\mathbb{C}) \oplus \ell^2(\mathbb{C})$,

$$\mathbf{g} = \begin{pmatrix} (\frac{1}{2}\sqrt{k}a_k)_{k\ge 1} \\ (\frac{1}{2}\sqrt{k}b_k)_{k\ge 1} \end{pmatrix}. \tag{11}$$

We can now state our main theorem which gives the asymptotics for the determinant (1) as $n \to \infty$.

Theorem 1.1 *Assume that the Jordan curve* γ *is* $C^{5+\alpha}$, $\alpha > 0$, *and that*

$$\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty. \tag{12}$$

We then have the asymptotic formula

$$D_n[e^g] = \frac{(2\pi)^n\ cap(\gamma)^{n^2}}{\sqrt{\det(I+K)}} \exp\left(na_0/2 + \mathbf{g}^t(I+K)^{-1}\mathbf{g} + o(1)\right), \tag{13}$$

as $n \to \infty$.

The theorem will be proved in the next section. We see that the geometry of the curve enters via the operator K, and also directly via ϕ since we have the composition of g with ϕ in (10). If γ is the unit circle, we have $K = 0$, $\mathrm{cap}(\gamma) = 1$, and we get the usual strong Szegő limit theorem. For the partition function (3) we obtain the following asymptotic formula,

$$Z_n(\gamma) = D_n[1] = \exp\left(n^2 \log \mathrm{cap}(\gamma) + n \log 2\pi - \frac{1}{2}\log\det(I+K) + o(1)\right), \tag{14}$$

as $n \to \infty$.

1.2 Discussion

It was proved in [6, Sec. 6.2] that if γ is an analytic curve then

$$\lim_{n\to\infty} \operatorname{cap}(\gamma)^{-2n-1}\frac{D_{n+1}[e^g]}{D_n[e^g]} = 2\pi \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi(e^{i\theta}))\,d\theta\right), \tag{15}$$

which is a consequence of (13) above. In [9, Sec. III] the following relative Szegő type theorem was proved. If g is $C^{1+\alpha}$ on γ and γ is $C^{10+\alpha}$ for some $\alpha > 0$, then

$$\frac{D_n[e^g]}{D_n[1]} = \exp[na_0/2 + \mathbf{g}^t(I+K)^{-1}\mathbf{g} + o(1)] \tag{16}$$

as $n \to \infty$. The expression for the constant term in the exponent in the right side was less clear in [9], see Theorem 7.1 and its Corollary. The form give here for the constant term is more elegant and satisfactory. A calculation shows that they are identical. We see that Theorem 1.1 is a strengthening of the earlier result. In particular it is not a relative Szegő theorem since we do not divide by $D_n[1]$, and hence we also get asymptotics for the partition function as in (14). The condition (12) in the theorem on g is natural since $(I+K)^{-1}$ is an operator on $\ell^2(\mathbb{C})$ we have to require that $\mathbf{g} \in \ell^2(\mathbb{C})$ which is exactly (12). Thus the condition on g is optimal. The condition in Theorem 1.1 that γ is $C^{5+\alpha}$ is certainly not optimal although it is not immediately clear what the optimal condition is. If we consider the case when $g = 0$ we can say more about the optimal condition on γ. This is the topic of the next subsection. At the end of that subsection we give a conjecture on the the optimal condition on γ in the theorem.

If γ has a cusp it is not a quasicircle and the Grunsky inequality (6) no longer holds with $\kappa < 1$. It would be interesting to see what the effect of a cusp is on the asymptotics of the determinant. Another question is to generalize the result to an arc instead of a Jordan curve. There are results in case when γ is an interval in $\mathbb{R}$ since in that case we get a Hankel determinant, see [7], [5, 9]. In the case of an arc on the unit circle there is an asymptotic formula due to Widom in [22]. See also [11] when we have Fisher-Hartwig singularities, and [3] for a relative Szegő theorem.

1.3 Convergence of the Partition Function and Weil-Petersson Quasicircles

Note that since $\det(I+K) = \det(I - B^*B)$, see (50), it follows that (14) can be written as

$$\lim_{n\to\infty} \log \frac{Z_n(\gamma)/\operatorname{cap}(\gamma)^{n^2}}{Z_n(\mathbb{T})/\operatorname{cap}(\mathbb{T})^{n^2}} = \lim_{n\to\infty} \log \frac{Z_n(\gamma)}{(2\pi)^n \operatorname{cap}(\gamma)^{n^2}} = -\frac{1}{2}\log\det(I - B^*B), \tag{17}$$

since $\mathrm{cap}(\mathbb{T}) = 1$. Interestingly, the right side of (17) has occurred in other contexts. In [18] it appears, up to a multiplicative constant, under the name universal Liouville action which is a Kähler potential for the Weil-Petersson metric on the $T_0(1)$ component of the universal Teichmüller space $T(1)$. It is also the so called *Loewner energy* of the Jordan curve, see [14, 21]. Motivated in part by connections to the Schramm-Loewner Evolution, the Loewner energy has been further studied in [19, 20]. Curves with the property that the Grunsky operator is a Hilbert-Schmidt operator are called *Weil-Petersson quasicircles*. There are many different characterizations and possible definitions of Weil-Petersson quasicircles, see [1] and [21, Sect. 8]. It is therefore natural to conjecture that (17) holds if and only if γ is a Weil-Petersson quasicircle. We will prove this but in order to state a theorem let us be a bit more precise.

Let γ be a Jordan curve and let $\phi(z)$ be the exterior mapping function for γ as above. Set $\phi_r(z) = \frac{1}{r}\phi(rz)$ for $|z| \geq \rho^{-1}$, where $1 < \rho < r$, and let γ_r be the image of $\mathbb{T}$ under ϕ_r. Note that ϕ_r is an analytic curve so in particular $Z_n(\gamma_r)$ is well-defined. Define the function

$$E_n(r) = \log \frac{Z_n(\gamma_r)}{(2\pi)^n \mathrm{cap}(\gamma)^{n^2}}, \tag{18}$$

which we informally can think of as a finite n Loewner energy of γ_r. As $n \to \infty$ it converges to a multiple of the Loewner energy by (17). The following lemma will be proved in Sect. 4.

Lemma 1.2 *The function $E_n(r)$ is decreasing in $(1, \infty)$ for every $n \geq 1$.*

The sequence $E_n(r)$ is also increasing in n for each fixed $r > 1$. This is the content of Lemma 4.1 below.

The determinant in (1) is not well-defined for a general Jordan curve. Therefore we define the n:th partition function for the Jordan curve γ by

$$Z_n(\gamma) = \lim_{r \to 1+} Z_n(\gamma_r) \tag{19}$$

with value in $\mathbb{R} \cup \{\infty\}$. The limit exists by Lemma 1.2 possibly equal to infinity. We can now formulate our theorem which gives a new characterization of Weil-Petersson quasicircles and a way to compute their Loewner energy.

Theorem 1.3 *The Jordan curve γ is a Weil-Petersson quasicircle if and only if*

$$\limsup_{n \to \infty} \frac{Z_n(\gamma)}{(2\pi)^n cap(\gamma)^{n^2}} < \infty \tag{20}$$

and in that case we have the limit (17).

In fact, by Lemma 4.1, the sequence in (20) is increasing in n so we could replace the upper limit with a proper limit. The theorem is proved in Sect. 4. In view of this theorem it is reasonable to conjecture that the optimal condition on the curve γ in theorem 1.1 is that γ is a Weil-Petersson quasicircle.

1.4 The β-Ensemble

Consider the β-ensemble corresponding to (2), i.e. consider

$$D_{n,\beta}[e^g] = \frac{1}{n!}\int_{\gamma^n} \prod_{1\le\mu\neq\nu\le n} |\zeta_\mu - \zeta_\nu|^{\beta/2} \prod_{\mu=1}^{n} e^{g(\zeta_\mu)} \prod_{\mu=1}^{n} |d\zeta_\mu|,$$

where $\beta > 0$. This is not a determinant when $\beta \neq 2$ but is the quantity analogous to (1). The corresponding partition function is $Z_{n,\beta}(\gamma) = D_{n,\beta}[1]$. If $\gamma = \mathbb{T}$, then

$$Z_{n,\beta}(\mathbb{T}) = \frac{(2\pi)^n}{n!}\frac{\Gamma(1+\beta n/2)}{\Gamma(1+\beta/2)^n}.$$

The expansion (5) gives

$$\log\phi'(z) = -\sum_{k=2}^{\infty}\left(\sum_{j=1}^{k-1} a_{j,k-j}\right) z^{-k}. \tag{21}$$

Define

$$\mathbf{g}_\beta = (\beta/2-1)\mathbf{d} + \mathbf{g},$$

where

$$\mathbf{d} = \begin{pmatrix} \left(\frac{1}{2}\sqrt{k}\mathrm{Re}\left(\sum_{j=1}^{k-1} a_{j,k-j}\right)\right)_{k\ge1} \\ \left(\frac{1}{2}\sqrt{k}\mathrm{Im}\left(\sum_{j=1}^{k-1} a_{j,k-j}\right)\right)_{k\ge1} \end{pmatrix}$$

comes from $\log|\phi'(z)|$. Here the $k=1$ component is $=0$, compare with (21). We *conjecture* that, if $a_0 = 0$, then

$$\lim_{n\to\infty}\frac{D_{n,\beta}[e^g]}{Z_{n,\beta}(\mathbb{T})\mathrm{cap}(\gamma)^{\beta n(n-1)/2+n}} = \frac{1}{\sqrt{\det(I+K)}}\exp\left(\frac{2}{\beta}\mathbf{g}_\beta^t(I+K)^{-1}\mathbf{g}_\beta\right). \tag{22}$$

We will give a heuristic argument for this in Section 5. If we let $\beta = 2$ in this argument it also gives the idea behind the proof of (13) given in the paper, although making it precise is more complicated. If the conjecture is correct, we see that the appearance of the Fredholm determinant $\det(I+K) = \det(I - B^*B)$ is not related to $\beta = 2$. The proof of (16) in [9] does not use the fact that $\beta = 2$ in an essential way, so by slightly modifying that proof it should be possible to show that

$$\lim_{n\to\infty} \frac{D_{n,\beta}[e^g]}{D_{n,\beta}[1]} = \exp\left(\frac{2}{\beta}\mathbf{g}^t(I+K)^{-1}\mathbf{g} + 2(1-\frac{2}{\beta})\mathbf{d}^t(I+K)^{-1}\mathbf{g}\right),$$

under the assumptions in [9].

2 Proof of the Main Theorem

In this section we will prove Theorem 1.1. An essential ingredient is Lemma 2.2 which is proved in the next section. Assume that γ is a $C^{5+\alpha}$-curve for some $\alpha > 0$. Then, by a theorem of Kellogg, see e.g. [4, Thm. II 4.3], the map ϕ can be extended to $|z| \geq 1$ in such a way that ϕ is $C^{5+\alpha}$ on $\mathbb{T}$, and $\phi'(z) \neq 0$ when $|z| \geq 1$. Clearly, the expansion (5) then holds for $|z|, |\zeta| \geq 1$. It follows from (2), by introducing the parametrization $\zeta_\mu = \operatorname{cap}(\gamma)\phi(e^{\mathrm{i}\theta_\mu})$, that

$$D_n[e^g] = \frac{\operatorname{cap}(\gamma)^{n^2}}{n!} \int_{[-\pi,\pi]^n} \exp\left[\sum_{\mu\neq\nu} \log|\phi(e^{\mathrm{i}\theta_\mu}) - \phi(e^{\mathrm{i}\theta_\nu})| \right.$$
$$\left. + \sum_\mu \log|\phi'(e^{\mathrm{i}\theta_\mu})| + g(\phi(e^{\mathrm{i}\theta_\mu}))\right] d\theta. \tag{23}$$

Combining (21) with (5) leads to the identity

$$\sum_{\mu\neq\nu} \log|\phi(e^{\mathrm{i}\theta_\mu}) - \phi(e^{\mathrm{i}\theta_\nu})| + \sum_\mu \log|\phi'(e^{\mathrm{i}\theta_\mu})| \tag{24}$$
$$= \sum_{\mu\neq\nu} \log|e^{\mathrm{i}\theta_\mu} - e^{\mathrm{i}\theta_\nu}| - \operatorname{Re} \sum_{k,\ell=1}^\infty a_{k\ell}\left(\sum_\mu e^{-\mathrm{i}k\theta_\mu}\right)\left(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu}\right).$$

Let

$$\mathbb{E}_n[(\cdot)] = \frac{1}{(2\pi)^n n!} \int_{[-\pi,\pi]^n} \exp\left(\sum_{\mu\neq\nu} \log|e^{\mathrm{i}\theta_\mu} - e^{\mathrm{i}\theta_\nu}|\right)(\cdot)\, d\theta, \tag{25}$$

be the expectation over the eigenvalues $e^{\mathrm{i}\theta_\mu}$ of a random unitary matrix with respect to normalized Haar measure, [12]. It follows from (23), (24) and (25) that

$$D_n[e^g] = (2\pi)^n \mathrm{cap}(\gamma)^{n^2} \mathbb{E}_n \left[\exp\left(-\operatorname{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \left(\sum_\mu e^{-\mathrm{i}k\theta_\mu}\right)\left(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu}\right) + \sum_\mu g(\phi(e^{\mathrm{i}\theta_\mu}))\right)\right]. \tag{26}$$

To proceed we will need some linear algebra. Let $b_{k\ell}$ be given by (8) and define

$$B_m = (b_{k\ell})_{1\le k,\ell\le m} = B_m^{(1)} + \mathrm{i}B_m^{(2)}, \tag{27}$$

so that $B_m^{(1)}$ and $B_m^{(2)}$ are real symmetric m times m matrices. Since B_m is a complex, symmetric matrix there is a unitary matrix $U_m = R_m + \mathrm{i}S_m$, with R_m and S_m real, such that

$$B_m = U_m \Lambda_m U_m^t, \tag{28}$$

where $\Lambda_m = \mathrm{diag}(\lambda_{m,1}, \dots, \lambda_{m,m})$ and $\lambda_{m,k}$, $1 \le k \le m$, are the singular values of B_m, [8, Sec. 4.4]. Define the real, symmetric $2m$ by $2m$ matrix

$$K_m = \begin{pmatrix} B_m^{(1)} & B_m^{(2)} \\ B_m^{(2)} & -B_m^{(1)} \end{pmatrix},$$

and the matrices

$$T_m = \begin{pmatrix} R_m & S_m \\ S_m & -R_m \end{pmatrix}, \quad \tilde{\Lambda}_m = \begin{pmatrix} \Lambda_m & 0 \\ 0 & -\Lambda_m \end{pmatrix}.$$

Lemma 2.1 *The matrix T_m is orthogonal and*

$$K_m = T_m \tilde{\Lambda}_m T_m^t, \tag{29}$$

so that K_m has eigenvalues $\pm\lambda_{m,k}$. These eigenvalues satisfy

$$|\lambda_{m,k}| \le \kappa < 1, \tag{30}$$

where κ is the constant in the Grunsky inequality (7). Furthermore, if $\mathbf{x} = (x_j)_{1\le j\le m}$, $\mathbf{y} = (y_j)_{1\le j\le m}$ are real column vectors , then

$$Re \sum_{k,\ell=1}^{m} b_{k\ell}(x_k - \mathrm{i}y_k)(x_\ell - \mathrm{i}y_\ell) = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^t K_m \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}. \tag{31}$$

Proof That T_m is orthogonal follows from $U_m U_m^* = I$. The identities (27), (28) and $U_m = R_m + \mathrm{i}S_m$ give

$$B_m^{(1)} = R_m \Lambda_m R_m^t - S_m \Lambda_m S_m^t,$$
$$B_m^{(2)} = R_m \Lambda_m S_m^t + S_m \Lambda_m R_m^t,$$

which translates into (29). We also see that

$$\operatorname{Re} \sum_{k,\ell=1}^{m} b_{k\ell}(x_k - \mathrm{i}y_k)(x_\ell - \mathrm{i}y_\ell) = \sum_{k,\ell=1}^{m} b_{k\ell}^{(1)}(x_k x_\ell - y_k y_\ell) + b_{k\ell}^{(2)}(x_k y_\ell + y_k x_\ell)$$
$$= \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^t \begin{pmatrix} B_m^{(1)} & B_m^{(2)} \\ B_m^{(2)} & -B_m^{(1)} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix},$$

which proves (31). It follows from (7) and (31) that

$$\left| \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^t K_m \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \right| \le \kappa \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^t \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix},$$

which shows that all the eigenvalues of K_m have absolute value $\le \kappa$.

Let

$$\mathbf{X} = \left(\frac{1}{\sqrt{k}} \sum_\mu \cos k\theta_\mu \right)_{k \ge 1}, \quad \mathbf{Y} = \left(\frac{1}{\sqrt{k}} \sum_\mu \sin k\theta_\mu \right)_{k \ge 1},$$

be infinite column vectors, and let P_m denote projection onto the first m components. It follows from (29) and (31) that

$$-\operatorname{Re} \sum_{k,\ell=1}^{m} a_{k\ell} \left(\sum_\mu e^{-\mathrm{i}k\theta_\mu} \right) \left(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu} \right) = \begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix}^t T_m \begin{pmatrix} -\Lambda_m & 0 \\ 0 & \Lambda_m \end{pmatrix} T_m^t \begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix}. \tag{32}$$

Define, for $\zeta \in \mathbb{C}$,

$$M_m(\zeta) = \begin{pmatrix} \zeta \Lambda_m^{1/2} & 0 \\ 0 & \Lambda_m^{1/2} \end{pmatrix} T_m^t \begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix}, \tag{33}$$

so that

$$-\operatorname{Re} \sum_{k,\ell=1}^{m} a_{k\ell} \left(\sum_\mu e^{-\mathrm{i}k\theta_\mu} \right) \left(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu} \right) = M_m(\mathrm{i})^t M_m(\mathrm{i}). \tag{34}$$

Without loss of generality we can assume that $a_0 = 0$ in (10) by just subtracting off the mean. For $\zeta \in \mathbb{C}$, we define

$$a_k(\zeta) = \frac{1}{2}\sqrt{k}(\operatorname{Re} a_k + \zeta \operatorname{Im} a_k), \quad b_k(\zeta) = \frac{1}{2}\sqrt{k}(\operatorname{Re} b_k + \zeta \operatorname{Im} b_k), \tag{35}$$

and the infinite column vectors

$$\mathbf{a}(\zeta) = (a_k(\zeta))_{k\geq 1}, \quad \mathbf{b}(\zeta) = (b_k(\zeta))_{k\geq 1},$$

which lie in $\ell^2(\mathbb{C})$ by the assumption (12). Set

$$\mathbf{g}(\zeta) = \begin{pmatrix} \mathbf{a}(\zeta) \\ \mathbf{b}(\zeta) \end{pmatrix},$$

so that $\mathbf{g}(\mathrm{i}) = \mathbf{g}$ given by (11). We see that (10) can be written

$$\sum_{\mu} g(\phi(e^{\mathrm{i}\theta_\mu})) = 2\mathbf{g}(\mathrm{i})^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}. \tag{36}$$

Define

$$w_m(\theta_1, \theta_2) = \operatorname{Re} \sum_{k\vee\ell>m} a_{k\ell} e^{-\mathrm{i}k\theta_1 - \mathrm{i}\ell\theta_2}, \tag{37}$$

and

$$W_m(\theta) = -\sum_{\mu,\nu} w_m(\theta_\mu, \theta_\nu). \tag{38}$$

Combining (34), (36), (37) and (38), we obtain the identity

$$\begin{aligned} &-\operatorname{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \left(\sum_{\mu} e^{-\mathrm{i}k\theta_\mu}\right) \left(\sum_{\nu} e^{-\mathrm{i}\ell\theta_\nu}\right) + \sum_{\mu} g(\phi(e^{\mathrm{i}\theta_\mu})) \\ &= M_m(\mathrm{i})^t M_m(\mathrm{i}) + 2\mathbf{g}(\mathrm{i})^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} + W_m(\theta), \end{aligned}$$

for every $m \geq 1$. Using this identity in (26) leads us to define the entire function

$$G_{m,n}(\zeta) = \mathbb{E}_n[\exp(M_m(\zeta)^t M_m(\zeta) + 2\mathbf{g}(\zeta) \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} + W_m(\theta))], \tag{39}$$

for $\zeta \in \mathbb{C}$, so that, for any $m \geq 1$,

$$D_n(e^g) = (2\pi)^n \mathrm{cap}(\gamma)^{n^2} G_{m,n}(\mathrm{i}). \tag{40}$$

Note that $G_{m,n}(\mathrm{i})$ is independent of m, but of course for other ζ the function $G_{m,n}(\zeta)$ does depend on m.

The expression $M_m(\zeta)^t M_m(\zeta)$ is a quadratic form in $\mathbf{X}$ and $\mathbf{Y}$ and we want instead to have a linear form in $\mathbf{X}$ and $\mathbf{Y}$. This can be achieved by using a Gaussian integral, an idea that was also used in [10, Sec. 6.5]. Let $\mathbf{u} = (u_k)_{1\leq k\leq m}$, $\mathbf{v} = (v_k)_{1\leq k\leq m}$ be real column vectors. Then,

$$\exp(M_m(\zeta)^t M_m(\zeta)) = \frac{1}{\pi^m}\int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\left(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} + 2\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t M_m(\zeta)\right)$$

and Fubini's theorem in (39) to get the formula

$$G_{m,n}(\zeta) = \frac{1}{\pi^m}\int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix})\mathbb{E}_n[\exp(2\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t M_m(\zeta) + 2\mathbf{g}(\zeta)\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} + W_m(\theta))]. \tag{41}$$

From the definition (33) we see that

$$\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t M_m(\zeta) = L_m(\zeta)^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix},$$

where

$$L_m(\zeta) = \begin{pmatrix}P_m & 0\\ 0 & P_m\end{pmatrix}^t T_m \begin{pmatrix}\zeta\Lambda_m^{1/2} & 0\\ 0 & \Lambda_m^{1/2}\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}. \tag{42}$$

Thus,

$$G_{m,n}(\zeta) = \frac{1}{\pi^m}\int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix})\mathbb{E}_n[\exp(2(L_m(\zeta)+\mathbf{g}(\zeta))^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} + W_m(\theta))]. \tag{43}$$

Note that by the definitions

$$|G_{m,n}(\zeta)| \leq G_{m,n}(\mathrm{Re}\,\zeta). \tag{44}$$

We will now state a lemma that will allow us to prove the theorem. The proof of the lemma will be given in the next section. Define the function

$$f_\zeta(\lambda) = \begin{cases} (1-\zeta^2\lambda)^{-1}\zeta^2\lambda, & \lambda \geq 0 \\ -(1+\lambda)^{-1}\lambda, & \lambda < 0 \end{cases} \tag{45}$$

on the real line. We can use spectral calculus to define $f_\zeta(K)$. Recall that K is a symmetric trace class operator with spectrum in $[-\kappa, \kappa]$. Define

$$G(\zeta) = \frac{1}{\sqrt{\det(I-\zeta^2|B|)\det(I-|B|)}} \exp\left(\mathbf{g}(\zeta)^t (I + f_\zeta(K))\mathbf{g}(\zeta)\right), \tag{46}$$

which is holomorphic in $|\zeta| < \kappa$. Note that B is a trace-class operator by Lemma 3.1 below.

Lemma 2.2 *Let $G_{m,n}(\zeta)$ be defined by (39). Then if $\rho \in (1, 1/\sqrt{\kappa})$ there is a constant C so that*

$$|G_{m,n}(\zeta)| \leq C \tag{47}$$

for all $|\zeta| \leq \rho$, $n \geq 1$ and m sufficiently large. Also, if ζ is real and $|\zeta| \leq \rho$, then

$$\lim_{m\to\infty}\lim_{n\to\infty} G_{m,n}(\zeta) = G(\zeta). \tag{48}$$

Assume Lemma 2.2. It follows from (47) that $\{G_{m,n}(\zeta)\}$ is a normal family in $|\zeta| < \rho$. Let $H_n = G_{m,n}(\mathrm{i})$, which is independent of m. By (47), $|H_n| \leq C$ for all $n \geq 1$. Let $\{H_{n_i}\}$ be any convergent subsequence. If we can show that

$$\lim_{i\to\infty} H_{n_i} = G(\mathrm{i}), \tag{49}$$

we are done since,

$$I + f_{\mathrm{i}}(K) = I - (I+K)^{-1}K = (I+K)^{-1}.$$

Also,

$$\det(I+|B|)\det(I-|B|) = \det(I-|B|^2) = \prod_{j=1}^{\infty}(1-\lambda_j^2) = \det(I+K), \tag{50}$$

where λ_j are the singular values of B. The fact that $\{G_{m,n}(\zeta)\}$ is a normal family and a diagonal argument shows that there is a subsequence $\{n_{i,i}\}$ of $\{n_i\}$ such that

$$\lim_{i\to\infty} G_{m,n_{i,i}}(\zeta) =: G_m(\zeta)$$

uniformly in $|\zeta| \le \rho' < \rho$ for each sufficiently large m. By (47), $\{G_m(\zeta)\}$ is a normal family in $|\zeta| < \rho$. Let $\{m_j\}$ be any sequence such that $G_{m_j}(\zeta)$ converges uniformly in $|\zeta| \le \rho'$, where $1 < \rho' < \rho$. Then, for ζ real, $|\zeta| \le \rho'$,

$$\lim_{j\to\infty} G_{m_j}(\zeta) = \lim_{j\to\infty}\lim_{i\to\infty} G_{m_j,n_{i,i}}(\zeta) = G(\zeta),$$

by (48). Since $\{m_j\}$ was arbitrary, we see that $\lim_{m\to\infty} G_m(\zeta) = G(\zeta)$ uniformly for $\zeta \in \mathbb{C}$, $|\zeta| \le \rho'$. For any fixed m,

$$\lim_{i\to\infty} H_{n_i} = \lim_{i\to\infty} H_{n_{i,i}} = \lim_{i\to\infty} G_{m,n_{i,i}}(\mathrm{i}) = G_m(\mathrm{i}).$$

We can now let $m \to \infty$ to get (49). This completes the proof of the theorem.

3 Proof of Lemma 2.2

We start by giving a technical lemma that we will use below.

Lemma 3.1 *Assume that γ is a $C^{5+\alpha}$ curve for some $\alpha > 0$ so that the extended exterior mapping function ϕ is $C^{5+\alpha}$ on $\mathbb{T}$. Let the operator B on $\ell^2(\mathbb{C})$ be defined by (9). Then B is a trace class operator. Also, if δ_m is defined by*

$$\delta_m = \left(\sum_{k\vee\ell>m} (k\ell)^{2+\epsilon}|b_{k\ell}|^2\right)^{1/2}, \tag{51}$$

we have that $\delta_m \to 0$ as $m \to \infty$. Furthermore there is a constant C so that

$$\sum_{k\vee\ell>m} (k^2+\ell^2)|a_{k\ell}| \le C \tag{52}$$

for all $m \ge 1$.

Proof Since ϕ is $C^{5+\alpha}$ it follows from the definition of the Grunsky coefficients that there is a constant C such that

$$|a_{k\ell}| \le \frac{C}{k^{2+\epsilon}\ell^{2+\epsilon}}, \tag{53}$$

$$|a_{k\ell}| \le \frac{C}{k^{3+\epsilon}\ell^{1+\epsilon}}, \tag{54}$$

for some $\epsilon > 0$. Consider the operators given by $K = (k^{1+\epsilon/2}\ell^{1+\epsilon/2}a_{k\ell})_{k,\ell\geq 1}$ and $D = (k^{-1/2-\epsilon/2}\delta_{k\ell})_{k,\ell\geq 1}$. Then K and D are Hilbert-Schmidt operators, and since $B = DKD$ we see that B is a trace class operator.

We see from (8), (51), and (53) that

$$\delta_m^2 = \sum_{k\vee\ell>m} k^{2+\epsilon}\ell^{2+\epsilon}|b_{k\ell}|^2 \leq \sum_{k\vee\ell>m} \frac{C}{k^{1+\epsilon}\ell^{1+\epsilon}},$$

which $\to 0$ as $m \to \infty$. Also, since $a_{k\ell}$ is symmetric (54) gives the estimate

$$\sum_{k\vee\ell>m} (k^2+\ell^2)|a_{k\ell}| = 2\sum_{k\vee\ell>m} k^2|a_{k\ell}| \leq C$$

for all $m \geq 1$. □

We turn now to the proof of the estimate (47). This proof will also give us an upper bound in (48). After that we will prove a lower bound in (48) which will coincide with the upper bound and hence prove the limit. First, we need an estimate of $W_m(\theta)$ defined by (38). Note that

$$W_m(\theta) \leq \sum_{k\vee\ell>m} |b_{k\ell}||X_k - \mathrm{i}Y_k||X_\ell - \mathrm{i}Y_\ell| = \lim_{M\to\infty} \sum_{\substack{k\vee\ell>m \\ k\wedge\ell\leq M}} |b_{k\ell}||X_k - \mathrm{i}Y_k||X_\ell - \mathrm{i}Y_\ell|.$$

Let $\epsilon > 0$ be fixed. By the Cauchy-Schwarz' inequality

$$\begin{aligned} &\sum_{\substack{k\vee\ell>m \\ k\wedge\ell\leq M}} |b_{k\ell}||X_k - \mathrm{i}Y_k||X_\ell - \mathrm{i}Y_\ell| \qquad (55)\\ &\leq \left(\sum_{\substack{k\vee\ell>m \\ k\wedge\ell\leq M}} (k\ell)^{2+\epsilon}|b_{k\ell}|^2\right)^{1/2} \left(\sum_{\substack{k\vee\ell>m \\ k\wedge\ell\leq M}} \frac{1}{k^{2+\epsilon}}|X_k - \mathrm{i}Y_k|^2 \frac{1}{\ell^{2+\epsilon}}|X_\ell - \mathrm{i}Y_\ell|^2\right)^{1/2} \\ &\leq \delta_m \left(\sum_{k=1}^{M} \frac{1}{k^{2+\epsilon}}(X_k^2 + Y_k^2)\right). \end{aligned}$$

We know from Lemma 3.1 that our assumptions on γ imply that $\delta_m \to 0$ as $m \to \infty$. Define the $2M$ times $2M$ matrix $D_{m,M}$ by

$$D_{m,M}^{-1} = \begin{pmatrix} \mathrm{diag}\,(\frac{\delta_m}{k^{2+\epsilon}})_{1\leq k\leq M} & 0 \\ 0 & \mathrm{diag}\,(\frac{\delta_m}{k^{2+\epsilon}})_{1\leq k\leq M} \end{pmatrix}.$$

Because of the inequality (44) we can assume that $\zeta \in \mathbb{R}$ which we will do from now on. We see from (43), Fatou's lemma and (55) that

$$G_{m,n}(\zeta) \le \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}) \tag{56}$$

$$\times \lim_{M\to\infty} \mathbb{E}_n \left[\exp\left(2(L_m(\zeta)^t + \mathbf{g}(\zeta))^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} + \sum_{\substack{k\vee\ell>m\\ k\wedge\ell\le M}} |b_{k\ell}||X_k - \mathrm{i}Y_k||X_\ell - \mathrm{i}Y_\ell| \right)\right]$$

$$\le \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix})$$

$$\times \lim_{M\to\infty} \mathbb{E}_n \left[\exp\left(2(L_m(\zeta)^t + \mathbf{g}(\zeta))^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} + \begin{pmatrix}P_M\mathbf{X}\\ P_M\mathbf{Y}\end{pmatrix}^t D_{m,M}^{-1} \begin{pmatrix}P_M\mathbf{X}\\ P_M\mathbf{Y}\end{pmatrix}\right)\right].$$

We now use the Gaussian integral

$$\exp\left(\begin{pmatrix}P_M\mathbf{X}\\ P_M\mathbf{Y}\end{pmatrix}^t D_{m,M}^{-1} \begin{pmatrix}P_M\mathbf{X}\\ P_M\mathbf{Y}\end{pmatrix}\right) = \frac{1}{\pi^M}\left(\prod_{k=1}^M \frac{k^{2+\epsilon}}{\delta_m}\right) \int_{\mathbb{R}^M} dp \int_{\mathbb{R}^M} dq$$

$$\times \exp\left(-\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}^t D_{m,M} \begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix} + 2\begin{pmatrix}P_M^t\mathbf{p}\\ P_M^t\mathbf{q}\end{pmatrix}^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right),$$

where $\mathbf{p}$ and $\mathbf{q}$ are column vectors in $\mathbb{R}^M$. If we use this identity in (56), we obtain the estimate

$$G_{m,n}(\zeta) \le \frac{1}{\pi^{m+M}}\left(\prod_{k=1}^M \frac{k^{2+\epsilon}}{\delta_m}\right) \int_{\mathbb{R}^M} dp \int_{\mathbb{R}^M} dq \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \tag{57}$$

$$\times \lim_{M\to\infty} \exp\left(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} - \begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}^t D_{m,M} \begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}\right)$$

$$\times \mathbb{E}_n\left[\exp\left(2\left(L_m(\zeta) + \mathbf{g}(\zeta) + \begin{pmatrix}P_M^t\mathbf{p}\\ P_M^t\mathbf{q}\end{pmatrix}\right)^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right)\right].$$

We will now make use of the following upper bound which is a consequence of the strong Szegő limit theorem, [9, p. 268], or the Geronimo-Case-Borodin-Okounkov identity, [10, Lemma 2.3].

Lemma 3.2 *We have the estimate*

$$\mathbb{E}_n\left[\exp\left(2\begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}^t\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right)\right]\le\exp\left(\begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}^t\begin{pmatrix}\mathbf{c}\\ \mathbf{d}\end{pmatrix}\right),\tag{58}$$

for infinite column vectors $\mathbf{c}=(c_k)_{k\ge1}$, $\mathbf{d}=(d_k)_{k\ge1}$ *in* $\ell^2(\mathbb{R})$.

The estimate (58) gives

$$\begin{aligned}&\mathbb{E}_n\left[\exp\left(2\left(L_m(\zeta)+\mathbf{g}(\zeta)+\begin{pmatrix}P_M^t\mathbf{p}\\ P_M^t\mathbf{q}\end{pmatrix}\right)^t\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right)\right]\qquad(59)\\ &\le\exp\left(\left(L_m(\zeta)+\mathbf{g}(\zeta)+\begin{pmatrix}P_M^t\mathbf{p}\\ P_M^t\mathbf{q}\end{pmatrix}\right)^t\left(L_m(\zeta)+\mathbf{g}(\zeta)+\begin{pmatrix}P_M^t\mathbf{p}\\ P_M^t\mathbf{q}\end{pmatrix}\right)\right)\\ &=\exp\left((L_m(\zeta)+\mathbf{g}(\zeta))^t(L_m(\zeta)+\mathbf{g}(\zeta))+2(L_m(\zeta)+\mathbf{g}(\zeta))^t\begin{pmatrix}P_M^t\mathbf{p}\\ P_M^t\mathbf{q}\end{pmatrix}+\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}^t\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}\right).\end{aligned}$$

Inserting this into (57), the pq-integral becomes

$$\begin{aligned}&\frac{1}{\pi^M}\left(\prod_{k=1}^M\frac{k^{2+\epsilon}}{\delta_m}\right)\int_{\mathbb{R}^M}dp\int_{\mathbb{R}^M}dq\exp\left(-\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}^t(D_{m,M}-I)\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}\right.\qquad(60)\\ &\left.+2(L_m(\zeta)+\mathbf{g}(\zeta))^t\begin{pmatrix}P_M&0\\ 0&P_M\end{pmatrix}^t\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}\right)\\ &=\left(\prod_{k=1}^M\frac{k^{2+\epsilon}/\delta_m}{k^{2+\epsilon}/\delta_m-1}\right)\\ &\times\exp\left((L_m(\zeta)+\mathbf{g}(\zeta))^t\begin{pmatrix}P_M&0\\ 0&P_M\end{pmatrix}^t(D_{m,M}-I)^{-1}\begin{pmatrix}P_M&0\\ 0&P_M\end{pmatrix}(L_m(\zeta)+\mathbf{g}(\zeta))\right)\\ &\le\prod_{k=1}^M\frac{1}{1-\delta_m/k^{2+\epsilon}}\exp\left(\frac{\delta_m}{1-\delta_m}(L_m(\zeta)+\mathbf{g}(\zeta))^t(L_m(\zeta)+\mathbf{g}(\zeta))\right),\end{aligned}$$

where the last inequality follows from the fact that all entries in $(D_{m,M}-I)^{-1}$ are $\le\delta_m(1-\delta_m)^{-1}$. If we assume that m is so large that $\delta_m\le1/2$ then there is a constant C_ϵ so that

$$\prod_{k=1}^M\frac{1}{1-\delta_m/k^{2+\epsilon}}\le e^{C_\epsilon\delta_m}.$$

Thus, (57), (59) and (60) give

$$G_{m,n}(\zeta) \le \frac{e^{C_\epsilon \delta_m}}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\Bigg(- \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^t \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} + \frac{1}{1-\delta_m} (L_m(\zeta) + \mathbf{g}(\zeta))^t (L_m(\zeta) + \mathbf{g}(\zeta)) \Bigg).$$

We now insert the definition (42) of $L_m(\zeta)$ into the right side. After some computation we get the estimate

$$G_{m,n}(\zeta) \le \frac{e^{C_\epsilon \delta_m}}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\Bigg(- \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^t \left(I - \begin{pmatrix} \frac{\zeta^2}{1-\delta_m}\Lambda_m & 0 \\ 0 & \frac{1}{1-\delta_m}\Lambda_m \end{pmatrix} \right) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \tag{61}$$

$$+ \frac{2}{1-\delta_m} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^t \begin{pmatrix} \zeta \Lambda_m^{1/2} & 0 \\ 0 & \Lambda_m^{1/2} \end{pmatrix} T_m^t \begin{pmatrix} P_m & 0 \\ 0 & P_m \end{pmatrix} \mathbf{g}(\zeta) + \frac{1}{1-\delta_m} \mathbf{g}(\zeta)^t \mathbf{g}(\zeta) \Bigg)$$

Since $0 \le \lambda_{m,k} \le \kappa$, we see that if $|\zeta| \le \rho < 1/\sqrt{\kappa}$ and m is so large that $\rho^2\kappa/(1-\delta_m) < 1$, then the matrix $I - \begin{pmatrix} \frac{\zeta^2}{1-\delta_m}\Lambda_m & 0 \\ 0 & \frac{1}{1-\delta_m}\Lambda_m \end{pmatrix}$ is positive definite. Hence, we can compute the Gaussian integral in (61) to get the estimate

$$G_{m,n}(\zeta) \le \frac{e^{C_\epsilon \delta_m}}{\sqrt{\det(I - \frac{\zeta^2}{1-\delta_m}|B_m|) \det(I - \frac{1}{1-\delta_m}|B_m|)}} \exp\Bigg(\frac{1}{1-\delta_m} \begin{pmatrix} P_m \mathbf{a}(\zeta) \\ P_m \mathbf{b}(\zeta) \end{pmatrix}^t T_m \tag{62}$$

$$\times \begin{pmatrix} (I - \frac{\zeta^2}{1-\delta_m}\Lambda_m)^{-1} \frac{\zeta^2}{1-\delta_m}\Lambda_m & 0 \\ 0 & (I - \frac{1}{1-\delta_m}\Lambda_m)^{-1} \frac{1}{1-\delta_m}\Lambda_m \end{pmatrix} T_m^t \begin{pmatrix} P_m \mathbf{a}(\zeta) \\ P_m \mathbf{b}(\zeta) \end{pmatrix} \tag{63}$$

$$+ \frac{1}{1-\delta_m} \mathbf{g}(\zeta)^t \mathbf{g}(\zeta) \Bigg)$$

for all $\zeta \in [-\rho, \rho]$ and m sufficiently large. Since T_m is an orthogonal matrix we see that the $\ell^2(\mathbb{R}) \oplus \ell^2(\mathbb{R})$-norm of $T_m^t \begin{pmatrix} P_m \mathbf{a}(\zeta) \\ P_m \mathbf{b}(\zeta) \end{pmatrix}$ is

$$\le \sum_{k=1}^m a(\zeta)_k^2 + b(\zeta)_k^2 \le \rho^2 \sum_{k=1}^\infty k(|a_k|^2 + |b_k|^2) < \infty,$$

by the assumption (12). Since $|\zeta| \le \rho$, $\lambda_{m,k}$, and $\rho^2\kappa/(1-\delta_m) < 1$, $\delta_m < 1/2$ for m sufficiently large, we see that the expression in the exponent in the right side of (62) is bounded by a constant. Note that it is proved in Lemma 3.1 that $\delta_m \to 0$ as $m \to \infty$ and hence $C_\epsilon \delta_m \le 1$ if m is sufficiently large. Also, since B is trace class

$$\det(I - \frac{\zeta^2}{1-\delta_m}|B_m|) \to \det(I - \zeta^2|B|)$$

as $m \to \infty$ for $|\zeta| \le \rho$. This proves (47) in Lemma 2.2. If we recall (45), we se that (62) gives

$$\overline{\lim_{n\to\infty}}\, G_{m,n}(\zeta) \le \frac{e^{C_\epsilon \delta_m}}{\sqrt{\det(I - \frac{\zeta^2}{1-\delta_m}|B_m|)\det(I - \frac{1}{1-\delta_m}|B_m|)}}$$
$$\times \exp\left(\frac{1}{1-\delta_m}\begin{pmatrix} P_m\mathbf{a}(\zeta) \\ P_m\mathbf{b}(\zeta)\end{pmatrix}^t f_\zeta\left(\frac{1}{1-\delta_m}K_m\right)\begin{pmatrix} P_m\mathbf{a}(\zeta) \\ P_m\mathbf{b}(\zeta)\end{pmatrix} + \frac{1}{1-\delta_m}\mathbf{g}(\zeta)^t\mathbf{g}(\zeta)\right),$$

for $\zeta \in \mathbb{R}$, $|\zeta| \le \rho$. We can let $m \to \infty$ in the right side to conclude

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\, G_{m,n}(\zeta) \le \frac{\exp\left(\mathbf{g}(\zeta)^t(I + f_\zeta(K))\mathbf{g}(\zeta)\right)}{\sqrt{\det(I-\zeta^2|B|)\det(I-|B|)}} = G(\zeta), \tag{64}$$

for $\zeta \in \mathbb{R}$, $|\zeta| \le \rho$.

In order to prove (48) we also need a lower bound. Fix $D > 0$ and let $\zeta \in \mathbb{R}$. We see from (43) that

$$G_{m,n}(\zeta) \ge \frac{1}{\pi^m}\int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix})$$
$$\times \mathbb{E}_n\left[\exp\left(2(L_m(\zeta) + \mathbf{g}(\zeta))^t\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} + W_m(\theta)\right)\right]. \tag{65}$$

Let $f(\theta)$ be such that

$$2(L_m(\zeta) + \mathbf{g}(\zeta))^t\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = \sum_\mu f(\theta_\mu). \tag{66}$$

Note that f is real-valued since $\zeta \in \mathbb{R}$. We want to estimate

$$\mathbb{E}_n[\exp(\sum_\mu f(\theta_\mu) + W_m(\theta))]$$

from below. To do this we will use an idea from [9, Lemma 2.3]. Let $h(\theta)$ be a given, smooth 2π-periodic, real-valued function, and let $C(h)$ denote a positive constant, whose exact meaning will change, that depends only on h but not on n or θ. Where it occurs below it can be bounded by $||h||_\infty$, $||h'||_\infty$ and $||h''||_\infty$. Write

$$S_n(\theta) = \sum_\mu f(\theta_\mu - \frac{1}{n}h(\theta_\mu)),$$

$$U_n(\theta) = -\frac{1}{n}\sum_{\mu\neq\nu}\frac{1}{2}\cot(\frac{\theta_\mu-\theta_\nu}{2})(h(\theta_\mu)-h(\theta_\nu)),$$

$$V_n(\theta) = -\frac{1}{n}\sum_\mu h'(\theta_\mu) - \frac{1}{n}\sum_{\mu\neq\nu}\frac{(h(\theta_\mu)-h(\theta_\nu))^2}{\sin^2(\frac{\theta_\mu-\theta_\nu}{2})}.$$

If we let

$$\phi_\mu = \theta_\mu - \frac{1}{n}h(\theta_\mu),$$

a Taylor expansion gives

$$\begin{aligned}
&\sum_{\mu\neq\nu}\log|e^{\mathrm{i}\phi_\mu}-e^{\mathrm{i}\phi_\nu}| + \sum_\mu f(\phi_\mu) \qquad (67)\\
&= \sum_{\mu\neq\nu}\log\left|2\sin\frac{\theta_\mu-\theta_\nu-\frac{1}{n}(h(\theta_\mu)-h(\theta_\nu))}{2}\right| + \sum_\mu f(\theta_\mu - \frac{1}{n}h(\theta_\mu))\\
&= \sum_{\mu\neq\nu}\log|e^{\mathrm{i}\theta_\mu}-e^{\mathrm{i}\theta_\nu}| + S_n(\theta)+U_n(\theta)+V_n(\theta)+\frac{1}{n}\sum_\mu h'(\theta_\mu)+R_n^{(1)}(\theta),
\end{aligned}$$

where

$$|R_n^{(1)}(\theta)| \le \frac{C(h)}{n}. \qquad (68)$$

We see from (65), (66) and (67) that

$$\begin{aligned}
G_{m,n}(\zeta) &\ge \frac{1}{\pi^m}\int_{[-D,D]^m}du\int_{[-D,D]^m}dv\exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}) \qquad (69)\\
&\times\mathbb{E}_n\left[\exp\left(S_n(\theta)+U_n(\theta)+V_n(\theta)+\sum_{\mu,\nu}w_m(\theta_\mu-\frac{h(\theta_\mu)}{n},\theta_\nu-\frac{h(\theta_\nu)}{n})+R_n^{(2)}(\theta)\right)\right],
\end{aligned}$$

where

$$R_n^{(2)}(\theta) = R_n^{(1)}(\theta) + \sum_{\mu} \log(1 - \frac{1}{n}h'(\theta_\mu)) + \frac{1}{n}h'(\theta_\mu)$$

by (68) satisfies

$$|R_n^{(2)}(\theta)| \leq \frac{C(h)}{n}. \tag{70}$$

The 1- and 2-point marginal densities for $\mathrm{e}_n[\cdot]$ are given by, [12],

$$p_{1,n}(\theta) = \frac{1}{2\pi} \tag{71}$$

$$p_{2,n}(\theta_1, \theta_2) = \frac{1}{4\pi^2 n(n-1)} \left[n^2 - \left| \sum_{j=0}^{n-1} e^{\mathrm{i} j(\theta_1 - \theta_2)} \right|^2 \right].$$

These formulas can be used to show that

$$\mathbb{E}_n[U_n(\theta)] = 0, \quad \text{and} \quad \mathbb{E}_n[V_n(\theta)] \geq -\sum_{k=1}^{\infty} k|h_k|^2, \tag{72}$$

where h_k are the complex Fourier coefficients of h. Hence, we can use Jensen's inequality in (69) to get the estimate

$$G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}) \tag{73}$$

$$\times \exp\left(\mathbb{E}_n[S_n(\theta)] - \sum_{k=1}^{\infty} k|h_k|^2 - \mathbb{E}_n\left[\sum_{\mu,\nu} w_m(\theta_\mu - \frac{h(\theta_\mu)}{n}, \theta_\nu - \frac{h(\theta_\nu)}{n}) \right] - \frac{C(h)}{n} \right).$$

Define

$$T_n^{(1)} = \mathbb{E}_n[S_n(\theta)] = \frac{n}{2\pi} \int_{-\pi}^{\pi} f(\theta - \frac{1}{n}h(\theta))\, d\theta, \tag{74}$$

$$T_n^{(2)} = -\frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w_m(\theta_1 - \frac{1}{n}h(\theta_1), \theta_2 - \frac{1}{n}h(\theta_2))\, d\theta_1 d\theta_2,$$

$$T_n^{(3)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w_m(\theta_1 - \frac{1}{n}h(\theta_1), \theta_2 - \frac{1}{n}h(\theta_2)) \left| \sum_{j=0}^{n-1} e^{\mathrm{i} j(\theta_1 - \theta_2)} \right|^2 d\theta_1 d\theta_2$$

$$- \frac{n}{2\pi} \int_{-\pi}^{\pi} w_m(\theta - \frac{1}{n}h(\theta), \theta - \frac{1}{n}h(\theta))\, d\theta.$$

Then, using (71) and (73), we find that for $\zeta \in \mathbb{R}$,

$$G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp\left(- \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^t \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} + T_n^{(1)} + T_n^{(2)} + T_n^{(3)} \right. \tag{75}$$

$$\left. - \sum_{k=1}^{\infty} k|h_k|^2 - \frac{C(h)}{n} \right).$$

Note that f depends on u and v, and we can choose h to depend on u and v also. Hence $T_n^{(j)}$ depends on u and v. Define, for n sufficiently large (depending on h),

$$r_n(\theta) = \theta - \frac{1}{n} h(\theta), \quad \text{and} \quad s_n(\theta) = r_n^{-1}(\theta). \tag{76}$$

Then, if we write

$$s_n'(\theta) = 1 + \frac{1}{n} h'(\theta) + \frac{1}{n^2} H_n(\theta), \tag{77}$$

we have the bound

$$|H_n(\theta)| \leq C(h). \tag{78}$$

By (74), (76), (77), and the fact that f has zero mean,

$$\begin{aligned} T_n^{(1)} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) h'(\theta)\, d\theta + \frac{1}{2\pi n} \int_{-\pi}^{\pi} f(\theta) H_n(\theta)\, d\theta \\ &= -\mathrm{i} \sum_{k \in \mathbb{Z}} k f_k h_{-k} + e_n^{(1)}, \end{aligned} \tag{79}$$

where

$$|e_n^{(1)}| \leq \frac{C(h)}{n} ||f||_1. \tag{80}$$

Also,

$$\begin{aligned} T_n^{(2)} &= -\frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w_m(\theta_1, \theta_2) s_n'(\theta_1) s_n'(\theta_2)\, d\theta_1 d\theta_2 \\ &= -\mathrm{Re} \sum_{k \vee \ell > m} a_{k\ell} \frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}k\theta_1 - \mathrm{i}\ell\theta_2} s_n'(\theta_1) s_n'(\theta_2)\, d\theta_1 d\theta_2. \end{aligned}$$

By (77) and (78),

$$\frac{n^2}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}k\theta_1-\mathrm{i}\ell\theta_2} s_n'(\theta_1)s_n'(\theta_2)\,d\theta_1 d\theta_2 = -k\ell h_k h_\ell + e_n^{(2)}(k,\ell),$$

where

$$|e_n^{(2)}(k,\ell)| \le \frac{C(h)}{n}.$$

Thus

$$T_n^{(2)} = \mathrm{Re}\sum_{k\vee\ell>m} k\ell a_{k\ell} h_k h_\ell - e_n^{(2)}, \tag{81}$$

where

$$|e_n^{(2)}| = \left|\mathrm{Re}\sum_{k\vee\ell>m} a_{k\ell} e_n^{(2)}(k,\ell)\right| \le \frac{C(h)}{n}, \tag{82}$$

by Lemma 3.1. We now consider $T_n^{(3)}$. By Taylor's theorem

$$w_m(\theta_1 - \frac{1}{n}h(\theta_1), \theta_2 - \frac{1}{n}h(\theta_2)) = \mathrm{Re}\sum_{k\vee\ell>m} a_{k\ell} e^{-\mathrm{i}k\theta_1-\mathrm{i}\ell\theta_2}$$
$$-\frac{1}{n}(h(\theta_1)+h(\theta_2))\mathrm{Re}\sum_{k\vee\ell>m} -\mathrm{i}k a_{k\ell} e^{-\mathrm{i}k\theta_1-\mathrm{i}\ell\theta_2} + e_n^{(3)}(\theta_1,\theta_2),$$

where

$$|e_n^{(3)}(\theta_1,\theta_2)| \le \frac{C(h)}{n^2}\sum_{k\vee\ell>m}(k^2+\ell^2)|a_{k\ell}| \le \frac{C(h)}{n^2}, \tag{83}$$

by Lemma 3.1. Thus, by the definition of $T_n^{(3)}$,

$$T_n^{(3)} = \mathrm{Re}\sum_{k\vee\ell>m} a_{k\ell}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}k\theta_1-\mathrm{i}\ell\theta_2}\left|\sum_{j=0}^{n-1} e^{\mathrm{i}j(\theta_1-\theta_2)}\right|^2 d\theta_1 d\theta_2 \tag{84}$$
$$-\frac{1}{n}\mathrm{Re}\sum_{k\vee\ell>m}\frac{-\mathrm{i}k a_{k\ell}}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(h(\theta_1)+h(\theta_2))e^{-\mathrm{i}k\theta_1-\mathrm{i}\ell\theta_2}\left|\sum_{j=0}^{n-1} e^{\mathrm{i}j(\theta_1-\theta_2)}\right|^2 d\theta_1 d\theta_2$$
$$+2\mathrm{Re}\sum_{k\vee\ell>m} a_{k\ell}\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)(-\mathrm{i}k e^{-\mathrm{i}(k+\ell)\theta})\,d\theta + e_N^{(3)} =: I_1 + I_2 + I_3 + e_n^{(3)},$$

where

$$e_n^{(3)} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e_n^{(3)}(\theta_1,\theta_2)\left|\sum_{j=0}^{n-1} e^{\mathrm{i}j(\theta_1-\theta_2)}\right|^2 d\theta_1 d\theta_2 - \frac{n}{2\pi}\int_{-\pi}^{\pi} e_n^{(3)}(\theta,\theta)\,d\theta.$$

Since

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\sum_{j=0}^{n-1} e^{\mathrm{i}j(\theta_1-\theta_2)}\right|^2 d\theta_1 d\theta_2 = n$$

it follows from the estimate (83) that

$$|e_n^{(3)}| \le \frac{C(h)}{n}. \tag{85}$$

Now,

$$I_1 = \mathrm{Re}\sum_{k\vee\ell>m} a_{k\ell}\sum_{j_1,j_2=0}^{n-1}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}k\theta_1-\mathrm{i}\ell\theta_2+\mathrm{i}(j_1-j_2)(\theta_1-\theta_2)}\,d\theta_1 d\theta_2 \tag{86}$$

$$= \mathrm{Re}\sum_{k\vee\ell>m} a_{k\ell}\sum_{j_1,j_2=0}^{n-1}\delta_{k,j_1-j_2}\delta_{\ell,j_2-j_1} = 0, \tag{87}$$

since non-zero Kronecker deltas require $j_1 - j_2 = k = -\ell$, which is not possible since $k, \ell \ge 1$. Next, we see that

$$I_2 = -\frac{1}{n}\mathrm{Re}\sum_{k\vee\ell>m} -\mathrm{i}ka_{k\ell}\sum_{j_1,j_2=0}^{n-1}(h_{k+j_2-j_1}\delta_{\ell,j_2-j_1} + h_{\ell+j_1-j_2}\delta_{k,j_1-j_2})$$

$$= -\frac{1}{n}\mathrm{Re}\sum_{k\vee\ell>m} -\mathrm{i}ka_{k\ell}(2n-(k+\ell))h_{k+\ell}.$$

Finally,

$$I_3 = 2\mathrm{Re}\sum_{k\vee\ell>m} -\mathrm{i}ka_{k\ell}h_{k+\ell}$$

and thus

$$I_2 + I_3 = -\frac{1}{2n}\mathrm{Re}\sum_{k\vee\ell>m}\mathrm{i}(k+\ell)^2 a_{k\ell}h_{k+\ell}.$$

Using Lemma 3.1 we see that

$$|I_2 + I_3| \leq \frac{C(h)}{n},$$

and thus by (84), (85) and (86),

$$|T_n^{(3)}| \leq \frac{C(h)}{n}.$$

We have shown that

$$T_n^{(1)} + T_n^{(2)} + T_n^{(3)} \geq -\mathrm{i} \sum_{k \in \mathbb{Z}} k f_k h_{-k} + \mathrm{Re} \sum_{k \vee \ell > m} k\ell a_{k\ell} h_k h_\ell - \frac{C(h)}{n}(1 + ||f||_1),$$

and inserting this estimate into (75) gives

$$G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp\left(- \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^t \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} - \sum_{k=1}^{\infty} k h_k h_{-k} \right. \tag{88}$$

$$\left. -\mathrm{i} \sum_{k \in \mathbb{Z}} k f_k h_{-k} + \mathrm{Re} \sum_{k \vee \ell > m} k\ell a_{k\ell} h_k h_\ell - \frac{C(h)}{n}(1 + ||f||_1) \right).$$

We now choose $h_k = -\mathrm{isgn}\,(k) f_k$, $1 \leq |k| \leq m$, $h_k = 0$ if $|k| > m$ or $k = 0$, so that h is a cut-off of the Fourier series for the conjugate function to f. Then

$$-\mathrm{i} \sum_{k \in \mathbb{Z}} k f_k h_{-k} = 2 \sum_{k=1}^{m} k |f_k|^2, \quad \text{and} \quad \sum_{k=1}^{\infty} k h_k h_{-k} = \sum_{k=1}^{m} k |f_k|^2,$$

and

$$\mathrm{Re} \sum_{k \vee \ell > m} k\ell a_{k\ell} h_k h_\ell = 0.$$

Hence, from (88) we see that for $\zeta \in \mathbb{R}$,

$$G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp\left(- \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}^t \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \right.$$

$$\left. + \sum_{k=1}^{m} k |f_k|^2 - \frac{C(h)}{n}(1 + ||f||_1) \right).$$

With m fixed and for $u, v \in [-D, D]^m$ with D fixed, and $|\zeta| \le \rho$, we see that $C(h)$ is bounded and thus

$$\varliminf_{n\to\infty} G_{m,n}(\zeta) \ge \frac{1}{\pi^m}\int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp\left(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} + \sum_{k=1}^m k|f_k|^2\right). \tag{89}$$

We see from (66) that

$$\sum_{k=1}^m k|f_k|^2 = (L_m(\zeta) + P_m\mathbf{g}(\zeta))^t(L_m(\zeta) + P_m\mathbf{g}(\zeta)).$$

In (89) we can let $D \to \infty$ so that the integration in the right side is over $\mathbb{R}^m$ and compute the Gaussian integral. The same computations that led to (62) then give

$$\varliminf_{n\to\infty} G_{m,n}(\zeta) \ge \frac{1}{\sqrt{\det(I - \frac{\zeta^2}{1-\delta_m}|B_m|)\det(I - \frac{1}{1-\delta_m}|B_m|)}} \exp\left(\begin{pmatrix}P_m\mathbf{a}(\zeta)\\ P_m\mathbf{b}(\zeta)\end{pmatrix}^t T_m\right.$$
$$\left.\times \begin{pmatrix}(I-\zeta^2\Lambda_m)^{-1}\zeta^2\Lambda_m & 0\\ 0 & (I-\Lambda_m)^{-1}\Lambda_m\end{pmatrix} T_m^t \begin{pmatrix}P_m\mathbf{a}(\zeta)\\ P_m\mathbf{b}(\zeta)\end{pmatrix} + \begin{pmatrix}P_m\mathbf{a}(\zeta)\\ P_m\mathbf{b}(\zeta)\end{pmatrix}^t \begin{pmatrix}P_m\mathbf{a}(\zeta)\\ P_m\mathbf{b}(\zeta)\end{pmatrix}\right).$$

We can now let $m \to \infty$, and the same computations as previously then give

$$\varliminf_{m\to\infty} \varliminf_{n\to\infty} G_{m,n}(\zeta) \ge G(\zeta),$$

for $\zeta \in [-\rho, \rho]$ which is what we wanted to prove.

