Some Connections Between Stochastic Mechanics, Optimal Control, and Nonlinear Schrödinger Equations

Sergio Albeverio, Francesco Carlo De Vecchi, and Stefania Ugolini

1 Introduction

Dedication
We are happy to have been given the opportunity to contribute a little paper to a publication in honor of Catriona Byrne. The first named author had the great luck of meeting Catriona already in the early 80s and has discussed with her many issues concerning our common passion, mathematics in all its multiform and fascinating aspects. She is a very special communicative person, full of enthusiasm. It is always a great pleasure to meet her and share with her impressions about not only mathematics but also the world of arts. She saw at an early stage how the still rather scattered attempts to create more bridges between probability theory, rather abstract aspects of the theory of stochastic processes and infinite-dimensional analysis on one hand, and apparently distant other areas of mathematics, from number theory to geometry and non-standard analysis on the other, could be enhanced, also through interactions coming from mathematical physics (especially quantum theory). Catriona joined, directly or through her coworkers, several scientific meetings, in particular those where S.A. was in some way involved (from Bielefeld, Bochum, Bonn and Oberwolfach to Levico, Verona, Warwick), and the informal discussions with her produced new interconnections between the participants. S.A. also remembers with gratitude the encouraging interest she expressed in work he was pursuing on the theory of Feynman path integrals. This

S. Albeverio (✉) · F. C. De Vecchi
Department for Applied Mathematics, Universität Bonn, Bonn, Germany
e-mail: albeverio@iam.uni-bonn.de; francesco.devecchi@uni-bonn.de

S. Ugolini
Dipartimento di Matematica, Università di Milano, Milano, Italy
e-mail: stefania.ugolini@unimi.it

J.-M. Morel, B. Teissier (eds.), *Mathematics Going Forward*, Lecture Notes in Mathematics 2313, https://doi.org/10.1007/978-3-031-12244-6_36

led in particular to a second edition of a *Lecture Notes in Mathematics* [11], with the late Raphael Høegh-Krohn and Sonia Mazzucchi as coauthors. Also on her initiative four lectures in the series "Saint-Flour Seminars in Probability" were republished in a Springer book with the title *Mathematical physics at Saint-Flour* [8] with, besides S.A., Hans Föllmer, Leonard Gross and Ed Nelson as authors. It is not by chance that these lectures happen to have a strong component in analysis and probability theory besides one in mathematical physics. The present paper relates to a number of arguments that have their roots in the topics treated in that book initiated by Catriona. Her influence in fact is also recognizable in many books of Proceedings edited by S.A. in various collaborations, in particular those emanating from activities of the Research Center BiBos. It is a great pleasure for all authors to thank Catriona for all she has done for the mathematical community and in particular for our areas of work. We do this with our heartfelt wishes for good health, happiness, enjoyment and success in all her future undertakings (it is really difficult to imagine a future for her not full of beautiful activities!).

The Topics Discussed

The topics we shall present in the present paper are connected, in several ways, with those of concern in the above mentioned books [8, 11]. The main motivations come from questions that arose in physics, namely on how to better understand certain phenomena appearing in nature, as manifestations of an underlying "quantum world".

In the first part of this exposé (Sects. 2–6) we shall concentrate on attempts to understand in a mathematical way some aspects of the particular complex phenomenon of Bose–Einstein condensation (BEC). In the second part (Sect. 7) we shall mention and briefly discuss future possible developments in this connection, but also more general issues connected with multiform and fascinating relations between quantum evolution and probabilistic evolution that still have not been brought to light.

Let us start by briefly mentioning what the physical phenomena of BEC is. BEC might be characterized by saying that it happens when a sufficiently diluted gas of bosons (i.e. consisting of identical particles with integer spin, in the case we shall consider with spin zero, called "bosons") confined to a box, is cooled down in an appropriate way to "very low temperatures" (close to absolute zero). In this case a large fraction of the number of bosons of the gas happen to get into the same lowest energy quantum state ("ground state"), and behaves as a single quantum object. Since the cooled down gas is often macroscopic, we have then a macroscopic system exhibiting quantum behavior. The phenomenon was predicted in the sense of theoretical physics for an "ideal boson gas" (without any interaction), using quantum statistical considerations, by S.N. Bose and A. Einstein already in 1924–1925. Its experimental verification for a "real gas" had to wait until 1995 (for this experimental work E. Cornell, W. Ketteler and C. Wiemann received the Nobel prize in 2001). Present day experimental techniques have been developed very much since then, and permit us to establish many detailed properties of BEC.

Of particular interest for us is that the quantum state associated with a BE condensate can be well described by a single quantum mechanical wave function satisfying a nonlinear Schrödinger equation with a cubic nonlinearity called the Gross–Pitaevskii equation (see below and for references, e.g., [64, 65] and [80]). The nonlinear term in this equation expresses a local self-interaction of each particle of the condensate and depends on the density of the particles (it is the collective result of the presence of 2-particle interactions between the particles of the gas).

The mathematical derivation of the Gross–Pitaevskii (GP) equation and other related equations from a quantum mechanical N-particle system, described by a Hamiltonian H_N (see (1) below), usually with a confining potential V and with two particle interactions given by a potential v_N, taking the limit as N tends to infinity, has been an important issue in mathematical physics for many years and there is still much research going on, as we shall indicate.

The derivation involves the choice of particular 2-particle interactions, scaled in a certain way depending on N and the dimension n of the underlying space in which particles move (here we shall mainly consider the case $n = 3$, but other values of n have been examined by similar methods). As we shall mention in detail in Sect. 2, in the case $n = 3$ essentially three choices of scaling, characterized by a parameter $0 \leq \beta \leq 1$, have been discussed: the mean-field one for $\beta = 0$ (where the limit equation, called the mean-field or Hartree equation, contains a cubic nonlinear and non-local term with a "good kernel"); the intermediate one for $0 < \beta < 1$ (where the limit equation is a nonlinear Schrödinger equation with a cubic local nonlinearity with a constant factor in front involving the integral of the original 2-particle potential); and for the value $\beta = 1$ the GP equation (with a local cubic nonlinearity and a constant in front depending on the scattering length of the 2-particle potential).

