Invariant KAM Tori: From Theory to Applications to Exoplanetary Systems

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Abstract We consider the classical problem of the construction of invariant tori exploiting suitable Hamiltonian normal forms. This kind of approach can be translated by means of the Lie series method into explicit computational algorithms, which are particularly suitable for applications in the field of Celestial Mechanics. First, the algorithm constructing the Kolmogorov normal form is described in detail. Then, the extension to lower-dimensional elliptic tori is provided. We adopt the same formalism and notations in both cases, with the aim of making the latter easier to understand. Finally, they are both used in a combined way in order to approximate carefully the secular dynamics of the extrasolar system hosting two planets orbiting around the HD 4732 star.

Keywords Elliptic lower-dimensional tori · KAM theory · Normal forms · Hamiltonian perturbation theory · Exoplanets · n-body planetary problem · Celestial mechanics

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1 Introduction

The birth of KAM[1] theory was marked by a famous article written in 1954 by A.N. Kolmogorov, i.e., [22]. At that epoch the great potential of KAM theorem in order to solve interesting problems in the field of Celestial Mechanics was immediately understood. In fact, it was applied just a few years later in order to prove the stability of the triangular Lagrangian points in the planar restricted problem of three bodies for almost all admissible mass ratios (see [25]). Since then, several applications have ensured the existence of invariant tori in the context of other Hamiltonian models that are of interest in Celestial Mechanics (see, e.g., [10]). Nevertheless, the applications of KAM theory to physically realistic models have never been straightforward. This is mainly due to a few severe constraints that appear in the hypotheses of KAM theorem (e.g., concerning the smallness on the parameter ruling the size of the perturbation).

In the last few decades, the successful applications of KAM theory to Celestial Mechanics introduced more and more refinements in the preliminary work to adapt the Hamiltonian model in such a way to bypass the aforementioned difficulties (see, e.g., [27, 28]). In some other works, the novelty concerns the design of a new approach strategy. In particular, this has been made by combining the results provided by two different theorems; for instance, in [18, 20] the estimates *à la* Nekhoroshev have been applied in the neighborhood of an invariant KAM torus, by following the proof scheme described in [31]. This kind of strategy can be implemented in a natural way by adopting an approach based on suitable *normal forms*. Indeed, different normal form algorithms can be applied one after the other. This work has the ambitious goal of fully explaining a very recent type of applications in the field of Celestial Mechanics, where the computational procedure leading to the Kolmogorov normal form is performed in the neighborhood of a periodic orbit. In turn, such an invariant manifold is preliminarily located by a corresponding normal form for an elliptic torus. The addition of this intermediate step is crucial in order to successfully apply our

[1] It is worth to repeat, here and once again, the story explaining the choice of the acronym KAM. In 1954, during the International Congress of Mathematicians in Amsterdam, Kolmogorov presented his version of the (KAM) theorem. In the same year, he also wrote the very short article [22], where he provided just a scheme of the proof. According to a few direct witnesses, a few years later Kolmogorov explained all the details of his proof in a cycle of lectures delivered at the Moscow University. This was based on a sequence of canonical transformations coherently defined on a so called scale of Banach spaces; a modern reformulation of the proof that should be very similar to the original one is included in [11]. In 1963, V.I. Arnold (who had been a student of Kolmogorov) published a complete proof of the theorem, based on a different approach able to ensure the existence of a Cantor set including many invariant tori and having positive Lebesgue measure (see the statement of Corollary 1 and [1]). In the meantime, the german mathematician J. Moser developed a completely independent version of the proof in the case of symplectic mappings (see [33]). Let us also recall that at the beginning the correctness of the Kolmogorov's approach was doubtful for Moser. Indeed, also because of a famous sentence included in the report he wrote for *Mathematical Reviews* on the Kolmogorov's article (see MR0097508, 20 n. 4066), for many years Arnold's approach was thought to be the only viable one, in order to prove KAM theorem for quasi-integrable Hamiltonian systems.

computational algorithm in its entirety to extrasolar planetary systems with rather eccentric orbits (i.e., whose eccentricity values are significantly larger than those observed for the gaseous planets of our Solar System).

The first theoretical results about the existence of elliptic tori go back to [13, 29, 36]. In the last two decades, similar statements have been proved also in the context of Hamiltonian planetary systems (see [3, 4, 19]). In the present notes, we aim to develop an approach that is far from being purely theoretical. Indeed, we will explain how to extract from the proof schemes the information that is fundamental in order to properly design a computational procedure, which allows to determine invariant manifolds that are in good agreement with the orbital motions of extrasolar planets.

In the following, Sect. 2 contains a quick introduction of a few elementary notions concerning the Hamiltonian perturbation theory and a careful description of the normal form method constructing KAM tori. In Sect. 3, we show how that approach can be adapted for the construction of lower-dimensional invariant manifolds of elliptic type. In the final Sect. 4 our new application to an exoplanetary system is explained in detail; this is designed by combining the two kind of normal forms previously discussed, whose constructions are performed one after each other.

2 Basics of KAM Theory

2.1 Near to the Identity Canonical Transformations by Lie Series

Let us consider two generic dynamical functions $f = f(\boldsymbol{p}, \boldsymbol{q})$ and $\chi = \chi(\boldsymbol{p}, \boldsymbol{q})$, that are defined on all the phase space endowed by n pairs of conjugate canonical variables $(\boldsymbol{p}, \boldsymbol{q}) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$. It is well known that the time evolution of f under the flow induced by χ is ruled by the Poisson bracket between these two functions, i.e., $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{p}(t), \boldsymbol{q}(t)) = \{f, \chi\}$, where

$$\{f\,,\,\chi\} = \sum_{j=1}^{n} \frac{\partial f}{\partial q_j}\frac{\partial \chi}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial \chi}{\partial q_j} \tag{1}$$

and the flow $(\boldsymbol{p}(t), \boldsymbol{q}(t)) = \Phi_{\chi}^{t}\big(\boldsymbol{p}(0), \boldsymbol{q}(0)\big)$ is defined by the solution of the corresponding Hamilton equations

$$\dot{p}_j = -\frac{\partial \chi}{\partial q_j}\,, \qquad \dot{q}_j = \frac{\partial \chi}{\partial p_j}\,, \qquad \forall\, j = 1, \ldots, n \tag{2}$$

(being $(\boldsymbol{p}(0), \boldsymbol{q}(0))$ regarded as initial conditions).

Let us now focus on the Taylor expansion with respect to time of the generic dynamical function f, i.e., $f + t\dot{f} + \frac{t^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\dot{f} + \ldots = f + t\{f, \chi\} + \frac{t^2}{2}\{\{f, \chi\}, \chi\} + \ldots$, that can be reformulated in terms of Lie series. First, let us introduce the so called Lie derivative operator: $\mathcal{L}_\chi \cdot = \{\cdot, \chi\}$; in the present context, it is usual to refer to χ as the generating function of the corresponding Lie derivative. Thus, the previous Taylor expansion in time can be expressed as $\exp\big(t\mathcal{L}_\chi\big)f = \sum_{j=0}^{\infty} \frac{t^j}{j!}\mathcal{L}_\chi^j f$. It is common to define the Lie series operator just in the case with $t = 1$, i.e., it acts on the generic dynamical function f in such a way that

$$\exp\big(\mathcal{L}_\chi\big)f = \sum_{j=0}^{\infty} \frac{1}{j!}\mathcal{L}_\chi^j f \ ;$$

let us emphasize that this formula must be interpreted at a purely formal level, in the sense that we do not wonder about the convergence of the series. However, it can be ensured if the sup-norm of the generating function χ is small enough,[2] as it is natural to expect.

Since any single canonical coordinate can be seen as a particular dynamical function, we can express the Hamiltonian flow at time 1 in terms of Lie series in the following way:

$$\Phi_\chi^1(\boldsymbol{p}, \boldsymbol{q}) = \exp\big(\mathcal{L}_\chi\big)(\boldsymbol{p}, \boldsymbol{q}), \tag{3}$$

where, for every pair of canonical variables $(p_i\, ,\, q_i)$ (being $i = 1, \ldots, n$), we put

$$\Phi_\chi^1 p_i = \exp\big(\mathcal{L}_\chi\big)p_i\ , \qquad \Phi_\chi^1 q_i = \exp\big(\mathcal{L}_\chi\big)q_i\ .$$

It is well known that the Hamiltonian flow is canonical, then we readily obtain that the map defined by the Lie series operator in the right hand side of (3) is canonical as well. Moreover, such a change of coordinates is obviously close to the identity in the limit of the generating functions shrinking to zero.

The canonical formalism makes very convenient the writing of the equations of motion in the new variables. Let us assume that the evolution in the original set of coordinates $(\boldsymbol{p}, \boldsymbol{q})$ is ruled by a single function $H = H(\boldsymbol{p}, \boldsymbol{q})$ entering the Hamilton equations (2) in place of χ; moreover, let $(\boldsymbol{p}, \boldsymbol{q}) = \mathcal{C}(\boldsymbol{P},\, \boldsymbol{Q})$ be a canonical transformation. Therefore, the new equations of motions can be written as follows:

$$\dot{P}_j = -\frac{\partial \mathcal{K}}{\partial Q_j}\ , \qquad \dot{Q}_j = \frac{\partial \mathcal{K}}{\partial P_j}\ , \qquad \forall\ j = 1, \ldots, n, \tag{4}$$

being $\mathcal{K}(\boldsymbol{P},\, \boldsymbol{Q}) = H\big(\mathcal{C}(\boldsymbol{P},\, \boldsymbol{Q})\big)$ the new Hamiltonian function. In such a context, the Lie series formalism makes automatic (and, then, somehow easier) the procedure

[2] The convergence of the Lie series is carefully discussed in [14, 15]; in particular, the explanatory notes in [15] contains also a rather self-consistent introduction to the Lie series formalism in the Hamiltonian framework.

of substitution, because of the so called "exchange theorem" (see [14]). In fact, if χ is a small enough generating function, the new Hamiltonian can be expressed as

$$\mathcal{K}(\boldsymbol{P},\,\boldsymbol{Q}) = \exp\left(\mathcal{L}_{\chi}\right)H\Big|_{(\boldsymbol{p},\boldsymbol{q})=(\boldsymbol{P},\,\boldsymbol{Q})}\,,$$

this means that we can apply the Lie series to the old Hamiltonian function so as to rename the variables, only at the end. For more detailed explanations we defer to the whole Sect. 4.1 of [15]. Of course, the same computational procedure holds also for the corresponding canonical transformation, that is given by

$$(\boldsymbol{p},\boldsymbol{q}) = \mathcal{C}(\boldsymbol{P},\,\boldsymbol{Q}) = \exp\left(\mathcal{L}_{\chi}\right)(\boldsymbol{p},\boldsymbol{q})\Big|_{(\boldsymbol{p},\boldsymbol{q})=(\boldsymbol{P},\,\boldsymbol{Q})}\,.$$

2.2 Statement(s) of KAM Theorem

First, let us recall the statement of KAM theorem as in its very first version introduced by Kolmogorov (see [22]).

Theorem 1 (KAM, according to the version due to Kolmogorov) *Consider a Hamiltonian function* $H : \mathcal{A}\times\mathbb{T}^n \mapsto \mathbb{R}$ *(being* $\mathcal{A}\subseteq\mathbb{R}^n$ *an open set) of the form* $H(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{\omega}\cdot\boldsymbol{p} + h(\boldsymbol{p}) + \varepsilon f(\boldsymbol{p},\boldsymbol{q})$ *where h is at least quadratic with respect to the actions* $\boldsymbol{p}$*, i.e.,* $h(\boldsymbol{p}) = \mathcal{O}(\|\boldsymbol{p}\|^2)$ *for* $\boldsymbol{p}\to\boldsymbol{0}$*. Moreover, let us assume the following hypotheses:*

(a) $\boldsymbol{\omega}$ *is Diophantine; this means that there are two positive constants*[3] γ *and* τ *such that* $|\boldsymbol{k}\cdot\boldsymbol{\omega}| \ge \frac{\gamma}{|\boldsymbol{k}|^{\tau}}\ \forall\,\boldsymbol{k}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}$*;*
(b) H *is analytic on its action–angle*[4] *domain of definition* $\mathcal{A}\times\mathbb{T}^n$*;*
(c) $h(\boldsymbol{p})$ *is non-degenerate, i.e.,* $\det\left(\frac{\partial^2 h}{\partial p_i\partial p_j}(\boldsymbol{p})\right)_{i,j} \neq 0\ \forall\,\boldsymbol{p}\in\mathcal{A}$*;*
(d) ε *is a small enough parameter.*

Therefore, there is a canonical transformation $(\boldsymbol{p},\boldsymbol{q}) = \psi_{\varepsilon}(\boldsymbol{P},\,\boldsymbol{Q})$*, leading H in the so called Kolmogorov normal form* $\mathcal{K}(\boldsymbol{P},\,\boldsymbol{Q}) = \omega\cdot\boldsymbol{P} + \mathcal{O}(\|\boldsymbol{P}\|^2)$*, being* $\mathcal{K} = H\circ\psi_{\varepsilon}$*.*

In our exposition of these topics, we do not consider all the very interesting mathematical work that has been done in the last fifty years in order to weaken the

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[3] Indeed, in order to satisfy the Diophantine inequality, it is essential that $\tau \ge n-1$.

[4] Although there exist formulations of the KAM theorem that are not dealing with action–angle canonical coordinates (see, e.g., [12]), we stress that this is a rather natural framework to assume. In fact, by definition a n–dimensional torus $\mathbb{T}^n$ is in a bijective correspondence with n angles, denoted as $(q_1,\ldots,q_n)$ in agreement with the text. Thus, they can be adopted as coordinates. Let us recall that in Hamiltonian mechanics the product between each conjugate pair of canonical variables has the physical dimension of an action, that is the same as an angular momentum. Therefore, $\forall\ j=1,\ldots n$, the conjugate momentum p_j is an action, because $(\boldsymbol{p},\boldsymbol{q})$ are assumed to be canonical coordinates.

assumptions on the KAM theorem. We prefer to focus on what makes the theorem suitable to apply to interesting physical problems. This is somehow hidden in the thesis of the statement and, mainly, in the proof scheme. Let us highlight such a content.

One can easily verify that, if the Hamiltonian is in the Kolmogorov normal form $\mathcal{K}(\boldsymbol{P},\,\boldsymbol{Q}) = \omega\cdot\boldsymbol{P} + \mathcal{O}(\|\boldsymbol{P}\|^2)$, then $t\mapsto\,(\boldsymbol{P}(t)=\boldsymbol{0}\,,\;\boldsymbol{Q}(t)=\boldsymbol{Q}_0+\boldsymbol{\omega}t)$ is the solution for the equations of motion (4) starting from the generic initial conditions $(\boldsymbol{P}(0)\,,\;\boldsymbol{Q}(0))=(\boldsymbol{0}\,,\;\boldsymbol{Q}_0)$. Since the canonical transformations enjoy the property of preserving solutions, this allows us to design the following integration scheme for the equations of motion (3), when the generic Hamiltonian χ is replaced by H, that describes the problem we are considering:

$$
\begin{array}{ccc}
\big(\boldsymbol{p}(0),\boldsymbol{q}(0)\big) & \overset{\psi_\varepsilon^{-1}}{\longrightarrow} & (\boldsymbol{P}(0)\,,\;\boldsymbol{Q}(0)) \\
 & & \big\downarrow \Phi^t_{\mathcal{K}} \\
\big(\boldsymbol{p}(t),\boldsymbol{q}(t)\big) & \overset{\psi_\varepsilon}{\longleftarrow} & (\boldsymbol{P}(t)\,,\;\boldsymbol{Q}(t))
\end{array}
\;. \qquad (5)
$$

In the scientific literature, this way to compute the motion law $t\mapsto\,(\boldsymbol{p}(t)\,,\;\boldsymbol{q}(t)) = \Phi^t_H(\boldsymbol{p}(0)\,,\;\boldsymbol{q}(0))$ is often said to be semi-analytic. Such a name is due to the fact that the schematic procedure above is usually performed after having determined the Fourier expansions of the canonical transformation ψ_ε, by using a software package designed for doing computer algebra manipulations.

In spite of the fact that the very first version of the KAM theorem ensures the existence of a single invariant torus, the statement can be extended so as to cover a very generic situation. Indeed, in his very short but incredibly seminal article [22], Kolmogorov recalled a well known result of number theory: almost all n–dimensional vectors are Diophantine. This remark jointly with the uniform non-degeneracy of the so called action-frequency map in the integrable approximation, i.e., $\boldsymbol{p}\mapsto\boldsymbol{\omega}(\boldsymbol{p}) = \big(\frac{\partial h}{\partial p_i}(\boldsymbol{p})\big)_{i=1,\ldots,n}$, allowed him to state the following result in [22].

Corollary 1 (KAM, according to the version proved by Arnold) *Consider a quasi-integrable Hamiltonian depending on action–angle variables, i.e.,* $H:\mathcal{A}\times\mathbb{T}^n\mapsto\mathbb{R}$ *(being* $\mathcal{A}\subseteq\mathbb{R}^n$ *an open set) of the form* $H(\boldsymbol{p},\boldsymbol{q}) = h(\boldsymbol{p}) + \varepsilon f(\boldsymbol{p},\boldsymbol{q})$*. If we assume the same hypotheses (b)–(d) of Theorem 1, then there is a set* $\mathcal{S}_\varepsilon$ *that is made by invariant tori and is such that its Lebesgue measure* $\mu\big(\mathcal{S}_\varepsilon\big)$ *is positive. Moreover,*

$$
\lim_{\varepsilon\to 0}\mu\Big(\big(\mathcal{A}\times\mathbb{T}^n\big)\setminus\mathcal{S}_\varepsilon\Big) = 0\;.
$$

Let us emphasize that this statement highlights one of the main merits of the KAM theorem: it shows that there is a sort of continuity (in terms of the Lebesgue measure) between integrable systems and quasi-integrable ones. From one hand, this sort of intuitive concept was (and still is) considered to be extremely natural; on the other hand, at that epoch such an expectation was in contrast with the famous theorem by Poincaré (that can be felt as somehow paradoxical, see [34]) on the non-existence of integrals of motion apart from the energy for a generic quasi-integrable Hamiltonian system.

Although the statement of the Corollary above can be easily deduced from the original version of the KAM theorem that is due to Kolmogorov, the proof scheme introduced by Arnold in [1] is extremely deep, because it provides a more global picture of the dynamics. This approach has been further extended, for instance, in [35], where it is proved that quasi-integrable Hamiltonian satisfying the usual hypotheses (b)–(d) of Theorem 1 can be conjugated to integrable ones via a canonical transformation that is not analytic, but it is $\mathcal{C}^{(\infty)}$.

2.3 Algorithmic Construction of the Kolmogorov Normal Form

These notes are focusing more on the applications based on the KAM theory rather than on the theory itself. Therefore, it is important to describe carefully the so called formal algorithm constructing the Kolmogorov normal form. The results about the convergence of such a computational procedure are very well established (see, e.g., [17]) and in the following we will just briefly recall them.