4 Proof of Theorem 1.3

Without loss of generality we can assume that $\mathrm{cap}(\gamma) = 1$ and we will do so in this section. Consider the function $E_n(r)$ defined by (18). We have the following lemma.

Lemma 4.1 *The sequence of functions $E_n(r)$, $n \ge 1$ is increasing.*

The lemma will be proved below. We will now prove Theorem 1.3 using Lemma 1.2 and Lemma 4.1.

Proof (Of Theorem 1.3) Let B_r, $r > 1$ be the Grunsky operator for the curve γ_r. Then for each $r > 1$, by (17),

$$E(r) := \lim_{n\to\infty} E_n(r) = -\frac{1}{2}\log\det(I - B_r^* B_r) \tag{90}$$

since γ_r satisfies the conditions of Theorem 1.1. Assume first that γ is a Weil-Petersson quasicircle. Then the Grunsky operator B for γ is a Hilbert-Schmidt operator so B^*B is a trace-class operator and consequently it follows from (90) that

$$E := \lim_{r\to 1+} E(r) = -\frac{1}{2}\log\det(I - B^*B) < \infty, \tag{91}$$

which can be seen by expressing the Grunsky coefficients for B_r in terms of the Grunsky coefficients of B, see below. Since $E_n(r)$ is increasing in n, we have that $E_n(r) \le E(r)$ and combining this with (91) we obtain

$$E_n := \log \frac{Z_n(\gamma)}{(2\pi)^n} = \lim_{r\to 1+} E_n(r) \le \lim_{r\to 1+} E(r) = E < \infty$$

for all $n \ge 1$. Hence

$$\limsup_{n\to\infty} \frac{Z_n(\gamma)}{(2\pi)^n} \le E < \infty, \tag{92}$$

which proves (20). It remains to prove that we also get the right limit. From the monotonicity in r we have that $E_n(r) \le E_n(r')$ if $1 < r < r'$, and letting $r' \to 1+$ gives $E_n(r) \le E_n$ for all $r > 1$, $n \ge 1$. Taking the limit $n \to \infty$ gives $E(r) \le \underline{\lim}_{n\to\infty} E_n$ for all $r > 1$. Finally, we can let $r \to 1+$ to obtain $E \le \underline{\lim}_{n\to\infty} E_n$, which combined with (92) gives,

$$\lim_{n\to\infty} \log \frac{Z_n(\gamma)}{(2\pi)^n} = E = -\frac{1}{2}\log\det(I - B^*B),$$

which is what we wanted to prove.

Next we want to prove that if (20) holds, then γ is a Weil-Petersson quasicircle. It follows from Lemma 1.2 and the definition (19) that

$$\frac{Z_n(\gamma_r)}{(2\pi)^n} \le \frac{Z_n(\gamma)}{(2\pi)^n} \tag{93}$$

for any $r > 1$. We can use (17) and take the limit $n \to \infty$ in (93) to obtain

$$(\det(I - B_r^*B_r))^{-1/2} \le \overline{\lim_{n\to\infty}} \frac{Z_n(\gamma)}{(2\pi)^n} =: A < \infty \tag{94}$$

for any $r > 1$. From (5) we see that if the Grunsky coefficients for γ are $b_{k\ell}$, $k, \ell \ge 1$, then the Grunsky coefficients for γ_r are $b_{k\ell}/r^{k+\ell}$, $k, \ell \ge 1$. Let $\lambda_j(r)$ be the singular values of B_r. Then (94) gives the inequality

$$\prod_{j=1}^{\infty}(1 - \lambda_j(r)^2) \ge A^{-2},$$

so

$$||B_r||_{HS}^2 = \sum_{j=1}^{\infty} \lambda_j(r)^2 \le -\sum_{j=1}^{\infty} \log(1-\lambda_j(r)^2) \le 2\log A$$

for all $r > 1$. Thus

$$\sum_{k,\ell=1}^{\infty} \frac{|b_{k\ell}|^2}{2r^{k+\ell}} \le 2\log A,$$

and letting $r \to 1+$ shows that B is a Hilbert-Schmidt operator, so γ is a Weil-Petersson quasicircle. □

Lemma 1.2 will follow from the following lemma.

Lemma 4.2 *The function $E_n(r)$ defined by (18) satisfies*

$$rE_n''(r) + E_n'(r) \ge 0 \tag{95}$$

for all $r > 1$. Furthermore

$$\lim_{r\to\infty} E_n(r) = 0. \tag{96}$$

Proof Note that by definition

$$Z_n(\gamma_r) = \frac{1}{n!r^{n(n-1)}} \int_{[-\pi,\pi]^n} \exp\Bigg(\mathrm{Re}\Bigg(\sum_{\mu\neq\nu} \log(\phi(re^{\mathrm{i}\theta_\mu}) - \phi(re^{\mathrm{i}\theta_\nu})) + \sum_{\mu} \log\phi'(re^{\mathrm{i}\theta_\mu}) \Bigg)\Bigg) d\theta, \tag{97}$$

where we used $\phi_r(z) = \phi(rz)/r$ and $\phi_r'(z) = \phi'(rz)$. Making the change of variables $\theta_\mu \mapsto \theta_\mu + \alpha$ for some real α in the right side of (97) does not change its value so

$$Z_n(\gamma_r) = \frac{1}{n!r^{n(n-1)}} \int_{[-\pi,\pi]^n} e^{F(r,\alpha,\theta)}\, d\theta, \tag{98}$$

where

$$F(r,\alpha,\theta) = \sum_{\mu\neq\nu} \log(\phi(re^{\mathrm{i}(\theta_\mu+\alpha)}) - \phi(re^{\mathrm{i}(\theta_\nu+\alpha)})) + \sum_{\mu} \log\phi'(re^{\mathrm{i}(\theta_\mu+\alpha)}). \tag{99}$$

From the definition of $E_n(r)$ we then obtain

$$E_n(r) = -\log((2\pi)^n n!) - n(n-1)\log r + \log \int_{[-\pi,\pi]^n} e^{F(r,\alpha,\theta)}\, d\theta. \tag{100}$$

Using this formula we can compute the derivatives of $E_n(r)$ which gives

$$E_n'(r) = -\frac{n(n-1)}{r} + \frac{\int (\mathrm{Re}\,\partial_r F) e^{\mathrm{Re}\,F}\, d\theta}{\int e^{\mathrm{Re}\,F}\, d\theta}, \tag{101}$$

and

$$E_n''(r) = \frac{n(n-1)}{r} + \frac{\int (\mathrm{Re}\,\partial_r^2 F + (\mathrm{Re}\,\partial_r F)^2) e^{\mathrm{Re}\,F}\, d\theta}{\int e^{\mathrm{Re}\,F}\, d\theta} - \left(\frac{\int (\mathrm{Re}\,\partial_r F) e^{\mathrm{Re}\,F}\, d\theta}{\int e^{\mathrm{Re}\,F}\, d\theta}\right)^2, \tag{102}$$

where the integrals are over $[-\pi,\pi]^n$. If we take the derivative with respect to α in (100) we get similarly

$$\frac{\int (\mathrm{Re}\,\partial_\alpha F) e^{\mathrm{Re}\,F}\, d\theta}{\int e^{\mathrm{Re}\,F}\, d\theta} = 0, \tag{103}$$

and

$$\frac{\int (\mathrm{Re}\,\partial_r^2 F + (\mathrm{Re}\,\partial_\alpha F)^2) e^{\mathrm{Re}\,F}\, d\theta}{\int e^{\mathrm{Re}\,F}\, d\theta} - \left(\frac{\int (\mathrm{Re}\,\partial_\alpha F) e^{\mathrm{Re}\,F}\, d\theta}{\int e^{\mathrm{Re}\,F}\, d\theta}\right)^2 = 0. \tag{104}$$

From the definition (99) we see that

$$\partial_r F = \sum_{\mu \neq \nu} \frac{e^{\mathrm{i}(\theta_\mu+\alpha)}\phi'(re^{\mathrm{i}(\theta_\mu+\alpha)}) - e^{\mathrm{i}(\theta_\nu+\alpha)}\phi'(re^{\mathrm{i}(\theta_\nu+\alpha)})}{\phi(e^{\mathrm{i}(\theta_\mu+\alpha)}) - \phi(e^{\mathrm{i}(\theta_\nu+\alpha)})} + \sum_\mu \frac{e^{\mathrm{i}(\theta_\mu+\alpha)}\phi''(re^{\mathrm{i}(\theta_\mu+\alpha)})}{\phi'(e^{\mathrm{i}(\theta_\mu+\alpha)})},$$

and

$$\begin{aligned}\partial_r^2 F = \sum_{\mu \neq \nu} \Bigg[&\frac{(e^{\mathrm{i}(\theta_\mu+\alpha)})^2\phi''(re^{\mathrm{i}(\theta_\mu+\alpha)}) - (e^{\mathrm{i}(\theta_\nu+\alpha)})^2\phi''(re^{\mathrm{i}(\theta_\nu+\alpha)})}{\phi(e^{\mathrm{i}(\theta_\mu+\alpha)}) - \phi(e^{\mathrm{i}(\theta_\nu+\alpha)})} \\ &- \left(\frac{e^{\mathrm{i}(\theta_\mu+\alpha)}\phi'(re^{\mathrm{i}(\theta_\mu+\alpha)}) - e^{\mathrm{i}(\theta_\nu+\alpha)}\phi'(re^{\mathrm{i}(\theta_\nu+\alpha)})}{\phi(e^{\mathrm{i}(\theta_\mu+\alpha)}) - \phi(e^{\mathrm{i}(\theta_\nu+\alpha)})}\right)^2\Bigg] \\ &+ \sum_\mu \left[\frac{(e^{\mathrm{i}(\theta_\mu+\alpha)})^2\phi'''(re^{\mathrm{i}(\theta_\mu+\alpha)})}{\phi'(e^{\mathrm{i}(\theta_\mu+\alpha)})} - \left(\frac{e^{\mathrm{i}(\theta_\mu+\alpha)}\phi''(re^{\mathrm{i}(\theta_\mu+\alpha)})}{\phi'(e^{\mathrm{i}(\theta_\mu+\alpha)})}\right)^2\right].\end{aligned}$$

An analogous computation gives

$$\partial_\alpha F = \mathrm{i} r \partial_r F, \quad \text{and} \quad \partial_\alpha^2 F = -r\partial_r F - r^2 \partial_r^2 F.$$

Consequently,

$$\mathrm{Re}\, \partial_\alpha F = -r \mathrm{Im}\, \partial_r F, \quad \text{and} \quad \mathrm{Re}\, \partial_\alpha^2 F = -r \mathrm{Re}\, \partial_r F - r^2 \mathrm{Re}\, \partial_r^2 F.$$

If we insert these relations into (104), we get

$$\frac{\int (-r\mathrm{Re}\, \partial_r F - r^2 \mathrm{Re}\, \partial_r^2 F + r^2 (\,\mathrm{Im}\, \partial_r F)^2) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} - \left(\frac{\int (r \mathrm{Im}\, \partial_r F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \right)^2 = 0,$$

which gives

$$\begin{aligned}
&\frac{r^2 \int (\mathrm{Re}\, \partial_r^2 F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \\
&= -\frac{r \int (\mathrm{Re}\, \partial_r F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} + \frac{r^2 \int (\mathrm{Im}\, \partial_r F)^2 e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} - r^2 \left(\frac{\int (\mathrm{Im}\, \partial_r F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \right)^2 \\
&= -r E_n'(r) - n(n-1) + \frac{r^2 \int (\mathrm{Im}\, \partial_r F)^2 e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} - r^2 \left(\frac{\int (\mathrm{Im}\, \partial_r F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \right)^2,
\end{aligned}$$

where the last equality follows from (101). We can use this identity in (102) to find

$$\begin{aligned}
r^2 E_n''(r) &= n(n-1) - r E_n'(r) - n(n-1) + r^2 \Bigg[\frac{\int (\mathrm{Im}\, \partial_r F)^2 e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \\
&- \left(\frac{\int (\mathrm{Im}\, \partial_r F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \right)^2 + \frac{\int (\mathrm{Re}\, \partial_r F)^2 e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} - \left(\frac{\int (\mathrm{Re}\, \partial_r F) e^{\mathrm{Re}\, F}\, d\theta}{\int e^{\mathrm{Re}\, F}\, d\theta} \right)^2 \Bigg].
\end{aligned}$$

This can be written

$$\begin{aligned}
&\frac{1}{r^2}(r^2 E_n''(r) + r E_n'(r)) \Big(\int e^{\mathrm{Re}\, F}\, d\theta \Big)^2 \\
&= \frac{1}{2} \int d\theta \int d\theta' \big[(\mathrm{Im}\, \partial_r F(r,\alpha,\theta) - \ \mathrm{Im}\, \partial_r F(r,\alpha,\theta'))^2 \\
&+ (\mathrm{Re}\, \partial_r F(r,\alpha,\theta) - \ \mathrm{Re}\, \partial_r F(r,\alpha,\theta'))^2 \big] e^{\mathrm{Re}\, F(r,\alpha,\theta) + \mathrm{Re}\, F(r,\alpha,\theta')} \geq 0,
\end{aligned}$$

and we have proved the inequality (95).

We have that

$$\frac{Z_n(\gamma_r)}{(2\pi)^n} = \frac{1}{n!}\int_{[-\pi,\pi]^n} \prod_{\mu\neq\nu} |\phi_r(e^{\mathrm{i}\theta_\mu}) - \phi_r(e^{\mathrm{i}\theta_\nu})| \prod_{\mu} |\phi_r'(e^{\mathrm{i}\theta_\mu})|\, d\theta. \tag{105}$$

Since the series (4) is absolutely convergent for $|z| > 1$, there is a constant C so that $|\phi_{-k}| \le C2^k$ for all $k \ge 1$. Now, by (4)

$$\phi_r(z) = z + \sum_{k=0}^{\infty} \frac{\phi_{-k}}{r^{k+1}z^k}, \quad \text{and} \quad \phi_r'(z) = 1 + \sum_{k=1}^{\infty} \frac{k\phi_{-k}}{r^{k+1}z^{k+1}}, \tag{106}$$

and consequently $\phi_r(z) \to z$ and $\phi_r'(z) \to 1$ uniformly for $z \in \mathbb{T}$ as $r \to \infty$. Hence, we can take the limit $r \to \infty$ in (105) to obtain

$$\lim_{r\to\infty} \frac{Z_n(\gamma_r)}{(2\pi)^n} = \frac{1}{(2\pi)^n n!}\int_{[-\pi,\pi]^n} \prod_{\mu\neq\nu} |e^{\mathrm{i}\theta_\mu} - e^{\mathrm{i}\theta_\nu}|\, d\theta = 1.$$

This proves (96) and we are done.

Now we can give the

Proof (Of Lemma 1.2) Assume that $E_n'(r_0) > 0$ for some $r_0 > 1$. From (95) we see that $rE_n'(r)$ is increasing and hence $rE_n'(r) \ge r_0 E_n'(r_0)$ for $r \ge r_0$. Thus,

$$E_n(r) \ge E_n(r_0) + r_0 E_n'(r_0)\int_{r_0}^{r} \frac{ds}{s} = E_n(r_0) + r_0 E_n'(r_0)\log(r/r_0).$$

If we let $r \to \infty$ this contradicts (96). Consequently, $E_n'(r) \le 0$ for all $r > 1$.

We turn now to the proof of Lemma 4.1.

Proof (Of Lemma 4.1) Let Π_n be the set of all polynomials of degree $\le n$ with leading coefficient $= 1$. Then, see [17, Sec. 16.2], [16], we have that

$$\frac{Z_{n+1}(\gamma_r)/(2\pi)^{n+1}}{Z_n(\gamma_r)/(2\pi)^n} = \frac{D_{n+1}(1)/(2\pi)^{n+1}}{D_n(1)/(2\pi)^n} = \frac{1}{\kappa_n^2} = \min_{p\in\Pi_n} \frac{1}{2\pi}\int_{\gamma_r} |p(\zeta)|^2\, |d\zeta|, \tag{107}$$

and the minimum is attained if and only if $p(\zeta) = \pi_n(\zeta) := \frac{1}{\kappa_n} p_n(\zeta) = \zeta^n + \dots$, where p_n are the orthonormal polynomials with respect to γ_r,

$$\int_{\gamma_r} p_m(\zeta)\overline{p_n(\zeta)}\, |d\zeta| = \delta_{mn}.$$

Hence,

$$E_{n+1}(r) - E_n(r) = \log\left(\frac{1}{2\pi}\int_{\gamma_r} |\pi_n(\zeta)|^2\,|d\zeta|\right). \tag{108}$$

Note that ϕ_r is analytic in $|z| > \rho^{-1}$ if $1 \le \rho < r$. Fix $\rho \in (1, \infty)$. Then, by (106),

$$z\phi_r(\frac{1}{z}) = 1 + \sum_{k=0}^{\infty} \frac{\phi_{-k}}{r^{k+1}} z^{k+1}, \quad \text{and} \quad h_r(z) := \phi_r(\frac{1}{z}) = 1 + \sum_{k=0}^{\infty} \frac{k\phi_{-k}}{r^{k+1}} z^{k+1}, \tag{109}$$

and these functions are analytic in $|z| < \rho$. By (108) and Jensen's inequality,

$$\begin{aligned} E_{n+1}(r) - E_n(r) &= \log\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\pi_n(\phi_r(e^{\mathrm{i}\theta}))|^2 |\phi_r'(e^{\mathrm{i}\theta})|\,d\theta\right) \\ &\ge \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\log|\pi_n(\phi_r(e^{\mathrm{i}\theta}))| + \log|\phi_r'(e^{\mathrm{i}\theta})|\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\log|\pi_n(\phi_r(e^{-\mathrm{i}\theta}))| + \log|\phi_r'(e^{-\mathrm{i}\theta})|\,d\theta. \end{aligned} \tag{110}$$

Note that

$$|\pi_n(\phi_r(e^{-\mathrm{i}\theta}))| = |(e^{\mathrm{i}\theta})^n \pi_n(\phi_r(e^{-\mathrm{i}\theta}))| = |\psi_{n,r}(e^{\mathrm{i}\theta})|, \tag{111}$$

where

$$\psi_{n,r}(z) = z^n \pi_n(\phi_r(\frac{1}{z})).$$

If $\pi_n(z) = \sum_{j=0}^n a_j z^j$, with $a_n = 1$, then

$$\psi_{n,r}(z) = z^n \sum_{j=0}^n a_j \phi_r(\frac{1}{z})^j = \sum_{j=0}^n a_j z^{n-j}\big(z\phi_r(\frac{1}{z})\big)^j, \tag{112}$$

so we see that $\psi_{n,r}$ is analytic in $|z| < \rho$. Hence, $\log|\psi_{n,r}(z)|$ and $\log|h_r(z)|$ are subharmonic functions in $|z| < \rho$, and we see from (110) and (111) that

$$\begin{aligned} E_{n+1}(r) - E_n(r) &\ge \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\log|\psi_{n,r}(e^{\mathrm{i}\theta})| + \log|h_r(e^{\mathrm{i}\theta})|\,d\theta \\ &\ge 2\log\psi_{n,r}(0) + \log|h_r(0)|. \end{aligned} \tag{113}$$

It follows from (109) that $h_r(0) = 1$, and from (109) and (112),

$$\psi_{n,r}(0) = \sum_{j=0}^{n} a_j 0^{n-j} \cdot 1^j = a_n = 1.$$

Consequently, (113) gives $E_{n+1}(r) - E_n(r) \geq 0$.

5 Heuristic Argument for the β-Ensemble

We will use the same notation, sometimes slightly modified, as in the previous sections and only sketch the argument. Let

$$\mathbb{E}_{n,\beta}[(\cdot)] = \frac{1}{Z_{n,\beta}(\mathbb{T})n!} \int_{[-\pi,\pi]^n} \prod_{\mu \neq \nu} |e^{\mathrm{i}\theta_\mu} - e^{\mathrm{i}\theta_\nu}|^{\beta/2} (\cdot)\, d\theta$$

denote expectation with respect to the β-ensemble on the unit circle. As in (26), we see that

$$\frac{D_{n,\beta}[e^g]}{Z_{n,\beta}(\mathbb{T})} = \mathrm{cap}(\gamma)^{\frac{\beta n^2}{2}} \mathbb{E}_{n,\beta} \Bigg[\exp\Bigg(-\frac{\beta}{2} \mathrm{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \Big(\sum_\mu e^{-\mathrm{i}k\theta_\mu} \Big) \Big(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu} \Big) \tag{114}$$

$$+(1-\frac{\beta}{2}) \sum_\mu \log|\phi'(e^{\mathrm{i}\theta_\mu})| + \sum_\mu g(\phi(e^{\mathrm{i}\theta_\mu})) \Bigg) \Bigg]$$

$$= \mathrm{cap}(\gamma)^{\frac{\beta n^2}{2}} \mathbb{E}_{n,\beta} \left[\exp\left(-\frac{\beta}{2} \mathrm{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \Big(\sum_\mu e^{-\mathrm{i}k\theta_\mu} \Big) \Big(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu} \Big) + 2\mathbf{g}_\beta^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \right) \right],$$

where we used (21). If we write

$$\sum_{k,\ell=1}^{\infty} a_{k\ell} \Big(\sum_\mu e^{-\mathrm{i}k\theta_\mu} \Big) \Big(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu} \Big) = \lim_{m\to\infty} \sum_{k,\ell=1}^{m} a_{k\ell} \Big(\sum_\mu e^{-\mathrm{i}k\theta_\mu} \Big) \Big(\sum_\nu e^{-\mathrm{i}\ell\theta_\nu} \Big)$$

in (114), take the limit outside the expectation and then interchange the order of the $m \to \infty$ and $n \to \infty$ limits, we are led to study the limit

$$\lim_{m\to\infty} \lim_{n\to\infty} \mathrm{e}_{n,\beta} \left[\exp\left(-\frac{\beta}{2} \mathrm{Re} \sum_{k,\ell=1}^{m} b_{k\ell} (X_k - \mathrm{i}Y_k)(X_\ell - \mathrm{i}Y_\ell) + 2\mathbf{g}_\beta^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \right) \right].$$

It seems that it is not easy to justify changing the order of the limits but doing so leads, as we will see, to the conjecture (22). Set $M_{m,\beta} = \sqrt{\frac{\beta}{2}} M_m(\mathrm{i})$, with $M_m(\mathrm{i})$ as in (33). We can then use a Gaussian integral to write

$$\mathbb{E}_{n,\beta}\left[\exp\left(-\frac{\beta}{2}\mathrm{Re}\sum_{k,\ell=1}^{m} b_{k\ell}(X_k - \mathrm{i}Y_k)(X_\ell - \mathrm{i}Y_\ell) + 2\mathbf{g}_\beta^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right)\right] \tag{115}$$

$$= \frac{1}{\pi^m}\int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix})\mathbb{E}_{n,\beta}\left[\exp\left(2\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t M_{m,\beta} + 2\mathbf{g}_\beta^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right)\right]$$

$$= \frac{1}{\pi^m}\int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix})\mathbb{E}_{n,\beta}\left[\exp\left(2(L_{m,\beta} + \mathbf{g}_\beta)^t \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}\right)\right],$$

where

$$L_{m,\beta} = \sqrt{\frac{\beta}{2}} L_m(\mathrm{i}) = \sqrt{\frac{\beta}{2}}\begin{pmatrix}P_m & 0\\ 0 & P_m\end{pmatrix} T_m \begin{pmatrix}\mathrm{i}\Lambda_m^{1/2} & 0\\ 0 & \Lambda_m^{1/2}\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}.$$

We can now use the strong Szegő limit theorem for the β-ensemble on the unit circle to take the $n \to \infty$ limit in the last expectation in (115). This gives the limit

$$\frac{1}{\pi^m}\int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp(-\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}) \exp\left(\frac{2}{\beta}(L_{m,\beta} + \mathbf{g}_\beta)^t (L_{m,\beta} + \mathbf{g}_\beta)\right). \tag{116}$$

Now,

$$\frac{2}{\beta} L_{m,\beta}^t L_{m,\beta} = -\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \left(I - \begin{pmatrix}-\Lambda_m & 0\\ 0 & \Lambda_m\end{pmatrix}\right)\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix},$$

and

$$\frac{4}{\beta} L_{m,\beta}\mathbf{g}_\beta = 2\sqrt{\frac{2}{\beta}}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}^t \begin{pmatrix}\mathrm{i}\Lambda_m^{1/2} & 0\\ 0 & \Lambda_m^{1/2}\end{pmatrix} T_m \begin{pmatrix}P_m & 0\\ 0 & P_m\end{pmatrix}\mathbf{g}_\beta.$$

We can now perform the Gaussian integrations in (116) to get

$$\frac{1}{\sqrt{\det(I + K_m)}} \exp\left(-\frac{2}{\beta}\mathbf{g}_\beta \begin{pmatrix}P_m & 0\\ 0 & P_m\end{pmatrix}(I + K_m)^{-1} K_m \begin{pmatrix}P_m & 0\\ 0 & P_m\end{pmatrix}\mathbf{g}_\beta + \frac{2}{\beta}\mathbf{g}_\beta^t \mathbf{g}_\beta\right).$$

If we take the $m \to \infty$ limit of this expression we obtain the right side of (22).

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Toeplitz Operators with Non-trivial Kernels and Non-dense Ranges on Weak Hardy Spaces

Oleksiy Karlovych and Eugene Shargorodsky

To the memory of Harold Widom—one of the pioneers of the theory of Toeplitz operators

Abstract The well known Coburn lemma can be stated as follows: a nonzero Toeplitz operator $T(a)$ with symbol $a \in L^\infty(\mathbb{T})$ has a trivial kernel or a dense range on the Hardy space $H^p(\mathbb{T})$ with $p \in (1, \infty)$. We show that an analogue of this result does not hold for the Hardy-Marcinkiewicz (weak Hardy) spaces $H^{p,\infty}(\mathbb{T})$ with $p \in (1, \infty)$: there exist continuous nonzero functions $a : \mathbb{T} \to \mathbb{C}$ depending on p such that $\dim(\operatorname{Ker} T(a)) = \infty$ and $\dim\left(H^{p,\infty}(\mathbb{T}) / \operatorname{clos}_{H^{p,\infty}(\mathbb{T})}\left(\operatorname{Ran} T(a)\right)\right) = \infty$.

Keywords Toeplitz operator · Kernel · Range · Coburn's lemma · Hardy-Marcinkiewicz space · Blaschke product

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1 Introduction and Main Result

Let X be a Banach space. For a bounded linear operator A on the space X, its kernel and range (image) are defined by

$$\operatorname{Ker} A := \{x \in X : Ax = 0\}, \quad \operatorname{Ran} A := \{Ax : x \in X\}.$$

If $\mathcal{S}$ is a subset of X, then $\operatorname{clos}_X(\mathcal{S})$ denotes the closure of $\mathcal{S}$ in X.

For $1 \le p \le \infty$, let $L^p(\mathbb{T})$ be the standard Lebesgue space on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$, equipped with the norm

$$\|f\|_p := \left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}, \quad 1 \le p < \infty,$$

and

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\theta \in [0,2\pi]} |f(e^{i\theta})|.$$

Further, let $C(\mathbb{T})$ denote the space of all continuous functions $f : \mathbb{T} \to \mathbb{C}$. For $f \in L^1(\mathbb{T})$, let

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi}\, d\varphi, \quad n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. For $1 \le p \le \infty$, the classical Hardy spaces $H^p(\mathbb{T})$ are defined by

$$H^p(\mathbb{T}) := \left\{ f \in L^p(\mathbb{T}) \ : \ \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\}. \tag{1}$$

Consider the operators S and P, defined for a function $f \in L^1(\mathbb{T})$ and an a.e. point $t \in \mathbb{T}$ by

$$(Sf)(t) := \frac{1}{\pi i} \,\text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\, d\tau, \quad (Pf)(t) := \frac{f(t) + (Sf)(t)}{2},$$

respectively, where the integral is understood in the Cauchy principal value sense. It is well known that the operators P and S are bounded on $L^p(\mathbb{T})$ if $p \in (1, \infty)$ and are not bounded on $L^p(\mathbb{T})$ if $p \in \{1, \infty\}$ (see, e.g., [2, Section 1.42]). For $a \in L^\infty(\mathbb{T})$, the Toeplitz operator $T(a)$ with symbol a is defined by

$$T(a)f := P(af), \tag{2}$$

where $f \in H^p(\mathbb{T})$. It easy to see that $T(a)$ is bounded on every Hardy space $H^p(\mathbb{T})$ with $1 < p < \infty$.

Lewis Coburn observed in the proof of [3, Theorem 4.1] (see also [5, Proposition 7.24]) that if $T(a)$ is a non-zero Toeplitz operator on $H^2(\mathbb{T})$, then

$$\operatorname{Ker} T(a) = \{0\} \quad \text{or} \quad \operatorname{Ker} T^*(a) = \operatorname{Ker} T(\overline{a}) = \{0\}. \tag{3}$$

This result remains true for $H^p(\mathbb{T})$ with $1 < p < \infty$, and it can be rephrased as follows (see, e.g., [2, Theorem 2.38]).

Theorem 1 *If $a \in L^\infty(\mathbb{T}) \setminus \{0\}$, then the Toeplitz operator $T(a)$ has a trivial kernel or a dense range on the Hardy space $H^p(\mathbb{T})$ with $1 < p < \infty$.*

The aim of this paper is to show that an analogue of the above theorem does not hold for the so-called Hardy-Marcinkiewicz spaces $H^{p,\infty}(\mathbb{T})$ with $1 < p < \infty$ (see Theorem 2 below). On the other hand, it can be shown that (3) does generalise to this setting (see [12] and Sect. 6 below).

Let us recall the definition of the Hardy-Marcinkiewicz spaces. The Lebesgue arc-length measure of a measurable set $E \subseteq \mathbb{T}$ will be denoted by $|E|$. The distribution function m_f of a measurable a.e. finite function $f : \mathbb{T} \to \mathbb{C}$ is given by

$$m_f(\lambda) := |\{t \in \mathbb{T} : |f(t)| > \lambda\}|, \quad \lambda \geq 0.$$

The non-increasing rearrangement of f is defined by

$$f^*(x) := \inf\{\lambda : m_f(\lambda) \leq x\}, \quad x \geq 0.$$

We refer to [1, Chap. 2, Section 1] for properties of distribution functions and non-increasing rearrangements. For $1 < p < \infty$, the Marcinkiewicz space (or the weak-L^p space) $L^{p,\infty}(\mathbb{T})$ consists of all measurable a.e. finite functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{p,\infty} := \sup_{x>0} \left(x^{1/p} f^*(x)\right)$$

is finite. Note that, by [9, Proposition 1.4.5(16)],

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \left(\lambda m_f(\lambda)^{1/p}\right). \tag{4}$$

Although $\|\cdot\|_{p,\infty}$ is not a norm, it is equivalent to a norm. More precisely, by [1, Chap. 4, Lemma 4.5], for every measurable a.e. finite function $f : \mathbb{T} \to \mathbb{C}$, one has

$$\|f\|_{p,\infty} \leq \|f\|_{(p,\infty)} \leq \frac{p}{p-1} \|f\|_{p,\infty},$$

where

$$\|f\|_{(p,\infty)} := \sup_{x>0} \left(x^{1/p} f^{**}(x)\right)$$

and

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(y)\,dy, \quad x > 0.$$

In view of [1, Chap. 4, Theorem 4.6], $L^{p,\infty}(\mathbb{T})$ is a Banach (function) space with respect to the norm $\|\cdot\|_{(p,\infty)}$. Marcinkiewicz spaces form a very interesting class of non-separable rearrangement-invariant Banach function spaces (see, e.g., [1, Chap. 4, Section 4]).

Since each Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$ is continuously embedded into $L^1(\mathbb{T})$, by analogy with (1), for every $p \in (1,\infty)$, one can define the Hardy-Marcinkiewicz (weak Hardy) space by

$$H^{p,\infty}(\mathbb{T}) := \left\{ f \in L^{p,\infty}(\mathbb{T}) \ : \ \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\}.$$

It follows from the boundedness of P on all Lebesgue spaces $L^p(\mathbb{T})$ with $p \in (1,\infty)$ and Calderón's extension of the Marcinkiewicz interpolation theorem (see, e.g., [1, Chap. 4, Theorem 4.13]), that the operator P is bounded on all Marcinkiewicz spaces $L^{p,\infty}(\mathbb{T})$ with $1 < p < \infty$. Thus, one can define the Toeplitz operator $T(a)$ with symbol $a \in L^\infty(\mathbb{T})$ on the Hardy-Marcinkiewicz space $H^{p,\infty}(\mathbb{T})$ by (2) for $f \in H^{p,\infty}(\mathbb{T})$.

Very little seems to be known about Toeplitz operators on abstract Hardy spaces built upon *non-separable* rearrangement-invariant Banach function spaces (this class of spaces includes all Hardy-Marcinkiewicz spaces $H^{p,\infty}(\mathbb{T})$ with $p \in (1,\infty)$). We were able to find only one paper [13] where such operators are treated. It contains, among other things, a version of the Brown-Halmos theorem in this setting (see [13, Theorem 4.3]).

Our main result says that a direct analogue of Theorem 1 does not hold on Hardy-Marcinkiewicz spaces $H^{p,\infty}(\mathbb{T})$ for $1 < p < \infty$.

Theorem 2 (Main Result) *For every $1 < p < \infty$, there exists a function $a \in C(\mathbb{T}) \setminus \{0\}$ such that the following equalities hold for the kernel and the closure of the range of the Toeplitz operator $T(a)$ acting on the Hardy-Marcinkiewicz space $H^{p,\infty}(\mathbb{T})$:*

$$\dim(\operatorname{Ker} T(a)) = \infty, \quad \dim\left(H^{p,\infty}(\mathbb{T}) / \operatorname{clos}_{H^{p,\infty}(\mathbb{T})}\left(\operatorname{Ran} T(a)\right)\right) = \infty.$$

The proof proceeds along the following lines. Suppose $a \in C(\mathbb{T})$ is such that $|a(\zeta)| = |\zeta + 1|^\alpha$ for $\zeta \in \mathbb{T}$, where $0 < \alpha < 1/p$. Take any $\varphi \in H^{p,\infty}(\mathbb{T})$. Then $T(a)\varphi = af$, where $f := \frac{1}{a}P(a\varphi)$. It follows from the Khvedelidze theorem and Calderón's extension of the Marcinkiewicz interpolation theorem that the operator $\frac{1}{a}PaI$ is bounded on the Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$ (see Sect. 3). So, every element of $\operatorname{Ran} T(a)$ is the product of a function in $L^{p,\infty}(\mathbb{T})$ and the function a, which vanishes at $\zeta = -1$. This allows one to prove the existence of linearly independent functions $f_l \in H^{p,\infty}(\mathbb{T})$, $l \in \mathbb{N}$ such that no nontrivial linear combination of them belongs to $\operatorname{clos}_{H^{p,\infty}(\mathbb{T})}\left(\operatorname{Ran} T(a)\right)$, which implies that

$$\dim\left(H^{p,\infty}(\mathbb{T})/\operatorname{clos}_{H^{p,\infty}(\mathbb{T})}\left(\operatorname{Ran} T(a)\right)\right)=\infty$$

(see Sect. 5). It is essential for this part of the proof that the underlying Banach function space is the non-separable Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$. The functions f_l are defined as outer functions whose moduli $g_l = |f_l| \in L^{p,\infty}(\mathbb{T})$ are constructed in Sect. 4. The part of the proof described above depends only on $|a|$. It is left to choose the argument of a in such a way that $\dim \operatorname{Ker} T(a) = \infty$. To this end, we take

$$a(z) := \overline{B(z)(z+1)^{\alpha}}\,,$$

where B is an infinite Blaschke product (see Section 2). The zeros of B converge to -1, so $B \in C(\mathbb{T} \setminus \{0\})$. Since $\alpha > 0$, one has $a(-1) = 0$ and $a \in C(\mathbb{T})$.

In the final Sect. 6, we state a theorem that extends (3) to Hardy-Marcinkiewicz spaces $H^{p,\infty}(\mathbb{T})$ (a more general result is proved in [12]).

2 Producing an Infinite-Dimensional Kernel

As usual, we denote by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$. Recall that a function F analytic in $\mathbb{D}$ is said to belong to the Hardy space $H^p(\mathbb{D})$, $1 \le p \le \infty$, if

$$\|F\|_{H^p(\mathbb{D})} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{i\theta})|^p\,d\theta\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|F\|_{H^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}}|F(z)| < \infty.$$

If $F \in H^p(\mathbb{D})$, $1 \le p \le \infty$, then the limit

$$f(e^{i\theta}) = \lim_{r\to 1-0} F(re^{i\theta})$$

exists for almost all $\theta \in [-\pi, \pi]$ (see, e.g., [6, Theorem 2.2]) and the boundary function $f = f(e^{i\theta})$ belongs to $L^p(\mathbb{T})$.

Theorem 3 *For every $p \in (1,\infty)$, there exists a function $a \in C(\mathbb{T}) \setminus \{0\}$ depending on p such that*

$$\dim \operatorname{Ker} T(a) = \infty \tag{5}$$

on the Hardy-Marcinkiewicz space $H^{p,\infty}(\mathbb{T})$.

Proof Let B be a convergent Blaschke product

$$B(z) = z \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j}\, z}, \quad z \in \mathbb{D} \tag{6}$$

with

$$z_j \in \mathbb{D} \setminus \{0\} \text{ for } j \in \mathbb{N}, \quad z_j \neq z_l \text{ if } j \neq l, \quad z_j \to -1 \text{ as } j \to \infty, \tag{7}$$

and

$$\sum_{j=1}^{\infty} (1 - |z_j|) < \infty. \tag{8}$$

By [7, Chap. II, Theorem 6.1], the function B admits an analytic continuation to

$$\mathbb{C} \setminus \left(\{-1\} \cup \{1/\overline{z_j} \ : \ j \in \mathbb{N}\}\right).$$

In particular, B is continuous on $\mathbb{T} \setminus \{-1\}$. It follows from [7, Chap. II, Theorem 2.2] that $|B| = 1$ on $\mathbb{T} \setminus \{-1\}$.

Let $\alpha \in (0, 1/p)$ and

$$a(z) := \overline{B(z)(z+1)^{\alpha}}, \quad z \in \mathbb{D} \cup \mathbb{T}, \tag{9}$$

where w^{α} denotes the branch that is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and positive on $(0, +\infty)$. Since $\alpha > 0$, we conclude that a is a continuous function on $\mathbb{T}$ with $a(-1) = 0$.

Let

$$B_k(z) := \prod_{j=1}^{k} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j}\, z}, \quad z \in \mathbb{C}, \quad k \in \mathbb{N}.$$

Then

$$a(\zeta) B_k(\zeta) = \overline{b_k(\zeta)}, \quad \zeta \in \mathbb{T},$$

where

$$b_k(z) = z \left(\prod_{j=k+1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j}\, z} \right) (z+1)^{\alpha}, \quad z \in \mathbb{D} \cup \mathbb{T}.$$

It follows from [7, Chap. II, Theorem 2.2] that the middle term of $b_k(z)$ belongs to $H^{\infty}(\mathbb{D})$. It is also clear that $(z+1)^{\alpha}$ belongs to $H^{\infty}(\mathbb{D})$. Thus $b_k(z)$ belongs to

$H^\infty(\mathbb{D})$ and the Taylor expansion of $b_k(z)$ does not contain a constant term. Taking into account [6, Theorem 3.4], we conclude that the boundary value of b_k belongs to $H^\infty(\mathbb{T})$ and $\widehat{b_k}(0) = 0$. Hence

$$T(a)B_k = P(aB_k) = P\left(\overline{b_k}\right) = 0.$$

So, $B_k \in \operatorname{Ker} T(a)$ for all $k \in \mathbb{N}$.

The finite Blaschke products B_k are linearly independent. Indeed, suppose they are linearly dependent. Then there exist $c_1, \ldots, c_N \in \mathbb{C}$ such that $c_N \neq 0$ and

$$\sum_{k=1}^{N} c_k B_k = 0.$$

So,

$$B_N = -\frac{1}{c_N} \sum_{k=1}^{N-1} c_k B_k,$$

where the left-hand side has a pole at $1/\overline{z_N}$, while the right-hand side does not. This contradiction shows that B_k are linearly independent and concludes the proof of (5). □

3 Boundedness of an Auxiliary Operator

Lemma 1 *Let $1 < p < \infty$, $0 < \alpha < 1/p$, let a convergent Blaschke product B be given by (6)–(8), and let $a \in C(\mathbb{T}) \setminus \{0\}$ be given by (9). Then the operator $\frac{1}{a} PaI$ is bounded on the Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$.*

Proof Take p_1 and p_2 such that $1 < p_1 < p < p_2 < \infty$ and $\alpha < 1/p_2$. It follows from (9) and [7, Chap. II, Theorem 2.2] that $|a(\zeta)| = |\zeta + 1|^\alpha$ for a.e. $\zeta \in \mathbb{T}$. Since

$$-1/p_1 < -1/p_2 < -\alpha < 0 < 1 - 1/p_1 < 1 - 1/p_2,$$

by Khvedelidze's theorem (see, e.g., [8, Chap. 1, Theorem 4.1]), the Riesz projection P is bounded on the weighted Lebesgue spaces

$$L^{p_j}\left(\mathbb{T}, |a|^{-1}\right) := \left\{f : \mathbb{T} \to \mathbb{C} \ : \ f|a|^{-1} \in L^{p_j}(\mathbb{T})\right\}$$

equipped with the norms $\|f\|_{p_j,|a|^{-1}} := \|f|a|^{-1}\|_{p_j}$ for $j = 1, 2$. Equivalently, the operator $\frac{1}{a} PaI$ is bounded on $L^{p_j}(\mathbb{T})$. Then it follows from Calderón's extension of the Marcinkiewicz interpolation theorem (see [1, Chap. 4, Theorem 4.13]) that $\frac{1}{a} PaI$ is bounded on $L^{p,\infty}(\mathbb{T})$. □

4 A Family of Auxiliary Functions

Lemma 2 *Let* $1 < p < \infty$ *and*

$$\gamma_{n,l} := \left\{ e^{i\theta} \ : \ \frac{n+l-1}{2^n n} \le \theta - \pi < \frac{n+l}{2^n n} \right\}, \quad n \in \mathbb{N}, \quad l = 1, \dots, n. \tag{10}$$

Then for every $l \in \mathbb{N}$ *the function*

$$g_l(\zeta) := \begin{cases} (2^n n)^{1/p}, \ \zeta \in \gamma_{n,l}, & n \ge l, \\ 1, \qquad \zeta \in \mathbb{T} \setminus \left(\bigcup_{n \ge l} \gamma_{n,l} \right) \end{cases} \tag{11}$$

belongs to the Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$.

Proof For any $\lambda \ge 2^{1/p}$, there exists a unique $n \in \mathbb{N}$ such that

$$\left(2^n n\right)^{1/p} \le \lambda < \left(2^{n+1}(n+1)\right)^{1/p}.$$

Then

$$m_{g_l}(\lambda) = |\{\zeta \in \mathbb{T} \ : \ |g_l(\zeta)| > \lambda\}| = \left| \bigcup_{m \ge \max\{n+1,l\}} \gamma_{m,l} \right| = \sum_{m=\max\{n+1,l\}}^{\infty} \left|\gamma_{m,l}\right|$$
$$\le \sum_{m=n+1}^{\infty} \frac{1}{2^m m} \le \frac{1}{n+1} \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n(n+1)} < 2\lambda^{-p}.$$

It follows from the above inequality and (4) that

$$\|g_l\|_{p,\infty} = \max \left\{ \sup_{0<\lambda<2^{1/p}} \left(\lambda m_{g_l}(\lambda)^{1/p}\right), \sup_{\lambda \ge 2^{1/p}} \left(\lambda m_{g_l}(\lambda)^{1/p}\right) \right\}$$
$$\le \max \left\{ (2\pi)^{1/p} \sup_{0<\lambda<2^{1/p}} \lambda, \ \sup_{\lambda \ge 2^{1/p}} \left(\lambda \frac{2^{1/p}}{\lambda}\right) \right\} = (4\pi)^{1/p} < \infty.$$

Hence $g_l \in L^{p,\infty}(\mathbb{T})$. □

5 Producing an Infinite-Dimensional Co-Kernel

Theorem 4 *Let* $1 < p < \infty$, $0 < \alpha < 1/p$, *let a convergent Blaschke product* B *be given by* (6)–(8)*, and let* $a \in C(\mathbb{T}) \setminus \{0\}$ *be given by* (9)*. Then*

$$\dim \left(H^{p,\infty}(\mathbb{T}) / \operatorname{clos}_{H^{p,\infty}(\mathbb{T})} \left(\operatorname{Ran} T(a)\right)\right) = \infty. \tag{12}$$

Proof For $l \in \mathbb{N}$, let the functions g_l be defined by (10)–(11). It follows from Lemma 2 that $g_l \in L^{p,\infty}(\mathbb{T}) \subset L^1(\mathbb{T})$. Since $g_l \geq 1$, we have $0 \leq \log g_l \leq g_l$, whence $\log g_l \in L^1(\mathbb{T})$. Consider the outer functions

$$F_l(z) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log g_l(e^{i\theta})\, d\theta\right), \quad z \in \mathbb{D}$$

(see [10, Chap. 5]). They belong to $H^1(\mathbb{D})$ and their non-tangential boundary values f_l satisfy

$$|f_l| = g_l \text{ a.e. on } \mathbb{T}, \qquad l \in \mathbb{N}. \tag{13}$$

The above equality immediately implies that $f_l \in L^{p,\infty}(\mathbb{T})$. On the other hand, by [6, Theorem 3.4], $f_l \in H^1(\mathbb{T})$. Hence the functions f_l belong to the Hardy-Marcinkiewicz space $H^{p,\infty}(\mathbb{T}) = L^{p,\infty}(\mathbb{T}) \cap H^1(\mathbb{T})$ for every $l \in \mathbb{N}$.

Let us show that for any $N \in \mathbb{N}$ and any $c_1, \dots, c_N \in \mathbb{C}$ with

$$\sum_{l=1}^{N} |c_l| > 0$$

we have

$$\sum_{l=1}^{N} c_l f_l \notin \operatorname{clos}_{H^{p,\infty}(\mathbb{T})} \left(\operatorname{Ran} T(a)\right).$$

Let $k \in \{1, \dots, N\}$ be such that

$$|c_k| = \max_{l=1,\dots,N} |c_l|.$$

Then

$$c_l' := \frac{c_l}{c_k}, \quad l = 1, \dots, N,$$

satisfy

$$|c_l'| \le 1, \quad l = 1, \dots, N. \tag{14}$$

It is sufficient to prove that

$$f_k + \sum_{l \in \{1,\dots,N\} \setminus \{k\}} c_l' f_l \notin \operatorname{clos}_{H^{p,\infty}(\mathbb{T})} \big(\operatorname{Ran} T(a)\big). \tag{15}$$

Take any $\varphi \in H^{p,\infty}(\mathbb{T})$. Then it follows from Lemma 1 that

$$f := \frac{1}{a} P(a\varphi) \in L^{p,\infty}(\mathbb{T})$$

and

$$T(a)\varphi = af. \tag{16}$$

For $n \ge N$, let

$$\gamma_{n,k}^0 := \left\{ \zeta \in \gamma_{n,k} \ : \ |(T(a)\varphi)(\zeta)| > \frac{(2^n n)^{1/p}}{3} \right\}.$$

For every

$$\zeta = e^{i\vartheta} \in \gamma_{n,k} \subset \left\{ e^{i\theta} : 2^{-n} \le \theta - \pi < 2^{-(n-1)} \right\},$$

one has

$$\begin{aligned} |a(\zeta)| = |a(e^{i\vartheta})| = |e^{i\vartheta} + 1|^\alpha = |e^{i(\vartheta-\pi)} - 1|^\alpha = \left| \int_0^{\vartheta-\pi} e^{i\tau} d\tau \right|^\alpha \\ \le |\vartheta - \pi|^\alpha < 2^{-(n-1)\alpha}. \end{aligned}$$

The above inequality and equalities (4) and (16) imply that for all $n \ge N$,

$$\begin{aligned} \left|\gamma_{n,k}^0\right| &= \left| \left\{ \zeta \in \gamma_{n,k} \ : \ |a(\zeta) f(\zeta)| > \frac{(2^n n)^{1/p}}{3} \right\} \right| \\ &\le \left| \left\{ \zeta \in \gamma_{n,k} \ : \ |f(\zeta)| > \frac{(2^n n)^{1/p}\, 2^{(n-1)\alpha}}{3} \right\} \right| \\ &= m_f \left(\frac{(2^n n)^{1/p} 2^{(n-1)\alpha}}{3} \right) \le \|f\|_{p,\infty}^p \, 3^p \, 2^{-(n-1)\alpha p} \, \frac{1}{2^n n}. \end{aligned} \tag{17}$$

Take $n_0 \geq N$ such that

$$\|f\|_{p,\infty}^p \, 3^p \, 2^{-(n_0-1)\alpha p} \leq \frac{1}{2}, \qquad N-1 < \frac{(2^{n_0} n_0)^{1/p}}{3}. \tag{18}$$

Equality (10), inequality (17) and the first inequality in (18) yield

$$\left|\gamma_{n,k}\right| = \frac{1}{2^n n}, \quad \left|\gamma_{n,k}^0\right| \leq \frac{1}{2^{n+1} n},$$

which implies that for all $n \geq n_0$,

$$\left|\gamma_{n,k} \setminus \gamma_{n,k}^0\right| = \left|\gamma_{n,k}\right| - \left|\gamma_{n,k}^0\right| \geq \frac{1}{2^n n} - \frac{1}{2^{n+1} n} = \frac{1}{2^{n+1} n}. \tag{19}$$

It follows from (11) and (13) that

$$|f_k(\zeta)| = (2^n n)^{1/p} \tag{20}$$

for all $\zeta \in \gamma_{n,k}$. If $l \in \{1, \dots, N\} \setminus \{k\}$, then (10) implies that

$$\left(\bigcup_{n \geq l} \gamma_{n,l}\right) \cap \left(\bigcup_{n \geq k} \gamma_{n,k}\right) = \emptyset.$$

Therefore, (11), (13), (14), and the second inequality in (18) allow us to conclude that for $n \geq n_0$ and $\zeta \in \gamma_{n,k}$,

$$\sum_{l \in \{1,\dots,N\} \setminus \{k\}} |c_l'| \, |f_l(\zeta)| \leq N - 1 < \frac{(2^n n)^{1/p}}{3}. \tag{21}$$

Let

$$h := f_k + \sum_{l \in \{1,\dots,N\} \setminus \{k\}} c_l' f_l - T(a)\varphi.$$

Then it follows from the definition of the sets $\gamma_{n,k}^0$ and from (20)–(21) that for all $n \geq n_0$ and $\zeta \in \gamma_{n,k} \setminus \gamma_{n,k}^0$,

$$\begin{aligned} |h(\zeta)| &\geq |f_k(\zeta)| - \sum_{l \in \{1,\dots,N\} \setminus \{k\}} |c_l'| \, |f_l(\zeta)| - |(T(a)\varphi)(\zeta)| \\ &> \left(2^n n\right)^{1/p} - \frac{(2^n n)^{1/p}}{3} - \frac{(2^n n)^{1/p}}{3} = \frac{(2^n n)^{1/p}}{3} =: \lambda_n. \end{aligned}$$

Then, taking into account (19), we see that for all $n \geq n_0$,

$$\begin{aligned} m_h(\lambda_n) &= |\{\zeta \in \mathbb{T} \,:\, |h(\zeta)| > \lambda_n\}| \\ &\geq |\{\zeta \in \gamma_{n,k} \setminus \gamma_{n,k}^0 \,:\, |h(\zeta)| > \lambda_n\}| \\ &= \left|\gamma_{n,k} \setminus \gamma_{n,k}^0\right| \geq \frac{1}{2^{n+1}n} = \frac{1}{3^p 2}\, \lambda_n^{-p}. \end{aligned}$$

This inequality and (4) imply that for $n \geq n_0$,

$$\|h\|_{p,\infty} \geq \lambda_n m_h(\lambda_n)^{1/p} \geq \frac{1}{2^{1/p}3}.$$

Hence, for every $\varphi \in H^{p,\infty}(\mathbb{T})$,

$$\left\| f_k + \sum_{l \in \{1,\ldots,N\}\setminus\{k\}} c_l' f_l - T(a)\varphi \right\|_{p,\infty} = \|h\|_{p,\infty} \geq \frac{1}{2^{1/p}3}$$

and (15) holds. This proves (12). □

Theorem 2 follows immediately from Theorems 3 and 4.

6 Concluding Remarks

Let $1 < p < \infty$. The associate space $(L^{p,\infty})'(\mathbb{T})$ of the Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$ is the collection of all measurable a.e. finite functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|'_{(p,\infty)} := \sup\left\{ \int_0^\infty f^*(x) g^*(x)\, dx \,:\, g \in L^{p,\infty}(\mathbb{T}),\ \|g\|_{(p,\infty)} \leq 1 \right\} < \infty$$

(see, e.g., [1, Chap. 1, Section 2 and Chap. 2, Proposition 4.2]). It is well known that the space $(L^{p,\infty})'(\mathbb{T})$ coincides up to equivalence of norms with the Lorentz space $L^{q,1}(\mathbb{T})$, where $1/p + 1/q = 1$, consisting of all measurable a.e. finite functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{(q,1)} := \int_0^\infty x^{1/q} f^{**}(x) \frac{dx}{x} < \infty$$

(see, e.g., [1, Chap. 4, Theorem 4.7]). Since the Marcinkiewicz space $L^{p,\infty}(\mathbb{T})$ is non-separable, its associate is canonically isometrically isomorphic to a *proper* subspace of the Banach dual space $(L^{p,\infty})^*(\mathbb{T})$ (see [1, Chap. 1, Theorem 2.9 and

Corollaries 4.3 and 5.6] and [4]). The Hardy-Lorentz space $H^{q,1}(\mathbb{T})$ is defined, as usual, by

$$H^{q,1}(\mathbb{T}) := \left\{ f \in L^{q,1}(\mathbb{T}) \ : \ \widehat{f}(n) = 0 \quad \text{for all} \quad n < 0 \right\}.$$

It follows from the boundedness of P on all Lebesgue spaces $L^r(\mathbb{T})$ with $r \in (1, \infty)$ and Calderón's extension of the Marcinkiewicz interpolation theorem (see, e.g., [1, Chap. 4, Theorem 4.13]), that the operator P is bounded on all Lorentz spaces $L^{q,1}(\mathbb{T})$ with $1 < q < \infty$.

A proof of the following version of Coburn's lemma (cf. (3)) will be given in [12] in the more general setting of abstract Hardy spaces built upon Banach function spaces on which the Riesz projection is bounded (see [11]).

Theorem 5 *Let $a \in L^\infty(\mathbb{T})$, $1 < p < \infty$ and $1/p + 1/q = 1$. Then the kernel of the Toeplitz operator*

$$T(a) : H^{p,\infty}(\mathbb{T}) \to H^{p,\infty}(\mathbb{T})$$

or the kernel of the Toeplitz operator

$$T(\overline{a}) : H^{q,1}(\mathbb{T}) \to H^{q,1}(\mathbb{T})$$

is trivial.

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Rényi Entropies of the Free Fermi Gas in Multi-Dimensional Space at High Temperature

Hajo Leschke, Alexander V. Sobolev, and Wolfgang Spitzer

In memory of Harold Widom (1932–2021)

Abstract We study the local and (bipartite) entanglement Rényi entropies of the free Fermi gas in multi-dimensional Euclidean space $\mathbb{R}^d$ in thermal equilibrium. We prove positivity of the entanglement entropies with Rényi index $\gamma \le 1$ for all temperatures $T > 0$. Furthermore, for general $\gamma > 0$ we establish the asymptotics of the entropies for large T and large scaling parameter $\alpha > 0$ for two different regimes—for fixed chemical potential $\mu \in \mathbb{R}$ and also for fixed particle density $\rho > 0$. In particular, we thereby provide the last remaining building block for a complete proof of our low- and high-temperature results presented (for $\gamma = 1$) in J. Phys. A: Math. Theor. **49**, 30LT04 (2016); [Corrigendum. **50**, 129501 (2017)], but being supported there only by the basic proof ideas.

Keywords Non-smooth functions of Wiener–Hopf operators · Asymptotic trace formulas · Rényi (entanglement) entropy of fermionic equilibrium states

Mathematics Subject Classification (2020) Primary 47G30, 35S05; Secondary 47B10, 47B35

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1 Introduction

This section briefly describes the physical background, introduces the most important mathematical definitions, and provides a summary of the main results.

1.1 Physical Background

Over the last decades the so-called entanglement entropy (EE) has become a useful single-number quantifier of non-classical correlations between subsystems of a composite quantum-mechanical system [1, 14]. For example, one may imagine a macroscopically large system consisting of a huge number of particles in the state of thermal equilibrium at some (absolute) temperature $T \geq 0$. All the particles inside some bounded spatial region Λ may then be considered to constitute one subsystem and the particles outside of Λ another one. The corresponding EE, more precisely the spatially bipartite thermal EE, now quantifies, to some extent, how strongly these two subsystems are correlated "across the interface" between Λ and its complement.

In the simplified situation where the particles do not dynamically interact with each other, such as in the ideal gas or, slightly more general, in the free gas, all possible correlations are entirely due to either the Bose–Einstein or the (Pauli–) Fermi–Dirac statistics by the assumed indistinguishability of the (point-like and spinless) particles.The present study is devoted to the latter case. Accordingly, we consider the free Fermi gas [3, 7, 10] infinitely extended in the Euclidean space $\mathbb{R}^d$ of an arbitrary dimension $d \geq 1$. Although the fermions neither interact with each other nor with any externally applied field, their EE remains a complicated function of the region $\Lambda \subset \mathbb{R}^d$ which is difficult to study by analytical methods. In general one can only hope for estimates and asymptotic results for its (physically interesting) growth when Λ is replaced with $\alpha\Lambda$ where the scaling parameter $\alpha > 0$ becomes large. A decisive progress towards the understanding of the growth of the EE at $T = 0$, in other words of the ground-state EE, is due to Gioev and Klich [11, 12]. They observed, remarkably enough, that this growth is related to a conjecture of Harold Widom [28, 31] about the quasi-classical Szegő-type asymptotics for traces of (smooth) functions of multi-dimensional versions of truncated Wiener–Hopf operators with discontinuous symbols. After Widom's conjecture had been proved by one of us [21, 23] the gate stood open to confirm [16] the precise (leading) large-scale growth conjectured in [12] and, in addition, to establish its extension from the von Neumann EE to the whole one-parameter family of (quantum) Rényi EE's.

