Lefschetz Fixed Point Theorems for Correspondences

Loring W. Tu

Dedicated to Catriona Byrne on the occasion of her retirement from Springer

2000 Mathematics Subject Classification Primary 58C30; Secondary 32Hxx

The classical Lefschetz fixed point theorem states that the number of fixed points, counted with multiplicity ± 1, of a smooth map f from a compact oriented manifold M to itself can be calculated as the alternating sum $\sum(-1)^k \operatorname{Tr} f^*|_{H^k(M)}$ of the trace of the induced homomorphism in cohomology.[1] This alternating sum is called the ***Lefschetz number*** $L(f)$ of the map f. As a corollary, if the Lefschetz number $L(f)$ is nonzero, then f has at least one fixed point.

In 1964, at the AMS Woods Hole Conference in Algebraic Geometry, Shimura conjectured an analogue for a holomorphic map of the Lefschetz fixed point theorem. Shimura's conjecture got the people at the conference all excited, and there was a workshop to prove it. At the end of the conference, there were two proofs—an algebraic proof by Verdier, Mumford, Hartshorne, and others, along more or less classical lines from the Grothendieck version of Serre duality, and an

[1] Throughout this article $H^*(M)$ denotes de Rham cohomology [4] and the fixed points are assumed to be nondegenerate.

L. W. Tu (✉)
Department of Mathematics, Tufts University, Medford, MA, USA
e-mail: loring.tu@tufts.edu

J.-M. Morel, B. Teissier (eds.), *Mathematics Going Forward*, Lecture Notes in Mathematics 2313, https://doi.org/10.1007/978-3-031-12244-6_5

analytic proof by Atiyah and Bott. Grothendieck generalized the algebraic proof in [9, Cor. 6.12, p. 131] and Atiyah and Bott generalized the analytic proof to the Atiyah–Bott fixed point theorem for an elliptic complex in [1, Th. 1, p. 246] and [2, Th. A, p 377].

There was a bit of controversy about this, because afterwards, Shimura's name disappeared from this theorem. It is now called the holomorphic Lefschetz fixed point theorem and the more general version is the Atiyah–Bott fixed point theorem. Shimura was quite upset about this. The principals in this story have all passed away, Atiyah and Shimura in the last 2 years. Fortunately, while they were still living, I was able to interview Michael Atiyah, Raoul Bott, Goro Shimura, and John Tate about the holomorphic Lefschetz fixed point theorem and in 2015 I published an article [13] in the hope of setting the history straight.

In Shimura's recollection, he had conjectured more than the holomorphic Lefschetz fixed point theorem. He said he had made a conjecture for an algebraic correspondence, which for a complex projective variety is the same as a holomorphic correspondence, but he could not remember the statement nor did he keep any notes. He believed that his conjecture for a holomorphic correspondence should have number-theoretic consequences for a Hecke correspondence and higher-dimensional automorphic forms. This article is an exploration of Shimura's forgotten conjecture, first for a smooth correspondence, then for a holomorphic correspondence in the form of two conjectures, and finally an open problem involving an extension to holomorphic vector bundles over two varieties and the calculation of the trace of a Hecke correspondence.

The coincidence locus of two set maps $f, g\colon N \to M$ is the subset of N on which they agree. A coincidence locus is sometimes the fixed-point set of a correspondence and vice versa, but the two types of sets are not the same. In Lefschetz's original paper [11] he obtained a coincidence locus formula for two continuous maps of manifolds. The fixed-point formula for a smooth correspondence (Theorem 5.1) in this article agrees with Lefschetz's coincidence formula when the coincidence is a correspondence. Thus, Theorem 5.1 is essentially already in Lefschetz [11]. It is also a special case of [5] for the trivial group action and of [6, Theorem 4.7, p. 15] for the trivial sheaf. Since Lefschetz's time, there have been many generalizations and variants of his coincidence and fixed-point formulas [5–7, 10, 12]. I offer this article in the hope that a simple-minded proof of a simple-minded statement in the smooth case may spur some interest in the holomorphic case.

At the end of the article, I include as historical documents some emails concerning the conjecture from Shimura to Atiyah and me in 2013. I would like to thank Jeffrey D. Carlson, Mark Goresky, Jacob Sturm, and the anonymous referee for many helpful comments and suggestions.