As we mentioned above, the GP regime ($\beta = 1$) is the most used in the study of BEC, but it is also the one that is most mathematically complex. The major results were obtained in a series of papers by Lieb, Seiringer and Yngvason (see, e.g., [63, 65] and the book [64], see also [24, 43]). The choice of the 2-particle, translation invariant, potentials is a point interaction one, that heuristically permits certain explicit calculations (typical of point interactions, see, e.g., [9, 13]) leading in particular to the presence of a local nonlinearity, but also already presents for $n = 3$ intriguing mathematical problems in the choice of the starting Hamiltonian (connected with the theory of self-adjoint extensions of symmetric operators and renormalization theory; these problems also arise in physical phenomena like the Efimov and Thomas effects, and not by chance their study, both theoretical and experimental, has strong connections with the work on BEC: see, e.g., [7], [42] and also the excellent exposition in [47], we shall say a bit more on this in Sect. 7). For a detailed explanation of the mean-field scaling limit $\beta = 0$ and its applications see, e.g., [61, 62]. For the intermediate case $0 < \beta < 1$ see, e.g., [83].

In our presentation in the first part of the present paper we shall stress a new approach to this circle of problems developed in the last decade, starting from [72], based on ideas of Nelson's stochastic mechanics (see, e.g., [23, 40, 76, 78, 79]), associating to a solution of the N-particle Schrödinger equation related to the N-body Hamiltonian H_N a certain diffusion process on $\mathbb{R}^{nN}$ having invariant measure whose drift is the logarithmic derivative of the solution of the original Schrödinger equation. It was shown in [72] that in the GP-limit one gets a process with drift depending on the wave function of the BE condensate. A further discussion can be found in [16]. Progress associated with this state is discussed and a probabilistic counterpart of the asymptotic localization of the interaction energy has been shown in [73] and chaotic properties have been established in [86] for this scaling limit. Other developments in this setting are discussed in Sect. 7. We shall also present, in Sect. 4, original results on the mean field limit. For this we shall use a new variational approach that is inspired by previous work of K. Yasue [87] and Guerra–Morato [51], starting from an N-particle approximation of the relative mean-field stochastic optimal control problem introduced in [4].

In Sect. 5 we present a Markovian N-particle approximation (based on our work in [4]) to the stochastic optimal control discussed in Sect. 4. With a suitable choice of potentials we prove two convergence results: one involving the invariant measure of the optimal controlled N-particle process, the other concerning the law of the process on the whole path space $C^0([0, T], \mathbb{R}^{nN})$ (for any arbitrary $T > 0$ fixed). In Theorem 5.1 the convergence to zero of the $\frac{1}{N}$-multiple of the entropy of $\rho_{0,N}$ (the invariant measure of the optimally controlled N-particle system) relative to $\rho_0^{\otimes N}$ (the tensor product of the invariant measure of the optimally controlled mean-field system) is proven. A corresponding result, Theorem 5.2, holds for the conditional entropy (with respect to the k-partial marginals, for any $k \in \mathbb{N}$) on the path space.

In Sect. 6 the case of a variational problem with a convoluted delta potential is studied for all values of $\beta \in (0, 1]$. The optimal control is discussed in relation to the methods used in Sects. 4 and 5 for the case $\beta = 0$. In the case $0 < \beta < 1$ both the convergence of the "value function" and the probability measure on the path space, with respect to the relative entropy, are considered (see Theorems 6.1 and 6.2 respectively), using the methods of [5]. In the case $\beta = 1$, a weak convergence result of the probability law on the path space, obtained in [6], is also mentioned.

In Sect. 7 we first discuss possible extensions of the work presented in the previous sections on stochastic optimal control, especially to a time-dependent case (rather than the stationary case studied before). We also broaden the perspective to other problems where the relations between hyperbolic problems and parabolic ones play an important role, e.g., we mention the truly infinite-dimensional problems one meets when one replaces particle quantum mechanics with relativistic quantum field theory. Here new problems arise and very little is known about extending optimal stochastic control to this area. We observe that much success in the study of quantum fields has been obtained by taking a "Euclidean, Wiener-like, path integral" method instead of the "hyperbolic path integral" (Feynman path integral). The latter corresponds in a sense to taking imaginary time in the Euclidean path integral, followed by an analytic continuation procedure. More direct methods have been developed to extend the existing rigorous mathematical work of Feynman path integrals (see [10, 11, 68]) from the "finite-dimensional case" of non-relativistic quantum mechanics to the "infinite-dimensional case" of quantum field theory. Additional connections between probability, analysis, and geometry are also briefly mentioned.

2 Quantum Mechanics and Bose–Einstein Condensation

For the sake of simplicity hereafter we consider the quantum mechanical description of $N \in \mathbb{N}$ identical Bosons of mass $m > 0$. More precisely, the N-body Hamiltonian used in the description of the experiments on Bose–Einstein Condensation (BEC) [37, 56, 69] is of the type

$$H_N = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \triangle_i + V(\mathbf{r_i}) \right) + \sum_{1 \le i < j \le N} v_N(\mathbf{r_i} - \mathbf{r_j}), \tag{1}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a confining potential, v_N a pair-wise repulsive (rotation invariant) interaction potential and $\mathbf{r}_i \in \mathbb{R}^3, i = 1, \dots, N$. H_N is realized (under suitable assumptions) as a self-adjoint operator in the complex $L^2_s(\mathbb{R}^{3N})$-space of permutation symmetric square-integrable functions ("wave functions"). We denote the scalar product in this space by $(\cdot, \cdot)$ and the norm by $||\cdot||$. $\hbar$ denotes the (reduced) planck's constant.

The state of the system is described by the wave function $\Psi_{N,t}$ solving the Schrödinger equation

$$i\hbar\partial_t \Psi_{N,t} = H_N \Psi_{N,t} = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \triangle_i \Psi_{N,t} + V(\mathbf{r_i})\Psi_{N,t} \right) + \sum_{1 \le i < j \le N} v_N(\mathbf{r_i} - \mathbf{r_j})\Psi_{N,t}. \tag{2}$$

with the initial condition $\Psi_{N,0} \in L^2(\mathbb{R}^{3N})$, whose modulus square $\rho_t^N(\mathbf{r}) = |\Psi_{N,t}(\mathbf{r})|^2, \mathbf{r} \in \mathbb{R}^{3N}$ gives (by Born's interpretation) the probability density (with respect to Lebesgue measure) associated with the system of N-particles. In the following we will focus on the stationary case, more precisely the ground state, i.e. the wave function $\Psi_{N,t}$ does not depend on t, and it is the eigenfunction of the lowest eigenvalue of H_N. In the study of BEC, the ground state plays the main role (physically this is due to the fact that the BEC phenomenon happens at very low temperatures). Let us characterize the ground state denoted by Ψ_0, by a variational principle: consider the functional

$$E[\Psi] := \frac{1}{2} \int_{\mathbb{R}^{3N}} \Psi(\mathbf{r}) H_N \Psi(\mathbf{r}) d\mathbf{r} = T_\Psi + \Phi_\Psi. \tag{3}$$