In the present study we only consider the case of a true thermal state characterized by a *strictly* positive temperature $T > 0$ (and a chemical potential $\mu \in \mathbb{R}$ or, equivalently, a spatial particle-number density $\rho > 0$). On the one hand, the case $T > 0$ is simpler, because the *Fermi function* $E \mapsto 1/\big(1 + \exp(E/T)\big)$ on the real line $\mathbb{R}$ is smooth in contrast to its zero-temperature limit, the Heaviside unit-step function. On the other hand, a reasonable definition of the EE (see (8)) is more complicated, because the thermal state is not a pure state. Nevertheless, due to the presence of the "smoothing parameter" $T > 0$ the leading asymptotic growth of the EE as $\alpha \to \infty$, is determined by an asymptotic coefficient again going back to Widom [27, 30], see also [8, 19]. We introduce it in (17) and denote it by $\mathcal{B}$.

From a physical point of view it is interesting to study the scaling asymptotics $\alpha \to \infty$ as the temperature T varies. The emerging double asymptotics of the EE and of the coefficient $\mathcal{B}$ are not simple to analyze and hard to guess by heuristic arguments. For low temperatures, that is, small $T > 0$, this analysis has been performed in [18] for $d = 1$ and in [25] for $d \geq 2$ yielding a result consistent with that for $T = 0$ in [12, 16].

At high temperatures quantum effects become weaker and the free Fermi gas should exhibit properties of the corresponding classical free gas without correlations (for fixed particle density). In particular, the ideal Fermi gas [3, 10] should behave like the Maxwell–Boltzmann gas, the time-honored "germ cell" of statistical mechanics. Hence the main purpose of our study is to determine the precise two-parameter asymptotics of the EE as $\alpha \to \infty$ and $T \to \infty$.

1.2 Pseudo-Differential Operators and Entropies

At first we introduce the translation invariant pseudo-differential operator[1]

$$\big(\mathrm{Op}_\alpha(a)u\big)(\mathbf{x}) := \frac{\alpha^d}{(2\pi)^d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\alpha\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{y})} a(\boldsymbol{\xi}) u(\mathbf{y})\, \mathrm{d}\mathbf{y}\mathrm{d}\boldsymbol{\xi}\,, \qquad \mathbf{x} \in \mathbb{R}^d\,. \tag{1}$$

Here the smooth real-valued function a is its underlying *symbol*, u is an arbitrary complex-valued Schwartz function, and $\alpha > 0$ is the scaling parameter. Informally, one may think of $\mathrm{Op}_\alpha(a)$ as the function $a(-(\mathrm{i}/\alpha)\boldsymbol{\nabla})$ of the gradient operator $\boldsymbol{\nabla} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_d})$, that is, the vector of partial derivatives with respect to $\mathbf{x} = (x_1, x_2, \ldots, x_d)$.

The main role will be played by the *truncated Wiener–Hopf operator*

$$W_\alpha(a, \Lambda) := \chi_\Lambda \,\mathrm{Op}_\alpha(a) \chi_\Lambda\,,$$

where χ_Λ is the (multiplication operator corresponding to the) indicator function of the "truncating" open set $\Lambda \subset \mathbb{R}^d$. Clearly, for a bounded symbol a the operators $\mathrm{Op}_\alpha(a)$ and $W_\alpha(a; \Lambda)$ are bounded on the Hilbert space $\mathsf{L}^2(\mathbb{R}^d)$. Given a *test function* $f : \mathbb{R} \to \mathbb{R}$, we are interested in the operator $f\big(W_\alpha(a, \Lambda)\big)$ and in the

[1] In [18, 26] the right-hand side of (1) is mistakenly multiplied by $(2\pi)^{d/2}$.

operator difference

$$D_\alpha(a,\Lambda;f) := \chi_\Lambda f\big(W_\alpha(a,\Lambda)\big)\chi_\Lambda - W_\alpha(f\circ a,\Lambda)\,, \tag{2}$$

where the symbol $f \circ a$ is the composition of f and a defined by $(f \circ a)(\boldsymbol{\xi}) := f\big(a(\boldsymbol{\xi})\big)$.

For a bounded Λ and suitable a and f both operators on the right-hand side of (2) belong individually to the trace class. Remarkably, its difference does so even for a large class of unbounded Λ, see Condition 2.1 and Proposition 2.6 below. Our analysis of the scaling behavior of the entropies will be based on the asymptotics for the trace of $D_\alpha(a,\Lambda;f)$ as $\alpha\to\infty$. The reciprocal parameter α^{-1} can be naturally viewed as the Planck constant, and hence the limit $\alpha\to\infty$ can be regarded as the quasi-classical limit. By a straightforward change of variables the operator (2) is seen to be unitarily equivalent to $D_1(a,\alpha\Lambda;f)$, so that $\alpha\to\infty$ can be interpreted also as a spatial large-scale limit. In our large-scale applications it is either α itself or a certain combination of α with the temperature T that will become large.

The macroscopic thermal equilibrium state of the free Fermi gas depends, first of all, on its (classical) single-particle Hamiltonian $h : \mathbb{R}^d \to \mathbb{R}$, sometimes also called the energy-momentum (dispersion) relation. The minimal conditions that we impose on h are as follows. We assume that h is smooth in the sense that $h \in \mathsf{C}^\infty(\mathbb{R}^d)$ and that it satisfies the bounds

$$\big|\partial_{\boldsymbol{\xi}}^n h(\boldsymbol{\xi})\big| \lesssim |\boldsymbol{\xi}|^{2m} \quad \text{for all } n\in\mathbb{N}_0^d\,, \text{ and } h(\boldsymbol{\xi}) \gtrsim |\boldsymbol{\xi}|^{2m} \text{ for } |\boldsymbol{\xi}|\gtrsim 1\,, \tag{3}$$

for some constant $m > 0$. Here $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and the notations $\partial_{\boldsymbol{\xi}}^n$, $\lesssim$, $\gtrsim$ are defined at the end of the Introduction.

For given h, temperature $T > 0$, and chemical potential $\mu \in \mathbb{R}$ we introduce the *Fermi symbol* as the composition of the Fermi function and the "shifted" Hamiltonian $h - \mu$ by

$$a_{T,\mu}(\boldsymbol{\xi}) := \frac{1}{1+\exp\big(\frac{h(\boldsymbol{\xi})-\mu}{T}\big)}\,, \quad \boldsymbol{\xi}\in\mathbb{R}^d\,. \tag{4}$$

Next, we introduce the (bounded and continuous) *entropy function* $\eta_\gamma : \mathbb{R} \to [0, \ln(2)]$ for each *Rényi index* $\gamma > 0$. For $\gamma \neq 1$ it is defined by

$$\eta_\gamma(t) := \begin{cases} \frac{1}{1-\gamma}\ln\big[t^\gamma + (1-t)^\gamma\big] & \text{if } t\in(0,1),\\ 0 & \text{if } t\notin(0,1), \end{cases} \tag{5}$$

and for $\gamma = 1$, the von Neumann case, it is given by the point-wise limit

$$\eta_1(t) := \lim_{\gamma\to1}\eta_\gamma(t) = \begin{cases} -t\ln(t) - (1-t)\ln(1-t) & \text{if } t\in(0,1),\\ 0 & \text{if } t\notin(0,1). \end{cases} \tag{6}$$

Finally, we define the (Rényi) *local entropy* associated with a bounded region Λ as the trace

$$\mathrm{S}_\gamma(T, \mu; \Lambda) := \operatorname{tr} \eta_\gamma\big(W_1(a_{T,\mu}, \Lambda)\big) \geq 0, \tag{7}$$

and the (Rényi) *entanglement entropy* (EE) for the bipartition $\mathbb{R}^d = \Lambda \cup (\mathbb{R}^d \setminus \Lambda)$ as

$$\mathrm{H}_\gamma(T, \mu; \Lambda) := \operatorname{tr} D_1(a_{T,\mu}, \Lambda; \eta_\gamma) + \operatorname{tr} D_1(a_{T,\mu}, \mathbb{R}^d \setminus \Lambda; \eta_\gamma)\,. \tag{8}$$

This definition is motivated by the notion of mutual information, see e.g. [1, Eq. (9)]. The conditions (3) guarantee that these entropies are well-defined, see the paragraph after Proposition 2.6. In formula (8) either Λ or its complement $\mathbb{R}^d \setminus \Lambda$ is assumed to be bounded, see Sect. 2 for details. It is useful to observe, as in (26) of [17], that the local entropy (7) can be expressed as

$$\mathrm{S}_\gamma(T, \mu; \Lambda) = s_\gamma(T, \mu)\,|\Lambda| + \operatorname{tr} D_1(a_{T,\mu}, \Lambda; \eta_\gamma)\,. \tag{9}$$

Here $|\Lambda|$ is the volume (Lebesgue measure) of the bounded region Λ and

$$s_\gamma(T, \mu) := \frac{\operatorname{tr} W_1(\eta_\gamma \circ a_{T,\mu}, \Lambda)}{|\Lambda|} = (2\pi)^{-d} \int_{\mathbb{R}^d} \eta_\gamma\big(a_{T,\mu}(\boldsymbol{\xi})\big)\, \mathrm{d}\boldsymbol{\xi} \tag{10}$$

is the spatial *entropy density*. For $\gamma = 1$ it is the usual thermal entropy density [3, 7].

1.3 Summary of the Main Results

Our first result is of non-asymptotic nature. In Sect. 3 we explore concavity properties of the function η_γ. First we notice that η_γ is concave on the unit interval $[0, 1]$ for all $\gamma \in (0, 2]$, so that for a bounded Λ one can use a Jensen-type trace inequality to establish a lower bound for the local entropy (7) in terms of the entropy density (10), see Theorem 3.3. For the EE (8) this argument is not applicable, but we observe that η_γ for $\gamma \in (0, 1]$ is even *operator concave* so that the Davis operator inequality (32) implies the positivity of the EE for $\gamma \in (0, 1]$ (including the most important case $\gamma = 1$ corresponding to the von Neumann EE), see Theorem 3.5. We do not know whether the EE is positive for $\gamma \in (1, 2]$. Later on however, we will see that the EE for $\gamma \in (1, 2]$ is positive at least for large T, see Remark 2 and 5 in Sect. 6.3.

The other main objective of the present paper is to study the asymptotics of $\mathrm{H}_\gamma(T, \mu; \alpha\Lambda)$ as $\alpha \to \infty$ and $T \to \infty$. For this, we have to impose conditions on the Hamiltonian h stronger than those in (3), see Sect. 5.1. In particular, h should be asymptotically homogeneous as $|\boldsymbol{\xi}| \to \infty$. We distinguish between two high-temperature cases: the case of a constant chemical potential $\mu \in \mathbb{R}$ and the case where the mean particle density

$$\varrho(T,\mu) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} a_{T,\mu}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,, \tag{11}$$

being finite by (3), is (asymptotically) fixed to a prescribed constant $\rho > 0$, as T becomes large. The latter case implies that the chemical potential effectively becomes a function of T (and ρ) in the sense that

$$\varrho(T, \mu_\rho(T)) \to \rho \quad \text{as} \quad T \to \infty\,. \tag{12}$$

This corresponds to the quasi-classical limit of the free Fermi gas at fixed particle density. Indeed, since the so-called integrated density of states

$$\mathcal{N}(T) := (2\pi)^{-d}\int_{h(\boldsymbol{\xi})<T} \mathrm{d}\boldsymbol{\xi} \tag{13}$$

of the single-particle Hamiltonian h, evaluated at T, tends to infinity as $T \to \infty$, one has $\rho/\mathcal{N}(T) \to 0$. In physical terms, the spatial density of the number of particles is much smaller than that of the number of occupiable single-particle states with (eigen)energies below T, when T is sufficiently large. Therefore the restrictions by the Pauli exclusion principle are negligible.

Our results on the high-temperature scaling asymptotics are presented in Theorem 6.1 (for constant μ) and in Theorem 6.3 (for constant ρ). We postpone the discussion of these theorems until Sect. 6. Here we only mention two important facts: a) in both regimes the asymptotics still hold if $T \to \infty$ and the scaling parameter α is fixed, e.g. $\alpha = 1$, b) the EE with Rényi index $\gamma > 2$ becomes *negative* for fixed particle density at high temperature; the same happens for fixed chemical potential at large γ. This suggests that values $\gamma > 2$ are only of limited physical interest.

The main asymptotic orders as $\alpha \to \infty$ and $T \to \infty$ have been presented (without precise pre-factors and proofs) in [17], in the cases of fixed μ and fixed ρ, for $\gamma = 1$ and all $d \geq 1$. Here we provide complete proofs for all $\gamma > 0$, but concentrate on the multi-dimensional case $d \geq 2$. The case $d = 1$ is not considered for lack of space.

The paper is organized as follows. In Sect. 2 we present the basic information such as our conditions on the truncating region and the definition of the asymptotic coefficient $\mathcal{B}$. Section 2 also contains the results, borrowed mostly from [26], that are used throughout the paper. In Sect. 3 we study the concavity of the function η_γ and investigate the positivity of the corresponding entanglement entropy. In Sect. 4 we derive elementary trace-class bounds for some abstract bounded (self-adjoint) operators. These bounds are based on estimates for *quasi-commutators* of the form $f(A)J - Jf(B)$ with bounded J, bounded self-adjoint A, B, and suitable functions

f. By using such bounds in Sect. 5 we obtain the large α and T asymptotics for the trace of the operator $D_\alpha(p_T, \Lambda; f)$ with a symbol p_T modeling the Fermi symbol (4) in the fixed μ or fixed ρ regimes. In Sect. 6 we collect our results on the high-temperature asymptotics for the EE (8), see Theorems 6.1 and 6.3. Their proofs are directly based on the formulas obtained in Sect. 5. In Sect. 6.3 we also comment on the asymptotics of the local entropy (7). The Appendix contains a short calculation clarifying the structure of the Fermi symbol when the mean particle density is fixed as $T \to \infty$.

Throughout the paper we adopt the following standard notations. For two positive numbers (or functions) X and Y, possibly depending on parameters, we write $X \lesssim Y$ (or $Y \gtrsim X$) if $X \le CY$ with some constant $C \ge 0$ independent of those parameters. If $X \lesssim Y$ and $X \gtrsim Y$, then we write $X \asymp Y$. To avoid confusion we often make explicit comments on the nature of the (implicit) constants in the bounds. For multiple partial derivatives we use the notation $\partial_{\boldsymbol{\xi}}^n := \partial_{\xi_1}^{n_1}\partial_{\xi_2}^{n_2}\cdots\partial_{\xi_d}^{n_d}$ for a vector $\boldsymbol{\xi} \in \mathbb{R}^d$ and a multi-index $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$ of order $|n| := n_1 + n_2 + \cdots + n_d$. By $B(\mathbf{z}, R)$ we mean the open ball in $\mathbb{R}^d$ with center $\mathbf{z} \in \mathbb{R}^d$ and radius $R > 0$. We also use the weight function $\langle \mathbf{v} \rangle := \sqrt{1 + |\mathbf{v}|^2}$ for any vector $\mathbf{v} \in \mathbb{R}^d$.

The notation $\mathfrak{S}_p$, $p \in (0, \infty)$, is used for the Schatten–von Neumann classes of compact operators on a complex separable Hilbert space $\mathcal{H}$, see e.g. [6, Chapter 11]. By definition, the operator A belongs to $\mathfrak{S}_p$ if $\|A\|_p := (\mathrm{tr}(A^*A)^{p/2})^{1/p} < \infty$. The functional $\|\cdot\|_p$ on $\mathfrak{S}_p$ is a norm if $p \ge 1$ and a quasi-norm if $p < 1$. Apart from Sect. 4, where the space $\mathcal{H}$ is arbitrary, we assume that $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^d)$.

We dedicate this paper to the memory of Harold Widom (1932–2021). His ground-breaking results on the asymptotic expansions for traces of pseudo-differential operators have been highly influential to many researchers including us. Without his results the present contribution and our previous ones to the study of fermionic entanglement entropy would have been unthinkable. We are deeply indebted to Widom's ingenious insights. All three of us had the honor and pleasure of meeting him at a memorable workshop in March 2017 hosted by the American Institute of Mathematics (AIM) in San Jose, California, USA.

2 Basic Definitions and Basic Facts

In this section we collect some definitions and facts from [26] concerning the trace of (2) and its asymptotic evaluation. They are instrumental in the proof of our main asymptotic results corresponding to $a = a_{T,\mu}$ and $f = \eta_\gamma$. Throughout the rest of the paper we always assume $d \ge 2$ for the spatial dimension.

2.1 Conditions on the Truncating Region Λ

We call an open and connected set $\Lambda \subset \mathbb{R}^d$ a Lipschitz domain, if it can be described locally as the epigraph of a Lipschitz function, see [22, 23] for details. We call Λ a *Lipschitz region*, if it is a union of finitely many Lipschitz domains such that their closures are pair-wise disjoint. From now on we always assume that Λ satisfies the following condition. Nevertheless, for convenience we will often mention it.

Condition 2.1

1. *The set $\Lambda \subset \mathbb{R}^d$ is a Lipschitz region, and either Λ or $\mathbb{R}^d \setminus \Lambda$ is bounded.*
2. *The boundary (surface) $\partial\Lambda$ is piece-wise C^1-smooth.*

We note that Λ and $\mathbb{R}^d \setminus \overline{\Lambda}$ satisfy Condition 2.1 simultaneously.

2.2 The Asymptotic Coefficient $\mathcal{B}$ and Its Basic Properties

We assume the real-valued symbol to be smooth in the sense that $a \in \mathsf{C}^\infty(\mathbb{R}^d)$ and satisfies the decay condition

$$\left|\partial_{\boldsymbol{\xi}}^n a(\boldsymbol{\xi})\right| \lesssim \langle\boldsymbol{\xi}\rangle^{-\beta} \quad \text{with some constant} \quad \beta > d\,, \tag{14}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $n \in \mathbb{N}_0^d$ with some implicit constants that may depend on n.

Before stating the leading asymptotic formula for $\operatorname{tr} D_\alpha(a, \Lambda;\, f)$ as $\alpha \to \infty$, we need to introduce the corresponding asymptotic coefficient. For a function $f : \mathbb{R} \to \mathbb{R}$ and any $u, v \in \mathbb{R}$ we consider the integral

$$U(u, v;\, f) := \int_0^1 \frac{f\big(tu + (1-t)v\big) - \big[tf(u) + (1-t)f(v)\big]}{t(1-t)}\, \mathrm{d}t\,. \tag{15}$$

It is well-defined for any Hölder continuous f. And it is positive if f is also concave.

For every unit vector $\mathbf{e} \in \mathbb{R}^d$ we define a functional of the symbol a by the principal-value integral:

$$\mathcal{A}(a, \mathbf{e};\, f) := \frac{1}{8\pi^2} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \int_{|t|>\varepsilon} \frac{U\big(a(\boldsymbol{\xi}), a(\boldsymbol{\xi} + t\mathbf{e});\, f\big)}{t^2}\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\xi}\,. \tag{16}$$

Finally we define the main asymptotic coefficient by

$$\mathcal{B}(a, \partial\Lambda;\, f) := \frac{1}{(2\pi)^{d-1}} \int_{\partial\Lambda} \mathcal{A}(a, \mathbf{n}_{\mathbf{x}};\, f)\, \sigma(\mathrm{d}\mathbf{x})\,, \tag{17}$$

where $\mathbf{n}_{\mathbf{x}}$ is the (unit outward) normal vector at the point $\mathbf{x} \in \partial\Lambda$ and σ is the canonical $(d-1)$-dimensional area measure on the boundary surface $\partial\Lambda$.

For functions $f \in \mathsf{C}^2(\mathbb{R})$ with bounded second derivative and for a symbol a obeying the condition (14) the integral (16) exists in the usual sense and is bounded uniformly in $\mathbf{e}$. Hence (17) is also finite, see [26, Section 3]. However, in order to accommodate the entropy function η_γ we allow for test functions being non-smooth in the sense of the following condition.

Condition 2.2 *The function f is in $\mathsf{C}^2(\mathbb{R}\setminus\mathcal{T})\cap\mathsf{C}(\mathbb{R})$, where $\mathcal{T} := \{t_1, t_2, \dots, t_N\}$ is a finite set of its singular points. Moreover, for some $\delta > 0$ and all $R > 0$ the function $f = f^{(0)}$ and its first two derivatives satisfy the bounds*

$$\left|f^{(k)}(t)\right| \lesssim \sum_{u\in\mathcal{T}} |t-u|^{\delta-k}, \quad k = 0, 1, 2, \tag{18}$$

for all $t \in [-R, R] \setminus \mathcal{T}$ with an implicit constant that may depend on R.

Under this condition $\mathcal{B}(a, \partial\Lambda; f)$ is finite:

Proposition 2.3 ([26, Corollary 3.4]) *Let the set Λ satisfy Condition 2.1 and let the function f satisfy Condition 2.2 with $\delta > 0$. Finally, let the symbol a satisfy (14), but this time with some $\beta > d \max\{1, 1/\delta\}$. Then the coefficient $\mathcal{B}(a, \partial\Lambda; f)$ is finite.*

We point out a few simple properties of this coefficient.

Remark 2.4

1. Since $\mathcal{A}(a, \mathbf{e}; f) = \mathcal{A}(a, -\mathbf{e}; f)$, the coefficients $\mathcal{B}$ for the regions Λ and $\mathbb{R}^d \setminus \overline{\Lambda}$ coincide.
2. By definition (15), the coefficient (17) is positive for concave functions f and negative for convex ones. For example, the function η_γ with $\gamma \in (0, 2]$ is concave on the interval $[0, 1]$ (see Lemma 3.1) so that $\mathcal{B}(a, \partial\Lambda; \eta_\gamma) \geq 0$ for symbols a taking values only in $[0, 1]$, like the Fermi symbol $a_{T,\mu}$.
3. If the symbol a is spherically symmetric (for example, by spherical symmetry of the Hamiltonian h in $a_{T,\mu}$), then the surface area $|\partial\Lambda|$ factors out of $\mathcal{B}(a, \partial\Lambda; f)$. Nevertheless, the remaining integral is still hard to compute for general a and f. See, however, Remark 2 in Sect. 6.3 for Gaussian a and quadratic f.

A less obvious property of the coefficient $\mathcal{B}(a, \partial\Lambda; f)$ is its continuity in the symbol a. Since it is important for our purposes, we state a corresponding result. In the next and subsequent assertions we consider a one-parameter family of symbols $\{a_0, a_\lambda\}$, $\lambda > 0$, all of them satisfying (14) with some $\beta > d \max\{1, \delta^{-1}\}$, uniformly in λ, and point-wise convergence $a_\lambda \to a_0$ as $\lambda \downarrow 0$.

Proposition 2.5 ([26, Corollary 3.5]) *Let the set Λ and the function f be as in Proposition 2.3. Then*

$$\mathcal{B}(a_\lambda, \partial\Lambda; f) \to \mathcal{B}(a_0, \partial\Lambda; f) \quad \text{as } \lambda \downarrow 0. \tag{19}$$

2.3 *The Asymptotics for* tr $D_\alpha(a, \Lambda; f)$ *as* $\alpha \to \infty$

Now we are in a position to state the required asymptotic facts.

Proposition 2.6 ([26, Theorem 2.3]) *Let the set* Λ*, the function* f*, and the symbol* a *be as in Proposition 2.3. Then the operator* $D_\alpha(a, \Lambda; f)$ *is of trace class and*

$$\lim_{\alpha\to\infty} \alpha^{1-d} \operatorname{tr} D_\alpha(a, \Lambda; f) = \mathcal{B}(a, \partial\Lambda; f)\,.$$

This limit is uniform in the class of symbols a *that satisfy* (14) *with the same implicit constants.*

Proposition 2.6 ensures the existence of the entropies (7) and (8). In fact, assume that Λ satisfies Condition 2.1 and that the Hamiltonian h in the Fermi symbol $a_{T,\mu}$ of (4) is as specified in (3). Then $a_{T,\mu}$ satisfies (14) for all $T > 0$ and $\mu \in \mathbb{R}$. Moreover, the function η_γ satisfies Condition 2.2 for all $\gamma > 0$ with arbitrary $\delta < \min\{1, \gamma\}$ and the set $\mathcal{T} = \{0, 1\}$. Thus, due to Proposition 2.6, the operators $D_1(a_{T,\mu}, \Lambda; \eta_\gamma)$ and $D_1(a_{T,\mu}, \mathbb{R}^d \setminus \Lambda; \eta_\gamma)$ are of trace class, so that the entanglement entropy (8) is finite. If we additionally assume that Λ is bounded, then by (9) also the local entropy (7) is finite.

Proposition 2.6 was also used in [26] to determine the scaling asymptotics for the entanglement entropy at fixed temperature. To study the high-temperature regime, we need the continuity of this result in the symbol a:

Corollary 2.7 *Let the set* Λ *and the function* f *be as in Proposition 2.3. Then*

$$\lim \alpha^{1-d} \operatorname{tr} D_\alpha(a_\lambda, \Lambda; f) = \mathcal{B}(a_0, \partial\Lambda; f)\,, \tag{20}$$

where the limits $\alpha \to \infty$ *and* $\lambda \to 0$ *are taken simultaneously.*

Proof According to Proposition 2.6,

$$\lim_{\alpha\to\infty} \alpha^{1-d} \operatorname{tr} D_\alpha(a_\lambda, \Lambda; f) = \mathcal{B}(a_\lambda, \partial\Lambda; f)\,,$$

uniformly in λ. Together with (19) this leads to (20). □

The next two propositions describe the asymptotics for "small" symbols.

Proposition 2.8 ([26, Theorem 2.5]) *Let the set* Λ *satisfy Condition 2.1 and let* f_0 *be the function defined by* $f_0(t) := M|t|^\delta$ *with real constants* M *and* $\delta > 0$*. Finally, suppose that the function* $f \in \mathsf{C}^2(\mathbb{R} \setminus \{0\})$ *satisfies the conditions*

$$\lim_{t\to 0} |t|^{k-\delta} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(f(t) - f_0(t)\big) = 0\,, \quad k = 0, 1, 2\,. \tag{21}$$

Then

$$\lim_{\alpha\to\infty}\lim_{\lambda\to 0}\big(\alpha^{1-d}\lambda^{-\delta}\operatorname{tr} D_\alpha(\lambda a_\lambda, \Lambda; f)\big) = \mathcal{B}(a_0, \partial\Lambda; f_0)\,.$$

In the next proposition we consider instead of the homogeneous function f_0 the function η defined by

$$\eta(t) := -t\ln(|t|)\,,\quad t\in\mathbb{R}\,, \tag{22}$$

which still leads to an asymptotically homogeneous behavior.

Proposition 2.9 ([26, Theorem 2.6]) *Let the set Λ satisfy Condition 2.1 and suppose that the function $f\in \mathsf{C}^2(\mathbb{R}\setminus\{0\})$ satisfies the conditions*

$$\lim_{t\to 0}|t|^{k-1}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(f(t)-\eta(t)\big) = 0\,,\quad k=0,1,2\,. \tag{23}$$

Then

$$\lim_{\alpha\to\infty}\lim_{\lambda\to 0}\big(\alpha^{1-d}\lambda^{-1}\operatorname{tr} D_\alpha(\lambda a_\lambda, \Lambda; f)\big) = \mathcal{B}(a_0, \partial\Lambda; \eta)\,.$$

We note that under any of the assumptions (21) and (23) the function f satisfies Condition 2.2 with $\mathcal{T}=\{0\}$. For assumption (21) (resp. (23)) the condition (18) holds with the constant δ from (21) (resp. arbitrary $\delta<1$).

The asymptotic results listed above are useful, but, as they stand, not directly applicable for our purposes. This is because our symbol of main interest, the Fermi symbol (4), depends on the two parameters T and μ, and in the course of our analysis in Sect. 5 we naturally come across certain "effective" symbols that do not satisfy conditions like (14) uniformly in these parameters. However, we overcome this problem by considering a wider class of symbols, called *multi-scale symbols* in [18, Section 3].

2.4 Multi-Scale Symbols

We consider symbols $a\in\mathsf{C}^\infty(\mathbb{R}^d)$ for which there exist two continuous functions τ and v on $\mathbb{R}^d$ with $\tau>0$, $v>0$, v bounded, and such that

$$\big|\partial_{\boldsymbol{\xi}}^k a(\boldsymbol{\xi})\big| \lesssim \tau(\boldsymbol{\xi})^{-|k|}v(\boldsymbol{\xi})\,,\ k\in\mathbb{N}_0^d,\quad \boldsymbol{\xi}\in\mathbb{R}^d\,, \tag{24}$$

with implicit constants independent of $\boldsymbol{\xi}$. It is natural to call τ the *scale (function)* and v the *amplitude (function)*. The scale τ is assumed to be globally Lipschitz continuous with Lipschitz constant $\mathcal{L} < 1$, that is,

$$|\tau(\boldsymbol{\xi}) - \tau(\boldsymbol{\xi}')| \leq \mathcal{L}|\boldsymbol{\xi} - \boldsymbol{\xi}'|, \quad \text{for all } \boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbb{R}^d. \tag{25}$$

Under this assumption the amplitude v is assumed to satisfy the bounds

$$v(\boldsymbol{\xi}') \asymp v(\boldsymbol{\xi}), \quad \text{for all } \boldsymbol{\xi}' \in B\big(\boldsymbol{\xi}, \tau(\boldsymbol{\xi})\big), \tag{26}$$

with implicit constants independent of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$. It is useful to think of τ and v as (functional) parameters. They, in turn, may depend on other parameters, e.g. numerical ones like α and T. For example, the results in the previous subsections are based on the assumption that a satisfies (14), which translates into (24) with $\tau(\boldsymbol{\xi}) = 1$ and $v(\boldsymbol{\xi}) = \langle\boldsymbol{\xi}\rangle^{-\beta}$. On the other hand, in Sect. 5 we encounter amplitudes and scales depending on the temperature T.
Actually, we will only need the following result involving multi-scale symbols. As mentioned in the Introduction, $\|\cdot\|_p$ denotes the (quasi-)norm in the Schatten-von Neumann class $\mathfrak{S}_p$ of compact operators. Below the underlying Hilbert space is $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^d)$.

Proposition 2.10 ([18, Lemma 3.4]) *Let the set Λ satisfy Condition 2.1 and let the functions τ and v be as described above. Suppose that the symbol a satisfies* (24) *and that the conditions*

$$\alpha\tau_{\text{inf}} \gtrsim 1, \quad \tau_{\text{inf}} := \inf_{\boldsymbol{\xi}\in\mathbb{R}^d} \tau(\boldsymbol{\xi}) > 0, \tag{27}$$

hold. Then for any $q \in (0, 1]$ we have

$$\|[\mathrm{Op}_\alpha(a), \chi_\Lambda]\|_q^q \lesssim \alpha^{d-1} \int_{\mathbb{R}^d} \frac{v(\boldsymbol{\xi})^q}{\tau(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi}. \tag{28}$$

This bound is uniform in the symbols a satisfying (24) *with the same implicit constants.*

We will make use of (28) in Sect. 5 by combining it with bounds obtained in Sect. 4.

3 The Positivity of Certain Entanglement Entropies

Given the organization of the paper, this section is a kind of interlude. It turns out that the property given by its title is present if the underlying entropy function η_γ, as defined in (5) and (6), is operator concave (when restricted from the real line $\mathbb{R}$ to its unit interval $[0, 1]$). Since this and related results are not of asymptotic character, we assume in this section $\alpha = 1$ for the scaling parameter.

3.1 Concavity of the Entropy Function η_γ for $\gamma \le 2$

We prove the property given by the title of this subsection and then establish consequences for the corresponding local and entanglement entropies. The next lemma is elementary.

Lemma 3.1 *The entropy function η_γ is concave on the interval* $[0, 1]$ *if* $\gamma \in (0, 2]$, *and neither convave nor convex if* $\gamma > 2$.

Proof By the continuity of η_γ on $[0, 1]$ its enough to check the sign of the second derivative of η_γ on the open interval $(0, 1)$. For $\gamma = 1$ we simply have $\eta_1''(t) = -t^{-1}(1-t)^{-1} < 0$ so that η_1 is concave. For $\gamma \neq 1$ we use the formula

$$\eta_\gamma''(t)\big[t^\gamma + (1-t)^\gamma\big]^2 = -\gamma[t(1-t)]^{\gamma-2} - \frac{\gamma}{1-\gamma}[t^{\gamma-1} - (1-t)^{\gamma-1}]^2 \,. \tag{29}$$

For $\gamma < 1$ the right-hand side is obviously negative for all $t \in (0, 1)$. For $\gamma = 2$ it simply equals $-8t(1-t) < 0$. If $\gamma > 2$, then (29) implies $\eta_\gamma''(0) = \gamma/(\gamma-1) > 0$ and $\eta_\gamma''(1/2) = -4\gamma < 0$. Hence η_γ is neither concave nor convex.

It remains to consider the case $\gamma \in (1, 2)$. We rewrite (29) as

$$\eta_\gamma''(t)[t^\gamma + (1-t)^\gamma]^2 = -\frac{\gamma}{\gamma-1} g_{\gamma-1}(t)\,,$$
$$g_p(t) := p[t(1-t)]^{p-1} - [t^p - (1-t)^p]^2\,,$$

for any $p := \gamma - 1 \in (0, 1)$. Our claim $g_p(t) \ge 0$ is equivalent to

$$[t(1-t)]^{1-p}[t^{2p} + (1-t)^{2p}] \le 2t(1-t) + p\,. \tag{30}$$

Using the abbreviation

$$M := 2^{p-1} \max_{t\in[0,1]} [t^{2p} + (1-t)^{2p}] = \begin{cases} 2^{-p} & \text{if} \quad 0 < p < 1/2 \\ 2^{p-1} & \text{if} \quad 1/2 \le p < 1 \end{cases}\,,$$

the (elementary example of the) Young inequality

$$ab \le \frac{a^u}{u} + \frac{b^v}{v}\,, \quad a, b \ge 0\,, \quad u, v > 1\,,\ \frac{1}{u} + \frac{1}{v} = 1$$

for $a = [t(1-t)]^{1-p}$, $u = (1-p)^{-1}$ and $b = 1$, $v = p^{-1}$ yields

$$[(t(1-t)]^{1-p}[t^{2p} + (1-t)^{2p}] \le M[2t(1-t)]^{1-p}$$
$$\le M(1-p)[2t(1-t)] + Mp \le 2t(1-t) + p\,.$$

Since this coincides with (30), the proof is complete. □

The just established concavity is useful to find a lower bound on the local entropy (7) with $\gamma \le 2$, which is greater than the obvious bound 0. To this end, we recall a formulation [15, Theorem A.1] of an abstract Jensen-type trace inequality dating back to Berezin [5].

Proposition 3.2 *Let $\mathcal{H}$ be a complex separable Hilbert space, P an orthogonal projection on $\mathcal{H}$, A a self-adjoint operator on $\mathcal{H}$ with its spectrum contained in the interval $I \subset \mathbb{R}$, and $f : I \to \mathbb{R}$ a concave function. Finally, let $\Delta := Pf(PAP)P - Pf(A)P$ be of trace class and PAP compact. Then* $\operatorname{tr}\Delta \ge 0$. *If Δ and PAP are of trace class, than also the following trace inequality is valid:*

$$\operatorname{tr}(Pf(PAP)P) \ge \operatorname{tr}(Pf(A)P)\,.$$

(If $0 \notin I$, then the operator $f(PAP)$ is understood to act on the subspace $P\mathcal{H}$).

The following result is a corollary to Proposition 3.2.

Theorem 3.3 *Let $\Lambda \subset \mathbb{R}^d$ be bounded and satisfy Condition 2.1. Assume that the Hamiltonian h satisfies* (3) *and that* $\gamma \in (0, 2]$. *Then the local entropy* (7) *obeys the inequality*

$$\mathrm{S}_\gamma(T, \mu; \Lambda) \ge s_\gamma(T, \mu)|\Lambda|\,, \tag{31}$$

where $s_\gamma(T, \mu)$ is the entropy density (10)*.*

Proof We use Proposition 3.2 with $A = \mathrm{Op}_1(a_{T,\mu})$, $P = \chi_\Lambda$, and the concave function $f = \eta_\gamma$. Since $0 \le A \le \mathbb{1}$ and PAP has $\varrho(T, \mu)|\Lambda|$, see (11), as its finite trace, Proposition 3.2 is indeed applicable and yields $\operatorname{tr} D_1(a, \Lambda; \eta_\gamma) \ge 0$. By (9) this entails (31). □

We stress that Proposition 3.2 cannot be applied if the set Λ is unbounded, since in this case the operator $\chi_\Lambda \mathrm{Op}_\alpha(a_{T,\mu})\chi_\Lambda$ is not necessarily compact. Thus Theorem 3.3 cannot be used to determine the sign of the entanglement entropy (8). But, fortunately, we can use the rather strong property as given by the title of the following subsection.

3.2 Operator Concavity of the Entropy Function η_γ for $\gamma \leq 1$

For the general background of this genre we recommend Simon's comprehensive book [20]. Let $\mathcal{H}$ be a complex separable Hilbert space of infinite dimension and $\{A, B\}$ an arbitrary pair of bounded self-adjoint operators on $\mathcal{H}$ with spectra in an interval $I \subset \mathbb{R}$. A continuous function $f : I \to \mathbb{R}$ is called (decreasing) *operator monotone* if the (operator) inequality $f(A) \geq f(B)$ holds whenever $A \geq B$. Likewise it is called *operator concave* if $f\big(\lambda A + (1-\lambda)B\big) \geq \lambda f(A) + (1-\lambda) f(B)$ holds for all $\lambda \in [0, 1]$. It is called *operator convex* if $-f$ is operator concave. Of course, every operator monotone (operator concave) function is monotone (concave). We are going to use the following standard examples, see [2]:

1. The function $t \mapsto t^p$, $t \in [0, \infty)$, is operator monotone and operator concave if $p \in (0, 1]$.
2. The function $t \mapsto \ln(t)$, $t \in (0, \infty)$, is operator monotone and operator concave.
3. The function $t \mapsto -t \ln(t)$, $t \in [0, \infty)$, is operator concave.

Any operator concave function f satisfies [9, 13] the following *Davis operator inequality* for all bounded self-adjoint operators A with spectrum in I and all orthogonal projections P on $\mathcal{H}$:

$$Pf\big(PAP\big)P \geq Pf(A)P\,. \tag{32}$$

If $0 \notin I$, then the operator $f(PAP)$ is understood to act on the subspace $P\mathcal{H}$. If $0 \in I$ and $f(0) = 0$, then (32) may be shortened to $f\big(PAP\big) \geq Pf(A)P$.

Lemma 3.4 *If* $\gamma \in (0, 1]$, *then* η_γ *is operator concave on the interval* $[0, 1]$.

Proof It suffices to consider self-adjoint operators A and B with $0 \leq A, B \leq \mathbb{1}$. Assume first that $\gamma < 1$. The function $g_\gamma(t) := t^\gamma + (1-t)^\gamma$ is operator concave on $[0, 1]$ (by example 1 above) and the logarithm is operator monotone on $(0, 1]$ (example 2). Thus for all $\lambda \in [0, 1]$ we have

$$\begin{aligned}\eta_\gamma\big(\lambda A + (1-\lambda)B\big) &= \frac{1}{1-\gamma}\,\ln\big[g_\gamma\big(\lambda A + (1-\lambda)B\big)\big] \\ &\geq \frac{1}{1-\gamma}\ln\big[\lambda g_\gamma\big(A\big) + (1-\lambda)g_\gamma\big(B\big)\big]\,.\end{aligned}$$

Now, since the logarithm is also operator concave on $(0, 1]$ (again example 2), the right-hand side is larger than or equal to

$$\frac{1}{1-\gamma}\big[\lambda \ln(g_\gamma(A)) + (1-\lambda)\ln(g_\gamma(B))\big] = \lambda\eta_\gamma(A) + (1-\lambda)\eta_\gamma(B)\,.$$

Hence η_γ, for $\gamma < 1$, is operator concave on $(0, 1]$ and, by continuity, on $[0, 1]$. For $\gamma = 1$ we proceed more directly and use that $g(t) := -t\ln(t)$ is operator concave on [0, 1] (example 3).This immediately implies that $\eta_1(t) = g(t) + g(1-t)$ is also operator concave on [0, 1]. □

Now we can use the inequality (32) with $f = \eta_\gamma$ (for $\gamma \in (0, 1]$), $P = \chi_\Lambda$, and $A = \mathrm{Op}_1(a_{T,\mu})$. Combining this with Proposition 2.6 yields the following result.

Theorem 3.5 *Let $\Lambda \subset \mathbb{R}^d$ satisfy Condition 2.1. Assume that the Hamiltonian h is as in (3) and that $\gamma \in (0, 1]$. Then both operators $D_1(a_{T,\mu}, \Lambda; \eta_\gamma)$ and $D_1(a_{T,\mu}, \mathbb{R}^d \setminus \Lambda; \eta_\gamma)$ are not only of trace class, but also positive. Hence the entanglement entropy (8) is positive.*

This method cannot be used for the EE with $\gamma > 1$ because of the following negative result.

Theorem 3.6 *If $\gamma > 1$, then η_γ is not operator concave on the interval* [0, 1].

Proof For convenience, instead of η_γ we consider the function

$$g_\gamma(u) := -\eta_\gamma\big(u + \tfrac{1}{2}\big) = \tfrac{1}{\gamma-1} \ln\big[(\tfrac{1}{2} - u)^\gamma + (\tfrac{1}{2} + u)^\gamma\big], \quad u \in \big[-\tfrac{1}{2}, \tfrac{1}{2}\big].$$

Our objective is to show that g_γ is not operator convex on $[-1/2, 1/2]$. If g_γ were operator convex, then by [4, Corollary 1], g_γ would be analytic on the complex plane with cuts along the half-lines $(-\infty, -1/2)$ and $(1/2, \infty)$. Let us prove that such an analytic continuation of g_γ is impossible. To this end, let $u = \mathrm{i}y/2$ with $y > 0$. Then

$$\tfrac{1}{2} \pm u = \tfrac{1}{2}\sqrt{1 + y^2}\, \exp[\pm \mathrm{i} \tan^{-1}(y)]$$

so that

$$(\tfrac{1}{2} - u)^\gamma + (\tfrac{1}{2} + u)^\gamma = 2^{1-\gamma}(1 + y^2)^{\gamma/2} \cos[\gamma \tan^{-1}(y)]\,. \tag{33}$$

Since $\gamma > 1$, there exists a finite $y_0 > 0$ such that $\gamma \tan^{-1}(y_0) = \pi/2$, so that the right-hand side of (33) changes sign at $y = y_0$. This implies that the function g_γ has a branching point at $u = \mathrm{i}y_0/2$, and hence cannot be analytic in the whole upper half-plane. This proves that g_γ is not operator convex, as claimed. □

4 Quasi-Commutator Bounds

In this section we collect some bounds for the Schatten–von Neumann classes $\mathfrak{S}_p$, $p \in (0, \infty)$, of compact operators on a complex separable Hilbert space $\mathcal{H}$, see e.g. [6, Chapter 11]. As mentioned at the end of the Introduction, the functional $\|A\|_p := (\mathrm{tr}(A^*A)^{p/2})^{1/p}$, $A \in \mathfrak{S}_p$, defines a norm for $p \geq 1$ and a quasi-norm for $p < 1$. It satisfies the following "triangle inequality":

$$\|A + B\|_p^p \le \|A\|_p^p + \|B\|_p^p, \quad 0 < p \le 1. \tag{34}$$

This inequality is used systematically in what follows. The main part is played by estimates for *quasi-commutators* $f(A)J - Jf(B)$ with bounded J and bounded self-adjoint A, B. The following fact is adapted from [24, Theorem 2.4].

Proposition 4.1 *Suppose that f satisfies Condition 2.2 with some $\delta > 0$. Let A, B be two bounded self-adjoint operators and let J be a bounded operator. Suppose that $AJ - JB \in \mathfrak{S}_q$ where q satisfies $0 < q < \min\{1, \delta\}$. Then*

$$\|f(A)J - Jf(B)\|_1 \lesssim \|J\|^{1-q}\big(1 + \|A\|^{\delta-q} + \|B\|^{\delta-q}\big)\|AJ - JB\|_q^q, \tag{35}$$

with a constant independent of the operators A, B, J. This constant in (35) may depend on the set $\mathcal{T}$ in Condition 2.2, and is uniform in the functions satisfying (18) with the same implicit constants.

Actually, [24, Theorem 2.4] provides bounds of the type (35) in arbitrary (quasi-)normed operator ideals of compact operators and gives a more precise dependence on the constants related to the function f. For the present paper (35) is sufficient.

All subsequent bounds involving the function f are uniform in f in the sense specified in Proposition 4.1. We are going to apply Proposition 4.1 to obtain various bounds for the operator difference $\mathcal{D}(A, P; f) := Pf(PAP)P - Pf(A)P$ with an orthogonal projection P.

Corollary 4.2 *Let the function f and the parameter q be as in Proposition 4.1. Let A, B be bounded self-adjoint operators and let J be a bounded operator such that*

$$[A, J] = [B, J] = 0, \quad (A - B)J = 0. \tag{36}$$

Then

$$\|\mathcal{D}(A, P; f)J\|_1 + \|J\mathcal{D}(A, P; f)\|_1 \lesssim \|[J, P]\|_q^q + \|[JA, P]\|_q^q, \tag{37}$$

and

$$\|\mathcal{D}(A, P; f)J - J\mathcal{D}(B, P; f)\|_1 \lesssim \|[J, P]\|_q^q + \|[J, P]\|_1. \tag{38}$$

The implicit constants in these bounds depend on the norms $\|A\|$, $\|B\|$, and $\|J\|$, but they are uniform in the operators A, B, J whose norms are bounded by the same constants. They are also uniform in the functions f in the sense specified in Proposition 4.1.

Proof The proof is based mainly on the bound (35). The assumption (36) considerably simplifies the calculations, and we use it without further comments.

For the proof of (37) we carry out the estimate for the first term on the left-hand side only, as the second one can be treated in the same way. We rewrite

$$\mathcal{D}(A, P; f)J = P\big(f(PAP)PJ - PJf(A)\big) - P[f(A), PJ]\,. \tag{39}$$

Then we use (35) and (34) to estimate the first term on the right-hand side,

$$\begin{aligned}\|Pf(PAP)PJ - PJf(A)\|_1 &\le \|f(PAP)PJ - PJf(A)\|_1 \\ &\lesssim \|PAPJ - PJA\|_q^q \lesssim \|P(AJ - JA)P\|_q^q + \|[P, J]\|_q^q + \|[JA, P]\|_q^q \\ &= \|[P, J]\|_q^q + \|[JA, P]\|_q^q\,.\end{aligned}$$

For the second term on the right-hand side of (39) we also use (35) and (34):

$$\begin{aligned}\|P[f(A), PJ]\|_1 &\lesssim \|APJ - PJA\|_q^q \\ &\lesssim \|(AJ - JA)P\|_q^q + \|[P, J]\|_q^q + \|[JA, P]\|_q^q \\ &= \|[P, J]\|_q^q + \|[JA, P]\|_q^q\,.\end{aligned}$$

Adding up the above two estimates we get (37).

In order to prove (38) we first estimate the difference

$$\begin{aligned}Pf(PAP)PJ - JPf(PBP)P = {} & P\big(f(PAP)J - Jf(PBP)\big)P \\ & + Pf(PAP)[P, J] - [J, P]f(PBP)P\,.\end{aligned} \tag{40}$$

Then we use (35) to estimate the first term on the right-hand side as follows

$$\|P\big(f(PAP)J - Jf(PBP)\big)P\|_1 \lesssim \|PAPJ - JPBP\|_q^q \le \|[J, P]\|_q^q\,.$$

To estimate the second and the third term on the right-hand side of (40), we notice that $\|f(A)\| \lesssim 1$, $\|f(B)\| \lesssim 1$ uniformly in the operators A, B and in the function f. Consequently,

$$\|Pf(PAP)[P, J] - [J, P]f(PBP)P\|_1 \le 2\|[J, P]\|_1\, \|f\|_{\mathsf{L}^\infty} \lesssim \|[J, P]\|_1\,.$$

To summarize, the difference (40) has an upper bound like the one in the claim (38).

Finally we are going to derive such a bound also for the difference analogous to (40):

$$\begin{aligned}Pf(A)PJ - JPf(B)P = {} & P\big(f(A)J - Jf(B)\big)P \\ & + Pf(A)[P, J] - [J, P]f(B)P\,.\end{aligned}$$

In view of (36), the first term on the right-hand side vanishes, and hence

$$\|Pf(A)PJ - JPf(B)P\|_1 \le 2\|[P, J]\|_1 \, \|f\|_{\mathsf{L}^\infty} \lesssim \|[J, P]\|_1 \,.$$

By combining this with the upper bound on (40) and using the triangle inequality for the trace norm we arrive at (38). □

Corollary 4.3 *Under the assumptions of Corollary 4.2 (with $\mathbb{1}$ being the identity operator) we have*

$$\|\mathcal{D}(A, P; f) - \mathcal{D}(B, P; f)\|_1 \lesssim \|[J, P]\|_q^q + \|[(\mathbb{1} - J)A, P]\|_q^q + \|[(\mathbb{1} - J)B, P]\|_q^q + \|[J, P]\|_1 \,. \tag{41}$$

This bound is uniform in A, B, J and in the function f in the same sense as in Corollary 4.2.

Proof We observe

$$\mathcal{D}(A, P; f) - \mathcal{D}(B, P; f) = \mathcal{D}(A, P; f)J - J\mathcal{D}(B, P; f) + \mathcal{D}(A, P; f)(\mathbb{1} - J) - (\mathbb{1} - J)\mathcal{D}(B, P; f)$$

and apply Corollary 4.2. □

5 High-Temperature Analysis

The purpose of this section is to obtain the large α and large T asymptotics for the trace of the operator $D_\alpha(p_T, \Lambda; f)$ with the symbol p_T of (46), modeling the Fermi symbol (4) for large T. Throughout the section we assume that the function f satisfies Condition 2.2 with some $\delta > 0$ and recall that this condition is guaranteed by assumption (21) as well as by assumption (23). We also continue to assume that the truncating region Λ satisfies Condition 2.1. Since Λ is always fixed, we omit it from the notation and simply write $D_\alpha(p_T; f)$ and $\mathcal{B}(p_T; f)$. Recall that $d \ge 2$ throughout.

5.1 Further Conditions on the Single-Particle Hamiltonian h

So far we assumed that the Hamiltonian h satisfies (3) with some $m > 0$. Now we need to impose on h more restrictive conditions. We assume that $h \in \mathsf{C}^\infty(\mathbb{R}^d)$ and that for some constant $m \in \mathbb{N}$, the following bounds hold:

$$\left|\partial_{\boldsymbol{\xi}}^n h(\boldsymbol{\xi})\right| \lesssim \langle \boldsymbol{\xi} \rangle^{2m-|n|}, \quad \text{for all } n \in \mathbb{N}_0^d \quad \text{and} \quad \boldsymbol{\xi} \in \mathbb{R}^d \,. \tag{42}$$

Furthermore, we assume that there exists a function $h_\infty : \mathbb{R}^d \to \mathbb{R}$ which is homogeneous of degree $2m$ (that is, $h_\infty(t\boldsymbol{\xi}) = t^{2m} h_\infty(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $t > 0$), such that

$$|\boldsymbol{\xi}|^{-2m} \big| h(\boldsymbol{\xi}) - h_\infty(\boldsymbol{\xi}) \big| \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \,. \tag{43}$$

The function h_∞ is assumed to be non-degenerate in the sense that

$$2\nu := \min_{|\boldsymbol{\xi}|=1} h_\infty(\boldsymbol{\xi}) > 0 \,. \tag{44}$$

Homogeneity and non-degeneracy of h_∞ imply that $h_\infty \geq 0$. It is clear that the conditions (42), (43) and (44) imply that h satisfies (3) with the constant m from (42). It is important to emphasize that from now on the constant $m > 0$ is supposed to be integer. This guarantees that the function h_∞ belongs to $\mathsf{C}^\infty(\mathbb{R}^d)$ which enables application of the results in Subsection 2.3 to the limiting symbols $(1 + \mathrm{e}^{h_\infty})^{-1}$ and e^{-h_∞} featured in Sect. 6.

5.2 *Modeling the Fermi Symbol*

Given two positive continuous functions $T \mapsto \phi_T \geq 0$ and $T \mapsto \omega_T > 0$ on the temperature half-line $[1, \infty)$ with the properties

$$\phi_T \to \phi_\infty \geq 0 \quad \text{and} \quad \omega_T \to \omega_\infty > 0 \quad \text{as} \quad T \to \infty \,, \tag{45}$$

we generalize the Fermi symbol $a_{T,\mu}$ of (4) to the symbol p_T by the definition

$$p_T(\boldsymbol{\xi}) := \frac{1}{\phi_T + \omega_T \exp\big(h(\boldsymbol{\xi})/T\big)} \,, \quad \boldsymbol{\xi} \in \mathbb{R}^d \,. \tag{46}$$

We also consider its "high-temperature limit" p_∞ naturally defined by

$$p_\infty(\boldsymbol{\xi}) := \frac{1}{\phi_\infty + \omega_\infty \exp\big(h_\infty(\boldsymbol{\xi})\big)} \,. \tag{47}$$

Theorem 5.1 *Let p_T be the symbol defined in (46). Then*

$$\lim \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \operatorname{tr} D_\alpha(p_T; f) = \mathcal{B}(p_\infty; f) \,, \tag{48}$$

as $\alpha T^{1/2m} \to \infty$ and $T \to \infty$.

We also consider the operator $D_\alpha(\lambda_T\, p_T; f)$ with the symbol $\lambda_T\, p_T$, where $\lambda_T > 0$ is, for the time being, an arbitrary continuous function of T that tends to zero as $T \to \infty$.

Theorem 5.2 *Let f_0 be as in Proposition 2.8 and η be as in (22). Assume that $\alpha T^{1/2m} \to \infty$ and $T \to \infty$. Then the following implications hold:*

1. If $f \in \mathsf{C}^2(\mathbb{R} \setminus \{0\}) \cap \mathsf{C}(\mathbb{R})$ satisfies (21), then

$$\lim \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \lambda_T^{-\delta} \operatorname{tr} D_\alpha(\lambda_T\, p_T; f) = \mathcal{B}(p_\infty; f_0)\,. \tag{49}$$

2. If $f \in \mathsf{C}^2(\mathbb{R} \setminus \{0\}) \cap \mathsf{C}(\mathbb{R})$ satisfies (23), then

$$\lim \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \lambda_T^{-1} \operatorname{tr} D_\alpha(\lambda_T\, p_T; f) = \mathcal{B}(p_\infty; \eta)\,. \tag{50}$$

To prove these two theorems we compare the operator D_α for two different symbols defined as follows. Firstly, we pick an arbitrary real-valued "cut-off" function $w \in \mathsf{C}^\infty(\mathbb{R}^d)$ with $w(\boldsymbol{\xi}) = 0$ if $|\boldsymbol{\xi}| \le 1$ and $w(\boldsymbol{\xi}) = 1$ if $|\boldsymbol{\xi}| \ge 2$. Moreover, we define two scaled versions of w by

$$w_T(\boldsymbol{\xi}) := w\big(\boldsymbol{\xi} T^{-\frac{1}{2m}}\big)\,, \quad \widetilde{w}_T(\boldsymbol{\xi}) := w_T(\boldsymbol{\xi}/2)\,, \tag{51}$$

so that $w_T \widetilde{w}_T = \widetilde{w}_T$. For a fixed number $r \in (0, 1]$ we now consider the operators

$$A = \operatorname{Op}_\alpha(p_T)\,, \quad B = \operatorname{Op}_\alpha(w_{rT}\, p_T)\,, \quad P = \chi_\Lambda\,, \quad J = \operatorname{Op}_\alpha(\widetilde{w}_{rT})\,.$$

They fulfill (36) and their (uniform) norms are uniformly bounded in T. Thus we can use Corollary 4.3 for the proof of the following "comparison lemma":

Lemma 5.3 *Assume that $T \gtrsim 1$ and $\alpha(rT)^{\frac{1}{2m}} \gtrsim 1$ for a fixed $r \in (0, 1]$. Then, using (51), we have the trace norm estimate*

$$\|D_\alpha(p_T; f) - D_\alpha(w_{rT}\, p_T; f)\|_1 \lesssim \alpha^{d-1}(rT)^{\frac{d-1}{2m}}\,, \tag{52}$$

with an implicit constant independent of α, T, and r.

Proof Let us estimate the right-hand side of (41) and start with a bound for $\|[J, P]\|_q$, $q \le 1$. Since $[J, P] = -[\mathbb{1} - J, P]$, we use Proposition 2.10 with $a = 1 - \widetilde{w}_{rT}$. This symbol satisfies (24) with scale and amplitude functions

$$\tau(\boldsymbol{\xi}) = (rT)^{\frac{1}{2m}}\,, \quad v(\boldsymbol{\xi}) = \langle \boldsymbol{\xi}(rT)^{-\frac{1}{2m}} \rangle^{-\beta}\,, \quad \boldsymbol{\xi} \in \mathbb{R}^d\,,$$

with an arbitrary $\beta > 0$. Now we assume that $\beta q > d$. The conditions (25) and (26) are obviously satisfied, and hence Proposition 2.10 is applicable. We estimate the integral on the right-hand side of (28) as follows:

$$\int_{\mathbb{R}^d} \frac{v(\boldsymbol{\xi})^q}{\tau(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} = (rT)^{-\frac{1}{2m}} \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(rT)^{-\frac{1}{2m}}\rangle^{-\beta q}\, \mathrm{d}\boldsymbol{\xi} \lesssim (rT)^{\frac{d-1}{2m}} .$$

Thus, under our assumptions on α, T, and r the condition (27) is satisfied, and hence, by (28), we have

$$\|[J, P]\|_q^q = \|[\mathrm{Op}_\alpha(a), \chi_\Lambda]\|_q^q \lesssim \alpha^{d-1}(rT)^{\frac{d-1}{2m}} .$$

Estimating $\|[(\mathbb{1} - J)A, P]\|_q$ and $\|[(\mathbb{1} - J)B, P]\|_q$ is somewhat trickier. For the first commutator we are going to use Proposition 2.10 with the symbol $a = (1 - \widetilde{w}_{rT})p_T$. At first we estimate the derivatives of $\mathrm{e}^{h(\boldsymbol{\xi})/T}$ for $|\boldsymbol{\xi}| \le 4(rT)^{\frac{1}{2m}}$ using (42):

$$\left|\partial_{\boldsymbol{\xi}}^k\, \mathrm{e}^{\frac{h(\boldsymbol{\xi})}{T}}\right| \lesssim \langle\boldsymbol{\xi}\rangle^{-|k|}\mathrm{e}^{\frac{h(\boldsymbol{\xi})}{T}} \lesssim \langle\boldsymbol{\xi}\rangle^{-|k|} , \quad k \in \mathbb{N}_0^d .$$

Furthermore,

$$\left|\partial_{\boldsymbol{\xi}}^k \big(1 - \widetilde{w}_{rT}(\boldsymbol{\xi})\big)\right| \lesssim (rT)^{-\frac{|k|}{2m}} \chi_{\{|\boldsymbol{\xi}| \le 4(rT)^{\frac{1}{2m}}\}}(\boldsymbol{\xi}) \lesssim \langle\xi\rangle^{-|k|}\, \chi_{\{|\boldsymbol{\xi}| \le 4(rT)^{\frac{1}{2m}}\}}(\boldsymbol{\xi}) .$$

Therefore, we obtain from (46) that

$$\left|\partial_{\boldsymbol{\xi}}^k\, a(\boldsymbol{\xi})\right| \lesssim \langle\boldsymbol{\xi}\rangle^{-|k|}\langle\boldsymbol{\xi}(rT)^{-\frac{1}{2m}}\rangle^{-\beta} .$$

with an arbitrary $\beta > d/q$. Consequently, the symbol a satisfies (24) with the scale and amplitude

$$\tau(\boldsymbol{\xi}) = \frac{1}{2}\langle\boldsymbol{\xi}\rangle , \quad v(\boldsymbol{\xi}) = \langle\boldsymbol{\xi}(rT)^{-\frac{1}{2m}}\rangle^{-\beta} , \quad \boldsymbol{\xi} \in \mathbb{R}^d .$$

Again the conditions (25), (26), and (27) are satisfied, and we can use Proposition 2.10 to produce the bound

$$\int_{\mathbb{R}^d} \frac{v(\boldsymbol{\xi})^q}{\tau(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} \le 2\int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{-1}\langle\boldsymbol{\xi}(rT)^{-\frac{1}{2m}}\rangle^{-\beta q}\, \mathrm{d}\boldsymbol{\xi} \lesssim (rT)^{\frac{d-1}{2m}} .$$

Thus by (28),

$$\|[(\mathbb{1} - J)A, P]\|_q^q = \|[\mathrm{Op}_\alpha(a), \chi_\Lambda]\|_q^q \lesssim \alpha^{d-1}(rT)^{\frac{d-1}{2m}} .$$

The bound for the commutator $[(\mathbb{1} - J)B, P]$ is proved in the same way. Substituting the above bounds into the statement (41) of Corollary 4.3, we get the claimed estimate (52). ◻

A useful consequence of this fact is the following continuity statement:

Corollary 5.4 *With the function w_r defined in* (51) *and the symbol p_∞ defined in* (47) *we have*

$$\lim_{r\to 0} \mathcal{B}(w_r p_\infty; f) = \mathcal{B}(p_\infty; f)\,. \tag{53}$$

Proof We apply Lemma 5.3 with $h = h_\infty$, $T = 1$, and the constant functions $\omega \equiv \omega_\infty$ and $\phi \equiv \phi_\infty$. Then, for $\alpha r \gtrsim 1$,

$$\|D_\alpha(p_\infty; f) - D_\alpha(w_r p_\infty; f)\|_1 \lesssim \alpha^{d-1} r^{\frac{d-1}{2m}}\,.$$

Therefore,

$$\left|\alpha^{1-d} \operatorname{tr} D_\alpha(p_\infty; f) - \alpha^{1-d} \operatorname{tr} D_\alpha(w_r p_\infty; f)\right| \le r^{\frac{d-1}{2m}}\,.$$

Now, we use Proposition 2.6 to obtain the estimate

$$\left|\mathcal{B}(p_\infty; f) - \mathcal{B}(w_r p_\infty; f)\right| \lesssim r^{\frac{d-1}{2m}}\,.$$

This leads to (53), as claimed. □

We already have a result on the continuity of the asymptotic coefficient, see Proposition 2.5. However, this proposition is not applicable to the truncated symbol $w_r\, p_T$, since its derivatives are not bounded uniformly in $r > 0$. This explains why we need Corollary 5.4.