For the sake of definiteness, we need to introduce some notations. For a fixed positive integer K we introduce the distinct classes of functions $\mathcal{P}_{\ell,sK}$, for all non-negative indexes $\ell,\ s \ge 0$. Any generic function $g \in \mathcal{P}_{\ell,sK}$ can be written as

$$g(\boldsymbol{p},\boldsymbol{q}) = \sum_{\substack{\boldsymbol{j}\in\mathbb{N}^n \\ |\boldsymbol{j}|=\ell}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^n \\ |\boldsymbol{k}|\le sK}} c_{\boldsymbol{j},\boldsymbol{k}}\boldsymbol{p}^{\boldsymbol{j}} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\right) , \tag{6}$$

where $(\boldsymbol{p},\boldsymbol{q})$ are action–angle canonical variables and the coefficients $c_{\boldsymbol{j},\boldsymbol{k}} \in \mathbb{C}$ satisfy the following relation: $c_{\boldsymbol{j},-\boldsymbol{k}} = \bar{c}_{\boldsymbol{j},\boldsymbol{k}}$ so that $g : \mathbb{R}^n \times \mathbb{T}^n \mapsto \mathbb{R}$. Moreover, in the previous formula, we have introduced the symbol $|\cdot|$ to denote the ℓ_1-norm (i.e., $|\boldsymbol{k}| = |k_1| + \ldots + |k_n|$) and we have adopted the multi-index notation, i.e., $\boldsymbol{p}^{\boldsymbol{j}} = p_1^{j_1} \cdot \ldots \cdot p_n^{j_n}$. In the following, we will adopt the usual notation for the average of a function g with respect to the generic angles $\boldsymbol{\vartheta} \in \mathbb{T}^n$, i.e., $\langle g\rangle_{\boldsymbol{\vartheta}} = \int_{\mathbb{T}^n} \mathrm{d}\vartheta_1 \ldots \mathrm{d}\vartheta_n\, g/(2\pi)^n$.

We will start the formal algorithm from a Hamiltonian of the following type:

$$\begin{aligned} H^{(0)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)}) \;=\; & E^{(0)} + \boldsymbol{\omega}^{(0)}\cdot\boldsymbol{p} + \sum_{s\ge0}\sum_{\ell\ge2} f_\ell^{(0,\,s)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)}) \\ & + \sum_{s\ge1}\sum_{\ell=0}^{1} f_\ell^{(0,\,s)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)}) , \end{aligned} \tag{7}$$

where $f_\ell^{(0,\,s)} \in \mathcal{P}_{\ell,sK}$, being the first upper index related to the normalization step, and $E^{(0)} \in \mathcal{P}_{0,0}$ is a constant meaning the energy level of the torus $\big\{(\boldsymbol{p},\boldsymbol{q})\ :\ \boldsymbol{p}=\boldsymbol{0},\ \boldsymbol{q}\in\mathbb{T}^n\big\}$ that is invariant in the integrable approximation. The occurrence of $\boldsymbol{\omega}^{(0)}$ at the end of the list of the arguments emphasizes that those functions depend also on that angular velocity vector in a parametric way. We also stress that the terms appearing in the second row of formula (7) have to be considered as the small perturbation we aim to remove in order to bring the Hamiltonian in Kolmogorov normal form. According to the definition given by Poincaré (see [34]), the *general problem of the dynamics* is described by a real analytic Hamiltonian of type $H(\boldsymbol{I},\boldsymbol{\varphi};\varepsilon)=h(\boldsymbol{I})+\varepsilon f(\boldsymbol{I},\boldsymbol{\varphi})$, being $(\boldsymbol{I},\boldsymbol{\varphi})$ action–angle coordinates and ε a small parameter. It is well known that such an Hamiltonian can be put in the form (7) provided that the Hessian of the integrable part h is non-degenerate on its open domain, say $\mathcal{A}\subseteq\mathbb{R}^n$. Indeed, it is just matter of performing a canonical change of coordinates that translates the origin of the actions in correspondence to $\boldsymbol{I}^\star\in\mathcal{A}$, because

$$\frac{\partial h\big(\boldsymbol{I}\big)}{\partial I_j}\bigg|_{\boldsymbol{I}=\boldsymbol{I}^\star} = \frac{\partial h\big(\boldsymbol{I}(\boldsymbol{p})\big)}{\partial p_j}\bigg|_{\boldsymbol{p}=\boldsymbol{0}} = \omega_j^{(0)} \quad \forall\ j=1,\ldots,n\,,$$

where $\boldsymbol{I}=\boldsymbol{p}+\boldsymbol{I}^\star$. Obviously, the so called action–frequency map in the integrable approximation, i.e., $\boldsymbol{I}^\star\mapsto\boldsymbol{\omega}^{(0)}$, can be inverted because the Hessian of h is non-degenerate. Therefore, the angular velocity vector $\boldsymbol{\omega}^{(0)}$ can be used instead of $\boldsymbol{I}^\star$ in order to parameterize the whole Hamiltonian. Moreover the Fourier decay of the coefficients with respect to the angles $\boldsymbol{q}=\boldsymbol{\vartheta}$ allows to perform the expansion (7) in such a way that $f_\ell^{(0,\,s)}=\mathcal{O}(\varepsilon^s)$. In other words, the positive integer parameter K can be chosen in such a way that the superscript s refers at the same time to both the order of magnitude and the trigonometric degree (being $f_\ell^{(0,\,s)}\in\mathcal{P}_{\ell,sK}$); more details about that can be found in [17].

We are now ready for the description of the (generic) r-th step of the normalization procedure, which defines the Hamiltonian $H^{(r)}$ starting from $H^{(r-1)}$, whose expansion is written as follows:

$$\begin{aligned} H^{(r-1)}(\boldsymbol{p},\boldsymbol{q}) \ =\ & E^{(r-1)}+\boldsymbol{\omega}^{(r-1)}\cdot\boldsymbol{p}+\sum_{s\ge0}\sum_{\ell\ge2} f_\ell^{(r-1,\,s)}(\boldsymbol{p},\boldsymbol{q}) \\ & +\sum_{s\ge r}\sum_{\ell=0}^{1} f_\ell^{(r-1,\,s)}(\boldsymbol{p},\boldsymbol{q})\,. \end{aligned} \tag{8}$$

Hereafter, we omit the dependence of the function from the parameters, unless it has some special meaning. Let us assume that some fundamental properties that hold true for $H^{(0)}$ are satisfied also for the expansion above of $H^{(r-1)}$, i.e., $f_\ell^{(r-1,\,s)}\in\mathcal{P}_{\ell,sK}$ and $f_\ell^{(r-1,\,s)}=\mathcal{O}(\varepsilon^s)$. Since the r-th normalization step aims to remove the main perturbing terms, that are $f_0^{(r-1,\,r)}$ and $f_1^{(r-1,\,r)}$, we introduce a first generating function $\chi_1^{(r)}$, that is determined by solving the following (first) homological equation:

$$\left\{\boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p},\ \chi_1^{(r)}\right\} + f_0^{(r-1,\,r)}(\boldsymbol{q}) = \langle f_0^{(r-1,r)}(\boldsymbol{q})\rangle_{\boldsymbol{q}} \ . \tag{9}$$

Since $f_0^{(r-1,\,r)} \in \mathcal{P}_{0,rK}$, its expansion is written as

$$f_0^{(r-1,\,r)}(\boldsymbol{q}) = \sum_{|\boldsymbol{k}|\le rK} c_{\boldsymbol{k}} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\right) ,$$

where the complex coefficients are such that $c_{-\boldsymbol{k}} = \bar{c}_{\boldsymbol{k}}$. Therefore, one can easily check that the first homological equation (9) is solved by putting $\langle f_0^{(r-1,r)}(\boldsymbol{q})\rangle_{\boldsymbol{q}} = c_{\boldsymbol{0}}$ and

$$\chi_1^{(r)}(\boldsymbol{q}) = \sum_{0<|\boldsymbol{k}|\le rK} \frac{c_{\boldsymbol{k}} \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\right)}{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}} \ . \tag{10}$$

In order to preserve the validity of the solution above, of course, we have to require that none of the divisors can eventually vanish; thus we assume the following non-resonance condition:

$$\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)} \neq 0 \quad \forall\, 0 < \boldsymbol{k} \le rK \ . \tag{11}$$

The first half of the r-th normalization step is completed by introducing $\hat{H}^{(r)} = \exp\left(\mathcal{L}_{\chi_1^{(r)}}\right) H^{(r-1)}$. Such an intermediate Hamiltonian can be written in a form similar to formula (8), i.e.,

$$\hat{H}^{(r)}(\boldsymbol{p},\boldsymbol{q}) = E^{(r)} + \boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p} + \sum_{s\ge 0}\sum_{\ell\ge 2} \hat{f}_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q}) + \sum_{s\ge r}\sum_{\ell=0}^{1} \hat{f}_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q}) \ , \tag{12}$$

where the recursive definitions of the new summands $\hat{f}_\ell^{(r,\,s)}$ (in terms of $f_\ell^{(r-1,\,s)}$) can be given by exploiting the linearity of the Lie series and by separating the functions according to the different classes $\mathcal{P}_{\ell,sK}$ they belong to. We think it is convenient to formulate these definitions in a rather unconventional way, by using a notation similar to that commonly used in the programming languages; in our opinion, such a choice should make easier the translation of the formal algorithm in any code to be executed in a computational environment. For this purpose, we first define[5] $\hat{f}_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q}) = f_\ell^{(r-1,\,s)}(\boldsymbol{p},\boldsymbol{q})\ \forall\,\ell \ge 0,\ s \ge 0$. Then, *by abuse of notation*, we update $\lfloor s/r \rfloor$ times the definition of the terms $\hat{f}_\ell^{(r,\,s)}$ appearing in the expansion of the new Hamiltonian according to the following rule:

$$\hat{f}_{\ell-j}^{(r,\,s+jr)} \hookleftarrow \frac{1}{j!}\mathcal{L}_{\chi_1^{(r)}}^{j} f_\ell^{(r-1,\,s)} \quad \forall\,\ell \ge 1,\ 1 \le j \le \ell,\ s \ge 0\ , \tag{13}$$

[5] We remark that $f_\ell^{(r-1,\,s)}$ do not enter in the expansion (8) if $\ell = 0,\ 1$ and $s < r$. The same applies to the terms $\hat{f}_\ell^{(r,\,s)}$ that do not make part of the expression of $\hat{H}^{(r)}$, which is written in (12). However, the recursive definitions described in the present subsection are such that $f_\ell^{(r-1,\,s)} = \hat{f}_\ell^{(r,\,s)} = 0$ $\forall\, 0 \le s < r$ when $\ell = 0,\ 1$.

where with the notation $a \hookleftarrow b$ we mean that the quantity a is redefined so as to be equal to $a+b$. Moreover, there is a last additional contribution that is due to the application of the Lie series to the Hamiltonian $H^{(r-1)}$, and in order to take it into account we write

$$\hat{f}_0^{(r,\,r)} \hookleftarrow \mathcal{L}_{\chi_1^{(r)}} \boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p} \,. \tag{14}$$

However, because of the homological equation (9), we can finally put $\hat{f}_0^{(r,\,r)} = 0$ and update the constant energy value so that

$$E^{(r)} = E^{(r-1)} + \langle f_0^{(r-1,\,r)} \rangle_{\boldsymbol{q}} \,. \tag{15}$$

At this point, it is important to remark that the angular average of the remaining perturbing term that is $\mathcal{O}(\varepsilon^r)$, i.e., $\langle \hat{f}_1^{(r,\,r)} \rangle_{\boldsymbol{q}}$ is exactly of the same type as $\boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p}$ (this means that both of them are linear with respect to the actions and do not depend on the angles). Therefore, it is useful to update also the angular velocity vector[6] by joining together these two terms. This can be done, by redefining

$$\boldsymbol{\omega}^{(r)} \cdot \boldsymbol{p} = \boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p} + \langle \hat{f}_1^{(r,\,r)} \rangle_{\boldsymbol{q}} \tag{16}$$

and

$$\hat{f}_1^{(r,\,r)} = \hat{f}_1^{(r,\,r)} - \langle \hat{f}_1^{(r,\,r)} \rangle_{\boldsymbol{q}} \,. \tag{17}$$

Let us recall that all the terms $\hat{f}_\ell^{(r,\,s)}$ that appear in formula (12) are organized so that they belong to different classes of functions. In order to prove that these structures are suitably preserved by the normalization algorithm, the following statement is essential.

Lemma 1 *Let us consider two generic functions $g \in \mathcal{P}_{\ell,sK}$ and $h \in \mathcal{P}_{m,rK}$, where K is a fixed positive integer number. Then, the following inclusion property holds true*[7]*:*

$$\big\{g, h\big\} = \mathcal{L}_h\, g \in \mathcal{P}_{\ell+m-1,(r+s)K} \quad \forall\, \ell,\ m,\ r,\ s \in \mathbb{N}\,.$$

The proof is omitted, because it can be obtained as a straightforward consequence of the definition of the Poisson brackets. By applying repeatedly the lemma above and a trivial induction argument to formulæ (13)–(17), one can easily prove that $E^{(r)} \in \mathcal{P}_{0,0}$ and $\hat{f}_\ell^{(r,\,s)} \in \mathcal{P}_{\ell,sK}$ for all the terms of type $\hat{f}_\ell^{(r,\,s)}$ that appear in formula (12). Moreover, it can be ensured that $\big|E^{(r)} - E^{(r-1)}\big| = \mathcal{O}(\varepsilon^r)$ and $\hat{f}_\ell^{(r,\,s)} = \mathcal{O}(\varepsilon^s)$, if the same relation is assumed to be true at the end of the previous normalization step, i.e., $f_\ell^{(r-1,\,s)} = \mathcal{O}(\varepsilon^s)$.

[6] We emphasize that this is one of the main differences with respect to the original proof scheme designed by Kolmogorov, where the angular velocity vector is kept fixed at every normalization step (see [5] for a fully consistent translation of such an approach, that is implemented by using the Lie series technique).

[7] The statement can be considered as valid also in the trivial case with $\ell = m = 0$, by enlarging the definition of the classes of functions so that $\mathcal{P}_{-1,sK} = \big\{0\big\}\ \forall\, s \in \mathbb{N}$.

In order to complete the r-th normalization step, we have to remove the remaining perturbing term that is $\mathcal{O}(\varepsilon^r)$ and appears in the expansion (12) of Hamiltonian $\hat{H}^{(r)}$, i.e., $\hat{f}_1^{(r,\,r)}$. For such a purpose, we determine a second generating function $\chi_2^{(r)}$, by solving the following (second) homological equation:

$$\left\{\boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p},\ \chi_2^{(r)}\right\}+\hat{f}_1^{(r,\,r)}(\boldsymbol{p},\boldsymbol{q})=0\,. \tag{18}$$

We can deal with the equation above in a very similar way with respect to what has been done for the first homological equation (9). In fact, the solution of (18) can be written as follows:

$$\chi_2^{(r)}(\boldsymbol{p},\boldsymbol{q})=\sum_{|\boldsymbol{j}|=1}\ \sum_{0<|\boldsymbol{k}|\le rK}\frac{c_{\boldsymbol{j},\boldsymbol{k}}\,\boldsymbol{p}^{\boldsymbol{j}}\exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\right)}{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r)}}\,, \tag{19}$$

where the expansion of the perturbing term $\hat{f}_1^{(r,\,r)}\in\mathcal{P}_{1,rK}$ is of type

$$\hat{f}_1^{(r,\,r)}(\boldsymbol{p},\boldsymbol{q})=\sum_{|\boldsymbol{j}|=1}\ \sum_{0<|\boldsymbol{k}|\le rK}c_{\boldsymbol{j},\boldsymbol{k}}\,\boldsymbol{p}^{\boldsymbol{j}}\exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\right)\,.$$

Let us recall that the angular average of $\hat{f}_1^{(r,\,r)}$ is equal to zero, because of the redefinition (17). Of course, the solution written in (19) is valid provided that the following non-resonance condition is satisfied:

$$\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r)}\neq 0\quad\forall\,0<\boldsymbol{k}\le rK\ . \tag{20}$$

Finally, $H^{(r)}=\exp\left(\mathcal{L}_{\chi_2^{(r)}}\right)\hat{H}^{(r)}$ is the new Hamiltonian that is defined by the canonical transformation of coordinates that is introduced by the r-th normalization step. Also the expansion of such a Hamiltonian can be written in a form similar to (8), i.e.,

$$H^{(r)}(\boldsymbol{p},\boldsymbol{q})=E^{(r)}+\boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p}+\sum_{s\ge0}\sum_{\ell\ge2}f_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q})+\sum_{s\ge r+1}\sum_{\ell=0}^{1}f_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q})\,. \tag{21}$$

In this case too, the recursive definitions of the new summands $f_\ell^{(r,\,s)}$ can be given by exploiting the linearity of the Lie series and by separating the functions according to the different classes they belong to. Let us start by introducing $f_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q})=\hat{f}_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q})\ \forall\,\ell\ge0,\ s\ge0$. By a new abuse of notation, we update many times the definition of the terms appearing in the expansion of Hamiltonian $H^{(r)}$ according to the following rule:

$$f_\ell^{(r,\,s+jr)}\hookleftarrow\frac{1}{j!}\mathcal{L}^j_{\chi_2^{(r)}}\hat{f}_\ell^{(r,\,s)}\quad\forall\,\ell\ge2,\ j\ge1,\ s\ge0\ \text{or}\ \forall\,\ell=0,1,\ j\ge1,\ s>r\,. \tag{22}$$

In order to take into account also the summands that are generated by the application of the Lie series $\exp\big(\mathcal{L}_{\chi_2^{(r)}}\big)$ to both the terms $\boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p}$ and $f_1^{(r,\,r)}(\boldsymbol{p},\boldsymbol{q})$, we add the prescription

$$f_1^{(r,\,(j+1)r)} \hookleftarrow \frac{j}{(j+1)!}\mathcal{L}^j_{\chi_2^{(r)}}\hat{f}_\ell^{(r,\,r)} \quad \forall\ j\geq 1\ , \tag{23}$$

where we make use of formula (18). Also the last redefinition, i.e.,

$$f_1^{(r,r)} = 0\ , \tag{24}$$

is a straightforward consequence of the second homological equation. By applying again Lemma 1 and a trivial induction argument to formulæ (22)–(23), one can easily prove that $f_\ell^{(r,\,s)}\in\mathcal{P}_{\ell,sK}$ for all the summands $f_\ell^{(r,\,s)}=\mathcal{O}(\varepsilon^s)$ that appear in formula (21).

This final remark ends the description of the r-th normalization step of the algorithm that can be iterated so as to determine the next Hamiltonian $H^{(r+1)}$, starting from $H^{(r)}$, and so on.

Let us add a few further comments about the algorithm constructing the Kolmogorov normal form in order to understand its applicability. In practice, one is often interested in determining an approximation up to a fixed order, say $R_{\rm I}\in\mathbb{N}$, of the motions travelling an invariant KAM torus. For this purpose, starting from $H^{(0)}$, one has to preliminarly compute the Taylor-Fourier truncated expansions of the following type, for all the Hamiltonian $H^{(r)}$ that are introduced by the normalization algorithm with $r=1,\ \ldots\ ,\ R_{\rm I}$:

$$H^{(r)}(\boldsymbol{p},\boldsymbol{q}) \simeq E^{(r)} + \boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p} + \sum_{s=0}^{R_{\rm I}}\sum_{\ell=0}^{\ell_{\max}} f_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q})\ , \tag{25}$$

where all the terms that are $o\big(\varepsilon^{R_{\rm I}}\big)$ or of polynomial degree larger than $\ell_{\max}$ with respect to the actions[8] have been neglected. Let us recall that the algorithm works in such a way to define $f_\ell^{(r,\,s)}=0\ \forall\ \ell=0,1\ ,\ 0\leq s\leq r$. When the first $R_{\rm I}$ normalization steps are performed, all the generating functions $\chi_1^{(r)}$ and $\chi_2^{(r)}\ \forall\ r=1,\ \ldots\ ,\ R_{\rm I}$ are fully determined. Their composition allows to compute the expansion of ψ_ε that enters in the definition of the semi-analytic scheme of integration (5) and is truncated,

[8] In the practical applications, it is very common to truncate this kind of Taylor series expansions up to a finite degree. In this framework, it is important to remark that the upper limit on the degree in actions is preserved by the Lie series having $\chi_1^{(r)}\in\mathcal{P}_{0,rK}$ and $\chi_2^{(r)}\in\mathcal{P}_{1,rK}$ as generating functions. This can be easily checked by applying repeatedly Lemma 1, that can be used also to prove that just functions of type $f_\ell^{(r,\,s)}$ with $\ell\leq R_{\rm I}+1$ are involved in the definitions of $\chi_1^{(r)}$ and $\chi_2^{(r)}\ \forall\ r=1,\ \ldots\ ,\ R_{\rm I}$. In other terms, this means that the request of determining an approximation up to a fixed order of magnitude $\mathcal{O}\big(\varepsilon^{R_{\rm I}}\big)$ (for what concerns the canonical transformation that conjugates some orbits to an invariant torus) yields in a fully consistent way also a truncation limit on the polynomial degree in the actions.

once again, so as to neglect all the summands that are $o\big(\varepsilon^{R_{\rm I}}\big)$. Therefore, the wanted approximation of the motions travelling an invariant KAM torus up to a fixed order of magnitude $\mathcal{O}\big(\varepsilon^{R_{\rm I}}\big)$ can be provided by the scheme (5) where also the normal form Hamiltonian $\mathcal{K}$ is replaced by $H^{(R_{\rm I})}$, which requires $\ell_{\max}\big(R_{\rm I}+1\big)^2$ functions of type $f_\ell^{(r,\,s)} \in \mathcal{P}_{\ell,sK}$ to be determined. Since their expansions in Taylor-Fourier series are finite (recall definition (6)), all their coefficients are *representable on a computer* (that is equipped with a large enough memory). Therefore, it is *finite* also the number of elementary operations that are defined by the Poisson brackets prescribed by normalization algorithm. The same conclusion applies also for the aforementioned expansion of the canonical transformation ψ_ε. As a whole, we can conclude that the wanted approximation of the motions travelling an invariant KAM torus is *explicitly computable*, because the total amount of operations that are defined by the normalization algorithm is *finite*.