$E[\Psi]$ is the mean quantum mechanical energy, where

$$T_\Psi := \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} |\nabla_i \Psi|^2 \mathrm{d}\mathbf{r}_1 \cdots \mathrm{d}\mathbf{r}_N$$

is the *"(mean) kinetic energy"* and

$$\Phi_\Psi = \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} V(\mathbf{r}_i)|\Psi|^2 \mathrm{d}\mathbf{r}_1 \cdots \mathrm{d}\mathbf{r}_N + \frac{1}{2}\sum_{i=2}^{N} \int v_N(\mathbf{r}_1 - \mathbf{r}_i)|\Psi|^2 \mathrm{d}\mathbf{r}_1 \cdots \mathrm{d}\mathbf{r}_N$$

the *(mean) potential energy* associated with $\Psi \in L^2_s(\mathbb{R}^{3N})$. If there exists a minimizing function Ψ_N^0 of $E[\Psi]$ with respect to the complex-valued functions Ψ in $L^2_s(\mathbb{R}^{3N})$ subject to the constraint $\|\Psi\|_2 = 1$, it is called a variational *ground state*. The corresponding energy $E[\Psi_N^0]$ given by

$$E[\Psi_N^0] := \inf\left\{E(\Psi) : \|\Psi\|^2 = 1\right\},$$

where Ψ in the previous set belongs to $L^2_{\mathrm{s}}(\mathbb{R}^{3N})$, is called *ground state energy*. Under suitable assumptions on the potentials V and v one can prove the existence and uniqueness of the ground state Ψ_N^0 for (1). By the minimax principle (see, e.g., [81, Thm. XIII.1]) one has $H_N\Psi_0^N = E[\Psi_N^0]\Psi_0^N$, i.e. Ψ_0^N is the eigenfunction corresponding to the lowest eigenvalue $E[\Psi_N^0]$ of H_N, as a self-adjoint operator acting in $L^2_{\mathrm{s}}(\mathbb{R}^{3N})$.

Remark 2.1 Uniqueness of the ground state is to be understood as uniqueness apart from an *overall phase*. Regularity conditions on V and v implying the strict positivity and the continuous differentiability of the ground state (wave function) are well known (indeed they follow by a suitable adaptation of the arguments in [81] (Thm.XIII.46 and XIII.47) and [81] (Thm.XIII.11)), respectively).

The mathematical notion of the quantum phenomenon of Bose–Einstein condensation can be introduced in quantum theory by starting from the one-particle density matrix, i.e. the operator in $L^2(\mathbb{R}^3)$ having kernel:

$$\gamma(\mathbf{r}, \mathbf{r}') = \int \Psi_N^0(\mathbf{r}, \mathbf{r}_2, \ldots \mathbf{r}_N) \cdot \Psi_N^0(\mathbf{r}', \mathbf{r}_2, \ldots \mathbf{r}_N)\mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N,$$

where Ψ_N^0 denotes the wave function of the ground state.

Definition 2.1 Complete BEC is defined by the property that

$$\lim_{N\uparrow\infty} \gamma(\mathbf{r}, \mathbf{r}') = \varphi(\mathbf{r})\varphi(\mathbf{r}')$$

for some $\varphi \in L^2(\mathbb{R}^3)$ and in some topology for density matrices.

One of the main problems in the mathematical physics literature of the subject consists in justifying the various non-linear one-particle approximation models for describing the Bose–Einstein condensate. This goal is pursued, in the ground state framework, starting from the N-body Hamiltonian for N Bose particles (1) and by performing a suitable limit of an infinite number of particles.

Under certain assumptions on V and v_N, it has been shown that, for $N \to +\infty$, there is a limit wave function φ in $L^2(\mathbb{R}^3)$ of norm 1 solving a suitable (nonlinear) Schrödinger equation with Hamiltonian of the form

$$H_{BE}(\varphi) = -\frac{\hbar^2}{2m}\triangle + V(\mathbf{r}) + \tilde{v}(|\varphi|^2, \mathbf{r}), \quad \mathbf{r} \in \mathbb{R}^3, \tag{4}$$

where $\tilde{v}(|\varphi|^2, \mathbf{r})$ is an $L^2(\mathbb{R}^3)$-operator depending on the probability density $|\varphi|^2$ on $\mathbb{R}^3$. The related energy functional is given by the expression

$$E_{BE}(\varphi) = \int_{\mathbb{R}^3} \left(\frac{1}{2m}|\nabla\varphi(\mathbf{r})|^2 + \frac{1}{2}V(\mathbf{r})|\varphi(\mathbf{r})|^2 + \frac{1}{4}\tilde{v}(|\varphi|^2, \mathbf{r})|\varphi(\mathbf{r})|^2 \right) \mathrm{d}\mathbf{r}. \tag{5}$$

The precise form of the operator $\tilde{v}$ is strongly dependent on the kind of scaling limit of the original interaction potential v_N. If we take v_N (in (1)) of the form

$$v_N(\mathbf{r}) = \frac{N^{3\beta}}{N-1} v_0(N^\beta \mathbf{r}), \qquad 0 \le \beta \le 1 \tag{6}$$

we can distinguish three regimes:

1. the *mean-field regime (also called Hartree)*, that is $\beta = 0$, in which
$$\tilde{v}(|\varphi|^2, \mathbf{r}) = \int_{\mathbb{R}^3} v_0(\mathbf{r} - y)|\varphi|^2(y)\mathrm{d}y = (v_0 * |\varphi|^2)(\mathbf{r});$$
2. the *intermediate regime* (also called nonlinear Schrödinger), i.e. $0 < \beta < 1$, in which
$$\tilde{v}(|\varphi|^2, \mathbf{r}) = \left(\int_{\mathbb{R}^3} v_0(y)\mathrm{d}y \right) |\varphi|^2(\mathbf{r}) = \left(\int_{\mathbb{R}^3} v_0(y)\mathrm{d}y \right) (\delta_0 * |\varphi|^2)(\mathbf{r})$$
(where δ_0 is the Dirac delta in 0);
3. the *Gross–Pitaevskii regime*, i.e. $\beta = 1$, in which $\tilde{v}(|\varphi|^2, \mathbf{r}) = \frac{4\pi\hbar^2 a}{m}(\delta_0 * |\varphi|^2)(\mathbf{r})$, where a is the scattering length of the potential v_0 (see, e.g., [64, Appendix C] for the definition of scattering length, and see also [9] for other physical contexts where it plays an important role).

