Valuing Option Under Double Heston Jump-Diffusion Model with Stochastic Interest Rate and Approximative Fractional Brownian Motion

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Abstract. In the light of the current research, we propose a more general and realistic model based on approximative fractional Brownian motion studies. This framework presents an option pricing model under the double Heston Jump-Diffusion model, including approximative fractional motion with stochastic interest rate and stochastic intensity. The stochastic interest rate is determined using a two-factor Vasicek model. The negative interest rate is allowed for this model. Therefore, we are constructing a multi-factor model with a stochastic interest rate structure. We derive a closed-form pricing formula with an analytical solution for European options. Finally, some numerical results are presented to illustrate the value of a European call option comparing to other classical models.

1 Introduction

In 1997 Black & Scholes [4] published a groundbreaking paper in which they proposed an elegant model focused on Brownian motion to explain the complexities of the underlying asset price and presented a closed-form formula for European options. According to Duan and Wei [7], the Black-Scholes model cannot explain the phenomena of the asymmetric leptokurtic and also the volatility smile that is observed in the real market. Since That point, academic researchers have created different models by joining in the Black-Scholes model the non-constant volatility . The Scott [19] model, Hull and White [13] model, the Stein and Stein [22] model and the Wiggins [26] model. However, the majority of these stochastic volatility models are unsuitable for use. In 1993 Heston [11] describe the variance (the square of volatility) by Cox-Ingersoll-Ross process [5] and deriving a closed-form formula for European options.

On the other side, a single factor model cannot describe the shapes of the volatility smile with precision. Multi-factor stochastic volatility models are useful for expressing return data in various ways, such as using a stylized effect or

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fitting the implied surface. We choose to investigate option pricing under two-factor stochastic volatility in this study since it is more appropriate for practical applications.

Otherwise, the financial market owns long-range persistence and self-similarity traits, and fractional Brownian motion has these two essential properties. Moreover, fractional Brownian motion is not a Markov process or semi-martingale; the classical Ito calculus cannot be used in this case. Wick products have been created by Hu and Oksendal [12] for analyzing it. In addition, Xiao and Al [27] used the Wick products to define a fractional stochastic integral. Björk and Hult [3] demonstrated that the model lacks an economic interpretation. To solve this problem is appropriate to use the mixed fractional Brownian motion [8,17,23,28]. Approximation Fractional Brownian motion [24] can also be used instead of fractional Brownian motion. Thao [24] showed that Approximation Fractional Brownian motion is a semi-martingale. Furthermore, many researchers (see [6]) adopted Approximation Fractional Brownian motion in building stochastic volatility models.

Many authors have worked on a hybrid model in recent years by incorporating the stochastic interest rate into stochastic models [9,10,14,21]. In addition, empirical studies show that using stochastic interest rates into option pricing models will contribute to improved model results [18].

Roughly speaking, permitting for changes in volatility and interest rate and the presence of jumps and the jump intensity changing over time indicate realistic asset return dynamics. In a parallel development, incorporating jump into models for pricing option also proposes describing the discontinuous behavior of the underlying asset (see [1,2,15,16,20]).

The rest of the paper is organized as follows. We adopt the double-Heston jump-diffusion (DHJD) model with approximative fractional Brownian motion, stochastic intensity, and interest rate follow a two-factor model in Sect. 2. In Sect. 3, we derive analytical pricing formula for European call option. In Sect. 4, we present some numerical illustrations. Finally, we conclude in Sect. 5.

2 The Model

We present some basic information on approximative fractional Brownian motion. At the first, we present an analysis of fractional Brownian motion $(B_t^H)_{t\geq 0}$ with the Hurst index $H \in (0,1)$. It is a Gaussian process with zero mean and the following covariance:

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\bigg(|t|^{2H} + |s|^{2H} - |Tt - s|^{2H}\bigg). \tag{1}$$

The decomposition of a fractional Brownian motion B is as follows:

$$B_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\bigg[Z_t + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s\bigg] \tag{2}$$

where

$$Z_t = \int_{-\infty}^{0} \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dW_s, \tag{3}$$