2.4 On the Convergence of the Algorithm Constructing the Kolmogorov Normal Form

In the present context, it is useful to introduce another version of the KAM theorem.

Proposition 1 *Consider the family of Hamiltonians $H^{(0)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)})$ of the type described in (7). Those functions are defined so that $H^{(0)}\,:\,\mathcal{A}\times\mathbb{T}^n\times\mathcal{U}\mapsto\mathbb{R}$, where both $\mathcal{A}$ and $\mathcal{U}$ are open subsets of $\mathbb{R}^n$, being $\mathbf{0}\in\mathcal{A}$ and $\mathcal{U}$ bounded. Therefore, $(\boldsymbol{p},\boldsymbol{q})$ are action-angle canonical coordinates and the family of Hamiltonians is parameterized with respect to $\boldsymbol{\omega}^{(0)}\in\mathcal{U}$. Let us also assume that for some fixed and positive values of $K\in\mathbb{N}$, $\varepsilon\in\mathbb{R}$ and $E\in\mathbb{R}$, the following inequalities are satisfied by the functions $f_\ell^{(0,s)}\in\mathcal{P}_{\ell,sK}$:*

$$\sup_{(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)})\in\mathcal{A}\times\mathbb{T}^n\times\mathcal{U}}\left|f_\ell^{(0,s)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)})\right|\le E\,\varepsilon^s \tag{26}$$

$\forall\,s\ge1,\ \ell\ge0$ *and* $\forall\,\ell\ge2$ when $s=0$.

Then, there is a positive $\varepsilon^\star$ such that for $0\le\varepsilon<\varepsilon^\star$ the following statement holds true: there exists a non-resonant set $\mathcal{U}^{(\infty)}\subset\mathcal{U}$ such that the Lebesgue measure μ of the complementary set $\mathcal{U}\setminus\mathcal{U}^{(\infty)}$ goes to zero for $\varepsilon\to0$ and for each $\boldsymbol{\omega}^{(0)}\in\mathcal{U}^{(\infty)}$ there is an analytic canonical transformation $(\boldsymbol{p},\boldsymbol{q})=\psi^{(\infty)}_{\varepsilon;\boldsymbol{\omega}^{(0)}}(\boldsymbol{P},\,\boldsymbol{Q})$ leading the Hamiltonian to the normal form

$$H^{(\infty)}(\boldsymbol{P},\,\boldsymbol{Q};\boldsymbol{\omega}^{(0)})=E^{(\infty)}+\boldsymbol{\omega}^{(\infty)}\cdot\boldsymbol{P}+\sum_{s\ge0}\sum_{\ell\ge2}f_\ell^{(\infty,\,s)}(\boldsymbol{P},\,\boldsymbol{Q};\boldsymbol{\omega}^{(0)})\,, \tag{27}$$

where $f_\ell^{(\infty,\,s)}\in\mathcal{P}_{\ell,sK}$ $\forall\ s\ge0,\ \ell\ge2$ and $E^{(\infty)}$ is a finite real value fixing the constant energy level that corresponds to the invariant torus $\big\{(\boldsymbol{P}=\mathbf{0},\ \boldsymbol{Q}\in\mathbb{T}^n)\big\}$.

Moreover, the canonical change of coordinates is close to the identity in the sense that $\left\|\psi^{(\infty)}_{\varepsilon;\boldsymbol{\omega}^{(0)}}(\boldsymbol{P},\boldsymbol{Q})-(\boldsymbol{P},\boldsymbol{Q})\right\|=\mathcal{O}(\varepsilon)$ *and the same applies also to both the energy level and the detuning of the angular velocity vector (that are* $\left|E^{(\infty)}-E^{(0)}\right|=\mathcal{O}(\varepsilon)$ *and* $\left\|\omega^{(\infty)}-\omega^{(0)}\right\|=\mathcal{O}(\varepsilon)$*, respectively).*

The statement above is substantially equivalent to that claimed in theorem C of [36] (which is considered as a classical version of the KAM theorem, in the very own words of the Author, J. Pöschel). The proof of Proposition 1 can be obtained by adapting the one described in [7] in such a way to prove the convergence of the normalization algorithm described in the previous Sect. 2.3. Indeed, both articles [7, 36] deal only with the more complicate proof of existence for invariant tori that are of dimension smaller than the number n of degrees of freedom and have elliptic character in the transverse directions. The construction of the normal form corresponding to such a type of invariant manifolds will be widely discussed in the next Sect. 3. As a main difference between the approaches developed in those two works, let us recall that the proof adopted in [36] is based on a fast convergence scheme of quadratic type (a so called Newton-like method, where perturbing terms of order of magnitude $\mathcal{O}\big(\varepsilon^{2^{r-1}}\big)$ are removed during the r-th normalization step). Such a technique has been adopted since the very first works in KAM theory, but the convergence of the normalization algorithm described in Sect. 2.3 is of linear type (because perturbing terms of order of magnitude $\mathcal{O}\big(\varepsilon^{r}\big)$ are removed during the r-th normalization step). The latter is in a better position for the applications[9] and a complete proof of the KAM theorem adopting a convergence method of linear type is available since the last decade of the past century (see [17]). Rather curiously, the best way to translate the algorithm constructing the Kolmogorov normal form in a computer-assisted proof requires to join the convergence scheme of linear type (in order to explicitly perform on a computer the largest possible number $R_{\rm I}$ of preliminary steps) with that of quadratic type (that provides a statement of KAM theorem that is very suitable to rigorously complete the proof). This is one of the main conclusions discussed in a recent work (see [40]).

The statement of Proposition 1 highlights that we are forced to provide a result which holds true with respect to the Lebesgue measure, because we have *chosen* to adopt a version of the normalization algorithm where the angular velocity vector is allowed to vary at each step (recall formula (16) that defines the detuning shift $\boldsymbol{\omega}^{(r)}-\boldsymbol{\omega}^{(r-1)}$). This means that such a statement has to be understood in a probabilistic sense, because we are not able to describe in detail the structure of the non-resonant set $\mathcal{U}^{(\infty)}$. In particular, for a fixed initial value of the angular velocity vector $\boldsymbol{\omega}^{(0)}$ we cannot establish whether the specific Hamiltonian $H^{(0)}(\boldsymbol{p},\boldsymbol{q};\boldsymbol{\omega}^{(0)})$ can be brought in Kolmogorov normal form or not. We can just claim that the normalization algorithm can converge with a rate of success (i.e., $\mu\big(\mathcal{U}\setminus\mathcal{U}^{(\infty)}\big)\,/\,\mu(\mathcal{U})$) that gets larger and larger when the small parameter ε which rules the size of the

[9] This is the main reason why the present work is focusing on approaches based on a convergence scheme of linear type. A very far from being exhaustive list of references to applications of KAM theorem has been discussed in the Introduction.

perturbation is decreasing. On the other hand, we can characterize very well the set of the final values of the angular velocities, i.e., $\big\{\boldsymbol{\omega}^{(\infty)}\big(\boldsymbol{\omega}^{(0)}\big) \,:\, \boldsymbol{\omega}^{(0)} \in \mathcal{U}^{(\infty)}\big\}$, because they are Diophantine. In the recent work [37], the problem of the convergence of this type of normalization algorithms is revisited so as to provide a KAM-like statement. It is proved by fixing since the beginning the final value $\boldsymbol{\omega}^{(\infty)}$ and its non-resonance properties (that allow to explicitly solve the homological equations at every step of the algorithm). Moreover, the total detuning $\boldsymbol{\omega}^{(\infty)} - \boldsymbol{\omega}^{(0)}$ is given in terms of series whose coefficients are defined in a recursive way. Therefore, the convergence of the normalization algorithm is ensured (provided that the perturbation is small enough), the total detuning is estimated explicitly, while the exact location of $\boldsymbol{\omega}^{(0)}$ remains partially unknown, because it can be determined just by iterating *ad infinitum* the computational procedure.

3 Construction of Invariant Elliptic Tori by a Normal Form Algorithm

Elliptic tori are compact invariant manifolds of dimension smaller than the maximal one, that is equal to the number n of degrees of freedom. In order to better imagine them, let us consider a phase space $\mathcal{F}$ that is endowed by the canonical coordinates $(\boldsymbol{P},\,\boldsymbol{Q},\,\boldsymbol{X},\,\boldsymbol{Y})$, where $(\boldsymbol{P},\,\boldsymbol{Q}) \in \mathbb{R}^{n_1} \times \mathbb{T}^{n_1}$ are action-angle variables and also $(\boldsymbol{X},\boldsymbol{Y}) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ denote pairs of conjugate (momenta and) coordinates, while $n = n_1 + n_2$ with both n_1 and n_2 positive integers. Let us consider a Hamiltonian of the following type:

$$\mathcal{H}(\boldsymbol{P},\,\boldsymbol{Q},\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{\omega}\cdot\boldsymbol{P} + \sum_{j=1}^{n_2}\frac{\Omega_j}{2}(X_j^2+Y_j^2) + \mathcal{R}(\boldsymbol{P},\,\boldsymbol{Q},\boldsymbol{X},\boldsymbol{Y})\,,$$

where $\boldsymbol{\Omega} \in \mathbb{R}^{n_2}$ and the remainder $\mathcal{R}$ is an analytic function with respect to its arguments and is such that $\mathcal{R}(\boldsymbol{P},\,\boldsymbol{Q},\boldsymbol{X},\boldsymbol{Y}) = o\big(\|\boldsymbol{P}\| + \|(\boldsymbol{X},\boldsymbol{Y})\|^2\big)$, when $(\boldsymbol{P},\boldsymbol{X},\boldsymbol{Y}) \to (\boldsymbol{0},\boldsymbol{0},\boldsymbol{0})$. It is easy to check that

$$(\boldsymbol{P}(t),\,\boldsymbol{Q}(t),\boldsymbol{X}(t),\boldsymbol{Y}(t)) = \big(\boldsymbol{0},\,\boldsymbol{Q}(0)+\boldsymbol{\omega}t,\boldsymbol{0},\boldsymbol{0}\big) \tag{28}$$

is a solution of Hamilton equations, since the function $\mathcal{H}$, except for its main part, contains terms of type $\mathcal{O}(\|\boldsymbol{P}\|^2)$, $\mathcal{O}(\|\boldsymbol{P}\|\|(\boldsymbol{X},\boldsymbol{Y})\|)$ and $\mathcal{O}(\|(\boldsymbol{X},\boldsymbol{Y})\|^3)$ only. Because of this remark, it is evident that the n_1–dimensional manifold $\big\{(\boldsymbol{P},\,\boldsymbol{Q},\boldsymbol{X},\boldsymbol{Y}) \,:\, \boldsymbol{P} = \boldsymbol{0},\ \boldsymbol{Q} \in \mathbb{T}^{n_1},\ \boldsymbol{X} = \boldsymbol{Y} = \boldsymbol{0}\big\}$ is invariant. The elliptical character is given by the fact that, in the remaining $n_2 = n - n_1$ degrees of freedom, the dynamics that is transverse with respect to such an invariant manifold is given by the composition of n_2 oscillatory motions whose periods tend to the values $2\pi/\Omega_1\,,\,\ldots\,,\,2\pi/\Omega_{n_2}$, in the limit of $(\boldsymbol{P},\boldsymbol{X},\boldsymbol{Y}) \to (\boldsymbol{0},\boldsymbol{0},\boldsymbol{0})$. Of course, this is due to the occurrence of the term

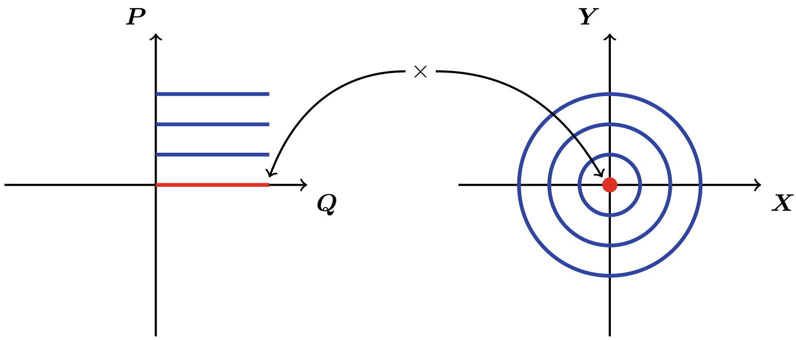

Fig. 1 Schematic representation of an elliptic torus. The orbit is given by the cartesian product of the two invariant surfaces that are marked in red, that are a torus (in the left panel) and a stable equilibrium point (on the right, resp.)

$\sum_{j=1}^{n_2} \Omega_j (X_j^2 + Y_j^2)/2$ which overwhelms the effect of the remainder $\mathcal{R}$ in the so called limit of small oscillations.

The name of elliptic torus is well justified by all the remarks discussed since the beginning of the present section. A schematic representation of such kind of invariant manifolds is sketched in Fig. 1.

3.1 Algorithmic Construction of the Normal Form for Elliptic Tori

Since we aim at introducing the algorithm constructing the normal form for invariant elliptic tori in a way that is as much as possible coherent with what we have already done in Sect. 2.3 for KAM tori, we prefer to not adopt canonical coordinates $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n_1} \times \mathbb{T}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ that are substantially the ones considered in the discussion at the beginning of the present section. Indeed, we think it is convenient to introduce the so called action-angle coordinates for harmonic oscillators, in order to replace the polynomial ones, that are $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$; this means that we define $(\boldsymbol{J}, \boldsymbol{\varphi}) \in \left(\mathbb{R}_+^{n_2} \cup \{\boldsymbol{0}\}\right) \times \mathbb{T}^{n_2}$ so that $x_j = \sqrt{2J_j}\cos\varphi_j$ and $y_j = \sqrt{2J_j}\sin\varphi_j$, where this change of coordinates is canonical $\forall\ j = 1, \ldots, n_2$.

We are now ready to introduce classes of functions depending on $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi}) \in \mathbb{R}^{n_1} \times \mathbb{T}^{n_1} \times \left(\mathbb{R}_+^{n_2} \cup \{\boldsymbol{0}\}\right) \times \mathbb{T}^{n_2}$ in a very similar way to what has been previously done. For some fixed positive integer K we introduce the distinct classes of functions $\widehat{\mathfrak{P}}_{\hat{m},\,\hat{\ell},\,sK}$, with integers $\hat{m},\ \hat{\ell},\ s \geq 0$; any generic function $g \in \widehat{\mathfrak{P}}_{\hat{m},\,\hat{\ell},\,sK}$ can be written as

$$g\,(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) = \sum_{\substack{\boldsymbol{m}\in\mathbb{N}^{n_1}\\ |\boldsymbol{m}|=\hat{m}}}\ \sum_{\substack{\boldsymbol{\ell}\in\mathbb{N}^{n_2}\\ |\boldsymbol{\ell}|=\hat{\ell}}}\ \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^{n_1}\\ |\boldsymbol{k}|\le sK}}\ \sum_{\substack{\hat{\ell}_j=-\ell_j,\,-\ell_j+2,\ldots,\,\ell_j\\ j=1,\,\ldots\,,n_2}} c_{\boldsymbol{m},\boldsymbol{\ell},\boldsymbol{k},\hat{\boldsymbol{\ell}}}\,\boldsymbol{p}^{\boldsymbol{m}}\big(\sqrt{\boldsymbol{J}}\big)^{\boldsymbol{\ell}}\exp\big[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}+\hat{\boldsymbol{\ell}}\cdot\boldsymbol{\varphi})\big]\,, \tag{29}$$

where the complex coefficients are such that $c_{\boldsymbol{m},\boldsymbol{\ell},-\boldsymbol{k},-\hat{\boldsymbol{\ell}}} = \bar{c}_{\boldsymbol{m},\boldsymbol{\ell},\boldsymbol{k},\hat{\boldsymbol{\ell}}}$, then the codomain of any $g \in \widehat{\mathfrak{P}}_{\hat{m},\,\hat{\ell},\,sK}$ is included in $\mathbb{R}$. Let us emphasize that, in each term appearing in the Taylor-Fourier expansion of a function belonging to a class of type $\widehat{\mathfrak{P}}_{\hat{m},\,\hat{\ell},\,sK}$, the indexes vector $(\hat{\ell}_1\,,\,\ldots\,,\,\hat{\ell}_{n_2})$ are subject to special restrictions that are inherited by the corresponding polynomial structure with respect to the variables $(\boldsymbol{x},\,\boldsymbol{y}) = \big(\sqrt{2\boldsymbol{J}}\cos\boldsymbol{\varphi},\,\sqrt{2\boldsymbol{J}}\sin\boldsymbol{\varphi}\big)$. In fact, they are such that $\forall\ j = 1,\,\ldots\,,n_2$ the j-th component of the Fourier harmonic $\hat{\ell}_j$ must have the same parity with respect to the corresponding degree ℓ_j of $\sqrt{J_j}$ and also the inequality $\big|\hat{\ell}_j\big| \le \ell_j$ must be satisfied.[10] Furthermore, we will say that $g \in \mathfrak{P}_{\ell,sK}$ if

$$g \in \bigcup_{\substack{\hat{m}\ge 0,\hat{\ell}\ge 0\\ 2\hat{m}+\hat{\ell}=\ell}} \widehat{\mathfrak{P}}_{\hat{m},\hat{\ell},sK}\ . \tag{30}$$

In other words, a function belonging to the class $\mathfrak{P}_{\ell,sK}$ depends on the actions so as to be homogeneous polynomials of total degree ℓ in the square roots of $\boldsymbol{p}$ and $\boldsymbol{J}$, while its Fourier expansion contain harmonics of total trigonometric degree in $\boldsymbol{q}$ that are not larger than sK.

In order to extend the approach described in Sect. 2.3 with the aim to design an efficient algorithm constructing the normal form in the case of elliptic tori, we are also forced to reformulate the Lemma 1 in a suitable version to describe the action of the Poisson brackets on these new classes of functions, that are defined thanks to formulæ (29)–(30). This is made as it follows.