The next lemma provides the same asymptotics as in Theorems 5.1 and 5.2, but this time for $w_{rT}\, p_T$ instead of p_T. We recall that $\lambda_T > 0$ obeys $\lambda_T \to 0$ as $T \to \infty$.

Lemma 5.5 *Assume that $r \in (0, 1]$ is fixed and that $\alpha T^{\frac{1}{2m}} \to \infty$, $T \to \infty$. Then*

$$\lim \left(\alpha T^{\frac{1}{2m}}\right)^{1-d} \operatorname{tr} D_\alpha(w_{rT}\, p_T; f) = \mathcal{B}(w_r\, p_\infty; f)\,. \tag{54}$$

If f satisfies (21)*, then*

$$\lim \left(\alpha T^{\frac{1}{2m}}\right)^{1-d} \lambda_T^{-\delta} \operatorname{tr} D_\alpha(\lambda_T w_{rT}\, p_T; f) = \mathcal{B}(w_r\, p_\infty; f_0)\,. \tag{55}$$

If f satisfies (23)*, then*

$$\lim \left(\alpha T^{\frac{1}{2m}}\right)^{1-d} \lambda_T^{-1} \operatorname{tr} D_\alpha(\lambda_T w_{rT}\, p_T; f) = \mathcal{B}(w_r\, p_\infty; \eta)\,. \tag{56}$$

Proof By a straightforward change of variables in definition (1), we obtain

$$\operatorname{Op}_\alpha(w_{rT}\, p_T) = \operatorname{Op}_L(b_T) \quad \text{and} \quad D_\alpha(w_{rT}\, p_T; f) = D_L(b_T; f)\,,$$

where

$$L := \alpha T^{\frac{1}{2m}} \quad \text{and} \quad b_T(\boldsymbol{\xi}) := w_r(\boldsymbol{\xi})\, p_T(T^{\frac{1}{2m}}\boldsymbol{\xi})\,, \quad \boldsymbol{\xi} \in \mathbb{R}^d\,.$$

Thanks to the condition (43), for all $\boldsymbol{\xi} \neq 0$, we have as $T \to \infty$:

$$T^{-1}h(T^{\frac{1}{2m}}\boldsymbol{\xi}) \to h_\infty(\boldsymbol{\xi}) \quad \text{and hence, by (45),} \quad b_T(\boldsymbol{\xi}) \to w_r(\boldsymbol{\xi})\, p_\infty(\boldsymbol{\xi})\,.$$

Assuming that $T \gtrsim 1$, an elementary calculation using (42) leads to the bounds

$$\left|\partial_{\boldsymbol{\xi}}^n b_T(\boldsymbol{\xi})\right| + \left|\partial_{\boldsymbol{\xi}}^n \big(w_r(\boldsymbol{\xi})\, p_\infty(\boldsymbol{\xi})\big)\right| \lesssim \mathrm{e}^{-\nu|\boldsymbol{\xi}|^{2m}}\,, \; n \in \mathbb{N}_0^d\,, \; \boldsymbol{\xi} \in \mathbb{R}^d\,, \tag{57}$$

with $\nu > 0$ from (44) and implicit constants depending on the number $r \in (0, 1]$. Since b_T satisfies (57) uniformly in $T \gtrsim 1$, we obtain by Corollary 2.7 that

$$\lim_{L\to\infty, T\to\infty} L^{1-d} D_L(b_T;\, f) = \mathcal{B}(w_r p_\infty;\, f)\,.$$

By the above change of variables, this leads to (54). Formulas (55) and (56) follow along the same lines from Propositions 2.8 and 2.9. □

Proof (Of Theorem 5.1) We begin by estimating as follows:

$$\begin{aligned}
&\left|\alpha^{-(d-1)} T^{-\frac{d-1}{2m}} \operatorname{tr} D_\alpha(p_T;\, f) - \mathcal{B}(p_\infty;\, f)\right| \\
&\quad \le \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \| D_\alpha(p_T;\, f) - D_\alpha(w_{rT}\, p_T;\, f)\|_1 \\
&\quad\quad + \left|\big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \operatorname{tr} D_\alpha(w_{rT}\, p_T;\, f) - \mathcal{B}(w_r\, p_\infty;\, f)\right| \\
&\quad\quad + |\mathcal{B}(w_r\, p_\infty;\, f) - \mathcal{B}(p_\infty;\, f)|\,.
\end{aligned}$$

By (52) and (54) we then obtain

$$\begin{aligned}
\limsup &\left|\alpha^{-(d-1)} T^{-\frac{d-1}{2m}} \operatorname{tr} D_\alpha(p_T;\, f) - \mathcal{B}(p_\infty;\, f)\right| \\
&\qquad \le r^{\frac{d-1}{2m}} + |\mathcal{B}(w_r\, p_\infty;\, f) - \mathcal{B}(p_\infty;\, f)|\,,
\end{aligned}$$

where the upper limit is taken as $\alpha T^{\frac{1}{2m}} \to \infty$, $T \to \infty$. Taking $r \to 0$ and using (53) we arrive at (48). □

Proof (Of Theorem 5.2) We recall that the only singular point of the function f is $t = 0$. We assume that f satisfies (21), so that for all $t \neq 0$,

$$|f^{(k)}(t)| \lesssim |t|^{\delta-k}, \quad k = 0, 1, 2\,. \tag{58}$$

Consequently, the function $\tilde{f}_T(t) := \lambda_T^{-\delta} f(\lambda_T t)$, $t \in \mathbb{R}$, satisfies the same inequalities with the same constants. Now we can apply the argument in the previous proof to the operator

$$D_\alpha(p_T; \tilde{f}_T) = \lambda_T^{-\delta} D_\alpha(\lambda_T p_T; f).$$

More precisely, we estimate as follows

$$\begin{aligned}
&\left|\alpha^{-(d-1)} T^{-\frac{d-1}{2m}} \lambda_T^{-\delta} \operatorname{tr} D_\alpha(\lambda_T p_T; f) - \mathcal{B}(p_\infty; f_0)\right| \\
&\quad \le \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \big\| D_\alpha(p_T; \tilde{f}_T) - D_\alpha(w_{rT} p_T; \tilde{f}_T)\big\|_1 \\
&\qquad + \left|\big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \lambda_T^{-\delta} \operatorname{tr} D_\alpha(\lambda_T w_{rT} p_T; f) - \mathcal{B}(w_r p_\infty; f_0)\right| \\
&\qquad + \left|\mathcal{B}(w_r p_\infty; f_0) - \mathcal{B}(p_\infty; f_0)\right|.
\end{aligned} \tag{59}$$

By (52) and (55) we then obtain

$$\begin{aligned}
\limsup &\left|\alpha^{-(d-1)} T^{-\frac{d-1}{2m}} \lambda_T^{-\delta} \operatorname{tr} D_\alpha(\lambda_T p_T; f) - \mathcal{B}(p_\infty; f_0)\right| \\
&\qquad \le r^{\frac{d-1}{2m}} + \left|\mathcal{B}(w_r p_\infty; f_0) - \mathcal{B}(p_\infty; f_0)\right|,
\end{aligned}$$

where lim sup is taken as $\alpha T^{\frac{1}{2m}} \to \infty$, $T \to \infty$. Taking $r \to 0$ and using (53) we obtain (49).

Now we assume that f satisfies (23). We use (59) with $\delta = 1$ and f_0 replaced by η. Then

$$\begin{aligned}
&\left|\alpha^{-(d-1)} T^{-\frac{d-1}{2m}} \lambda_T^{-1} \operatorname{tr} D_\alpha(\lambda_T p_T; f) - \mathcal{B}(p_\infty; \eta)\right| \\
&\quad \le \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \big\| D_\alpha(p_T; \tilde{f}_T) - D_\alpha(w_{rT} p_T; \tilde{f}_T)\big\|_1 \\
&\qquad + \left|\big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \lambda_T^{-1} \operatorname{tr} D_\alpha(\lambda_T w_{rT} p_T; f) - \mathcal{B}(w_r p_\infty; \eta)\right| \\
&\qquad + \left|\mathcal{B}(w_r p_\infty; \eta) - \mathcal{B}(p_\infty; \eta)\right|.
\end{aligned}$$

As before, the last term on the right-hand side tends to zero due to (53). The second term vanishes as $\alpha T^{\frac{1}{2m}} \to \infty$, $T \to \infty$ due to (56). To estimate the first term we represent $f = \eta + g$, so that g satisfies (58) with $\delta = 1$. Therefore,

$$\begin{aligned}
D_\alpha(p_T; \tilde{f}_T) - D_\alpha(w_{rT} p_T; \tilde{f}_T) &= \big[D_\alpha(p_T; \tilde{\eta}_T) - D_\alpha(w_{rT} p_T; \tilde{\eta}_T)\big] \\
&\quad + \big[D_\alpha(p_T; \tilde{g}_T) - D_\alpha(w_{rT} p_T; \tilde{g}_T)\big],
\end{aligned}$$

where $\tilde{\eta}_T(t) := \lambda_T^{-1} \eta(\lambda_T t)$ and $\tilde{g}_T(t) := \lambda_T^{-1} g(\lambda_T t)$. As in the previous calculation, the second term is estimated with the help of (52) by $r^{\frac{d-1}{2m}}$. Since

$\tilde{\eta}_T(t) = \eta(t) - t\ln(\lambda_T)$ and the operator difference (2) vanishes for linear functions f, we have

$$D_\alpha(p_T; \tilde{\eta}_T) - D_\alpha(w_{rT}\, p_T; \tilde{\eta}_T) = D_\alpha(p_T; \eta) - D_\alpha(w_{rT}\, p_T; \eta)\,.$$

The function η from (22) satisfies (58) for all $\delta < 1$, and hence by (52) the above difference is again estimated by $r^{\frac{d-1}{2m}}$. This entails (50) and the proof of Theorem 5.2 is complete. □

6 Main Results on the High-Temperature Asymptotics

In this section we adapt Theorems 5.1 and 5.2 to two different asymptotic regimes of the entanglement entropy (8), when the temperature becomes large. This is straightforward for the (first) regime of a fixed chemical potential μ, since we work from the outset within the grand-canonical formalism [3, 7] for an indefinite number of particles. For the (second) regime of a fixed particle density ρ it is slightly more involved, but physically often more interesting. Both results will be discussed in some detail in Sect. 6.3.

6.1 Case of a Fixed Chemical Potential μ

Since the Fermi symbol $a_{T,\mu}$ of (4) equals p_T of (46) with $\phi_T = 1$ and $\omega_T = \exp(-\mu/T)$, the limit symbol in this case is obviously $p_\infty = (1 + \mathrm{e}^{h_\infty})^{-1}$. For the function f we take η_γ which satisfies Condition 2.2 with $\mathcal{T} = \{0, 1\}$ and an arbitrary $\delta < \min\{1, \gamma\}$. Thus, by combining definition (8), Remark 2.4(1), and Theorem 5.1 we obtain:

Theorem 6.1 *Let the truncating region Λ satisfy Condition 2.1. Then we have,*

$$\lim \big(\alpha T^{\frac{1}{2m}}\big)^{1-d}\, \mathrm{H}_\gamma(T, \mu; \alpha\Lambda) = 2\,\mathcal{B}\big((1 + \mathrm{e}^{h_\infty})^{-1}, \partial\Lambda; \eta_\gamma\big) \tag{60}$$

for any fixed $\mu \in \mathbb{R}$, as $\alpha T^{\frac{1}{2m}} \to \infty$ and $T \to \infty$.

6.2 Case of a Fixed Particle Density ρ

In this case we have to find a function $T \mapsto \mu_\rho(T)$ satisfying (12) for a fixed constant $\rho > 0$. According to the Appendix we have for any such function

$$\exp\big(-\mu_\rho(T)/T\big) = \lambda_T^{-1}\big(1 + o(1)\big) \quad \text{as} \quad T \to \infty\,, \tag{61}$$

with the function $T \mapsto \lambda_T$ given explicitly by

$$\lambda_T := \rho T^{-\frac{d}{2m}}/\varkappa \quad \text{where} \quad \varkappa := (2\pi)^{-d}\int_{\mathbb{R}^d} \exp(-h_\infty(\boldsymbol{\xi}))\,\mathrm{d}\boldsymbol{\xi}\,. \tag{62}$$

This implies for the Fermi symbol at fixed ρ the formula

$$a_{T,\mu_\rho(T)} = \lambda_T\, p_T\,, \tag{63}$$

where p_T is given by (46) with

$$\phi_T = \lambda_T \quad \text{and} \quad \omega_T = \lambda_T \exp\big(-\mu_\rho(T)/T\big)\,. \tag{64}$$

By (61) and (62) we clearly have $\phi_T \to 0$ and $\omega_T \to 1$ as $T \to \infty$. Hence the limit symbol is the classical "Boltzmann factor" corresponding to h_∞:

$$p_\infty(\boldsymbol{\xi}) = \mathrm{e}^{-h_\infty(\boldsymbol{\xi})}\,, \quad \boldsymbol{\xi} \in \mathbb{R}^d\,. \tag{65}$$

To study the symbol $a_{T,\mu_\rho(T)}$ we use Theorem 5.2 with $f = \eta_\gamma$. For a start we need to understand the behavior of $\eta_\gamma(t)$ for small t. Since this behavior depends on γ, we need to distinguish five different regimes for its value. To state the result in a unified way for all values, we define a parameter δ_γ and a pair of functions $\{f_\gamma, \eta_\gamma^{\text{eff}}\}$ on the interval [0,1] in Table 1.

The next lemma shows that η_γ^{eff} describes the effective asymptotic contribution of f_γ as $t \downarrow 0$.

Table 1 The parameter δ_γ and the functions $\{f_\gamma, \eta_\gamma^{\text{eff}}\}$ for different values of the Rényi index γ

	δ_γ	$f_\gamma(t)$	$\eta_\gamma^{\text{eff}}(t)$
$0 < \gamma < 1$	γ	$\eta_\gamma(t)$	$\frac{1}{1-\gamma}t^\gamma$
$\gamma = 1$	1	$\eta_1(t) - t$	$-t\ln(t)$
$1 < \gamma < 2$	γ	$\eta_\gamma(t) - \frac{\gamma}{\gamma-1}t$	$\frac{1}{1-\gamma}t^\gamma$
$\gamma = 2$	3	$\eta_2(t) - 2t$	$-\frac{4}{3}t^3$
$\gamma > 2$	2	$\eta_\gamma(t) - \frac{\gamma}{\gamma-1}t$	$\frac{\gamma}{2(\gamma-1)}t^2$

Lemma 6.2 *Let the parameter* δ_γ *and the functions* f_γ, η_γ^{eff} *on the interval* $(0, 1)$ *be as defined in Table 1. Then the following three asymptotic relations hold:*

$$\lim_{t\to 0} t^{k-\delta_\gamma} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(f_\gamma(t) - \eta_\gamma^{\text{eff}}(t)\big) = 0\,,\ k = 0, 1, 2\,. \tag{66}$$

Proof For $\gamma \neq 1$ we expand η_γ at $t = 0$ to obtain

$$(1-\gamma)\eta_\gamma(t) = t^\gamma - \gamma t - \frac{\gamma}{2}t^2 + O(t^3) + O(t^{2\gamma}) + O(t^{1+\gamma})\,. \tag{67}$$

The notation $g(t) = O(t^b)$ means that $|g^{(k)}(t)| \lesssim t^{b-k}$, for all $k \in \mathbb{N}_0$. This expansion leads to the claim (66) for all $\gamma \notin \{1, 2\}$. For $\gamma = 2$ the expansion (67) is insufficient since the terms with t^2 cancel out. By including the third-order term explicitly we find $\eta_2(t) = 2t - \frac{4}{3}t^3 + O(t^4)$ and obtain (66) for $\gamma = 2$. Finally, for $\gamma = 1$ we have $\eta_1(t) = -t\ln(|t|) + t + O(t^2)$ which again leads to (66). □

Now we are in a position to state and prove our second main result for the scaling of the entanglement entropy.

Theorem 6.3 *Let the truncating region* Λ *satisfy Condition 2.1. For a fixed number* $\rho > 0$ *let* $T \mapsto \mu_\rho(T)$ *be a function satisfying* (12) *and* $\varkappa$ *be as defined in* (62)*. Finally, let* δ_γ *and* η_γ^{eff} *be as defined in Table 1. Then we have the asymptotic relation*

$$\lim \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \big(\varkappa\, T^{\frac{d}{2m}}/\rho\big)^{\delta_\gamma}\, \mathrm{H}_\gamma(T, \mu_\rho(T); \alpha\Lambda) = 2\,\mathcal{B}\big(\mathrm{e}^{-h_\infty}, \partial\Lambda; \eta_\gamma^{\text{eff}}\big)\,, \tag{68}$$

as $\alpha T^{\frac{1}{2m}} \to \infty$ and $T \to \infty$.

Proof We replace the symbol $a_{T,\mu_\rho(T)}$ with $\lambda_T\, p_T$ as specified in (64). Furthermore, since the operator difference (2) vanishes for linear functions f, we have

$$D_\alpha(\lambda_T\, p_T, \Lambda; \eta_\gamma) = D_\alpha(\lambda_T\, p_T, \Lambda; f_\gamma)\,.$$

As the function f_γ satisfies the condition (66), we can use Theorem 5.2 which gives

$$\lim \big(\alpha T^{\frac{1}{2m}}\big)^{1-d} \lambda_T^{-\delta_\gamma}\, \mathrm{tr}\, D_\alpha(\lambda_T\, p_T, \Lambda; f_\gamma) = \mathcal{B}(p_\infty, \partial\Lambda; \eta_\gamma^{\text{eff}})\,,$$

as $\alpha T^{\frac{1}{2m}} \to \infty$ and $T \to \infty$, where p_∞ is given by (65). The same formula holds for the region $\mathbb{R}^d \setminus \overline{\Lambda}$. The claimed formula (68) now follows from definition (8), Remark 2.4(1), (62), and (63).

□

6.3 Concluding Remarks

1. We have proved the positivity of the EE for Rényi index $\gamma \le 1$, see Theorem 3.5. For bounded Λ we have actually a somewhat stronger statement, namely:

$$0 \le \mathrm{S}_\gamma(T, \mu; \Lambda) - s_\gamma(T, \mu)|\Lambda| \le \mathrm{H}_\gamma(T, \mu; \Lambda), \quad \gamma \le 1 .$$

Here the first inequality for the local entropy is due to Theorem 31 and holds even for $\gamma \le 2$. The second one follows by combining (8), (9), and Theorem 3.5. Although Theorem 3.5 ensures the positivity of the EE only for $\gamma \le 1$, the asymptotic coefficient $\mathcal{B}(a_{T,\mu}, \partial\Lambda; \eta_\gamma)$ is positive for all $\gamma \le 2$ and all $\mu \in \mathbb{R}$, $T > 0$, see Remark 2.4(2). It is an open question however whether the positivity of the EE extends to all $\gamma \le 2$.

2. In case of a fixed ρ (Theorem 6.3) and $\gamma \le 2$, the function $\eta_\gamma^{\mathrm{eff}}$ in Table 1 is strictly concave on [0, 1], and since $h_\infty > 0$, the coefficient $\mathcal{B}\big(\mathrm{e}^{-h_\infty}, \partial\Lambda; \eta_\gamma^{\mathrm{eff}}\big)$ is strictly positive. On the other hand, if $\gamma > 2$, then $\eta_\gamma^{\mathrm{eff}}$ is strictly convex so that this coefficient is strictly *negative*. This change of sign, unexpected by us, suggests that our definition (8) of the EE is somewhat unphysical for $\gamma > 2$. In passing we note that $\mathcal{B}\big(\mathrm{e}^{-h_\infty}, \partial\Lambda; \eta_\gamma^{\mathrm{eff}}\big)$ for $\gamma > 2$ can be computed rather explicitly. Indeed, since $\eta_\gamma^{\mathrm{eff}}(t) = \gamma(2(\gamma-1))^{-1}\, t^2$ it is easy to calculate the function (15):

$$U\big(u, v; \eta_\gamma^{\mathrm{eff}}\big) = -\frac{\gamma}{2(\gamma-1)}(u-v)^2, \quad \gamma > 2 .$$

This observation and the use of Parseval's identity, as in the proof of Proposition 1 in [29], enables us to find the coefficient (16) and hence (17) in terms of the Fourier transform of $\exp(-h_\infty)$. In particular, for the quadratic "asymptotic" Hamiltonian $h_\infty(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2/2$ (which includes the Hamiltonian of the ideal Fermi gas) we obtain the formula

$$\mathcal{B}\big(\mathrm{e}^{-h_\infty}, \partial\Lambda; \eta_\gamma^{\mathrm{eff}}\big) = -\frac{\gamma}{\gamma-1}2^{-d-3}\pi^{-(d+1)/2}|\partial\Lambda|, \quad \gamma > 2 .$$

3. For $\gamma < 2$ the function $\eta_\gamma^{\mathrm{eff}}$ is exactly the classical entropy function of the Maxwell–Boltzmann gas. This confirms our expectations stated in the Introduction. This conclusion does not hold for $\gamma \ge 2$.
4. In case of a fixed μ (Theorem 6.1), and $\gamma \le 2$, we know that the coefficient $\mathcal{B}_\gamma := \mathcal{B}(p_\infty, \partial\Lambda; \eta_\gamma)$, $p_\infty = (1 + \exp(h_\infty))^{-1}$, on right-hand side of (60) is positive. Here we indicate that $\mathcal{B}_\gamma$ should become negative for large γ. Indeed, we calculate the point-wise limit:

$$\eta_\infty(t) := \lim_{\gamma\to\infty} \eta_\gamma(t) = \min\big\{-\ln(1-t), -\ln(t)\big\}, \quad t \in [0, 1] .$$

This function is (strictly) convex on [0, 1/2] and also on [0, 1/2], but not on the whole interval [0, 1]. Since by our assumptions in Sect. 5.1 $h_\infty \geq 0$, we have $p_\infty \leq 1/2$, and hence

$$\mathcal{B}_\infty = \mathcal{B}(p_\infty, \partial\Lambda; -\ln(1-\cdot)) < 0\,.$$

Thus, assuming continuity of $\mathcal{B}_\gamma$ as a function of $\gamma > 0$ we can claim that there is a value $\gamma_0 > 2$ such that $\mathcal{B}_{\gamma_0} = 0$ and $\mathcal{B}_\gamma < 0$ for all finite $\gamma > \gamma_0$.

One can be more specific about the value of $\mathcal{B}_\infty$. Namely, for $u, v \in [0, 1/2]$ we have

$$\begin{aligned} U\big(u, v; \eta_\infty\big) &= -U\big(u, v; \ln(1-\cdot)\big) = -U\big(1-u, 1-v; \ln(\cdot)\big) \\ &= -\frac{1}{2}\big(\ln(1-u) - \ln(1-v)\big)^2\,. \end{aligned}$$

The last step is an elementary calculation (see [29]) that reconfirms the negativity of U. With $b(\boldsymbol{\xi}) := \ln\big(1+\exp(-h_\infty(\boldsymbol{\xi}))\big)$ we get

$$U\big(p_\infty(\boldsymbol{\xi}), p_\infty(\boldsymbol{\xi}+t\mathbf{e}); \eta_\infty\big) = -\frac{1}{2}\big(b(\boldsymbol{\xi}) - b(\boldsymbol{\xi}+t\mathbf{e})\big)^2\,.$$

Similarly to Remark 2, the coefficient $\mathcal{B}_\infty$ can be found in terms of the Fourier transform of the symbol b. In particular, in the case $h_\infty(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2/2$ the coefficient $\mathcal{B}_\infty$ can be computed explicitly:

$$\mathcal{B}((1+\exp(h_\infty))^{-1}, \partial\Lambda; \eta_\infty) = -\frac{1}{2}(2\pi)^{-(d+1)/2}\Sigma(d)|\partial\Lambda|\,,$$

where $\Sigma(d) := \sum_{n,m\geq 1}(-1)^{n+m}(nm)^{-1/2}(n+m)^{-(1+d)/2}$ [Numerically we find e.g. $\Sigma(2) \approx 0.19798$ and $\Sigma(3) \approx 0.15419$]. We omit the details.

5. In both high-temperature regimes (fixed μ or fixed ρ) the scaling parameter α may be fixed to, say, $\alpha = 1$. Thus the truncating set $\alpha\Lambda$ is fixed and only the temperature T tends to infinity.
6. Using the relation (9) and Theorem 5.1 one can also derive appropriate asymptotic formulas for the local entropy. For example, assuming that μ is fixed, as in Theorem 6.1, we easily obtain the asymptotic relation

$$\lim\big(\alpha T^{\frac{1}{2m}}\big)^{1-d}\big[\mathrm{S}_\gamma(T,\mu;\alpha\Lambda) - \alpha^d s_\gamma(T,\mu)\,|\Lambda|\big] = \mathcal{B}\big((1+\mathrm{e}^{h_\infty})^{-1}, \partial\Lambda; \eta_\gamma\big)\,,$$

as $\alpha T^{\frac{1}{2m}} \to \infty$ and $T \to \infty$. However, in order to obtain from this formula a proper asymptotic expansion for the local entropy, one needs to find an expansion for the entropy density $s_\gamma(T,\mu)$ as $T \to \infty$. A convenient starting point for its derivation could be, for example, the representation (10.8) in [18].

An analogous formula can be written down for the case of a fixed ρ. The inequality $\mathcal{B}\big(\mathrm{e}^{-h_\infty}, \partial\Lambda; \eta_\gamma^{\mathrm{eff}}\big) < 0$ for $\gamma > 2$ would then imply that the local entropy obeys for large T the bound $\mathrm{S}_\gamma(T, \mu_\rho(T); \Lambda) < s_\gamma(T, \mu_\rho(T))|\Lambda|$, instead of (31) which is valid for all $\gamma \le 2$ and all $T > 0$.

Appendix: The Fermi Symbol for Fixed ρ and Large T

Our main aim is to derive formula (61) for a Hamiltonian h as specified in Sect. 5.1. After a change of variables and replacing μ with $\mu_\rho(T)$ formula (11) for the mean particle density takes the form

$$\varrho\big(T, \mu_\rho(T)\big) = \frac{T^{\frac{d}{2m}}}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{\xi}}{1 + \exp\big(-\mu_\rho(T)/T\big) \exp\big(h(\boldsymbol{\xi} T^{\frac{1}{2m}})/T\big)} .$$

Since $T^{-1}h(T^{\frac{1}{2m}}\boldsymbol{\xi}) \to h_\infty(\boldsymbol{\xi})$ as $T \to \infty$, for each $\boldsymbol{\xi} \ne 0$, the condition (12) requires that $\exp\big(-\mu_\rho(T)/T\big) \to \infty$. Consequently,

$$\varrho\big(T, \mu_\rho(T)\big) = \varkappa T^{\frac{d}{2m}} \exp\big(\mu_\rho(T)/T\big)\big(1 + o(1)\big),$$

where $\varkappa$ is defined in (62). Using (12), this leads to (61).

As explained in the Introduction, the high-temperature limit under the condition (12) corresponds to the Maxwell–Boltzmann gas limit. This fact can be conveniently restated in terms of the so-called fugacity $z_\rho(T) := \exp\big(\mu_\rho(T)/T\big)$ as follows. By (43) the integrated density of states $\mathcal{N}(T)$ of (13), satisfies for large T the relation $\mathcal{N}(T) \asymp T^{\frac{d}{2m}}$. So (61) implies $z_\rho(T) \asymp \rho\,\mathcal{N}(T)^{-1} \to 0$.

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Spectral Asymptotics for Toeplitz Matrices Having Certain Piecewise Continuous Symbols

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Dedicated to the memory of Harold Widom, 1932–2021

Abstract The limiting behavior of the eigenvalues of the Toeplitz matrices $T_n[\sigma] = (\hat{\sigma}(i-j))$, where $0 \le i, j \le n$, as $n \to \infty$, is investigated in the case of complex valued functions σ defined on the unit circle $\mathbb{T}$ and having exactly one point of discontinuity. It is found that if $\sigma(z) = (-z)^{\beta}\tau(z)$, β not an integer and τ satisfying certain smoothness conditions, then $\det T_n[\sigma] = \mathbf{G}[\tau]^{n+1} n^{-\beta^2} E[\tau, \beta](1 + o(1))$ as $n \to \infty$, where $\mathbf{G}[\tau]$ denotes the geometric mean of τ and E is a constant independent of n. A value for E is found in terms of the Fourier coefficients of τ and an analytic function of β. These results were known previously in the case that $\Re\beta$, the real part of β, was sufficiently small. A corollary of this result is a determination of the limiting set and limiting distributions for the eigenvalues of $T_n[\sigma]$.

Keywords Toeplitz operator · Toeplitz determinant · Spectral asymptotics

1 Introduction

A classical result of Szegő describes the limiting behavior of the eigenvalues for the Toeplitz matrices $T_n[\sigma] = (\hat{\sigma}(i-j))$, $0 \le i, j \le n$, for bounded, measurable, real valued functions σ defined on the unit circle $\mathbb{T}$ as $n \to \infty$. Here $\hat{\sigma}(k)$ denotes the kth Fourier coefficient of σ. Let m denote Lebesgue measure on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$. Define μ_σ as the measure given by $\mu_\sigma(A) = m(\sigma^{-1}(A))$ for measurable sets A. Let $\mu_{n,\sigma}$ denote the discrete measure assigning to each point λ in the spectrum of $T_n[\sigma]$ measure $\frac{1}{n+1}$ times the multiplicity of λ. Szegő's well known result is that the measure $\mu_{n,\sigma}$ tends weakly to μ_σ as $n \to \infty$; i.e., for any

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continuous function F,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} F(\lambda_{i,n}) = \frac{1}{2\pi} \int_0^{2\pi} F(\sigma(e^{i\theta}))\, d\theta \tag{1}$$

where $\lambda_{0,n}, \ldots, \lambda_{n,n}$ are the eigenvalues of $T_n[\sigma]$, counted according to multiplicity. For general $\sigma : \mathbb{T} \to \mathbb{C}$ such that (1) holds, the eigenvalues of $T_n[\sigma]$ are said to be *canonically distributed* (see [35]).

For the moment, consider also the *limiting set* of the eigenvalues of $T_n[\sigma]$, namely the set of limit points of sequences having the form $\{\lambda_{i_j,n_j} : j = 1, 2, 3, \ldots, j < k \Rightarrow n_j < n_k\}$. If σ is continuous, bounded, and real valued, then from an application of the *finite section method*, as developed by Böttcher and Silbermann [10], and a theorem of Hartman and Wintner [20, pp. 179–183], one finds that the limiting set of the eigenvalues is equal to the range of σ.

These results need not hold for σ complex-valued. A trivial example is provided by the function $\sigma(z) = z$, for which the finite Toeplitz matrices are strictly lower triangular. The measures $\mu_{n,\sigma}$ equal the Dirac measure at $\{0\}$ and clearly do not converge in any meaningful way to μ_σ, which in this case is just our normalized Lebesgue measure m. Canonical distribution has been shown to fail in general for Laurent polynomials [28] and for rational functions with poles off $\mathbb{T}$ [15]. The limiting sets, too, behave differently from the real valued case. Canonical distribution is known to hold for certain classes of symbols σ that have (among other features) the property that σ cannot be extended analytically to any open annulus either containing $\mathbb{T}$ or having $\mathbb{T}$ as a component of its boundary. It is an outstanding conjecture of Widom that this last condition is sufficient for canonical distribution to hold (see [35]).

In order to obtain information concerning the eigenvalue distributions of these matrices, one begins with the asymptotic nature of their determinants $D_n[\sigma] = \det T_n[\sigma]$. Note that if σ is positive and bounded away from 0 and if (1) holds, then from the case $F(x) = \log(x)$ one obtains

$$\lim_{n\to\infty} \frac{1}{n+1} \log D_n[\sigma] = \log \mathbf{G}[\sigma], \tag{2}$$

where as before, $\mathbf{G}[\sigma]$ denotes the geometric mean of σ, namely

$$\mathbf{G}[\sigma] = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \sigma(e^{i\theta})\, d\theta \right).$$

Under certain conditions on σ, a technique that goes back to at least Grenander and Szegő [26] and has been used by Widom as well, allows one to obtain (1) for general F from the special case (2). Much research beginning with Szegő's original work has been devoted to proving results similar to (2) for other classes of functions, and the theory of Toeplitz determinants has also been extended to cases where σ itself is matrix valued. The oldest of these results considers the cases where σ is sufficiently smooth, real valued, and has no zeroes. In 1952 Szegő [30] showed that if σ' satisfies an appropriate Lipschitz condition, then

$$D_n[\sigma] = \mathbf{G}[\sigma]^{n+1} E[\sigma]\,(1 + o(1))\,,$$

where

$$E[\sigma] = \exp\left(\sum_{k=1}^{\infty} k \cdot \widehat{\log\sigma}(k)\,\widehat{\log\sigma}(-k)\right).$$

This result does not hold if σ has zeroes or discontinuities, the case under present consideration. Relevant in this case is the (now largely proven) conjecture of Fisher and Hartwig [23], who considered functions with a finite number of zeroes and discontinuities. These functions can be written as

$$\sigma(e^{i\theta}) = \tau(e^{i\theta}) \prod_{r=1}^{R} (2 - 2\cos(\theta - \theta_r))^{\alpha_r} \exp\left(i\beta_r \arg(-e^{i(\theta-\theta_r)})\right)$$

where β_r is not an integer for any r and τ is a sufficiently smooth non-vanishing function with winding number zero. The argument in the last term is chosen in $(-\pi, \pi]$. Note that this term is a function with a jump discontinuity at $\theta = \theta_r$. By considering special cases where $D_n[\sigma]$ is explicitly calculable, they conjectured that

$$D_n[\sigma] = \mathbf{G}[\tau]^{n+1} n^{\sum(\alpha_r^2 - \beta_r^2)} E[\tau, \alpha_1, \ldots, \alpha_R, \beta_1, \ldots, \beta_R]\,(1 + o(1))\,,$$

where E does not depend on n. Early research of this conjecture in the 1970s and 1980s includes the work of Widom [34], Basor [3, 4], Böttcher and Silbermann [10–12], and others, providing verification of the conjecture in several cases. In 1973 Widom proved the conjecture in the case that $\beta_r = 0$ and $\Re\alpha_r > -1/2$ for all r, τ is continuously differentiable and of winding number zero, and τ' satisfies a Lipschitz condition with positive exponent. A value for E was also obtained. He also found a proof in the case $R = 1$, $|\Re\alpha| < \frac{1}{2}$, $|\Re\beta| < \frac{1}{2}$, without specifying the value for E. In 1979 Basor verified the conjecture in the case $\alpha_r = 0$ for all r, $|\Re\beta| < \frac{1}{2}$ for all r, obtaining an expression for E as well. In the 1980s Böttcher and Silbermann verified the conjecture for several other cases, for example, when $|\Re(\alpha_r)| < 1/2$ and $|\Re(\beta_r)| < 1/2$ for all r. In these cases E was expressed as $E[\tau]$ per Szegő's definition, multiplied by an explicit analytic function of the α_r's and β_r's.

Building on these lines of development the present work examines the conjecture in the case $R = 1$ and $\alpha_1 = 0$. It can be assumed without loss of generality that $\theta_1 = 0$. The significance of this case lies in the fact that the winding number of σ will not be assumed to be bounded. The main result obtained is that if β is not an integer and

$$\sigma(e^{i\theta}) = \tau(e^{i\theta})e^{i\beta(\theta-\pi)}, \qquad 0 < \theta < 2\pi, \tag{3}$$

where τ is continuous, non-vanishing of winding number zero, and satisfies a certain smoothness condition, then

$$D_n[\sigma] = \mathbf{G}[\tau]^{n+1} n^{-\beta^2} E[\tau, \beta]\,(1 + o(1))\,. \tag{4}$$

Here $\mathbf{G}[\tau]$ again denotes the geometric mean, and $E[\tau, \beta]$ is defined below in (7). The smoothness conditions requires that τ is C^∞ away from $\theta = 0$ and has one-sided derivatives of any order at $\theta = 0$. From this result one finds that the eigenvalues of the matrices $T_n[\sigma]$ are canonically distributed as $n \to \infty$ and that the limiting set of the eigenvalues is the closure of the range of σ.

The main idea behind the proof is as follows. We start with Widom's result for the case $|\Re\beta| < \frac{1}{2}$ in which he constructs a pair of operator equations via Wiener-Hopf factorization, which allows us to describe the asymptotic behavior of certain elements of the inverse matrices $T_n[\sigma]^{-1}$ as $n \to \infty$. This information yields the desired asymptotic formula of the main result, by way of Jacobi's generalization of Cramer's rule. This technique allows us to determine the general nature of the asymptotic formula for almost all β. An application of the Poisson-Jensen formula and careful estimates for the behavior of the determinants as β approaches the remaining set of measure zero show that the asymptotic formula holds there as well. The product of the Barnes G-function and $E[\tau]$ in (4) is the same as is found by Basor and by Böttcher and Silbermann, via use of Vitali's convergence theorem, making the extension of Widom's result minus the restriction on β complete. Much of the machinery for this result was developed in the author's Ph.D. thesis [27] to prove the result when $|\Re\beta| < \frac{5}{2}$; the goal of the present work was to remove this last restriction. The author completed this research in advance of, and presented these results at, a conference organized in 1992 in honor of Harold Widom's 60th birthday.

Investigations into the Fisher-Hartwig conjecture have continued from the 1990s to the present. In 1999 Ehrhardt [21] proved the Fisher-Hartwig conjecture in all cases of parameters α_r and β_r for which it is true and under the assumption $\tau \in C^\infty$ (see also [22] for the case of one singularity). Earlier it had been found that the Fisher-Hartwig conjecture cannot be true for certain parameters and a generalized conjecture was formulated by Basor and Tracy [6] (see also [9]). This generalized conjecture was proved by Deift et al. in 2011 [16]. For an overview of this development see the survey [18] and the monograph [13]. The smoothness condition on τ was also discussed in [19, 22]. In these papers it is required that the Fourier coefficients of τ decay power-like, depending on the parameters. This condition is notably different from the condition imposed in this paper, and it seems that the asymptotics of the eigenvalue distribution obtained in this paper cannot be directly inferred in the same manner from the results in [19, 22] (except possibly when $|\Re\beta|$ is small).

Apart from symbols with one jump discontinuity, the asymptotic eigenvalue distribution has also been studied subsequently to some extent in other cases. For various aspects see the papers of Widom [33], Basor and Morrison [5], Böttcher et al. [14], as well as Deift et al. [17].

2 Solutions to Finite Toeplitz Systems

2.1 Preliminaries

For complex valued β we set $z^\beta = \exp(\beta \log|z| + i\beta \arg(z))$ where arg takes its values in the interval $(-\pi, \pi]$. Since we have assumed without loss of generality that $\theta_1 = 0$ in (3) we may write

$$\sigma(z) = (-z)^\beta \tau(z). \tag{5}$$

This function has a jump discontinuity at $z = 1$. The minus sign in this expression simplifies many of the expressions which follow.

Definition 1 Let C_β denote the class of functions $\sigma : \mathbb{T} \to \mathbb{C}$ of the form $\sigma(z) = (-z)^\beta \tau(z)$, where τ satisfies the following conditions:

(i) τ is continuous on $\mathbb{T}$,
(ii) $0 \notin \text{Range}(\tau)$,
(iii) $\Delta_{0\le\theta\le2\pi} \arg\left(\tau(e^{i\theta})\right) = 0$,
(iv) τ is C^∞ away from $\theta = 0$ and the left and right hand limits

$$\lim_{\theta\to0^+} \frac{d^k}{d\theta^k}\tau(e^{i\theta}) \quad \text{and} \quad \lim_{\theta\to2\pi^-} \frac{d^k}{d\theta^k}\tau(e^{i\theta})$$

exist for all $k > 0$.

Let $|\Re\beta| < \frac{1}{2}$ and suppose $\sigma \in C_\beta$. It is known ([34] and [4]) that

$$D_n[\sigma] = \mathbf{G}[\tau]^{n+1} n^{-\beta^2} E[\tau, \beta] \left(1 + o(1)\right), \qquad \text{as } n \to \infty, \tag{6}$$

where $\mathbf{G}[\tau]$ is the geometric mean,

$$E[\tau, \beta] = G(1+\beta)G(1-\beta)E[\tau]\tau_+(1)^\beta \tau_-(1)^{-\beta}, \tag{7}$$

$$E[\tau] = \exp\left(\sum_{k=1}^{\infty} k \cdot \widehat{\log\tau}(k)\,\widehat{\log\tau}(-k)\right),$$

and $G(\cdot)$ denotes the Barnes G-function [2]

$$G(z+1) = (2\pi)^{z/2} e^{-[z^2(\gamma+1)+z]/2} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^n e^{z^2/(2n)-z} \right],$$

γ being Euler's constant. The Barnes G-function is perhaps best understood in terms of its functional equation $G(z+1) = \Gamma(z)G(z)$ and its value $G(1) = 1$. The functions τ_+ and τ_- are the factors of the Wiener-Hopf factorziation of τ defined below in (12).

We make use of the following facts [10, pp. 26–39]. Let **PC** denote the algebra of all bounded, measurable, complex valued functions σ on $\mathbb{T}$ that are continuous except for finitely many points, such that the right and left hand limits of σ exist at these points of discontinuity. For $\sigma \in \mathbf{PC}$ let R_σ denote the continuous curve obtained by adjoining to the range of σ the straight line segments connecting the right and left hand limits of each discontinuity of σ. Let $w(R_\sigma)$ denote the winding number of R_σ about the origin [1, pp. 114–117], provided it exists. Let $\mathbf{H}^2(\mathbb{T}) \subset \mathbf{L}^2(\mathbb{T})$ denote the Hardy space of square integrable functions on $\mathbb{T}$ whose negative Fourier coefficients vanish; define for any $\sigma \in \mathbf{L}^\infty(\mathbb{T})$, the *Toeplitz operator with symbol* σ on $\mathbf{H}^2(\mathbb{T})$, as $T[\sigma] = PM[\sigma]$, where $M[\sigma]$ denotes multiplication by σ and P is the standard projection of $\mathbf{L}^2(\mathbb{T})$ onto $\mathbf{H}^2(\mathbb{T})$. Let $\mathbf{P}_n$ denote the projection of $\mathbf{H}^2(\mathbb{T})$ onto the subspace spanned by the functions $\{1, e^{i\theta}, \ldots, e^{in\theta}\}$. With respect to this basis, the operator $\mathbf{P}_n M[\sigma]\mathbf{P}_n$ has matrix representation $T_n[\sigma]$. Taking a minor liberty with operator and matrix notation we can examine the nature of any convergence of operators $T_n[\sigma] \to T[\sigma]$ by imagining the matrices $T_n[\sigma]$ growing without bound to a semi-infinite matrix representing $T[\sigma]$.

Theorem 1 *For any* $\sigma \in \mathbf{PC}$*,* $T[\sigma]$ *is a Fredholm operator if and only if* $w(R_\sigma)$ *exists, in which case the index of* $T[\sigma]$ *is equal to* $-w(R_\sigma)$*.*

Theorem 2 (Coburn) *For any* $\sigma \in \mathbf{L}^\infty(\mathbb{T})$ *not identically zero, either* $T[\sigma]$ *or* $T[\bar{\sigma}]$ *has trivial kernel.*

Here $\bar{\sigma}$ denotes the complex conjugate of σ.

By imposing the restriction $|\Re\beta| < \frac{1}{2}$ it easily follows from these two theorems that $\sigma \in C_\beta$ implies $T[\sigma]$ is invertible on $\mathbf{H}^2(\mathbb{T})$. From an application of the *finite section method*, it follows that $T_n[\sigma]$ is invertible for n sufficiently large (the main focus of this method being the suitability of $P_n T[\sigma]^{-1} P_n$ as an approximate inverse for $T_n[\sigma]$; see [10], ch. 3). For what follows we will assume n to be thus sufficiently large.

For $p \leq n$ let X denote the $p \times p$ matrix with (i, j) entry $x_{i,j}$ equal to the $(n-p+i+1, j)$ entry of $T_n[\sigma]^{-1}$. Jacobi's theorem concerning minors of inverse matrices (extending Cramer's rule, see [24, p. 20]) implies that

$$\det X = \frac{(-1)^{(n+1)p} \det \tilde{T}}{D_n[\sigma]}$$

where $\tilde{T}$ is the matrix obtained from $T_n[\sigma]$ by deleting the last p columns and the first p rows. An easy inspection of these matrices shows that

$$\tilde{T} = T_{n-p}[z^{-p}\sigma] = (-1)^{-(n-p+1)p}T_{n-p}[(-z)^{-p}\sigma],$$

so that

$$D_{n-p}[(-z)^{-p}\sigma] = (-1)^p \det X \cdot D_n[\sigma]. \tag{8}$$

It follows that if a first order asymptotic expression for det X is found, then a first order asymptotic expansion is obtainable for the determinants of related Toeplitz matrices with symbols not subject to the restriction $|\Re\beta| < \frac{1}{2}$. To this end, since X is a submatrix of $T_n[\sigma]^{-1}$, we determine the entries of X by investigating the solution to finite Toeplitz systems of equations.

2.2 Wiener-Hopf Factorization

For a starting point it will be most convenient to consider $T_n[\sigma]$ as acting on the space of polynomials in the variable z of degree at most n. The equation

$$T_n[\sigma]p = q \tag{9}$$

will be taken to mean that

$$\hat{q}(i) = \sum_{j=0}^{n} \hat{\sigma}(i-j)\hat{p}(j)$$

where

$$p(z) = \sum_{i=0}^{n} \hat{p}(i)z^i \quad \text{and} \quad q(z) = \sum_{i=0}^{n} \hat{q}(i)z^i.$$

Setting $q_i(z) = z^i$ and $T_n[\sigma]p_i = q_i$ it follows that the (i, j) entry in the matrix $T_n[\sigma]^{-1}$ is given by $\widehat{p_j}(i)$. We obtain from these definitions and that of X that

$$x_{i,j} = \hat{p}(n-p+1+1). \tag{10}$$

Equation (10) and the condition on σ yield the equation

$$\sigma p = q + \phi + z^n\psi \tag{11}$$

where $\phi \in \overline{z\mathbf{H}^1}$ and $\psi \in z\mathbf{H}^1$. (Here $\mathbf{H}^s = \{f \in \mathbf{L}^s(\mathbb{T}) : n < 0 \Rightarrow \hat{f}(n) = 0\}$, $1 \leq s \leq \infty$; the variable z takes values in $\mathbb{T}$.) The solution of (11) proceeds by means of the *Wiener-Hopf factorization* of σ and the introduction of certain projection operators. The notation $f \sim \sum_{i=-\infty}^{\infty} \hat{f}(i)z^i$ will be used to denote the representation of a function by its Fourier series.

For $g \in \mathbf{L}^2(\mathbb{T})$ define

$$P^+g(z) = \sum_{i=0}^{\infty} \hat{g}(i)z^i, \qquad P^-g(z) = \sum_{i=-\infty}^{0} \hat{g}(i)z^i,$$

$$P_+g(z) = g(z) - P^-g(z), \qquad P_-g(z) = g(z) - P^+g(z),$$

and

$$\tilde{g}(z) = -i\,(P_+g(z) - P_-g(z))$$

The operator P^+ is the standard orthogonal projection of $\mathbf{L}^2[\mathbb{T}]$ onto $\mathbf{H}^2[\mathbb{T}]$. P^- is the projection onto $\overline{\mathbf{H}^2[\mathbb{T}]}$. The two operators P_+ and P_- are a simple way of excluding zero from sums defining P^+ and P^-, respectively. If g satisfies an additional Lipschitz condition with exponent greater than zero, then it follows that $\log g$ and $\widetilde{\log g}$ are continuous [36, theorem III.13.27].

Set

$$g_\pm = \exp\left(\frac{1}{2}\left(\log g \pm i\,\widetilde{\log g}\right)\right), \tag{12}$$

so that $g = g_-g_+$, the Wiener-Hopf factorization of g. The function g_+ (respectively, g_-) extends analytically and is nonzero inside (respectively, outside) the unit circle of the complex plane. Taking the Wiener-Hopf factorization of the function τ from Eq. (5) we define

$$\sigma_+(z) = (1-z)^\beta \tau_+(z),$$
$$\sigma_-(z) = (1-z^{-1})^{-\beta} \tau_-(z),$$

so that $\sigma = \sigma_-\sigma_+$ and the function σ_+ (respectively, σ_-) also extends analytically and is nonzero inside (respectively, outside) the unit circle. Equation (11) may now be written as a pair of equations

$$\begin{aligned} \sigma_+ p &= \frac{q}{\sigma_-} + \frac{\phi}{\sigma_-} + \frac{z^n\psi}{\sigma-}, \\ z^{-n}\sigma_- p &= \frac{q}{z^n\sigma_+} + \frac{\phi}{z^n\sigma_+} + \frac{\psi}{\sigma_+}. \end{aligned} \tag{13}$$

The condition $|\Re\beta| < \frac{1}{2}$ implies that $\phi \in \overline{z\mathbf{H}^2}$, $\psi \in z\mathbf{H}^2$, $\sigma_+^{\pm 1} \in \mathbf{H}^2$, and $\sigma_-^{\pm 1} \in \overline{\mathbf{H}^2}$. As p is a polynomial of finite degree, $\sigma_+ p \in \mathbf{H}^2$ and $z^{-1}\sigma_- p \in \overline{\mathbf{H}^2}$. We apply the operators P_- and P_+ to these equations, obtaining

$$0 = P_-\left(\frac{q}{\sigma_-}\right) + \frac{\phi}{\sigma_-} + P_-\left(\frac{z^n\psi}{\sigma_-}\right),$$
$$0 = P_+\left(\frac{q}{z^n\sigma_+}\right) + P_+\left(\frac{\phi}{z^n\sigma_+}\right) + \frac{\psi}{\sigma_+}.$$

Let $u(z) = \frac{\sigma_-(z)}{\sigma_+(z)}$ and let $v(z) = 1/u(z)$. Setting $z = e^{i\theta}$ we have

$$u(z) = (2 - 2\cos\theta)^{-\beta}\frac{\tau_-(\theta)}{\tau_+(\theta)}.$$

Define operators U and V by

$$U(g) = P_+(z^{-n}ug) \quad \text{and} \quad V(g) = P_-(z^n vg).$$

Our pair of equations can now be written as a matrix equation:

$$\begin{bmatrix} I & V \\ U & I \end{bmatrix}\begin{bmatrix} \frac{\phi}{\sigma_-} \\ \frac{\psi}{\sigma_+} \end{bmatrix} = \begin{bmatrix} -P_-\left(\frac{q}{\sigma-}\right) \\ -P_+\left(\frac{q}{z^n\sigma_+}\right) \end{bmatrix}.$$

Multiplying on the left by the matrix $\begin{bmatrix} I & -V \\ -U & I \end{bmatrix}$ yields

$$\begin{bmatrix} I - VU & O \\ O & I - UV \end{bmatrix}\begin{bmatrix} \frac{\phi}{\sigma_-} \\ \frac{\psi}{\sigma_+} \end{bmatrix} = \begin{bmatrix} -P_-\left(z^n vP^-\left(\frac{q}{z^n\sigma_+}\right)\right) \\ -P_+\left(z^{-n}uP^+\left(\frac{q}{\sigma_-}\right)\right) \end{bmatrix}.$$

This last equation has a solution if the matrix on the left is invertible, which in turn yields a solution of (11) for p. As it happens, we need only consider the invertibility of $I - VU$, namely the solution of the equation

$$(I - VU)\left(\frac{\phi}{\sigma_-}\right) = -P_-\left(z^n vP^-\left(\frac{q}{z^n\sigma_+}\right)\right). \tag{14}$$

In keeping with our identification of $\mathbf{L}^2$ functions with their Fourier series, the above equation can be interpreted as a semi-infinite matrix equation on the space of series indexed by the negative integers. The operator VU has the matrix representation with (i, j) entry

$$(VU)_{i,j} = \sum_{k=1}^{\infty} \hat{u}(k+n-j)\hat{v}(i-n-k), \tag{15}$$

the convergence of the series depending on the restriction $|\Re\beta| < \frac{1}{2}$.

The estimation of $\det X$ is obtained from this information in two steps. The first step consists of finding a complete asymptotic expansion for the entries $x_{i,j}$ of X as $n \to \infty$. The second step is the use of this expansion to find a first order expression for $\det X$. To achieve step one we first approximate VU by an integral operator acting on a particular function space, the approximation being in the context of finding an estimate for (14) and relying on a simple identification of a sequence of complex numbers with a function on the real line that is constant between consecutive integers. Under this identification a matrix acting on sequences behaves like an integral operator with kernel consisting of a function in the plane which is constant on squares with unit length edges and integer-valued coordinate vertices. The operator $I - VU$ is first approximated by an operator with more easily obtainable asymptotic information, and the approximation is then improved using a Neumann expansion and the Euler-Maclaurin summation formula. By keeping track of pertinent details of the resulting asymptotic expansions of the entries of X a relatively straightforward attack on $\det X$ is possible, yielding a solution for step two.

3 Invertibility of $I - VU$

We start with a consideration of the asymptotics for the entries in the matrix VU.

Lemma 1 *As $n \to \infty$ we have the asymptotic expansions*

$$\hat{u}(n) \sim \sum_{m=0}^{\infty} c_m n^{-1+2\beta-m} \quad \text{and} \quad \hat{v}(-n) \sim \sum_{m=0}^{\infty} c'_m n^{-1-2\beta-m}.$$

These expansions follow directly from Erdélyi's method of integration by parts (see, for example, [7, pp. 89–91]).

Definition 2 For $x \in \mathbb{R}$ let $\{x\}$ denote the smallest integer greater than or equal to x, called the *ceiling* of x.

Let $M : \mathbb{R}^2 \to \mathbb{C}$ be given by

$$M(x, y) = VU_{\{-x\},\{-y\}}. \tag{16}$$

We formally define the integral operator

$$\mathbf{M}f(x) = \int_0^\infty M(x,y)f(y)\,dy.$$

Lemma 2 *As $n \to \infty$ and for any $\delta > 0$ we have*

$$M(x,y) = c_0c_0' \int_0^\infty (n+\{x\}+z)^{-1-2\beta}(n+\{y\}+z)^{-1+2\beta}\,dz$$
$$+ o\left((n+\{x\})^{-\frac{1}{2}-\beta-\delta}(n+\{y\})^{-\frac{1}{2}+\beta-\delta}\right).$$

Proof Apply Lemma 1, (16), and the Euler-Maclaurin summation formula ([32, pp. 127–128]; the calculation is carried out in full in [27, pp. 17–18]).

Definition 3 We make use of the following function spaces and their norms:

$$\mathbf{L}^{2,\beta}(0,\infty) = \{f(x) : (1+x)^{-\beta}f(x) \in \mathbf{L}^2(0,\infty)\},$$
$$\|f(x)\|_{2,\beta} = \|(1+x)^{-\beta}f(x)\|_2,$$

$$\mathbf{L}^{2,\beta,n}(0,\infty) = \{f(x) : f(nx) \in \mathbf{L}^{2,\beta}(0,\infty)\},$$
$$\|f(x)\|_{2,\beta,n} = \|f(nx)\|_{2,\beta}.$$

$\mathbf{L}^{2,\beta}$ and $\mathbf{L}^{2,\beta,n}$ with the given norms are easily shown to be Banach spaces.