W_t indicates standard Brownian motion, and Γ indicates the gamma function. It is sufficient to focus exclusively on the term:

$$B_t = \int_0^t (t-s)^{H-\frac{1}{2}} \tag{4}$$

that has a long-range memory. Note that The approximation of B_t is $\tilde{B}_t^{\epsilon,H}$ which can be expressed as [26]

$$\tilde{B}_t^{\varepsilon,H} = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s \tag{5}$$

where H is a long-memory parameter, ε is non negative approximation factor. Thao [24] proved that for $\varepsilon \to 0$, $(B_t^{\varepsilon,t})_\varepsilon$ converges uniformly to a non-Markov process. In addition, if $\varepsilon > 0$ then $B_t^{\varepsilon,t}$ is a semi-martingale [24]

$$d\tilde{B}_t^{\varepsilon,H} = (H - \frac{1}{2})\psi_t dt + \varepsilon^{H-\frac{1}{2}} dW_t^v \tag{6}$$

ψ_t is a stochastic processes expressed as

$$\psi_t = \int_0^t (t-s+\varepsilon)^{H-\frac{3}{2}} dW_s^\psi, \tag{7}$$

where$(W_t^\psi)_{t\in[0,T]}$ and $(W_t^v)_{t\in[0,T]}$, are independent standard Brownian motions.

Let$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ be a complete probability space with a filtration and $\mathbb{Q}$ presents a risk-neutral measure.The stock price S_t is expressed by the following dynamic system:

$$\begin{cases} \frac{dS_t}{S_t} = (r_1 + r_2 - \lambda_t \mu_J)\mathrm{dt} + \sqrt{v_t} dW_t^s + \sqrt{\hat{v}} d\hat{W}_t^s + (J-1)dN_t \\ dv_t = k_v(\theta - v_t)dt + \sigma_v \sqrt{v_t} d\tilde{B}_t^{\epsilon,H} \\ d\hat{v}_t = \hat{k}(\hat{\theta} - \hat{v})dt + \sigma_{\hat{v}} \sqrt{\hat{v}} dW_t^{\hat{v}} \\ d\lambda_t = k_\lambda(\theta_\lambda - \lambda_1)dt + \sigma_\lambda dW_t^\lambda \\ dr_1 = \alpha_1(\beta_1 - r_1)dt + \sigma_1 dW_t^{r_1} \\ dr_2 = \alpha_2(\beta_2 - r_1)dt + \sigma_2 dW_t^{r_2} \end{cases} \tag{8}$$

where $W_1^s, \hat{W}_t^s, W_t^v, W_t^{r_1}, W_t^{r_2}$ and W_t^λ are the standard Brownian motions. We assume that W_t^s is correlated with W_t^v, $dW_t^s.dW_t^v = \rho_1 dt$,$\hat{W}_t^s$ correlated with $W_t^{\hat{v}}$, $d\hat{W}_t^s dW_t^{\hat{v}} = \rho_2 dt$ and $W_t^{r_1}$ correlated with $W_t^{r_2}$, $dW_t^{r_1}.dW_t^{r_2} = \rho_r dt$. Any other Brownian motions are pairwise independent.

$v_t, \hat{v}_t$ are variances, and λ_t is the jump intensity. $k, \hat{k}$ and k_λ are mean reversion rates, θ, $\hat{\theta}$ and θ_λ are mean reversion levels, σ_v, $\sigma_{\hat{v}}$ and σ_λ are the volatilities of the variances. and the short rate is follow two-factor Vasicek model where the short rate is given as a sum of two factors r_1 and r_2, where β_1, β_2 are their mean-reversion , α_1, α_2 are theire mean-reversion speed, σ_1, σ_2 are their volatilities, N_t represents Poisson process with intensity λ_t and J represents the jump size, and we suppose that lnJ has an asymmetric double exponential distribution with density function $pdf_u(z)$:

$$pdf_u(z) = p\eta_1 e^{\eta_1 z} 1_{\geq 0} + q\eta_2 e^{\eta_2 z} 1_{z<0}, \tag{9}$$

where $\eta_1 > 1, \eta_2 > 0, p, q > 0$, and $p + q = 1$, where q and p represent the probabilities for positive and negative jumps, respectively. As a result we can obtain that $\mu_J = \mathbb{E}^{\mathbb{Q}}(J-1) = (p\eta_1/\eta_1 - 1) + (q\eta_2/\eta_2 + 1) - 1$.