Lemma 2 *Let us consider two generic functions $g \in \mathfrak{P}_{\ell,sK}$ and $h \in \mathfrak{P}_{m,rK}$, where K is a fixed positive integer number. Then,*[11]

$$\big\{g,h\big\} = \mathcal{L}_h\,g \in \mathfrak{P}_{\ell+m-2,(r+s)K} \quad \forall\,\ell,\ m,\ r,\ s \in \mathbb{N}\ .$$

[10] When there are variables such that they appear in the Taylor-Fourier expansions of a function so that they follow this kind of restrictions, then they are often said to be of D'Alembert type. This name is given by analogy, because in Celestial Mechanics the secular part of the Hamiltonian perturbing terms due to the interactions between planets shows the same kind of expansions, since they satisfy the so called D'Alembert rules.

[11] The statement can be considered as valid also in the trivial cases with $\ell + m = 0,\,1$, by enlarging the definition of the classes of functions so that $\mathfrak{P}_{-2,sK} = \mathfrak{P}_{-1,sK} = \big\{0\big\}\ \forall\,s \in \mathbb{N}$.

Also in this case the proof is omitted, because it can be obtained by simply applying[12] the definition of the Poisson brackets.

As an environment where it is natural to properly define the algorithm constructing the normal form for elliptic tori, let us start to consider a Hamiltonian $\mathcal{H}^{(0)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi})$ that can be written in the following way:

$$\begin{aligned}\mathcal{H}^{(0)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi}) = & \,\mathcal{E}^{(0)} + \boldsymbol{\omega}^{(0)} \cdot \boldsymbol{p} + \boldsymbol{\Omega}^{(0)} \cdot \boldsymbol{J} + \sum_{s\ge 0}\sum_{\ell\ge 3} f_{\ell}^{(0,\,s)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi}) \\ & + \sum_{s\ge 1}\sum_{\ell=0}^{2} f_{\ell}^{(0,\,s)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi})\,,\end{aligned} \tag{31}$$

where $\mathcal{E}^{(0)} \in \mathfrak{P}_{0,0}$ is a constant[13] and $f_{\ell}^{(0,\,s)} \in \mathfrak{P}_{\ell,sK}$, being the first upper index related to the normalization step. For instance, in [8] it is shown how to bring an FPU chain of $N+1$ particles in the form above, by following a procedure that is valid for a generic Hamiltonian problem in the neighborhood of a stable equilibrium point. In other words, the Hamiltonian describing that model can be expanded as $\mathcal{H}^{(0)}$ in (31), with $f_{\ell}^{(0,\,s)} = 0$ when $s \ge 3$ and $f_{\ell}^{(0,\,1)} \in \mathfrak{P}_{\ell,K}$, $f_{\ell}^{(0,\,2)} \in \mathfrak{P}_{\ell,2K}$ $\forall\ \ell \ge 0$, being[14] $K = 2$. This holds true, both for the so called α–model and the β one. Let us also emphasize that the energy value $\mathcal{E}^{(0)}$, the angular velocity vector $\boldsymbol{\Omega}^{(0)} \in \mathbb{R}^{n_2}$ and all the functions $f_{\ell}^{(0,\,s)}$ depend on $\boldsymbol{\omega}^{(0)} \in \mathbb{R}^{n_1}$ in a parametric way. In order to keep the notation so that it does not get too cumbersome, in the present subsection we do not include $\boldsymbol{\omega}^{(0)}$ among the arguments of the terms appearing in the expansions of the Hamiltonians. Moreover, for a generic problem in the neighborhood of a stable equilibrium point one can also easily show that $f_{\ell}^{(0,\,s)} = \mathcal{O}(\varepsilon^s)$, where ε is the natural

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[12] Actually, it looks natural to be doubtful about the fact that Poisson brackets always preserve the restrictions on the Fourier harmonics that must be satisfied by variables of D'Alembert type. However, one can immediately realize that the only tricky case occurs when the Poisson brackets include also the following terms:

$$\begin{aligned}& \frac{\partial\big(\sqrt{J_j}\big)^{|\hat{\ell}_j|}\exp(\mathrm{i}\hat{\ell}_j\varphi_j)}{\partial\varphi_j}\,\frac{\partial\big(\sqrt{J_j}\big)^{|\hat{m}_j|}\exp(\mathrm{i}\hat{m}_j\varphi_j)}{\partial J_j} \\ & -\frac{\partial\big(\sqrt{J_j}\big)^{|\hat{\ell}_j|}\exp(\mathrm{i}\hat{\ell}_j\varphi_j)}{\partial J_j}\,\frac{\partial\big(\sqrt{J_j}\big)^{|\hat{m}_j|}\exp(\mathrm{i}\hat{m}_j\varphi_j)}{\partial\varphi_j} \\ & = \frac{\mathrm{i}}{2}\big(\hat{\ell}_j|\hat{m}_j| - \hat{m}_j|\hat{\ell}_j|\big)\big(\sqrt{J_j}\big)^{|\hat{\ell}_j|+|\hat{m}_j|-2}\exp\big(\mathrm{i}(\hat{\ell}_j+\hat{m}_j)\varphi_j\big)\,.\end{aligned}$$

However, if $\hat{\ell}_j$ and $\hat{m}_j$ have opposite signs then $\big|\hat{\ell}_j + \hat{m}_j\big| \le \big|\hat{\ell}_j\big| + \big|\hat{m}_j\big| - 2$ (let us remark that the term above vanishes if $\hat{\ell}_j = 0$ or $\hat{m}_j = 0$). In the remaining case (i.e., $\hat{\ell}_j \ne 0$ and $\hat{m}_j \ne 0$ have the same sign), the coefficient $\hat{\ell}_j|\hat{m}_j| - \hat{m}_j|\hat{\ell}_j|$ is always equal to zero.

[13] $\mathcal{E}^{(0)}$ denotes the energy level of the elliptic torus that is invariant in the approximation given by the angular average, i.e., when $f_{\ell}^{(0,\,s)} = 0\ \forall\, s > 0$.

[14] Setting $K = 2$ is quite natural for Hamiltonian systems close to stable equilibria, see, e.g., [20].

small parameter for this kind of models, because it denotes the first approximation of the distance (expressed in terms of the actions) between the wanted elliptic torus and the stable equilibrium point.

In a strict analogy with what has been done to construct the Kolmogorov normal form, here our main purpose is to eliminate from the Hamiltonian all the terms having total degree less than three in the square root of the actions; by referring to the paradigmatic form described in (31), the unwanted terms are appearing in its last row. Actually, such a goal can be achieved by performing an infinite sequence of canonical transformations, so as to bring the Hamiltonian to the following final normal form:

$$\mathcal{H}^{(\infty)}(\boldsymbol{P},\,\boldsymbol{Q},\,\boldsymbol{\Xi},\,\boldsymbol{\Theta}) = \mathcal{E}^{(\infty)} + \boldsymbol{\omega}^{(\infty)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(\infty)}\cdot\boldsymbol{\Xi} + \sum_{s\ge 0}\sum_{\ell\ge 3} f_\ell^{(\infty,\,s)}(\boldsymbol{P},\,\boldsymbol{Q},\,\boldsymbol{\Xi},\,\boldsymbol{\Theta})\,, \tag{32}$$

with $f_\ell^{(\infty,\,s)} \in \mathfrak{P}_{\ell,sK}$ and $\mathcal{E}^{(\infty)} \in \mathfrak{P}_{0,0}$. The motion law $(\boldsymbol{P}(t),\,\boldsymbol{Q}(t),\,\boldsymbol{\Xi}(t),\,\boldsymbol{\Theta}(t)) = \big(\boldsymbol{0},\,\boldsymbol{Q}_0 + \boldsymbol{\omega}^{(\infty)}t,\,\boldsymbol{0},\,\boldsymbol{\Theta}\big)$ is a solution of the Hamilton equations related to the normal form $\mathcal{H}^{(\infty)}$ and is equivalent[15] to (28). Such a motion law is generated by the initial condition $(\boldsymbol{0},\,\boldsymbol{Q}_0,\,\boldsymbol{0},\,\boldsymbol{\Theta})$, is quasi-periodic with an angular velocity vector equal to $\boldsymbol{\omega}^{(\infty)}$ and the corresponding orbit lies on the n_1-dimensional invariant torus $\boldsymbol{P}=\boldsymbol{0}$, $\boldsymbol{\Xi}=\boldsymbol{0}$. The energy level of such a manifold is $H^{(\infty)}(\boldsymbol{0},\,\boldsymbol{Q},\,\boldsymbol{0},\,\boldsymbol{\Theta}) = \mathcal{E}^{(\infty)}$. Moreover, it is elliptic in the sense that the transverse dynamics in a neighborhood of the invariant torus itself is given by oscillations whose corresponding angular velocity vector is approaching $\boldsymbol{\Omega}^{(\infty)}$ in the limit of $\|(\boldsymbol{P},\,\boldsymbol{\Xi})\|$ going to zero.

Also in the present case, that is concerning the elliptic tori, the formal algorithm for the construction of the normal form is composed by a sequence of canonical transformations, defined using the formalism of Lie series. We can summarize the r-th normalization step, by giving the formula defining the canonical change of coordinates that transforms the intermediate Hamiltonian $\mathcal{H}^{(r-1)}$ into the subsequent $\mathcal{H}^{(r)}$. The expansion of the former is of the following type:

$$\begin{aligned}\mathcal{H}^{(r-1)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) =& \mathcal{E}^{(r-1)} + \boldsymbol{\omega}^{(r-1)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(r-1)}\cdot\boldsymbol{J} + \sum_{s\ge 0}\sum_{\ell\ge 3} f_\ell^{(r-1,s)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) \\ &+ \sum_{s\ge r}\sum_{\ell=0}^{2} f_\ell^{(r-1,\,s)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi})\,,\end{aligned} \tag{33}$$

[15] We remark that $\dot{\boldsymbol{P}} = \big\{\boldsymbol{P}\,,\,\mathcal{H}^{(\infty)}\big\} = \boldsymbol{0}$ and $\dot{\boldsymbol{\Xi}} = \big\{\boldsymbol{\Xi}\,,\,\mathcal{H}^{(\infty)}\big\} = \boldsymbol{0}$ when $\boldsymbol{P}=\boldsymbol{0}$ and $\boldsymbol{\Xi}=\boldsymbol{0}$. Because of the well known degeneracy of the change of coordinates $(\boldsymbol{X},\boldsymbol{Y}) = \big(\sqrt{2\boldsymbol{\Xi}}\cos\boldsymbol{\Theta},\,\sqrt{2\boldsymbol{\Xi}}\sin\boldsymbol{\Theta}\big)$, all the set $\big\{\big(\boldsymbol{\Xi}=\boldsymbol{0},\,\boldsymbol{\Theta}\in\mathbb{T}^{n_2}\big)\big\}$ correspond to a single point $\big\{(\boldsymbol{X}=\boldsymbol{0},\,\boldsymbol{Y}=\boldsymbol{0})\big\}$ of the reduced phase space that considers just the last n_2 degrees of freedom. By the way, we emphasize that such a degeneracy is completely harmless in the framework we have adopted. In order to conclude the check of the solution of the Hamilton equations related to the normal form $\mathcal{H}^{(\infty)}$ when $\boldsymbol{P}=\boldsymbol{0}$ and $\boldsymbol{\Xi}=\boldsymbol{0}$, it is enough to remark that $\dot{\boldsymbol{Q}} = \big\{\boldsymbol{Q}\,,\,\mathcal{H}^{(\infty)}\big\} = \boldsymbol{\omega}^{(\infty)}$.

being $f_{\ell}^{(r-1,\,s)} \in \mathfrak{P}_{\ell,sK}$ and $\mathcal{E}^{(r-1)} \in \mathfrak{P}_{0,0}$, i.e., it is a constant referring to the level of the energy in the approximation that is valid up to terms $\mathcal{O}\big(\varepsilon^r\big)$. Let us emphasize that the starting Hamiltonian $\mathcal{H}^{(0)}$ written in Eq. (31) is exactly in the form (33) with $r = 1$. The conjugacy relation which allows to write the Hamiltonian defined at the end of the r-th normalization step as a function of the previous one is given by

$$\mathcal{H}^{(r)} = \mathcal{H}^{(r-1)} \circ \exp\left(\mathcal{L}_{\chi_0^{(r)}}\right) \circ \exp\left(\mathcal{L}_{\chi_1^{(r)}}\right) \circ \exp\left(\mathcal{L}_{\chi_2^{(r)}}\right) \circ \mathfrak{D}^{(r)} \,, \tag{34}$$

where the Lie series[16] operator $\exp\left(\mathcal{L}_{\chi_j^{(r)}}\right)\cdot$ removes the Hamiltonian terms with total degree in the square root of the actions equal to j and with trigonometric degree in the angles $\boldsymbol{q}$ up to rK. Moreover, by a linear canonical transformation $\mathfrak{D}^{(r)}$, the terms that are quadratic in $\sqrt{\boldsymbol{J}}$ and do not depend on both the actions $\boldsymbol{p}$ and the angles $\boldsymbol{q}$ are brought to a diagonal form. At the end of this r-th normalization step, the ineliminable terms that are independent on the angles $\boldsymbol{q}$ and linear either in $\boldsymbol{p}$ or in $\boldsymbol{J}$ are added to the normal form part. This requires to update the angular velocities from $\big(\boldsymbol{\omega}^{(r-1)}, \boldsymbol{\Omega}^{(r-1)}\big)$ to $\big(\boldsymbol{\omega}^{(r)}, \boldsymbol{\Omega}^{(r)}\big)$, that is why in (32) the Hamiltonian in Kolmogorov normal form has new frequency vectors $\boldsymbol{\omega}^{(\infty)}$ and $\boldsymbol{\Omega}^{(\infty)}$.

All the details that properly define how the algorithm actually works are exhaustively described in the following.

First Stage of the *r*-th Normalization Step

In the context of the r-th normalization step, the first stage aims to remove the terms depending just on the angles $\boldsymbol{q}$ up to the trigonometrical degree rK, i.e. the terms collected in $f_0^{(r-1,r)} = \mathcal{O}(\varepsilon^r)$. We determine the generating function $\chi_0^{(r)}$ by solving the homological equation

$$\left\{\boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p},\, \chi_0^{(r)}\right\} + f_0^{(r-1,\,r)}(\boldsymbol{q}) = \langle f_0^{(r-1,r)}(\boldsymbol{q})\rangle_{\boldsymbol{q}} \,. \tag{35}$$

Let us remark that the equation above is perfectly equivalent to that in formula (9), because $f_0^{(r-1,\,r)} \in \mathfrak{P}_{0,rK}$ depends on $\boldsymbol{q}$ only and, therefore, $f_0^{(r-1,\,r)} \in \mathcal{P}_{0,rK} = \mathfrak{P}_{0,rK}$. Thus, we can write the solution of this new (first) homological equation (35) exactly in the same way as we have done for what concerns (10), i.e., we put $\langle f_0^{(r-1,r)}(\boldsymbol{q})\rangle_{\boldsymbol{q}} = c_{\boldsymbol{0}}$ and

$$\chi_0^{(r)}(\boldsymbol{q}) = \sum_{0<|\boldsymbol{k}|\le rK} \frac{c_{\boldsymbol{k}} \exp\big(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\big)}{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}} \,, \tag{36}$$

[16] Because of the so called "exchange theorem" (see [14]), the new Hamiltonian $H^{(r)}$ is obtained from the old one, by applying the Lie series to $H^{(r-1)}$ in reverse order with respect to what is written in (34). This is consistent with the order of the discussion in the following subsections: the first stage of the r-th normalization step deals with the canonical transformation generated by $\chi_0^{(r)}$, the second one with $\chi_1^{(r)}$ and the last one with both $\chi_2^{(r)}$ and $\mathfrak{D}^{(r)}$.

being $f_0^{(r-1,r)}(\boldsymbol{q}) = \sum_{|\boldsymbol{k}|\le rK} c_{\boldsymbol{k}} \exp\big(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\big)$. Of course, such a solution is certainly valid provided the non-resonance condition (11) is satisfied.

Now, we apply the canonical transformation $\exp\mathcal{L}_{\chi_0^{(r)}}$ to the Hamiltonian which is defined at the end of the $r-1$-th normalization step. By the usual abuse of notation, we choose to rename the new variables as the old ones. This allows to write the transformed Hamiltonian $H^{(\mathrm{I};\,r)} = \exp\big(\mathcal{L}_{\chi_0^{(r)}}\big)H^{(r-1)}$ as follows:

$$\begin{aligned}\mathcal{H}^{(\mathrm{I};\,r)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) =&\mathcal{E}^{(r)} + \boldsymbol{\omega}^{(r-1)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(r-1)}\cdot\boldsymbol{J} + \sum_{s\ge0}\sum_{\ell\ge3} f_\ell^{(\mathrm{I};\,r,\,s)} \\ &+\sum_{s\ge r}\sum_{\ell=0}^{2} f_\ell^{(\mathrm{I};\,r,\,s)}\ ,\end{aligned} \tag{37}$$

where for the sake of brevity we have omitted to list the arguments of the functions $f_\ell^{(\mathrm{I};\,r,\,s)}$. Let us introduce them in the same unconventional way we have adopted in Sect. 2.3 to describe the algorithm constructing the Kolmogorov normal form. First, we define[17] $f_\ell^{(\mathrm{I};\,r,\,s)} = f_\ell^{(r-1,\,s)}$ $\forall\ \ell\ge0,\ s\ge0$. By further abuses of notation, we update many times the definition of the terms appearing in the expansion of the new Hamiltonian according to the following rule:

$$f_{\ell-2i}^{(\mathrm{I};\,r,\,s+jr)} \hookleftarrow \frac{1}{j!}\mathcal{L}^{j}_{\chi_0^{(r)}} f_\ell^{(r-1,\,s)} \quad \forall\,\ell\ge0,\ 1\le j\le\lfloor\ell/2\rfloor,\ s\ge0\ . \tag{38}$$

By applying repeatedly Lemma 2 and a trivial induction argument to the formula above, one can easily prove that $f_\ell^{(\mathrm{I};\,r,\,s)}\in\mathfrak{P}_{\ell,sK}$ $\forall\ \ell\ge0,\ s\ge0$. In order to end the description of the first stage of the r-th normalization step, we have to take into account also the effects induced by the homological equation (35). For such a purpose, we finally set $f_0^{(\mathrm{I};\,r,\,r)} = 0$ and we update the approximated value referring to the energy of the wanted elliptic torus exactly in the same way we have done to write formula (15), i.e., we put $\mathcal{E}^{(r)} = \mathcal{E}^{(r-1)} + \langle f_0^{(r-1,\,r)}\rangle_{\boldsymbol{q}}$.

Second Stage of the *r*-th Normalization Step

The second stage of the r-th normalization step acts on the Hamiltonian that is initially expanded as in (37), with the goal to remove the perturbing term which is linear in $\sqrt{\boldsymbol{J}}$ and independent of $\boldsymbol{p}$, i.e., $f_1^{(\mathrm{I};\,r,\,r)}$. Thus, we have to solve the following homological equation:

$$\Big\{\boldsymbol{\omega}^{(r-1)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(r-1)}\cdot\boldsymbol{J},\ \chi_1^{(r)}\Big\} + f_1^{(\mathrm{I};\,r,\,r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) = 0\ . \tag{39}$$

[17] We remark that the terms $f_\ell^{(r-1,\,s)}$ do not enter in the expansion (33) when $\ell=0,\ 1,\ 2$ and $s<r$. However, the recursive definitions described in the present subsection are such that all those functions are equal to zero. Keeping in mind this fact allows to write in a rather compact way both formula (38) and the analogous ones in the following.

