A Design Model for a (Grid)shell Based on a Triply Orthogonal System of Surfaces

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Abstract. We present a design model for a (grid)shell that is an assembly of 3D components ('rational-voxels') fabricated from planar/developable faces. This rationalization was achieved thanks to the geometric properties of principal patches arising from a Triply Orthogonal system of Surfaces (TOS). By using such a system we were able to generate a curvilinear coordinate system where the coordinate lines are principal curves on the respective surfaces in the TOS. Next, generate 3D components (voxels) where each voxel is a curvilinear cube where its sides are principal patches, and its edges are principal curves obtained by intersecting two neighboring surfaces from each of the three families in the TOS. These voxels are then rationalized into rational-voxels having planar/developable faces and straight/planar edges. The design model allows for five degrees of design freedom for choosing: (1) the shell-slice type in the TOS, (2) the shell-slice thickness, (3) the voxel-assembly method, (4) the rational-voxel type and (5) being either a solid or a hollow voxel-assembly. A design to build process of a large scale pavilion is presented as a demonstration of the proposed design model.

Keywords: Principal network · Triply orthogonal surfaces · (Grid)shell

1 Introduction

In this work we present a design model for a (grid)shell composed from an assembly of 3D components fabricated from planar and/or developable faces. Our approach rests fundamentally on the so-called *Triply Orthogonal system of Surfaces* (TOS) giving rise to explicit parameterizations of three families of surfaces by *principal curves*. These principal-curves parameterizations are orthogonal conjugate networks, which make them optimal in the discretization of doubly-curved surfaces using planar and developable pieces. Using a TOS thus provide us a three-dimensional *principal network* in the sense of a curvilinear coordinate system which will be used to define the 3D component (or voxel) of the assembly forming the (grid)shell. The six sides of a voxel are given (by pieces of) surfaces from the three families in the TOS, hence are principal patches. There follows that we can transform our voxel into a rational-voxel whose sides are

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either planar or developable, allowing for a feasible construction by simple methods. In Sect. 2, we present the geometry of principal networks and the TOS, in Sect. 3, we present the design model and degrees of design freedom, in Sect. 4, we present a realized (grid)shell based on our design model.

2 Geometry

We will now briefly recall few facts from differential geometry (cf. [4] and [5]).

2.1 Principal Patches

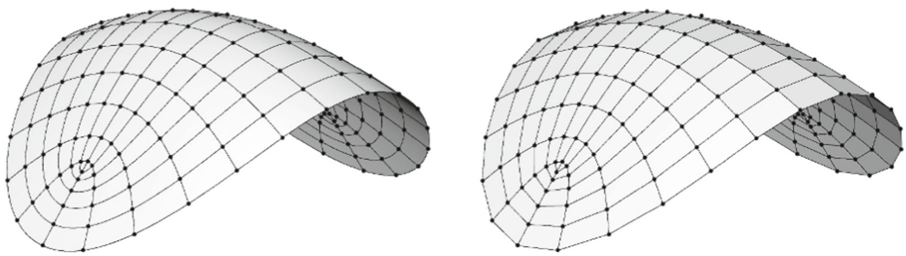
Consider a surface *S* given by a smooth parameterization *X(u,v)* where the domain of *(u,v)* is an open set in the plane and the coordinate lines form a network on *S*. We will thus refer to *X(u,v)* as a patch or as a network. We denote by X_u, X_v the partial derivatives generating the tangent plane T_pS and by N the normal vector. The first and second fundamental forms are denoted by *I, II*, the shape operator is given by S = *II/I*.

Let *a,b* be tangent vectors in T_pS then their directions are called

$$\begin{cases} \text{Orthogonal if } \langle a, b\rangle = 0 \text{ that is } a^T \cdot I \cdot b = 0 \\ \text{Conjugate if } \langle a, S(b)\rangle = 0 \text{ that is } a^T \cdot II \cdot b = 0. \end{cases}$$

The eigenvalues of the shape operator are the *principal curvatures* k_1, k_2 and the corresponding (necessarily orthogonal) eigenvectors define the *principal directions.*

- A curve in *S* is a *principal curve* if at every point it is tangent to a principal direction that is, it is an integral curve for that principal direction field.
- A patch *X(u,v)* on *S* is a *principal patch (network)* if its coordinate lines are principal curves, that is the directions of X_u, X_v are both orthogonal and conjugate.
- A mesh on *S* is a *principal mesh* if it is constituted of quadrilateral faces obtained by joining the intersection points of a principal network.

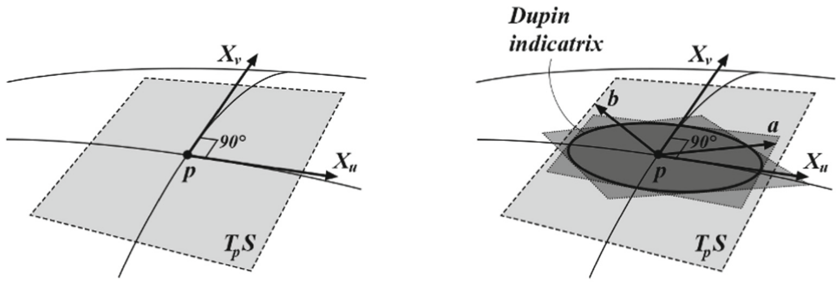


(a) Principal network. (b) Principal mesh.