Definition 4 Let

$$K(x,y) = c_0c_0' \int_0^\infty (n+x+z)^{-1-2\beta}(n+y+z)^{-1+2\beta}\,dz.$$

We define the following operators on $\mathbf{L}^{2,-\beta,n}(0,\infty)$:

$$\mathbf{K}f(x) = \int_0^\infty K(x,y)f(y)\,dy,$$
$$\mathbf{K_e}f(x) = \int_0^\infty [M(x,y)-K(x,y)]f(y)\,dy.$$

Lemma 3 *The operator $\mathbf{I}-\mathbf{K}$ is bounded and invertible on $\mathbf{L}^{2,-\beta,n}(0,\infty)$. The norm of $\mathbf{I}-\mathbf{K}$ does not depend on n.*

Proof Let $\tilde{\mathbf{K}} = \mathbf{A}\mathbf{K}\mathbf{A}^{-1}$, where

$$\mathbf{A}g(x) = e^{(\frac{1}{2}+\beta)x}g\left(n(e^x-1)\right).$$

Direct calculation shows that

$$\tilde{\mathbf{K}} f(x) = \int_0^\infty k(x-y) f(y)\, dy,$$

where

$$k(x) = c_0 c_0' e^{\left(\frac{1}{2}+\beta\right)x} \int_0^\infty (z+1)^{-1+2\beta} (z+e^x)^{-1-2\beta}\, dz.$$

Calculation also shows that $\mathbf{A}$ is a norm-preserving linear isomorphism of $\mathbf{L}^{2,-\beta,n}$ onto $\mathbf{L}^2$, and that $\tilde{\mathbf{K}}$ is a *Wiener-Hopf operator* (namely, an operator of the form $\mathbf{W}[\sigma]f = \mathcal{F}^{-1}P(\sigma \mathcal{F} f)$, where $\mathcal{F}$ denotes the Fourier transform; see [33, p. 111]) with symbol given by the Fourier transform of k:

$$\hat{k}(\xi) = c_0 c_0' \frac{\pi^2 \csc\left(\pi\left(\frac{1}{2}-\beta+i\xi\right)\right) \csc\left(\pi\left(\frac{1}{2}-\beta-i\xi\right)\right)}{\Gamma(1+2\beta)\Gamma(1-2\beta)}.$$

The values of the constants c_0 and c_0' are useful at this point, being obtained from an integration by parts in each case:

$$c_0 = \frac{\Gamma(1-2\beta)\sin\pi\beta}{\pi} \cdot \frac{\tau_-(1)}{\tau_+(1)},$$
$$c_0' = \frac{\Gamma(1+2\beta)\sin\pi\beta}{\pi} \cdot \frac{\tau_+(1)}{\tau_-(1)}.$$

These formulas yield

$$\hat{k}(\xi) = -\frac{\sin^2 \pi\beta}{\cosh^2 \pi\xi - \sin^2 \pi\beta}.$$

Since $|\Re\beta| < \frac{1}{2}$ it follows that $\|\tilde{\mathbf{K}}\|_2 \le \|\hat{k}(\xi)\|_\infty < \infty$, implying that $\tilde{\mathbf{K}}$ is a bounded operator. Also, the curve $\{1 - \hat{k}(\xi) : \xi \in \mathbb{R}\}$ never vanishes and has winding number zero about the origin. These facts imply that the operator $\mathbf{I} - \tilde{\mathbf{K}}$ is invertible on $\mathbf{L}^2(0,\infty)$ and consequently that the operator $\mathbf{I} - \mathbf{K}$ is invertible on $\mathbf{L}^{2,-\beta,n}$ (see [25, p. 41]).

Finally, the norm of $\mathbf{I} - \mathbf{K}$ is seen to be independent of n since $\mathbf{A}$ is norm-preserving and $\tilde{\mathbf{K}}$ is independent of n.

Lemma 4 *We have* $\|\mathbf{K_e}\|_{2,-\beta,n} = o(n^{-\delta})$.

Proof The kernel $K_e(x,y) = o\left((n+x)^{-\frac{1}{2}-\beta-\delta}(n+y)^{-\frac{1}{2}+\beta-\delta}\right)$, from the definition and from Lemma 2. The Schwarz inequality for the spaces $\mathbf{L}^{2,\beta,n}$ is given by

$$\|fg\|_1 \leq \|f\|_{2,\beta,n}\|g\|_{2,-\beta,n},$$

hence

$$\begin{aligned}\|\mathbf{K_e} f(x)\|_{2,-\beta,n} &= \|(1+x)^{\beta}\mathbf{K_e} f(nx)\|_2 \\ &\leq \int_0^\infty (1+x)^{\beta}(n+nx)^{-\frac{1}{2}-\beta-\delta}dx \cdot n^{\frac{1}{2}+\beta-\delta} \cdot \|f\|_{2,-\beta,n} \\ &= c'n^{-2\delta}\|f\|_{2,-\beta,n}.\end{aligned}$$

Proposition 1 *The operator* $\mathbf{I}-\mathbf{M}$ *is invertible on* $\mathbf{L}^{2,-\beta,n}$ *for n sufficiently large.*

Proof $\mathbf{I}-\mathbf{M} = \mathbf{I}-\mathbf{K}-\mathbf{K_e}$. Apply Lemmas 3 and 4 and the fact that the set of invertible operators is open.

The conclusion to be drawn from Proposition 1 is that for n sufficiently large, the operator $I-VU$ is invertible on the space of sequences $l^{2,-\beta,n}(\mathbb{Z}^+)$ obtained from $\mathbf{L}^{2,-\beta,n}(0,\infty)$ by considering the subspace of functions constant on open intervals between successive integers.

4 Asymptotics of a Section of $T_n[\sigma]^{-1}$

We state first an important step towards the desired result of this section.

Proposition 2 $\det X = (-1)^p\mathbf{G}[\tau]^{-p}n^{-p^2+2\beta p}c\,(1+o(1))$, *where c is a constant.*

The proof of this identity is divided into three parts. The first part is a factorization, essentially due to Widom, for which the evaluation of the determinants of the individual terms is facilitated.

Lemma 5 $X = -T_{p-1}[1/\sigma_-]\,Y\,T_{p-1}[1/\sigma_-]$, *where* Y *is the* $p\times p$ *matrix with* (i,j) *entry*

$$y_{i,j} = \left(z^{-n}u(I-VU)^{-1}z^j\right)\hat{}\,(-i). \tag{17}$$

Proof Recall X has (i, j) entry

$$\begin{aligned} x_{i,j} &= \widehat{p_j}(n-p+i+1) \\ &= \sum_{k=1}^{p-1} \widehat{\sigma_-^{-1}}(i-k)\widehat{\sigma_- p_j}(n-p+k+1) \end{aligned}$$

since $\sigma_-^{-1} \in \overline{\mathbf{H}^2}$. From (13) we obtain

$$z^{-n}\sigma_- p_i = z^{i-n}\sigma_+^{-1} + z^{-n}u\frac{\phi}{\sigma_-} + \frac{\psi}{\sigma_+}.$$

Now

$$\frac{\phi}{\sigma_-} = -(I-VU)^{-1}\left(P^+\left(\frac{z^i}{\sigma_-}\right)\right) - \frac{z^i}{\sigma_-}$$

so that

$$z^{-n}\sigma_- p_i = -z^{-n}u(I-VU)^{-1}\left(P^+\left(\frac{z^i}{\sigma_-}\right)\right),$$

as $\frac{\psi}{\sigma_+} \in z\mathbf{H}^1$. Putting the above identities together yields the desired matrix identity.

An auxiliary fact is the

Corollary 1 $\det X = (-1)^p \mathbf{G}[\tau]^{-p} \det Y$.

Using the identity $(I-VU)^{-1} = I + (I-VU)^{-1}VU$ we write

$$y_{i,j} = \hat{u}(n-i-j) + \sum_{k=0}^{\infty} \hat{u}(n-i+k)\left[(I-VU)^{-1}VUz^j\right]\hat{}\,(-k). \tag{18}$$

The second part of the proof of Proposition 2 establishes the following asymptotic expansion.

Lemma 6 *$y_{i,j} \sim \sum_{k=0}^{\infty} p_k(i,j)n^{-1+2\beta-k}$, where p_k is a polynomial of degree k.*

Proof We use the Euler-Maclaurin summation formula to obtain terms in the asymptotic expansion of $y_{i,j}$. As in the previous section we utilize an approximation of VU by an operator with smooth kernel. As it happens, the particular operator used previously is not suitable for obtaining a complete asymptotic expansion. We alter the given operators as follows. Define complex-valued functions $\zeta_1(x)$ for

$0 \leq x < \infty$ and $\zeta_2(x)$ for $-\infty < x \leq 0$ by the formulas

$$\zeta_1(x) = \sum_{m=0}^{M} c_m x^{-1+2\beta-m} \quad \text{and} \quad \zeta_2(-x) = \sum_{m=0}^{M} c'_m x^{-1-2\beta-m}$$

for $x \geq 0$, where the constants c_m and c'_m are defined previously by Lemma 1 and M is as large as we like (for any fixed value of β, we require only finitely many terms in any of these expansions, the number growing larger as the modulus of β increases). From these definitions and Lemma 1 we immediately conclude that

$$\hat{u}(n) - \zeta_1(n) = o(n^{-1+2\beta-M}), \qquad \hat{v}(-n) - \zeta_2(-n) = o(n^{-1-2\beta-M}),$$

and that

$$(VU)_{i,j} = \sum_{k=1}^{\infty} \zeta_1(n-j+k)\zeta_2(-n+i-k) + o(n^{-1-M}).$$

Let

$$W(x,y) = \sum_{k=1}^{\infty} \zeta_1(n+\{y\}+k)\zeta_2(n-\{x\}-k),$$

and let $\mathbf{W}$ denote the integral operator on $\mathbf{L}^{2,-\beta,n}(0,\infty)$ with kernel $W(x,y)$. Letting o notation here be in the context of operator norm, it follows that $\mathbf{M} = \mathbf{W} + o(n^{-M})$ and hence by Proposition 1 that $\mathbf{I}-\mathbf{W}$ is invertible on $\mathbf{L}^{2,-\beta,n}(0,\infty)$ and that $(\mathbf{I}-\mathbf{M})^{-1} = (\mathbf{I}-\mathbf{W})^{-1} + o(n^{-M})$. Replacing our old definitions of $\mathbf{K}$ and $\mathbf{K_e}$ we write

$$\begin{aligned} K(x,y) &= \int_0^{\infty} \zeta_1(n+y+z)\zeta_2(-n-x-z)\,dz, \\ K_e(x,y) &= W(x,y) - K(x,y), \\ g_k &= K(x,-k), \\ g_{k,e} &= K_e(x,-k), \end{aligned}$$

and let $\mathbf{K}$ and $\mathbf{K_e}$ denote the integral operators on $\mathbf{L}^{2,-\beta,n}(0,\infty)$ with kernels $K(x,y)$ and $K_e(x,y)$, respectively. We obtain

$$(\mathbf{I}-\mathbf{M})^{-1}\mathbf{M}z^k = (\mathbf{I}-\mathbf{K}-\mathbf{K_e})^{-1}(g_k + g_{k,e}) + o(n^{-M}),$$

where again, $o(n^{-M})$ refers to a function with this norm on $\mathbf{L}^{2,-\beta,n}(0,\infty)$. $\mathbf{K}$ is just a perturbation of our previous operator of this name; it is easy to show that $\mathbf{I}-\mathbf{K}$ is invertible for n sufficiently large and that the norm of the new $\mathbf{K}$ is the same as the old, up to a term of norm $o(1)$. The operator $\mathbf{K_e}$, too, behaves like its previous version; in particular we have $\|\mathbf{K_e}\| = o(n^{-1})$ as $n \to \infty$. We therefore obtain a Neumann expansion for the inverse:

$$(\mathbf{I} - \mathbf{K} - \mathbf{K_e})^{-1} = (\mathbf{I} - \mathbf{K})^{-1} \sum_{i=0}^{\infty} \left[\mathbf{K_e}(\mathbf{I} - \mathbf{K})^{-1}\right]^i .$$

Applying Euler-Maclaurin summation to each term in this series, we obtain an expansion

$$(\mathbf{I} - \mathbf{K} - \mathbf{K_e})^{-1}(g_k + g_{k,e})(x) \sim \sum_{i=0}^{\infty} (n-k)^{-1-i} h_i \left(\frac{j+k}{n-j}\right),$$

where the functions h_i do not depend on n or k. Using Euler-Maclaurin summation on the expansion

$$\sum_{k=0}^{\infty} \hat{u}(n-i+k) \sum_{l=0}^{\infty} (n-j)^{-1-l} h_l \left(\frac{j+k}{n-j}\right),$$

Lemma 1 and the binomial theorem applied to $\hat{u}(n-i-j)$, counting carefully the resulting powers of the i and j terms, yield the desired result.

The third part of the proof of Proposition 1 now uses the above information to compute the desired determinant.

Lemma 7 $\det Y = cn^{-p^2+2\beta p}\,(1+o(1))$, *where c is a constant.*

Proof Given the expansion

$$y_{i,j} = \sum_{k=0}^{M} p_k(i,j) n^{-1+2\beta-k} + o(n^{-1+2\beta-M}),$$

we compute the determinant of Y directly. For the computation that follows we shall use for the sake of convenience the definition $0^0 = 1$. We have

$$\det Y = \sum_{k_0=0}^{M} \cdots \sum_{k_{p-1}=0}^{M} \det\left[\left(p_{k_i}(i,j)\right)_{0\le i,j<p}\right] n^{-p+2\beta p-k_0-\cdots-k_{p-1}}.$$

Writing the polynomials in the above expression as sums of monomials and expanding the determinant we obtain a sum of terms of the form

$$c \det\left[\left(i^{k_{1,j}} j^{k_{2,j}}\right)_{0\le i,j<p}\right] n^{-p+2\beta p-k_0-\cdots-k_{p-1}}$$

where $k_{1,i} + k_{2,i} \leq k_i$. This last expression equals

$$c \prod_{i=0}^{p-1} i^{k_{1,i}} \det\left[\left(j^{k_{2,i}}\right)_{0 \leq i,j < p}\right] n^{-p+2\beta p - k_0 - \cdots - k_{p-1}}.$$

In collecting these terms to obtain an expression for $\det Y$ one finds considerable algebraic cancellation. Note that if $k_{2,i_1} = k_{2,i_2}$ for some $0 \leq i_1 \neq i_2 < p$, then the determinant of the matrix $\left[\left(j^{k_{2,i}}\right)_{i,j}\right]$ is zero. Furthermore, if $k_{1,i_1} = k_{2,i_2}$ for $0 \leq i_1 \neq i_2 < p$, then the collection of terms constituting $\det Y$ will contain two terms corresponding to the permutations of the set $\{i_1, i_2\}$; these terms cancel each other as they differ by a factor of (-1). From these observations we conclude that nonzero contributions to a first order asymptotic expansion of $\det Y$ arise from the case in which $k_{1,i_0}, \ldots, k_{1,i_{p-1}}$ are distinct and $k_{2,i_0}, \ldots, k_{2,i_{p-1}}$ are distinct. Having these two sets of distinct elements implies in turn that

$$\begin{aligned} k_0 + \cdots + k_{p-1} &\geq k_{1,i_0}, \ldots, k_{1,i_{p-1}} + k_{2,i_0}, \ldots, k_{2,i_{p-1}} \\ &\geq 2 \sum_{i=0}^{p-1} i = p^2 - p. \end{aligned}$$

The leading term in the asymptotic expansion of $\det Y$ is therefore of the form

$$cn^{-p+2\beta p-(p^2-p)} = cn^{-p^2+2\beta p},$$

yielding $\det Y = cn^{-p^2+2\beta p}\,(1 + o(1))$, as desired.

Proof (Of Proposition 2)

$$\begin{aligned} \det X &= (-1)^p \mathbf{G}[\tau]^{-p} \det Y \\ &= (-1)^p \mathbf{G}[\tau]^{-p} cn^{-p^2+2\beta p}\,(1 + o(1)), \end{aligned}$$

as desired.

Having this result we may now partially extend Eq. (6) for values of β outside of the region of the complex plane $|\Re\beta| < \frac{1}{2}$, along the lines of the remarks following Eq. (8).

5 Asymptotics of $D_n[\sigma]$ and Eigenvalue Distributions

Proposition 3 *For $\sigma \in C_\beta$, $|\Re\beta| < \frac{1}{2}$, $D_n[(-z)^p\sigma] = \mathbf{G}[\tau]^{n+1} n^{-(p+\beta)^2}$ $c\,(1 + o(1))$ for any integer p.*

Proof The case $p = 0$ is just Eq. (5). The case $p < 0$ follows from Eqs. (5), (8), and Proposition 2. The case $p > 0$ is obtained from the case $p < 0$ by matrix transposition.

We have obtained our first order asymptotic expression for $D_n[\sigma]$ for $\sigma \in C_\beta$, provided $\beta \notin \mathbb{Z} + \frac{1}{2}$. It remains to remove this last condition and to determine the value of the constant c in the above proposition. To this end, we use the following corollary of the Poisson-Jensen formula ([1, p. 208]; see also [34, p. 358]).

Lemma 8 *Suppose h is an analytic function on the disk $|z| \le 1$ and satisfies there $|h(z)| \le |\Re z|^{-c}$ for some constant $c > 0$. Then for each subdisk $|z| \le \rho < 1$ we have $|h(z)| \le A$ where A is a constant depending only on c and ρ.*

A proof of this lemma appears in [27]. We now come to our main results.

Theorem 3 *For $\sigma(z) = (-z)^\beta \tau(z) \in C_\beta$ we have*

$$D_n[\sigma] = \mathbf{G}[\tau]^{n+1} n^{-\beta^2} E[\tau, \beta]\,(1 + o(1))\,,$$

as $n \to \infty$, where $E[\tau, \beta]$ is given by (7).

Proof The proof of this theorem is in several steps. We first determine the behavior of the coefficients $y_{i,j}$ as $|\Re\beta| < \frac{1}{2}$, $|\Re\beta| \to \frac{1}{2}$. The idea, with Lemma 8 in mind, is to show that the formula for $y_{i,j}$ at most blows up only polynomially at the boundary $|\Re\beta| = \frac{1}{2}$. In [27, pp. 49–53], the estimate $|y_{i,j}| \le d_\beta^{-M_1} n^{-1+2\Re\beta}$ is obtained from equation (18), where $d_\beta = \min\{\frac{1}{2} - \Re\beta, \frac{1}{2} + \Re\beta\}$ and M_1 is a constant. From this result one demonstrates that $\det Y$ itself at most blows up only polynomially at $|\Re\beta| = \frac{1}{2}$, the formula being

$$|\det Y| \le c d_\beta^{-M_2} n^{-p^2+2\Re\beta p}, \tag{19}$$

where c is a constant depending on τ, Y is $p \times p$, and M_2 is a constant. The means by which these results are obtained are as follows. In [27] it is shown that $p^2 - p + 1$ terms of the asymptotic series for the coefficients $y_{i,j}$ are required to obtain the first order term for $\det Y$, due to the large number of cancelling terms, along the lines of the proof of Lemma 7. Writing $y_{i,j} = w_{i,j} + \varepsilon_{i,j}$, where

$$w_{i,j} = \sum_{k=0}^{p^2-p} p_k(i, j) n^{-1+2\beta-k}$$

denotes the first $p^2 - p + 1$ terms in the expansion of $y_{i,j}$, we consider the expansion

$$\begin{aligned}\det Y &= \det\left[\left(w_{i,j}\right)_{0\le i,j<p} + \left(\varepsilon_{i,j}\right)_{0\le i,j<p}\right]\\ &= \det\left[\left(w_{i,j}\right)_{0\le i,j<p}\right] + \varepsilon\end{aligned}$$

where ε denotes the error obtained by the multilinear expansion of the determinant. This expansion gives the exponent of n of Eq. (19); the polynomial growth of d_β arises from the polynomial growth of the corresponding term in $y_{i,j}$ and from the fact that the coefficients of the polynomials $p_k(i, j)$ are also polynomially bounded; see [27, Lemma 5.6].

We now make use of the estimate $|D_n[(-z)^\beta \tau]| n^{\beta^2} \le c d_\beta^{-3}$, essentially done in [34, §XIII], the details of which are found in [27, Lemma 5.7]. In combination with the previous result, we obtain the estimate

$$|D_n[(-z)^\beta \tau]| n^{\beta^2} \le c d_\beta^{-M_3},$$

where we now take $d_\beta = \text{dist}(\beta, \mathbb{Z} + \frac{1}{2})$ and M_3 is a constant. Applying Lemma 8 we conclude that

$$D_n[(-z)^\beta \tau] n^{-\beta^2} = O(1)$$

uniformly on compact subsets of the complex plane. It follows that

$$D_n[\sigma] = \mathbf{G}[\tau]^{n+1} n^{-\beta^2} c\,(1 + o(1))$$

where c depends on τ and β. If $|\Re\beta| < \frac{1}{2}$ then a result due to Basor [4] and Böttcher [8] states that

$$c = E[\tau, \beta]$$

Since the foregoing results demonstrate that, for fixed τ but variable β, $D_n[\sigma]$ is an analytic function of β, Vitali's convergence theorem [31, p. 168] implies that the formula for c holds for all β, proving Theorem 3.

The result for the asymptotic eigenvalue distribution of $T_n[\sigma]$ will now be established for functions from the following class $\mathbf{PC}_1$, consisting of all functions $\sigma : \mathbb{T} \to \mathbb{C}$ which are C^∞ away form $\theta = 0$ and have left and right limits

$$\lim_{\theta\to 0^+} \frac{d^k}{d\theta^k}\sigma(e^{i\theta}) \quad \text{and} \quad \lim_{\theta\to 2\pi^-} \frac{d^k}{d\theta^k}\sigma(e^{i\theta})$$

exist for all $k > 0$, while on the other hand,

$$\lim_{\theta \to 0^+} \sigma(e^{i\theta}) \neq \lim_{\theta \to 2\pi^-} \sigma(e^{i\theta}). \tag{20}$$

In other words, these are piecewise continuous functions with precisely one jump discontinuity that satisfy in addition appropriate smoothness conditions.

Theorem 4 *For $\sigma \in \mathbf{PC}_1$ the eigenvalues of $T_n[\sigma]$ are canonically distributed as $n \to \infty$. Moreover, the limiting set L of the eigenvalues of $T_n[\sigma]$ equals the closure of the range of σ.*

Proof If λ does not belong to the closure of the range of σ, then $\sigma - \lambda$ satisfies the assumption of Definition 1, i.e., there exists a (uniquely determined) $\beta \notin \mathbb{Z}$ such that $\sigma - \lambda \in C_\beta$. Hence we can apply Theorem 3 to the function $\sigma - \lambda$ and obtain

$$\lim_{n\to\infty} \frac{1}{n+1} \log |D_n[\sigma - \lambda]| = \log |\mathbf{G}[\sigma - \lambda]| = \log \mathbf{G}[|\sigma - \lambda|],$$

which thus holds in the sense of measure for $\lambda \in \mathbb{C}$, as the constant term $E[\tau, \beta]$ of Theorem 3 is nonzero. By a result of Widom [35, Lemma 5.1], this fact implies canonical distribution of the eigenvalues. Now let $\{\lambda_{0,n}, \ldots, \lambda_{n,n}\}$ denote the eigenvalues of $T_n[\sigma]$, counted according to multiplicity. Let λ be a point in the closure of the range of σ and for $\varepsilon > 0$ let F_ε be a continuous function, positive near λ and zero outside the open disk of radius ε centered at λ. We have $\int (F_\varepsilon \circ \sigma) d\theta > 0$; by the above discussion it follows that for any n sufficiently large, there is an i_n such that $\text{dist}(\lambda_{i_n}, \lambda) < \varepsilon$. Thus λ is a limit point of a sequence of eigenvalues and therefore is in L. L therefore contains the closure of the range of σ. For the reverse inclusion, suppose λ is not in the closure of the range of σ. By Theorem 3, $D_n[\sigma - \lambda]$ is bounded away from zero for n sufficiently large and it easily follows that the estimate holds uniformly for any $\tilde{\lambda}$ in a small neighborhood of λ. It follows that no infinite sequence $\{\lambda_{i_k,n_k}\}_{k=0}^\infty$ tends to λ, so $\lambda \notin L$.

Corollary 2 *For any $\varepsilon > 0$ the number of eigenvalues $\lambda_{i,n}$ within ε distance of a given point in the range of σ is $O(n)$.*

Corollary 3 *For any $\varepsilon > 0$ there is a number N such that the eigenvalues of $T_n[\sigma]$ are within ε distance of the range of σ whenever $n > N$.*

Proof Suppose not, i.e., that there exists a sequence $\{\lambda_{i_k,n_k}\}_{k=0}^\infty$, with $n_0 < n_1 < \cdots$, outside the set of points within ε distance of the range of σ. As the eigenvalues of $T_n[\sigma]$ are uniformly bounded in absolute value by the (finite) operator norm of $T[\sigma]$ on $\mathbf{H}^2$, it follows that $\{\lambda_{i_k,n_k}\}_{k=0}^\infty$ has a subsequence which converges to a value λ, which by construction is not in the range of σ, a contradiction of Theorem 4.

We conclude by noting that the condition (20) is necessary, as the counterexample $\sigma(z) = -z$ easily demonstrates. Indeed, it leads us to the situation where the jump parameter β is a nonzero integer.

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Global and Local Scaling Limits for Linear Eigenvalue Statistics of Jacobi β-Ensembles

Chao Min and Yang Chen

Dedicated to the memory of Harold Widom

Abstract We study the moment-generating functions (MGF) for linear eigenvalue statistics of Jacobi unitary, symplectic and orthogonal ensembles. By expressing the MGF as Fredholm determinants of kernels of finite rank, we show that the mean and variance of the suitably scaled linear statistics in these Jacobi ensembles are related to the sine kernel in the bulk of the spectrum, whereas they are related to the Bessel kernel at the (hard) edge of the spectrum. The relation between the Jacobi symplectic/orthogonal ensemble (JSE/JOE) and the Jacobi unitary ensemble (JUE) is also established.

Keywords Linear eigenvalue statistics · Moment-generating function · Jacobi β-ensembles · Mean and variance · Sine kernel · Bessel kernel

1 Introduction

In random matrix theory (RMT), the joint probability density function for the (real) eigenvalues $\{x_j\}_{j=1}^N$ of $N \times N$ Hermitian matrices from a matrix ensemble is given by [19]

$$P_N^{(\beta)}(x_1, x_2, \ldots, x_N) = \frac{1}{Z_N} \prod_{1 \le j < k \le N} \left|x_j - x_k\right|^\beta \prod_{j=1}^N w(x_j), \tag{1}$$

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where $\beta = 1, 2$ and 4 (the Dyson index) correspond to the orthogonal, unitary and symplectic ensembles respectively, $w(x)$ is a weight function and Z_N is a normalization constant. If $w(x) = \mathrm{e}^{-x^2},\ x \in \mathbb{R}$ and $w(x) = x^{\alpha}\mathrm{e}^{-x},\ x \in \mathbb{R}^+, \alpha > -1$, these are the Gaussian β-ensembles (GβE) and Laguerre β-ensembles (LβE). See also [29] on the relation between orthogonal, symplectic and unitary ensembles.

Linear statistics is an important research object in RMT and has various applications; see, e.g., [2, 5–9, 13, 16, 18, 28]. In previous works [21, 22], the authors studied the large N asymptotics for the moment-generating functions (MGF) of the suitably scaled linear statistics in GβE and LβE, from which the mean and variance of the linear statistics are derived. In the present paper, we focus on the problem in Jacobi β-ensembles (JβE). In this case, the weight function is $w(x) = (1-x)^a(1+x)^b,\ x \in [-1, 1],\ a, b > -1$.

The MGF of the linear statistics $\sum_{j=1}^{N} F(x_j)$ in JβE is given by the mathematical expectation with respect to the joint probability density function (1),

$$\mathbb{E}\left(\mathrm{e}^{-\lambda\sum_{j=1}^{N}F(x_j)}\right) = \frac{\int_{[-1,1]^N}\prod_{1\le j<k\le N}\left|x_j - x_k\right|^{\beta}\prod_{j=1}^{N} w(x_j)\mathrm{e}^{-\lambda F(x_j)}dx_j}{\int_{[-1,1]^N}\prod_{1\le j<k\le N}\left|x_j - x_k\right|^{\beta}\prod_{j=1}^{N} w(x_j)dx_j}, \tag{2}$$

where λ is a parameter and $F(\cdot)$ is a sufficiently well-behaved function to make the integral well-defined. Similarly as in [21, 22], we write the right-hand side of (2) in the form

$$G_N^{(\beta)}(f) := \frac{\int_{[-1,1]^N}\prod_{1\le j<k\le N}\left|x_j - x_k\right|^{\beta}\prod_{j=1}^{N} w(x_j)\left(1 + f(x_j)\right)dx_j}{\int_{[-1,1]^N}\prod_{1\le j<k\le N}\left|x_j - x_k\right|^{\beta}\prod_{j=1}^{N} w(x_j)dx_j}, \tag{3}$$

where

$$f(x) = \mathrm{e}^{-\lambda F(x)} - 1. \tag{4}$$

The denominator in (2) or (3) is known as Selberg's integral, which has closed form expression [19, (17.6.1)]. We are interested in the large N asymptotics of the MGF. It is well known that the distributions of linear statistics in random matrix ensembles are Gaussian; see, e.g., [8, 24].

We first consider the $\beta = 2$ case, which is the simplest among the three cases. From the well-known result of Tracy and Widom [27] by expressing $G_N^{(2)}(f)$ as a Fredholm determinant, we obtain its large N asymptotics. With the relation of $f(x)$ and $F(x)$, we compute the mean and variance of linear statistics $\sum_{j=1}^{N} F(x_j)$ in the bulk of the spectrum and at the edge respectively. It can be seen that in the bulk of the spectrum the results are related to the sine kernel, while at the edge they are related to the Bessel kernel. The mean and variance of the linear statistics in unitary ensembles have been studied a lot; see, e.g., [3, 4, 7, 20]. So the mean and variance in the $\beta = 2$ case can also be obtained by using other approaches. Our main goal of this paper is to obtain the mean and variance in the $\beta = 4$ and $\beta = 1$ cases for Jacobi ensembles. We show the results of the $\beta = 2$ case for reference and apply the method to the $\beta = 4$ and $\beta = 1$ cases.

For the $\beta = 4$ case, we apply the previous results for general weight functions in [21, 22] to the Jacobi weight. By making use of the skew orthogonal polynomials for the Jacobi weight [1], we express $G_N^{(4)}(f)$ as a Fredholm determinant involving the Christoffel-Darboux kernel. The large N asymptotics of $G_N^{(4)}(f)$ is derived by using the trace-log expansions. The mean and variance of the scaled linear statistics $\sum_{j=1}^{N} F(x_j)$ then follows and the relation between the $\beta = 4$ case and $\beta = 2$ case is built.

The $\beta = 1$ case is more difficult to deal with, and we only consider the case when N is even. Usually in this situation the weight is taken to be the square root of the weight considered in the $\beta = 2$ case, so we let $w(x) = (1-x)^{a/2}(1+x)^{b/2},\ x \in [-1, 1],\ a, b > -2$. The following development is similar to the $\beta = 4$ case, but with more complicated computations. Finally we obtain the mean and variance of the scaled linear statistics $\sum_{j=1}^{N} F(x_j)$ and establish the relation between the $\beta = 1$ case and $\beta = 2$ case. Note that as in the $\beta = 2$ case we also consider the $\beta = 4$ and $\beta = 1$ cases in the bulk of the spectrum and at the edge, and the results are related to the sine kernel and Bessel kernel, respectively.

We would like to point out that in this paper some calculations on the asymptotics are heuristic. To be specific, we always substitute the asymptotic expressions of the traces into the trace-log expansions for the MGF and do not care much about the error terms. So the error estimates in the asymptotic analysis should be made more precisely, such as the errors in the mean and variance formulas of the scaled linear statistics obtained in the following sections.

2 Jacobi Unitary Ensemble (JUE)

In this section, we consider the $\beta = 2$ case, which is the simplest case and provides a comparison to the $\beta = 4$ and $\beta = 1$ cases.

2.1 Finite N Case for the MGF in JUE

Recall that the weight function is $w(x) = (1-x)^a(1+x)^b,\ x \in [-1, 1],\ a, b > -1$. Let $\{\varphi_j(x)\}_{j=0}^{\infty}$ be the sequence obtained by orthonormalizing the sequence $\{x^j(1-x)^{a/2}(1+x)^{b/2}\}$ in $L^2[-1, 1]$ and

$$K_N^{(2)}(x, y) := \sum_{j=0}^{N-1} \varphi_j(x)\varphi_j(y). \tag{5}$$

In fact,

$$\varphi_j(x) = \frac{1}{\sqrt{h_j^{(a,b)}}} P_j^{(a,b)}(x)(1-x)^{a/2}(1+x)^{b/2},$$

where $P_j^{(a,b)}(x)$ is the Jacobi polynomial of degree j with the orthogonality [15, 25]

$$\int_{-1}^{1} P_j^{(a,b)}(x) P_k^{(a,b)}(x)(1-x)^a(1+x)^b dx = h_j^{(a,b)} \delta_{jk}, \qquad j,k = 0,1,2,\ldots$$

and

$$h_j^{(a,b)} = \frac{2^{a+b+1}\Gamma(j+a+1)\Gamma(j+b+1)}{j!(2j+a+b+1)\Gamma(j+a+b+1)}.$$

Tracy and Widom [27] proved that $G_N^{(2)}(f)$ can be expressed as a Fredholm determinant

$$G_N^{(2)}(f) = \det\left(I + K_N^{(2)} f\right),$$

where $K_N^{(2)}$ is the operator on $L^2[-1,1]$ with kernel $K_N^{(2)}(x,y)$ given by (5), and f denotes the operator of multiplication by f. In addition, it is well known that

$$\begin{aligned}\log\det\left(I + K_N^{(2)} f\right) &= \operatorname{Tr}\log\left(I + K_N^{(2)} f\right)\\ &= \operatorname{Tr} K_N^{(2)} f - \frac{1}{2}\operatorname{Tr}\left(K_N^{(2)} f\right)^2 + \frac{1}{3}\operatorname{Tr}\left(K_N^{(2)} f\right)^3 - \cdots. \qquad (6)\end{aligned}$$

This formula will help us to analyze the large N asymptotics of $G_N^{(2)}(f)$ in the following subsections.

2.2 *Scaling in the Bulk of the Spectrum in JUE*

In this subsection, we study the large N asymptotics of $G_N^{(2)}(f)$ in the bulk of the spectrum for the JUE, and obtain the mean and variance of the suitably scaled linear statistics. We state a theorem before our discussion.

Theorem 1 *For $x, y \in \mathbb{R}$, we have as $N \to \infty$,*

$$\frac{1}{N} K_N^{(2)}\left(\frac{x}{N}, \frac{y}{N}\right) = K_{\text{sine}}(x,y) + O(N^{-1}),$$

where $K_{\text{sine}}(x, y)$ is the sine kernel defined by

$$K_{\text{sine}}(x, y) := \frac{\sin(x - y)}{\pi(x - y)}. \tag{7}$$

The error term is uniform for x and y in compact subsets of $\mathbb{R}$.

Proof By using the Christoffel-Darboux formula, we have

$$\begin{aligned} K_N^{(2)}(x, y) = &\frac{\Gamma(N + 1)\Gamma(N + a + b + 1)}{2^{a+b}(2N + a + b)\Gamma(N + a)\Gamma(N + b)} \\ &\times \frac{P_N^{(a,b)}(x)P_{N-1}^{(a,b)}(y) - P_{N-1}^{(a,b)}(x)P_N^{(a,b)}(y)}{x - y} \\ &\times (1 - x)^{a/2}(1 + x)^{b/2}(1 - y)^{a/2}(1 + y)^{b/2}. \end{aligned} \tag{8}$$

Taking advantage of the large n asymptotic formula of the Jacobi polynomials [25, p. 196]

$$\begin{aligned} P_n^{(a,b)}(\cos\theta) = &\frac{1}{\sqrt{\pi n}}\left(\sin\frac{\theta}{2}\right)^{-a-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-b-\frac{1}{2}} \\ &\times \cos\left[\left(n + \frac{a + b + 1}{2}\right)\theta - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right] + O(n^{-3/2}), \end{aligned} \tag{9}$$

where $0 < \theta < \pi$, we find that $K_N^{(2)}(\cos\theta, \cos\phi)$ equals

$$\begin{aligned} &\frac{\Gamma(N + 1)\Gamma(N + a + b + 1)}{2^{a+b}(2N + a + b)\Gamma(N + a)\Gamma(N + b)(\cos\theta - \cos\phi)}\Bigg\{\frac{2^{a+b+1}}{\pi\sqrt{N(N - 1)\sin\theta\sin\phi}} \\ &\times \Bigg[\cos\left(\left(N + \frac{a + b + 1}{2}\right)\theta - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\cos\left(\left(N + \frac{a + b - 1}{2}\right)\phi - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right) \\ &- \cos\left(\left(N + \frac{a + b - 1}{2}\right)\theta - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\cos\left(\left(N + \frac{a + b + 1}{2}\right)\phi - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\Bigg] \\ &+ O(N^{-2})\Bigg\}, \qquad 0 < \theta,\ \phi < \pi. \end{aligned}$$

The above error terms are uniform for θ and ϕ in compact subsets of $(0, \pi)$. Note that the expression in the square brackets $[\cdots]$ can be written in the form

$$\begin{aligned} &2\Bigg[\cos\left(\left(N + \frac{a + b}{2}\right)\theta - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\sin\left(\left(N + \frac{a + b}{2}\right)\phi - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\cos\frac{\theta}{2}\sin\frac{\phi}{2} \\ &- \sin\left(\left(N + \frac{a + b}{2}\right)\theta - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\cos\left(\left(N + \frac{a + b}{2}\right)\phi - \frac{\pi}{2}\left(a + \frac{1}{2}\right)\right)\sin\frac{\theta}{2}\cos\frac{\phi}{2}\Bigg]. \end{aligned}$$

Replacing $\cos\theta$ and $\cos\phi$ by x/N and y/N respectively and taking a large N limit, we establish the theorem with the aid of Stirling's formula. See also [12, 23]. □

Remark 1 When $x = y$, we have as $N \to \infty$,

$$\frac{1}{N} K_N^{(2)}\left(\frac{x}{N}, \frac{x}{N}\right) = K_{\text{sine}}(x, x) + O(N^{-1}),$$

where

$$K_{\text{sine}}(x, x) = \frac{1}{\pi}.$$

The error term is uniform for x in compact subsets of $\mathbb{R}$.

Using Theorem 1, we compute (6) term by term as $N \to \infty$, and we change $f(x)$ to $f(Nx)$ in the computations. The first term is

$$\begin{aligned}\mathrm{Tr} K_N^{(2)} f &= \int_{-1}^{1} K_N^{(2)}(x, x) f(Nx) dx \\ &= \int_{-N}^{N} \frac{1}{N} K_N^{(2)}\left(\frac{x}{N}, \frac{x}{N}\right) f(x) dx \\ &= \int_{-\infty}^{\infty} K_{\text{sine}}(x, x) f(x) dx + O(N^{-1}), \qquad N \to \infty.\end{aligned}$$

Remark 2 We assume that $f(\cdot)$ is a continuous real-valued function belonging to $L^1(\mathbb{R})$ and vanishes at $\pm\infty$.

The second term gives

$$\begin{aligned}\mathrm{Tr}\left(K_N^{(2)} f\right)^2 &= \int_{-1}^{1}\int_{-1}^{1} K_N^{(2)}(x, y) f(Ny) K_N^{(2)}(y, x) f(Nx) dx dy \\ &= \frac{1}{N^2}\int_{-N}^{N}\int_{-N}^{N} K_N^{(2)}\left(\frac{x}{N}, \frac{y}{N}\right) f(y) K_N^{(2)}\left(\frac{y}{N}, \frac{x}{N}\right) f(x) dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\text{sine}}^2(x, y) f(x) f(y) dx dy + O(N^{-1}), \qquad N \to \infty.\end{aligned}$$

Hence, we find heuristically from (6) that $\log\det\left(I + K_N^{(2)} f\right)$ equals

$$\int_{-\infty}^{\infty} K_{\text{sine}}(x, x) f(x) dx - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\text{sine}}^2(x, y) f(x) f(y) dx dy + \cdots + O(N^{-1}). \tag{10}$$

Now we are able to derive the mean and variance of the scaled linear statistics $\sum_{j=1}^{N} F(Nx_j)$. Taking account of the relation of $f(x)$ and $F(x)$ in (4), we have

$$f(x) = -\lambda F(x) + \frac{\lambda^2}{2} F^2(x) - \cdots . \tag{11}$$

Substituting (11) into (10) gives

$$\log\det\left(I + K_N^{(2)} f\right) = -\lambda \int_{-\infty}^{\infty} K_{\text{sine}}(x,x)F(x)dx + \frac{\lambda^2}{2}\left[\int_{-\infty}^{\infty} K_{\text{sine}}(x,x)F^2(x)dx \right.$$
$$\left. - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\text{sine}}^2(x,y)F(x)F(y)dxdy\right] + \cdots + O(N^{-1}).$$

From the coefficients of λ and λ^2 and in view of $\log G_N^{(2)}(f) = \log\det\left(I + K_N^{(2)} f\right)$, we get the following results.

Theorem 2 *Let* $\mu_N^{(\text{JUE})}$ *and* $\mathcal{V}_N^{(\text{JUE})}$ *be the mean and variance of the scaled linear statistics* $\sum_{j=1}^{N} F(Nx_j)$*, respectively. We have as* $N \to \infty$,

$$\mu_N^{(\text{JUE})} = \int_{-\infty}^{\infty} K_{\text{sine}}(x,x)F(x)dx + O(N^{-1}), \tag{12}$$

$$\mathcal{V}_N^{(\text{JUE})} = \int_{-\infty}^{\infty} K_{\text{sine}}(x,x)F^2(x)dx - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\text{sine}}^2(x,y)F(x)F(y)dxdy$$
$$+ O(N^{-1}), \tag{13}$$

where $K_{\text{sine}}(x,y)$ *is the sine kernel defined by (7).*

Remark 3 The result of the above theorem can also be derived by using the method in the paper [3], and it is consistent with the one for the Gaussian unitary ensemble in that paper.

2.3 Scaling at the Edge of the Spectrum in JUE

Contrasting to the previous subsection, we rescale the JUE at the (hard) edge of the spectrum in this subsection. It will be seen that the Bessel kernel arises.

Theorem 3 *For* $x, y \in \mathbb{R}^+$*, we have as* $N \to \infty$,

$$\frac{1}{2N^2} K_N^{(2)}\left(1 - \frac{x}{2N^2}, 1 - \frac{y}{2N^2}\right) = K_{\text{Bessel}}^{(a)}(x,y) + O(N^{-1}),$$

where $K_{\text{Bessel}}^{(a)}(x, y)$ is the Bessel kernel of order a defined by

$$K_{\text{Bessel}}^{(a)}(x, y) := \frac{J_a(\sqrt{x})\sqrt{y}J_a'(\sqrt{y}) - J_a'(\sqrt{x})\sqrt{x}J_a(\sqrt{y})}{2(x - y)}, \tag{14}$$

and $J_a(\cdot)$ is the Bessel function of the first kind of order a [17, p. 102]. The error term is uniform for x and y in compact subsets of $\mathbb{R}^+$.

Proof Taking account of (8) and using the Hilb-type asymptotic formula of the Jacobi polynomials [25, p. 197]

$$\left(\sin\frac{\theta}{2}\right)^a \left(\cos\frac{\theta}{2}\right)^b P_n^{(a,b)}(\cos\theta) = \left(n + \frac{a+b+1}{2}\right)^{-a} \frac{\Gamma(n+a+1)}{n!} \left(\frac{\theta}{\sin\theta}\right)^{1/2} \times J_a\left(\left(n + \frac{a+b+1}{2}\right)\theta\right) + \theta^{1/2} O(n^{-3/2}), \tag{15}$$

where $0 < \theta < \pi$, we find that $\frac{1}{2N^2} K_N^{(2)}\left(1 - \frac{x}{2N^2}, 1 - \frac{y}{2N^2}\right)$ equals

$$\frac{\Gamma(N+1)\Gamma(N+a+b+1)}{(2N+a+b)\Gamma(N+a)\Gamma(N+b)(x-y)} \left\{\left[J_a\left(\frac{N+\frac{a+b-1}{2}}{N}\sqrt{x}\right) J_a\left(\frac{N+\frac{a+b+1}{2}}{N}\sqrt{y}\right) - J_a\left(\frac{N+\frac{a+b+1}{2}}{N}\sqrt{x}\right) J_a\left(\frac{N+\frac{a+b-1}{2}}{N}\sqrt{y}\right)\right] + O(N^{-2})\right\},$$

uniformly for x and y in compact subsets of $\mathbb{R}^+$. By writing the formula in the square brackets $[\cdots]$ as

$$J_a\left(\frac{N+\frac{a+b-1}{2}}{N}\sqrt{x}\right)\left(J_a\left(\frac{N+\frac{a+b+1}{2}}{N}\sqrt{y}\right) - J_a\left(\frac{N+\frac{a+b-1}{2}}{N}\sqrt{y}\right)\right) - J_a\left(\frac{N+\frac{a+b-1}{2}}{N}\sqrt{y}\right)\left(J_a\left(\frac{N+\frac{a+b+1}{2}}{N}\sqrt{x}\right) - J_a\left(\frac{N+\frac{a+b-1}{2}}{N}\sqrt{x}\right)\right),$$

we finally obtain the desired result by taking a large N limit together with the aid of Stirling's formula. □

Remark 4 When $x = y$, we have as $N \to \infty$,

$$\frac{1}{2N^2} K_N^{(2)}\left(1 - \frac{x}{2N^2}, 1 - \frac{x}{2N^2}\right) = K_{\text{Bessel}}^{(a)}(x, x) + O(N^{-1}),$$

where

$$K_{\text{Bessel}}^{(a)}(x,x) = \frac{(J_a(\sqrt{x}))^2 - J_{a+1}(\sqrt{x})J_{a-1}(\sqrt{x})}{4},$$

which is obtained by letting $y \to x$ in (14). The error term is uniform for x in compact subsets of $\mathbb{R}^+$. The Bessel kernel also arises in the Laguerre unitary ensemble when scaling at the hard edge of the spectrum [11]; see also [26].

Similarly as in the previous subsection, we use Theorem 3 to compute (6) term by term as $N \to \infty$, and replace $f(x)$ by $f(2N^2(1-x))$ in the computations. The first term reads

$$\begin{aligned}
\text{Tr} K_N^{(2)} f &= \int_{-1}^{1} K_N^{(2)}(x,x) f(2N^2(1-x))dx \\
&= \int_0^{4N^2} \frac{1}{2N^2} K_N^{(2)}\left(1 - \frac{x}{2N^2}, 1 - \frac{x}{2N^2}\right) f(x)dx \\
&= \int_0^{\infty} K_{\text{Bessel}}^{(a)}(x,x) f(x)dx + O(N^{-1}), \qquad N \to \infty.
\end{aligned}$$

Remark 5 We assume that $f(\cdot)$ is a continuous real-valued function belonging to $L^1(\mathbb{R}^+)$ and vanishes at $+\infty$.

The second term gives

$$\begin{aligned}
\text{Tr}\left(K_N^{(2)} f\right)^2 &= \int_{-1}^{1}\int_{-1}^{1} K_N^{(2)}(x,y) f(2N^2(1-y)) K_N^{(2)}(y,x) f(2N^2(1-x))dxdy \\
&= \int_0^{\infty}\int_0^{\infty} \left(K_{\text{Bessel}}^{(a)}(x,y)\right)^2 f(x)f(y)dxdy + O(N^{-1}), \quad N \to \infty.
\end{aligned}$$

It follows, again heuristically, from (6) that $\log\det\left(I + K_N^{(2)} f\right)$ equals

$$\begin{aligned}
&\int_0^{\infty} K_{\text{Bessel}}^{(a)}(x,x) f(x)dx - \frac{1}{2}\int_0^{\infty}\int_0^{\infty} \left(K_{\text{Bessel}}^{(a)}(x,y)\right)^2 f(x)f(y)dxdy \\
&+ \cdots + O(N^{-1}).
\end{aligned}$$

Substituting (11) into the above gives

$$\log\det\left(I+K_N^{(2)}f\right)=-\lambda\int_0^\infty K_{\text{Bessel}}^{(a)}(x,x)F(x)dx$$
$$+\frac{\lambda^2}{2}\Big[\int_0^\infty K_{\text{Bessel}}^{(a)}(x,x)F^2(x)dx-\int_0^\infty\int_0^\infty\left(K_{\text{Bessel}}^{(a)}(x,y)\right)^2F(x)F(y)dxdy\Big]$$
$$+\cdots+O(N^{-1}).$$

Then, the following theorem follows.

Theorem 4 *Let* $\tilde{\mu}_N^{(\text{JUE})}$ *and* $\tilde{\mathcal{V}}_N^{(\text{JUE})}$ *be the mean and variance of the scaled linear statistics* $\sum_{j=1}^N F(2N^2(1-x_j))$*, respectively. Then as* $N\to\infty$*,*

$$\tilde{\mu}_N^{(\text{JUE})}=\int_0^\infty K_{\text{Bessel}}^{(a)}(x,x)F(x)dx+O(N^{-1}),\tag{16}$$

$$\tilde{\mathcal{V}}_N^{(\text{JUE})}=\int_0^\infty K_{\text{Bessel}}^{(a)}(x,x)F^2(x)dx-\int_0^\infty\int_0^\infty\left(K_{\text{Bessel}}^{(a)}(x,y)\right)^2F(x)F(y)dxdy$$
$$+\,O(N^{-1}),\tag{17}$$

where $K_{\text{Bessel}}^{(a)}(x,y)$ *is the Bessel kernel defined by (14).*

Remark 6 The result of the above theorem is consistent with the one for the Laguerre unitary ensemble by scaling at the hard edge of the spectrum [3].

3 Jacobi Symplectic Ensemble (JSE)

3.1 Finite N Case for the MGF in JSE

In this case, $w(x)=(1-x)^a(1+x)^b,\quad x\in[-1,1],\ a,b>0$. The authors [21] expressed $G_N^{(4)}(f)$ as a Fredholm determinant based on the work of Dieng and Tracy [10] and Tracy and Widom [27]. Define

$$\psi_j^{(4)}(x):=\pi_j(x)\sqrt{w(x)},\quad j=0,1,2,\ldots,$$

where $\pi_j(x)$ is an arbitrary polynomial of degree j, and

$$M^{(4)}:=\left[\int_{-1}^1\left(\psi_j^{(4)}(x)\frac{d}{dx}\psi_k^{(4)}(x)-\psi_k^{(4)}(x)\frac{d}{dx}\psi_j^{(4)}(x)\right)dx\right]_{j,k=0}^{2N-1}$$

with its inverse denoted by

$$\left(M^{(4)}\right)^{-1} =: (\mu_{jk})_{j,k=0}^{2N-1}.$$

It was shown in [21] that

$$\left[G_N^{(4)}(f)\right]^2 = \det\left(I + 2K_N^{(4)} f - K_N^{(4)} \varepsilon f'\right), \tag{18}$$

where $K_N^{(4)}$ and ε are integral operators with kernel

$$K_N^{(4)}(x, y) := -\sum_{j,k=0}^{2N-1} \mu_{jk} \psi_j^{(4)}(x) \frac{d}{dy} \psi_k^{(4)}(y) \tag{19}$$

and

$$\varepsilon(x, y) := \frac{1}{2}\operatorname{sgn}(x - y),$$

respectively. We require that $f \in C^1[-1, 1]$ and vanishes at the endpoints ± 1.

Remark 7 If $g(x)$ is an integrable function on $[-1, 1]$, then

$$\varepsilon g(x) = \int_{-1}^{1} \varepsilon(x, y) g(y) dy = \frac{1}{2}\left(\int_{-1}^{x} g(y)dy - \int_{x}^{1} g(y)dy\right), \qquad x \in [-1, 1].$$

In addition, it is easy to see that $\varepsilon(y, x) = -\varepsilon(x, y)$, i.e., $\varepsilon^t = -\varepsilon$, where t denotes the transpose.

The fundamental theorem of calculus implies the following result [10]; see also [21].

Lemma 1 *Let D be the operator that acts by differentiation. Then for any function $g \in C^1[-1, 1]$ and $g(-1) = g(1) = 0$, we have $D\varepsilon g(x) = \varepsilon D g(x) = g(x)$, i.e., $D\varepsilon = \varepsilon D = I$.*

Similarly to the discussions in [10, 21, 27], we choose a special $\psi_j^{(4)}$ to simplify $M^{(4)}$ as much as possible. To proceed, let

$$\psi_{2j+1}^{(4)}(x) := \frac{1}{\sqrt{2}}(1 - x^2)\varphi_{2j+1}^{(4)}(x), \quad \psi_{2j}^{(4)}(x) := -\frac{1}{\sqrt{2}} \varepsilon \varphi_{2j+1}^{(4)}(x), \quad j = 0, 1, 2, \ldots,$$

where $\varphi_j^{(4)}(x)$ is given by

$$\varphi_j^{(4)}(x) = \frac{P_j^{(a-1,b-1)}(x)}{\sqrt{h_j^{(a-1,b-1)}}}(1-x)^{\frac{a}{2}-1}(1+x)^{\frac{b}{2}-1},$$

and $P_j^{(a-1,b-1)}(x)$, $j = 0, 1, \ldots$ are the usual Jacobi polynomials with the orthogonality condition

$$\int_{-1}^{1} P_j^{(a-1,b-1)}(x) P_k^{(a-1,b-1)}(x)(1-x)^{a-1}(1+x)^{b-1}dx = h_j^{(a-1,b-1)}\delta_{jk}.$$

It can be shown that $\psi_j^{(4)}(x)$ is equal to $(1-x)^{a/2}(1+x)^{b/2}$ multiplied by a polynomial of degree j. Similarly as the Laguerre symplectic ensemble case studied in [21, Theorem 3.10], $M^{(4)}$ is computed to be the direct sum of N copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by using the orthogonality, namely

$$M^{(4)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}_{2N\times 2N}.$$

It follows that $\left(M^{(4)}\right)^{-1} = -M^{(4)}$, so $\mu_{2j,2j+1} = -1$, $\mu_{2j+1,2j} = 1$, $j = 0, 1, \ldots, N-1$, and $\mu_{jk} = 0$ for other cases.

Lemma 2 *We have*

$$K_N^{(4)}(x, y) = \frac{1}{2}S_N^{(4)}(x, y) + \frac{1}{2}C_{2N}^{(4)}\varepsilon\varphi_{2N+1}^{(4)}(x)\varphi_{2N}^{(4)}(y), \tag{20}$$

where

$$C_{2N}^{(4)} = \sqrt{\frac{(2N+1)(2N+a)(2N+b)(2N+a+b-1)}{(4N+a+b+1)(4N+a+b-1)}}$$

and

$$S_N^{(4)}(x, y) = \sum_{j=0}^{2N}(1-x^2)\varphi_j^{(4)}(x)\varphi_j^{(4)}(y).$$

Proof From (19) we find that $K_N^{(4)}(x, y)$ equals

$$\sum_{j=0}^{N-1} \psi_{2j}(x)\psi'_{2j+1}(y) - \sum_{j=0}^{N-1} \psi_{2j+1}(x)\psi'_{2j}(y)$$
$$= \frac{1}{2}\sum_{j=0}^{N-1}(1-x^2)\varphi_{2j+1}^{(4)}(x)\varphi_{2j+1}^{(4)}(y) - \frac{1}{2}\sum_{j=0}^{N-1} \varepsilon\varphi_{2j+1}^{(4)}(x)\left[(1-y^2)\varphi_{2j+1}^{(4)}(y)\right]'.$$

By using the recurrence formulas for the Jacobi polynomials [25, Sec. 4.5]

$$(2n+a+b)(1-x^2)\frac{d}{dx}P_n^{(a,b)}(x)$$
$$= n\left[a-b-(2n+a+b)x\right]P_n^{(a,b)}(x) + 2(n+a)(n+b)P_{n-1}^{(a,b)}(x) \quad (21)$$

and

$$(2n+a+b+1)[(2n+a+b)(2n+a+b+2)x+a^2-b^2]P_n^{(a,b)}(x)$$
$$= 2(n+1)(n+a+b+1)(2n+a+b)P_{n+1}^{(a,b)}(x)$$
$$+ 2(n+a)(n+b)(2n+a+b+2)P_{n-1}^{(a,b)}(x), \quad (22)$$

we obtain

$$\left[(1-y^2)\varphi_{2j+1}^{(4)}(y)\right]' = C_{2j}^{(4)}\varphi_{2j}^{(4)}(y) - C_{2j+1}^{(4)}\varphi_{2j+2}^{(4)}(y),$$

where

$$C_j^{(4)} := \sqrt{\frac{(j+1)(j+a)(j+b)(j+a+b-1)}{(2j+a+b+1)(2j+a+b-1)}}.$$

It follows that

$$K_N^{(4)}(x, y) = \frac{1}{2}\sum_{j=0}^{N-1}(1-x^2)\varphi_{2j+1}^{(4)}(x)\varphi_{2j+1}^{(4)}(y)$$
$$+ \frac{1}{2}\sum_{j=0}^{N}\left[C_{2j-1}^{(4)}\varepsilon\varphi_{2j-1}^{(4)}(x) - C_{2j}^{(4)}\varepsilon\varphi_{2j+1}^{(4)}(x)\right]\varphi_{2j}^{(4)}(y)$$
$$+ \frac{1}{2}C_{2N}^{(4)}\varepsilon\varphi_{2N+1}^{(4)}(x)\varphi_{2N}^{(4)}(y). \quad (23)$$

Using (21) and (22) again, we find

$$\left[(1-x^2)\varphi_{2j}^{(4)}(x)\right]' = C_{2j-1}^{(4)}\varphi_{2j-1}^{(4)}(x) - C_{2j}^{(4)}\varphi_{2j+1}^{(4)}(x).$$

Then from Lemma 1 we have

$$(1-x^2)\varphi_{2j}^{(4)}(x) = \varepsilon\left[(1-x^2)\varphi_{2j}^{(4)}(x)\right]' = C_{2j-1}^{(4)}\varepsilon\varphi_{2j-1}^{(4)}(x) - C_{2j}^{(4)}\varepsilon\varphi_{2j+1}^{(4)}(x). \tag{24}$$

The combination of (23) and (24) gives

$$\begin{aligned} K_N^{(4)}(x,y) &= \frac{1}{2}\sum_{j=0}^{N-1}(1-x^2)\varphi_{2j+1}^{(4)}(x)\varphi_{2j+1}^{(4)}(y) + \frac{1}{2}\sum_{j=0}^{N}(1-x^2)\varphi_{2j}^{(4)}(x)\varphi_{2j}^{(4)}(y) \\ &\quad + \frac{1}{2}C_{2N}^{(4)}\varepsilon\varphi_{2N+1}^{(4)}(x)\varphi_{2N}^{(4)}(y) \\ &= \frac{1}{2}\sum_{j=0}^{2N}(1-x^2)\varphi_j^{(4)}(x)\varphi_j^{(4)}(y) + \frac{1}{2}C_{2N}^{(4)}\varepsilon\varphi_{2N+1}^{(4)}(x)\varphi_{2N}^{(4)}(y). \end{aligned}$$

The proof is complete. □

Theorem 5 *For the Jacobi symplectic ensemble, we have*

$$\left[G_N^{(4)}(f)\right]^2 = \det(I + T_{\mathrm{JSE}}), \tag{25}$$

where

$$T_{\mathrm{JSE}} := S_N^{(4)}f - \frac{1}{2}S_N^{(4)}\varepsilon f' + C_{2N}^{(4)}\left(\varepsilon\varphi_{2N+1}^{(4)}\right)\otimes\varphi_{2N}^{(4)}f + \frac{1}{2}C_{2N}^{(4)}\left(\varepsilon\varphi_{2N+1}^{(4)}\right)\otimes\left(\varepsilon\varphi_{2N}^{(4)}\right)f'.$$

Proof Substituting $K_N^{(4)}$ with the kernel given by (20) into (18), we obtain the desired result with the aid of the property $(u\otimes v)A = u\otimes(A^t v)$ for integral operators. □

Finally we mention the following expansion formula

$$\log\det(I+T_{\mathrm{JSE}}) = \mathrm{Tr}\log(I+T_{\mathrm{JSE}}) = \mathrm{Tr}T_{\mathrm{JSE}} - \frac{1}{2}\mathrm{Tr}T_{\mathrm{JSE}}^2 + \frac{1}{3}\mathrm{Tr}T_{\mathrm{JSE}}^3 - \cdots, \tag{26}$$

which will be used in the asymptotic analysis in the next subsections.