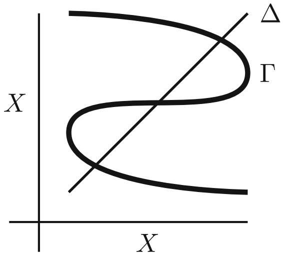

Fig. 1 A correspondence Γ on X

1 Correspondences

Definition 1.1 Let X be a topological space. A ***correspondence*** on X is a subspace $\Gamma \subset X \times X$ such that the two projections $\pi_i : \Gamma \subset X \to X$, $i = 1, 2$, are covering maps of finite degree (Fig. 1).

A correspondence Γ on X may be viewed as the graph of a multivalued function from X to X whose value at $p \in X$ is the set $\pi_2 \pi_1^{-1}(p)$. By symmetry, it can also be the multivalued function $\pi_1 \pi_2^{-1}$.

We have defined a correspondence in the continuous category. Clearly, it can also be defined in the categories of smooth manifolds and smooth maps, complex manifolds and holomorphic maps, and algebraic varieties and regular maps.

2 Lefschetz Number of a Smooth Correspondence

Suppose $\pi : N \to M$ is a C^∞ covering map of degree r. Denote by $\mathcal{A}^k(N)$ the vector space of smooth k-forms on N. For $\omega \in \mathcal{A}^k(N)$ and $p \in M$, define a k-covector $(\pi_*\omega)_p$ at p on M by

$$(\pi_*\omega)_p(v_1, \dots, v_k) = \sum_{q_i \in \pi^{-1}(p)} \omega_{q_i}(v_1^i, \dots, v_k^i),$$

where $v_1, \dots, v_k \in T_pM$ and $v_1^i, \dots, v_k^i$ are the unique tangent vectors in $T_{q_i}(N)$ such that $\pi_* v_j^i = v_j$. As p varies over M, the k-covector $(\pi_*\omega)_p$ becomes a k-form $\pi_*\omega$ on M. This defines a pushforward map $\pi_* : \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ of smooth k-forms on N. Since $\pi_* d = d\pi_*$, the pushforward induces a linear map $H^k(N) \to H^k(M)$ in cohomology, also denoted by π_*.

A smooth correspondence induces a linear map on the cohomology of the manifold M by

$$\pi_{1*}\pi_2^* : H^*(M) \to H^*(M).$$

Definition 2.1 The ***Lefschetz number*** $L(\Gamma)$ of a smooth correspondence Γ is defined to be the alternating sum of the traces of the linear map $\pi_{1*}\pi_2^*$ on $H^k(M)$:

$$L(\Gamma) = \sum_{k=0}^{n} (-1)^k \operatorname{Tr} \pi_{1*}\pi_2^* \colon H^k(M) \to H^k(M).$$

3 Fixed Points of a Smooth Correspondence

A ***fixed point*** of a smooth correspondence Γ on a manifold M is a point p in M such that $(p, p) \in \Gamma \cap \Delta$ in $M \times M$, where Δ is the diagonal. The correspondence is called ***transversal*** if Γ intersects Δ transversally in $M \times M$. In this case, the fixed points are said to be ***nondegenerate***. Nondegenerate fixed points are isolated.

When the manifold M is oriented and the correspondence is transversal, we can assign a ***multiplicity*** or ***index*** to each fixed point p in the usual way: $\iota_\Gamma(p) = \pm 1$ depending on whether the orientation on the tangent space $T_{(p,p)}(M \times M)$ agrees or disagrees with the orientation on the direct sum $T_{(p,p)}\Gamma \oplus T_{(p,p)}\Delta$. The intersection number $\#(\Gamma, \Delta)$ is then the sum $\sum \iota_\Gamma(p)$, where the sum runs over all fixed points p of the correspondence Γ. When the manifold M is compact, the number of nondegenerate fixed points is finite and the intersection number is defined.

4 The Trace of a Smooth Correspondence

We show how to calculate the trace of a correspondence in terms of differential forms.