We remark that when $\beta = 0$, which corresponds properly to the mean-field approximation, the potential range is fixed and the intensity of the interaction potential decreases as $1/N$ for $N \to \infty$. In the regime corresponding to $0 < \beta < 1$

the interaction potential goes to a delta function in the sense of the convergence of measures. This intermediate (or general) mean-field case is not very well studied and it is usually called the nonlinear Schrödinger limit. There are many results both for the mean-field and for the intermediate case. For the latter there are some quantitative estimates of the convergence rate for small values of β (see [62, 82] and references therein). In [4] the general mean-field convergence problem ($0 < \beta < 1$) is faced by using the hard results for the case $\beta = 1$ and the convergence of the one-particle ground-state energy to the ground-state energy of the nonlinear Schrödinger functional for the case of purely repulsive interaction potential is proved.

We finally stress that the case $\beta = 1$ cannot be considered as a mean-field regime and it involves the scattering length of the interaction potential. The convergence of the ground state energy in this setting has been provided by Lieb and Seiringer [63] and Lieb et al. [65] and, recently, in [75].

In the time-dependent framework one of the main problems is that of controlling whether the Bose–Einstein condensation is preserved by the time evolution, that is, whether at time $t > 0$ for N large enough the one-particle density $\gamma^1_{N,t}$ is, in some approximation, equal to $|\varphi_t|^2$, where φ_t is the solution of the nonlinear (time-dependent) Schrödinger or Gross–Pitaevskii equation. More precisely, starting from a factorized initial wave function for the N-body Hamiltonian (1) and introducing the time evolution $\Psi_{N,t}$ of the initial wave function, the goal is to prove that the one-particle density associated to $\Psi_{N,t}$ converges to $|\varphi_t|^2$, with φ_t playing the role of the time-dependent wave function of the Bose–Einstein condensate (see, e.g., [1, 20, 27, 43]). The techniques used in the time-dependent setting are different from those of the stationary one, in particular instead of the mean quantum energies the Schrödinger hierarchies are used. Many other problems, such as the study of the fluctuations around the limit, are actually faced in the more general time-dependent framework, see for instance [26, 28].

3 Nelson's Stochastic Mechanics

One of the main problems in giving a stochastic representation of solutions to the Schrödinger equation is the reversibility in time of the quantum evolution (which is given by a one-parameter unitary group, and not by a contraction semigroup). Indeed the time marginal probability of, for example, a diffusion Markov process is a solution to the Fokker–Planck equation, which is a parabolic (and thus non-time-reversible) equation. A possible solution to this problem, at least in the one particle case, is given by Nelson's Stochastic Mechanics, introduced by Edward Nelson in 1966. It intends to study certain quantum phenomena using a well-determined class of diffusion processes (see [23, 30, 40, 76, 78, 79]). See [29] for a relatively recent review on Nelson's Stochastic Mechanics. Here we recall only the basic elements of the theory in order to present a suggestive variational approach (due to K. Yasue, see [87], and F. Guerra and L. Morato, see [51], see below and Sect. 7 for other references) that motivated our own approach in Sect. 4.

Consider a quantum particle of mass m moving on $\mathbb{R}^n$, subject to a force of potential V (and thus having Hamiltonian $H = -\frac{\hbar^2}{2m}\Delta + V(x)$). Nelson associates to it a Markovian process which is a solution to the following SDE:

$$\mathrm{d}X_t = b(X_t, t)\mathrm{d}t + \nu \mathrm{d}W_t, \tag{7}$$

where $\nu = \sqrt{\frac{\hbar}{m}}$, W is a standard Wiener process in $\mathbb{R}^n$, and $b : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is a measurable vector field whose regularity will be made more precise below. The core of the kinetic part of the theory is the fundamental pair of stochastic derivatives. The forward stochastic derivative is on smooth real functions f on $\mathbb{R}^n$ defined by:

$$\mathrm{D}f(X_t) := \lim_{h\downarrow 0} \mathbb{E}_t\left[\frac{f(X_{t+h}) - f(X_t)}{h}\right]$$

(where $\mathbb{E}_t$ is the conditional expectation with respect to X_t) and has the property that:

$$\mathrm{D}X_t = b(X_t, t).$$

Nelson also introduced the backward stochastic derivative:

$$\mathrm{D}_* f(X_t) := \lim_{h\downarrow 0} \mathbb{E}_t\left[\frac{f(X_t) - f(X_{t-h})}{h}\right]$$

which gives:

$$\mathrm{D}_* X_t = b_*(X_t, t),$$

for a certain vector field b_* on $\mathbb{R}^n \times \mathbb{R}_+$. The literature on time reversal of diffusion processes is quite large (see, e.g., [71] and references therein, see also [74]). Foellmer [46] individuated, in the context of Stochastic Mechanics presented here, a sufficient condition for the existence of the backward derivative: $\mathbb{E}[|b(X_t, t)|^2] < \infty$. If the vector field b is such that the probability density ρ_t for the solution X_t to the SDE (7) is strictly positive and differentiable, we have the relation

$$b_*(x, t) = b(x, t) - \frac{\nu^2}{2}\nabla \log \rho_t(x). \tag{8}$$

As for the dynamic, Nelson introduced the stochastic Newton equation

$$\frac{1}{2}[\mathrm{D}\mathrm{D}_* + \mathrm{D}_*\mathrm{D}]X_t = -\nabla V(X_t), \tag{9}$$

where V is the potential in which the particle of mass m is moving. Using the relation between b_* and b, writing $u(x,t) = \frac{1}{2}(b(x,t) - b_*(x,t))$ and $v(x,t) = \frac{1}{2}(b_*(x,t) + b(x,t))$ we get

$$\partial_t v = -\nabla V(x) + u \cdot \nabla u + v \cdot \nabla v + \frac{\nu^2}{2} \Delta u, \tag{10}$$

$$\partial_t u = -\nabla(u \cdot v) - \frac{\nu^2}{2} \nabla(\nabla \cdot v), \tag{11}$$

which are reversible (in time) equations. It is possible to prove, by choosing the initial conditions in a suitable way, that the previous system of PDEs (10), (11) admits solutions which, by an important result of Carlen (see [29–32], permit us to solve Eq. (7) and to associate a stochastic process to the quantum system. The solution process is then associated with the Schrödinger equation with the potential V (appearing in Eq. (9)). In the stationary case Eq. (10) reduces to an equation for u which is of the form $V_u = \frac{1}{2}(|u|^2 + \nu^2 \mathrm{div}(u))$.