3.2 Scaling in the Bulk of the Spectrum in JSE

Similarly as Theorem 1, we have the following theorem.

Theorem 6 *For $x, y \in \mathbb{R}$, we have as $N \to \infty$,*

$$\frac{1}{2N} S_N^{(4)}\left(\frac{x}{2N}, \frac{y}{2N}\right) = K_{\mathrm{sine}}(x, y) + O(N^{-1}),$$

where $K_{\mathrm{sine}}(x, y)$ is the sine kernel given by (7). The error term is uniform for x and y in compact subsets of $\mathbb{R}$.

Theorem 7 *For $x \in \mathbb{R}$, we have as $N \to \infty$,*

$$\varphi_{2N}^{(4)}\left(\frac{x}{2N}\right) = \sqrt{\frac{2}{\pi}} \sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(4N - 1 + a + b) \arccos \frac{x}{2N}\right)\right] + O(N^{-1}),$$

$$\varphi_{2N+1}^{(4)}\left(\frac{x}{2N}\right) = \sqrt{\frac{2}{\pi}} \sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(4N + 1 + a + b) \arccos \frac{x}{2N}\right)\right] + O(N^{-1}),$$

$$\begin{aligned}\varepsilon\varphi_{2N}^{(4)}\left(\frac{x}{2N}\right) = \frac{1}{2N\sqrt{2\pi}} \Bigg\{ &\sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(4N + 1 + a + b) \arccos \frac{x}{2N}\right)\right] \\ &- \sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(4N - 3 + a + b) \arccos \frac{x}{2N}\right)\right]\Bigg\} + O(N^{-2}),\end{aligned}$$

$$\begin{aligned}\varepsilon\varphi_{2N+1}^{(4)}\left(\frac{x}{2N}\right) = \frac{1}{2N\sqrt{2\pi}} \Bigg\{ &\sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(4N + 3 + a + b) \arccos \frac{x}{2N}\right)\right] \\ &- \sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(4N - 1 + a + b) \arccos \frac{x}{2N}\right)\right]\Bigg\} + O(N^{-2}).\end{aligned}$$

The error terms are uniform for x in compact subsets of $\mathbb{R}$.

Proof By using the asymptotic formula of the Jacobi polynomials (9), we obtain the desired results after direct calculations. □

Remark 8 It is easy to see from the above theorem that as $N \to \infty$,

$$\varphi_{2N}^{(4)}\left(\frac{x}{2N}\right) = O(1), \qquad \varphi_{2N+1}^{(4)}\left(\frac{x}{2N}\right) = O(1),$$

$$\varepsilon\varphi_{2N}^{(4)}\left(\frac{x}{2N}\right) = O(N^{-1}), \qquad \varepsilon\varphi_{2N+1}^{(4)}\left(\frac{x}{2N}\right) = O(N^{-1}),$$

uniformly for x in compact subsets of $\mathbb{R}$.

We now use Theorems 6 and 7 to compute (26) as $N \to \infty$, and change $f(x)$ to $f(2Nx)$ in the calculations (in this case $f'(x)$ becomes $2Nf'(2Nx)$). We first consider $\mathrm{Tr}\, T_{\mathrm{JSE}}$:

$$\mathrm{Tr}\, T_{\mathrm{JSE}} = \mathrm{Tr}\, S_N^{(4)} f - \mathrm{Tr}\, \frac{1}{2} S_N^{(4)} \varepsilon f' + \mathrm{Tr}\, C_{2N}^{(4)} \left(\varepsilon \varphi_{2N+1}^{(4)}\right) \otimes \varphi_{2N}^{(4)} f$$
$$+ \mathrm{Tr}\, \frac{1}{2} C_{2N}^{(4)} \left(\varepsilon \varphi_{2N+1}^{(4)}\right) \otimes \left(\varepsilon \varphi_{2N}^{(4)}\right) f'. \tag{27}$$

The first term gives

$$\begin{aligned}
\mathrm{Tr}\, S_N^{(4)} f &= \int_{-1}^{1} S_N^{(4)}(x,x) f(2Nx) dx \\
&= \int_{-2N}^{2N} \frac{1}{2N} S_N^{(4)} \left(\frac{x}{2N}, \frac{x}{2N}\right) f(x) dx \\
&= \int_{-\infty}^{\infty} K_{\mathrm{sine}}(x,x) f(x) dx + O(N^{-1}), \qquad N \to \infty.
\end{aligned}$$

Remark 9 We assume that $f(\cdot)$ is smooth and sufficiently decreasing at $\pm\infty$ to make the integrals well-defined.

It can be shown that the rest terms have contributions of $O(N^{-1})$, where we have used the fact

$$\int_x^{\infty} K_{\mathrm{sine}}(x,y) dy - \int_{-\infty}^{x} K_{\mathrm{sine}}(x,y) dy = 0$$

in the calculation of $\mathrm{Tr}\, \frac{1}{2} S_N^{(4)} \varepsilon f'$. Hence,

$$\mathrm{Tr}\, T_{\mathrm{JSE}} = \int_{-\infty}^{\infty} K_{\mathrm{sine}}(x,x) f(x) dx + O(N^{-1}). \tag{28}$$

Next, we consider $\mathrm{Tr}\, T_{\mathrm{JSE}}^2$:

$$\begin{aligned}
\mathrm{Tr}\, T_{\mathrm{JSE}}^2 = {} & \mathrm{Tr} \left(S_N^{(4)} f\right)^2 - \mathrm{Tr}\, S_N^{(4)} f S_N^{(4)} \varepsilon f' + \mathrm{Tr}\, 2 C_{2N}^{(4)} S_N^{(4)} f \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varphi_{2N}^{(4)}\right) f \\
& + \mathrm{Tr}\, C_{2N}^{(4)} S_N^{(4)} f \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varepsilon \varphi_{2N}^{(4)}\right) f' + \mathrm{Tr}\, \frac{1}{4} \left(S_N^{(4)} \varepsilon f'\right)^2 \\
& - \mathrm{Tr}\, C_{2N}^{(4)} S_N^{(4)} \varepsilon f' \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varphi_{2N}^{(4)}\right) f - \mathrm{Tr}\, \frac{1}{2} C_{2N}^{(4)} S_N^{(4)} \varepsilon f' \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varepsilon \varphi_{2N}^{(4)}\right) f' \\
& + \mathrm{Tr} \left(C_{2N}^{(4)}\right)^2 \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varphi_{2N}^{(4)} f\right)^2 + \mathrm{Tr}\, \frac{1}{4} \left(C_{2N}^{(4)}\right)^2 \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varepsilon \varphi_{2N}^{(4)} f'\right)^2 \\
& + \mathrm{Tr} \left(C_{2N}^{(4)}\right)^2 \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varphi_{2N}^{(4)} f\right) \left(\varepsilon \varphi_{2N+1}^{(4)} \otimes \varepsilon \varphi_{2N}^{(4)} f'\right).
\end{aligned} \tag{29}$$

We find as $N \to \infty$,

$$\mathrm{Tr}\left(S_N^{(4)} f\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\mathrm{sine}}^2(x, y) f(x) f(y) dx dy + O(N^{-1}),$$

$$\mathrm{Tr}\, S_N^{(4)} f S_N^{(4)} \varepsilon f' = -\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Si}^2(x-y) f'(x) f'(y) dx dy + O(N^{-1}),$$

$$\mathrm{Tr}\,\frac{1}{4}\left(S_N^{(4)} \varepsilon f'\right)^2 = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Si}^2(x-y) f'(x) f'(y) dx dy + O(N^{-1}),$$

where we have used integration by parts to obtain the second equality and the formula

$$\int_{-\infty}^{x} K_{\mathrm{sine}}(y, z) dz - \int_{x}^{\infty} K_{\mathrm{sine}}(y, z) dz = \frac{2}{\pi}\,\mathrm{Si}(x-y),$$

and $\mathrm{Si}(x)$ is the sine integral defined by

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t} dt,$$

which can be found in Lebedev's famous book [17, Sec. 3.3].

The rest terms in (29) are proven to have contributions of $O(N^{-1})$ after some elaborate computations. Hence,

$$\mathrm{Tr}\, T_{\mathrm{JSE}}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\mathrm{sine}}^2(x, y) f(x) f(y) dx dy + \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Si}^2(x-y) f'(x) f'(y) dx dy + O(N^{-1}). \tag{30}$$

Proceeding as in the JUE case, we substitute (11) into (28) and (30) and finally heuristically obtain from (26) that

$$\begin{aligned}\log\det(I + T_{\mathrm{JSE}}) = -\lambda \int_{-\infty}^{\infty} K_{\mathrm{sine}}(x, x) F(x) dx + \frac{\lambda^2}{2}\Big[\int_{-\infty}^{\infty} K_{\mathrm{sine}}(x, x) F^2(x) dx \\ - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\mathrm{sine}}^2(x, y) F(x) F(y) dx dy \\ - \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Si}^2(x-y) F'(x) F'(y) dx dy\Big] + \cdots + O(N^{-1}).\end{aligned}$$

Since we have $\log G_N^{(4)}(f) = \frac{1}{2}\log\det(I + T_{\mathrm{JSE}})$ from (25), the following theorem follows.

Theorem 8 *Let $\mu_N^{\text{(JSE)}}$ and $\mathcal{V}_N^{\text{(JSE)}}$ be the mean and variance of the scaled linear statistics $\sum_{j=1}^N F(2Nx_j)$. We have as $N \to \infty$,*

$$\mu_N^{\text{(JSE)}} = \frac{1}{2}\mu_N^{\text{(JUE)}} + O(N^{-1}),$$

$$\mathcal{V}_N^{\text{(JSE)}} = \frac{1}{2}\mathcal{V}_N^{\text{(JUE)}} - \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Si}^2(x-y)F'(x)F'(y)dxdy + O(N^{-1}),$$

where $\mu_N^{\text{(JUE)}}$ and $\mathcal{V}_N^{\text{(JUE)}}$ are given by (12) and (13), respectively.

3.3 Scaling at the Edge of the Spectrum in JSE

Similarly as Theorem 3, we have the following result.

Theorem 9 *For $x, y \in \mathbb{R}^+$, we have as $N \to \infty$,*

$$\frac{1}{8N^2}S_N^{(4)}\left(1-\frac{x}{8N^2}, 1-\frac{y}{8N^2}\right) = \sqrt{\frac{x}{y}}\,K_{\text{Bessel}}^{(a-1)}(x,y) + O(N^{-1}),$$

where $K_{\text{Bessel}}^{(a-1)}(x,y)$ is the Bessel kernel of order $a-1$ given by

$$K_{\text{Bessel}}^{(a-1)}(x,y) = \frac{J_{a-1}(\sqrt{x})\sqrt{y}J'_{a-1}(\sqrt{y}) - J'_{a-1}(\sqrt{x})\sqrt{x}J_{a-1}(\sqrt{y})}{2(x-y)}.$$

The error term is uniform for x and y in compact subsets of $\mathbb{R}^+$.

Theorem 10 *For $x \in \mathbb{R}^+$, we have as $N \to \infty$,*

$$\varphi_{2N}^{(4)}\left(1-\frac{x}{8N^2}\right) = (2N)^{3/2}\,\frac{J_{a-1}(\sqrt{x})}{\sqrt{x}} + O(N^{1/2}),$$

$$\varphi_{2N+1}^{(4)}\left(1-\frac{x}{8N^2}\right) = (2N)^{3/2}\,\frac{J_{a-1}(\sqrt{x})}{\sqrt{x}} + O(N^{1/2}),$$

$$\varepsilon\varphi_{2N}^{(4)}\left(1-\frac{x}{8N^2}\right) = 2^{-3/2}N^{-1/2}\left(1-2\mathbf{J}_{a-1}(\sqrt{x})\right) + O(N^{-3/2}),$$

$$\varepsilon\varphi_{2N+1}^{(4)}\left(1-\frac{x}{8N^2}\right) = 2^{-3/2}N^{-1/2}\left(1-2\mathbf{J}_{a-1}(\sqrt{x})\right) + O(N^{-3/2}),$$

where

$$\mathbf{J}_{a-1}(x) := \int_0^x J_{a-1}(t)dt. \tag{31}$$

The error terms are uniform for x in compact subsets of $\mathbb{R}^+$.

Proof The results come from direct computations by using the large n Hilb-type asymptotic formula of the Jacobi polynomials (15), and the formula

$$\int_x^\infty \frac{J_{a-1}(\sqrt{y})}{\sqrt{y}}dy - \int_0^x \frac{J_{a-1}(\sqrt{y})}{\sqrt{y}}dy = 2\left(1 - 2\mathbf{J}_{a-1}(\sqrt{x})\right),$$

where use has been made of the fact that $\int_0^\infty J_{a-1}(t)dt = 1$ (see, e.g., [14, p. 659]). □

Using Theorems 9 and 10 to compute (27) term by term and changing $f(x)$ to $f(8N^2(1-x))$, we find

$$\mathrm{Tr}\, S_N^{(4)} f = \int_0^\infty K_{\mathrm{Bessel}}^{(a-1)}(x,x) f(x)dx + O(N^{-1}),$$

$$\mathrm{Tr}\, \frac{1}{2} S_N^{(4)} \varepsilon f' = -\frac{1}{4}\int_0^\infty L^{(a-1)}(x,x) f'(x)dx + O(N^{-1}),$$

$$\mathrm{Tr}\, C_{2N}^{(4)}\left(\varepsilon\varphi_{2N+1}^{(4)}\right) \otimes \varphi_{2N}^{(4)} f = \frac{1}{8}\int_0^\infty \frac{J_{a-1}(\sqrt{x})}{\sqrt{x}}\left(1 - 2\mathbf{J}_{a-1}(\sqrt{x})\right) f(x)dx + O(N^{-1}),$$

$$\mathrm{Tr}\, \frac{1}{2} C_{2N}^{(4)}\left(\varepsilon\varphi_{2N+1}^{(4)}\right) \otimes \left(\varepsilon\varphi_{2N}^{(4)}\right) f' = -\frac{1}{16}\int_0^\infty \left(1 - 2\mathbf{J}_{a-1}(\sqrt{x})\right)^2 f'(x)dx + O(N^{-1}),$$

where

$$L^{(a-1)}(x,y) := \int_0^x \sqrt{\frac{y}{z}}\, K_{\mathrm{Bessel}}^{(a-1)}(y,z)dz - \int_x^\infty \sqrt{\frac{y}{z}}\, K_{\mathrm{Bessel}}^{(a-1)}(y,z)dz. \tag{32}$$

Remark 10 We assume that $f(\cdot)$ is smooth and sufficiently decreasing at infinity to make the integrals well-defined.

Through integration by parts, we have the formula

$$\int_0^\infty \left(1 - 2\mathbf{J}_{a-1}(\sqrt{x})\right)^2 f'(x)dx = 2\int_0^\infty \frac{J_{a-1}(\sqrt{x})}{\sqrt{x}}\left(1 - 2\mathbf{J}_{a-1}(\sqrt{x})\right) f(x)dx.$$

It follows that

$$\mathrm{Tr}\, T_{\mathrm{JSE}} = \int_0^\infty K_{\mathrm{Bessel}}^{(a-1)}(x,x) f(x) dx + \frac{1}{4}\int_0^\infty L^{(a-1)}(x,x) f'(x) dx + O(N^{-1}). \tag{33}$$

Similarly, we obtain from (29) by changing $f(x)$ to $f(8N^2(1-x))$ that Tr T_{JSE}^2 equals

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left(K_{\mathrm{Bessel}}^{(a-1)}(x,y)\right)^2 f(x) f(y) dx dy \\
&+\frac{1}{2}\int_0^\infty \int_0^\infty \sqrt{\frac{x}{y}}\, K_{\mathrm{Bessel}}^{(a-1)}(x,y) L^{(a-1)}(x,y) f'(x) f(y) dx dy \\
&+\frac{1}{4}\int_0^\infty \int_0^\infty \frac{J_{a-1}(\sqrt{x})}{\sqrt{y}} K_{\mathrm{Bessel}}^{(a-1)}(x,y) \left(1-2\mathbf{J}_{a-1}(\sqrt{x})\right) f(x) f(y) dx dy \\
&+\frac{1}{16}\int_0^\infty \int_0^\infty L^{(a-1)}(x,y) L^{(a-1)}(y,x) f'(x) f'(y) dx dy \\
&-\frac{1}{16}\int_0^\infty \int_0^\infty \frac{J_{a-1}(\sqrt{x})}{\sqrt{x}} \left(1-2\mathbf{J}_{a-1}(\sqrt{y})\right) \left(L^{(a-1)}(x,y) - L^{(a-1)}(y,x)\right) f(x) f'(y) dx dy \\
&+\frac{1}{32}\int_0^\infty \int_0^\infty \left(1-2\mathbf{J}_{a-1}(\sqrt{x})\right)\left(1-2\mathbf{J}_{a-1}(\sqrt{y})\right) L^{(a-1)}(x,y) f'(x) f'(y) dx dy \\
&+O(N^{-1}),
\end{aligned} \tag{34}$$

where we have used integration by parts to simplify the results.

Similarly as in Sect. 3.2, by substituting (11) into (33) and (34) and using the fact that $\log G_N^{(4)}(f) = \frac{1}{2}\log\det(I + T_{\mathrm{JSE}})$, we heuristically obtain the following theorem.

Theorem 11 *Denoting by $\tilde{\mu}_N^{(\mathrm{JSE})}$ and $\tilde{\mathcal{V}}_N^{(\mathrm{JSE})}$ the mean and variance of the linear statistics $\sum_{j=1}^N F(8N^2(1-x_j))$, we have as $N \to \infty$,*

$$\tilde{\mu}_N^{(\mathrm{JSE})} = \frac{1}{2}\tilde{\mu}_N^{(\mathrm{JUE},a-1)} + \frac{1}{8}\int_0^\infty L^{(a-1)}(x,x) F'(x) dx + O(N^{-1}),$$

$$\begin{aligned}
\tilde{\mathcal{V}}_N^{(\mathrm{JSE})} = \frac{1}{2}\tilde{\mathcal{V}}_N^{(\mathrm{JUE},a-1)} &- \frac{1}{4}\int_0^\infty \int_0^\infty \sqrt{\frac{x}{y}}\, K_{\mathrm{Bessel}}^{(a-1)}(x,y) L^{(a-1)}(x,y) F'(x) F(y) dx dy \\
&- \frac{1}{8}\int_0^\infty \int_0^\infty \frac{J_{a-1}(\sqrt{x})}{\sqrt{y}} K_{\mathrm{Bessel}}^{(a-1)}(x,y) \left(1-2\mathbf{J}_{a-1}(\sqrt{x})\right) F(x) F(y) dx dy \\
&+ \frac{1}{32}\int_0^\infty \int_0^\infty \frac{J_{a-1}(\sqrt{x})}{\sqrt{x}} \left(1-2\mathbf{J}_{a-1}(\sqrt{y})\right) \left(L^{(a-1)}(x,y) - L^{(a-1)}(y,x)\right) \\
&\quad F(x) F'(y) dx dy - \frac{1}{32}\int_0^\infty \int_0^\infty L^{(a-1)}(x,y) L^{(a-1)}(y,x) F'(x) F'(y) dx dy
\end{aligned}$$

$$- \frac{1}{64} \int_0^\infty \int_0^\infty \left(1 - 2\mathbf{J}_{a-1}(\sqrt{x})\right) \left(1 - 2\mathbf{J}_{a-1}(\sqrt{y})\right) L^{(a-1)}(x, y)$$
$$F'(x)F'(y)dxdy + \frac{1}{4} \int_0^\infty L^{(a-1)}(x, x)F(x)F'(x)dx + O(N^{-1}),$$

where $\tilde{\mu}_N^{(\mathrm{JUE},a-1)}$ and $\tilde{\mathcal{V}}_N^{(\mathrm{JUE},a-1)}$ are given by (16) and (17) with a replaced by $a-1$, and $\mathbf{J}_{a-1}(x)$ and $L^{(a-1)}(x, y)$ are defined by (31) and (32) respectively.

4 Jacobi Orthogonal Ensemble (JOE)

4.1 Finite N Case for the MGF in JOE

In the JOE case, we take the weight $w(x)$ to be $w(x) = (1-x)^{a/2}(1+x)^{b/2}, \;\; x \in [-1, 1], \;\; a, b > -2$ for convenience. We assume that N is even. The authors [21] expressed $G_N^{(1)}(f)$ as a Fredholm determinant based on [10, 27]. Let

$$\psi_j^{(1)}(x) := \pi_j(x)w(x), \quad j = 0, 1, 2, \ldots,$$

where $\pi_j(x)$ is an arbitrary polynomial of degree j, and

$$M^{(1)} := \left(\int_{-1}^{1} \psi_j^{(1)}(x)\varepsilon\psi_k^{(1)}(x)dx \right)_{j,k=0}^{N-1}$$

with its inverse denoted by

$$\left(M^{(1)}\right)^{-1} =: (\nu_{jk})_{j,k=0}^{N-1}.$$

It was shown in [21] that

$$\left[G_N^{(1)}(f)\right]^2 = \det\left(I + K_N^{(1)}(f^2 + 2f) - K_N^{(1)}\varepsilon f' - K_N^{(1)} f\varepsilon f'\right), \tag{35}$$

where $K_N^{(1)}$ is the integral operator with kernel

$$K_N^{(1)}(x, y) := \sum_{j,k=0}^{N-1} \nu_{jk}\varepsilon\psi_j^{(1)}(x)\psi_k^{(1)}(y). \tag{36}$$

We also require that $f \in C^1[-1, 1]$ and vanishes at the endpoints ± 1.

Similarly as the discussions in Sect. 3.1, we can choose a special $\psi_j^{(1)}$ to simplify $M^{(1)}$ as the direct sum of $N/2$ copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let

$$\psi_{2j+1}^{(1)}(x) = \frac{d}{dx}\left[(1-x^2)\varphi_{2j}^{(1)}(x)\right], \qquad \psi_{2j}^{(1)}(x) = \varphi_{2j}^{(1)}(x), \quad j = 0, 1, 2, \ldots,$$

where $\varphi_j^{(1)}(x)$ is given by

$$\varphi_j^{(1)}(x) = \frac{P_j^{(a+1,b+1)}(x)}{\sqrt{h_j^{(a+1,b+1)}}}(1-x)^{a/2}(1+x)^{b/2},$$

and $P_j^{(a+1,b+1)}(x)$, $j = 0, 1, \ldots$ are the Jacobi polynomials with the orthogonality condition

$$\int_{-1}^{1} P_j^{(a+1,b+1)}(x) P_k^{(a+1,b+1)}(x)(1-x)^{a+1}(1+x)^{b+1} dx = h_j^{(a+1,b+1)} \delta_{jk}.$$

It is easy to see that $\psi_j^{(1)}(x)$ is equal to $(1-x)^{a/2}(1+x)^{b/2}$ multiplied by a polynomial of degree j. Moreover, $M^{(1)}$ is computed to be

$$M^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}_{N \times N}.$$

It follows that $\left(M^{(1)}\right)^{-1} = -M^{(1)}$, so $\nu_{2j,2j+1} = -1$, $\nu_{2j+1,2j} = 1$ and $\nu_{jk} = 0$ for other cases.

Lemma 3 *We have*

$$K_N^{(1)}(x,y) = S_N^{(1)}(x,y) + C_N^{(1)} \varepsilon \varphi_N^{(1)}(x) \varphi_{N-1}^{(1)}(y), \tag{37}$$

where

$$C_N^{(1)} = \sqrt{\frac{N(N+a+1)(N+b+1)(N+a+b+2)}{(2N+a+b+1)(2N+a+b+3)}}$$

and

$$S_N^{(1)}(x,y)=\sum_{j=0}^{N-1}(1-x^2)\varphi_j^{(1)}(x)\varphi_j^{(1)}(y).$$

Proof According to (36), we find that $K_N^{(1)}(x,y)$ equals

$$\sum_{j=0}^{\frac{N}{2}-1}\varepsilon\psi_{2j+1}^{(1)}(x)\psi_{2j}^{(1)}(y)-\sum_{j=0}^{\frac{N}{2}-1}\varepsilon\psi_{2j}^{(1)}(x)\psi_{2j+1}^{(1)}(y)$$
$$=\sum_{j=0}^{\frac{N}{2}-1}(1-x^2)\varphi_{2j}^{(1)}(x)\varphi_{2j}^{(1)}(y)-\sum_{j=0}^{\frac{N}{2}-1}\varepsilon\varphi_{2j}^{(1)}(x)\left[(1-y^2)\varphi_{2j}^{(1)}(y)\right]'. \tag{38}$$

In view of the recurrence formulas for the Jacobi polynomials (21) and (22), we find

$$\left[(1-y^2)\varphi_{2j}^{(1)}(y)\right]'=C_{2j}^{(1)}\varphi_{2j-1}^{(1)}(y)-C_{2j+1}^{(1)}\varphi_{2j+1}^{(1)}(y),$$

where

$$C_j^{(1)}:=\sqrt{\frac{j(j+a+1)(j+b+1)(j+a+b+2)}{(2j+a+b+1)(2j+a+b+3)}}.$$

Then (38) becomes

$$K_N^{(1)}(x,y)=\sum_{j=0}^{\frac{N}{2}-1}(1-x^2)\varphi_{2j}^{(1)}(x)\varphi_{2j}^{(1)}(y)+C_N^{(1)}\varepsilon\varphi_N^{(1)}(x)\varphi_{N-1}^{(1)}(y)$$
$$+\sum_{j=1}^{\frac{N}{2}}\left[C_{2j-1}^{(1)}\varepsilon\varphi_{2j-2}^{(1)}(x)-C_{2j}^{(1)}\varepsilon\varphi_{2j}^{(1)}(x)\right]\varphi_{2j-1}^{(1)}(y). \tag{39}$$

Using (21) and (22) again, we have

$$\left[(1-x^2)\varphi_{2j-1}^{(1)}(x)\right]'=C_{2j-1}^{(1)}\varphi_{2j-2}^{(1)}(x)-C_{2j}^{(1)}\varphi_{2j}^{(1)}(x).$$

It follows from Lemma 1 that

$$(1-x^2)\varphi_{2j-1}^{(1)}(x)=\varepsilon\left[(1-x^2)\varphi_{2j-1}^{(1)}(x)\right]'=C_{2j-1}^{(1)}\varepsilon\varphi_{2j-2}^{(1)}(x)-C_{2j}^{(1)}\varepsilon\varphi_{2j}^{(1)}(x). \tag{40}$$

The combination of (39) and (40) produces

$$K_N^{(1)}(x,y) = \sum_{j=0}^{\frac{N}{2}-1}(1-x^2)\varphi_{2j}^{(1)}(x)\varphi_{2j}^{(1)}(y) + \sum_{j=1}^{\frac{N}{2}}(1-x^2)\varphi_{2j-1}^{(1)}(x)\varphi_{2j-1}^{(1)}(y)$$
$$+ C_N^{(1)}\varepsilon\varphi_N^{(1)}(x)\varphi_{N-1}^{(1)}(y)$$
$$= \sum_{j=0}^{N-1}(1-x^2)\varphi_j^{(1)}(x)\varphi_j^{(1)}(y) + C_N^{(1)}\varepsilon\varphi_N^{(1)}(x)\varphi_{N-1}^{(1)}(y).$$

The theorem is then established. □

Theorem 12 *For the Jacobi orthogonal ensemble, we have*

$$\left[G_N^{(1)}(f)\right]^2 = \det(I + T_{\text{JOE}}),$$

where

$$T_{\text{JOE}} := S_N^{(1)}(f^2+2f) - S_N^{(1)}\varepsilon f' - S_N^{(1)} f\varepsilon f' + C_N^{(1)}\left(\varepsilon\varphi_N^{(1)}\right)\otimes\varphi_{N-1}^{(1)}(f^2+2f)$$
$$+ C_N^{(1)}\left(\varepsilon\varphi_N^{(1)}\right)\otimes\left(\varepsilon\varphi_{N-1}^{(1)}\right)f' - C_N^{(1)}\left(\varepsilon\varphi_N^{(1)}\right)\otimes\varphi_{N-1}^{(1)} f\varepsilon f'. \tag{41}$$

Proof Substituting (37) into (35), we obtain the desired result. □

The amenable expression of $G_N^{(1)}(f)$ in the above theorem will allow us to study its large N asymptotics in the next subsections by using the expansion formula

$$\log\det(I+T_{\text{JOE}}) = \text{Tr}\log(I+T_{\text{JOE}}) = \text{Tr}T_{\text{JOE}} - \frac{1}{2}\text{Tr}T_{\text{JOE}}^2 + \frac{1}{3}\text{Tr}T_{\text{JOE}}^3 - \cdots. \tag{42}$$

4.2 Scaling in the Bulk of the Spectrum in JOE

Similarly as Theorems 6 and 7, we have the following two theorems.

Theorem 13 *For $x, y \in \mathbb{R}$, we have as $N \to \infty$,*

$$\frac{1}{N}S_N^{(1)}\left(\frac{x}{N}, \frac{y}{N}\right) = K_{\text{sine}}(x,y) + O(N^{-1}),$$

uniformly for x and y in compact subsets of $\mathbb{R}$.

Theorem 14 *For $x \in \mathbb{R}$, we have as $N \to \infty$,*

$$\varphi_N^{(1)}\left(\frac{x}{N}\right) = -\sqrt{\frac{2}{\pi}}\sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(2N+3+a+b)\arccos\frac{x}{N}\right)\right] + O(N^{-1}),$$

$$\varphi_{N-1}^{(1)}\left(\frac{x}{N}\right) = -\sqrt{\frac{2}{\pi}}\sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(2N+1+a+b)\arccos\frac{x}{N}\right)\right] + O(N^{-1}),$$

$$\begin{aligned}\varepsilon\varphi_N^{(1)}\left(\frac{x}{N}\right) = \frac{1}{N\sqrt{2\pi}}\Bigg\{ &\sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(2N+1+a+b)\arccos\frac{x}{N}\right)\right] \\ &- \sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(2N+5+a+b)\arccos\frac{x}{N}\right)\right]\Bigg\} + O(N^{-2}),\end{aligned}$$

$$\begin{aligned}\varepsilon\varphi_{N-1}^{(1)}\left(\frac{x}{N}\right) = \frac{1}{N\sqrt{2\pi}}\Bigg\{ &\sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(2N-1+a+b)\arccos\frac{x}{N}\right)\right] \\ &- \sin\left[\frac{1}{4}\left(\pi + 2\pi a - 2(2N+3+a+b)\arccos\frac{x}{N}\right)\right]\Bigg\} + O(N^{-2}).\end{aligned}$$

The error terms are uniform for x in compact subsets of $\mathbb{R}$.

Using Theorems 13 and 14 to compute $\operatorname{Tr} T_{\mathrm{JOE}}$ and $\operatorname{Tr} T_{\mathrm{JOE}}^2$ from (41), and changing $f(x)$ to $f(Nx)$, we find that $\operatorname{Tr} T_{\mathrm{JOE}}$ and $\operatorname{Tr} T_{\mathrm{JOE}}^2$ equal

$$\begin{aligned}&\int_{-\infty}^{\infty} K_{\mathrm{sine}}(x,x)\left(f^2(x) + 2f(x)\right)dx \\ &- \frac{1}{2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1 - 2\chi_{(-\infty,x)}(y)\right)K_{\mathrm{sine}}(x,y)f(y)dy\right]f'(x)dx + O(N^{-1})\end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\mathrm{sine}}^2(x,y)\left(f^2(x)+2f(x)\right)\left(f^2(y)+2f(y)\right)dxdy \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\mathrm{sine}}(x,y)\left[\int_{-\infty}^{\infty}\left(1-2\chi_{(-\infty,x)}(z)\right)K_{\mathrm{sine}}(y,z)f(z)dz\right] \\ &f'(x)\left(f^2(y)+2f(y)\right)dxdy + \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{Si}^2(x-y)f'(x)f'(y)(2f(y)+1)dxdy \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{Si}(x-y)\left[\int_{-\infty}^{\infty}\left(1-2\chi_{(-\infty,x)}(z)\right)K_{\mathrm{sine}}(y,z)f(z)dz\right]f'(x)f'(y)dxdy \\ &+ \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1-2\chi_{(-\infty,x)}(z)\right)K_{\mathrm{sine}}(y,z)f(z)dz\right] \\ &\left[\int_{-\infty}^{\infty}\left(1-2\chi_{(-\infty,y)}(u)\right)K_{\mathrm{sine}}(x,u)f(u)du\right]f'(x)f'(y)dxdy + O(N^{-1})\end{aligned}$$

respectively, where we have used integration by parts and $\chi_J(x)$ is the characteristic function of the interval J, i.e., $\chi_J(x) = 1$ for $x \in J$ and 0 otherwise.

Remark 11 We assume that $f(\cdot)$ is smooth and sufficiently decreasing at $\pm\infty$ to make the integrals well-defined.

With the relation of $f(x)$ and $F(x)$ in (11), we find from (42) that

$$\begin{aligned}\log\det(I + T_{\text{JOE}}) = &-2\lambda \int_{-\infty}^{\infty} K_{\text{sine}}(x, x)F(x)dx + \frac{\lambda^2}{2}\Big\{4\int_{-\infty}^{\infty} K_{\text{sine}}(x, x)F^2(x)dx \\ &- \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1 - 2\chi_{(-\infty,x)}(y)\right) K_{\text{sine}}(x, y)F(y)dy\right] F'(x)dx \\ &- 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{\text{sine}}^2(x, y)F(x)F(y)dxdy \\ &- \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Si}^2(x - y)F'(x)F'(y)dxdy\Big\} + \cdots + O(N^{-1}).\end{aligned}$$

Taking account of the fact that $\log G_N^{(1)}(f) = \frac{1}{2}\log\det(I + T_{\text{JOE}})$ from Theorem 12, we have the following heuristic result.

Theorem 15 *Letting $\mu_N^{(\text{JOE})}$ and $\mathcal{V}_N^{(\text{JOE})}$ be the mean and variance of the scaled linear statistics $\sum_{j=1}^{N} F(Nx_j)$, we have as $N \to \infty$,*

$$\mu_N^{(\text{JOE})} = \mu_N^{(\text{JUE})} + O(N^{-1}),$$

$$\begin{aligned}\mathcal{V}_N^{(\text{JOE})} = 2\mathcal{V}_N^{(\text{JUE})} &- \frac{1}{2}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1 - 2\chi_{(-\infty,x)}(y)\right) K_{\text{sine}}(x, y)F(y)dy\right] F'(x)dx \\ &- \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Si}^2(x - y)F'(x)F'(y)dxdy + O(N^{-1}),\end{aligned}$$

where $\mu_N^{(\text{JUE})}$ and $\mathcal{V}_N^{(\text{JUE})}$ are given by (12) and (13), respectively.

4.3 Scaling at the Edge of the Spectrum in JOE

Similarly as Theorems 9 and 10, we have the following results.

Theorem 16 *For $x, y \in \mathbb{R}^+$, we have as $N \to \infty$,*

$$\frac{1}{2N^2} S_N^{(1)}\left(1 - \frac{x}{2N^2}, 1 - \frac{y}{2N^2}\right) = \sqrt{\frac{x}{y}}\, K_{\text{Bessel}}^{(a+1)}(x, y) + O(N^{-1}),$$

where $K_{\text{Bessel}}^{(a+1)}(x,y)$ is the Bessel kernel of order $a+1$ given by

$$K_{\text{Bessel}}^{(a+1)}(x,y) = \frac{J_{a+1}(\sqrt{x})\sqrt{y}J'_{a+1}(\sqrt{y}) - J'_{a+1}(\sqrt{x})\sqrt{x}J_{a+1}(\sqrt{y})}{2(x-y)}.$$

The error term is uniform for x and y in compact subsets of $\mathbb{R}^+$.

Theorem 17 *For $x \in \mathbb{R}^+$, we have as $N \to \infty$,*

$$\varphi_N^{(1)}\left(1 - \frac{x}{2N^2}\right) = N^{3/2}\,\frac{J_{a+1}(\sqrt{x})}{\sqrt{x}} + O(N^{1/2}),$$

$$\varphi_{N-1}^{(1)}\left(1 - \frac{x}{2N^2}\right) = N^{3/2}\,\frac{J_{a+1}(\sqrt{x})}{\sqrt{x}} + O(N^{1/2}),$$

$$\varepsilon\varphi_N^{(1)}\left(1 - \frac{x}{2N^2}\right) = 2^{-1}N^{-1/2}\left(1 - 2\mathbf{J}_{a+1}(\sqrt{x})\right) + O(N^{-3/2}),$$

$$\varepsilon\varphi_{N-1}^{(1)}\left(1 - \frac{x}{2N^2}\right) = 2^{-1}N^{-1/2}\left(1 - 2\mathbf{J}_{a+1}(\sqrt{x})\right) + O(N^{-3/2}),$$

where

$$\mathbf{J}_{a+1}(x) = \int_0^x J_{a+1}(t)dt. \tag{43}$$

The error terms are uniform for x in compact subsets of $\mathbb{R}^+$.

Using Theorems 16 and 17 to compute Tr T_{JOE} and Tr T_{JOE}^2 from (41), and changing $f(x)$ to $f(2N^2(1-x))$, we obtain the next theorem following the similar heuristic procedure in Sect. 4.2. (We assume that $f(\cdot)$ is smooth and sufficiently decreasing at infinity.)

Theorem 18 *Denoting by $\tilde{\mu}_N^{(\text{JOE})}$ and $\tilde{\mathcal{V}}_N^{(\text{JOE})}$ the mean and variance of the linear statistics $\sum_{j=1}^N F(2N^2(1-x_j))$, we have as $N \to \infty$,*

$$\tilde{\mu}_N^{(\text{JOE})} = \tilde{\mu}_N^{(\text{JUE},a+1)} + \frac{1}{4}\int_0^\infty L^{(a+1)}(x,x)F'(x)dx + O(N^{-1}),$$

$$\begin{aligned}\tilde{\mathcal{V}}_N^{(\text{JOE})} &= 2\tilde{\mathcal{V}}_N^{(\text{JUE},a+1)} + \frac{1}{2}\int_0^\infty L^{(a+1)}(x,x)F(x)F'(x)dx \\ &\quad - \frac{1}{2}\int_0^\infty\left[\int_0^\infty \left(1 - 2\chi_{(0,x)}(y)\right)\sqrt{\frac{x}{y}}\,K_{\text{Bessel}}^{(a+1)}(x,y)F(y)dy\right]F'(x)dx \\ &\quad - \frac{1}{16}\int_0^\infty\left[\int_0^\infty \left(1 - 2\chi_{(0,x)}(y)\right)\frac{J_{a+1}(\sqrt{y})}{\sqrt{y}}F(y)dy\right]\frac{J_{a+1}(\sqrt{x})}{\sqrt{x}}F(x)dx\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^\infty\int_0^\infty \frac{J_{a+1}(\sqrt{x})}{\sqrt{y}} K_{\text{Bessel}}^{(a+1)}(x,y)\left(1-2\mathbf{J}_{a+1}(\sqrt{y})\right) F(x)F(y)dxdy \\
&+\frac{1}{8}\int_0^\infty\int_0^\infty \frac{J_{a+1}(\sqrt{x})}{\sqrt{x}}\left(1-2\mathbf{J}_{a+1}(\sqrt{y})\right)\left(L^{(a+1)}(x,y)-L^{(a+1)}(y,x)\right) \\
&\quad F(x)F'(y)dxdy-\frac{1}{8}\int_0^\infty\int_0^\infty L^{(a+1)}(x,y)L^{(a+1)}(y,x)F'(x)F'(y)dxdy \\
&-\frac{1}{16}\int_0^\infty\int_0^\infty\left(1-2\mathbf{J}_{a+1}(\sqrt{x})\right)\left(1-2\mathbf{J}_{a+1}(\sqrt{y})\right)L^{(a+1)}(x,y)F'(x)F'(y)dxdy \\
&-\int_0^\infty\int_0^\infty\sqrt{\frac{x}{y}}\,K_{\text{Bessel}}^{(a+1)}(x,y)L^{(a+1)}(x,y)F'(x)F(y)dxdy+O(N^{-1}),
\end{aligned}
$$

where

$$
L^{(a+1)}(x,y)=\int_0^x\sqrt{\frac{y}{z}}\,K_{\text{Bessel}}^{(a+1)}(y,z)dz-\int_x^\infty\sqrt{\frac{y}{z}}\,K_{\text{Bessel}}^{(a+1)}(y,z)dz,
$$

$\mathbf{J}_{a+1}(x)$ *is defined in (43) and* $\tilde{\mu}_N^{(\text{JUE},a+1)}$ *and* $\tilde{\mathcal{V}}_N^{(\text{JUE},a+1)}$ *are given by (16) and (17) with a replaced by* $a+1$.

Remark 12 We have used integration by parts to simplify the results in the above theorem.

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On the Limit of Some Variable-Coefficient Toeplitz-Like Determinants

Bin Shao

In Memory of Harold Widom

Abstract The focus of this paper is to extend a limit theorem due to Tracy and Widom which describes the asymptotics of the determinants of a certain class of finite-rank perturbation of Toeplitz matrices. More precisely, we compute the asymptotics of the determinants of the $(m+N+1)\times(m+N+1)$ variable-coefficient Toeplitz-like matrices whose (j,k)-entries are given by

$$\sigma_{p_j-q_k}\left(\frac{r_j}{m+N}\right) := \frac{1}{2\pi}\int_0^{2\pi}\sigma\left(\frac{r_j}{m+N}, e^{i\theta}\right)e^{-i(p_j-q_k)\theta}\,d\theta.$$

Here $\{p_j\}_{j=0}^{\infty}$, $\{q_j\}_{j=0}^{\infty}$ and $\{r_j\}_{j=0}^{\infty}$ are sequences of integers satisfying $p_j = q_j = r_j = j$ for j sufficiently large, say $j \geq m$, as well as $r_j \geq 0$ for all $0 \leq j < m$. Under some smoothness assumption on the function $\sigma : [0,1]\times\mathbb{T}\to\mathbb{C}$, we determine the limit of the ratio between the determinants of these matrices and the usual variable-coefficient Toeplitz matrices ($m = 0$):

$$\lim_{N\to\infty}\frac{\det\left(\sigma_{p_j-q_k}\left(\frac{r_j}{m+N}\right)\right)_{j,k=0,1,\ldots,m+N}}{\det\left(\sigma_{j-k}\left(\frac{j}{N}\right)\right)_{j,k=0,1,\ldots,N}}.$$

In comparison to the pure Toeplitz case of Tracy and Widom, a new feature appears in the limit.

Keywords Variable-coefficient Toeplitz matrices · Toeplitz determinants · Szegő-Widom limit theorem

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1 Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane, let σ be a complex-valued, essentially bounded function defined on $[0, 1] \times \mathbb{T}$ with the identification $\sigma(x, \theta) := \sigma(x, e^{i\theta})$, and let

$$\sigma_j(x) = \frac{1}{2\pi} \int_0^{2\pi} \sigma(x, \theta) e^{-ij\theta}\, d\theta, \qquad x \in [0, 1],\ j \in \mathbb{Z},$$

denote the j-th variable-dependent Fourier coefficient of σ. For a given integer $m \geq 0$, and given sequences of integers $\{p_j\}_{j=0}^{\infty}$, $\{q_j\}_{j=0}^{\infty}$, and $\{r_j\}_{j=0}^{\infty}$, satisfying the conditions

(i) $p_j = q_j = r_j = j$ for all $j \geq m$ and
(ii) $r_j \geq 0$ for all $0 \leq j < m$,

we consider the following $(m + N + 1) \times (m + N + 1)$ matrices

$$M_{m+N}^{\sigma} := \left(\sigma_{p_j - q_k} \left(\frac{r_j}{m + N} \right) \right)_{j,k=0,1,\ldots,m+N},$$

whose determinant asymptotics will be the subject of this paper. We note that the above conditions imply that $0 \leq \frac{r_j}{m+N} \leq 1$ for all $0 \leq j \leq m + N$ whenever N is sufficiently large, say $N \geq N_0$. Hence these matrices are well-defined for all $N \geq N_0$.[1]

In the case $m = 0$, the matrix M_{m+N}^{σ} reduces to the ordinary $(N + 1) \times (N + 1)$ variable-coefficient Toeplitz matrix[2]

$$\mathrm{op}_N \sigma := \left(\sigma_{j-k} \left(\frac{j}{N} \right) \right)_{j,k=0,1,\ldots,N}.$$

If σ is independent of x, i.e., $\sigma(\theta) = \sigma(\theta, x)$, then $\mathrm{op}_N \sigma$ further reduces to the pure $(N + 1) \times (N + 1)$ Toeplitz matrix[3]

$$T_N(\sigma) = \left(\sigma_{j-k} \right)_{j,k=0,1,\ldots,N},$$

[1] To ensure that the matrices are well-defined for all $N \geq 1$, one needs to replace the condition (ii) by the requirement that $0 \leq r_j \leq m + 1$ for all $0 \leq j < m$.

[2] The matrix $\mathrm{op}_N \sigma$ can be interpreted as the discrete analogue of a class of pseudo-differential operators. The notation $\mathrm{op}_\alpha \sigma$ (for a large parameter α) appears in the early work of Widom on asymptotic expansions for pseudo-differential operators on bounded domains (see [8–10] for historical references).

[3] In many (but not all) publications, including [6], the notation $T_N(\sigma)$ is used to denote the Toeplitz matrix of size $N \times N$. Here we deviate from this convention in order to view $T_N(\sigma)$ as a special case of $\mathrm{op}_N \sigma$.

whose determinant has been the subject of numerous investigations (see, e.g., [2] for historical background and details). In particular, for sufficiently smooth nonvanishing functions σ with winding number zero, the asymptotics of $\det T_N(\sigma)$ is given by the well-known Szegő-Widom limit theorem,

$$\det T_N(\sigma) \sim G[\sigma]^{N+1} E[\sigma] \quad \text{as } N \to \infty,$$

where

$$G[\sigma] = \exp([\log \sigma]_0), \qquad E[\sigma] = \exp\left(\sum_{k=1}^{\infty} k[\log \sigma]_k [\log \sigma]_{-k}\right). \tag{1}$$

The asymptotics of Toeplitz determinants have important applications in statistical physics and in random matrix theory.

The asymptotics of the determinants of M^{σ}_{m+N} in the case when σ is independent of x were considered by Tracy and Widom [6] in 2002 for the first time. Notice that the Fourier coefficients $\sigma_j = \sigma_j(x)$ are also independent of x, and the matrices M^{σ}_{m+N} become

$$M^{\sigma}_{m+N} = \left(\sigma_{p_j - q_k}\right)_{j,k=0,1,\ldots,m+N},$$

where the integer sequences $\{p_j\}$ and $\{q_j\}$ are subject to condition (i) above. More precisely, Tracy and Widom proved that the ratio between these determinants and ordinary Toeplitz determinants converges to a limit,

$$\lim_{N\to\infty} \frac{\det M^{\sigma}_{m+N}}{\det T_N(\sigma)} = \det\left(\sum_{k=1}^{\infty} \sigma^{-}_{p_i+k-m}\ \sigma^{+}_{-q_j-k+m}\right)_{i,j=0,1,\ldots,m-1}, \tag{2}$$

provided that σ belongs to a suitable class of function on the unit circle. Therein, $\sigma = \sigma^-\sigma^+$ is the Wiener-Hopf factorization of σ, and the Fourier coefficients of the Wiener-Hopf factors $\sigma^{\pm}$ are denoted by $\sigma^{\pm}_j$. Notice that the sum in (2) has only finitely many terms since the Fourier coefficients of the Wiener-Hopf factors vanish for k sufficiently large.

The asymptotics of the above determinants of M^{σ}_{m+N} in the x-independent case has attracted the attention of further researchers. Indeed, using a different approach the same asymptotics stated in a very different form was independently established by Bump and Diaconis [3].

The aim of this paper is to obtain a variable-coefficient generalization of the Tracy-Widom limit formula (2). Our main result will be Theorem 2.1, which states that under certain assumptions on σ,

$$\lim_{N\to\infty} \frac{\det M^{\sigma}_{m+N}}{\det \operatorname{op}_N \sigma} = K_m[\sigma] \cdot F_m[\sigma], \tag{3}$$

where the constant

$$F_m[\sigma] = \det\left(\sum_{k=1}^{\infty} \sigma^-_{p_i+k-m}(0)\, \sigma^+_{-q_j-k+m}(0) \right)_{0\le i,j\le m-1} \tag{4}$$

resembles the one in (2) (see (17) below for the definition of the Wiener-Hopf factors), while the constant

$$K_m[\sigma] = \exp\left(\frac{m}{2\pi} \int_0^{2\pi} \int_0^1 \log \frac{\sigma(x,\theta)}{\sigma(0,\theta)}\, dx\, d\theta \right) \tag{5}$$

appears as a new feature.

The asymptotics of the determinant of $\mathrm{op}_N\sigma$ has been determined by the author and Ehrhardt [4]. Under certain assumptions on σ, it is given by the following variable-coefficient version of the strong Szegő-Widom limit theorem,

$$\det \mathrm{op}_N\sigma \sim G[\sigma]^{N+1} E[\sigma] \quad \text{as } N \to \infty, \tag{6}$$

where, in contrast to (1),

$$\begin{aligned}
G[\sigma] &= \exp\left(\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \log \sigma(x,\theta)\, d\theta\, dx \right),\\
E[\sigma] &= \exp\left(\frac{1}{2} \sum_{k=1}^{\infty} k\, [\log \sigma(0,\theta)]_k [\log \sigma(0,\theta)]_{-k} \right)\\
&\quad \times \exp\left(\frac{1}{2} \sum_{k=1}^{\infty} k\, [\log \sigma(1,\theta)]_k [\log \sigma(1,\theta)]_{-k} \right)\\
&\quad \times \exp\left(\frac{1}{2} \int_0^1 \left(\sum_{k=-\infty}^{\infty} k\, [\log \sigma(x,\theta)]_k [(\partial_x \log \sigma)(x,\theta)]_{-k} \right) dx \right)\\
&\quad \times \exp\left(\frac{1}{4\pi} \int_0^{2\pi} (\log \sigma(0,\theta) + \log \sigma(1,\theta))\, d\theta \right)\\
&\quad \times \exp\left(-\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \log \sigma(x,\theta)\, d\theta\, dx \right).
\end{aligned}$$

Combining the asymptotic formulas (3) and (6) gives the complete asymptotics of the determinant $\det M^{\sigma}_{m+N}$ as $N \to \infty$.

The rest of the paper is organized as follows. We will give the statement of the main result (Theorem 2.1) and outline its proof in Sect. 2. We will also give the definitions for the various function spaces in which the function σ will be considered. Section 3 is devoted to some norm estimates for the matrices $\mathrm{op}_N\sigma$

and $D_N^{m\sigma}$ and the convergence of their inverses. The matrix $D_N^{m\sigma}$ (to be defined in Sect. 2) plays a key role in splitting the limit (3) in two parts. In Sect. 4 we give the proof of the main result by evaluating the two parts of the limit, the first one being a direct analogue of the Tracy-Widom formula, the second one involving trace norm computations. Finally, in Sect. 5 we comment on a possible application of the main result to the computation of the second-order asymptotics as $N \to \infty$ of $\operatorname{tr} f(M_{m+N}^{\sigma})$ for suitable functions f.

2 Preliminaries and Main Result

In this section we introduce some preliminaries necessary to state and prove our result. We begin with introducing some notation describing the smoothness condition that has to be imposed on the symbol σ.

For each nonnegative integer n and $0 \le \mu < 1$, let $\mathcal{C}^{n+\mu}$ stand for the set of all n times differentiable functions defined on $\mathbb{T}$ whose n-th derivative is continuous (if $\mu = 0$) or satisfies a Hölder condition with exponent μ (if $0 < \mu < 1$), respectively. It is well known that, for $0 \le \alpha < \infty$, one can define a norm $\|\cdot\|_\alpha$ such that $\mathcal{C}^\alpha$ becomes a Banach algebra. It is also well known that for $f \in \mathcal{C}^\alpha$ with $0 \le \alpha < \infty$, the Fourier coefficients f_k of the function f satisfy the estimate

$$|f_k| \le K_\alpha (1 + |k|)^{-\alpha} \|f\|_\alpha \tag{7}$$

for all $k \in \mathbb{Z}$, where the constant K_α depends only on α.

Now let $\Sigma^{\alpha,0}$ denote the set of all functions σ defined on $[0, 1] \times \mathbb{T}$ such that for each fixed $x \in [0, 1]$, the function $\sigma_{[x]}(e^{i\theta}) := \sigma(x, e^{i\theta})$ belongs to $\mathcal{C}^\alpha$ and such that the mapping $x \in [0, 1] \mapsto \sigma_{[x]} \in \mathcal{C}^\alpha$ is continuous. It is easy to see that the set $\Sigma^{\alpha,0}$ forms a Banach algebra with the norm

$$\|\sigma\|_{\Sigma^{\alpha,0}} = \max_{x\in[0,1]} \|\sigma_{[x]}(e^{i\theta})\|_\alpha .$$

Furthermore, let $\Sigma^{\alpha,1}$ denote the set of all functions $\sigma \in \Sigma^{\alpha,0}$ such that the partial derivative $\partial_x \sigma$ exists and belongs to $\Sigma^{\alpha,0}$. The set $\Sigma^{\alpha,1}$ is a Banach algebra equipped with the norm

$$\|\sigma\|_{\Sigma^{\alpha,1}} = \|\sigma\|_{\Sigma^{\alpha,0}} + \|(\partial_x \sigma)\|_{\Sigma^{\alpha,0}} .$$

Finally, let $\Sigma_0^{\alpha,1}$ be the subset of $\Sigma^{\alpha,1}$ consisting of all functions possessing a logarithm $\log \sigma \in \Sigma^{\alpha,1}$. In fact, one can show that $\Sigma_0^{\alpha,1}$ is the set of all functions $\sigma \in \Sigma^{\alpha,1}$ which do not vanish on $[0, 1] \times \mathbb{T}$ and for which the functions $\sigma_{[x]}$ have winding number zero for each $x \in [0, 1]$ (see [4, Lemma 2]).

Now we are prepared to state the main result of this paper.

Theorem 2.1 *Let $\sigma \in \Sigma_0^{\alpha,1}$ with $\alpha > \frac{3}{2}$. Then formula (3) holds with the constants given by (4) and (5).*

To give an outline of the proof, we introduce another $(N+1) \times (N+1)$ matrix related to M^{σ}_{m+N},

$$D_N^{m\sigma} = \left(\sigma_{j-k}\left(\frac{m+j}{m+N}\right)\right)_{j,k=0,1,\dots,N}.$$

The proof of the main result requires two steps. We are going to show that on the one hand (Theorem 4.2)

$$\lim_{N\to\infty} \frac{\det M^{\sigma}_{m+N}}{\det D_N^{m\sigma}} = F_m[\sigma], \tag{8}$$

while on the other hand (Theorem 4.5 and Proposition 4.6)

$$\lim_{N\to\infty} \frac{\det D_N^{m\sigma}}{\det \mathrm{op}_N \sigma} = K_m[\sigma]. \tag{9}$$

The computation of the limit (8) follows the original ideas of Tracy and Widom [6] and is based on the fact that $D_N^{m\sigma}$ appears as a sub-matrix in a suitable partition of M^{σ}_{m+N}. A well-known Schur-complement type formula from matrix theory [11] states that

$$\det\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det D \cdot \det(A - BD^{-1}C) \tag{10}$$

for a matrix partitioned in such a way that A and D are square matrices and D is invertible. The matrix M^{σ}_{m+N} can be partitioned in this way with $D_N^{m\sigma}$ taking the place of D. As this approach suggest, it is necessary to address the invertibility of $D_N^{m\sigma}$ (for sufficiently large N). In fact, we need to prove the uniform invertibility of $D_N^{m\sigma}$ and the convergence of the inverses $(D_N^{m\sigma})^{-1}$. This will be done in the next section.

Let us recall at this point some of the underlying basic notions. First of all, the matrices considered in this paper can be thought of as operators acting on the Hilbert space ℓ^2 of all one-sided infinite square-summable sequences. More precisely, let

$$P_N : (x_0, x_1, \dots) \to (x_0, x_1, \dots, x_N, 0, \dots)$$

denotes the finite rank projection on ℓ^2. An $(N+1) \times (N+1)$ matrix A_N will be identified with the matrix representation of the corresponding operator A_N : $\mathrm{im} P_N \to \mathrm{im} P_N$, and the latter can be thought of as the compression of an operator $A_N : \ell^2 \to \ell^2$ onto the image of P_N, i.e., $A_N = P_N A_N P_N$.

One says that a sequence $\{A_N\}_{N=1}^{\infty}$ of matrices or operators A_N is *uniformly invertible* (or *stable*) if there exists an N_0 such that matrices or operators A_N are invertible for all $N \geq N_0$ and

$$\sup_{N \geq N_0} \|A_N^{-1}\|_\infty < \infty.$$

Here and in what follows, $\|\cdot\|_\infty$ denotes the operator norm (with respect to the Hilbert space ℓ^2).

We recall the notion of strong convergence of operators on ℓ^2. As sequence of bounded linear operators A_N is said to *converges strongly on* ℓ^2 to a bounded linear operator A as $N \to \infty$, if $A_N x \to Ax$ in the norm of ℓ^2 for each $x \in \ell^2$.

A special class of bounded linear operators of ℓ^2 are *trace class operators* and *Hilbert-Schmidt operators*. We refer to [2, Chap. 5] or [5, Chap. III] for definitions and basic properties. The trace norm and the Hilbert-Schmidt norm of an operator A will be denoted by

$$\|A\|_1 \quad \text{and} \quad \|A\|_2,$$

respectively. When analyzing convergence, we will frequently use the following estimates involving a bounded and a trace class operator,

$$\|AB\|_1 \leq \|A\|_1 \|B\|_\infty, \qquad \|BA\|_1 \leq \|A\|_1 \|B\|_\infty.$$

as well as involving two Hilbert-Schmidt operators,

$$\|AB\|_1 \leq \|A\|_2 \|B\|_2.$$

Note that for trace class operators A, the determinant $\det(I + A)$ and the trace $\operatorname{tr} A$ are well-defined and depend continuously on A in the trace norm.

3 Boundedness and Invertibility of $\mathrm{op}_N \sigma$ and $D_N^{m\sigma}$

3.1 Boundedness

We begin with the following lemma concerning estimates in operator norm and Hilbert-Schmidt norm.

Lemma 3.1 *Let $\psi \in \Sigma^{\alpha,0}$ and consider the matrices*

$$A_N = \left(\psi_{j-k}(\xi_{jk}^{(N)})\right)_{j,k=0,1,2,\ldots,N}$$

where $\{\xi_{jk}^{(N)}\}_{0\le j,k\le N}$ *is any collection of real numbers in* $[0,1]$. *Then,*

(a) $\|A_N\|_\infty \le C_\alpha \|\psi\|_{\Sigma^{\alpha,0}}$ *if* $\alpha > 1$,
(b) $\|A_N\|_2 \le C_\alpha \sqrt{N}\, \|\psi\|_{\Sigma^{\alpha,0}}$ *if* $\alpha > \frac{1}{2}$.