Proposition 4.1 *Let $\Gamma \subset M \times M$ be a smooth correspondence on a compact oriented smooth manifold M, $\psi_1, \dots, \psi_m$ closed $(n-k)$-forms on M representing a basis for $H^{n-k}(M)$, and $\psi_1^*, \dots, \psi_m^*$ closed k-forms representing the dual basis for $H^k(M)$. Then on $H^k(M)$,*

$$\operatorname{Tr} \pi_{1*}\pi_2^* = \sum_{i=1}^{m} \int_\Gamma \pi_1^*\psi_i \wedge \pi_2^*\psi_i^*.$$

Proof Let $[a_j^i]$ be the matrix of the linear operator $\pi_{1*}\pi_2^*$ on $H^k(M)$:

$$\pi_{1*}\pi_2^*(\psi_j^*) = \sum a_j^i \psi_i^*.$$

Then

$$\begin{aligned} a^i_j &= \int_M \psi_i \wedge \pi_{1*}\pi_2^*\psi_j^* \\ &= \frac{1}{r}\int_\Gamma \pi_1^*\psi_i \wedge \pi_1^*\pi_{1*}\pi_2^*\psi_j^* \qquad \left(\text{because } \int_M \tau = \frac{1}{r}\int_\Gamma \pi_1^*\tau\right) \\ &= \int_\Gamma \pi_1^*\psi_i \wedge \pi_2^*\psi_j^* \qquad \left(\text{because } \omega \wedge \pi_1^*\pi_{1*}\tau = r\omega \wedge \tau\right). \end{aligned}$$

Therefore,

$$\operatorname{Tr} \pi_{1*}\pi_2^* = \sum_i a^i_i = \sum_i \int_\Gamma \pi_1^*\psi_i \wedge \pi_2^*\psi_i^*. \qquad \square$$

5 The Lefschetz Fixed Point Theorem for a Smooth Correspondence

Theorem 5.1 (Lefschetz Fixed Point Theorem for a Smooth Correspondence) *Suppose Γ is a transversal smooth correspondence on a compact, oriented smooth n-manifold M. Then the Lefschetz number of Γ is*

$$L(f) = \sum_{\text{fixed points } p} \iota_\Gamma(p).$$

Our proof largely emulates the approach of Griffiths and Harris in their account of the Lefschetz fixed point formula for a smooth self-map [8, Chap. 3, Sec. 4, pp. 419–422], but generalized to a smooth correspondence. The main idea is quite simple. By Poincaré duality, the intersection number $\#(\Gamma, \Delta)$ of the correspondence Γ with the diagonal Δ can be calculated as the integral of the wedge product of the differential forms representing their Poincaré duals. On the other hand, with the trace formula of Proposition 4.1, the Lefschetz number of the correspondence Γ can also be calculated in terms of differential forms. The two expressions in differential forms turn out to be equal.

Proof Let $\psi_1, \ldots, \psi_s$ be closed forms on M representing a basis for $H^*(M)$, and $\psi_1^*, \ldots, \psi_s^*$ closed forms representing the dual basis for $H^*(M)$. Note that the forms ψ_i, ψ_j^* run over all degrees, but ψ_i and ψ_i^* have complementary degrees in n. By the Künneth formula, $\pi_1^*\psi_i \wedge \pi_2^*\psi_j$ represent a basis for the cohomology $H^*(M \times M)$. It is proven in [8, p. 420] that the Poincaré dual of the diagonal Δ is given by

$$\eta_\Delta = \sum_i (-1)^{\deg \psi_i^*} \pi_1^*\psi_i \wedge \pi_2^*\psi_i^*.$$

Then

$$
\begin{aligned}
L(\Gamma) &= \sum_k (-1)^k \operatorname{Tr} \pi_{1*}\pi_2^*|_{H^k(M)} \\
&= \sum_k (-1)^k \sum_{\deg \psi_i = n-k} \int_\Gamma \pi_1^*\psi_i \wedge \pi_2^*\psi_i^* && \text{(Proposition 4.1)} \\
&= \int_\Gamma \sum_i (-1)^{\deg \psi_i^*} \pi_1^*\psi_i \wedge \pi_2^*\psi_i^* && (\psi_i \text{ runs over all degrees}) \\
&= \int_\Gamma \eta_\Delta && (\text{by the formula for } \eta_\Delta) \\
&= \int_M \eta_\Gamma \wedge \eta_\Delta && (\text{def. of the Poincaré dual } \eta_\Gamma) \\
&= \#(\Gamma \cdot \Delta) = \sum_{\text{fixed points } p} \iota_\Gamma(p). && \square
\end{aligned}
$$

6 A Conjecture for a Holomorphic Correspondence

Let Γ be a ***holomorphic correspondence*** on a complex manifold M of complex dimension n, that is, a complex submanifold of $M \times M$ such that the two projections $\pi_i : \Gamma \to M$ are holomorphic covering maps. As for a smooth correspondence, a fixed point of the holomorphic correspondence Γ is a point $p \in M$ such that (p, p) is in the intersection $\Gamma \cap \Delta$ in $M \times M$, where Δ is the diagonal in $M \times M$. The correspondence Γ is said to be ***transversal*** if Γ intersects the diagonal Δ transversally in $M \times M$.