Let us write the expansion of $f_1^{(\mathrm{I};\,r,\,r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi})$ as follows:

$$f_1^{(\mathrm{I};\,r,\,r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) = \sum_{0\le\boldsymbol{k}\le rK}\sum_{j=1}^{n_2}\sqrt{J_j}\left[c_{\boldsymbol{k},\,j}^{(+)}\,e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}+\varphi_j)} + c_{\boldsymbol{k},\,j}^{(-)}\,e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}-\varphi_j)}\right]\,, \qquad (40)$$

where every coefficients $c_{\boldsymbol{k},\,j}^{(+)}\in\mathbb{C}$ is equal to the complex conjugate of $c_{-\boldsymbol{k},\,j}^{(-)}$ $\forall\, 0\le \boldsymbol{k}\le rK,\ 1\le j\le n_2$. Therefore, the generating function $\chi_1^{(r)}$ solving Eq. (39) is determined in such a way that

$$\chi_1^{(r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) = \sum_{0\le\boldsymbol{k}\le rK}\sum_{j=1}^{n_2}\frac{\sqrt{J_j}}{\mathrm{i}}\left[\frac{c_{\boldsymbol{k},\,j}^{(+)}\,e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}+\varphi_j)}}{\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}+\Omega_j^{(r-1)}} + \frac{c_{\boldsymbol{k},\,j}^{(-)}\,e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}-\varphi_j)}}{\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}-\Omega_j^{(r-1)}}\right]. \qquad (41)$$

This expression is well-defined, provided that the frequency vector $\boldsymbol{\omega}^{(r-1)}$ satisfies the so-called first Melnikov non-resonance condition up to order rK (see [29]), i.e.,

$$\min_{\substack{0<|\boldsymbol{k}|\le rK,\\ |\boldsymbol{\ell}|=1}}\left|\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}+\boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right|\ge\frac{\gamma}{(rK)^{\tau}} \quad\text{and}\quad \min_{|\boldsymbol{\ell}|=1}\left|\boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right|\ge\gamma\,, \qquad (42)$$

for some fixed values of both $\gamma>0$ and $\tau>n_1-1$. By applying the Lie series $\exp\left(\mathcal{L}_{\chi_1^{(r)}}\right)$ to the old Hamiltonian $H^{(\mathrm{I};\,r)}$, we have a new one, which we denote as $H^{(\mathrm{II};\,r)}=\exp\left(\mathcal{L}_{\chi_1^{(r)}}\right)H^{(\mathrm{I};\,r)}$ and have the same structure as that described in (37), i.e.,

$$\begin{aligned}\mathcal{H}^{(\mathrm{II};\,r)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) =& \mathcal{E}^{(r)}+\boldsymbol{\omega}^{(r-1)}\cdot\boldsymbol{p}+\boldsymbol{\Omega}^{(r-1)}\cdot\boldsymbol{J}+\sum_{s\ge0}\sum_{\ell\ge3}f_\ell^{(\mathrm{II};\,r,\,s)}\\ &+\sum_{s\ge r}\sum_{\ell=0}^{2}f_\ell^{(\mathrm{II};\,r,\,s)}\,,\end{aligned} \qquad (43)$$

The functions $f_\ell^{(\mathrm{II};\,r,\,s)}$ that compose the new Hamiltonian can be determined with calculations similar to those listed during the description of the first stage of normalization. This means that we initially define $f_\ell^{(\mathrm{II};\,r,\,s)}=f_\ell^{(\mathrm{I};\,r,\,s)}$ $\forall\,\ell\ge0,\ s\ge0$. Then, (by abuse of notation) we redefine them many times according to the following rules:

$$\begin{aligned} f_{\ell-j}^{(\mathrm{II};\,r,\,s+jr)} &\hookleftarrow \frac{1}{j!}\mathcal{L}_{\chi_1^{(r)}}^{j} f_\ell^{(\mathrm{I};\,r,\,s)} \quad \forall\,\ell\ge0\,,\ 1\le j\le\ell\,,\ s\ge0\,,\\ f_0^{(r,\,2r)} &\hookleftarrow \frac{1}{2}\mathcal{L}_{\chi_1^{(r)}}^{2}\left(\boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p}+\boldsymbol{\Omega}^{(r)}\cdot\boldsymbol{J}\right).\end{aligned} \qquad (44)$$

Because of the homological equation (39), we add also a further redefinition so that $f_1^{(\mathrm{II};\,r,\,r)}=0$. By applying Lemma 2 to formula (44), it is easy to check that $f_\ell^{(\mathrm{II};\,r,\,s)}\in\mathfrak{P}_{\ell,sK}$ $\forall\,\ell\ge0,\ s\ge0$.

Third Stage of the *r*-th Normalization Step

The third and last stage of normalization is more elaborated. It aims to remove terms belonging to two different classes: first, those linear in $\boldsymbol{p}$ and independent of $(\boldsymbol{J}, \boldsymbol{\varphi})$, moreover, other terms that are quadratic in $\sqrt{\boldsymbol{J}}$ and independent of $\boldsymbol{p}$. Such a part of the perturbation is removed by the composition of two canonical transformations expressed by Lie series, being the corresponding generating functions $X_2^{(r)}(\boldsymbol{p}, \boldsymbol{q}) \in \widehat{\mathfrak{P}}_{1,0,rK}$ and $Y_2^{(r)}(\boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi}) \in \widehat{\mathfrak{P}}_{0,2,rK}$, respectively. Moreover, the third stage is ended by a linear canonical transformation $\mathfrak{D}^{(r)}$ that leaves the pair $(\boldsymbol{p}, \boldsymbol{q})$ unchanged and it aims to diagonalize the terms that are quadratic in $\sqrt{\boldsymbol{J}}$ and independent of the angles $\boldsymbol{q}$. Let us detail all these changes of coordinates, so that the algorithm will be unambiguously defined at the end of our discussion.

The generating functions $X_2^{(r)}$ is in charge to remove terms that are linear in $\boldsymbol{p}$ and do depend on the angles $\boldsymbol{q}$ up to the trigonometric degree rK. Therefore, it is a solution of the following homological equation:

$$\left\{\boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p},\ X_2^{(r)}\right\} + f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q}) - \langle f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q})\rangle_{\boldsymbol{q}} = 0\,. \tag{45}$$

Let us recall that $f_2^{(\mathrm{II};\, r,\, r)} \in \mathfrak{P}_{2,rK} = \widehat{\mathfrak{P}}_{1,0,rK} \cup \widehat{\mathfrak{P}}_{0,2,rK}$; indeed, such a function does depend on all the canonical variables, i.e., $f_2^{(\mathrm{II};\, r,\, r)} = f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi})$. Therefore, we denote with $f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q})$ the subpart of $f_2^{(\mathrm{II};\, r,\, r)}$ that is depending just on $(\boldsymbol{p}, \boldsymbol{q})$. Analogously, in the following $f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi})$ will denote the subpart of $f_2^{(\mathrm{II};\, r,\, r)}$ that does depend on all the canonical variables but the actions $\boldsymbol{p}$ and so on also for what concerns $f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{J}, \boldsymbol{\varphi})$. For the sake of clarity, this highly non-standard notation will be maintained up to the end of the present subsection. Let us here emphasize that the term $\langle f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q})\rangle_{\boldsymbol{q}}$ will be added to the part in normal form, by updating the angular velocity vector $\boldsymbol{\omega}$, in agreement with what has been done in the context of the construction of the Kolmogorov normal form. We can deal with the homological equation (45) in the same way as for (18). Indeed, the solution writes as

$$X_2^{(r)}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{|\boldsymbol{j}|=1} \sum_{0<|\boldsymbol{k}|\le rK} \frac{c_{\boldsymbol{j},\boldsymbol{k}}\, \boldsymbol{p}^{\boldsymbol{j}} \exp\left(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{q}\right)}{\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\omega}^{(r)}}\,, \tag{46}$$

where the expansion of the perturbing term $f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q}) \in \widehat{\mathfrak{P}}_{1,0,rK}$ is such that $f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{|\boldsymbol{j}|=1} \sum_{0<|\boldsymbol{k}|\le rK} c_{\boldsymbol{j},\boldsymbol{k}}\, \boldsymbol{p}^{\boldsymbol{j}} \exp\left(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{q}\right)$. Once again, the solution written in (46) is valid provided that the non-resonance condition (11) is satisfied.

The generating function $Y_2^{(r)}$ aims to remove the part of the term of $f_2^{(\mathrm{II};\, r,\, r)}$ that is quadratic in $\sqrt{\boldsymbol{J}}$ and does depend on the angles $\boldsymbol{q}$. Therefore, $Y_2^{(r)}$ has to solve the following homological equation:

$$\left\{\boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p} + \boldsymbol{\Omega}^{(r-1)} \cdot \boldsymbol{J},\ Y_2^{(r)}\right\} + f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi}) - \langle f_2^{(\mathrm{II};\, r,\, r)}(\boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi})\rangle_{\boldsymbol{q}} = 0\,. \tag{47}$$

In order to describe the solution of such an equation, it is convenient to write the explicit expansion of the perturbing term $f_2^{(\mathrm{II};\,r,\,r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi})$. For instance, this can be done in the following way:

$$f_2^{(\mathrm{II};\,r,\,r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) = \sum_{0\le\boldsymbol{k}\le rK}\ \sum_{i,\,j=1}^{n_2} c_{\boldsymbol{k},\,i,\,j}^{(\pm,\pm)}\sqrt{J_iJ_j}\exp\big[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}\pm\varphi_i\pm\varphi_j)\big]\,, \tag{48}$$

where $c_{\boldsymbol{k},\,i,\,j}^{(+,+)}$ and $c_{\boldsymbol{k},\,i,\,j}^{(+,-)}$ are the coefficients referring to the Fourier harmonics $\boldsymbol{k}\cdot\boldsymbol{q}+\varphi_i+\varphi_j$ and $\boldsymbol{k}\cdot\boldsymbol{q}+\varphi_i-\varphi_j$, respectively, and so on. Thus, the generating function $Y_2^{(r)}$ is determined by Eq. (47) in such a way that

$$Y_2^{(r)}(\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) = \sum_{0<\boldsymbol{k}\le rK}\ \sum_{i,\,j=1}^{n_2}\frac{c_{\boldsymbol{k},\,i,\,j}^{(\pm,\pm)}\sqrt{J_iJ_j}\exp\big[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{q}\pm\varphi_i\pm\varphi_j)\big]}{\mathrm{i}\big(\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}\pm\Omega_i^{(r-1)}\pm\Omega_j^{(r-1)}\big)}\,, \tag{49}$$

which is well defined provided that the angular velocity vector $\boldsymbol{\omega}^{(r-1)}$ satisfies both the already mentioned Diophantine inequality (11) and the so-called second Melnikov non-resonance condition up to order rK (see [29]), i.e.,

$$\min_{\substack{0<|\boldsymbol{k}|\le rK,\\ |\boldsymbol{\ell}|=2}}\left|\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}+\boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right|\ge\frac{\gamma}{(rK)^{\tau}} \tag{50}$$

with fixed values of both parameters $\gamma>0$ and $\tau>n_1-1$.

After having performed these two changes of coordinates, we still may have terms that do not depend on $\boldsymbol{q}$ and are either linear in $\boldsymbol{p}$ or quadratic in $\sqrt{\boldsymbol{J}}$. The former ones can be directly added to the part in normal form, whereas the latter have to be preliminarily put in diagonal form. This can be done by means of a canonical transformation $\mathfrak{D}^{(r)}$ such that

$$\left(\boldsymbol{\Omega}^{(r-1)}\cdot\boldsymbol{J}+f_2^{(\mathrm{II};\,r,\,r)}(\boldsymbol{J},\boldsymbol{\varphi})\right)\bigg|_{(\boldsymbol{J},\boldsymbol{\varphi})=\mathfrak{D}^{(r)}(\bar{\boldsymbol{J}},\bar{\boldsymbol{\varphi}})} = \boldsymbol{\Omega}^{(r)}\cdot\bar{\boldsymbol{J}}\,. \tag{51}$$

Such an equation in the unknown transformation $\mathfrak{D}^{(r)}$ can be solved provided that

$$\min_{|\boldsymbol{\ell}|=2}\left|\boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right|\ge\gamma \tag{52}$$

and $f_2^{(\mathrm{II};\,r,\,r)}$ is small enough, as it is explained, e.g., in Sect. 7 of [16] (where this problem is considered in the equivalent case dealing with polynomial canonical coordinates). In practical implementations, such a change of coordinates $\mathfrak{D}^{(r)}$ can be conveniently defined by composing a subsequence of Lie series, each of them being related to a quadratic generating function $\mathcal{D}_2^{(r;\,m)}(\boldsymbol{J},\boldsymbol{\varphi})\in\widehat{\mathfrak{P}}_{0,2,0}$ with $m\in\mathbb{N}\setminus\{0\}$. All these new generating functions can be determined by adopting the following computational (sub)procedure of iterative type. First, we introduce the new angular velocity vector $\boldsymbol{\Omega}^{(r;\,0)}$ so that

$$\boldsymbol{\Omega}^{(r;\,0)} \cdot \boldsymbol{J} = \boldsymbol{\Omega}^{(r-1)} \cdot \boldsymbol{J} + \langle f_2^{(\mathrm{II};\,r,\,r)}(\boldsymbol{J}, \boldsymbol{\varphi}) \rangle_{\boldsymbol{\varphi}} \tag{53}$$

and the new function

$$\mathfrak{g}_2^{(r;\,0)}(\boldsymbol{J}, \boldsymbol{\varphi}) = f_2^{(\mathrm{II};\,r,\,r)}(\boldsymbol{J}, \boldsymbol{\varphi}) - \langle f_2^{(\mathrm{II};\,r,\,r)}(\boldsymbol{J}, \boldsymbol{\varphi}) \rangle_{\boldsymbol{\varphi}} \,. \tag{54}$$

The general m-th step of this iterative (sub)procedure starts by solving the following homological equation:

$$\left\{ \boldsymbol{\Omega}^{(r;\,m-1)} \cdot \boldsymbol{J},\, \mathcal{D}_2^{(r;\,m)}(\boldsymbol{J}, \boldsymbol{\varphi}) \right\} + \mathfrak{g}_2^{(r;\,m-1)}(\boldsymbol{J}, \boldsymbol{\varphi}) = 0 \,, \tag{55}$$

where $\mathfrak{g}_2^{(r;\,m-1)} \in \widehat{\mathfrak{P}}_{0,2,0}$ is such that $\langle \mathfrak{g}_2^{(r;\,m-1)} \rangle_{\boldsymbol{\varphi}} = 0$ (and, therefore, also the new generating function $\mathcal{D}_2^{(r;\,m)}$ is sharing these same properties with $\mathfrak{g}_2^{(r;\,m-1)}$). Let us now initially introduce $\mathfrak{g}_2^{(r;\,m)} = 0$ and (by the usual abuse of notation) we redefine it many times according to the following rule:

$$\mathfrak{g}_2^{(r;\,m)} \hookleftarrow \frac{j}{(j+1)!} \mathcal{L}^{j}_{\mathcal{D}_2^{(r;\,m)}} \mathfrak{g}_2^{(r;\,m-1)} \quad \forall\, j \geq 1 \,. \tag{56}$$

Actually, at this point one can easily check that

$$\exp\left(\mathcal{L}_{\mathcal{D}_2^{(r;\,m)}}\right) \left(\boldsymbol{\Omega}^{(r;\,m-1)} \cdot \boldsymbol{J} + \mathfrak{g}_2^{(r;\,m-1)} \right) = \boldsymbol{\Omega}^{(r;\,m-1)} \cdot \boldsymbol{J} + \mathfrak{g}_2^{(r;\,m)} \,,$$

by using homological equation (55). Furthermore, we set

$$\boldsymbol{\Omega}^{(r;\,m)} \cdot \boldsymbol{J} = \boldsymbol{\Omega}^{(r;\,m-1)} \cdot \boldsymbol{J} + \langle \mathfrak{g}_2^{(r;\,m)}(\boldsymbol{J}, \boldsymbol{\varphi}) \rangle_{\boldsymbol{\varphi}} \tag{57}$$

and we redefine one last time $\mathfrak{g}_2^{(r;\,m)}$ so that

$$\mathfrak{g}_2^{(r;\,m)}(\boldsymbol{J}, \boldsymbol{\varphi}) = \mathfrak{g}_2^{(r;\,m)}(\boldsymbol{J}, \boldsymbol{\varphi}) - \langle \mathfrak{g}_2^{(r;\,m)}(\boldsymbol{J}, \boldsymbol{\varphi}) \rangle_{\boldsymbol{\varphi}} \,. \tag{58}$$

By applying repeatedly Lemma 2 to formulæ (53)–(58), it is easy to check that both functions $\mathcal{D}_2^{(r;\,m)}$ and $\mathfrak{g}_2^{(r;\,m)}$ belong to the class $\widehat{\mathfrak{P}}_{0,2,0}$ (also because they depend on neither $\boldsymbol{p}$ nor $\boldsymbol{q}$) and their angular average is equal to zero. In principle, these remarks would allow to iterate infinitely many times this computational (sub)procedure, that we are using to solve Eq. (51). However, in practical implementations, we have to set a criterion to stop the iterations so to ensure that the algorithm can be worked out in a finite number of operations. This can be done, for instance, in such a way to end the computations when the angular velocity vector does not modify anymore. This means that the final value $\bar{m}$ of the normalization step for this iterative (sub)procedure is such that the equation $\boldsymbol{\Omega}^{(r;\,\bar{m})} = \boldsymbol{\Omega}^{(r;\,\bar{m}-1)}$ holds true *in the framework of the numbers that are representable on a computer*[18] (for instance, the `double precision` type). By setting $\boldsymbol{\Omega}^{(r)} = \boldsymbol{\Omega}^{(r;\,\bar{m})}$ and the canonical transformation $\mathfrak{D}^{(r)}$ equal to composition of all the Lie series generated by the *finite* sequence of functions $\left\{\mathcal{D}_2^{(r;\,m)}\right\}_{m=1}^{\bar{m}}$, we determine a solution[19] of (51) that is valid up to the numerical round-off errors.

Finally, we need to understand how all these generating functions (that have been defined during the third stage of the r-th normalization step) give their contributions to the Hamiltonian terms appearing in the following expansion:

$$\begin{aligned}\mathcal{H}^{(r)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) =& \mathcal{E}^{(r)} + \boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(r)}\cdot\boldsymbol{J} + \sum_{s\ge 0}\sum_{\ell\ge 3} f_\ell^{(r,s)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi}) \\ &+ \sum_{s\ge r+1}\sum_{\ell=0}^{2} f_\ell^{(r,\,s)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi})\,,\end{aligned} \tag{59}$$

where $\mathcal{H}^{(r)}$ is defined in (34). In order to describe the definitions of those new summands, it is convenient to introduce the intermediate functions $g_\ell^{(r,\,s)}$, $g'^{(r,\,s)}_\ell$ in the following way. First, we define $g_\ell^{(r,\,s)} = f_\ell^{(\mathrm{II};\,r,\,s)}$ for all non-negative values of the indexes ℓ and s; then, we consider the effects induced by the application of the Lie series with generating function $X_2^{(r)}$ to the Hamiltonian. In order to do that, (by abuse of notation) we redefine many times the new intermediate functions $g_\ell^{(r,\,s)}$ according to the following rules:

$$\begin{aligned} g_\ell^{(r,\,s+jr)} &\hookleftarrow \frac{1}{j!}\mathcal{L}^j_{X_2^{(r)}} f_\ell^{(\mathrm{II};\,r,\,s)} \quad \forall\ j\ge 1,\ \ell\ge 0,\ s\ge 0\,, \\ g_2^{(r,\,jr)} &\hookleftarrow \frac{1}{j!}\mathcal{L}^j_{X_2^{(r)}}\big(\boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(r)}\cdot\boldsymbol{J}\big) \quad \forall\ j\ge 1\,. \end{aligned} \tag{60}$$

As usual, the prescriptions above have been set so to gather the new terms generated by the Lie series $\exp\big(\mathcal{L}_{X_2^{(r)}}\big)$ according to both their total degree in the square root of the actions and the trigonometric degree in the angles. In analogous way, we first introduce $g'^{(r,\,s)}_\ell = g'^{(r,\,s)}_\ell$ $\forall\ \ell\ge 0,\ s\ge 0$; then we apply many times the following redefinitions:

$$\begin{aligned} g'^{(r,\,s+jr)}_\ell &\hookleftarrow \frac{1}{j!}\mathcal{L}^j_{Y_2^{(r)}} g_\ell^{(r,\,s)} \quad \forall\ j\ge 1,\ \ell\ge 0,\ s\ge 0\,, \\ g'^{(r,\,jr)}_2 &\hookleftarrow \frac{1}{j!}\mathcal{L}^j_{Y_2^{(r)}}\big(\boldsymbol{\omega}^{(r)}\cdot\boldsymbol{p} + \boldsymbol{\Omega}^{(r)}\cdot\boldsymbol{J}\big) \quad \forall\ j\ge 1\,. \end{aligned} \tag{61}$$

[18] A similar criterion is adopted to determine a maximum value of the index j at which the redefinitions (56) must be stopped.