We set $\tau = T - t$, $X_t = lnS_t$, $Y = lnJ$, the interest rate r are determined by the sum of the two factors r_1 and r_2 ($r = r_1 + r_2$) and $k = lnK$, where T is the maturity date, and K is the strike price. In the risk-neutral world, the price of a call option $C(S, V1, V2, r, \lambda, t)$ at time $t \in [0, T]$ with strike price K and maturity date T is given by

$$C(S, v, \hat{v}, r_1, r_2, \lambda, t) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^t r_s ds} max(S_T - K, 0)|\mathcal{F}_t\right) \tag{10}$$

we convert measure $\mathbb{Q}$ to the measure $\mathbb{Q}^S$ and the T forward measure $\mathbb{Q}^T$. By applying Radon-Nikodym derivatives,

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{e^X}{e^{-\int_0^T r_s ds + X_T}} \tag{11}$$

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^T} = \frac{P(t,T)}{e^{-\int_0^t r_s ds}} \tag{12}$$

where

$$S = e^X = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds + X_T}|\mathcal{F}_t\right), \tag{13}$$

$P(t,T) := \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds +}|\mathcal{F}_t\right)$, is the price at time t of a zero-coupon bond which matures at time T (see appendix). Then, we can have the following expression:

$$C(S, v, \hat{v}, r_1, r_2, \lambda, t) = S\mathbb{E}^{\mathbb{Q}^S}(1_{\{X_T > k\}}|\mathcal{F}_t) - KP(t,T)\mathbb{E}^{\mathbb{Q}^T}(1_{\{X_T > k\}}|\mathcal{F}_t) \tag{14}$$

we define

$$\varphi_S(u) := \mathbb{E}^{\mathbb{Q}^S}(e^{iuX_T}|\mathcal{F}_t), \tag{15}$$

$$\varphi_T(u) := \mathbb{E}^{\mathbb{Q}^T}(e^{iuX_T}|\mathcal{F}_t), \tag{16}$$

$$\varphi(u) := \mathbb{E}^{\mathbb{Q}}(e^{\int_t^T r_s ds + iuX_T}|\mathcal{F}_t), \tag{17}$$

where $\varphi_S(u)$ denotes the characteristic function under $\mathbb{Q}^S$, $\varphi_T(u)$ denotes the characteristic function under $\mathbb{Q}^T$, and $\varphi(u)$ denotes the discounted characteristic function under $\mathbb{Q}$. Furthermore, by using Radon-Nikodym derivatives we can have the following expression:

$$C(S,v,\hat{v},r_1,r_2,\lambda,t) = S\Big(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty R\Big(\frac{e^{-iuk}\varphi(u-i)}{iu\varphi(-i)}\Big)du\Big) - KP(t,T)\Big(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty R\Big(\frac{e^{-iuk}\varphi(u)}{iuP(t,T)}\Big)du\Big) \tag{18}$$

all we need to do is to derive the formula of $\varphi(u)$ to have the pricing formula.

Theorem 1. *If the asset price is governed by the dynamic system (1), the discounted characteristic function $\varphi(u; X, v, \hat{v}, r_1, r_2, \lambda, \tau)$ takes the following form:*

$$\varphi(u; X, v, \hat{v}, r_1, r_2, \lambda, \tau) = e^{C(u,\tau)+D_v(u,\tau)v+D_{\hat{v}}(u,\tau)\hat{v}+E(u,\tau)r_1+F(u,\tau)r_2+G(u,\tau)\lambda+iuX} \tag{19}$$

where

$$C(u,\tau) = \frac{2k_v\theta_v}{\sigma_v^2\varepsilon^{2H-1}}\Big[\frac{(k_v - iu\rho_1\sigma_v\varepsilon^{H-\frac{1}{2}} - d)\tau}{2} + ln\frac{2d}{2d+(k_v - iu\rho_1\sigma_v\varepsilon^{H-\frac{1}{2}} - d)(1-e^{-d\tau})}\Big]$$