Denote by $\mathcal{O}$ the sheaf of holomorphic functions and $\mathcal{A}^{p,q}$ the sheaf of C^∞ (p, q)-forms on M. Let $\Gamma(M, \mathcal{A}^{p,q})$ be the space of global sections of $\mathcal{A}^{p,q}$; these are simply the C^∞ (p, q)-forms on M. The sheaf $\mathcal{O}$ has an acyclic resolution

$$0 \to \mathcal{O} \to \mathcal{A}^{0,0} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \xrightarrow{\bar\partial} \mathcal{A}^{0,2} \xrightarrow{\bar\partial} \cdots$$

and the cohomology $H^k(M, \mathcal{O})$ is the cohomology of the differential complex of global sections

$$\Gamma(M, \mathcal{A}^{0,0}) \xrightarrow{\bar\partial} \Gamma(M, \mathcal{A}^{0,1}) \xrightarrow{\bar\partial} \Gamma(M, \mathcal{A}^{0,2}) \xrightarrow{\bar\partial} \cdots .$$

(For background on sheaf cohomology, see [14].)

For a holomorphic covering map $f : N \to M$, both the pullback f^* and the pushforward f_* of C^∞ $(0, k)$-forms are cochain maps of the complexes $\Gamma(N, \mathcal{A}^{0,\bullet})$

and $\Gamma(M, \mathcal{A}^{0,\bullet})$. Since the projection maps $\pi : \Gamma \to M$ are holomorphic covering maps, both the pullback $\pi_2^* \colon H^*(M, \mathcal{O}) \to H^*(\Gamma, \mathcal{O})$ and the pushforward $\pi_1^* H^*(\Gamma, \mathcal{O}) \to H^*(M, \mathcal{O})$ in cohomology are well-defined. Thus, the holomorphic correspondence Γ induces linear maps of cohomology groups

$$\pi_{1*}\pi_2^* \colon H^k(M, \mathcal{O}) \to H^k(M, \mathcal{O}), \quad k = 0, \dots, n.$$

The ***holomorphic Lefschetz number*** $L(\Gamma, \mathcal{O})$ of Γ is defined to be an alternating sum of traces as before:

$$L(\Gamma, \mathcal{O}) = \sum_{k=0}^{n} (-1)^k \operatorname{Tr} \pi_{1*}\pi_2^* \colon H^k(M, \mathcal{O}) \to H^k(M, \mathcal{O}).$$

The holomorphic Lefschetz number is a global invariant. Next we define the local contribution at each fixed point. Since a correspondence is a holomorphic covering map of M via π_1, locally it is the graph of a holomorphic function f. At a fixed point p, let $J(\Gamma)$ be the Jacobian matrix of the holomorphic function f with respect to any holomorphic coordinate system.

Conjecture 6.1 *If Γ is a transversal holomorphic correspondence on a compact complex manifold M, then the holomorphic Lefschetz number of Γ is given by*

$$L(\Gamma, \mathcal{O}) = \sum_{\textit{fixed points } p} \frac{1}{1 - \det J(\Gamma)_p}.$$

I do not have any evidence for this conjecture other than that it specializes to the correct formula when the correspondence Γ is the graph of a holomorphic map $f \colon M \to M$. Of course, the simplicity of the statement plays in its favor.

7 Extension to Holomorphic Vector Bundles

In their seminal paper on the fixed point theorem for elliptic complexes [3], Atiyah and Bott extended, as a corollary of their general theorem, the Lefschetz fixed point theorem to a holomorphic vector bundle for a self-map of a compact complex manifold.