[19] As an alternative computational method, when one is dealing with the estimates needed to prove the convergence of the algorithm, in [19] the use of the Lie transforms (that are equivalent to the composition of *infinite* sequences of Lie series) has been found to be very suitable.

By applying Lemma 2 to formulæ (60)–(61), it is easy to check that $g'^{(r,\,s)}_{\ell} \in \mathfrak{P}_{\ell,sK}$ $\forall\ \ell \ge 0,\ s \ge 0$. Let us now remark that each class of type $\mathfrak{P}_{\ell,sK}$ is preserved[20] by the diagonalization transformation $\mathfrak{D}^{(r)}$, for all non-negative values of the indexes ℓ and s. Therefore, it is natural to put

$$f_{\ell}^{(r,s)} = g'^{(r,\,s)}_{\ell} \circ \mathfrak{D}^{(r)} \ . \tag{62}$$

for all indexes $\ell \ge 0$ and $s \ge 0$.

At the end of the r-th normalization step, it is convenient that the terms linearly depending just on $\boldsymbol{p}$ or $\boldsymbol{J}$ are included in the main part of the Hamiltonian, because all of them belong to the same class of functions, i.e. $\mathfrak{P}_{2,0}$. For this purpose, we introduce the new angular velocity vector $\boldsymbol{\omega}^{(r)}$, in such a way that

$$\boldsymbol{\omega}^{(r)} \cdot \boldsymbol{p} = \boldsymbol{\omega}^{(r-1)} \cdot \boldsymbol{p} + f_2^{(\mathrm{II};\,r,\,0)}(\boldsymbol{p}) \ , \tag{63}$$

while the new values of the components of $\boldsymbol{\Omega}^{(r)}$ are defined by Eq. (51), that also allows us to put $f_2^{(r,r)} = 0$. This ends the justification of the fact that the Hamiltonian $\mathcal{H}^{(r)}$ can be written as in formula (59) with new terms such that $f_{\ell}^{(r,\,s)} \in \mathfrak{P}_{\ell,sK}$ and $\mathcal{E}^{(r)} \in \mathfrak{P}_{0,0}$. Therefore, $\mathcal{H}^{(r)}$ has the same structure of $\mathcal{H}^{(r-1)}$ in (33); this also mean that the normalization algorithm can be iterated to the next ($r + 1$-th) step. As a final comment ending the present subsection, let us also remark that the new perturbative terms $f_{\ell}^{(r,s)}$ with $\ell = 0,\ 1,\ 2$ are expected to be smaller with respect to the previous ones; this is because of the Fourier decay of the coefficients jointly with the fact that we removed the part of perturbation up to the trigonometric degree rK.

3.2 On the Convergence of the Algorithm Constructing the Normal Form for Elliptic Tori

As we have discussed since the introduction, in the present work we make the choice of adopting the same approach to construct two different normal forms, that are related to KAM invariant manifolds and elliptic tori, respectively. For what concerns the analysis of the convergence, such a choice now allows us to use arguments that are very similar to those described in the previous Sect. 2. In particular, also for what concerns the motion on elliptic tori, we emphasize that it can be approximated within a precision up to a fixed order of magnitude by using our procedure that is *explicitly computable*, because the total amount of operations that are defined also by this normalization algorithm is *finite*.

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[20] This statement can be justified, by referring also to the definition of the canonical transformation $\mathfrak{D}^{(r)}$ as composition of all the Lie series generated by the set of functions $\big\{\mathcal{D}_2^{(r;\,m)}\big\}_{m=1}^{\bar{m}}$. In fact, it can be easily done by applying Lemma 2 to all the contributions due to the repeated application of the Lie derivative with generating functions $\mathcal{D}_2^{(r;\,m)} \in \mathfrak{P}_{2,0}$.

The non-resonance conditions we have assumed in (11), (42), (50) and (52) can be summarized in the following way:

$$\min_{\substack{0<|\boldsymbol{k}|\le rK,\\ 0\le|\boldsymbol{\ell}|\le 2}} \left|\boldsymbol{k}\cdot\boldsymbol{\omega}^{(r-1)}+\boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right| \ge \frac{\gamma}{(rK)^{\tau}} \quad \text{and} \quad \min_{0<|\boldsymbol{\ell}|\le 2}\left|\boldsymbol{\ell}\cdot\boldsymbol{\Omega}^{(r-1)}\right|\ge\gamma\,, \tag{64}$$

with $\gamma > 0$ and $\tau > n_1 - 1$. Let us here resume the parametric dependence of all the Hamiltonian terms on the initial value of the angular velocity vector $\boldsymbol{\omega}^{(0)}$, as it has been introduced at the beginning of the previous Sect. 3.1 (see the discussion following the statement of Lemma 2). In particular, in the Diophantine inequalities reported in (64) the angular velocity vectors at the r-th normalization step are functions of $\boldsymbol{\omega}^{(0)}$, i.e., $\boldsymbol{\omega}^{(r-1)} = \boldsymbol{\omega}^{(r-1)}(\boldsymbol{\omega}^{(0)})$ and $\boldsymbol{\Omega}^{(r-1)} = \boldsymbol{\Omega}^{(r-1)}(\boldsymbol{\omega}^{(0)})$. Let us recall that we do not try to keep a full control on the way for what concerns the angular velocity vectors that are modified passing from the $r-1$-th normalization step to the next one. Therefore, let us recall also here that such an approach is in contrast with the original proof scheme that was designed to construct the Kolmogorov normal form for *maximal* invariant tori, where the angular velocities are kept fixed (see [22] or, e.g., [17]), but it is somehow unavoidable because of the occurrence of the transversal angular velocities $\boldsymbol{\Omega}^{(r-1)}(\boldsymbol{\omega}^{(0)})$ that in general cannot remain constant along the normalization procedure. This seems to prevent the complete construction of the normal form and so also for what concerns the proof of the existence of an elliptic torus. Nevertheless, following the approach designed by Pöschel in [36], it can be proved that the Lebesgue measure of the resonant regions where the Melnikov conditions are not satisfied shrinks to zero with the size of the perturbation. Therefore, the chances of success in constructing the normal form for elliptic tori are described by the following statement.

Theorem 2 *Consider the family of real Hamiltonians $\mathcal{H}^{(0)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi};\boldsymbol{\omega}^{(0)})$ of the type described in (31). Those functions are defined so that $\mathcal{H}^{(0)}:\mathcal{O}_1\times\mathbb{T}^{n_1}\times\mathcal{O}_2\times\mathbb{T}^{n_2}\times\mathcal{U}\mapsto\mathbb{R}$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ open neighborhoods of the origin in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}_{+}\cup\{\mathbf{0}\}$, respectively, while $\boldsymbol{\omega}^{(0)}\in\mathcal{U}$, being $\mathcal{U}$ an open subset of $\mathbb{R}^{n_1}$. Moreover, let a special class of functions include each of the terms that are of type $f_{\ell}^{(0,s)}$ and appear in the expansion (31), in such a way that $f_{\ell}^{(0,s)}\in\mathfrak{P}_{\ell,sK}$ for a fixed positive integer K. We also assume that*

(a) all the functions $\mathcal{E}^{(0)}:\mathcal{U}\mapsto\mathbb{R}$, $\boldsymbol{\Omega}^{(0)}:\mathcal{U}\mapsto\mathbb{R}^{n_1}$ and $f_{\ell}^{(0,s)}:\mathcal{O}_1\times\mathbb{T}^{n_1}\times\mathcal{O}_2\times\mathbb{T}^{n_2}\times\mathcal{U}\mapsto\mathbb{R}$, appearing in (31), are analytic functions with respect to $\boldsymbol{\omega}^{(0)}\in\mathcal{U}$;

(b) $\Omega_i^{(0)}(\boldsymbol{\omega}^{(0)})\neq\Omega_j^{(0)}(\boldsymbol{\omega}^{(0)})$ and $\Omega_{i_2}^{(0)}(\boldsymbol{\omega}^{(0)})\neq 0$ for $\boldsymbol{\omega}^{(0)}\in\mathcal{U}$ and $1\le i<j\le n_2$, $1\le i_2\le n_2$;

(c) for some fixed and positive values of ε and E, one has

$$\sup_{(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi};\boldsymbol{\omega}^{(0)})\in\mathcal{O}_1\times\mathbb{T}^{n_1}\times\mathcal{O}_2\times\mathbb{T}^{n_2}\times\mathcal{U}}\left|f_{\ell}^{(0,s)}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{J},\boldsymbol{\varphi};\boldsymbol{\omega}^{(0)})\right|\le\varepsilon^s E \tag{65}$$

$\forall\, s\ge 1,\ \ell\ge 0$ and $\forall\,\ell\ge 3$ when $s=0$.

Then, there is a positive $\varepsilon^{\star}$ *such that for* $0 \le \varepsilon < \varepsilon^{\star}$ *the following statement holds true: there exists a non-resonant set* $\mathcal{U}^{(\infty)} \subset \mathcal{U}$ *of positive Lebesgue measure and with the measure of* $\mathcal{U} \setminus \mathcal{U}^{(\infty)}$ *tending to zero for* $\varepsilon \to 0$ *for bounded* $\mathcal{U}$*, such that for each* $\boldsymbol{\omega}^{(0)} \in \mathcal{U}^{(\infty)}$ *there exists an analytic canonical transformation* $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{J}, \boldsymbol{\varphi}) = \psi^{(\infty)}_{\varepsilon;\boldsymbol{\omega}^{(0)}}(\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{\Xi}, \boldsymbol{\Theta})$ *leading the Hamiltonian to the normal form written in* (32)*, where* $\mathcal{E}^{(\infty)}(\boldsymbol{\omega}^{(0)})$ *is a finite real value fixing the constant energy level that corresponds to the invariant elliptic torus* $\big\{(\boldsymbol{P} = \boldsymbol{0},\ \boldsymbol{Q} \in \mathbb{T}^{n_1}, \boldsymbol{\Xi} = \boldsymbol{0}, \boldsymbol{\Theta} = \boldsymbol{0})\big\}$*. Moreover, the canonical change of coordinates is close to the identity in the sense that* $\big\|\psi^{(\infty)}_{\varepsilon;\boldsymbol{\omega}^{(0)}}(\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{\Xi}, \boldsymbol{\Theta}) - (\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{\Xi}, \boldsymbol{\Theta})\big\| = \mathcal{O}(\varepsilon)$ *and the same applies also to both the energy level and the detunings of the angular velocity vectors (that are* $\big|\mathcal{E}^{(\infty)}(\boldsymbol{\omega}^{(0)}) - \mathcal{E}^{(0)}(\boldsymbol{\omega}^{(0)})\big| = \mathcal{O}(\varepsilon)$*,* $\big\|\boldsymbol{\omega}^{(\infty)}(\boldsymbol{\omega}^{(0)}) - \boldsymbol{\omega}^{(0)}\big\| = \mathcal{O}(\varepsilon)$ *and* $\big\|\boldsymbol{\Omega}^{(\infty)}(\boldsymbol{\omega}^{(0)}) - \boldsymbol{\Omega}^{(0)}(\boldsymbol{\omega}^{(0)})\big\| = \mathcal{O}(\varepsilon)$*, respectively).*

The complete proof of theorem above is reported in [7], where it is ensured the convergence of a normalization algorithm that is substantially the same with respect to the one described in the previous Sect. 3.1 apart some very minor modifications.[21] Therefore, the approach of that paper is based on a convergence scheme of linear type. Nevertheless, the more geometrical part of that work (which deals with the estimates of the volume covered by the resonant region) is borrowed from [36], where a statement nearly equivalent to Theorem 2 is proved by adopting a fast convergence scheme of quadratic type.

In the present case studying the elliptic tori, the choice to let the angular velocity vectors change at every normalization step is somehow more natural with respect to the original proof scheme designed by Kolmogorov. This is due to the fact that here the procedure allowing to keep fixed the angular velocities is not complete, because it involves less free parameters than the number of degrees of freedom. This is a major difference with respect to the algorithm constructing the normal form for KAM tori, where those two integer numbers are equal. For what concerns the case of the elliptic tori too, some work[22] is in progress in order to revisit the problem of the convergence of this type of normalization algorithms so as to provide a statement where the final result is not expressed in a probabilistic sense (i.e., by referring to the Lebesgue measure). This can be done by fixing since the beginning the final value of the angular velocity vectors $\big(\boldsymbol{\omega}^{(\infty)}, \boldsymbol{\Omega}^{(\infty)}\big)$ and their non-resonance properties; we emphasize that this allow to explicitly solve all the homological equations that are introduced at every step of the algorithm. Also here, the total detunings $\boldsymbol{\omega}^{(\infty)} - \boldsymbol{\omega}^{(0)}$ and $\boldsymbol{\Omega}^{(\infty)} - \boldsymbol{\Omega}^{(0)}$ are given in terms of series whose coefficients are defined in a recursive way. Such an approach is also inspired by the need to revisit what was successfully done in order to show the existence of elliptic tori in PDEs problems (see [6]).

[21] For instance, in order to describe the transverse dynamics with respect to the elliptic tori, the complex canonical coordinates $(z, \mathrm{i}\bar{z})$ instead of the action-angle ones are used, where $z_j = J_j e^{\mathrm{i}\varphi_j}$ $\forall\, j = 1, \ldots, n_2$.

[22] Danesi, V., Locatelli, U.: work in progress (2022).

4 Construction of Invariant KAM Tori in Exoplanetary Systems with Rather Eccentric Orbits

In order to properly introduce a Cauchy problem which includes the ordinary differential equations (ODE) for a planetary system, the initial conditions at a given time are needed and so also for the positions and the velocities in an astrocentric frame. It is well known that they can be replaced by the orbital elements

$$\Big\{\big(a_j\,,\,e_j\,,\,\iota_j\,,\,M_j\,,\,\omega_j\,,\,\Omega_j\big)\;:\;\forall\,j=1,\,\ldots\,,N\Big\}\,,$$

being N the number of the planets that are considered in the system. Orbital elements refer to the so called osculating Keplerian ellipse, which describes a fictitious motion having the same instantaneous values of both position and velocity with respect to the planet. For what concerns the Keplerian ellipse of the j-th planet, the symbols a_j, e_j, ι_j, M_j, ω_j, Ω_j denote the semi-major axis, the eccentricity, the inclination,[23] the mean anomaly, the argument of the pericenter[24] and the longitude of the ascending node, respectively. Of course, also the values of the masses $m_j\ \forall\ j=0,\ 1,\ \ldots\ ,\ N$ (being m_0 the stellar mass) are needed in order to properly introduce the Cauchy problem for a planetary system, because they enter in the definitions of the momenta, the kinetic energy and the potential one. Unfortunately, none of the detection methods that are nowadays available to discover extrasolar planets is able to measure all the orbital elements and the masses that completely define the ODE problem (see, e.g., [2]). For the sake of simplicity, instead of considering a generic planetary problem with $N+1$ bodies, let us focus on a specific case, i.e., the extrasolar system hosting two planets orbiting around the star named HD 4732[25] (the value of its mass is reported in the caption of the following table). The values of the known orbital elements of those exoplanets as they are given by the radial velocity detection method are reported in Table 1. Let us recall that such a detection technique is unable to provide a complete information about the mass of every j-th planet; instead, it gives its minimum value $m_j \sin(\iota_j)$.

Let us now explain how we have decided to complete the initial conditions, by also giving the motivations of our choice. Since we are interested in studying the planetary dynamics of the HD 4732 system in the framework of a secular model, we expect that its dependence on the initial values of the mean anomalies is weak. We emphasize that such an assumption does not hold true in general (see, e.g., [26]), but it is rather natural in the case of the HD 4732 planetary system because the revolution

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[23] ι_j is the inclination of the Keplerian ellipse with respect to the plane orthogonal to the line of sight (i.e., the direction pointing to the object one is observing), that is usually said to be "tangent to the celestial sphere".

[24] Unfortunately, the same symbol (namely, ω) is used to denote both the angular velocity in KAM theory and the pericenter argument in astronomy. Hereafter, when the symbol ω appears *without* superscripts, it will refer just to the latter quantity.

[25] Since the detection of a fainter stellar companion in 2019 (see [32]) HD 4732 has been renamed as HD 4732A. For brevity, in the present paper we refer to such a star with the old name.

Table 1 Known orbital elements and minimal masses of the detected exoplanets orbiting around the HD 4732 star, whose mass is 1.74 times bigger than the solar one. The following data are taken from the central values of the ranges given in Table 5 of [39]. The corresponding units of measure are reported in every column between pairs of square brackets; in particular, we recall that the eccentricity of an ellipse is a pure number ranging in (0, 1) and M_{Jup} means "Jupiter mass". Since the initial time is irrelevant for an autonomous system, we have set it equal to zero in the parentheses following the orbital elements

Planet name	Planet index j	$a_j(0)$ [AU]	$e_j(0)$	$\omega_j(0)$ [°]	$m_j \sin(\iota_j(0))$ [M_{Jup}]
HD 4732b	1	1.19	0.13	85	2.37
HD 4732c	2	4.60	0.23	118	2.37

periods are far from mean-motion resonances and they are much shorter with respect to those corresponding to the remaining angles that appear in the orbital elements list. Therefore, we simply set[26]

$$M_1(0) = M_2(0) = 0^\circ \ . \tag{66}$$

For what concerns the extrasolar system HD 4732, we plan to start a study of the dependence of its orbital dynamics on the mutual inclination $i_{\rm mut}$. The present section deals with the beginning of such a research project, that will be extended in a forthcoming work. For this purpose, it is convenient to consider orbital planes initially located in such a way they are symmetric with respect to the line of sight that is also orthogonal to their intersection. As an example of this particular configuration, we can consider the case with $\iota_1(0) = 89^\circ$, $\iota_2(0) = 91^\circ$ and

$$\Omega_1(0) = \Omega_2(0) = 0^\circ \ . \tag{67}$$

In view of the general relation

$$\cos i_{\rm mut} = \cos \iota_1 \cos \iota_2 + \sin \iota_1 \sin \iota_2 \cos(\Omega_1 - \Omega_2) \ ,$$

we readily obtain that $i_{\rm mut} = 2^\circ$. More in general, we introduce the following set of initial conditions

[26] Since the times of passage at the pericenter are given by the radial velocity detection methods and they are different, we stress that our choice of defining the initial values of the mean anomalies so that $M_1(0) = M_2(0) = 0^\circ$ is not coherent with the observations about the two planets orbiting around HD 4732. However, we consider that this small inconsistency of our settings should be harmless, just because of the expectation that its secular dynamics should be very weakly affected by the initial values of the mean anomalies.

$$\mathcal{I}_{i_{\rm mut}(0)} = \Big\{ \big(a_1(0)\,,\, a_2(0)\,,\ e_1(0)\,,\ e_2(0)\,, \\ \iota_1(0) = 90^\circ - \frac{i_{\rm mut}(0)}{2}\,,\ \iota_2(0) = 90^\circ + \frac{i_{\rm mut}(0)}{2}\,, \\ M_1(0)\,,\ M_2(0)\,,\omega_1(0)\,,\ \omega_2(0)\,,\Omega_1(0)\,,\Omega_2(0)\big)\Big\}\,, \tag{68}$$

where the inclinations are parameterized with respect to $i_{\rm mut}(0)$, while the values of all the remaining orbital elements are defined according to Table 1, jointly with formulæ (66) and (67). Of course, the values of the planetary masses m_1 and m_2 can be recovered multiplying the minimal masses (that appear in the last column of Table 1) by the increasing factor $1\big/\sin\big(\iota_j(0)\big)$. This remark helps us to understand that all the parameters and the initial conditions have been properly defined and they can eventually depend just on the value of $i_{\rm mut}(0)$. This way to parameterize the model has been introduced to better understand the properties of our (new) algorithm constructing invariant tori as a function of the mutual inclinations. A previous approach to the same problem was described in [41] and it was shown to be successful just for systems with rather small eccentricities of the exoplanets, being their initial values less than 0.1 . This is not the case of the exoplanets in the system HD 4732, because both their initial values of the eccentricities (reported in Table 1) are larger than 0.1 . We emphasize that this choice has been made with the purpose to show that our following new formulation of the constructing algorithms applies to a more extended range of models with respect to the previous approach.