$$+ \frac{2\hat{k}\hat{\theta}}{\sigma_{\hat{v}}^2}\Big[\frac{(\hat{k} - iu\rho_2\sigma_{\hat{v}} - \hat{d})\tau}{2} + ln\frac{2\hat{d}}{2d+(\hat{k} - iu\rho_2\sigma_{\hat{v}} - \hat{d})(1-e^{-\hat{d}\tau})}\Big]$$

$$+ (iu-1)\Big((\frac{\theta_1}{k_1}(k_1t - 1 - e^{-k_1t}) + \frac{\theta_2}{k_2}(k_2t - 1 - e^{-k_2t})\Big)$$

$$- \frac{\sigma_1^2}{4k_1^3}(iu-1)^2\Big(e^{-2k_1t} - 4e^{-k_1t} - 2k_1t + 3\Big) - \frac{\sigma_2^2}{4k_2^3}(iu-1)^2\Big(e^{-2k_2t} - 4e^{-k_2t} - 2k_2t + 3\Big)$$

$$+ \rho_r\sigma_1\sigma_2(iu-1)^2\frac{1}{k_1k_2}\Big(t + \frac{1}{k_2}e^{-k_2t} + \frac{1}{k_1}e^{-k_1t} - \frac{1}{k_{2+k_1}}e^{-(k_2+k_1)t} - \frac{1}{k_1} - \frac{1}{k_2}$$
$$+ \frac{1}{k_1+k_2}\Big)$$

$$+ \frac{2k_\lambda\theta_\lambda}{\sigma_\lambda^2}\Big[\frac{(k_\lambda - iu\rho_2\sigma_\lambda - \varsigma)\tau}{2} + ln\frac{2\varsigma}{2\varsigma+(k_\lambda - iu\rho_2\sigma_\lambda - \varsigma)(1-e^{-\hat{d}\tau})}\Big]$$

$$D_v(u,\tau) = ((iu)^2 - iu)\frac{1-e^{-d\tau}}{2d+(k_v - iu\rho_1\sigma_v\varepsilon^{H-\frac{1}{2}} - d)(1-e^{-d\tau})}$$

$$D_{\hat{v}}(u,\tau) = ((iu)^2 - iu)\frac{1-e^{-\hat{d}\tau}}{2d_{\hat{v}}+(\hat{k} - iu\rho_2\sigma_{\hat{v}} - \hat{d})(1-e^{-\hat{d}\tau})}$$

$$G(u,\tau) = 2\omega(u)\frac{1-e^{-\varsigma\tau}}{2\varsigma + (k_\lambda - \varsigma)(1-e^{-\varsigma\tau})}$$

$$E(u,\tau) = \frac{1}{k_1}(iu-1)(1-e^{-k_1\tau})$$

$$F(u,\tau) = \frac{1}{k_2}(iu-1)(1-e^{-k_2\tau})$$

$$d = \sqrt{(k_v - iu\rho_1\sigma_v\varepsilon^{H-\frac{1}{2}})^2 - \sigma^2\varepsilon^{2H-1}((iu)^2 - iu)},$$

$$\hat{d} = \sqrt{(\hat{k} - iu\rho_2\sigma_{\hat{v}})^2 - \sigma_{\hat{v}}^2((iu)^2 - iu)}$$

$$M(u) = \frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1$$

$$\omega(u) = M(u) - iu\mu_J, \varsigma = \sqrt{k_\lambda^2 - 2\sigma_\lambda^2\omega(u)}$$