To get an idea of what needs to be generalized for a holomorphic correspondence, we give here a brief summary of the Atiyah–Bott result for a holomorphic vector bundle. For more details, consult [3, Section 4, pp. 455–459]. Let E be a holomorphic vector bundle over a compact complex manifold M and $f \colon M \to M$ a holomorphic map. Denote by $\Gamma(E)$ the vector space of C^∞ sections of E over M and by $\Lambda^{p,q}$ the C^∞ vector bundle of (p, q)-covectors on M. The smooth sections of $E \otimes \Lambda^{p,q}$ are the E-valued (p, q)-forms on M. The $\bar{\partial}$-operator on (p, q)-forms extends to E-valued (p, q)-forms by acting as the identity on E and as $\bar{\partial}$ on the forms. There is then a differential complex

$$\Gamma(E) \xrightarrow{\bar{\partial}} \Gamma(E \otimes \Lambda^{0,1}) \xrightarrow{\bar{\partial}} \Gamma(E \otimes \Lambda^{0,2}) \xrightarrow{\bar{\partial}} \cdots .$$

The cohomology $H^*\big(M, \mathcal{O}(E)\big)$ of M with coefficients in E is defined to be the cohomology of this complex of E-valued (p, q)-forms.

Now let F be a holomorphic vector bundle over the complex manifold M and let f^*F be its pullback under the holomorphic map $f: M \to M$. The map $f: M \to M$ induces a linear map of C^∞ sections $f^*: \Gamma(F) \to \Gamma(f^*F)$ by sending a section $s \in \Gamma(F)$ to

$$(f^*s)(x) = (s \circ f)(x) = s\big(f(x)\big) \in F_{f(x)} = (f^*F)_x, \quad x \in M$$

where $F_{f(x)}$ is the fiber of F at $f(x)$. In order to obtain an endomorphism of $\Gamma(F)$, Atiyah and Bott introduced the notion of a ***lifting*** of the map f to the bundle F. It is a holomorphic bundle map $\varphi: f^*F \to F$ over M. A lifting φ induces a linear map $\varphi_*: \Gamma(f^*F) \to \Gamma(F)$ by composition: $\varphi_*(s) = \varphi \circ s$. The holomorphic map $f: M \to M$ and a lifting $\varphi: f^*F \to F$ together define an endomorphism of $\Gamma(F)$:

$$\Gamma(F) \xrightarrow{f^*} \Gamma(f^*F) \xrightarrow{\varphi_*} \Gamma(F).$$

Applied to $F = E \otimes \Lambda^{0,k}$, this will then induce an endomorphism

$$(f, \varphi)^*: H^*\big(M, \mathcal{O}(E)\big) \to H^*\big(M, \mathcal{O}(E)\big)$$

and the Lefschetz number of the triple (f, φ, E) is defined to be

$$L(f, \varphi, E) := \sum_{k=1}^{n} (-1)^k \operatorname{Tr} (f, \varphi)^*\big|_{H^k(M, \mathcal{O}(E))}, \quad n = \dim_{\mathbb{C}} M. \tag{1}$$

Theorem 7.1 (Atiyah and Bott [3, Theorem 4.12, p. 458]) *Let E be a holomorphic vector bundle over a compact complex manifold M, $f: M \to M$ a transversal holomorphic self-map, and $\varphi: f^*E \to E$ a holomorphic bundle map. Then*

$$L(f, \varphi, E) = \sum_{f(p)=p} \frac{\operatorname{Tr} \varphi_p}{\det(1 - f_{*,p})}.$$

In this theorem, a ***transversal*** map is one whose graph intersects the diagonal transversally in $M \times M$, $\varphi_p: E_{f(p)} = E_p \to E_p$ is a complex linear map, and $f_{*,p}$ is the differential of f on the holomorphic tangent space of M at p.

For a holomorphic correspondence Γ and a holomorphic vector bundle E over M, the lifting of a self-map needs to be replaced by some notion of a lifting of the correspondence Γ to the bundle E, which should be a holomorphic bundle map over Γ. Then a plausible conjecture should have the same form as Theorem 7.1.

Conjecture 7.1 *Let E be a holomorphic vector bundle over a compact complex manifold M, $\Gamma \subset M \times M$ a transversal holomorphic correspondence, φ a suitably defined lifting of Γ to E, and $L(\Gamma, \varphi, E)$ a suitably defined Lefschetz number. Then the Lefschetz number $L(\Gamma, \varphi, E)$ satisfies*

$$L(\Gamma, \varphi, E) = \sum_{f(p)=p} \frac{\operatorname{Tr} \varphi_p}{\det\big(1 - J(\Gamma)_p\big)},$$

where $J(\Gamma)_p$ is the Jacobian matrix of Γ at (p, p).

In Shimura's emails to Michael Atiyah and Loring Tu in June 2013 (see Appendix), he actually claimed more. He said he had conjectured at Woods Hole in 1965 a Lefschetz fixed point formula for an algebraic correspondence between two holomorphic vector bundles on two algebraic varieties of the same dimension. The statement of this forgotten conjecture remains a mystery.