Let us also recall that, in a three-body planetary problem, the longitudes of the nodes are always opposite, if they are measured with respect to the so called Laplace plane, that is invariant because it is orthogonal to the total angular momentum, by definition (see, e.g., Sect. 6.2 of [24]). Moreover, the Hamiltonian does not depend on the sum of $\Omega_1 + \Omega_2$, because of the invariance with respect to the rotations. In the following subsection, we will explain why it is preferable to consider expansions of the Hamiltonian in a frame where the Laplace plane is the horizontal one. In Celestial Mechanics the word "inclination" often refers to the angle (say, $\mathrm{i}_j \in [0^\circ, 180^\circ]$) between the angular momentum of the j-th planet and the total one. With this notation, the following relation holds true: $i_{\rm mut} = \mathrm{i}_1 + \mathrm{i}_2$.

4.1 Secular Model at Order Two in the Masses

In the present subsection, we are going to introduce a model describing the secular dynamics of a planetary system, in a way that provides results more reliable with respect to a simple average over the revolution angles (see, e.g., [38]). We emphasize that we derive the secular model at order two in the masses, by applying an approach inspired to the construction of the Kolmogorov normal form. This is a major difference with respect to other approaches providing the same level of accuracy for a secular model (see, e.g., [23] and references therein). Here, in order to introduce our secular model, we will adopt the approach described in [41], that is summarized as follows.

A three-body Hamiltonian problem has nine degrees of freedom, but three of them can be easily separated so as to describe the uniform motion of the center of mass in an inertial frame. The untrivial part of the dynamics is represented in astrocentric canonical coordinates and its degrees of freedom can be further reduced by two using the conservation of the total angular momentum $\boldsymbol{C}$. As it is shown in Sect. 6 of [24], this allows us to write the Hamiltonian in Poincaré canonical variables, that are

$$\begin{aligned} \Lambda_j =& \frac{m_0 m_j}{m_0 + m_j}\sqrt{G(m_0+m_j)a_j}\ , \qquad & \xi_j =& \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\cos(\omega_j)\ , \\ \lambda_j =& M_j + \omega_j\ , & \eta_j =& -\sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\sin(\omega_j)\ . \end{aligned} \tag{69}$$

The reduction of the total angular momentum makes implicit the dependence on the inclinations i_j and on the longitudes of the nodes Ω_j. In the Laplace reference frame the mutual inclination is the sum of the two inclinations and so is given by a rather simple relation involving the Poincaré variables, i.e.,

$$i_{\rm mut} = \mathrm{i}_1 + \mathrm{i}_2 = \arccos\left(\frac{C^2 - \Lambda_1^2(1-e_1^2) - \Lambda_2^2(1-e_2^2)}{2\Lambda_1\Lambda_2\sqrt{1-e_1^2}\sqrt{1-e_2^2}}\right)\ , \tag{70}$$

being $C = \sum_{k=1}^{2}\Lambda_k\sqrt{1-e_k^2}\cos\mathrm{i}_k$, that is the (constant) module of the total angular momentum. Moreover, we introduce a translation $L_j = \Lambda_j - \Lambda_j^*$, where Λ_j^* is defined in order to obtain that in the Keplerian approximation of the motion the values of the semi-major axes are in agreement with the observations. Indeed, the expansions of a Hamiltonian representing a planetary model are usually made around the average values of the semi-major axes or their initial values. For the sake of simplicity, we will adopt this latter option. Such expansions are actually made with respect to these Poincaré variables[27] and the parameter D_2, that measures the difference between the total angular momentum of the system and the one of a similar system with circular and coplanar orbits; i.e., it is defined as $D_2 = \left[(\Lambda_1^* + \Lambda_2^*)^2 - C^2\right]\big/(\Lambda_1^*\Lambda_2^*)$; therefore, it is of the same order as $e_1^2 + \mathrm{i}_1^2 + e_2^2 + \mathrm{i}_2^2$. Thus, we can write the Hamiltonian of the three–body problem as

$$H_{\rm 3BP} = \sum_{j_1=1}^{\infty} h_{j_1,0}^{(\rm Kep)}(\boldsymbol{L}) + \mu\sum_{s=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} D_2^s\, h_{s;j_1,j_2}^{(\mathcal{P})}(\boldsymbol{L},\boldsymbol{\lambda},\boldsymbol{\xi},\boldsymbol{\eta}) \tag{71}$$

[27] The computation of the coefficients appearing in the expansion (71) is not straightforward. For a detailed discussion of the method we have used for doing such a calculation we refer to [24].

where $\mu = \max\{m_1/m_0, m_2/m_0\}$. Moreover,

- $\mathcal{K}(\boldsymbol{L}) = \sum_{j_1=1}^{\infty} h_{j_1,0}^{(\mathrm{Kep})}(\boldsymbol{L})$ is the Keplerian part and $h_{j_1,0}^{(\mathrm{Kep})}$ is a homogeneous polynomial of degree j_1 in $\boldsymbol{L}$; in particular, $h_{1,0}^{(\mathrm{Kep})} = \boldsymbol{n}^* \cdot \boldsymbol{L}$, where the components of the angular velocity vector $\boldsymbol{n}^*$ are defined by the third Kepler law;
- $h_{s;j_1,j_2}^{(\mathcal{P})}$ is a homogeneous polynomial of degree j_1 in $\boldsymbol{L}$, degree j_2 in $(\boldsymbol{\xi}, \boldsymbol{\eta})$ and with coefficients that are trigonometric polynomials in $\boldsymbol{\lambda}$ and are related to the term D_2^s.

Clearly, in the applications we deal with finite expansions; the truncation parameters will be discussed in the following.

The expression of the Hamiltonian of the three-body problem in (71) highlights the distinction between the so called *fast variables* $(\boldsymbol{L}, \boldsymbol{\lambda})$ and the *secular variables* $(\boldsymbol{\xi}, \boldsymbol{\eta})$. Indeed, if we consider the corresponding Hamilton equations, we have that $\dot{\boldsymbol{\lambda}} = \mathcal{O}(1)$. This means that the motion of the planet along the orbit, that is in first approximation a Keplerian ellipse, has a different timescale with respect to the secular variables, whose variation is due to the interaction between the planets and, therefore, is of $\mathcal{O}(\mu)$. Since we are interested in the study of the long-time stability of the system, a common procedure consists on considering just the evolution of the secular variables, by averaging the Hamiltonian with respect to the fast angles $\boldsymbol{\lambda}$. With a simple average of $H_{3\mathrm{BP}}$ we would obtain a secular approximation with terms of order μ, namely at order 1 in the masses. Here, we consider terms up to order 2 in the masses, averaging with a close to the identity canonical change of coordinates inspired by the algorithm for the construction of the Kolmogorov normal form. Indeed, we focus on the torus corresponding to $\boldsymbol{L} = 0$. The first transformation of coordinates that we define aims at removing the perturbative terms that depend on the angles $\boldsymbol{\lambda}$ but do not depend on the actions $\boldsymbol{L}$, being $\dot{L}_j = \partial H/\partial \lambda_j$ for $j = 1, 2$. This is done by using the term linear in the actions, i.e., $\boldsymbol{n}^* \cdot \boldsymbol{L}$, to define a generating function $\chi_1^{(\mathcal{O}2)}(\boldsymbol{\lambda})$ as the solution of the following homological equation:

$$\left\{\chi_1^{(\mathcal{O}2)},\ \boldsymbol{n}^* \cdot \boldsymbol{L}\right\} + \mu \sum_{\substack{s=0,\ j_2=0 \\ 2s+j_2 \le N_S}} \left\lceil D_2^s\, h_{s;0,j_2}^{(\mathcal{P})} \right\rceil_{\boldsymbol{\lambda}:K_F} = \mu \sum_{\substack{s=0,\ j_2=0 \\ 2s+j_2 \le N_S}} D_2^s \left\langle h_{s;0,j_2}^{(\mathcal{P})} \right\rangle_{\boldsymbol{\lambda}}, \qquad (72)$$

being $\langle \cdot \rangle_{\boldsymbol{\lambda}}$ the average with respect to the angles $\boldsymbol{\lambda}$, while with the notation $\lceil \cdot \rceil_{K_F}$ we mean that the expansions are truncated at the trigonometrical degree K_F in the angles $\boldsymbol{\lambda}$. Let us add a few comments about the truncations parameters K_F and N_S. The value of K_F is defined so as to take into account the main mean-motion quasi-resonances of the system considered. For example, if the system is close to the resonance $k_1^* : k_2^*$, then K_F is defined as $K_F \ge |k_1^*| + |k_2^*|$. In the same spirit, the value N_S of the truncation of the expansions in eccentricity and inclination is set in order to consider the quasi-resonance. Let us assume that the quasi-resonant angular terms are of type $(k_1^* \lambda_1 - k_2^* \lambda_2)$, then in principle it would be convenient to consider expansions up to an order in eccentricity and inclination such that $N_S \ge 2(|k_1^*| - |k_2^*|)$, because of the D'Alembert rules (see [24]). Therefore, in the specific case of the extrasolar system HD 4732, it is rather natural to set $K_F = 9$, because the periods of the two planets are about 0.986 yr and 7.48 yr, respectively. However, since the ratio of the angular velocities n_1^*/n_2^* is not so close to the resonance 7 : 1 or to 8 : 1 and the terms of high degree in eccentricities are not so relevant, we have found convenient to limit our expansions to $N_S = 8$, in order to reduce the computational cost of the whole procedure.

Now we have to apply the transformation of coordinates defined by the application of the Lie series operator $\exp(\mathcal{L}_{\chi_1^{(\mathcal{O}2)}}) \cdot = \sum_{j=0}^{\infty}(1/j!)\mathcal{L}^j_{\chi_1^{(\mathcal{O}2)}} \cdot$ to the Hamiltonian. Recalling that in our secular model we will not consider terms depending on $\boldsymbol{L}$ or of order greater than μ^2, the only terms we need to compute are included in the following expansion:

$$\begin{aligned}\widetilde{H} = H_{3\text{BP}} + \frac{1}{2}\left\{\chi_1^{(\mathcal{O}2)},\ \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} h_{2,0}^{(\text{Kep})}\right\}_{\boldsymbol{L},\boldsymbol{\lambda}} \\ +\mu \sum_{\substack{s\geq 0\,,\ j_2\geq 0 \\ 2s+j_2\leq N_S}} D_2^s \left\{\chi_1^{(\mathcal{O}2)}, h_{s;1,j_2}^{(\mathcal{P})}\right\}_{\boldsymbol{L},\boldsymbol{\lambda}} + \frac{\mu}{2} \sum_{\substack{s\geq 0\,,\ j_2\geq 0 \\ 2s+j_2\leq N_S}} D_2^s \left\{\chi_1^{(\mathcal{O}2)}, h_{s;0,j_2}^{(\mathcal{P})}\right\}_{\boldsymbol{\xi},\boldsymbol{\eta}}\ ,\end{aligned} \tag{73}$$

where $\{\cdot,\cdot\}_{\boldsymbol{L},\boldsymbol{\lambda}}$ and $\{\cdot,\cdot\}_{\boldsymbol{\xi},\boldsymbol{\eta}}$ are the terms of the Poisson bracket involving only the derivatives with respect to the pairs of conjugate variables $(\boldsymbol{L},\boldsymbol{\lambda})$ and $(\boldsymbol{\xi},\boldsymbol{\eta})$, respectively. Then, according to [27], we have that

$$\langle H^{(\mathcal{O}2)}\rangle_{\boldsymbol{\lambda}}\Big|_{\boldsymbol{L}=\boldsymbol{0}} = \langle \widetilde{H}\,\rangle_{\boldsymbol{\lambda}}\Big|_{\boldsymbol{L}=\boldsymbol{0}} + \mathcal{O}(\mu^3)\,,$$

being $H^{(\mathcal{O}2)} = \exp(\mathcal{L}_{\chi_1^{(\mathcal{O}2)}})H_{3\text{BP}}$. Let us remark that for the definition of this model it is not necessary to compute the effects induced by the second generating function $\chi_2^{(\mathcal{O}_2)}(\boldsymbol{L},\boldsymbol{\lambda})$ for removing terms linear in $\boldsymbol{L}$, because the additional terms due to the application of such a Lie series operator are neglected in the secular approximation.

We can finally introduce our secular model up to order 2 in the masses by setting

$$H^{(\text{sec})}(D_2,\boldsymbol{\xi},\boldsymbol{\eta}) = \left\lceil\ \langle \widetilde{H}\,\rangle_{\boldsymbol{\lambda}}\Big|_{\boldsymbol{L}=\boldsymbol{0}}\ \right\rceil_{N_S}\ , \tag{74}$$

i.e., we take the averaged expansion (over the fast angles $\boldsymbol{\lambda}$) of the part of $\widetilde{H}$ that is both independent from the actions $\boldsymbol{L}$ and truncated up to a total order of magnitude N_S in eccentricity and inclination. Since D_2 is $\mathcal{O}\big(e_1^2+\mathrm{i}_1^2+e_2^2+\mathrm{i}_2^2\big)$, this means that we keep the Hamiltonian terms $h_{s;0,j_2}^{(\mathcal{P})}$ with $2s+j_2\leq N_S$. From now on, the parameter D_2 is replaced by its explicit value that is calculated as a function of the initial conditions; thus, we can write the Hamiltonian as follows:

$$H^{(\text{sec})}(\boldsymbol{\xi},\boldsymbol{\eta}) = \sum_{s=1}^{N_S/2} h_{2s}^{(\text{sec})}(\boldsymbol{\xi},\boldsymbol{\eta})\ , \tag{75}$$

where h_{2s} is an homogeneous polynomial of degree $2s$. This means that the expansion contains just terms of even degree, as a further consequence of the well known

D'Alembert rules. To fix the ideas, in the case of the extrasolar system HD 4732 let us emphasize that our secular model at order two in the masses is defined by a Hamiltonian $H^{(\mathrm{sec})}$ that is a simple (even) polynomial of maximal degree 8 in the four canonical variables $(\boldsymbol{\xi}, \boldsymbol{\eta})$.

We have explicitly performed all the computations of Poisson brackets (required by Lie series formalism to express canonical transformations) and all the expansions described in the present subsection and in in the next one, by using $X\varrho\acute{o}\nu o\varsigma$. It is a software package especially designed for doing computer algebra manipulations into the framework of Hamiltonian perturbation theory (see [21] for an introduction to its main concepts).

4.2 Semi-analytic Computations of Invariant Tori

In the framework of Hamiltonian theory for dynamical systems, often intuition can be fruitfully helped by numerical investigations. In particular, in the case of the extrasolar system HD 4732, they allow to easily motivate the new approach that is based on normal forms and we are going to describe. In the present section, we will discuss some results provided by direct numerical integrations of the secular model $H^{(\mathrm{sec})}$ that is defined in (75); all of them have been produced by simply applying the RK4 method.

A few dynamical features of the Hamiltonian model defined by $H^{(\mathrm{sec})}$ are summarized in the plots reported in Fig. 2. They refer, as an example, to the initial conditions corresponding to the set of values $\mathcal{I}_{4^\circ}$, defined in (68). The difference of the arguments of the pericenters $\omega_2 - \omega_1$ is plotted in the bottom-right panel of such

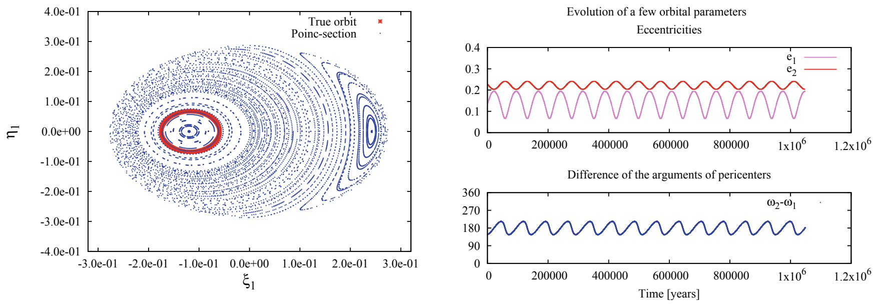

Fig. 2 On the left, Poincaré sections that are corresponding to the hyperplane $\eta_2 = 0$ (with the additional condition $\xi_2 > 0$) and are generated by the flow of the Hamiltonian secular model $H^{(\mathrm{sec})}$, which is given in (75) at order two in the masses for the exoplanetary system HD 4732; the orbit in red refers to the motion starting from the initial conditions corresponding to the set $\mathcal{I}_{4^\circ}$, that is described in (68). On the right, evolution of secular orbital elements: the eccentricities of both the exoplanets (that are e_1 and e_2) and the difference of the arguments of the pericenters (i.e., $\omega_2 - \omega_1$) are plotted on top and bottom, respectively

a figure; then, we can easily appreciate that this angle is librating around 180°. By taking into account of the fact that the nodes are opposite in the Laplace frame, this means that the pericenters of HD 4732b and HD 4732c are in the so called "apsidal locking" regime in the vicinity of the alignment of the pericenters. This phenomenon is expected to play a major role in making stable the orbits for systems where the Keplerian part of the Hamiltonian is strongly affected by the interactions between planets (see, e.g., [30] or [9]). The Poincaré sections of the motions starting from the initial conditions corresponding to $\mathcal{I}_{4^\circ}$ are plotted in red in the panel on the left of Fig. 2 and it is easy to remark that they are orbiting around a fixed point. Moreover, it looks rather close to those sections marked in red, when their distance from such a fixed point is compared with that from the orbits that are enclosing another fixed point. Let us recall that all the Poincaré sections reported in Fig. 2 refer to the same level of energy, say E, corresponding to the set of initial conditions $\mathcal{I}_{4^\circ}$. Since $H^{(\mathrm{sec})}$ is a two degrees of freedom Hamiltonian, the manifold labeled by such a value of the energy will be three-dimensional; in other words, by plotting the Poincaré sections, we automatically reduce by one the dimensions of the orbits. This is the reason why a fixed point actually corresponds to a periodic orbit. Since the fixed point with negative value of the abscissa is surrounded by closed curves, then we can argue that such a periodic orbit is linearly stable for what concerns the transverse dynamics. This means that it is a one-dimensional elliptic torus, in the terminology we have adopted in the present work. Therefore, we can conclude that the orbit generated by the set $\mathcal{I}_{4^\circ}$ of initial conditions is winding around a linearly stable periodic orbit, by remaining in its vicinity. This explains why we are going to adopt a strategy based on two different algorithms: the first one refers to the elliptic torus (that corresponds to a fixed point in the Poincaré sections) and provides a good enough approximation to start the second computational procedure that constructs the final KAM torus (which shall include also the points marked in red in Fig. 2).