Proof. $\varphi(u; X, v, \hat{v}, r_1, r_2, \lambda, \tau)$ satisfies a PIDE by applying the Feynman-Kac theorem:

$$-\frac{\partial\varphi}{\partial\tau} + (r_1 + r_2 - \lambda\mu_J - \frac{1}{2}(v+\hat{v}))\frac{\partial\varphi}{\partial x} + \frac{1}{2}(v+\hat{v})\frac{\partial^2\varphi}{\partial x^2} + (k_v(\theta_v - v) + (H-\frac{1}{2})\sigma_v\sqrt{v})\frac{\partial\varphi}{\partial v}$$
$$+ \frac{1}{2}\sigma_v^2\varepsilon^{2H-1}v\frac{\partial^2\varphi}{\partial v^2} + \hat{k}(\hat{\theta} - \hat{v})\frac{\partial\varphi}{\partial\hat{v}} + \frac{1}{2}\sigma_{\hat{v}}^2\hat{v}\frac{\partial^2\varphi}{\partial\hat{v}^2} + \rho_1\sigma_v v\varepsilon^{H-\frac{1}{2}}\frac{\partial^2}{\partial x\partial v} + \rho_2\sigma_{\hat{v}}\hat{v}\frac{\partial^2\varphi}{\partial x\partial\hat{v}} + k_1(\theta_1 - r_1)\frac{\partial\varphi}{\partial r_1}$$
$$+ \frac{1}{2}\sigma_1^2\frac{\partial^2\varphi}{\partial r_1^2} + k_2(\theta_1 - r_2)\frac{\partial\varphi}{\partial r_2} + \frac{1}{2}\sigma_2^2\frac{\partial^2\varphi}{\partial r_2^2} + \sigma_1\sigma_2\rho_r\frac{\partial^2\varphi}{\partial r_1\partial r_2} + k_\lambda(\theta_\lambda - \lambda)\frac{\partial\varphi}{\partial\lambda} + \frac{1}{2}\sigma_\lambda^2\frac{\partial^2\varphi}{\partial\lambda^2}$$
$$+ \lambda\int_{-\infty}^{+\infty}(\varphi(x+y) - \varphi(x))f(y)dy - r\varphi = 0 \tag{20}$$

If we assume that $\varphi(u; X, v, \hat{v}, r_1, r_2, \lambda, \tau)$ takes the form of

$$\varphi(u; X, v, \hat{v}, r_1, r_2, \lambda, \tau) = e^{C(u,\tau) + D_v(u,\tau)v + D_{\hat{v}}(u,\tau)\hat{v} + E(u,\tau)r_1 + F(u,\tau)r_2 + G(u,\tau)\lambda + iuX} \tag{21}$$

and substitute into Eq. (20), we can obtain

$$\begin{cases} \frac{\partial C}{\partial \tau} = k_v\theta_v D_v + \hat{k}\hat{\theta} D_{\hat{v}} + k_1\theta_1 E + k_2\theta_2 F + \frac{1}{2}\sigma_1^2 E^2 + \frac{1}{2}\sigma_2^2 F^2 + \rho_r\sigma_1\sigma_2 EF + Gk_\lambda\theta_\lambda \\ \frac{\partial D_v}{\partial \tau} = \frac{1}{2}\sigma_v^2\epsilon^{2H-1}D_v^2 + (\rho_1\sigma_v\epsilon^{H-\frac{1}{2}}iu - k_v)D_v + \frac{1}{2}iu(iu-1) \\ \frac{\partial D_{\hat{v}}}{\partial \tau} = \frac{1}{2}\sigma_{\hat{v}}^2 D_{\hat{v}}^2 + (\rho_2\sigma_{\hat{v}}iu - k_{\hat{v}})D_{\hat{v}} + \frac{1}{2}iu(iu-1) \\ \frac{\partial G}{\partial \tau} = \frac{1}{2}\sigma_\lambda^2 G^2 - k_\lambda G + M(u) - \mu_J iu \\ \frac{\partial E}{\partial \tau} = -k_1 E + iu - 1 \\ \frac{\partial F}{\partial \tau} = -k_1 F + iu - 1 \end{cases} \tag{22}$$

with boundary conditions $C(u,0) = D_v(u,0) = D_{\hat{v}}(u,0) = E(u,0) = F(u,0) = G(u,0) = 0$. by applying some algebraic calculations, we will obtain the result.

3 Numerical Discussion

We'll analyze European option prices under DHJDF with two-factor stochastic interest rate model parameters in this section. The parameters we use are listed in Table 1.