Fig. 1. a) Principal network. (b) Principal mesh.

2.2 Geometric Properties of Principal Networks

Let *X(U,v)* be a principal patch (network), that is X_u, X_v are principal directions (orthogonal and conjugate). Orthogonality is clear as seen in Fig. 1(a), as for conjugation, we recall that conjugate directions a and b are parallel to the edges of a parallelogram tangent to the DUpin's indicatrix, as seen in Fig. 1(B). Naturally, there is an infinity of such parallelograms and if we impose orthogonality between *a, b* then we have a rectangle with edges parallel to the principal directions (Fig. 2).



(a) Orthogonality property of X_u, X_v. (b) Conjugation property of X_u, X_v.

Fig. 2. A) Orthogonality property of X_u, X_v. (b) Conjugation property of X_u, X_v.

We now define two important types of ruled surfaces or 'strips' that will be useful (Fig. 3).

The *Normal strips* for a fixed u_o, v_o given by:

$$\begin{cases} N_1(s,u) = X(u, v_o) + sN(u, v_o) \\ N_2(s,v) = X(u_o, v) + sN(u_o, v). \end{cases}$$

The *'Tangent' strips* for a fixed u_o, v_o given by:

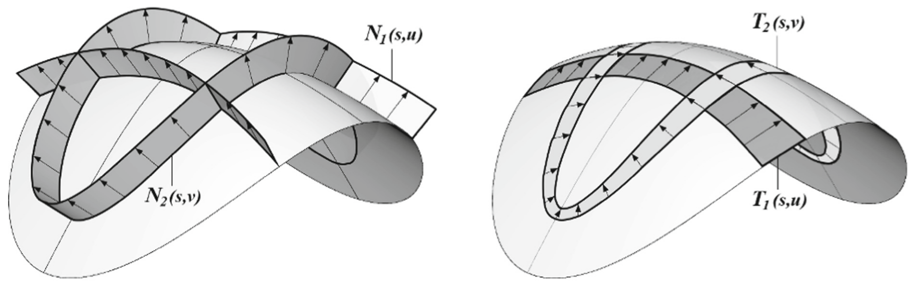
$$T_1(s,u) = X(u, v_o) + s(X(u, v_o + \varepsilon) - X(u, v_o))$$
$$T_2(s,v) = X(u_o, v) + s(X(u_o + \varepsilon, v) - X(u_o, v)).$$

2.3 Continuous Principal Networks and Discrete Circular PQ Meshes

Deviation from Planarity and Developability

It is important to point out that our principal meshes (arising from generic principal networks) are not technically circular PQ meshes. However, they are very close to being so, in the sense that if Q is a quad from a principal mesh it deviates from planarity by a very small error (see [3]) as seen in Fig. 4.

Hence, non-linear optimization problems of approximating surfaces by circular PQ meshes, see [2, 7] and [9] are much more likely to work if initiated from principal networks. Another (somehow inverted way) of looking at the deviation from planarity



(a) Normal strips. (b) Tangent strips.

Fig. 3. a) Normal strips. (b) Tangent strips.

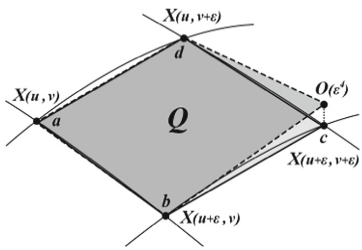


Fig. 4. *Q* deviate from planarity by O (ε^4) as opposed to O (ε^3) in other networks.

of principal meshes, which also shows the deviation from developability of the tangent strips $T_1(s,u)$, $T_2(s,v)$ is that:

$$\begin{array}{lcl} \text{Circular PQ meshes} & \xrightarrow[\text{refinement}]{\text{at the limit of}} & \text{Principal networks} \\ \text{Sequence of PQ faces} & & \text{Developable strips} \end{array}$$