Here C_α *is a constant depending only on* α.

Proof We decompose $A_N = \sum_{k=-N}^{N} A_N^k$, where A_N^k has nonzero values only on the k-th diagonal. Using estimate (7), it follows that

$$\begin{aligned}\|A_N\|_\infty \le \sum_{k=-N}^{N} \|A_N^k\|_\infty \;&\le \sum_{k=-N}^{N} \max_{x\in[0,1]} |\psi_k(x)|\\ &\le K_\alpha \max_{x\in[0,1]} \|\psi_{[x]}(\theta)\|_\alpha \sum_{k=-N}^{N} (1+|k|)^{-\alpha}.\end{aligned}$$

Similarly, for the Hilbert-Schmidt norm,

$$\begin{aligned}\|A_N\|_\infty &\le K_\alpha \max_{x\in[0,1]} \|\psi_{[x]}(\theta)\|_\alpha \left(\sum_{j,k=0}^{N} (1+|j-k|)^{-2\alpha}\right)^{\frac{1}{2}}\\ &\le K_\alpha \|\psi\|_{\Sigma^{\alpha,0}} \sqrt{N+1} \left(\sum_{k=-N}^{N} (1+|k|)^{-2\alpha}\right)^{\frac{1}{2}}.\end{aligned}$$

This implies (a) and (b). □

Since both matrices $\mathrm{op}_N\sigma$ and $D_N^{m\sigma}$ are of the form of the matrices appearing in the previous lemma, a direct consequence is the uniform boundedness in the operator norm as well as an estimate for the Hilbert-Schmidt norm. Note that statement (a) concerning $\mathrm{op}_N\sigma$ has appeared before in [4, Prop. 2].

Proposition 3.2 *Suppose* $\sigma \in \Sigma^{\alpha,0}$. *Then,*

(a) $\left\|\mathrm{op}_N\sigma\right\|_\infty \le C_\alpha \|\sigma\|_{\Sigma^{\alpha,0}}$ *and* $\left\|D_N^{m\sigma}\right\|_\infty \le C_\alpha \|\sigma\|_{\Sigma^{\alpha,0}}$ *if* $\alpha > 1$,
(b) $\left\|\mathrm{op}_N\sigma\right\|_2 \le C_\alpha \sqrt{N}\, \|\sigma\|_{\Sigma^{\alpha,0}}$ *and* $\left\|D_N^{m\sigma}\right\|_2 \le C_\alpha \sqrt{N}\, \|\sigma\|_{\Sigma^{\alpha,0}}$ *if* $\alpha > \frac{1}{2}$.

Here C_α *is a constant depending only on* α.

The matrices $D_N^{m\sigma}$ are a slight modification of the matrices $\mathrm{op}_N\sigma$. Therefore, our next concern is the difference of these two matrices. Let us introduce the matrices

$$S_N^{m\sigma} := D_N^{m\sigma} - \mathrm{op}_N\sigma, \tag{11}$$

$$W_N^m := \mathrm{diag}\left(\left\{\frac{m(N-j)}{N(m+N)}\right\}_{j=0,1,\ldots,N}\right). \tag{12}$$

It is easily seen that

$$\|W_N^m\|_\infty = O(N^{-1}), \quad \|W_N^m\|_2 = O(N^{-\frac{1}{2}}), \quad \|W_N^m\|_1 = O(1), \tag{13}$$

as $N \to \infty$, which we record here also for further reference.

Proposition 3.3 *Suppose $\sigma \in \Sigma^{\alpha,1}$. Then,*

(a) $\left\| S_N^{m\sigma} \right\|_\infty \le C_\alpha N^{-1} \|\sigma\|_{\Sigma^{\alpha,1}}$ *if* $\alpha > 1$,
(b) $\left\| S_N^{m\sigma} \right\|_2 \le C_\alpha N^{-\frac{1}{2}} \|\sigma\|_{\Sigma^{\alpha,1}}$ *if* $\alpha > \frac{1}{2}$,
(c) $\left\| S_N^{m\sigma} \right\|_1 \le C_\alpha \|\sigma\|_{\Sigma^{\alpha,1}}$ *if* $\alpha > \frac{1}{2}$.

Here C_α is a constant depending only on α.

Proof The (j,k)-entry of $S_N^{m\sigma}$ is given by

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\sigma\left(\frac{m+j}{m+N}, \theta \right) - \sigma\left(\frac{j}{N}, \theta \right) \right) e^{-i(j-k)\theta} \, d\theta.$$

Therefore, by the mean value theorem, there exists $\xi_{jk}^{(N)} \in [0,1]$ depending on j, k, N (and m) and satisfying $\frac{j}{N} \le \xi_{jk}^{(N)} \le \frac{m+j}{m+N}$ such that this entry is equal to

$$\frac{m(N-j)}{N(m+N)} \cdot \frac{1}{2\pi} \int_0^{2\pi} (\partial_x \sigma)(\xi_{jk}^{(N)}, \theta) e^{-i(j-k)\theta} \, d\theta.$$

Hence we can write $S_N^{m\sigma} = W_N^m A_N$ where

$$A_N = \left((\partial_x \sigma)_{j-k}(\xi_{jk}^{(N)}) \right)_{j,k=0,1,2,\dots,N}$$

is a matrix to which we can apply Lemma 3.1 with $\psi = \partial_x \sigma \in \Sigma^{\alpha,0}$. We are also going to use (13). In particular, part (a) follows from $\|A_N\|_\infty = O(1)$ and $\|W_N^m\|_\infty = O(N^{-1})$, while part (b) follows from $\|A_N\|_2 = O(N^{\frac{1}{2}})$ and again $\|W_N^m\|_\infty = O(N^{-1})$. Finally, part (c) follows from $\|A_N\|_2 = O(N^{\frac{1}{2}})$ and $\|W_N^m\|_2 = O(N^{-\frac{1}{2}})$. Alternatively, we could use (b) and the fact that $A_N = P_N A_N$ with $\|P_N\|_2 = \sqrt{N+1}$. □

In the previous proposition, it is the convergence in the operator norm,

$$\left\| S_N^{m\sigma} \right\|_\infty = \left\| D_N^{m\sigma} - \mathrm{op}_N \sigma \right\|_\infty = o(1), \qquad N \to \infty,$$

which will be of importance to us.

3.2 Uniform Invertibility

We are now going to examine the uniform invertibility of the sequence of matrices $D_N^{m\sigma}$ and the strong convergence of their inverses. It is based on the corresponding result for the sequence of matrices $\mathrm{op}_N\sigma$ established in [4].

For an x-independent symbol $\psi \in L^\infty(\mathbb{T})$ define the semi-infinite Toeplitz matrix,

$$T_\psi = \left(\psi_{j-k}\right)_{0\le j,k<\infty}\,, \tag{14}$$

as bounded linear operator on ℓ^2. Furthermore, let

$$Q_N : (x_0, x_1, \ldots) \to (x_N, x_{N-1}, \ldots, x_0, 0, \ldots) \tag{15}$$

be a flip operator on ℓ^2. Note that $Q_N^2 = P_N$.

The following result is based on Theorem 2 and Proposition 4 of [4].

Proposition 3.4 *Suppose $\sigma \in \Sigma_0^{\alpha,1}$ with $\alpha > \frac{3}{2}$. Then both Toeplitz operators $T_{\sigma(0,\theta)}$ and $T_{\widetilde{\sigma}(1,\theta)}$ are invertible on ℓ^2 and the matrices $\mathrm{op}_N\sigma$ are invertible for sufficiently large N. Moreover,*

$$\left(\mathrm{op}_N\sigma\right)^{-1} = \mathrm{op}_N(\sigma^{-1}) + P_N K_1 P_N + Q_N K_2 Q_N + C_N,$$

where $\|C_N\|_\infty = o(1)$ as $N \to \infty$ and both

$$K_1 = T^{-1}_{\sigma(0,\theta)} - T_{\sigma(0,\theta)^{-1}}, \qquad K_2 = T^{-1}_{\widetilde{\sigma}(1,\theta)} - T_{\widetilde{\sigma}(1,\theta)^{-1}}, \tag{16}$$

are trace class operators on ℓ^2.

The corresponding result for $D_N^{m\sigma}$ can now be derived.

Proposition 3.5 *Suppose $\sigma \in \Sigma_0^{\alpha,1}$ with $\alpha > \frac{3}{2}$. Then the matrices $D_N^{m\sigma}$ are invertible for sufficiently large N. Moreover,*

$$\left(D_N^{m\sigma}\right)^{-1} = \mathrm{op}_N(\sigma^{-1}) + P_N K_1 P_N + Q_N K_2 Q_N + \widehat{C}_N,$$

with K_1 and K_2 as in (16) and $\|\widehat{C}_N\|_\infty = o(1)$ as $N \to \infty$.

Proof Notice first that Proposition 3.4 in connection with Proposition 3.2(a) implies that is $\mathrm{op}_N\sigma$ is uniformly invertible, i.e., it is invertible for sufficiently large N and $\|(\mathrm{op}_N\sigma)^{-1}\|_\infty = O(1)$. Using that $D_N^{m\sigma} = \mathrm{op}_N\sigma + S_N^{m\sigma}$ it follows that

$$(\mathrm{op}_N\sigma)^{-1} D_N^{m\sigma} = P_N + (\mathrm{op}_N\sigma)^{-1} S_N^{m\sigma}$$

where $\|(\mathrm{op}_N\sigma)^{-1}S_N^{m\sigma}\|_\infty = o(1)$ by Proposition 3.3(a). Hence $D_N^{m\sigma}$ is also invertible for sufficiently large N and

$$(D_N^{m\sigma})^{-1} = \left(P_N + (\mathrm{op}_N\sigma)^{-1}S_N^{m\sigma}\right)^{-1}(\mathrm{op}_N\sigma)^{-1},$$

which implies that

$$\left\|(D_N^{m\sigma})^{-1} - (\mathrm{op}_N\sigma)^{-1}\right\|_\infty = o(1),$$

and now the assertion follows easily. □

Corollary 3.6 *Suppose $\sigma \in \Sigma_0^{\alpha,1}$ with $\alpha > \frac{3}{2}$. Then both*

$$P_N(\mathrm{op}_N\sigma)^{-1}P_N \to T^{-1}_{\sigma(0,\theta)}, \qquad P_N(D_N^{m\sigma})^{-1}P_N \to T^{-1}_{\sigma(0,\theta)},$$

strongly on ℓ^2 as $N \to \infty$.

Proof The first statement has already been shown in [4, Corollary 1]. Both statements follow easily by taking into account that $Q_N \to 0$ weakly on ℓ^2 and $\mathrm{op}_N\sigma \to T_{\sigma(0,\theta)}$ strongly on ℓ^2. The latter has been shown in [4, Prop. 3] and holds for $\sigma \in \Sigma^{\alpha,0}$. □

4 Proof of the Main Result

4.1 First Limit Computation

We are first going to compute the limit (8). We start with an auxiliary result established by Tracy and Widom [6], which assumes that σ is independent of x.

Proposition 4.1 (Tracy-Widom [6]) *For $\sigma \in L^\infty(\mathbb{T})$, let*

$$X = (\sigma_{p_i-m-k})_{i,k} \text{ and } Y = (\sigma_{m+k-q_j})_{k,j}, \quad (i,j = 0,\ldots,m-1; k = 0,1,\ldots)$$

be the semi-infinite $m \times \infty$ and $\infty \times m$ matrices, respectively, and assume that the infinite Toeplitz matrix $T_\sigma = (\sigma_{i-j})_{i,j}$ $(i,j = 0,1,\ldots)$ is invertible on ℓ^2. Then the (i,j)-entry of the square matrix $XT_\sigma^{-1}Y$ is

$$\sigma_{p_i-q_j} - \sum_{k=1}^{\infty}\sigma^-_{p_i-m+k}\,\sigma^+_{-k-q_j+m}.$$

Here, $\sigma = \sigma^-\sigma^+$ is the Wiener-Hopf factorization of σ.

We remark that the variable-coefficient analogue of the usual Wiener-Hopf factorization also exists (see [4, Sect. 7]). Indeed, for $\sigma \in \Sigma_0^{\alpha,1}$ with $\alpha > 0$, we have $\sigma(x,\theta) = \sigma^-(x,\theta)\sigma^+(x,\theta)$, where

$$\begin{aligned} \sigma^+(x,\theta) &:= \exp\left(\sum_{k=0}^{\infty} [\log\sigma(x,\theta)]_k e^{ik\theta}\right), \\ \sigma^-(x,\theta) &:= \exp\left(\sum_{k=1}^{\infty} [\log\sigma(x,\theta)]_{-k} e^{-ik\theta}\right). \end{aligned} \tag{17}$$

We will need it in what follows only for $x = 0$.

Theorem 4.2 *Let $\sigma \in \Sigma_0^{\alpha,1}$ with $\alpha > \frac{3}{2}$. Then*

$$\lim_{N\to\infty} \frac{\det M_{m+N}^{\sigma}}{\det D_N^{m\sigma}} = \det\left(\sum_{k=1}^{\infty} \sigma^-_{p_i+k-m}(0)\ \sigma^+_{-q_j-k+m}(0)\right)_{i,j=0,1,\ldots,m-1}.$$

Proof We consider a matrix partition

$$M_{m+N}^{\sigma} = \left(\begin{array}{c|c} A_m^{N\sigma} & B_{mN}^{\sigma} \\ \hline C_{Nm}^{\sigma} & D_N^{m\sigma} \end{array}\right),$$

where all entries of the partition are given as follows:

$$\begin{aligned} A_m^{N\sigma} &= \left(\sigma_{p_j-q_k}\left(\frac{r_j}{m+N}\right)\right)_{0\,\le\, j,\ k\,\le\, m-1} \\ B_{mN}^{\sigma} &= \left(\sigma_{p_j-k-m}\left(\frac{r_j}{m+N}\right)\right)_{0\,\le\, j\,\le\, m-1,\ \ 0\le k\le N} \\ C_{Nm}^{\sigma} &= \left(\sigma_{j+m-q_k}\left(\frac{m+j}{m+N}\right)\right)_{0\,\le\, j\,\le\, N,\ \ 0\le k\le m-1} \\ D_N^{m\sigma} &= \left(\sigma_{j-k}\left(\frac{m+j}{m+N}\right)\right)_{0\,\le j,\ k\,\le\, N} \end{aligned}$$

From Proposition 3.5 it follows that $D_N^{m\sigma}$ is invertible for sufficiently large N, and we can therefore use the matrix identity (10),

$$\det M_{m+N}^{\sigma} = \det D_N^{m\sigma} \cdot \det\left(A_m^{N\sigma} - B_{mN}^{\sigma}\left(D_N^{m\sigma}\right)^{-1} C_{Nm}^{\sigma}\right).$$

The last determinant is that of an $m \times m$ matrix, where throughout the paper m is given and fixed. We now claim that this matrix converges entry-wise (or, equivalently, in operator norm) to the following $m \times m$ matrix, as $N \to \infty$,

$$A_m^{\infty\sigma} - B_{m\infty}^{\sigma} T_{\sigma(0,\theta)}^{-1} C_{\infty m}^{\sigma},$$

where

$$A_m^{\infty\sigma} := \left(\sigma_{p_j - q_k}(0)\right)_{0 \le j,k \le m-1},$$
$$B_{m\infty}^{\sigma} := \left(\sigma_{p_j - k - m}(0)\right)_{0 \le j \le m-1,\ 0 \le k < \infty},$$
$$C_{\infty m}^{\sigma} := \left(\sigma_{j+m-q_k}(0)\right)_{0 \le j < \infty,\ 0 \le k < m-1},$$

and $T_{\sigma(0,\theta)} = \left(\sigma_{j-k}(0)\right)_{0 \le j,k < \infty}$ is the infinite Toeplitz matrix on ℓ^2. Notice that $B_{m\infty}^{\sigma}$ and $C_{\infty m}^{\sigma}$ are $m \times \infty$ and $\infty \times m$ matrices.

Clearly $A_m^{N\sigma} \to A_m^{\infty\sigma}$ entry-wise. Indeed, the difference of the entries can be estimated by

$$\left|\sigma_\ell\left(\frac{r_j}{m+N}\right) - \sigma_\ell(0)\right| \le K_\alpha (1 + |\ell|)^{-\alpha} \left\|\sigma_{[\frac{r_j}{m+N}]}(\theta) - \sigma_{[0]}(\theta)\right\|_\alpha$$

taking $\ell = p_j - q_k$. Note that this convergence will hold under the sole assumption $\sigma \in \Sigma^{0,\alpha}$ with $\alpha > 0$, where we use the fact that the map $x \in [0,1] \mapsto \sigma_{[x]} \in \mathcal{C}^\alpha$ is continuous.

From Corollary 3.6 we know that $P_N (D_N^{m\sigma})^{-1} P_N \to T_{\sigma(0,\theta)}^{-1}$ strongly on ℓ^2. Actually, we only need the weak convergence of $P_N (D_N^{m\sigma})^{-1} P_N$ and the uniform boundedness in the operator norm.

In order to proceed we need the following convergence results,

$$\|B_{mN}^{\sigma} - B_{m\infty}^{\sigma} P_N\|_\infty \to 0, \qquad \|C_{Nm}^{\sigma} - P_N C_{\infty m}^{\sigma}\|_\infty \to 0, \tag{18}$$

as $N \to \infty$, which we are going to prove below. Taking these for granted, we can continue to argue as follows

$$\begin{aligned} B_{mN}^{\sigma} (D_N^{m\sigma})^{-1} C_{Nm}^{\sigma} &= \left(B_{m\infty}^{\sigma} P_N + o(1)\right) (D_N^{m\sigma})^{-1} \left(P_N C_{\infty m}^{\sigma} + o(1)\right) \\ &= B_{m\infty}^{\sigma} P_N (D_N^{m\sigma})^{-1} P_N C_{\infty m}^{\sigma} + o(1) \\ &= B_{m\infty}^{\sigma} T_{\sigma(0,\theta)}^{-1} C_{\infty m}^{\sigma} + o(1). \end{aligned} \tag{19}$$

Therein $o(1)$ stands for a sequence of operators or matrices converging to zero in the operator norm. In the first step of the argument we only need the fact that the inverses $(D_N^{m\sigma})^{-1}$ are uniformly bounded in the operator norm and that the matrices $B_{m\infty}^{\sigma}$ and $C_{\infty m}^{\sigma}$ can be considered as bounded linear operators on ℓ^2. In the last step of the argument we need the weak convergence of $P_N (D_N^{m\sigma})^{-1} P_N$. Notice that the m rows of $B_{m\infty}^{\sigma}$ and the m columns of $C_{\infty m}^{\sigma}$ are in ℓ^2.

Having established (19) we can now conclude that

$$\frac{\det\ M^{\sigma}_{m+N}}{\det\ D^{m\sigma}_{N}} \ \to\ \det\left(A^{\infty\sigma}_{m} - B^{\sigma}_{m\infty}T^{-1}_{\sigma(0,\theta)}C^{\sigma}_{\infty m}\right).$$

Using Proposition 4.1 with $X = B^{\sigma}_{m\infty}$ and $Y = C^{\sigma}_{\infty m}$ it follows that the (i, j)-entry of the matrix in this determinant is given by

$$\sum_{k=1}^{\infty} \sigma^{-}_{p_i+k-m}(0)\ \sigma^{+}_{-q_j-k+m}(0).$$

Thus we arrive the conclusion of the theorem.

Let us now come back to the proof of the two outstanding claims (18) about convergence. Since $B^{\sigma}_{mN} - B^{\sigma}_{m\infty}P_N$ and $C^{\sigma}_{Nm} - P_N C^{\sigma}_{\infty m}$ have m of rows and columns, respectively, and m is fixed, it suffices to show that these rows and columns converge to zero in the ℓ^2-norm as $N \to \infty$. For $B^{\sigma}_{mN} - B^{\sigma}_{m\infty}P_N$, we can fix the row index $j = 0, \dots, m-1$ and consider the row vector

$$\left\{\sigma_{p_j-k-m}\left(\tfrac{r_j}{m+N}\right) - \sigma_{p_j-k-m}(0)\right\}_{0\le k\le N}$$

whose ℓ^2-norm can be estimated by

$$K_\alpha\left(\sum_{k=0}^{N}(1+|p_j-k-m|)^{-2\alpha}\right)^{\frac{1}{2}}\left\|\sigma_{[\frac{r_j}{m+N}]}(\theta)-\sigma_{[0]}(\theta)\right\|_{\alpha}$$

see estimate (7). The sum therein is bounded and the last term converges to zero as $N \to \infty$ because of the continuity of the map $x \mapsto \sigma_{[x]} \in \mathcal{C}^{\alpha}$.

For the term $C^{\sigma}_{Nm} - P_N C^{\sigma}_{m\infty}$, fix the column index $k = 0, \dots, m-1$ and consider the column vector

$$\left\{\sigma_{j+m-q_k}\left(\tfrac{m+j}{m+N}\right) - \sigma_{j+m-q_k}(0)\right\}_{0\le j\le N}$$

whose ℓ^2 can be estimated by

$$K_\alpha\left(\sum_{j=0}^{N}(1+|j+m-q_k|)^{-2\alpha}\left\|\sigma_{[\frac{m+j}{m+N}]}(\theta)-\sigma_{[0]}(\theta)\right\|^2_{\alpha}\right)^{\frac{1}{2}}.$$

We can now split the sum, e.g., into the parts $0 \le j < N^{\frac{1}{2}}$ and $N^{\frac{1}{2}} \le j \le N$, and estimates the first one by a constant times

$$\max_{0 \le x \le \frac{m+N^{\frac{1}{2}}}{m+N}} \left\| \sigma_{[x]}(\theta) - \sigma_{[0]}(\theta) \right\|_{\alpha}^{2},$$

while we estimate the last term by

$$2 \, \|\sigma\|^2_{\Sigma^{0,\alpha}} \sum_{j \ge N^{\frac{1}{2}}} (1 + |j + m - q_k|)^{-2\alpha}.$$

Both terms converge to zero as $N \to \infty$. This finishes the proof of (18), which only requires the assumption $\sigma \in \Sigma^{\alpha,0}$ with $\alpha > \frac{1}{2}$. □

This concludes the computation of the first limit.

4.2 Second Limit Computation

We are now going to compute the second limit (9). This means that again we have to relate the matrices $D_N^{m\sigma}$ to the matrices $\mathrm{op}_N \sigma$. What we will need is a refinement of Proposition 3.3. Recall the definition of the matrices $S_N^{m\sigma}$ and W_N^m given in (11) and (12). We start with a technical lemma.

Lemma 4.3 *Suppose $\psi \in \Sigma^{\alpha,0}$ with $\alpha > \frac{1}{2}$. Let B_N be a sequence of $(N+1) \times (N+1)$ matrices whose (j,k)-entries are equal to*

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\psi\left(\xi_{jk}^{(N)}, \theta\right) - \psi\left(\frac{j}{N}, \theta\right) \right) e^{-i(j-k)\theta} d\theta$$

where $\xi_{jk}^{(N)}$ satisfies $\frac{j}{N} \le \xi_{jk}^{(N)} \le \frac{m+j}{m+N}$. Then $\|B_N\|_2 = o(N^{\frac{1}{2}})$ as $N \to \infty$.

Proof Using estimate (7), the (j,k)-entry of B_N can be estimated by

$$|B_{jk}^{(N)}| \le K_\alpha (1 + |j-k|)^{-\alpha} \cdot \left\| \psi_{[\xi_{jk}^{(N)}]}(\theta) - \psi_{[\frac{j}{N}]}(\theta) \right\|_{\alpha}.$$

Taking into account that

$$0 \le \xi_{jk}^{(N)} - \frac{j}{N} \le \frac{m(N-j)}{N(m+N)} \le \frac{m}{m+N}$$

we can estimate the Hilbert-Schmidt norm of B_N as follow

$$\|B_N\|_2 \le K_\alpha \left(\sum_{j,k=0}^{N} (1+|j-k|)^{-2\alpha} \right)^{\frac{1}{2}} \sup_{|x-y|\le \frac{m}{m+N}} \left\| \psi_{[x]}(\theta) - \psi_{[y]}(\theta) \right\|_\alpha$$
$$\le \widehat{K}_\alpha \sqrt{N+1} \sup_{|x-y|\le \frac{m}{m+N}} \left\| \psi_{[x]}(\theta) - \psi_{[y]}(\theta) \right\|_\alpha .$$

Because the mapping $x \in [0,1] \mapsto \sigma_{[x]} \in \mathcal{C}^\alpha$ is continuous (hence uniformly continuous), we can conclude that the supremum converges to zero as $N \to \infty$. From this the statement follows. □

Lemma 4.4 *Let* $\sigma \in \Sigma^{\alpha,1}$ *with* $\alpha > \frac{1}{2}$. *Then*

$$\left\| S_N^{m\sigma} - W_N^m \left(\mathrm{op}_N \partial_x \sigma \right) \right\|_1 \to 0, \qquad N \to \infty.$$

Proof In the proof of Proposition 3.3 we have already shown that $S_N^{m\sigma} = W_N^m A_N$ where

$$A_N = \left((\partial_x \sigma)_{j-k} (\xi_{jk}^{(N)}) \right)_{j,k=0,1,2,\ldots,N}$$

with $\xi_{j,k}^{(N)}$ satisfying $\frac{j}{N} \le \xi_{jk}^{(N)} \le \frac{m+j}{m+N}$. Therefore

$$S_N^{m\sigma} - W_N^m \left(\mathrm{op}_N \partial_x \sigma \right) = W_N^m \left(A_N - \mathrm{op}_N \partial_x \sigma \right).$$

From (13) we know that the Hilbert-Schmidt norm $\|W_N^m\|_2 = O(N^{-\frac{1}{2}})$. Now the matrix $B_N := A_N - \mathrm{op}_N \partial_x \sigma$ has (j,k)-entry

$$\frac{1}{2\pi} \int_0^{2\pi} \left((\partial_x \sigma) \left(\xi_{jk}^{(N)}, \theta \right) - (\partial_x \sigma) \left(\frac{j}{N}, \theta \right) \right) e^{-i(j-k)\theta} d\theta.$$

Notice that $\psi := \partial_x \sigma \in \Sigma^{\alpha,0}$. Therefore, we can apply Lemma 4.3, which implies that the Hilbert-Schmidt norm $\|B_N\|_2 = o(N^{\frac{1}{2}})$. The lemma is established. □

Theorem 4.5 *Let* $\sigma \in \Sigma_0^{\alpha,1}$ *with* $\alpha > \frac{3}{2}$. *Then*

$$\frac{\det D_N^{m\sigma}}{\det \mathrm{op}_N \sigma} = \exp \left(\mathrm{tr} \left(W_N^m \, \mathrm{op}_N \left(\frac{\partial_x \sigma}{\sigma} \right) \right) + o(1) \right), \qquad N \to \infty.$$

Proof It follows from Proposition 3.4 that $\mathrm{op}_N \sigma$ is invertible for sufficiently large N and that its inverse is uniformly bounded in the operator norm,

$$\|(\mathrm{op}_N \sigma)^{-1}\|_\infty = O(1),$$

as $N \to \infty$. Hence, by the definition of $S_N^{m\sigma}$ we obtain

$$(\mathrm{op}_N\sigma)^{-1} D_N^{m\sigma} = P_N + (\mathrm{op}_N\sigma)^{-1} S_N^{m\sigma}$$

for all sufficiently large N. Taking the determinant,

$$\frac{\det D_N^{m\sigma}}{\det \mathrm{op}_N\sigma} = \det(P_N + A_N)$$

where we set

$$A_N := S_N^{m\sigma}(\mathrm{op}_N\sigma)^{-1}.$$

Since

$$\|S_N^{m\sigma}\|_1 = O(1), \qquad \|S_N^{m\sigma}\|_\infty = o(1), \qquad N \to \infty,$$

by Proposition 3.3 it follows that

$$\|A_N\|_1 = O(1), \qquad \|A_N\|_\infty = o(1), \qquad N \to \infty.$$

We conclude that $\|A_N^2\|_1 = o(1)$, and therefore, the regularized determinant

$$\begin{aligned}\det(P_N + A_N)\exp(-A_N) &= \det\left((P_N + A_N)\left(P_N - A_N + \frac{A_N^2}{2!} - \frac{A_N^3}{3!} + \cdots\right)\right)\\ &= \det\left(P_N - \frac{A_N^2}{2} + \frac{A_N^3}{3} - \cdots\right)\end{aligned}$$

tends to 1 as $N \to \infty$. It follows that

$$\det(P_N + A_N) \sim \det(\exp(A_N)) = \exp(\mathrm{tr}\, A_N), \qquad N \to \infty.$$

Thus we are left with analyzing the asymptotics of the trace of A_N. We will do this by approximating A_N in the trace norm. For convenience we let $o_1(1)$ stand for any sequence of matrices converging to zero in the trace norm (as $N \to \infty$).

From Proposition 3.4 we know that

$$(\mathrm{op}_N\sigma)^{-1} = \mathrm{op}_N(\sigma^{-1}) + B_N + C_N$$

with $\|B_N\|_1 = O(1)$, $\|C_N\|_\infty = o(1)$ and $\|\mathrm{op}_N(\sigma^{-1})\|_\infty = O(1)$ by Proposition 3.2(a). On the other hand,

$$S_N^{m\sigma} = W_N^m\,(\mathrm{op}_N\partial_x\sigma) + o_1(1)$$

by Lemma 4.4, and $\|W_N^m\|_\infty = O(N^{-1})$, $\|W_N^m\|_1 = O(1)$, and $\|(\mathrm{op}_N \partial_x \sigma)\|_\infty = O(1)$. Taking the product we see that

$$\begin{aligned} A_N &= \left(W_N^m \,(\mathrm{op}_N \partial_x \sigma) + o_1(1)\right)\left(\mathrm{op}_N(\sigma^{-1}) + B_N + C_N\right) \\ &= W_N^m \,(\mathrm{op}_N \partial_x \sigma)\left(\mathrm{op}_N(\sigma^{-1}) + B_N + C_N\right) + o_1(1) \\ &= W_N^m \,(\mathrm{op}_N \partial_x \sigma)\left(\mathrm{op}_N(\sigma^{-1})\right) + o_1(1). \end{aligned}$$

Therefore,

$$\mathrm{tr}(A_N) = \mathrm{tr}\left(W_N^m \,(\mathrm{op}_N \partial_x \sigma)\left(\mathrm{op}_N(\sigma^{-1})\right)\right) + o(1).$$

However, before proceeding with the trace calculation we are going to simplify the product of the matrices $\mathrm{op}_N \partial_x \sigma$ and $\mathrm{op}_N(\sigma^{-1})$. Notice that $\sigma^{-1} \in \Sigma^{\alpha,1}$ while $\partial_x \sigma \in \Sigma^{\alpha,0}$.

In order to deal with the product, we need more detailed results from [4]. The following formula,

$$\mathrm{op}_N(\phi\psi) - (\mathrm{op}_N\phi)(\mathrm{op}_N\overline{\psi})^* = -T_N(\phi,\psi) + H_N[\phi]H_N[\widetilde{\psi}]^t + J_N[\phi]J_N[\widetilde{\psi}]^t \quad (20)$$

was provided as formula (2.6) in [4] for variable-coefficient symbols. Therein, $\widetilde{\psi}(x,\theta) = \psi(x,-\theta)$, which in terms of Fourier coefficients means that $\widetilde{\psi}_j(x) = \psi_{-j}(x)$. Moreover, $T_N(\phi,\psi)$ is a new kind of matrix with (j,k)-entry

$$\frac{1}{2\pi}\int_0^{2\pi} \phi\left(\frac{j}{N},\theta\right)\left(\psi\left(\frac{k}{N},\theta\right) - \psi\left(\frac{j}{N},\theta\right)\right) e^{-i(j-k)\theta}\, d\theta,$$

$0 \le j,k \le N$, and the Hankel-type matrices

$$\begin{aligned} H_N[\phi] &= \left(\phi_{1+j+k}\left(\frac{j}{N}\right)\right)_{0\le j\le N,\ 0\le k<\infty}, \\ J_N[\phi] &= \left(\phi_{-1-N+j-k}\left(\frac{j}{N}\right)\right)_{0\le j\le N,\ 0\le k<\infty}, \end{aligned}$$

$J_N[\phi] = Q_N H_N[\widetilde{\phi}(1-x,\theta)]$ with the flip operator Q_N defined in (15) occur along with their transposes.[4] Finally,

$$(\mathrm{op}_N\overline{\psi})^* = \left(\psi_{j-k}\left(\frac{k}{N}\right)\right)_{0\le j,k\le N}$$

is the transpose of $\mathrm{op}_N\widetilde{\psi}$ (or the adjoint of $\mathrm{op}_N\overline{\psi}$). Note that

$$(\mathrm{op}_N\overline{\psi})^* = \mathrm{op}_N\psi + T_N(1,\psi).$$

There are several properties known for these operators and matrices. For instance in [4, Prop. 7] it is proved that

$$\left\|\mathrm{op}_N\psi - (\mathrm{op}_N\overline{\psi})^*\right\|_1 = o(N), \quad N\to\infty \tag{21}$$

whenever $\psi \in \Sigma^{\alpha,0}$, $\alpha > \frac{1}{2}$. Using this with $\psi = \sigma^{-1}$, we can conclude that

$$W_N^m(\mathrm{op}_N\partial_x\sigma)(\mathrm{op}_N(\sigma^{-1})) = W_N^m(\mathrm{op}_N\partial_x\sigma)(\mathrm{op}_N\overline{\sigma}^{-1})^* + o_1(1).$$

Notice here that $\|W_N^m\|_\infty = O(N^{-1})$ by (13) and $\|\mathrm{op}_N\partial_x\sigma\|_\infty = O(1)$ as observed earlier. Thus we arrive at

$$A_N = W_N^m(\mathrm{op}_N\partial_x\sigma)(\mathrm{op}_N\overline{\sigma}^{-1})^* + o_1(1).$$

Now we are going to apply formula (20). Our desired formula,

$$A_N = W_N^m\,\mathrm{op}_N\Big((\partial_x\sigma)(\sigma^{-1})\Big) + o_1(1) \tag{22}$$

will follow once we have shown that the product of W_N^m with

$$-T_N(\phi,\psi) + H_N[\phi]H_N[\tilde{\psi}]^t + J_N[\phi]J_N[\tilde{\psi}]^t, \tag{23}$$

where $\phi = \partial_x\sigma \in \Sigma^{\alpha,0}$ and $\psi = \sigma^{-1} \in \Sigma^{\alpha,1}$, tends to zero in the trace norm. Indeed, from [4, Lemma 1] it follows immediately that

$$\|T_N(\phi,\psi)\|_2 = o(N^{\frac{1}{2}})$$

[4] Formula (20) generalizes the familiar Widom's formula,

$$T_N(\phi\psi) = T_N(\phi)T_N(\psi) + P_N H(\phi)H(\widetilde{\psi})P_N + Q_N H(\widetilde{\phi})H(\psi)Q_N$$

for x-independent symbols.

whenever $\phi, \psi \in \Sigma^{\alpha,0}$, $\alpha > \frac{1}{2}$, which we combine with $\|W_N^m\|_2 = O(N^{-\frac{1}{2}})$ from (13). Furthermore, in [4, Prop. 5] it has been established that

$$\|H_N[\phi]\|_2 = O(1), \qquad \|H_N[\widetilde{\psi}]^t\|_2 = O(1)$$

whenever $\phi, \psi \in \Sigma^{\alpha,0}$, $\alpha > 1$. The same holds also for the other kind of Hankel operators. Combining this with $\|W_N^m\|_\infty = O(N^{-1})$, it follows finally that the product of W_N^m with (23) (where $\phi = \partial_x \sigma \in \Sigma^{\alpha,0}$ and $\psi = \sigma^{-1} \in \Sigma^{\alpha,1}$) tends to zero in the trace norm.

Therefore, the convergence (22) is proved and it follows

$$\operatorname{tr} A_N = \operatorname{tr}\left(W_N^m \operatorname{op}_N\left((\partial_x\sigma)(\sigma^{-1})\right)\right) + o(1), \tag{24}$$

which prove the theorem. □

What remains is to evaluate the trace expression in Theorem 4.5, which is done in what follows.

Proposition 4.6 *Let* $\sigma \in \Sigma_0^{\alpha,1}$ *with* $\alpha > \frac{3}{2}$. *Then*

$$\operatorname{tr}\left(W_N^m \operatorname{op}_N\left(\frac{\partial_x\sigma}{\sigma}\right)\right) = \frac{m}{2\pi}\int_0^{2\pi}\int_0^1 \log\frac{\sigma(x,\theta)}{\sigma(0,\theta)}\,dx\,d\theta + o(1)$$

as $N \to \infty$.

Proof Observing that $\partial_x\sigma/\sigma = \partial_x \log\sigma$ is continuous on $[0,1]\times\mathbb{T}$, we have

$$\operatorname{tr}\left(W_N^m \operatorname{op}_N(\partial_x\sigma/\sigma)\right) = \operatorname{tr}\left(W_N^m \operatorname{op}_N(\partial_x \log\sigma)\right).$$

The trace of the matrix on the right side is

$$\begin{aligned}
&\sum_{j=0}^{N} \frac{m(N-j)}{N(m+N)}\frac{1}{2\pi}\int_0^{2\pi}(\partial_x\log\sigma)\left(\frac{j}{N},\theta\right)d\theta \\
&\quad= \frac{mN}{m+N}\sum_{j=0}^{N}\frac{1}{N}\left(1-\frac{j}{N}\right)\frac{1}{2\pi}\int_0^{2\pi}(\partial_x\log\sigma)\left(\frac{j}{N},\theta\right)d\theta \\
&\quad= (m+o(1))\left(\frac{1}{2\pi}\int_0^1\int_0^{2\pi}(1-x)(\partial_x\log\sigma)(x,\theta)\,d\theta\,dx + o(1)\right) \\
&\quad= \frac{m}{2\pi}\int_0^1\int_0^{2\pi}(1-x)(\partial_x\log\sigma)(x,\theta)\,d\theta\,dx + o(1)
\end{aligned}$$

as $N \to \infty$. By switching the order of integration and performing integration by parts, the integral with respect to x in the last expression can be written as

$$-\log \sigma(0, \theta) + \int_0^1 \log \sigma(x, \theta)\, dx = \int_0^1 \log \frac{\sigma(x, \theta)}{\sigma(0, \theta)}\, dx.$$

This completes the proof. □

The second limit computation (9) is now established by the previous proposition with Theorem 4.5. Together with Theorem 4.2 this proves the main theorem (Theorem 2.1).

5 Miscellaneous Remarks

It is interesting to note that for a suitable class of analytic functions f it is possible to define $f(\mathrm{op}_N \sigma)$. Indeed, from Proposition 3.2(a) we know that

$$\|\mathrm{op}_N \sigma\|_\infty \le C_\alpha \|\sigma\|_{\Sigma^{\alpha,0}}.$$

Therefore, if Ω is a bounded open subset of the complex plane containing the spectra of $\mathrm{op}_N \sigma$ for all N, and f is an analytic function defined on Ω, then $f(\mathrm{op}_N \sigma)$ can be defined via a functional calculus.

Ideas of Widom's paper [9] can be used in order to improve the norm estimate under the smoothness condition $\sigma \in \Sigma^{\alpha,1}$, $\alpha > \frac{3}{2}$. That is, for each $\epsilon > 0$, one can obtain

$$\|\mathrm{op}_N \sigma\|_\infty \le (1 + \epsilon)\|\sigma\|_{\Sigma^{\alpha,1}} \tag{25}$$

for sufficiently large N. One can further generalize this estimate to

$$\|M^\sigma_{m+N}\|_\infty \le (1 + \epsilon)\|\sigma\|_{\Sigma^{\alpha,1}} \tag{26}$$

for sufficiently large N.

There exists another direction of improving the spectral norms. Indeed, a more delicate estimate of the uniform boundedness of $\mathrm{op}_N \sigma$ with weaker conditions on σ can be found in the work of Böttcher and Grudsky [1]. This is of importance if one is interested in the first order spectral asymptotics for quite general symbols σ.

The main remark we wish to make here is based on estimates (25) and (26) for defining $f(\mathrm{op}_N \sigma)$ and $f(M^\sigma_{m+N})$, while bearing in mind that Ω can be conveniently chosen as suitable prescribed set, on which f is required to be analytic. Let $\partial\Omega$ denote the boundary of Ω. Then one has

$$\mathrm{tr}\, f\left(\mathrm{op}_N \sigma\right) = G_f(\sigma) \cdot (N + 1) + E_f(\sigma) + o(1) \tag{27}$$

as $N \to \infty$, where

$$G_f(\sigma) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} f(\sigma(x,\theta))\, d\theta dx, \tag{28}$$

$$E_f(\sigma) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \log E[\sigma - \lambda]\, d\lambda. \tag{29}$$

The main result in this paper was the asymptotics of the determinants of the matrices M^{σ}_{m+N} as $N \to \infty$ (see Theorem 2.1 and the asymptotics (6)),

$$\det M^{\sigma}_{m+N} \sim G[\sigma]^{N+1} E[\sigma] K_m[\sigma] F_m[\sigma]$$

as $N \to \infty$. In much the same way, we have

$$\operatorname{tr} f(M^{\sigma}_{m+N}) = G_f(\sigma) \cdot (N+1) + E_f(\sigma) + K_f(m;\sigma) + F_f(m;\sigma) + o(1) \tag{30}$$

as $N \to \infty$, where

$$K_f(m;\sigma) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \log K_m[\sigma - \lambda]\; d\lambda, \tag{31}$$

$$F_f(m;\sigma) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \log F_m[\sigma - \lambda]\; d\lambda. \tag{32}$$

This can be viewed as the new feature in the generalization of the spectral asymptotics of $\mathrm{op}_N \sigma$.

We further remark that formulas (27) and (30) are the analogue of the classical Toeplitz case for the first order asymptotic formula for quite general symbols σ. It also bears a resemblance to the Wiener-Hopf or pseudodifferential operator case, which can be described by a general principle (see Widom [10]).

Lastly, in view of the classical Toeplitz case, we point out that formulas (29), (31) and (32) can be made more explicit without the contour integrals by the method of Widom [7]. For example, formula (31) is expected to take the form:

$$\frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \log K_m[\sigma - \lambda]\; d\lambda$$
$$= \frac{m}{2\pi} \int_0^{2\pi} \int_0^1 \left(f(\sigma)(x,\theta) - f(\sigma)(0,\theta) \right)\; dx\, d\theta.$$

This is seen to be the contribution of $K_m[\sigma]$ to the proposed $\operatorname{tr} f(M^{\sigma}_{m+N})$ formula.

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On Diagonalizable Quantum Weighted Hankel Matrices

František Štampach and Pavel Šťovíček

To the memory of Harold Widom (1932–2021)

Abstract A semi-infinite weighted Hankel matrix with entries defined in terms of basic hypergeometric series is explicitly diagonalized as an operator on $\ell^2(\mathbb{N}_0)$. The approach uses the fact that the operator commutes with a diagonalizable Jacobi operator corresponding to Al-Salam–Chihara orthogonal polynomials. Yet another weighted Hankel matrix, which commutes with a Jacobi operator associated with the continuous q-Laguerre polynomials, is diagonalized. As an application, several new integral formulas for selected quantum orthogonal polynomials are deduced. In addition, an open research problem concerning a quantum Hilbert matrix is also mentioned.

Keywords Weighted Hankel matrix · Jacobi matrix · Quantum Hilbert matrix · Al-Salam–Chihara polynomials · q-Laguerre polynomials

1 Introduction

A great account of research of Harold Widom was devoted to Hankel matrices [13, 14]. A prominent Hankel matrix of significant interest is the famous Hilbert matrix

$$H_\nu := \begin{pmatrix} \frac{1}{\nu} & \frac{1}{\nu+1} & \frac{1}{\nu+2} & \cdots \\ \frac{1}{\nu+1} & \frac{1}{\nu+2} & \frac{1}{\nu+3} & \cdots \\ \frac{1}{\nu+2} & \frac{1}{\nu+3} & \frac{1}{\nu+4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \nu \in \mathbb{R} \setminus (-\mathbb{N}_0), \tag{1}$$

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which, when regarded as an operator on $\ell^2(\mathbb{N}_0)$, is one of a very few non-trivial examples of Hankel matrices that admit an explicit diagonalization. With the aid of certain previously known identities due to Magnus and Shanker, the diagonalization of H_ν was done by Rosenblum [9], who found an integral operator whose matrix representation with respect to a suitably chosen orthonormal basis coincides with H_ν and diagonalizes the integral operator. The passing to the integral representation is not necessary though. Alternatively, the diagonalization of H_ν can be treated by noting that H_ν commutes with a Jacobi operator with an explicitly solvable spectral problem [7], an approach sometimes referred to as *the commutator method* [15].

Let us explain the basic idea of the commutator method in more detail. To a given operator H, whose spectral analysis is the ultimate goal, we seek a commuting operator J with simple spectrum and solvable spectral problem. Suppose that λ is an eigenvalue of J and $\phi \in \operatorname{Ker}(J-\lambda)$ an eigenvector. Then from equations $J\phi = \lambda\phi$ and $JH = HJ$, one infers that $H\phi \in \operatorname{Ker}(J-\lambda)$. Since the eigenvalue λ is simple, there is a number $h = h(\lambda)$ such that $H\phi = h\phi$. If we can assure that $\phi_0 \neq 0$, we may suppose $\phi_0 = 1$, then the eigenvalue h can be computed as follows:

$$h = h\phi_0 = (H\phi)_0 = \sum_{n=0}^{\infty} H_{0,n}\phi_n. \tag{2}$$

The Askey scheme of hypergeometric orthogonal polynomials and their q-analogues [8] can serve as a rich source of diagonalizable tridiagonal matrix operators, which follows from the well-known relation between spectral properties of Jacobi operators and orthogonal polynomials [1, 6]. The simple tridiagonal structure of the Jacobi matrix is helpful when trying to find a commuting Jacobi operator J in the commutant of H (i.e., in the space of commuting operators). Moreover, since the off-diagonal entries of J are non-vanishing, the spectrum of J is always simple. Of course, the spectrum of J need not be only discrete, and therefore the basic idea of the commutator method described in the preceding paragraph is to be generalized.

Suppose J is a self-adjoint Jacobi operator determined by the tridiagonal matrix

$$J = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3}$$

with $\beta_n \in \mathbb{R}$ and $\alpha_n \in \mathbb{R} \setminus \{0\}$, and $\{\phi_n\}_{n=0}^{\infty}$ is the sequence of corresponding orthonormal polynomials defined recursively by the equations

$$\begin{gathered} (\beta_0 - x)\phi_0(x) + \alpha_0\phi_1(x) = 0, \\ \alpha_{n-1}\phi_{n-1}(x) + (\beta_n - x)\phi_n(x) + \alpha_n\phi_{n+1}(x) = 0, \quad n \geq 1, \end{gathered} \tag{4}$$

and normalization $\phi_0(x) = 1$. Due to the self-adjointness of J, the polynomials $\{\phi_n \mid n \in \mathbb{N}_0\}$ form an orthonormal basis of $L^2(\mathbb{R}, \mathrm{d}\mu)$, where μ is a unique probability measure on $\mathbb{R}$,

$$\int_{\mathbb{R}} \phi_m(x)\phi_n(x)\mathrm{d}\mu(x) = \delta_{m,n}, \quad m, n \in \mathbb{N}_0.$$

Moreover, if we denote by $\{e_n \mid n \in \mathbb{N}_0\}$ the standard basis of $\ell^2(\mathbb{N}_0)$, then the unitary mapping

$$U : \ell^2(\mathbb{N}_0) \to L^2(\mathbb{R}, \mathrm{d}\mu) : e_n \mapsto \phi_n$$

diagonalizes J, i.e, $UJU^{-1} = T_{\mathrm{id}}$, where T_{id} is the operator of multiplication by the independent variable acting on $L^2(\mathbb{R}, \mathrm{d}\mu)$.

Now, if J commutes with a self-adjoint operator H, then there exists a measurable function h such that $H = h(J)$; see, for example [3, Thm. 1.4, p. 414]. Having the spectral representation of J, a determination of h is the last step of the spectral analysis of H. Since $UHU^{-1} = T_h$, where T_h is the multiplication operator by h, the function h can be found using the equation

$$h = T_h 1 = UHU^{-1}\phi_0 = UHe_0 = \sum_{n=0}^{\infty} H_{0,n}\phi_n, \tag{5}$$

cf. (2). Notice that $1 \in L^2(\mathbb{R}, \mathrm{d}\mu)$ since μ is a probability measure.

The article is organized as follows. In Sect. 2, a brief summary of the current state of the art of the research focused on diagonalizable weighted Hankel matrices is given. Recall the Askey scheme [8] is divided into two parts: the hypergeometric Askey scheme and their q-hypergeometric analogues. First, we explain that the Hilbert matrix is, in a sense, the only Hankel matrix which can be diagonalized by applying the commutator method to Jacobi operators associated to polynomial families from the hypergeometric Askey scheme. When more degrees of freedom are introduced to the problem by adding non-trivial weights, several explicitly diagonalizable weighted Hankel matrices have already been found. When passing to the q-Askey scheme, the applicability of the commutator method is fairly unexplored. As a related and interesting research project, we mention in Sect. 2 an open problem concerning diagonalizable quantum analogues of the Hilbert matrix.

Next, in Sect. 3, we initiate a study on diagonalizable weighted Hankel matrices commuting with Jacobi operators associated to polynomial families from the q-Askey scheme. First, as the main result of this article, we diagonalize a three-parameter family of weighted Hankel matrices that are found in the commutant of the Jacobi matrix associated to the Al-Salam–Chihara polynomials (Theorem 1). We use this result to diagonalize another weighted Hankel matrix corresponding to the continuous q-Laguerre polynomials (Theorem 2). As an application, we conclude Sect. 3 by deriving several integral formulas for the aforementioned quantum orthogonal polynomials. Finally, selected identities for the q-hypergeometric series, which are needed in proofs, are listed in the Appendix for reader's convenience.

2 The State of the Art and an Open Problem

There are only very few examples of Hankel matrices, regarded as operators on $\ell^2(\mathbb{N}_0)$, whose spectral problem is solvable explicitly or in terms of standard families of special functions. This contrasts the situation for other well known classes of special operators such as Jacobi, Schrödinger, Toeplitz, CMV, etc., where many solvable models exist and find various applications. This lack of concrete solvable models with Hankel matrices or their weighted generalizations served as a motivation for a research whose recent achievements are briefly summarized below.

2.1 The State of the Art

In [7], the authors observed that the three-parameter matrix $B = B(a, b, c)$ with entries

$$B_{m,n} = \frac{\Gamma(m+n+a)}{\Gamma(m+n+b+c)}\sqrt{\frac{\Gamma(m+b)\Gamma(m+c)\Gamma(n+b)\Gamma(n+c)}{\Gamma(m+a)\,m!\,\Gamma(n+a)\,n!}},$$

for $m, n \in \mathbb{N}_0$, regarded as an operator on $\ell^2(\mathbb{N}_0)$, commutes with the Jacobi matrix (3), where

$$\alpha_n = -\sqrt{n(n-1+a)(n-1+b)(n-1+c)}$$

and

$$\beta_n = n(n-1+c) + (n+a)(n+b).$$

For parameters a, b, c from a suitable domain, this interesting observation yields an explicit diagonalization of B since the commuting Jacobi operator is diagonalizable with the aid of a family of hypergeometric orthogonal polynomials from the Askey scheme called the continuous dual Hahn polynomials. As the entries of B are of the form

$$B_{m,n} = w_m h_{m+n} w_n, \quad m, n \in \mathbb{N}_0,$$

for

$$w_n = \sqrt{\frac{\Gamma(n+b)\Gamma(n+c)}{\Gamma(n+a)\,n!}} \quad \text{and} \quad h_n = \frac{\Gamma(m+n+a)}{\Gamma(m+n+b+c)},$$

B is a so-called weighted Hankel matrix. In particular, if $a = b$ and $c = 1$, $w_n = 1$ for all $n \in \mathbb{N}_0$ then B becomes a Hankel matrix. In fact, $B(\nu, \nu, 1)$ coincides with the Hilbert matrix (1) and hence the commutator method worked out in detail in [7] provides an alternative way for the diagonalization of the Hilbert matrix.

A natural question is whether there are other Hankel matrices commuting with the diagonalizable Jacobi matrices from the hypergeometric Askey scheme. Unfortunately, the answer is negative. More precisely, it was proven in [11] that, up to an inessential alternating factor, a scalar multiple of the Hilbert matrix is the only Hankel matrix with ℓ^2-columns and rank greater than 1 that can be found in commutants of Jacobi matrices from the Askey scheme. This fact emphasizes even more the prominent role of the Hilbert matrix.

On the other hand, if the class of considered Jacobi operators is slightly extended by adding Jacobi operators diagonalizable with the aid of the Stieltjes–Carlitz polynomials [4], four more diagonalizable Hankel matrices were found in [12, Thm. 6.1] only recently. Stieltjes–Carlitz polynomials do not belong to the hypergeometric Askey scheme since they are not given by terminating hypergeometric series. Rather than that, Stieltjes–Carlitz polynomials are intimately related to Jacobian elliptic functions.

When weighted Hankel matrices are considered, several more matrices, in addition to the above mentioned matrix B, were successfully diagonalized by applying the commutator method to Jacobi matrices from the Askey scheme. Namely, in [10], four families of weighted Hankel matrices were diagonalized with the aid of Hermite, Laguerre, Meixner, Meixner–Pollaczek, and dual Hahn polynomials.

When passing to the q-Askey scheme, i.e., quantum analogues of the classical orthogonal polynomials and corresponding diagonalizable Jacobi operators, the above problems have not been explored yet. In Sect. 3, we initiate the study by diagonalizing two weighted Hankel matrices that commute with Jacobi operators associated to Al-Salam–Chihara and continuous q-Laguerre polynomials. Another interesting question is whether a certain quantum analogue to the Hilbert matrix commutes with a tridiagonal matrix and possibly can be diagonalized with the aid of the q-Askey scheme. This open problem is partly discussed in the next subsection.

2.2 An Open Problem: The Quantum Hilbert Matrix

By the quantum Hilbert matrix, one may understand a Hankel matrix with entries dependent on a parameter q which, possibly after a suitable scaling, tend to the entries of the Hilbert matrix as $q \to 1$. Such a q-analogue of finite order has already appeared in the literature. In [2], the authors derived formulas for the determinant and the inverse of the finite quantum Hilbert matrix whose (m, n)-th entry equals

$$\frac{[\nu]_q}{[m+n+\nu]_q},$$

where

$$[\alpha]_q := \frac{q^{\alpha/2} - q^{-\alpha/2}}{q^{1/2} - q^{-1/2}}$$

is the symmetric q-deformation of a complex number α. Notice that $[\alpha]_q \to \alpha$, as $q \to 1$.

In greater generality, a reasonable candidate for the quantum analogue of the Hilbert matrix can be found in the three-parameter family of semi-infinite Hankel matrices $\mathcal{H}_\nu = \mathcal{H}_\nu(q; \epsilon)$ defined by

$$(\mathcal{H}_\nu)_{m,n} := \frac{q^{\epsilon(m+n)}}{1 - q^{m+n+\nu}}, \quad m, n \in \mathbb{N}_0,$$

where $q \in (0, 1)$, $\nu \in \mathbb{R} \setminus (-\mathbb{N}_0)$, and $\epsilon > 0$. Up to an unimportant multiplicative factor, $\mathcal{H}_\nu(q; 1/2)$ is a semi-infinite version of the quantum Hilbert matrix from [2]. One can check that $\mathcal{H}_\nu$ determines a compact operator on $\ell^2(\mathbb{N}_0)$, for example, by applying Widom's criterion [13, Thm. 3.2]. In fact, it is not difficult to see that $\mathcal{H}_\nu$ is actually trace class. More concrete results on spectral properties of $\mathcal{H}_\nu$ are definitely of interest. The most accessible cases seem to be $\epsilon = 1/2$ and $\epsilon = 1$.

Further, let us consider the specific case when $\epsilon = \nu = 1$ for simplicity and denote $\mathcal{G} := \mathcal{H}_1(q; 1/2)$. Hence, we consider the quantum analogue of the classical Hilbert matrix whose entries are the reciprocal quantum integers

$$\mathcal{G}_{m,n} = \frac{q^{m+n}}{1 - q^{m+n+1}}, \quad m, n \in \mathbb{N}_0.$$

Hoping for a diagonalization of $\mathcal{G}$ possibly in terms of the basic hypergeometric series, one may try to apply the commutator method. Surprisingly, $\mathcal{G}$ commutes with the Jacobi operator $\mathcal{J}$ given by (3) and sequences

$$\alpha_n = -\left(q^{-(n+1)/2} - q^{(n+1)/2}\right)^2$$

and

$$\beta_n = -4 + \left(q^{-1/2} + q^{1/2}\right)\left(q^{-n-1/2} + q^{n+1/2}\right).$$

Indeed, the commutation relation $\mathcal{G}\mathcal{J} = \mathcal{J}\mathcal{G}$ can be straightforwardly verified. The operator $\mathcal{J}$ does not correspond to any polynomial family listed in the q-Askey scheme, however. Moreover, to our best knowledge, properties of this operator or the corresponding family of orthogonal polynomials have not been studied yet. Such properties, as for example generating function formulas for the orthogonal polynomials, would be of interest on their own regardless the connection to the quantum Hilbert matrix.

Without going into details, let us remark that $\mathcal{J}$ determines an unbounded self-adjoint Jacobi operator (i.e. $\mathcal{J}$ restricted to the span of $\{e_n \mid n \in \mathbb{N}_0\}$ is essentially self-adjoint) which is positive, invertible, and has discrete spectrum. The matrix entries of the inverse read

$$\left(\mathcal{J}^{-1}\right)_{m,n} = \sum_{k=\max(m,n)}^{\infty} \frac{1}{\left(q^{-(k+1)/2} - q^{(k+1)/2}\right)^2}, \quad m, n \in \mathbb{N}_0.$$

Nevertheless, whether it is possible to analyze spectral properties of $\mathcal{G}$ or $\mathcal{J}$ in a greater detail possibly in terms of commonly known special functions remains an open problem.

3 Two Diagonalizable Quantum Weighted Hankel Matrices

We diagonalize a three-parameter family of weighted Hankel matrices with entries given in terms of the q-hypergeometric ${}_0\phi_1$-function. Recall the definition of the general q-hypergeometric series [5]:

$${}_p\phi_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| q;\, z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_p; q)_n}{(b_1, \dots, b_q; q)_n} (-1)^{(1+q-p)n} q^{(1+q-p)n(n-1)/2} \frac{z^n}{(q; q)_n},$$

where $(a_1, \dots, a_p; q)_n := (a_1; q)_n \dots (a_p; q)_n$ and

$$(a; q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right)$$

is the q-Pochhamer symbol. The index n can be taken ∞, the convergence of the infinite product is guaranteed by the assumption $|q| < 1$. In the notation, we follow the book of Gasper and Rahman [5].