Table 1. Values of parameters.

Parameter	Value	Parameter	value
k_v	9.9772k1	$\hat{k}$	2.3388
θ_v	0.0189	$\hat{\theta}$	0.001
σ_v	0.8379	$\sigma_{\hat{v}}$	0.9957
ρ_1	−0.9764	ρ_2	−0.8178
v	0.0002	$\hat{v}$	0.0633
ε	0.00005	ρ_r	1
α_1	0.3322	α_2	0.26594
β_1	0.1	β_2	0.1
σ_1	0.02	σ_2	0.02
r_1	0.001	r_2	0.012
k_λ	2	σ_λ	0.1
θ_λ	0.001	λ	0.001
k_{r_1}	0.02	k_{r_2}	0.02
η_1	1.0333	η_2	19.7482
S	100	K	100

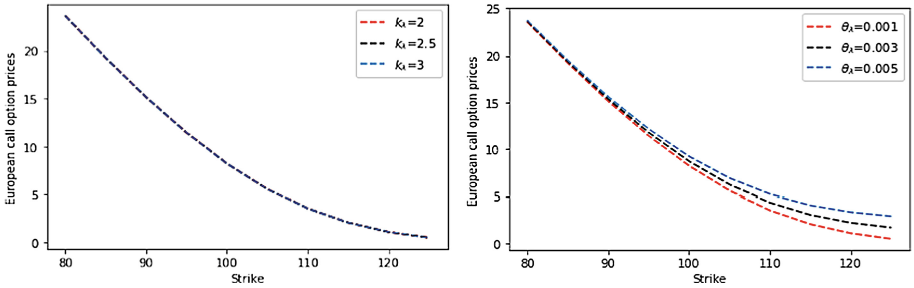


Fig. 1. The impact of k_λ and θ_λ on call option prices for $T = 1$.

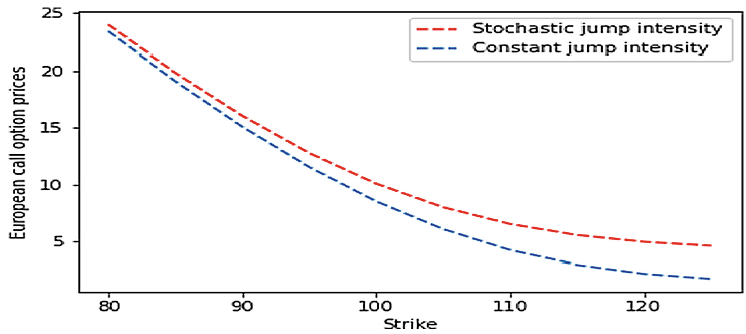
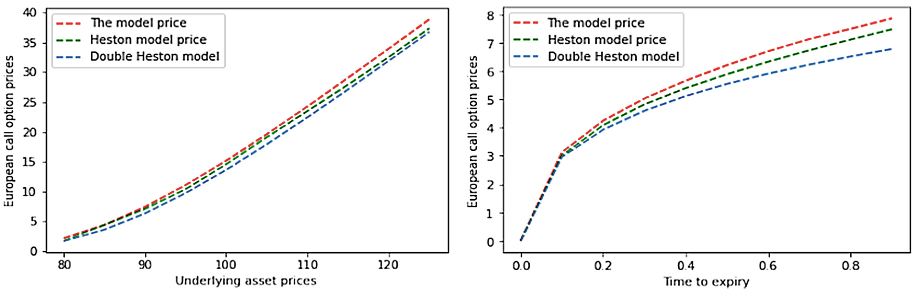


Fig. 2. The impact of the existence of the jump intensity process on call option prices for $T = 1$.



(a) (b)

Fig. 3. The model price, the Heston price and double Heston price with respect to the underlying asset price (a) and time to expiry (b).

Figure 1 shows that changes in the mean-reversion level θ_λ have a significant effect on call option prices, while changes in the mean-reversion rate k_λ have little effect on call option prices. The obtained results show that an increase in the value of θ_λ leads to an increase in the value of the call option price.