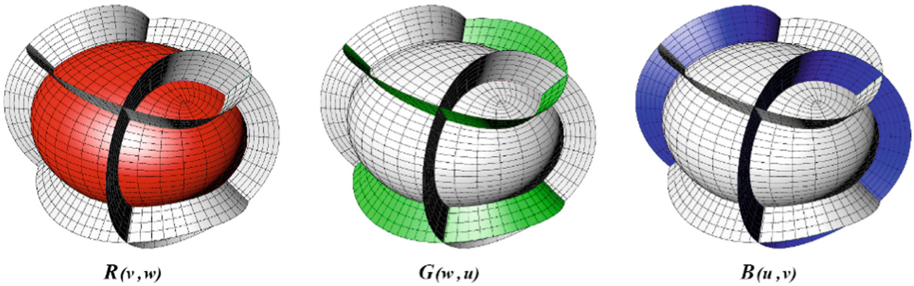
Error Tolerance in our 3D Models

In our approach we start by principal networks hence the normal strips $N_1(s,u)$, $N_2(s,v)$ are truly developable. Next, we produce discrete principal meshes (hence almost planar) and semi-discrete tangent strips (hence almost developable), as explained above. However, the errors are so small, that if we consider the realization of the model using *Rhinoceros* with its tolerance set to 0.001. Then, planarity of the principal meshes and the developability of the tangent strips in question are completely within this tolerance. Thus for all purposes of our architectural application:

- The principal meshes will be considered planar
- The tangent strips $T_1(s,u)$, $T_2(s,v)$ will be considered developable.

2.4 Triply Orthogonal System of Surfaces

As mentioned above the *Triply Orthogonal system of Surfaces* (TOS) are systems consisting of three families of surfaces that are mutually orthogonal. A theorem of Dupin asserts that given such system then the curves of intersection of the surfaces are principal curves on each surface, see [6]. Similar to discrete principal networks, there is a discrete version of TOS, see [1]. However, we will focus on smooth TOS giving rise to *elliptic curvilinear coordinates,* as seen in Fig. 5.



(a) Ellipsoids R. (b) One-sheet hyperboloids G. (c) Two-sheet hyperboloids B.

Fig. 5. a) Ellipsoids *R*. (b) One-sheet hyperboloids *G*. (c) Two-sheet hyperboloids *B*.

Explicit Parametric Definition of the TOS
For *a, b, c* real suitably bounding the parameters *u, v, w* we obtain the patches:

$$X(u, v, w) = \left\{ \pm\sqrt{\frac{(a-u)(a-v)(a-w)}{(a-b)(a-c)}}, \pm\sqrt{\frac{(b-u)(b-v)(b-w)}{(b-a)(b-c)}}, \pm\sqrt{\frac{(c-u)(c-v)(c-w)}{(b-c)(a-c)}} \right\}$$

parameterizing the elliptic TOS (R, G, B) where R are ellipsoids, G are one-sheeted hyperboloids and B are two-sheeted hyperboloids respectively, by principal patches:

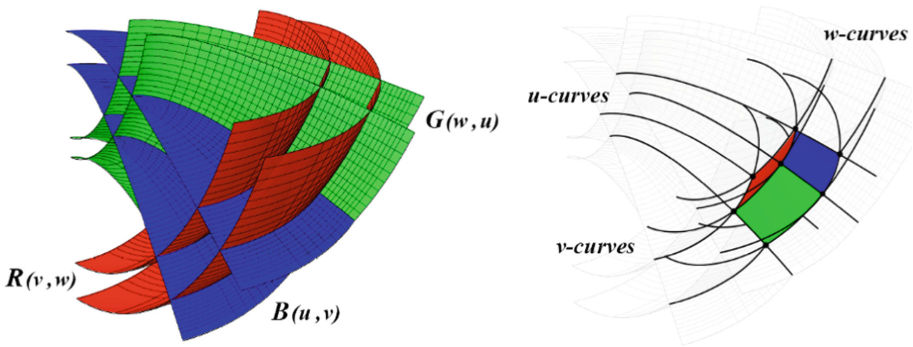
$$\begin{cases} R(v, w) = X(u_o, v, w) \\ G(w, u) = X(u, v_o, w) \\ B(u, v) = X(u, v, w_o) \end{cases}$$

where u in $[-c,c]$, v is in $[c,b]$ and w in $[b,a]$.

Important Advantages of the TOS

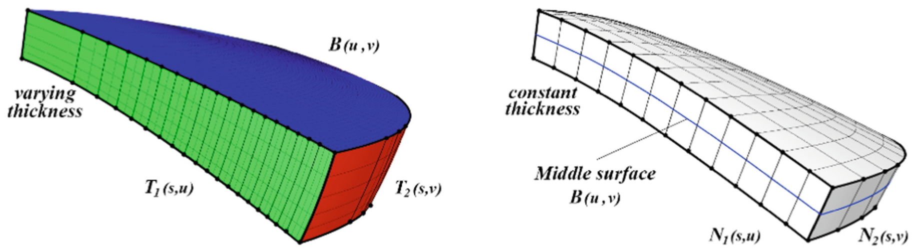
(1) We obtain an explicit expression for the parameterizations of the surfaces in question therefore, bypassing the need for solving the differential equation of principal curves in order to obtain the principal networks.
(2) Exploring the third dimension through the elliptic curvilinear coordinates and the generation of the 3D component (voxel), as seen in Fig. 6.