K. Yasue initiated a heuristic variational formulation of the association of X_t to the Schrödinger equation by introducing a Lagrangian function $\mathcal{L}$ associated with the quantum Hamiltonian H

$$\mathcal{L}(\mathrm{D}X_t, \mathrm{D}^* X_t, X_t) = \frac{1}{4}\left(|\mathrm{D}^* X_t|^2 + |\mathrm{D}X_t|^2\right) - V(X_t). \tag{12}$$

An alternative action functional, proposed by Guerra and Morato, is given by the expression

$$\tilde{\mathcal{L}}(\mathrm{D}X_t, \mathrm{D}^* X_t, X_t) = \frac{1}{2}\left(\mathrm{D}^* X_t \cdot \mathrm{D}X_t\right) - V(X_t). \tag{13}$$

By the relations (8) and (9) the Lagrangian (12) can be thought of as a function of the vector field b, the process X_t and the probability density ρ_t associated with it. Thanks to this observation we can use the Lagrangian $\mathcal{L}$ to formulate an optimal control problem for the controlled SDE (7) (where the vector field b plays the role of control parameter). We consider the finite horizon optimal control problem and the ergodic control problem associated with the Lagrangian $\mathcal{L}$, i.e. we have the respective cost functions

$$J^{\mathrm{fh}}(\rho_0, b, \rho_t) = \mathbb{E}_{X_0 \sim \rho_0}\left[\int_0^T \mathcal{L}(\mathrm{D}X_t, \mathrm{D}^* X_t, X_t)\mathrm{d}t\right], \tag{14}$$

$$J^{\mathrm{e}}(\rho_0, b) = \mathbb{E}_{X_0 \sim \rho_0}\left[\limsup_{T \to +\infty} \frac{1}{T} \int_0^T \mathcal{L}(\mathrm{D}X_t, \mathrm{D}^* X_t, X_t)\mathrm{d}t\right] \tag{15}$$

(the suffixes fh and e in J stand for "finite horizon" and "ergodic", respectively, see, e.g., [44, 45] for a reference on stochastic optimal control i the notation $X_0 \sim \rho_0$ stands for X_0 having law ρ_0). The reason for the choice of the cost functionals is that the optimal controls of the previous problems satisfy the Schrödinger equation. The same optimal control problems can be obtained replacing the Lagrangian $\mathcal{L}$ with the functional $\tilde{\mathcal{L}}$, given in Eq. (13), in the definition of the cost functionals J^{fh} and J^{e}. More precisely, if b is an optimal control to the problem (14) where the optimal solution process X_t has density ρ_t, then there is a unique (up to a complex multiplicative constant) function $\Psi_t : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that

$$b(x,t) = \mathrm{Re}\left(\frac{\nabla\Psi_t(x)}{\Psi_t(x)}\right) + \mathrm{Im}\left(\frac{\nabla\Psi_t(x)}{\Psi_t(x)}\right), \quad \rho_t(x) = |\Psi_t(x)|^2,$$

and the function Ψ_t satisfies the Schrödinger associated to the Hamiltonian H. Carlen proved the existence of Nelson diffusions also in the general case in which there are nodes of the wave function [29], [30] under a finiteness condition on the Fisher information. In the ergodic case the optimal control b and the related probability density ρ *do not depend on the time t* and they are of the form

$$b(x) = \nabla \log(\Psi^0(x)), \quad \rho(x) = |\Psi^0(x)|^2,$$

where the function Ψ^0 is the (real) ground state of the Hamiltonian H. Let us mention, finally, that the variational formulations by Yasue, as well as by Guerra–Morato, have important connections with the entropic optimal transport problem (see [38] and [35] for studies on this connection in a rigorous probabilistic setting related to the heat rather than the Schrödinger equation). See also Sect. 7 for other variational approaches.

4 Non-linear Stochastic Mechanics

We want to take inspiration from the above sketched variational formulation of stochastic mechanics and the methods used in the convergence proof of the Bose–Einstein condensation to study some stochastic optimal control problems of McKean–Vlasov type (namely where the cost function depends not only on the solution process X_t to the controlled equation but also on its law).

Following [4], let us start with the autonomous stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \nu \mathrm{d}W_t; \quad t \geq 0, \tag{16}$$

where b is a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}^n$, $\nu > 0$ is a constant, and W_t, $t \geq 0$, is an n-dimensional standard Brownian motion. The starting point for X_t at $t = 0$ is $x_0 \in \mathbb{R}^n$. We look here at b as a "control vector field" and we associate to (16) the following "cost functional"

$$J(b, x_0) = \limsup_{T\to+\infty} \frac{1}{T} \left(\int_0^T \mathbb{E}_{x_0} \left[\frac{|b(X_t)|^2}{2} + \mathcal{V}(X_t, \mathrm{Law}(X_t)) \right] \mathrm{d}t \right), \quad x_0 \in \mathbb{R}^n. \tag{17}$$

Let $\mathcal{P}(\mathbb{R}^n)$ be the space of probability measures on $\mathbb{R}^n$ endowed with the topology given by weak convergence, $\mathcal{V} : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ where $\mathcal{P}(\mathbb{R}^n)$ is the set of probability measures on $\mathbb{R}^n$ is a regular function (hereafter called "potential") satisfying some technical hypotheses (see Hypothesis $\mathcal{V}$ below) and $\mathbb{E}_{x_0}$ denotes the expectation with respect to the solution X_t to the SDE (16) such that $X_0 = x_0 \in \mathbb{R}^n$.

In [4] we proved existence and uniqueness of the optimal control $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for the problem given by (16) and (17). Here we give a simplified proof of these results. We remark that the action functional explicitly depends on the law of X_t through the potential $\mathcal{V}$ but we find that the optimal control itself can be expressed in terms of the same law.

We define the following "value function":

$$\mathfrak{I} := \operatorname{ess\,sup}_{x_0\in\mathbb{R}^n} \left(\inf_{b\in C^1(\mathbb{R}^n,\mathbb{R}^n)} J(b, x_0) \right), \tag{18}$$

where ess sup is the essential supremum over $x_0 \in \mathbb{R}^n$ and J is the cost functional (17).

Remark 4.1 There are two important observations to make about the initial conditions chosen in the definition of the value function (18). The first one is that the function $x_0 \longmapsto \inf_{\alpha\in C^1(\mathbb{R}^n,\mathbb{R}^n)} J(\alpha, x_0)$ is almost surely constant in x_0 with respect to the Lebesgue measure (see Theorem 4.1 below). This means that the $\operatorname{ess\,sup}_{x_0\in\mathbb{R}^n}$ is used only to exclude a set of measure zero with respect to x_0.