The weighted Hankel matrix to be diagonalized is found in the commutant of the Jacobi matrix associated to the Al-Salam–Chihara polynomials. Next, we also diagonalize another weighted Hankel matrix with more explicit entries and commuting with the Jacobi matrix associated to the continuous q-Laguerre polynomials. Lastly, as an application, we obtain several integral formulas for the aforementioned orthogonal polynomials that seem to be new.

3.1 The Case of Al-Salam–Carlitz Polynomials

We diagonalize the weighted Hankel matrix H with entries $H_{m,n} = w_m h_{m+n} w_n$, where the weight reads

$$w_n = \frac{(-a)^n q^{n(n-1)/2}}{\sqrt{(q, ab; q)_n}} \tag{6}$$

and the Hankel part is determined by the basic hypergeometric series

$$h_n = {}_0\phi_1\left(\begin{matrix} - \\ qb/a \end{matrix}\,\middle|\, q;\, \frac{q^{2-n}}{a^2}\right), \tag{7}$$

for $n \in \mathbb{N}_0$. Hence

$$H_{m,n} = \frac{(-a)^{n+m} q^{n(n-1)/2+m(m-1)/2}}{\sqrt{(q, ab; q)_m (q, ab; q)_n}}\, {}_0\phi_1\left(\begin{matrix} - \\ qb/a \end{matrix}\,\middle|\, q;\, \frac{q^{2-m-n}}{a^2}\right), \quad m, n \in \mathbb{N}_0. \tag{8}$$

If needed, we will write $H = H(a, b)$ to emphasize the dependence on the parameters a and b and similarly for $h_n = h_n(a, b)$ and $w_n = w_n(a, b)$. The dependence on q is always suppressed in the notation. The range for the parameters is restricted to $q \in (0, 1)$, $0 < |a| < 1$, and $|b| < 1$.

Let us also introduce the Jacobi operator $J = J(a, b)$ of the form (3) with entries determined by the sequences

$$\alpha_n = \sqrt{(1 - q^{n+1})(1 - abq^n)} \quad \text{and} \quad \beta_n = (a + b)q^n, \tag{9}$$

for $n \in \mathbb{N}_0$.

Proposition 1 *The matrices H and J commute.*

Proof Recall that the second Jackson q-Bessel function

$$J_\nu(x; q) := \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(\begin{matrix} - \\ q^{\nu+1} \end{matrix}\,\middle|\, q;\, -\frac{x^2 q^{\nu+1}}{4}\right)$$

solves the q-difference equation [6, Eq. (14.1.23)]

$$J_\nu(q^{-1/2}x;q) - \left(q^{-\nu/2} + q^{\nu/2}\right) J_\nu(x;q) + \left(1 + \frac{x^2}{4}\right) J_\nu(q^{1/2}x;q) = 0.$$

It follows that sequence (7) satisfies the recurrence

$$(ab - q^{1-k})h_{k-1} - a(a+b)h_k + a^2 h_{k+1} = 0,$$

for $k \in \mathbb{Z}$. Next, by writing $k = m+n$ in the above equation, one verifies the identity

$$(\beta_m - \beta_n)H_{m,n} + \alpha_{m-1}H_{m-1,n} + \alpha_m H_{m+1,n} - \alpha_{n-1}H_{m,n-1} - \alpha_n H_{m,n+1} = 0,$$

for the matrix entries $H_{m,n} = w_m h_{m+n} w_n$, where w_n is as in (6) and α_n and β_n given by (9); by convention, we also put $H_{-1,n} = H_{n,-1} := 0$ for any $n \in \mathbb{N}_0$. This means nothing but the matrix equality $JH - HJ = 0$. □

The orthogonal polynomials determined by the Jacobi parameters (9) are the Al-Salam–Chihara polynomials $Q_n(x;a,b\,|\,q)$ since they are given by the recurrence [8, Eq. (14.8.4)]

$$(1-q^n)(1-abq^{n-1})Q_{n-1}(x;a,b\,|\,q) + \big((a+b)q^n - 2x\big)Q_n(x;a,b\,|\,q) + Q_{n+1}(x;a,b\,|\,q) = 0$$

and $Q_{-1}(x;a,b\,|\,q) = 0$, $Q_0(x;a,b\,|\,q) = 1$. The corresponding orthonormal polynomials fulfill

$$\phi_n(2x) = \frac{1}{\sqrt{(q,ab;q)_n}} Q_n(x;a,b\,|\,q), \quad n \in \mathbb{N}_0, \tag{10}$$

and form an orthonormal basis in the Hilbert space $L^2((-1,1),\mathrm{d}\mu)$, where μ is the absolutely continuous orthogonality measure determined by the density

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(\cos\theta) = \frac{(q,ab;q)_\infty}{2\pi\sin\theta}\left|\frac{(e^{2\mathrm{i}\theta};q)_\infty}{(ae^{\mathrm{i}\theta},be^{\mathrm{i}\theta};q)_\infty}\right|^2, \tag{11}$$

for $x = \cos\theta$ and $\theta \in (0,\pi)$; see [8, § 14.8].

Thus, by the commutator method, there exists a Borel function h on $(-1,1)$ such that $UHU^{-1} = T_h$, where the unitary mapping $U : \ell^2(\mathbb{N}_0) \to L^2((-1,1),\mathrm{d}\mu)$ is determined by the correspondence $U : e_n \mapsto \phi_n(2\,\cdot)$, $n \in \mathbb{N}_0$. The function h satisfies

$$h(x) = h(x)\phi_0(2x) = \sum_{n=0}^{\infty} H_{0,n}\phi_n(2x) = w_0 \sum_{n=0}^{\infty} h_n w_n \phi_n(2x), \tag{12}$$

which means that

$$h(x)=\sum_{n=0}^{\infty}\frac{(-a)^n q^{n(n-1)/2}}{(q,ab;q)_n}\,{}_0\phi_1\!\left(\begin{matrix}-\\ qb/a\end{matrix}\,\middle|\,q;\ \frac{q^{2-n}}{a^2}\right)Q_n(x;a,b\,|\,q), \tag{13}$$

where we have substituted from (6), (7), and (10). At this point, the diagonalization of H is a matter of a possible simplification of the expression on the right-hand side in (13), which miraculously simplifies, indeed.

Proposition 2 *For $\theta\in(0,\pi)$, we have*

$$h(\cos\theta)=\frac{(ae^{-\mathrm{i}\theta},ae^{\mathrm{i}\theta},qe^{-\mathrm{i}\theta}/a,qe^{\mathrm{i}\theta}/a;q)_\infty}{(ab,qb/a;q)_\infty}. \tag{14}$$

Proof The starting point is the generating function formula for the Al-Salam–Carlitz polynomials [8, Eq. (14.8.16)]

$$\sum_{n=0}^{\infty}\frac{(\gamma;q)_n\,t^n}{(q;ab;q)_n}\,Q_n(x;a,b\,|\,q)=\frac{(\gamma e^{\mathrm{i}\theta}t;q)_\infty}{(e^{\mathrm{i}\theta}t;q)_\infty}\,{}_3\phi_2\!\left(\begin{matrix}\gamma,ae^{\mathrm{i}\theta},be^{\mathrm{i}\theta}\\ ab,\gamma e^{\mathrm{i}\theta}t\end{matrix}\,\middle|\,q;\ e^{-\mathrm{i}\theta}t\right),$$

where $|t|<1$ and $\gamma\in\mathbb{C}$. Here and everywhere below, $x=\cos\theta$ with $\theta\in(0,\pi)$ fixed. By putting $t=z/\gamma$ and sending $\gamma\to\infty$ in the above formula, we obtain

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(-z)^n}{(q;ab;q)_n}\,Q_n(x;a,b\,|\,q)=(ze^{\mathrm{i}\theta};q)_\infty\,{}_2\phi_2\!\left(\begin{matrix}ae^{\mathrm{i}\theta},be^{\mathrm{i}\theta}\\ ab,ze^{\mathrm{i}\theta}\end{matrix}\,\middle|\,q;\ ze^{-\mathrm{i}\theta}\right),$$

for $z\in\mathbb{C}$. Next, by setting $z=aq^{-m}$, multiplying both sides by

$$\frac{q^{m(m+1)}}{a^{2m}\,(q;qb/a;q)_m},$$

and summing up for $m=0,1,\dots$, we deduce from (13) the formula

$$h(x)=\sum_{m=0}^{\infty}\frac{(aq^{-m}e^{\mathrm{i}\theta};q)_\infty}{(q;qb/a;q)_m}\frac{q^{m(m+1)}}{a^{2m}}\,{}_2\phi_2\!\left(\begin{matrix}ae^{\mathrm{i}\theta},be^{\mathrm{i}\theta}\\ ab,aq^{-m}e^{\mathrm{i}\theta}\end{matrix}\,\middle|\,q;\ aq^{-m}e^{-\mathrm{i}\theta}\right).$$

Further, we apply formula (28) with

$$a\leftarrow ae^{\mathrm{i}\theta},\quad b\leftarrow\frac{a}{b}q^{-m},\quad c\leftarrow aq^{-m}e^{\mathrm{i}\theta},\quad z\leftarrow be^{-\mathrm{i}\theta},$$

which yields

$$h(x) = \frac{(be^{-i\theta};q)_\infty}{(ab;q)_\infty}\sum_{m=0}^{\infty}\frac{(aq^{-m}e^{i\theta};q)_\infty}{(q;qb/a;q)_m}\frac{q^{m(m+1)}}{a^{2m}}\,{}_2\phi_1\left(\begin{matrix}ae^{i\theta}, aq^{-m}/b\\ aq^{-m}e^{i\theta}\end{matrix}\,\middle|\, q;\, be^{-i\theta}\right). \tag{15}$$

As the next step, we apply identity (29) to the ${}_2\phi_1$-function in (15). Moreover, the coefficients given in terms of q-Pochhammer symbols slightly simplify with the aid of the identity

$$(\alpha q^{-m}, q^{m+1}/\alpha; q)_\infty = (-\alpha)^m q^{-m(m+1)/2}(\alpha, q/\alpha; q)_\infty,$$

which holds true for all $\alpha \in \mathbb{C}\setminus\{0\}$ and $m \in \mathbb{N}_0$. The resulting expression reads

$$\begin{aligned} h(x) = {} & \frac{(be^{-i\theta}, ae^{-i\theta}, ae^{i\theta}, qe^{-i\theta}/a; q)_\infty}{(ab, qb/a, e^{-2i\theta}; q)_\infty} \\ & \times \sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(q;q)_m}\left(-\frac{e^{i\theta}}{a}\right)^m {}_2\phi_1\left(\begin{matrix}be^{i\theta}, qe^{i\theta}/a\\ qe^{2i\theta}\end{matrix}\,\middle|\, q;\, q^{m+1}\right) \\ & + \text{c.c.}, \end{aligned} \tag{16}$$

where the abbreviation c.c. stands for the term which equals the complex conjugate of the previous term.

Using the definition of the ${}_2\phi_1$-function in (16) and interchanging the order of summation, we observe that the first term in (16), up to the multiplicative factor, is equal to

$$\sum_{n=0}^{\infty}\frac{(be^{i\theta}, qe^{i\theta}/a; q)_n}{(q, qe^{2i\theta}; q)_n}q^n\,{}_0\phi_0\left(\begin{matrix}-\\ -\end{matrix}\,\middle|\, q;\, \frac{e^{i\theta}q^{n+1}}{a}\right) = (qe^{i\theta}/a;q)_\infty\,{}_2\phi_1\left(\begin{matrix}be^{i\theta}, 0\\ qe^{2i\theta}\end{matrix}\,\middle|\, q;\, q\right),$$

where we have used (27). Hence we have

$$\begin{aligned} h(x) = {} & \frac{(ae^{-i\theta}, ae^{i\theta}, qe^{-i\theta}/a, qe^{i\theta}/a; q)_\infty}{(ab, qb/a; q)_\infty} \\ & \times \left[\frac{(be^{-i\theta};q)_\infty}{(e^{-2i\theta};q)_\infty}\,{}_2\phi_1\left(\begin{matrix}be^{i\theta}, 0\\ qe^{2i\theta}\end{matrix}\,\middle|\, q;\, q\right) + \text{c.c.}\right]. \end{aligned}$$

Finally, it suffices to notice that, by (30), the expression in the square brackets equals 1. □

In total, we have deduced a full spectral representation of H which is summarized in the next theorem.

Theorem 1 *For $a, b \in \mathbb{R}$ such that $0 < |a| < 1$ and $|b| < 1$, the operator H with matrix entries (8) is unitarily equivalent to the operator of multiplication by the function*

$$h(x) = \frac{(ae^{-\mathrm{i}\theta}, ae^{\mathrm{i}\theta}, qe^{-\mathrm{i}\theta}/a, qe^{\mathrm{i}\theta}/a; q)_\infty}{(ab, qb/a; q)_\infty}, \quad x = \cos\theta,$$

acting on $L^2((-1,1), \mathrm{d}\mu)$, where the measure μ is given by (11). In particular, the spectrum of H is simple, purely absolutely continuous, and fills the interval

$$\sigma_{\mathrm{ac}}(H) = \frac{1}{(ab, qb/a; q)_\infty}\left[(|a|, q/|a|; q)_\infty^2, (-|a|, -q/|a|; q)_\infty^2\right].$$

Consequently, the operator norm of H reads

$$\|H\| = \frac{(-|a|, -q/|a|; q)_\infty^2}{|(ab, qb/a; q)_\infty|}.$$

3.2 The Case of Continuous q-Laguerre Polynomials

With the aid of the results of the previous subsection, we diagonalize the weighted Hankel matrix

$$\tilde{H}_{m,n} = \tilde{H}_{m,n}(\alpha; q) := \frac{q^{(m-n)^2/2}\,(q^{\alpha+1}; q)_{m+n}}{\sqrt{(q^2, q^{2\alpha+2}; q^2)_m (q^2, q^{2\alpha+2}; q^2)_n}}, \quad m, n \in \mathbb{N}_0, \tag{17}$$

where $q \in (0, 1)$ and $\alpha > -1$. In the course of the diagonalization, we will work with the closely related matrix

$$G_{m,n} = G_{m,n}(a; q) := \frac{q^{(m-n)^2/4}\,(aq^{1/4}; q^{1/2})_{m+n}}{\sqrt{(q, a^2q^{1/2}; q)_m (q, a^2q^{1/2}; q)_n}}, \quad m, n \in \mathbb{N}_0, \tag{18}$$

rather than $\tilde{H}$. Notice that $G(q^{\alpha+1/2}; q^2) = \tilde{H}(\alpha; q)$. The relation between the matrices G and H from (8) reveals the following statement.

Proposition 3 *One has*

$$G(a; q) = AH(a, aq^{1/2}) + BH(aq^{1/2}, a), \tag{19}$$

where

$$A = -\frac{q^{1/4}}{a(1-q^{1/2})(q^{1/4}/a;q^{1/2})_\infty} \quad \textit{and} \quad B = \frac{1}{(q^{1/4}/a;q^{1/2})_\infty}.$$

Consequently, the matrices $G(a;q)$ and $J(a,aq^{1/2})$ commute.

Proof Equation (19) means that

$$AH_{m,n}(a,aq^{1/2}) + BH_{m,n}(aq^{1/2},a) = G_{m,n}(a;q), \quad \forall m,n \in \mathbb{N}_0,$$

which, when we use (6) and (18), gets the form

$$A\left(-aq^{-1/2}\right)^k h_k(a,aq^{1/2}) + B\,(-a)^k\, h_k(aq^{1/2},a) = q^{-k^2/4}(q^{1/4}a;q^{1/2})_k,$$

for $k := m+n$. Using also (7), we see that the claim holds provided the identity

$$A\left(-aq^{-1/2}\right)^k {}_0\phi_1\left(\begin{matrix}-\\ q^{3/2}\end{matrix}\,\middle|\, q;\ \frac{q^{2-k}}{a^2}\right) + B\,(-a)^k\, {}_0\phi_1\left(\begin{matrix}-\\ q^{1/2}\end{matrix}\,\middle|\, q;\ \frac{q^{1-k}}{a^2}\right) = q^{-k^2/4}(q^{1/4}a;q^{1/2})_k \tag{20}$$

is true for all $k \in \mathbb{N}_0$.

It is straightforward to decompose the q-exponential (27) into the sum of its odd and even part. It results in the identity

$$-\frac{z}{1-q}\,{}_0\phi_1\left(\begin{matrix}-\\ q^3\end{matrix}\,\middle|\, q^2;\ q^3z^2\right) + {}_0\phi_1\left(\begin{matrix}-\\ q\end{matrix}\,\middle|\, q^2;\ qz^2\right) = (z;q)_\infty,$$

for $z \in \mathbb{C}$. By substituting $z = q^{-k+1/2}/a$ and using that

$$q^{-k^2/2}\,(aq^{1/2};q)_k = \frac{(q^{-k+1/2}/a;q)_\infty}{(q^{1/2}/a;q)_\infty}\,(-a)^k,$$

for any $k \in \mathbb{N}_0$, we arrive at identity (20) with q replaced by q^2. This proves the first claim.

To verify the second claim, it suffices to note that $J(a,b)$ commutes with $H(a,b)$ as well as with $H(b,a)$, which follows from Proposition 1 and the symmetry $J(a,b) = J(b,a)$, see (9). Then it follows from (19) that $G(a;q)$ and $J(a,aq^{1/2})$ commute. □

The Jacobi matrix $J(a,aq^{1/2})$ with $a = q^{\alpha/2+1/4}$ corresponds to the continuous q-Laguerre polynomials $P_n^{(\alpha)}(\cdot \mid q)$ that are a special case of the Al-Salam–Chihara polynomials, see [8, § 14.19]. More precisely, one has

$$P_n^{(\alpha)}(x \mid q) = \frac{a^n}{(q;q)_n} Q_n(x; a, aq^{1/2} \mid q),$$

for $a = q^{\alpha/2+1/4}$. Hence, by (10) and (11), the functions

$$\phi_n(2x) = \frac{1}{\sqrt{(q, a^2q^{1/2}; q)_n}} Q_n(x; a, aq^{1/2} \,|\, q) = \sqrt{\frac{(q;q)_n}{(a^2q^{1/2};q)_n}}\, a^{-n} P_n^{(\alpha)}(x \mid q), \tag{21}$$

for $n \in \mathbb{N}_0$, form an orthonormal basis in the Hilbert space $L^2((-1,1), \mathrm{d}\mu)$, where

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(\cos\theta) = \frac{(q, a^2q^{1/2}; q)_\infty}{2\pi \sin\theta} \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{\mathrm{i}\theta}; q^{1/2})_\infty} \right|^2, \quad x = \cos\theta, \tag{22}$$

and $a = q^{\alpha/2+1/4}$ (here, we do not designate the dependence on a and $b = aq^{1/2}$ in the notation of ϕ and μ).

Analogously to the case of H, the unitary mapping

$$U : \ell^2(\mathbb{N}_0) \to L^2((-1,1), \mathrm{d}\mu) : e_n \mapsto \phi_n(2\,\cdot)$$

diagonalizes G, i.e., $UGU^{-1} = T_g$, where

$$g(x) = \sum_{n=0}^{\infty} G_{0,n}\phi_n(2x). \tag{23}$$

Proposition 4 *For $\theta \in (0, \pi)$, we have*

$$g(\cos\theta) = \frac{(q^{1/2}; q)_\infty (-q^{1/4}e^{\mathrm{i}\theta}, -q^{1/4}e^{-\mathrm{i}\theta}; q^{1/2})_\infty}{(-aq^{1/4}; q^{1/2})_\infty}.$$

Proof It follows from (23) and Proposition 3 that

$$g(x) = A\sum_{n=0}^{\infty} H_{n,0}(a, aq^{1/2})\phi_n(2x) + B\sum_{n=0}^{\infty} H_{n,0}(aq^{1/2}, a)\phi_n(2x),$$

which, when compared to (12), yields

$$g(x) = Ah(x; a, aq^{1/2}) + Bh(x; aq^{1/2}, a)$$

where we have designated the dependence on the parameters a, b in the notation $h(x) = h(x; a, b)$ for obvious reasons. Using Proposition 2, we obtain

$$g(\cos\theta) = \frac{1}{(a^2q^{1/2}, q^{1/2}; q)_\infty (q^{1/4}/a; q^{1/2})_\infty} \times \Big[(aq^{1/2}e^{\mathrm{i}\theta}, aq^{1/2}e^{-\mathrm{i}\theta}, q^{1/2}e^{\mathrm{i}\theta}/a, q^{1/2}e^{-\mathrm{i}\theta}/a; q)_\infty - \frac{q^{1/4}}{a}(ae^{\mathrm{i}\theta}, ae^{-\mathrm{i}\theta}, qe^{\mathrm{i}\theta}/a, qe^{-\mathrm{i}\theta}/a; q)_\infty \Big],$$

for $\theta \in (0, \pi)$. Finally, applying identity (31) to the expression in the square brackets, we arrive at the formula from the statement. □

Recalling that $G(q^{\alpha+1/2}; q^2) = \tilde{H}(\alpha; q)$, we may summarize the obtained results on the diagonalization of $\tilde{H}$ as follows.

Theorem 2 *For $\alpha > -1$, the operator $\tilde{H}$ with matrix entries (17) is unitarily equivalent to the operator of multiplication by the function*

$$\tilde{h}(x) = \frac{(q; q^2)_\infty (q^{1/2}e^{\mathrm{i}\theta}, q^{1/2}e^{-\mathrm{i}\theta}; q)_\infty}{(-q^{\alpha+1}; q)_\infty}, \quad x = \cos\theta, \tag{24}$$

acting on $L^2((-1, 1), \mathrm{d}\mu)$, where μ is given by (22) with q replaced by q^2. In particular, the spectrum of $\tilde{H}$ is simple, purely absolutely continuous, and fills the interval

$$\sigma_{\mathrm{ac}}(\tilde{H}) = \frac{(q; q^2)_\infty}{(-q^{\alpha+1}; q)_\infty} \Big[(q^{1/2}; q)^2_\infty, (-q^{1/2}; q)^2_\infty \Big].$$

Consequently, the operator norm of $\tilde{H}$ reads

$$\|\tilde{H}\| = \frac{(q; q^2)_\infty (-q^{1/2}; q)^2_\infty}{(-q^{\alpha+1}; q)_\infty}.$$

3.3 Application: Integral Formulas for Quantum Orthogonal Polynomials

Recall that, in Theorem 1, we have diagonalized H in the sense that

$$UHU^{-1} = T_h,$$

where T_h is the operator of multiplication by h acting on $L^2((-1,1),\mathrm{d}\mu)$ and the unitary mapping $U:\ell^2(\mathbb{N}_0)\to L^2((-1,1),\mathrm{d}\mu)$ is unambiguously determined by the correspondence $Ue_n=\phi_n(2\,\cdot)$ for all $n\in\mathbb{N}_0$. The measure μ is given by density (11) and the polynomials ϕ_n by (10). It follows that

$$H_{m,n}=\langle e_m,He_n\rangle_{\ell^2(\mathbb{N}_0)}=\langle\phi_m(2\,\cdot),T_h\phi_n(2\,\cdot)\rangle_{L^2((-1,1),\mathrm{d}\mu)}$$
$$=\int_{-1}^{1}h(x)\phi_m(2x)\phi_n(2x)\mathrm{d}\mu(x),$$

for all $m,n\in\mathbb{N}_0$. Substituting for $x=\cos\theta$ in the integral and using formulas (8), (10), (11), and (14), we obtain the following non-trivial integral identity for Al-Salam–Chihara polynomials:

$$\frac{(q;q)_\infty}{2\pi(qb/a;q)_\infty}\int_0^\pi Q_m(\cos\theta;a,b\mid q)\,Q_n(\cos\theta;a,b\mid q)\left|\frac{(e^{2\mathrm{i}\theta},qe^{\mathrm{i}\theta}/a;q)_\infty}{(be^{\mathrm{i}\theta};q)_\infty}\right|^2\mathrm{d}\theta$$
$$=(-a)^{n+m}\,q^{\frac{m(m-1)+n(n-1)}{2}}\,{}_0\phi_1\!\left(\begin{matrix}-\\ qb/a\end{matrix}\,\middle|\,q;\,\frac{q^{2-m-n}}{a^2}\right),\tag{25}$$

which holds true for all $m,n\in\mathbb{N}_0$, $q\in(0,1)$, and $a,b\in\mathbb{R}$ such that $0<|a|<1$ and $|b|<1$.

Analogously, using formulas (17), (21), (22), and (24) obtained in the course of the diagonalization of $\tilde{H}$, one deduces the integral formula for the continuous q-Laguerre polynomials:

$$\frac{(q;q^{\alpha+1};q)_\infty}{2\pi}\int_0^\pi P_m^{(\alpha)}(\cos\theta\mid q^2)\,P_n^{(\alpha)}(\cos\theta\mid q^2)\left|\frac{(e^{\mathrm{i}\theta},-e^{\mathrm{i}\theta},q^{1/2}e^{\mathrm{i}\theta};q)_\infty}{(q^{\alpha+1/2}e^{\mathrm{i}\theta};q)_\infty}\right|^2\mathrm{d}\theta$$
$$=\frac{q^{(\alpha+1/2)(m+n)+(m-n)^2/2}\,(q^{\alpha+1};q)_{m+n}}{(q^2;q^2)_m(q^2;q^2)_n},\tag{26}$$

for all $m,n\in\mathbb{N}_0$, $q\in(0,1)$, and $\alpha>-1$. In fact, there is another q-analogue to the Laguerre polynomials, see [8, Eq. (14.19.17)], that are related to the continuous q-Laguerre polynomials by the quadratic transformation

$$P_n^{(\alpha)}(x;q)=q^{-\alpha n}P_n^{(\alpha)}(x\mid q^2).$$

Identity (26) written in terms of polynomials $P_n^{(\alpha)}(\,\cdot\,;q)$ becomes

$$\frac{(q;q^{\alpha+1};q)_\infty}{2\pi}\int_0^\pi P_m^{(\alpha)}(\cos\theta;q)\,P_n^{(\alpha)}(\cos\theta;q)\left|\frac{(e^{\mathrm{i}\theta},-e^{\mathrm{i}\theta},q^{1/2}e^{\mathrm{i}\theta};q)_\infty}{(q^{\alpha+1/2}e^{\mathrm{i}\theta};q)_\infty}\right|^2\mathrm{d}\theta = \frac{q^{(m+n)/2+(m-n)^2/2}\,(q^{\alpha+1};q)_{m+n}}{(q^2;q^2)_m(q^2;q^2)_n},$$

where $m,n\in\mathbb{N}_0$, $q\in(0,1)$, and $\alpha>-1$.

Yet another similar identity can be deduced for the continuous q-Laguerre polynomials directly from (25) by using the equation

$$Q_n(x;q^{\frac{\alpha}{2}+\frac{1}{4}},q^{\frac{\alpha}{2}+\frac{3}{4}}\mid q)=\frac{(q;q)_n}{q^{\left(\frac{\alpha}{2}+\frac{1}{4}\right)n}}P_n^{(\alpha)}(x\mid q),$$

see the first limit relation in [8, § 14.19]. Thus, putting $a=q^{\frac{\alpha}{2}+\frac{1}{4}}$ and $b=q^{\frac{\alpha}{2}+\frac{3}{4}}$ in (25), one obtains

$$\frac{(q;q)_\infty}{2\pi(q^{3/2};q)_\infty}\int_0^\pi P_m^{(\alpha)}(\cos\theta\mid q)\,P_n^{(\alpha)}(\cos\theta\mid q)\left|\frac{(e^{2\mathrm{i}\theta},q^{\frac{3}{4}-\frac{\alpha}{2}}e^{\mathrm{i}\theta};q)_\infty}{(q^{\frac{3}{4}+\frac{\alpha}{2}}e^{\mathrm{i}\theta};q)_\infty}\right|^2\mathrm{d}\theta$$
$$=(-1)^{m+n}\,\frac{q^{(\alpha+1/2)(m+n)+m(m-1)/2+n(n-1)/2}}{(q;q)_m(q;q)_n}\,{}_0\phi_1\!\left(\begin{matrix}-\\ q^{3/2}\end{matrix}\,\middle|\,q;\,q^{-m-n-\alpha+3/2}\right),$$

for $m,n\in\mathbb{N}_0$, $q\in(0,1)$, and $\alpha>-1$.

We also mention another special case of (25) related to a q-analogue of Hermite polynomials. If $b=0$, the Al-Salam–Chihara polynomials becomes the continuous big q-Hermite polynomials

$$H_n(x;a\mid q)=Q_n(x;a,0\mid q),$$

see [8, § 14.18]. In this particular case, we have

$$\frac{(q;q)_\infty}{2\pi}\int_0^\pi H_m(\cos\theta;a\mid q)\,H_n(\cos\theta;a\mid q)\left|(e^{2\mathrm{i}\theta},qe^{\mathrm{i}\theta}/a;q)_\infty\right|^2\mathrm{d}\theta = (-a)^{n+m}\,q^{\frac{m(m-1)+n(n-1)}{2}}\,{}_0\phi_1\!\left(\begin{matrix}-\\ 0\end{matrix}\,\middle|\,q;\,\frac{q^{2-m-n}}{a^2}\right),$$

where $m,n\in\mathbb{N}_0$, $q\in(0,1)$, and $0<|a|<1$.

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Appendix

For the reader's convenience, we list 5 selected identities for basic hypergeometric series and admissible parameters that are used in the proofs. All of them are borrowed directly from [5].

One of the q-exponential functions is [5, Eq. (II.2)]

$$ {}_0\phi_0\left(\begin{matrix} - \\ 0 \end{matrix}\,\middle|\, q;\, z\right) = (z;q)_\infty. \tag{27}$$

Jackson's q-analogue of the Pfaff–Kummer formula reads [5, Eq. (1.5.4)]

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, q;\, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left(\begin{matrix} a,c/b \\ c,az \end{matrix}\,\middle|\, q;\, bz\right). \tag{28}$$

Three term transformation [5, Eq. (III.31)] together with Heine's transformation formula [5, Eq. (1.4.1)] yields the identity

$$ \begin{aligned} {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, q;\, z\right) &= \frac{(abz/c,q/c;q)_\infty}{(az/c,q/a;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/a,cq/abz \\ cq/az \end{matrix}\,\middle|\, q;\, bq/c\right) \\ &\quad - \frac{q}{az}\frac{(b,c/a,az/q,q^2/az;q)_\infty}{(c,q/a,c/az,z;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} q/b,z \\ aqz/c \end{matrix}\,\middle|\, q;\, bq/c\right). \end{aligned} \tag{29}$$

The particular case of the non-terminating q-Vandermonde identity with $b = 0$ reads

$$ \frac{(aq/c;q)_\infty}{(q/c;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} a,0 \\ c \end{matrix}\,\middle|\, q;\, q\right) + \frac{(a;q)_\infty}{(c/q;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} aq/c,0 \\ q^2/c \end{matrix}\,\middle|\, q;\, q\right) = 1, \tag{30}$$

see [5, Eq. (II.23)]. Finally, the particular case of the identity from [5, Ex. 2.16(ii)] with $\mu = e^{i\theta}$ and $\lambda = a$ yields the equality

$$ \begin{aligned} &(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta}, q^{1/2}e^{i\theta}/a, q^{1/2}e^{-i\theta}/a;q)_\infty \\ &\qquad - \frac{q^{1/4}}{a}(ae^{i\theta}, ae^{-i\theta}, qe^{i\theta}/a, qe^{-i\theta}/a;q)_\infty \\ &= (q^{1/2}, q^{1/2}, aq^{1/4}, aq^{3/4}, q^{1/4}/a, q^{3/4}/a;q)_\infty \\ &\qquad \times (-q^{1/4}e^{i\theta}, -q^{3/4}e^{i\theta}, -q^{1/4}e^{-i\theta}, -q^{3/4}e^{-i\theta};q)_\infty. \end{aligned} \tag{31}$$

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On the Product Formula for Toeplitz and Related Operators

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In memory of Harold Widom

Abstract In this note known formulas for the product of Toeplitz operators are revisited in the context of their applications to the study of Fredholmness, boundedness of Toeplitz products, and the Berezin-Toeplitz quantization. A few open problems are also mentioned.

Keywords Toeplitz operator · Hankel operator · Hardy space · Bergman space · Fock space

1 Introduction

Given two bounded Toeplitz operators T_f and T_g on the Hardy space H^2, their product can be written as

$$T_f T_g = T_{fg} - H_f H_{\tilde{g}}, \tag{1}$$

where H_f and $H_{\tilde{g}}$ are Hankel operators acting on H^2. As stated in [2], this identity was established by Widom [20], while it had been known and used for a long time in other forms, such as

$$PfPgP = PfgP - PfQgP,$$

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where P is the orthogonal projection of L^2 onto H^2 and $Q = I - P$. What resulted from Widom's use of this identity was a very ingenious way of dealing with the asymptotics of block Toeplitz determinants in [20], now known as the Szegö-Widom asymptotics, via operator theoretic methods and Schatten class properties of Hankel operators. Paper [2] is embarking on this topic.

Going back to the identity in (1) and its original intent to show that certain Toeplitz operators are Fredholm, I will discuss extensions of this formula in the context of other function spaces, such as Bergman and Fock spaces, and show how it leads to interesting questions about the properties of Hankel operators. What we lack in these other function spaces, however, are effective matrix representations of Toeplitz and Hankel operators, which creates an obstacle to obtaining Widom type identities for the products of truncated Toeplitz matrices.

For simplicity, we limit the discussion to function spaces defined over domains in $\mathbb{C}$, except for Sect. 5, and note that the generalizations to the n-dimensional setting can be easily found in the literature.

2 Preliminaries

For $0 < p < \infty$, $\Omega \subset \mathbb{C}$, and μ a positive measure on Ω, denote by $L^p(\Omega, d\mu)$ the space of all complex measurable functions f on Ω for which

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} < \infty.$$

For a complex measurable function f on Ω, define $\|f\|_\infty$ to be the essential supremum of $|f|$ and denote by $L^\infty(\Omega, d\mu)$ all f for which $\|f\|_\infty < \infty$. The set of all analytic functions in an open set Ω is denoted by $H(\Omega)$.

In terms of domains Ω, the usual three model cases consist of the unit circle $\mathbb{T}$, the unit disk $\mathbb{D}$, and the complex plane $\mathbb{C}$. When $\Omega = \mathbb{T}$, we write $L^p(\mathbb{T})$ for $L^p(\mathbb{T}, d\theta)$ and define the Hardy space H^p by

$$H^p = \{f \in L^p(\mathbb{T}) : f_k = 0 \text{ for } k < 0\}.$$

Let $dA = dxdy$ be the usual area measure on $\mathbb{C}$. We write $L^p(\mathbb{D})$ for $L^p(\mathbb{D}, dA)$ and define the Bergman space A^p by

$$A^p = H(\mathbb{D}) \cap L^p(\mathbb{D}).$$

When $\Omega = \mathbb{C}$, define the Fock space F^p by

$$F^p = H(\mathbb{C}) \cap L^p(\mathbb{C}, e^{-\frac{p}{2}|z|^2} dA).$$

Let $X^2(\Omega) \in \{H^2, A^2, F^2\}$. Then $X^2(\Omega)$ is a Hilbert space and the orthogonal projection of $L^2(\Omega)$ onto $X^2(\Omega)$ is denoted by P. We write $Q = I - P$ for the complementary projection. Given a bounded function f on Ω, the Toeplitz operator $T_f : X^p(\Omega) \to X^p(\Omega)$ with symbol f is defined by

$$T_f g = P(fg).$$

When $1 < p < \infty$, since P extends to a bounded projection on $L^p(\Omega)$, T_f is clearly bounded on $X^p(\Omega)$ if f is bounded.

Defining Hankel operators is less straightforward. Indeed, the Hankel operators that appear in (1) act on the Hardy space while the Hankel operators on Bergman spaces A^p or Fock spaces F^p map into the corresponding $L^p(\Omega)$. More precisely, define the flip operator $J : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ by

$$Jf(t) = \bar{t} f(\bar{t})$$

for $t \in \mathbb{T}$. For a bounded symbol f, the Hankel operator H_f is defined on H^p by

$$H_f g = PM_f QJf,$$

where M_f is the multiplication operator. When $\Omega \in \{\mathbb{D}, \mathbb{C}\}$, we define the Hankel operator $H_f : X^p(\Omega) \to L^p(\Omega)$ by

$$H_f g = Q(fg).$$

Again, it is easy to see that the Hankel operator H_f is bounded in all the three cases if $1 < p < \infty$ and f is bounded.

3 Fredholm Properties of Toeplitz Opeators

In this section the Fredholm properties of Toeplitz operators acting on Hardy, Bergman and Fock spaces are considered using (1) and its generalizations. Recall that an operator A on a Banach space is said to be Fredholm if $\ker A$ and $X/A(X)$ are both finite dimensional, in which case the index $\operatorname{ind} A$ is defined by

$$\operatorname{ind} A = \dim \ker A - \dim X/A(X).$$

Equivalently, A is Fredholm if and only if $A + K(X)$ is invertible in the Calkin algebra $B(X)/K(X)$, where $B(X)$ and $K(X)$ denote the sets of all bounded and compact operators on X, respectively. The essential spectrum of A is defined by

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}.$$

3.1 The Hardy Space Case

Let f, g be bounded on $\mathbb{T}$ and write $\tilde{f}(t) = f(\bar{t})$ for $t \in \mathbb{T}$. Then

$$\begin{aligned} T_{fg} &= PM_{fg}P = PM_f M_g P = PM_f PM_b P + PM_f QM_g P \\ &= PM_f P^2 M_g P + PM_f QJ^2 QM_g P, \end{aligned} \tag{2}$$

which is (1).

Suppose now that f is continuous and has no zeros on $\mathbb{T}$. Then $g = 1/f$ is also continuous and has no zeros. By (1), since H_f is known to be compact,

$$T_f T_g = I + H_f H_{\tilde{g}} = I + K$$

for some compact operator K. Similarly, $T_g T_f - I$ is compact, and hence T_f is Fredholm. In situations when Hankel operators are compact, the identity in (1) is tailor-made for proving that Toeplitz operators are Fredholm. In other words, whenever the Hankel operators are compact, the corresponding Toeplitz operators commute modulo compact operators. A similar approach also applies to symbols in the Douglas algebra $C + \overline{H^\infty}$ but the use of (1) is no longer as effective with more general classes of symbols.

Let $f \in L^\infty(\mathbb{T})^{N\times N}$ and consider the block Toeplitz operator T_f on $H^p_N = \{(f_1, \ldots, f_N)^\top : f_j \in H^p\}$. Suppose that $f \in (C + \overline{H^\infty})^{N\times N}$ and $\det f$ is invertible in $C + \overline{H^\infty}$. Choose $h \in (\mathcal{R} + \overline{H^\infty})^{N\times N}$, where $\mathcal{R}$ is the set of all rational functions, sufficiently close to f in the norm of $L^\infty_{N\times N}(\mathbb{T})$. Then

$$\mathrm{ind} T_f = \mathrm{ind} T_h \quad \text{and} \quad \mathrm{ind} T_{\det f} = \mathrm{ind} T_{\det h}.$$

Since H_h has finite rank, (1) implies that the entries of T_h commute modulo finite-rank operators, and hence $\mathrm{ind} T_h = \mathrm{ind} T_{\det h}$ (see Theorem 1.15 of [7]), which reduces the index computation to that of the scalar-valued symbols. For more general symbols, such as piecewise continuous symbols, no such reductions are possible.

Although the projection P is unbounded on $L^1(\mathbb{T})$, a Fredholm theory for Toeplitz operators on H^1 can still be developed. In particular, when f is a continuous function of logarithmic vanishing mean oscillation, the Hankel operator H_f is compact on H^1, so (1) is readily available, and the Fredholm properties can be described as in the reflexive case $1 < p < \infty$ (see [19]). However, as recently observed, there are continuous symbols f that generate bounded Toeplitz operators on H^1 and for which H_f is not compact (see [10]). This makes the study of Fredholmness of T_f with such continuous symbols considerably more difficult in H^1 because (1) no longer produces a desired conclusion.

There are many other aspects of Toeplitz operators on the Hardy space whose proofs benefit from (1), such as invertibility with analytic symbols, the applicability of local principles, the study of Toeplitz algebras and Fisher-Hartwig symbols, but we refrain from further details (all of which can be found in [7]) and keep our focus only on the Fredholm properties in this section.

3.2 The Bergman Space Case

As mentioned above, for $1 < p < \infty$ and $f \in L^\infty(\mathbb{D})$, the Hankel operator H_f is defined by $H_f g = Q(fg)$ for $g \in A^p$, and so it maps into $L^p(\mathbb{D})$ instead of A^p. However, we can still obtain formulas similar to (1) as follows. For two bounded functions f, g on Ω, using the inner product in A^2, it is easy to see that

$$T_f T_g = T_{fg} - H^*_{\bar{f}} H_g \tag{3}$$

when $p = 2$, which shows that

$$T_{|f|^2} - T_{\bar{f}} T_f = H^*_{\bar{f}} H_f,$$

and hence compactness of H_f is equivalent to compactness of the semi-self-commutator $T_{|f|^2} - T_{\bar{f}} T_f$. In addition, the formulas

$$T_f T_g = PM_f PM_g = PM_f(I - Q)M_g = T_{fg} - PM_f H_g \tag{4}$$

$$= I - P(I - M_{fg}) - PM_f H_g = I - T_{1-fg} - PM_f H_g \tag{5}$$

are useful. For example, in [14], the identity in (4) was used to show that the Toeplitz operator T_f with $f \in C(\overline{\mathbb{D}})$ is Fredholm on A^2 if and only if f has no zeros on the boundary. A similar approach, using (5), can be used to treat symbols in the Douglas algebra $C(\overline{\mathbb{D}}) + H^\infty$ and symbols of vanishing mean oscillation.

Let $f \in L^\infty(\mathbb{D})^{N\times N}$ and consider the block Toeplitz operator T_f on $A^p_N = \{(f_1, \ldots, f_N)^\top : f_j \in A^p\}$. Fredholmness of block Toeplitz operators with symbols in the Douglas algebra $(C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D}))^{N\times N}$ can be handled as in the Hardy space case but now with the identities in (4) and (5). However, the index formula for these symbols cannot be derived as easily as in the Hardy space case because the formula $\operatorname{ind} A = \operatorname{ind} \det A$, which holds for operator matrices A whose entries commute modulo trace class operators, fails to reach all of $C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D})$ via (4). For an alternate approach to the computation of the index of block Toeplitz operators T_f on the Bergman spaces on the unit ball, see [6]. Similar comments can be made about symbols in $(L^\infty(\mathbb{D}) \cap \mathrm{VMO})^{N\times N}$, where VMO is the space of functions of vanishing mean oscillation, and in particular the approach in [6] should produce an index formula for this symbol class, too. We return to this topic in the next section when dealing with Toeplitz operators in the Fock space setting, where an analogous problem still remains open.

3.3 The Fock Space Setting

As in the Bergman space setting, for Toeplitz operators on F^2, we again have

$$T_f T_g = T_{fg} - H^*_{\bar{f}} H_g. \tag{6}$$

To my knowledge, this identity was first used to describe the Fredholm properties of T_f on the Fock space in [17]. It was shown that, when $f \in L^\infty(\mathbb{C})$ and H_f is compact, we have

$$\sigma_{\text{ess}}(T_f) = \bigcap_{r>0} \operatorname{cl} \tilde{f}(\mathbb{C} \setminus \mathbb{D}_r), \tag{7}$$

where $\operatorname{cl} E$ stands for the closure of E in $\mathbb{C}$, $\mathbb{D}_r = \{|z| < r\}$, and $\tilde{f}$ is the Berezin transform of f defined by

$$\tilde{f}(z) = \frac{1}{2\pi} \int_{\mathbb{C}} f(w) e^{-\frac{1}{2}|z-w|^2} dA(w) \tag{8}$$

for $z \in \mathbb{C}$. In the proof of (7), identity (6) comes into play as follows. Suppose that $\xi \notin \operatorname{cl} f(\mathbb{C} \setminus \mathbb{D}_r)$ for some $r > 0$. To show that $T_{f-\xi}$ is Fredholm, define

$$g(z) = \begin{cases} (f(z) - \xi)^{-1} & \text{if } z \in \mathbb{C} \setminus \mathbb{D}_r, \\ 1 & \text{if } z \in \mathbb{D}_r. \end{cases}$$

Then $g \in L^\infty(\mathbb{C})$, and an application of (6) shows that

$$T_g T_{f-\xi} = I - H^*_{\bar{g}} H_f - T_{(f-\xi-1)\chi_{\mathbb{D}_r}}.$$

Notice that $(f - \xi - 1)\chi_{\mathbb{D}_r}$ has compact support and hence $T_{(f-\xi-1)\chi_{\mathbb{D}_r}}$ is compact. Since H_f is compact, it follows that $T_{f-\xi} + K(F^2)$ is left-invertible in $B(F^2)/K(F^2)$. That $T_{f-\xi} + K(F^2)$ is also right-invertible follows from $T_{f-\xi} = T^*_{\bar{f}-\bar{\xi}}$ and the fact that $H_{\bar{f}}$ is compact whenever H_f is compact (see, e.g., [4] or [13]). Therefore, $T_{f-\xi} = T_f - \xi$ is Fredholm, that is, $\xi \notin \sigma_{\text{ess}}(T_f)$, and so $\sigma_{\text{ess}}(T_f) \subset \operatorname{cl} f(\mathbb{C} \setminus D_r)$ for all $r > 0$. Further, since $T_{f-\tilde{f}}$ is known to be compact, $\sigma_{\text{ess}}(T_f) = \sigma_{\text{ess}}(T_{\tilde{f}}) \subset \operatorname{cl} \tilde{f}(\mathbb{C} \setminus D_r)$ for all $r > 0$ by the above argument applied to $\tilde{f}$. For the other inclusion (which involves no product formulas), see [17].

For an extension to other Fock spaces

$$F^p_\varphi = \left\{ f \in H(\mathbb{C}) : \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} \, dA(z) < \infty \right\}$$

with more general weights φ and $0 < p < \infty$, see [12], which deals with the so-called doubling weights. These are very general weights that include all standard weights (i.e., $\varphi(z) = -\frac{\alpha}{2}|z|^2$ with $\alpha > 0$), the so-called Fock-Sobolev weights, and the weights φ for which there are positive constants m and M (depending on φ) such that

$$m \leq \Delta\varphi \leq M \tag{9}$$

on $\mathbb{C}$, where Δ is the Laplacian. It is worth noting that, unlike in these other Fock spaces, we do not currently know whether Fredholmness of Toeplitz operators on doubling Fock spaces can be extended to $\mathbb{C}^n$ due to the lack of suitable estimates for the reproducing kernel.

Let $f \in L^\infty(\mathbb{C})^{N\times N}$ and consider the block Toeplitz operator T_f on $F_N^2 = \{(f_1, \ldots, f_N)^\top : f_j \in F^2\}$. As in the previous two function spaces, the study of Fredholmness of block Toeplitz operators can be reduced to the scalar-valued case using (6). However, similarly to T_f on A_N^2 with $f \in (C(\overline{\mathbb{D}}) + H^\infty)^{N\times N}$, the index computation in the Fock space setting cannot be reduced to the scalar-valued case and it remains an open problem—perhaps the approach in [6] can be adapted to this case.

A partial answer to the index computation can be derived from a recent result in [10], in which the Schatten class properties of H_f are described in terms of integral distance to analytic functions. More precisely, for $f \in L^2_{\text{loc}}(\mathbb{C})$, define

$$G_r(f)(z) = \inf_{h\in H(D(z,r))} \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |f-h|^2 dA\right)^{\frac{1}{2}} \quad (z \in \mathbb{C}),$$

where $D(z,r)$ is the disk centered at z with radius r. For $0 < s \leq \infty$, we say $f \in \text{IDA}^s$ if $\|G_r(f)\|_{L^s(\mathbb{C})} < \infty$ for some $r > 0$. Notice that the space IDA^s is independent of r. In [10], for $0 < p < \infty$, it was shown that H_f is in the Schatten class S_p if and only if $f \in \text{IDA}^p$. Let $f \in (L^\infty(\mathbb{C}) \cap \text{IDA}^1)^{N\times N}$ and suppose that $\widetilde{\det f}$ is bounded away from zero on $\mathbb{C}\setminus\mathbb{D}_R$ for some $R > 0$. Then (6) can be used to show that the entries of T_f commute modulo trace class operators, and hence using the scalar-valued case (see [4]), we conclude that

$$\text{ind}T_f = \text{ind}T_{\det f} = -\text{wind}(\det f|_{|z|=R}).$$

This result is unsatisfactory because there are bounded symbols that generate compact Hankel operators but do not belong to IDA^1, and further work is required as indicated above.

4 Sarason's Product Problem

In [16], Sarason proposed the problem of characterizing the pairs of functions f, g in H^2 such that the operator $T_f T_{\bar{g}}$ is bounded on H^2. Related to the present work, he remarked that the identity

$$H^*_{\bar{f}} H_{\bar{g}} = T_{f\bar{g}} - T_f T_{\bar{g}} \tag{10}$$

reduces the problem to the question of when $H^*_{\bar{f}} H_{\bar{g}}$ is bounded under the assumption that fg is bounded. When the boundedness assumption on fg is dropped, it can be easily seen that the latter problem is more general (e.g., choose an unbounded f such that $H_{\bar{f}}$ is bounded and take $g = f$). The following conjecture is often referred to as Sarason's conjecture: *For two functions f, g in H^2, $T_f T_{\bar{g}}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \widehat{|f|^2}(z)\widehat{|g|^2}(z) < \infty, \tag{11}$$

where $\widehat{h}$ is defined as the Poisson extension of $h \in L^1(\mathbb{T})$. In fact, Treil had communicated an argument showing that (11) is necessary to Sarason (see Comment 6 in [16]) and subsequently Zheng [21] proved that (11) with 2 replaced by $2+\epsilon$ is sufficient. Finally, in the well-known unpublished manuscript of Nazarov [15], it was shown that Sarason's conjecture fails.

A related conjecture was formulated in the Bergman space setting: *For $f, g \in A^2$, $T_f T_{\bar{g}}$ is bounded on A^2 if and only if*

$$\sup_{z\in\mathbb{D}} \widetilde{|f|^2}(z)\widetilde{|g|^2}(z) < \infty. \tag{12}$$

This conjecture was also shown to be false by Aleman et al. [1] using harmonic analysis. However, Stroethoff and Zheng [18] showed that if we consider the question of whether $T_f T_{\bar{g}}$ is both bounded and invertible, then (11) and (12) provide the right conditions in the settings of H^2 and A^2, respectively. More precisely, they showed that for $f, g \in A^2$, $T_f T_{\bar{g}}$ is bounded and invertible on A^2 if and only (12) holds and $\inf\{|f(z)||g(z)| : z \in \mathbb{D}\} > 0$. They also remarked that a similar approach yields an analogous result for Toeplitz operators on the Hardy space, that is, for $f, g \in H^2$, $T_f T_{\bar{g}}$ is bounded and invertible on H^2 if and only (11) holds and $\inf\{|f(z)||g(z)| : z \in \mathbb{D}\} > 0$. It should be noted that the latter result was proved earlier for a pair of outer functions $f, g \in H^2$ by Cruz-Uribe [9] using a characterization of invertible Toeplitz operators due to Devinatz and Widom (see, e.g., Theorem 2.23 of [7]).

Finally, using a number of product identities for Toeplitz operators, Stroethoff and Zheng [18] proved that $T_f T_{\bar{g}}$ is bounded and Fredholm on A^2 if and only if (12) holds and $\inf_{z\in\mathbb{D}\setminus r\mathbb{D}}|f(z)g(z)| > 0$ for some $r < 1$. Again, the same is true in the setting of the Hardy space—just replace (12) by (11).

Above we have considered Sarason's problem only rather superficially, and while the product formula in (10) gives a more general problem involving Hankel operators, the product formulas do not contribute to the two important counterexamples. It is also worth noting that, despite the considerable progress, Sarason's product problem still remains open in the Hardy and Bergman space settings.

We now turn our attention to the Fock space, where Sarason's problem has a simple solution. Indeed, in [8], for $f, g \in F^2$, it is shown that $T_f T_{\bar{g}}$ is bounded on F^2 if and only if there are $a, b, c \in \mathbb{C}$ such that $f(z) = e^{a+cz}$ and $g(z) = e^{b-cz}$ for all $z \in \mathbb{C}$. One of the key observations is that, when $a \in \mathbb{C}$, $f(z) = e^{\frac{1}{2}\bar{a}z}$, $g(z) = e^{-\frac{1}{2}\bar{a}z}$, we have

$$T_f T_{\bar{g}} = e^{\frac{1}{4}|a|^2} U_a,$$

where U_a is the unitary operator on F^2 defined by

$$U_a f(z) = f(z-a)k_a(z)$$

and k_a is the normalized reproducing kernel of F^2 defined by

$$k_a(z) = e^{\frac{1}{2}\bar{a}z-\frac{1}{4}|a|^2}.$$

As weighted Fock spaces F^2_φ have received significant attention recently, it would be interesting to know whether something similar holds true for more general weights than those considered in [5, 8]. A possible starting point may be the weights φ whose Laplacians are bounded above and below (see (9) and [11]). What makes Sarason's product problem interesting in this generalized setting is that the reproducing kernel of F^2_φ has no explicit representation (unlike in F^2) and the unitary operators U_a can no longer be employed. The former obstacle may be possible to overcome with the use of estimates for the (normalized) reproducing kernel, but overall the generalized Sarason's product problem seems nontrivial in generalized Fock spaces and requires new ideas.

5 Quantization

As an application of product formula (6) and recent work on Hankel operators, we consider deformation quantization (in the sense of Rieffel) and one of its essential ingredients involving the limit condition

$$\lim_{t\to 0}\left\| T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} \right\|_{F_t^2(\varphi)\to F_t^2(\varphi)} = 0, \tag{13}$$

where the Toeplitz operators $T_f^{(t)}$ and the Fock spaces $F_t^2(\varphi)$ are defined as follows. For $t > 0$, we set

$$d\mu_t(z) = \frac{1}{t^n} \exp\left\{-2\varphi\left(\frac{z}{\sqrt{t}}\right)\right\} dv(z)$$

and denote by $L_t^2(\varphi)$ the space of all Lebesgue measurable functions f in $\mathbb{C}^n$ such that

$$\|f\|_t = \left\{\int_{\mathbb{C}^n} |f|^2 \, d\mu_t(z)\right\}^{\frac{1}{2}}.$$

Further, we let $F_t^2(\varphi) = L_t^2(\varphi) \cap H(\mathbb{C}^n)$ and define the Toeplitz operator $T_f^{(t)}$ on $F_t^2(\varphi)$ by

$$T_f^{(t)} = P^{(t)} M_f,$$

where $P^{(t)}$ is the orthogonal projection of $L_t^2(\varphi)$ onto $F_t^2(\varphi)$.

Using the dilation $U_t : f \mapsto f(\cdot\sqrt{t})$, it can be easily shown that

$$\|H_f^{(t)}\|_{F_t^2(\varphi)\to L_t^2(\varphi)} = \|H_{f(\cdot\sqrt{t})}\|_{F^2(\varphi)\to L^2(\varphi)}, \tag{14}$$

where $H_f^{(t)} = (I - P^{(t)})M_f$ is the Hankel operator. To study the limit condition in (13), define for $f \in L^2_{\text{loc}}$, $z \in \mathbb{C}^n$, and $r > 0$,

$$MO_{2,r}(f)(z) = \left(\frac{1}{|B(z,r)|}\int_{B(z,r)} \left|f - f_{B(z,r)}\right|^2 dv\right)^{\frac{1}{2}}$$

where $B(z,r) = \{w \in \mathbb{C}^n : |z - w| < r\}$, $f_S = \frac{1}{|S|}\int_S f dv$ for $S \subset \mathbb{C}^n$ measurable and dv is the usual Lebesgue measure on $\mathbb{C}^n$. Now, let $f \in L^2_{\text{loc}}$. We say that $f \in \text{VMO}$ if

$$\lim_{r\to 0} \sup_{z\in\mathbb{C}^n} MO_{2,r}(f)(z) = 0.$$

Further, we say that $f \in \text{VDA}_*$ if

$$\lim_{r\to 0} \sup_{z\in\mathbb{C}^n} G_{2,r}(f)(z) = 0.$$

In [13], it was shown that, given $f \in L^\infty$, then for all $g \in L^\infty$, the limit condition in (13) holds if and only if $f \in \overline{\text{VDA}_*}$.

To verify this, notice first that (6) gives

$$T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} = -\left(H_{\overline{f}}^{(t)}\right)^* H_g^{(t)}.$$

for all $f, g \in L^\infty$. Let $f \in \overline{\mathrm{VDA}}_*$. Then, for all $g \in L^\infty$,

$$\begin{aligned}\left\| T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} \right\|_{F_t^2(\varphi)\to F_t^2(\varphi)} &\le \|g\|_{L^\infty} \left\| \left(H_{\overline{f}}^{(t)}\right)^* \right\|_{L_t^2(\varphi)\to F_t^2(\varphi)} \\ &\le C\|G_{2,1}(f(\cdot\sqrt{t}))\|_{L^\infty} = C\|G_{2,\sqrt{t}}(f)(\cdot\sqrt{t})\|_{L^\infty} \to 0\end{aligned}$$

as $t \to 0$, where we used the norm estimate for Hankel operators given in Theorem 1.1 of [13]. For the converse, again by product formula (6), we have

$$\begin{aligned}\lim_{t\to 0} \left\| H_{\overline{f}}^{(t)} \right\|^2_{F_t^2(\varphi)\to L_t^2(\varphi)} &= \lim_{t\to 0} \left\| \left(H_{\overline{f}}^{(t)}\right)^* H_{\overline{f}}^{(t)} \right\|_{F_t^2(\varphi)\to F_t^2(\varphi)} \\ &= \lim_{t\to 0} \|T_f^{(t)} T_{\overline{f}}^{(t)} - T_{|f|^2}^{(t)}\|_{F_t^2(\varphi)\to F_t^2(\varphi)} = 0,\end{aligned}$$

and it remains to notice that

$$\frac{1}{C}\|G_{2,1}(f(\cdot\sqrt{t}))\|_{L^\infty} \le \left\| \left(H_{\overline{f}}^{(t)}\right)^* \right\|_{L_t^2(\varphi)\to F_t^2(\varphi)},$$

which follows from the estimate for Hankel operators mentioned above.

Combining the characterization for (13) with the observation that $\mathrm{VMO} = \mathrm{VDA}_* \cap \overline{\mathrm{VDA}}_*$ gives the main result of [3] (where it was assumed that $\varphi(z) = \frac{1}{8}|z|^2$ is the standard weight), that is, given $f \in L^\infty$, then, for all $g \in L^\infty$, it holds that

$$\lim_{t\to 0} \left\| T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} \right\| = 0 \quad \text{and} \quad \lim_{t\to 0} \left\| T_g^{(t)} T_f^{(t)} - T_{fg}^{(t)} \right\| = 0 \tag{15}$$

if and only if $g \in \mathrm{VMO}$. Here $\|\cdot\| = \|\cdot\|_{F_t^2(\varphi)\to F_t^2(\varphi)}$. For further details, see [13].

As for an open problem in this line of work, it would be interesting to characterize those symbols $f \in L^\infty(\mathbb{C}^n)$ for which (13) holds for all $g \in L^\infty(\mathbb{C}^n)$ when the operator norm is replaced by the Hilbert-Schmidt (or other Schatten class) norm.

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