Explicit Construction of the Normal Form for Elliptic Tori in the Case of the Secular Model Representing the Planetary System HD 4732

The discussion above has highlighted that it is convenient to adopt a suitable set of coordinates including also a resonant angle, that is the difference of the arguments of the pericenters. In view of such a target, we first introduce the set of action-angle variables $(\boldsymbol{\mathcal{J}}, \boldsymbol{\psi})$ via the canonical transformation

$$\xi_j = \sqrt{2\mathcal{J}_j}\cos\psi_j\ , \qquad \eta_j = \sqrt{2\mathcal{J}_j}\sin\psi_j\ , \qquad \forall\ j = 1, 2, \tag{76}$$

being $(\boldsymbol{\xi}, \boldsymbol{\eta})$ the variables appearing as arguments of the secular Hamiltonian $H^{(\mathrm{sec})}$ defined in (75). It is important to recall that the angles (ψ_1, ψ_2) associated to these secular variables are nearly equal to the arguments of the pericenters (ω_1, ω_2), apart from a small correction due to the transformation of coordinates induced by the application of the Lie series $\exp\mathcal{L}_{\chi_1^{(\mathcal{O}2)}}$ to the Hamiltonian of the three-body planetary problem. Then, it is convenient to introduce a new set of variables $(\boldsymbol{I}, \boldsymbol{\vartheta})$ such that

$$\vartheta_1 = \psi_1 - \psi_2\ , \quad \vartheta_2 = \psi_2\ , \quad I_1 = \mathcal{J}_1\ , \quad I_2 = \mathcal{J}_2 + \mathcal{J}_1\ . \tag{77}$$

We now introduce the new canonical polynomial variables $(\boldsymbol{x}, \boldsymbol{y})$ defined as

$$x_j = \sqrt{2I_j}\cos\vartheta_j\ , \qquad y_j = \sqrt{2I_j}\sin\vartheta_j\ , \qquad \forall\ j = 1, 2\ . \tag{78}$$

Let us also remark that making Poincaré sections with respect to the hyperplane $\eta_2 = 0$, when $\xi_2 > 0$ is equivalent to impose $\psi_2 = 0$, because of the definitions in (76). Therefore, looking at formulæ (77)–(78), one can easily realize that the drawing in the left panel of Fig. 2 can be seen as a plot of the Poincaré sections in coordinates $(x_1\ , \ y_1)$ with respect to $y_2 = 0$ and with the additional condition $x_2 > 0$. Revisiting the plot in the bottom–right box of Fig. 2 in the context of the new canonical variables is interesting, because it makes clear that ϑ_1 is librating around 180°. In fact, we have that $\vartheta_1 = \psi_1 - \psi_2 \simeq \omega_1 - \omega_2$, because the relation between these differences of angles is given by the transformation induced by the application of the Lie series $\exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}}$, that is close to the identity.

By a numerical method,[28] we can easily determine the initial condition $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ that is in correspondence with a Poincaré section and generates a periodic solution. We can now subdivide the variables in two different couples. The first one is given by $(p, q) \in \mathbb{R} \times \mathbb{T}$, i.e., the action-angle couple describing the periodic motion. Thus, we rename the angle φ_2 as q, while the action is obtained by translating the origin of I_2 so that $p = I_2 - I^\star$, where at the first trial[29] the shift value $I^\star$ is fixed so that $I^\star = \big((x_2^\star)^2 + (y_2^\star)^2\big)/2$. For what concerns the second couple of canonical coordinates, we start from the polynomial variables (x_1, y_1) in order to describe the motion transverse to the periodic orbit. The last preliminary translation is on x_1, in order to have expansions around the value $x_1^\star$, given by the initial condition computed numerically. Let us emphasize that, since the fixed point we are trying to approximate in Fig. 2 corresponds to $\varphi_1 = 180°$, we have that $y_1^\star = 0$ and here a translation is not needed. It is now convenient to rescale the transverse variables $(\bar{x}_1, y_1)$, being $\bar{x}_1 = x_1 - x_1^\star$, in such a way that the Hamiltonian part which is quadratic in the new variables (x, y) and

[28] Let us imagine to start from an initial condition denoted by $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ that is close enough to the periodic orbit generated by the wanted solution $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$; typically, at the beginning one can put $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ equal to the values assumed by the canonical variables $(\boldsymbol{x}, \boldsymbol{y})$ in correspondence with the set $\mathcal{I}_{i_{\rm mut}(0)}$, defined in (68). During a long enough numerical integration of the Hamilton equations related to $H^{(\rm sec)}$, one can easily determine $\hat{x}_{1,-}$ and $\hat{x}_{1,+}$ that are the minimum value assumed by the variable x_1 in correspondence with the Poincaré sections and the maximum one, resp. If the difference $\hat{x}_{1,+} - \hat{x}_{1,-}$ is below a prescribed (small) threshold of tolerance, then we assume to know the solution with a good enough level of approximation and we stop this computational procedure by setting $(\boldsymbol{x}^\star, \boldsymbol{y}^\star) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. If such a "way out condition" is not satisfied, then we define $x_1^\star = (\hat{x}_{1,+} + \hat{x}_{1,-})/2$, $y_1^\star = 0$, $y_2^\star = 0$ and we determine the positive value of $x_2^\star$ so that the energy level of this new approximation of the final solution, i.e., $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$, is still equal to the value E corresponding to the set $\mathcal{I}_{i_{\rm mut}(0)}$. Let us remark that in the (re)definition of $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ we are exploiting both the definition of the Poincaré sections and their symmetry with respect to the axis of the abscissas. At this point, we put $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = (\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ and we restart the computational procedure by performing another numerical integration so to determine new values of $\hat{x}_{1,-}$ and $\hat{x}_{1,+}$ and so on, until the "way out condition" will be satisfied.

[29] See the discussion about the solution of the implicit equation (79) by using the Newton method, which is reported at the end of these explanations.

does not depend on (p, q) is in the form $\Omega^{(0)}(x^2 + y^2)/2$. This rescaling can be done by a canonical transformation as the quadratic part does not have any mixed term $\bar{x}_1 y_1$ and the coefficients of $\bar{x}_1^2$ and y_1^2 have the same sign, because of the proximity to an elliptic equilibrium point. Thus, since such a quadratic part is in the preliminary form $a\bar{x}_1^2 + by_1^2$, it suffices to define the new variables (x, y) as $x = \sqrt[4]{\frac{a}{b}}\,\bar{x}_1,\ y = \sqrt[4]{\frac{b}{a}}\,y_1$. Finally, we introduce the second pair of canonical coordinates $(J, \varphi) \in \mathbb{R}_+ \cup \{0\} \times \mathbb{T}$ so that $x = \sqrt{2J}\cos\varphi$ and $y = \sqrt{2J}\sin\varphi$.

In the case of the secular dynamics of the planetary system HD 4732, starting from $H^{(\mathrm{sec})}$ in (75), we have applied all the canonical transformations listed above and we have expanded the Hamiltonian $\mathcal{H}^{(0)}(p, q, J, \varphi)$ up to degree 16 in the square roots of the actions (p, J). Since $\mathcal{H}^{(0)}(p, q, J, \varphi)$ is in a suitable form to apply the algorithm fully described in Sect. 3.1 in the case with $n_1 = n_2 = 1$ (this is the reason why all the variables (p, q, J, φ) are here denoted as scalar quantities instead of vectorial ones), we have applied such a computational procedure. We have performed 19 steps of the normalization algorithm so producing $\mathcal{H}^{(19)}(p, q, J, \varphi)$. During those computations, the Fourier expansions in q of all the Hamiltonians defined by the algorithm have been truncated at a maximal trigonometric degree equal to 40; since $K = 2$, this choice allows to properly determine the generating functions for the first 20 normalization steps. For the sake of brevity, we omit to report the graphs of the norms of all the generating functions that are defined by the normalization procedure, also because those plots are similar to the corresponding ones included in [8, 9]. Indeed, they show that the convergence to the identity of the canonical transformations defined at the r-th step of the algorithm is very fast with respect to r. This fact also allows to iterate a few times all the normalization procedure constructing the normal form for an elliptic torus with a computational cost which is not too expensive. We are interested in doing that in order to refine the choice of the initial shift value $I^\star$. Since all other canonical transformations are unambiguously defined, we have some remaining arbitrariness just on the translation $p = I_2 - I^\star$. We finally determine $I^\star$ in such a way that

$$\mathcal{E}^{(19)}\big(I^\star\big) = E\ , \tag{79}$$

where E is the energy level of the Poincaré sections and $\mathcal{E}^{(19)}\big(I^\star\big)$ is the energy of the elliptic torus in the approximation provided after 19 steps of normalization. The implicit equation above can be numerically solved in the unknown $I^\star$ by iterating a few times the Newton method; this is done starting from the initial guess $\big((x_2^\star)^2 + (y_2^\star)^2\big)/2$, according with the discussion above.

For brevity, we omit also the tests showing that there is an excellent agreement between the wanted periodic orbit and the nearly invariant curve, which is provided by the last execution of the normalization algorithm, that is launched during the final iteration of the Newton method targeting the solution of (79). Actually, it corresponds to the counter-image of the set $(p = 0,\ q \in \mathbb{T},\ J = 0,\ \varphi = 0)$ and is expressed in the coordinates $(\boldsymbol{\xi}, \boldsymbol{\eta})$, after having composed all the previous canonical transformations.

Explicit Construction of the Normal Form for KAM Tori in the Case of the Secular Model Representing the Planetary System HD 4732

Since the Hamiltonian $\mathcal{H}^{(19)}(p, q, J, \varphi)$ is very close to the normal form related to the wanted elliptic torus, we use it as the starting point to construct a semi-analytic solution that should provide a good approximation of the orbits generated by the initial conditions corresponding to the set $\mathcal{I}_{4^\circ}$. For such a purpose, first we translate once again the coordinates. This is made in such a way that the new invariant torus we are going to construct will be located in the proximity of these initial conditions; therefore, we define two new pairs of action-angle coordinates $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^2 \times \mathbb{T}^2$. It is convenient to set $p_2 = J - J^\star$, being $J^\star$ the value of the momentum J computed in correspondence with the initial conditions related to the set $\mathcal{I}_{4^\circ}$, that generate the Poincaré sections marked in red in Fig. 2. We also introduce $p_1 = p - p^\star$, with $p^\star = -\big(\Omega^{(19)}/\omega^{(19)}\big)J^\star$, being $2\pi/\omega^{(19)}$ approximately equal to the period of the motion on the previously determined one-dimensional elliptic torus, while the angular velocity of the transverse (small) oscillations in its vicinity is close to $\omega^{(19)}$. We recall that the values of both $\omega^{(19)}$ and $\Omega^{(19)}$ appear in the expansion (59) of the Hamiltonian $\mathcal{H}^{(19)}$, that is provided at the end of the previous normalization algorithm. Moreover, we rename the angles (q, φ) as (q_1, q_2), respectively; then, we perform the two translations described just above, by expanding the new Hamiltonian $H^{(0)}(\boldsymbol{p}, \boldsymbol{q})$ up to degree 8 in the actions $\boldsymbol{p}$. By considering just the integrable approximations of $\mathcal{H}^{(19)}$ and $H^{(0)}$ (this means that the terms depending by the angles are temporarily neglected), one can easily realize that the energy constant $E^{(0)}$ corresponding to the new Hamiltonian is such that $E^{(0)} \simeq E$, because of the equation $\omega^{(19)}p^\star + \Omega^{(19)}J^\star = 0$ that is due to the definitions of the shift values $\big(p^\star, J^\star\big)$. Since $H^{(0)}(\boldsymbol{p}, \boldsymbol{q})$ is in a suitable form to apply the algorithm fully described in Sect. 2.3, we have performed 19 steps of such a computational procedure too, so producing $H^{(19)}(\boldsymbol{p}, \boldsymbol{q})$. During these computations, the Fourier expansions in q of all the Hamiltonians defined by the normalization algorithm have been truncated at a maximal trigonometric degree equal to 40. This choice allows to properly determine the generating functions $\chi_1^{(r)}$ and $\chi_2^{(r)}$ for the first 20 normalization steps.

It is convenient to define the norms of the generating functions as the sum of the absolute values of the coefficients appearing in their (finite) Taylor-Fourier expansions. In the left panel of Fig. 3, we report the plot of $\big\|\chi_2^{(r)}\big\|$ in a semi-log scale and as a function of the normalization step r, while we have decided to not include also $\big\|\chi_1^{(r)}\big\|$, because for every r it is definitely smaller than $\big\|\chi_2^{(r)}\big\|$. One can appreciate that the geometrical decrease of the generating functions is very sharp and regular; therefore, this shows that the normalization algorithm constructing the Kolmogorov normal form is convergent in a quite rapid way.

We can now check the quality of our results. Let us denote with $\mathcal{C}$ the canonical transformation we obtain by composing all the changes of coordinates we have discussed in the present Sect. 4.2. Therefore, we have that $(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathcal{C}(\boldsymbol{p}, \boldsymbol{q})$, where $(\boldsymbol{\xi}, \boldsymbol{\eta})$ are the canonical coordinates referring to the Hamiltonian secular model $H^{(\mathrm{sec})}$, that is defined in (75), while $(\boldsymbol{p}, \boldsymbol{q})$ are the action-angle variables that are introduced at the end of the previously described computational procedure. Inspired

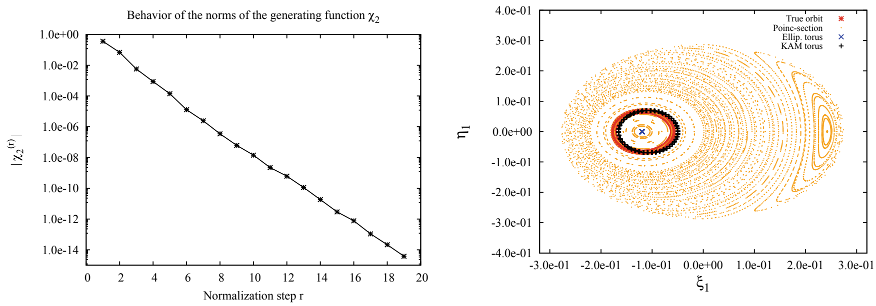

Fig. 3 On the left, study of the decrease of $\|\chi_2^{(r)}\|$ as a function of the normalization step r. On the right, comparisons between the Poincaré sections generated by two different initial conditions, that correspond to the set $\mathcal{I}_{4^\circ}$ and a point on the (nearly) invariant torus $\boldsymbol{p} = \boldsymbol{0}$ related to the Hamiltonian $H^{(19)}$, respectively. The former ones are marked in red as in the left panel of Fig. 2, while the latter ones are in black. The Poincaré sections are defined in the same way as those reported in Fig. 2; in particular, the dots plotted in blue there are located exactly in the same positions as those marked in orange here. The blue symbol $\times$ refers to the motion on the elliptic torus corresponding to the Hamiltonian $\mathcal{H}^{(19)}$

by the semi-analytic scheme (5), which provides a way to integrate the Hamilton equations, we start by computing $(\boldsymbol{\xi}(0), \boldsymbol{\eta}(0)) = \mathcal{C}(\boldsymbol{0}, \boldsymbol{0})$. Since $H^{(19)}$ is very close to be in Kolmogorov normal form and $H^{(\mathrm{sec})} \simeq H^{(19)}\big(\mathcal{C}(\boldsymbol{p}, \boldsymbol{q})\big)$ (the discrepancies are mainly due to the unavoidable truncations that are made on the expansions of the Hamiltonians), then $\big(\boldsymbol{\xi}(t), \boldsymbol{\eta}(t)\big) = \big(\mathcal{C}(\boldsymbol{0}, \boldsymbol{\omega}^{(19)}t)\big)$ provides a good approximation of the flow induced by $H^{(\mathrm{sec})}$. We also recall that the values of the angular velocity vector $\boldsymbol{\omega}^{(19)}$ appear in the expansion (21) of the Hamiltonian $H^{(19)}$. Computing the Poincaré sections of the motion law $\big(\mathcal{C}(\boldsymbol{0}, \boldsymbol{\omega}^{(19)}t)\big)$ is not very comfortable; therefore, it is convenient to refer to its approximation which is given by the numerical solution of the Hamilton equations for $H^{(\mathrm{sec})}$ starting form the initial conditions $(\boldsymbol{\xi}(0), \boldsymbol{\eta}(0)) = \mathcal{C}(\boldsymbol{0}, \boldsymbol{0})$. The Poincaré sections we have obtained in this way are plotted in black on the right panel of Fig. 3. They are in good agreement with the the Poincaré sections marked in red in both Figs. 2 and 3, that refer again to the flow induced by $H^{(\mathrm{sec})}$, but starting from the initial conditions related to the set $\mathcal{I}_{4^\circ}$. This confirms that we are able to obtain reliable approximations of the secular motions for extrasolar planetary systems, by using computational procedures based on the construction of suitable (Kolmogorov-like) normal forms.

Final Comments About Our Semi-analytic Results

Looking closely at the right panel of Fig. 3, one can observe that the Poincaré sections plotted in black goes from the part internal to the orbit in red to the external one and vice versa. This provides a clear indication that the energy level of the final KAM torus (that is $\simeq E^{(19)}$) is not very close to that of all the Poincaré sections plotted in Fig. 2 (being E its value). Indeed, the relative error $\big|E^{(19)} - E\big|/|E|$ is about 12 %. The agreement between the results produced by the purely numerical integrations or by adopting our semi-analytical approach can be strongly improved by a suitable further refinement of our computational procedure. The description of such an extension goes beyond the scopes of the present work, but we stress that it can be done so as to ensure also that the condition on the coherence with the energy of the Poincaré sections, i.e.,

$$E^{(R_{\rm I})} = E \ , \tag{80}$$

is satisfied within a tolerance range that is acceptable for a numerical solution of the equation above, where $R_{\rm I}$ is the number of steps that are explicitly performed in order to construct the final Kolmogorov normal form. Here, we limit ourselves to anticipate some of the results that can be obtained by implementing that further refinement, in order to let the reader appreciate the power of this kind of methods. For what concerns the planetary system HD 4732 we already have studied the motions starting from the following sets of initial conditions: $\mathcal{I}_{2^\circ}$, $\mathcal{I}_{4^\circ}$, $\mathcal{I}_{6^\circ}$, ... $\mathcal{I}_{40^\circ}$. We can construct invariant KAM tori well approximating the orbits for all these cases, except those corresponding to the sets $\mathcal{I}_{32^\circ}$ and $\mathcal{I}_{34^\circ}$. We emphasize that these limitations are due to real dynamical phenomena. The Poincaré sections generated by those initial conditions clearly shows that between 34° and 36° there is the transition from the librations to the circulation regime, for what concerns the difference of the argument of the pericenters. Moreover, this kind of orbits are observed in stable situations up to initial values of the mutual inclinations that are about 40°, while for even larger angles there are robust configurations just inside the Lidov-Kozai resonance, which has different dynamical features (see [42]). As we have already mentioned above, we plan to describe these new results in a forthcoming work.

The evolution of the eccentricities plotted in the right panel of Fig. 2 clearly shows that their average value is larger than 0.1 for both the exoplanets orbiting around HD 4732. Therefore, the new approach that we have introduced in the present work behaves definitely better with respect to the previous one, which was described in [41] and was shown to be successful just for systems with exoplanetary eccentricities smaller than 0.1 . In our opinion the main source of improvement is due to the new strategy, because it combines the preliminary construction of the normal form for a suitable elliptic torus with the final one, which is performed in its vicinity for a KAM torus whose shape is a good approximation of the secular orbits. In order to mention another relevant success of our new approach, let us stress that in [9] we applied it also to the delicate case of a system including both the two largest exoplanets orbiting around υ Andromedæ A and the star itself.

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