The second observation is that it is possible to extend our analysis by considering

$$\bar{J}(b, \rho) := \limsup_{T\to+\infty} \frac{1}{T} \left(\int_0^T \mathbb{E}_{X_0\sim\rho(x)dx} \left[\frac{|b(X_t)|^2}{2} + \mathcal{V}(X_t, \mathrm{Law}(X_t)) \right] \mathrm{d}t \right), \tag{19}$$

where the process X_t now has an initial probability law $\mathrm{Law}(X_0)$ which is absolutely continuous with respect to Lebesgue measure of the form $\rho(x)\mathrm{d}x$. Indeed in both Theorem 4.1 and Lemma 4.1, below, we can replace the deterministic initial condition with a random one, of the previous type, obtaining the corresponding statement. This fact proves that

$$\mathfrak{I} = \inf_{b\in C^1(\mathbb{R}^n,\mathbb{R}^n)} \bar{J}(b, \rho),$$

for any $\rho \in L^1(\mathbb{R}^n)$. In this paper we consider deterministic initial conditions in order to simplify the treatment of the general problem.

Definition 4.1 If $\mathcal{K} : \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ is a function we say that $\mathcal{K}$ is Gâteaux differentiable if for any $\mu, \mu' \in \mathcal{P}(\mathbb{R}^n)$ there exists a bounded continuous function $\partial_\mu \mathcal{K}(\cdot, \mu) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\epsilon \to 0^+} \frac{\mathcal{K}(\mu + \epsilon(\mu - \mu')) - \mathcal{K}(\mu)}{\epsilon} = \int_{\mathbb{R}^n} \partial_\mu \mathcal{K}(y, \mu)(\mu(\mathrm{d}y) - \mu'(\mathrm{d}y)), \tag{20}$$

and we can choose the normalization condition given by

$$\int_{\mathbb{R}^n} (\partial_\mu \mathcal{K})(y, \mu)\mu(\mathrm{d}y) = 0.$$

When a function $\bar{\mathcal{K}} : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ depends also on $x \in \mathbb{R}^n$ we say that $\bar{\mathcal{K}}$ is Gâteaux differentiable if $\bar{\mathcal{K}}(x, \cdot)$ is Gâteaux differentiable for any $x \in \mathbb{R}^n$. In this case we write

$$\lim_{\epsilon \to 0^+} \frac{\bar{\mathcal{K}}(x, \mu + \epsilon(\mu - \mu')) - \bar{\mathcal{K}}(x, \mu)}{\epsilon} = \int_{\mathbb{R}^n} \partial_\mu \bar{\mathcal{K}}(x, y, \mu)(\mu(\mathrm{d}y) - \mu'(\mathrm{d}y)).$$

After these remarks, let us make precise the hypothesis on the functional $\mathcal{V}$ entering in the cost functional (17):

- **Hypotheses** $\mathcal{V}$:
  - (i) *The map $\mathcal{V}$ is continuous from $\mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n)$ to $\mathbb{R}$ (where we recall that $\mathcal{P}(\mathbb{R}^n)$ is equipped with the weak topology of convergence of measures).*
  - (ii) *There is a positive function V and there are three positive constants c_1, c_2, c_3, with $c_2 > 0$, such that for any $\mu \in \mathcal{P}(\mathbb{R}^n)$:*

$$V(x) - c_1 \leq \mathcal{V}(x, \mu) \leq c_2 V(x) + c_3, \ \ x \in \mathbb{R}^n. \tag{21}$$

Furthermore, we assume that V is such that

$$|\partial^\alpha V(x)| \leq C_\alpha V(x) \qquad\qquad V(x) \leq C_1 V(y) \exp(C_2 |x - y|), \ \ x, y \in \mathbb{R}^n \tag{22}$$

where $\alpha \in \mathbb{N}^n$ is a multiindex of length at most $|\alpha| \leq 2$, and C_α, C_1 and C_2 are positive constants; V is also assumed to be growing to $+\infty$ as $|x| \to +\infty$.

  - (iii) *The map $\mathcal{V}$ is Gâteaux differentiable and $\partial_\mu \mathcal{V}(x, y, \mu)$ is uniformly bounded from below and we have*

$$\partial_\mu \mathcal{V}(x, y, \mu) \leq D_1 + D_2 V(x) V(y), \ \ x, y \in \mathbb{R}^n \tag{23}$$

for some constants $D_1, D_2 \geq 0$. *Furthermore, whenever*

$$\partial_\mu \tilde{\mathcal{V}}(y, \mu) = \mathcal{V}(y, \mu) + \int_{\mathbb{R}^n} \partial_\mu \mathcal{V}(x, y, \mu)\mu(\mathrm{d}x)$$

is well defined (namely when $\int_{\mathbb{R}^n} V(x)\mu(\mathrm{d}x) < +\infty$*), we require that* $\partial_\mu \tilde{\mathcal{V}}(\cdot, \mu)$ *is a* $C^{\frac{n}{2}+\delta}(\mathbb{R}^n, \mathbb{R})$ *Hölder function for some* $\delta > 0$.

- **Hypothesis C**$\mathcal{V}$ *the functional* $\tilde{\mathcal{V}}$ *is convex.*
- **Hypothesis Q**V*: the function V, in Hypotheses* $\mathcal{V}$*, is radially symmetric* $V(x) = \bar{V}(|x|)$*, where* $\bar{V}$ *is a* $C^1(\mathbb{R}_+, \mathbb{R})$ *increasing function for which there are constants* $e_1, \epsilon > 0, e_2, e_3 \geq 0$ *such that:*
  (i) $\bar{V}(r) \geq e_1 r^{2+\epsilon} - e_2$,
  (ii) $\bar{V}'(r) \leq e_3(\bar{V}(r))^{\frac{3}{2}}, r := |x|$.