(3) The TOS gives us a way to simultaneously have caps-surfaces that are principal patches with varying thickness all-the-while maintaining that the lateral-surfaces are developable. This is done by having the cap-surfaces given by two successive surfaces from one family in the TOS and the lateral-surfaces developable tangent strips on surfaces from the remaining two families, as seen in Fig. 7(b). This gives a larger variety of shell designs (of varying thickness) with the geometric advantages of constant-thickness shells. Note that, the principal patches cap-surfaces used here are not *parallel* neither are their principal meshes, by contrast to the *parallel meshes* addressed in [8].



(a) Principal patches *R,G,B*. (b) Elliptic curvilinear coordinate lines *u,v,w*.

Fig. 6. a) Principal patches *R, G, B*. (b) Elliptic curvilinear coordinate lines *u, v, w*.



(a) Constant thickness using offset. (b) Varying thickness using TOS.

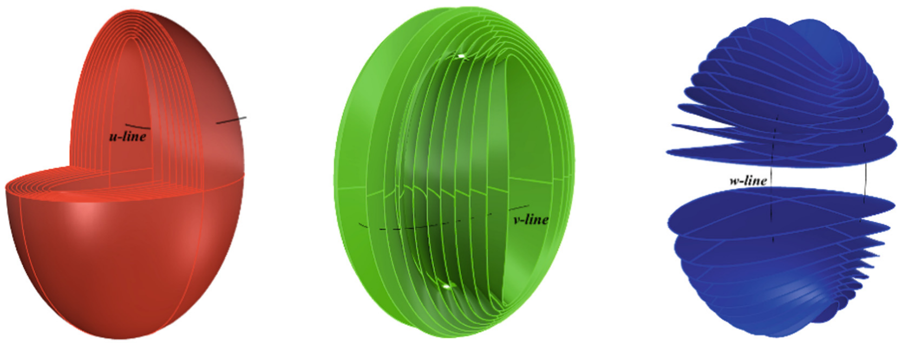
Fig. 7. a) Constant thickness using offset. (b) Varying thickness using TOS.

3 Design Model

Let us outline the five degrees of design freedom constituting the design model.

3.1 Choice of Shell-Slice Type

The first degree of design-freedom is in the choice of the leaves determining the caps-surfaces of the shell-slice. More precisely, by fixing one curvilinear direction and considering the continuum of leaves given by the family of surfaces orthogonal to that direction as seen in the Fig. 8.

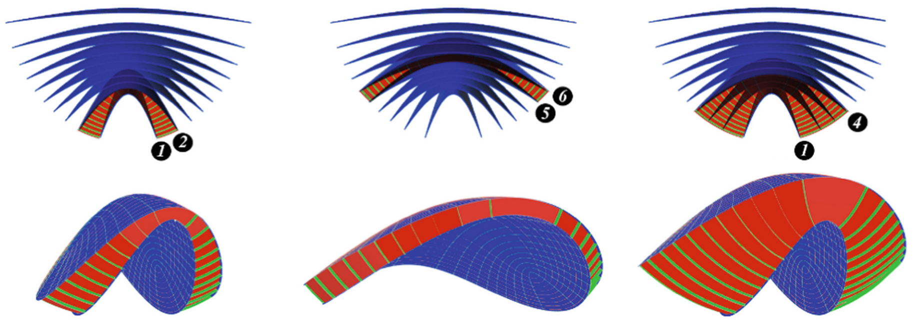


(a) *R*-leaves. (b) *G*-leaves. (c) *B*-leaves.

Fig. 8. a) *R*-leaves. (b) *G*-leaves. (c) *B*-leaves.

3.2 Choice of Shell-Slice Thickness

The second degree of design-freedom is in the choice of the specific two leaves determining the caps-surfaces of the shell-slice, as seen in Fig. 9.

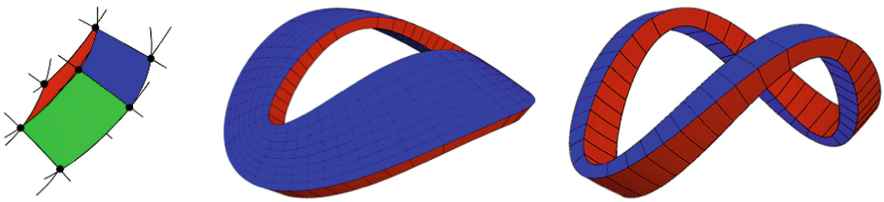


(a) Highly curved thick. (b) Medium curved thin. (c) Highly curved very thick.

Fig. 9. a) Highly curved thick. (b) Medium curved thin. (c) Highly curved very thick.