Stated more generally, Shimura's intention might have been the following (as formulated by Mark Goresky in a recent private communication):

> *Find and prove a holomorphic Lefschetz fixed point theorem that can be used to calculate the trace of a Hecke correspondence on the holomorphic cohomology, coherent cohomology, or $\bar{\partial}$-cohomology, of a Hermitian locally symmetric space.*

Appendix

Email from Goro Shimura to Loring Tu, June 13, 2013

Dear Loring,

It is nice to hear from you. I remember that you sent me your book in collaboration with Bott. Here is my belated thanks for the book!

As for that fixed point formula I can say the following.

In the case of Riemann surfaces, Eichler's result is quite general, and so it was definitely meaningless to conjecture something only for Riemann surfaces.

What I conjectured was a formula for an algebraic correspondence, not just for a map, between two algebraic varieties of the same dimension, so that it generalizes Eichler's formula. (Naturally, we have to (I had to) formulate it in terms of holomorphic bundles.) I thought it might be applicable to automorphic forms on the higher-dimensional spaces.

⋮

As I understand it, the Atiyah–Bott formula deals with only a map, not a correspondence, and so it does not include Eichler's formula, nor does it prove my conjecture. Therefore I think it is an open problem to prove it for a correspondence. Am I wrong?

⋮

With best regards,
Goro Shimura

Email from Goro Shimura to Michael Atiyah, June 19, 2013

Dear Michael,

⋮

Frankly I am incapable of telling you what exactly my conjecture was. Probably I made notes, but I don't think I can find them.

I can tell you that it concerned an algebraic correspondence between two holomorphic bundles on two base algebraic varieties of the same dimension, consistent with an algebraic correspondence on the base varieties. I formulated it so that it becomes Eichler's formula in the one-dimensional case, and also it becomes a special case of the Lefschetz fixed point formula when the bundles are trivial. I was not considering real analyticity.

⋮

With very best regards,
Goro

Note Added in Proofs

Mark Stern proves both Conjectures 6.1 and 7.1 for compact Kähler manifolds in his paper [15, Th. 3.4 and 3.11]. His lifting in Conjecture 7.1 Is a holomorphic bundle map $\varphi\colon \pi_2^*E \to \pi_1^*E$ over the correspondence Γ, where $\pi_i\colon \Gamma \to M$ are the two projections. Shimura's conjecture on two holomorphic bundles on two varieties and its number-theoretic applications remain open.

References

1. M. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic differential operators, Bull. Amer. Math. Soc. 72 (1966), 245–250.

2. M. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: I, Ann. of Math. 86 (1967), 374–407.
3. M. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: II, Applications, Ann. of Math. 88 (1968), 531–545.
4. R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, third corrected printing, Springer, New York, 1995.
5. I. Dell'Ambrogio, H. Emerson, R. Meyer, An equivariant Lefschetz fixed-point formula for correspondences, Doc. Math. 19 (2014), 141âĂŞ-194.
6. M. Goresky, R. MacPherson, Local contribution to the Lefschetz fixed point formula, Invent. Math. 111 (1993), no. 1, 1–33.
7. M. Goresky, R. MacPherson, The topological trace formula, J. Reine Angew. Math. 560 (2003), 77–150.
8. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley, New York, 1978.
9. A. Grothendieck and L. Illusie, Formule de Lefschetz, Exp. III, in SGA 5 (1966–67), Lecture Notes in Math. 589, Springer, 1977, 73–137.
10. M. Kuga and J. H. Sampson, A coincidence formula for locally symmetric spaces, Amer. J. Math. 94 (1972), 486–500.
11. L. Lefschetz, Intersections and transformations of complexes and manifolds, Transactions of A.M.S. 28 (1926), 1–49.
12. L. Taelman, *Sheaves and functions modulo p*, *Lectures on the Woods Hole trace formula*, London Math. Soc. Lecture Note Series vol. 429, Cambridge University Press, Cambridge, 2016.
13. L. W. Tu, On the genesis of the Woods Hole fixed point theorem, Notices of the AMS 62 (2015), 1200–1205.
14. L. W. Tu, Introduction to sheaf cohomology, ArXiv Math.AT preprint, 26 pages.
15. M. Stern, Fixed point theorems from a de Rham perspective, Asian J. Math. 13 (2009), pp. 65–88.