Remark 4.2 The previous hypotheses on the functional $\mathcal{V}$ cover the mean-field scaling regime of the interacting potential v_0 in the Bose–Einstein Condensation (BEC) (see Eq. (6) for $\beta = 0$). Indeed in the mean-field BEC the functional $\mathcal{V}$ in Eq. (17)) has the form

$$\mathcal{V}(x, \mu) = V_0(x) + \int_{\mathbb{R}^n} v_0(x - y)\mu(\mathrm{d}y), \tag{24}$$

where $V_0, v_0 \in C^{\frac{n}{2}+\epsilon}(\mathbb{R}^n)$, $\epsilon > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R}^n)$ (where $\mathcal{M}_c(\mathbb{R}^n)$ is the space of signed measures on $\mathbb{R}^n$ having total mass less than $c \in \mathbb{R}_+$). Furthermore, we require that V_0 grows to plus infinity as $|x| \to +\infty$, and there is a function V, satisfying the relation (22) and Hypothesis QV, such that $V_0(x) \sim V(x)$ as $|x| \to +\infty$ (where $\sim$ stands for $V_0(x)$ is bounded from above and below by positive constants times $V(x)$ as $|x| \to +\infty$). We also assume that v_0 is bounded, reflection symmetric, i.e., $v_0(x) = v_0(-x)$, and that there exists a positive measure π on $\mathbb{R}^n$ such that, for any $x \in \mathbb{R}^n$, $v_1(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-ikx}\pi(\mathrm{d}k)$ (i.e. v_1 is the Fourier transform of a positive measure). The class of functionals (24) satisfy the above Hypotheses $\mathcal{V}$ and C$\mathcal{V}$.

First we have that if the vector field b in Eq. (16) is such that $J(b, x_0) < +\infty$, then there is a unique invariant measure ρ_b of Eq. (16).

Lemma 4.1 *Under hypotheses* $\mathcal{V}$*(i) and* $\mathcal{V}$*(ii), if* $J(b, x_0)$ *as given by* (17) *(with* $b \in C^1$*) is not equal to* $+\infty$ *there exists an unique and ergodic invariant probability density measure* $\rho_b \in W^{1,\frac{n}{2}}(\mathbb{R}^n)$ *for the SDE* (16) *so that* $\mu_b(\mathrm{d}x) = \rho_b(x)\mathrm{d}x$ *is the invariant ergodic probability measure for the SDE* (16). *Furthermore, we have*

$$\tilde{J}(b, \rho_b) \leq J(b, x_0)$$

for almost all $x_0 \in \mathbb{R}^n$ *with respect to Lebesgue measure, where*

$$\tilde{J}(b, \rho_b) := \int_{\mathbb{R}^n} \left(\frac{|b(x)|^2}{2} + \mathcal{V}(x, \rho_b) \right) \rho_b(x) \mathrm{d}x. \tag{25}$$

Proof The proof is given in [4]. □

In order to minimize the cost functional (25) with respect to ρ, for $\rho \in W^{1,\frac{n}{2}}(\mathbb{R}^n)$, $\rho(x) \geq 0$ and $\int_{\mathbb{R}^n} \rho(x)\mathrm{d}x = 1$, we set

$$\mathcal{C}_\rho = \{b \in C^1(\mathbb{R}^n, \mathbb{R}^n),\ L_b^*(\rho) = 0 \text{ and } |\tilde{J}(b, \rho)| < +\infty\}. \tag{26}$$

Then $\mathcal{C}_\rho$ is the subset of $C^1(\mathbb{R}^n, \mathbb{R}^n)$ vector fields $b_\rho \in \mathcal{C}_\rho$ such that $L_{b_\rho}^*(\rho) = 0$ (where $L_{b_\rho}^*$ is the adjoint of the infinitesimal generator L_{b_ρ} for the solution process X_t of Eq. (16) and the equality, in the definition of $\mathcal{C}_\rho$ in (26), is understood in a distributional sense) and $|\tilde{J}(b_\rho, \rho)| < +\infty$.

Remark 4.3 Suppose that $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $J(b, x_0) < +\infty$, then by Lemma 4.1 there is a unique positive probability density ρ_b which is invariant and thus, since $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ by well-known results (see Proposition 3.1 in [4]), it satisfies the equation $L_b^*(\rho_b) = 0$. This implies that $b \in \mathcal{C}_{\rho_b}$, where $\mathcal{C}_{\rho_b}$ is defined by Eq. (26) with $\rho = \rho_b$.

We now introduce the following energy functional, for $\rho \in W^{1,\frac{n}{2}}(\mathbb{R}^n)$,

$$\mathcal{E}(\rho) := \mathcal{E}_K(\rho) + \mathcal{E}_P(\rho) = \int_{\mathbb{R}^n} \frac{|\nabla\rho|^2}{2\rho}\mathrm{d}x + \int_{\mathbb{R}^n} \mathcal{V}(x, \rho)\rho(x)\mathrm{d}x, \tag{27}$$

where the two terms on the right-hand side correspond by definition to the kinetic $\mathcal{E}_K(\rho)$ and potential $\mathcal{E}_P(\rho)$ energies, respectively. The kinetic term is also called the Fisher information.

The next lemma states a useful monotonicity property of the cost functional $\tilde{J}$.

Lemma 4.2 *For any given* $\rho \in W^{1,\frac{n}{2}}(\mathbb{R}^n)$ *we have*

$$\mathcal{E}(\rho) = \tilde{J}\left(\frac{\nabla\rho}{\rho}, \rho\right) \leq \inf_{b \in \mathcal{C}_\rho} \tilde{J}(b, \rho),$$

where $\tilde{J}(b, \rho)$ *is defined in* (25).

Proof By [25, Chapter 3, Theorem 3.1.2], if ρ is the density of the invariant measure of the SDE (16) we have that

$$\int_{\mathbb{R}^n} \frac{|\nabla\rho(x)|^2}{\rho^2(x)}\rho(x)\mathrm{d}x \leq \int_{\mathbb{R}^n} |b(x)|^2\rho(x)\mathrm{d}x,$$

for any $b \in \mathcal{C}_\rho$, with the equality holding if and only if $b = \frac{\nabla\rho}{2\rho}$. Since $\int_{\mathbb{R}^n} \mathcal{V}(x,\mu)\rho(x)\mathrm{d}x$ depends only on the invariant measure $\rho(x)\mathrm{d}x$, the lemma is proved. □

Let us now minimize the function $\mathcal{E}(\rho)$ given by (27) under the condition $\int_{\mathbb{R}^n} \rho(x)\mathrm{d}x = 1$. Introducing the variable $\varphi = \sqrt{\rho}$ the energy functional (27) becomes

$$\mathcal{E}(\varphi^2) = \int_{\mathbb{R}^n} \left(\frac{|\nabla\varphi|^2}{2} + \mathcal{V}(x,\varphi^2)\varphi^2(x) \right) \mathrm{d}x, \tag{28}$$

with $\varphi \in L^2(\mathbb{R}^n)$ satisfying the condition $\int_{\mathbb{R}^n} \varphi^2(x)\mathrm{d}x = 1$.