3.3 Choice of Voxel Assembly

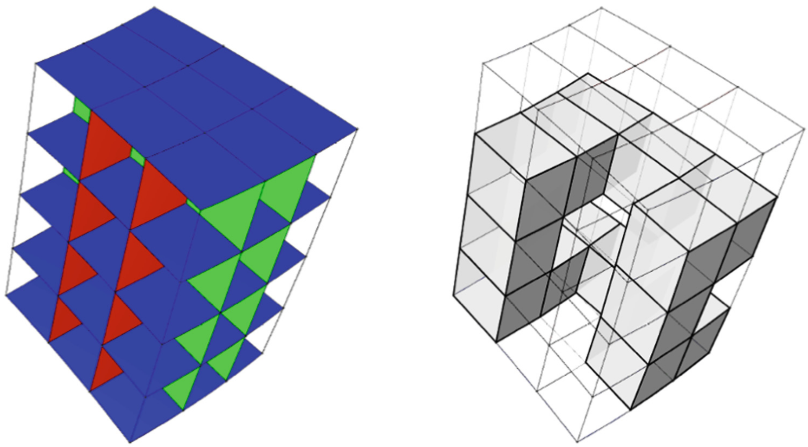
Consider the voxel bounded by two *R*-leaves, two *G*-leaves and two *B*-leaves as seen in Fig. 10(a). This voxel has (doubly-curved) faces which are principal patches and edges which are principal curves. The third degree of design-freedom is in the way we can choose a configuration of these voxels, for example, the chosen shell-slice can be seen as a two-dimensional assembly of these voxels, as seen in Fig. 10(b). This generalizes the assemblies on based revolution surfaces and normal offsets, seen in [10].



(a) One voxel. (b) Two-dimensional assembly. (c) One-dimensional assembly.

Fig. 10. a) One voxel. (b) Two-dimensional assembly. (c) One-dimensional assembly.

Three-Dimensional Assembly of Voxels
It is important to note that by construction of the TOS, this degree of design-freedom can be extended in three directions in the sense of three-dimensional assemblies, as seen in Fig. 11(b).



(a) Discrete number of (*R*,*G*,*B*)-sheets. (b) Three-dimensional assembly of voxels.

Fig. 11. a) Discrete number of (*R*, *G*, *B*)-sheets. (b) Three-dimensional assembly of voxels.

3.4 Choice of Rational-Voxel Type

Since a voxel has six sides all of which are principal patches then from discussion above, we can then rationalize the voxel into one whose faces are either planar or developable. The resulting rationalization of the voxel is a what we call the 'rational-voxel' and the choice of planar and/or developable faces in the voxel is the fourth degree of design-freedom, as seen in Fig. 12. We will call a rational-voxel with:

(1) Only planar faces a *P-voxel*, its assembly is called a *P-assembly*.
(2) Only developable faces a *D-voxel*, its assembly is called a *D-assembly*.
(3) Both planar and developable faces a *PD-voxel*, its assembly is a *PD-assembly*.

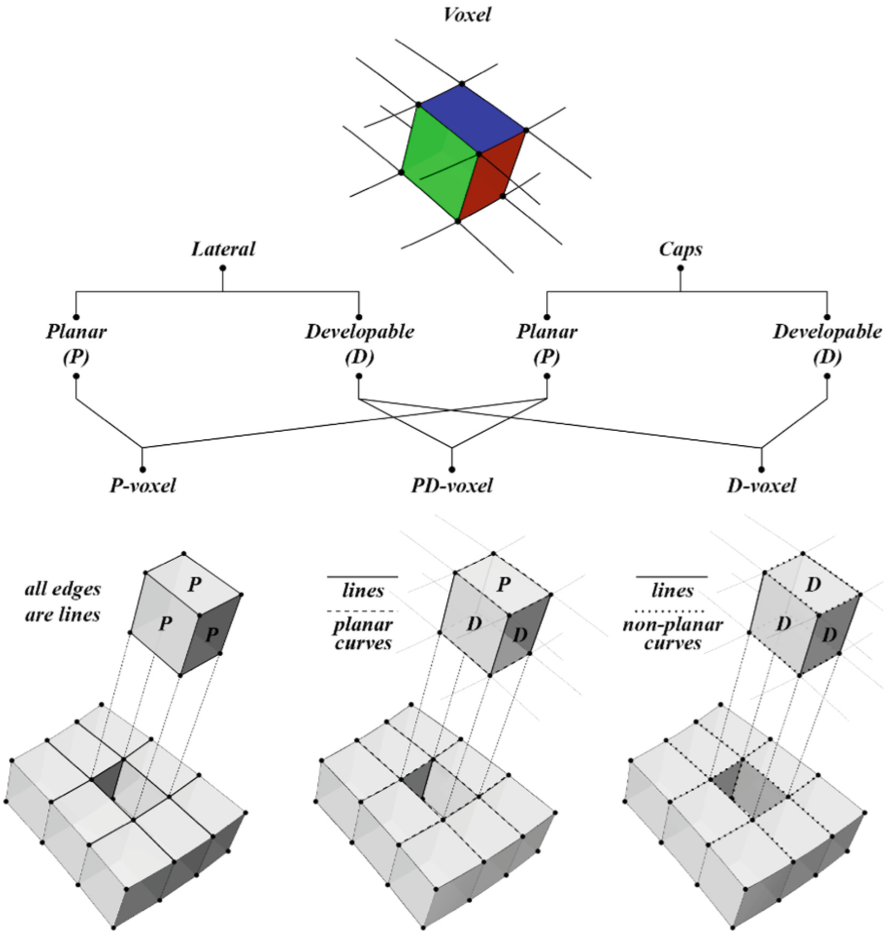


Fig. 12. Rationalization of the voxel into the rational-voxel.