Figure 2 illustrate the effect of the presence of the jump intensity process on call option prices. It shows that the price of a call option with stochastic jump intensity is greater than the price of a call option with a constant jump intensity.

On the other hand. By using theoretical results of pricing formula, we can investigate the impact of incorporating a two-factor stochastic interest rate into DHJD model with approximative fractional Brownian motion and stochastic intensity under the chosen set of parameters. It can be distinctly observed that our price model's is high that the Heston's price. Specifically, depicted in Fig. 3 is the option prices with different time to expiry. Clearly, our price and the price of Heston are about the same when the time of expiry increases, the gap between our price and the Heston price increases. The reason that this phenomenon happens is increasing time to expiry implies a longer period of time for the interest rate changes which can thus definitely rate that can reflect the widened divide.

4 Conclusion

This paper introduces the European option under double Heston jump-diffusion hybrid model based on approximative fractional Brownian motion by adding interest rate follow two-factor Vasicek model and jump intensity follow a stochastic process. We derived a closed pricing formula for European option under this model by used the Radon-Nikodym derivative. The numerical results show that European call option prices under this model are higher than those under the double Heston model and Heston model.

Appendix

If the risk-free interest rate follows the Two-Vasicek model, then $P(r_1, r_2, t, T)$ should satisfy the following PDE problem:

$$\begin{cases} \frac{\partial P}{\partial t} + k_1(\theta_2 - r_1)\frac{\partial P}{\partial r_1} + k_2(\theta_2 - r_2)\frac{\partial P}{\partial r_1} + \frac{1}{2}\sigma_1^2\frac{\partial^2 P}{\partial r_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 P}{\partial r_2^2} + \rho_r\sigma_1\sigma_2\frac{\partial^2 P}{\partial r_1 \partial r_2} - (r_2 + r_2)P = 0 \\ P(r_1, r_2, T, T) = 1 \end{cases} \tag{23}$$

If we assume that $P(r_1, r_2, t, T)$ takes the form of

$$P(r_1, r_2, t, T) = e^{[A(\tau) - B_1(\tau)r_1 - B_1(\tau)_{r2}]} \tag{24}$$

and substitute it into PDE (23), we can obtain:

$$\begin{cases} \frac{\partial B_1}{\partial t} = 1 - k_1 B_1 \\ \frac{\partial B_2}{\partial t} = 1 - k_2 B_2 \\ \frac{\partial A}{\partial t} = -k_1\theta_1 B_1 - k_2\theta_2 B_2 + \frac{1}{2}\sigma_1^2 B_1^2 + \frac{1}{2}\sigma_2^2 B_2^2 + \rho_r\sigma_1\sigma_2 B_1 B_2 \end{cases} \tag{25}$$

with the terminal condition $B_1(0) = B_2(0) = A(0) = 0$ Then we have :

$$B_1(\tau) = \frac{1}{k_1}(1 - e^{k_1 \tau}) \tag{26}$$

$$B_2(\tau) = \frac{1}{k_1}(1 - e^{k_1 \tau}) \tag{27}$$

$$\begin{aligned} A(\tau) = & -\theta_1(\tau + \frac{1}{k_1}e^{-k_1\tau} - \frac{1}{k_1}) - \theta_2(\tau + \frac{2}{k_2}e^{-k_2\tau} - \frac{1}{k_2}) + \frac{\sigma_1^2}{k_1^2}(t + \frac{2}{k_1}e^{-k_1 t} - \frac{1}{2k_1}e^{-2k_1 t} - \frac{3}{2k_1}) \\ & + p_r\sigma_2\sigma_2\frac{1}{k_1 k_2}(t + \frac{1}{k_1}e^{-k_1 t} + \frac{1}{k_2}e^{k_2 t} - \frac{1}{k_1 + k_2}e^{-(k_1+k_2)t} + \frac{1}{k_1 + k_2} - \frac{1}{k_1} - \frac{1}{k_2}) \\ & + \frac{\sigma_2^2}{k_2^2}(t + \frac{2}{k_2}e^{-k_2 t} - \frac{1}{2k_2}e^{-2k_2 t} - \frac{3}{2k_2}). \end{aligned} \tag{28}$$

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