Note that the choice of planarity or developability of the faces of the rational-voxel determines its continuity with its neighboring voxel, as seen in Fig. 13. In particular, the case of a D-assembly the results in series of developable strips amenable to feasible fabrication techniques seen for instance in [11].

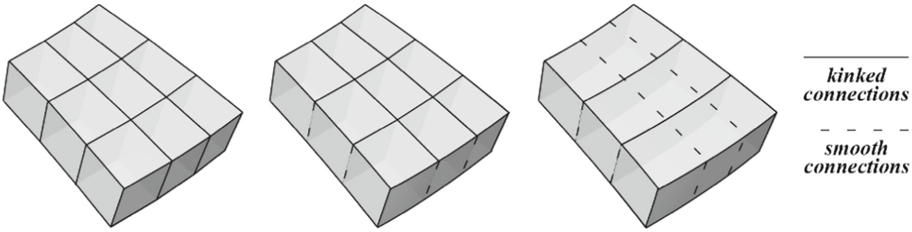


Fig. 13. P, PD and D-assemblies with the continuity with neighbors indicated.

3.5 Choice of Solid or Hollow Voxel-Assembly

The fifth degree of design-freedom is the choice of solid (shell) or hollow (grid-shell), as seen in Fig. 14. Illustrating the some of the design possibilities arising from the five degrees of design-freedom.

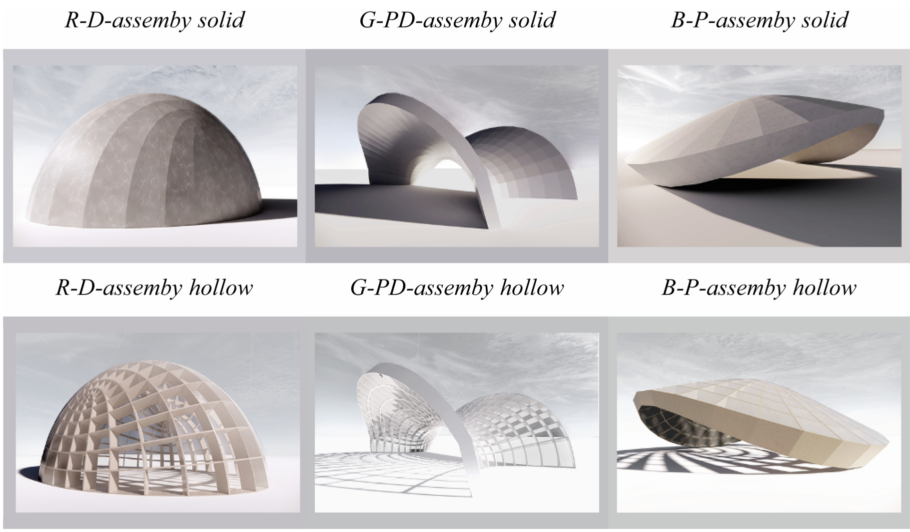


Fig. 14. Matrix of some of the prototypes based on the five degrees of design-freedom.

4 Realized Prototype

Next, we present a realized architectural prototype based on the design model above (Fig. 15).

Fig. 15. The final realized realized pavilion.

A. Form: We started by manipulating the five degrees of design-freedom. This has allowed us to explore different architectural interpretations. Next we narrowed them down to the following proposals seen in Fig. 16 (P-assembly Hollow).

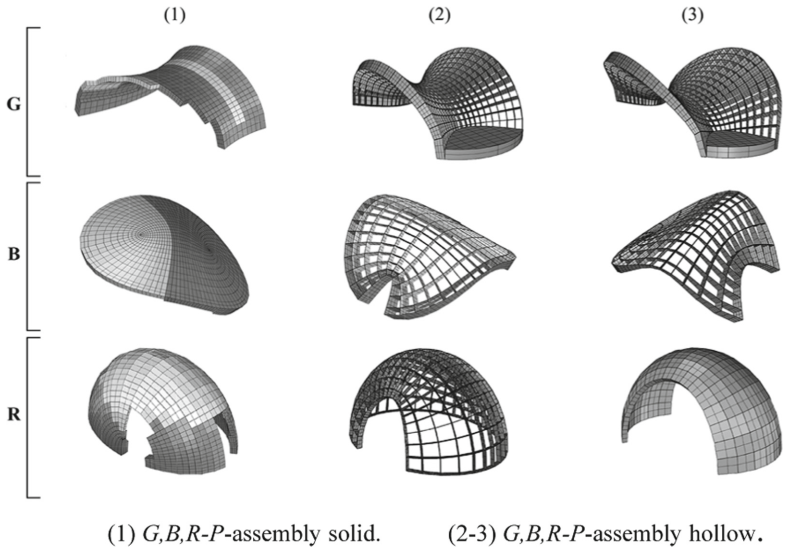


Fig. 16. (1) *G, B, R-P*-assembly solid. (2–3) *G, B, R-P*-assembly hollow.

From these proposals we decided on proposal (2B): the *B-P*-assembly hollow (version 1) as a base form upon which some adjustments will be made taking into account the size of the object, its spatial quality, and the needs of the project, as seen in Fig. 17.

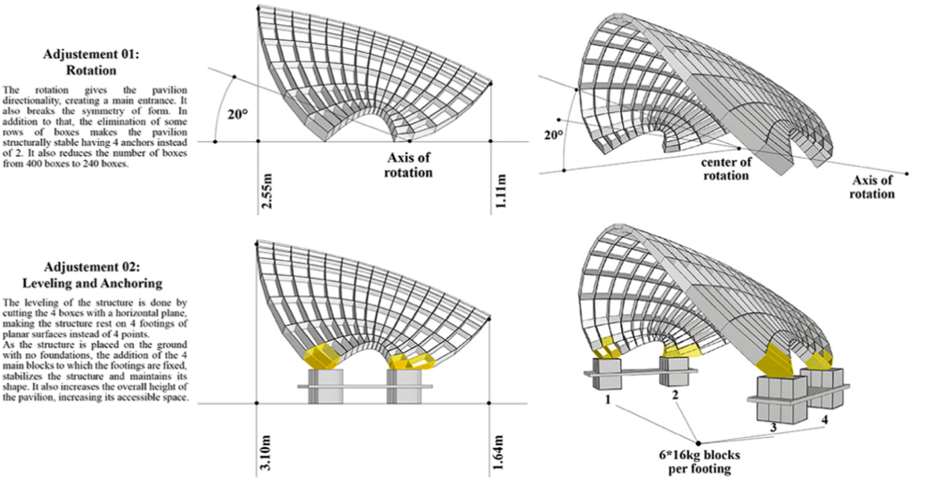


Fig. 17. Adjustments made to the B-P-assembly hollow version 01.

Once the exact final form was finalized, we made a 1:10 physical model in cardboard to test the form and the box-folding technique (Fig. 18).

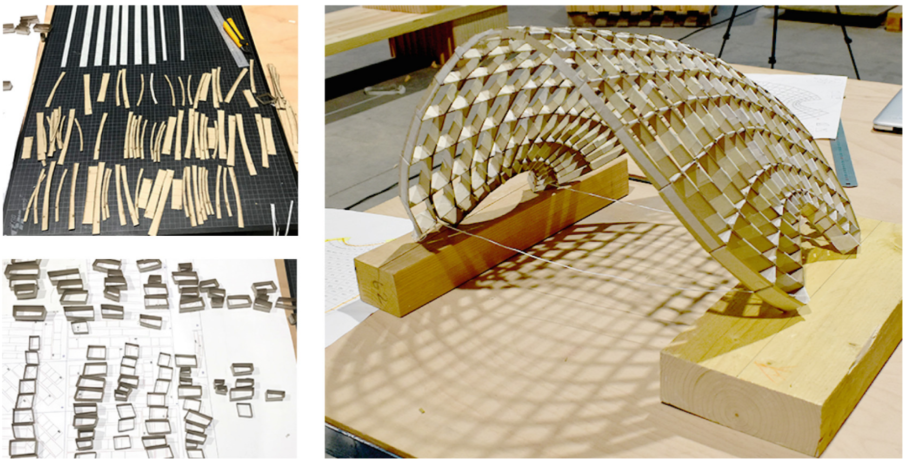


Fig. 18. 1:10 cardboard model of the final form.

B. Material: As we mentioned above, the choice of the P-assembly was fixed earlier on given the material we chose (27 sheets of plywood of 2.5 m length x 1.2 m width x

5 mm thickness) where we considered it to be mostly adapted to realizing planar surfaces given its limited bending capacity.

C. Fabrication and assembly: The advantage in working with P-assembly is that we were able to realize all the boxes (P-voxels) simply by folding flat pieces of plywood. These pieces were cut using the 3-axis milling and then folded as shown in Fig. 19.

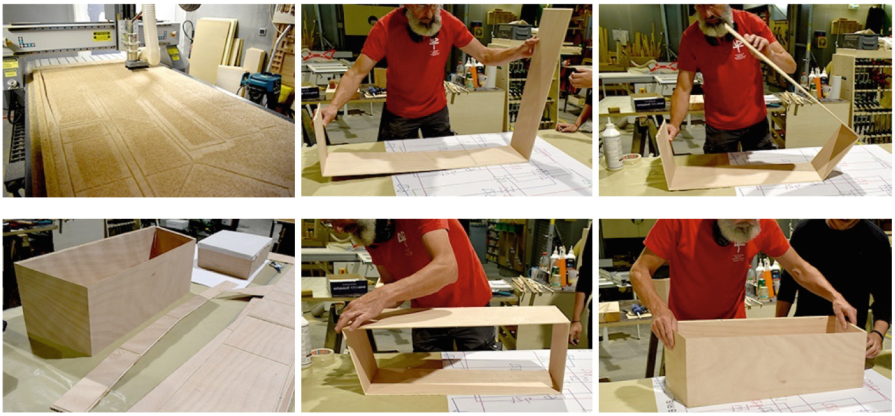


Fig. 19. The folding technique employed for the assembly of the boxes.

Once the boxes were assembled, they were then grouped into partitions according to their position in the structure (Fig. 20).

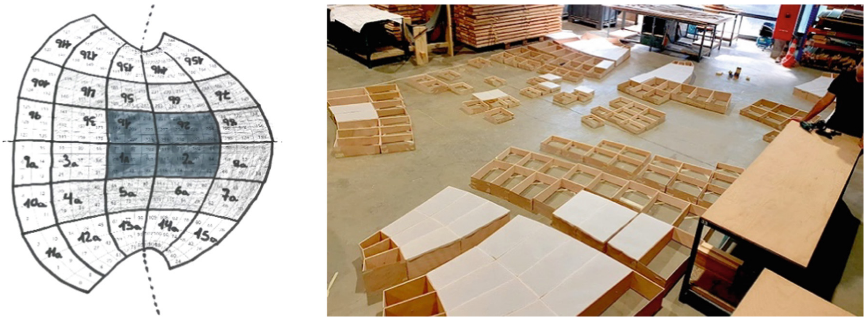


Fig. 20. The grouping of the assembled boxes per partition.

The boxes of each partition were bolted together, then the partitions themselves were bolted together to form the final assembled structure as shown in Fig. 21. Notice the use formwork to maintains the levels of the partitions while they're being joined and the footings fixed to the blocks.

Fig. 21. The assembling of the partitions into the fully assembled model.

5 Conclusion

In this paper, we gave an exposition of a TOS showing its potential as a mathematical basis for generating design models. We showed that the geometric properties of the principal patches constituting the TOS can be used as tools to generate a large variety of architectural prototypes realized as assemblies of rational-voxels made from planar and developable surfaces. To end the paper, we would like to mention two ways for furthering this work; the first being, exploring the three-dimensional assemblies, in the sense of going into ‘volumes’ and not just shells. Next, it could be seen that the TOS presented here is to certain extent limiting in terms of architectural form, because of its symmetry and geometric logic. Thus, an interesting continuation for us, would be to explore certain deformations of TOS (and principal patches) that preserve the principal networks. Hence, allowing for more freeform shapes while maintaining the geometric properties presented here and their advantages for fabrication.

Acknowledgement. The realized prototype was the result of work done within the framework of a workshop for the master students at the national school of architecture and landscape in Lille. The supply of materials, production, and transportation of the pavilion was funded by the school. The pavilion was exhibited the 11th of June 2022 at Bazaar St-So during the 10th edition of the "*braderie de l'architecture*".

References

1. Bobenko, A., Matthes, D., Suris, Y.: Discrete and smooth orthogonal systems: C^{∞}-approximation. Available from arXiv:math/0303333 (2003)
2. Bobenko, A., Suris, Y.: Discrete differential geometry. Consistency as integrability. Available from arXiv:math/0504358 (2005)
3. Bobenko, A., Tsarev, S.: Curvature line parameterization from circle patterns. Available from arXiv: 0706.3221 (2007)
4. DoCarmo, M.: Differential Geometry of Curves and Surfaces. Prentice-Hall Inc, Englewood Cliffs, New Jersey (1976)
5. Eisenhart, L.: A Treatise on Differential Geometry of Curves and Surfaces. Ginn and Company, Boston (1909)
6. Gray, A., Abbena, E., Salamon, S.: Modern differential geometry of curves and surfaces with Mathematica, 3rd edn. Chapman & Hall/CRC (2006)
7. Liu, Y., Pottmann, H., Wallner, J., Yang, Y., Wang, W.: Geometric modeling with conical meshes and developable surfaces. ACM Tr., Proc. SIGGRAPH **25**(3), 681–689 (2006)
8. Pottmann, H., Liu, Y., Bobenko, A., Wallner, J., Wang, W.: Geometry of multi-layer freeform structures for architecture. ACM Tr., Proc. SIGGRAPH **26**(65), 1–11 (2007)
9. Pottmann, H., et al.: Freeform surfaces from single curved panels. ACM Tr. **27**(3), 1–10 (2008)
10. Filz, G., Schiefer, S.: Rapid assembly of planar quadrangular, self-interlocking modules to anticlastically curved forms. Available on researchgate.net/publication/ 271436191 (2014)
11. Kilian, A.: Fabrication of partially double-curved surfaces out of flat sheet material through a 3D puzzle approach. Available on researchgate.net/publication/30871581 (2003)