Remark 4.4 The above well-known energy functional admits a unique minimizer which is strictly positive (see [4] and references therein). Furthermore, in the case where $\mathcal{V}$ is of the form (24), it coincides with the functional (5) of Bose–Einstein condensation in the mean field regime ($\beta = 0$ in relation (6)).

Lemma 4.3 *Under hypotheses $\mathcal{V}$ and $C\mathcal{V}$ the variational problem* (27)*, with $\varphi \in L^2(\mathbb{R}^n)$ satisfying the condition $\int_{\mathbb{R}^n} \varphi^2(x)\mathrm{d}x = 1$, admits a unique minimizer $\rho_0 = \varphi_0^2$. Furthermore, φ_0 is $C^{2+\epsilon}(\mathbb{R}^n)$ for some $\epsilon > 0$, it is strictly positive and satisfies (weakly) the equation*

$$-\Delta\varphi_0(x) + 2\mathcal{V}(x,\varphi_0^2)\varphi_0(x) + 2\int_{\mathbb{R}^n} \partial_\mu \mathcal{V}(y,x,\varphi_0^2)\varphi_0^2(y)\mathrm{d}y\varphi_0(x) = \mu_0\varphi_0(x), \tag{29}$$

where the uniquely determined constant μ_0 is given by

$$\mu_0 = 2\mathcal{E}(\varphi_0^2) + \int_{\mathbb{R}^n} \partial_\mu \mathcal{V}(y,x,\varphi_0^2)\varphi_0^2(y)\varphi_0^2(x)\mathrm{d}y\mathrm{d}x. \tag{30}$$

Remark 4.5 Under Hypotheses $\mathcal{V}$ and C$\mathcal{V}$ we have that $J\left(\frac{\nabla\rho_0}{\rho_0}, x_0\right) = \mathcal{E}(\rho_0)$, where $\rho_0 = \varphi_0^2$ is the unique minimizer of $\mathcal{E}$.

Finally we obtain our generalization, via an optimal control approach, of stochastic mechanics versus non-linear quantum models:

Theorem 4.1 *Under Hypotheses $\mathcal{V}$ and $C\mathcal{V}$, the logarithmic gradient of the unique minimizer $\rho_0 = \varphi_0^2$ of $\mathcal{E}$, that is $b = \frac{\nabla\rho_0}{2\rho_0}$, is the optimal control for the problem* (17) *for almost every $x_0 \in \mathbb{R}^n$ with respect to the Lebesgue measure.*

Proof (of Theorem 4.1) By Remark 4.5, and the definition of $\mathfrak{J}$ (given in Eq. (18)) we have that

$$\mathfrak{J} \leq \text{ess sup}_{x_0 \in \mathbb{R}^n} J\left(\frac{\nabla \rho_0}{\rho_0}, x_0\right) = \mathcal{E}(\rho_0). \tag{31}$$

In order to prove the statement of the theorem, it is sufficient to prove that $\mathcal{E}(\rho_0) \leq \mathfrak{J}$, indeed, by Lemma 4.5 and inequality (31), this implies that

$$\mathfrak{J} \leq \text{ess sup}_{x_0 \in \mathbb{R}^n} J\left(\frac{\nabla \rho_0}{\rho_0}, x_0\right)$$

and thus the thesis. By Lemma 4.1, we have $\tilde{J}(b, \rho_b) \leq J(b, x_0)$ and by Lemma 4.2, and since, by Remark 4.3, $b \in \mathcal{C}_{\rho_b}$, we get, for any fixed $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $J(b, x_0) < +\infty$,

$$\mathcal{E}(\rho_b) = \tilde{J}\left(\frac{\nabla \rho_b}{\rho_b}, \rho_b\right) \leq \inf_{\hat{b} \in \mathcal{C}_{\rho_b}} \tilde{J}(\hat{b}, \rho_b) \leq \tilde{J}(b, \rho_b).$$

Combining the previous two inequalities and Lemma 4.3, we obtain that, for any $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $J(b, x_0) < +\infty$,

$$\mathcal{E}(\rho_0) \leq \mathcal{E}(\rho_b) \leq \tilde{J}(b, \rho_b) \leq \text{ess sup}_{x_0 \in \mathbb{R}^n} J(b, x_0).$$

Taking the inf over $b \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ from the previous inequality we get $\mathcal{E}(\rho_0) \leq \mathfrak{J}$. □

Remark 4.6 An important consequence of Theorem (4.1) is that under Hypotheses $\mathcal{V}$ and C$\mathcal{V}$ we have that

$$\mathfrak{J} = \mathcal{E}(\rho_0) = \inf_{\varphi \in H^1(\mathbb{R}^n), \int \varphi^2 dx = 1} \mathcal{E}(\varphi^2),$$

where $\mathfrak{J}$ is the value function associated with the problem (16) and the cost functional (17), defined by (18).

Summarizing we proved that the (stationary) Nelson diffusion with drift of gradient type solves the ergodic optimal control problem with cost functional (17), with $\mathcal{V}$ satisfying Hypothesis $\mathcal{V}$, C$\mathcal{V}$ and Q$\mathcal{V}$. In particular our result contains the mean-field nonlinear Schrödinger model for the Bose–Einstein condensate (in this case where the potential $\mathcal{V}$ in the cost functional (17) is given by (24)).

5 Convergence of Markovian N-Particle Approximation

We are interested in studying a Markovian N-particle approximation to the stochastic optimal control problem given by (16) and (17). This approximation is inspired by the variational version of stochastic mechanics presented in Sect. 3.

We consider the process $X_{N,t} = (X^1_{N,t}, \ldots, X^N_{N,t}) \in \mathbb{R}^{nN}$ satisfying the SDE

$$\mathrm{d}X^i_{N,t} = b^i_N(X_{N,t})\mathrm{d}t + \nu \mathrm{d}W^i_t, \quad i = 1, \ldots, N \tag{32}$$

where $b_N := (b^1_N, \ldots, b^N_N) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ is a $C^{1+\epsilon}$ function, for some $\epsilon > 0$, and the $W^i_t, i = 1, \ldots, N$ are independent Brownian motions taking values in $\mathbb{R}^n$. If $\mathcal{V}$ is a functional satisfying Hypotheses $\mathcal{V}$, we introduce the sequence

$$\mathcal{V}_N(x) = \sum_{i=1}^{N} \mathcal{V}\left(x_i, \frac{1}{N-1}\sum_{k=1,k\neq i}^{N} \delta_{x^i}\right),$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^{nN}$, $N \geq 2$, and δ_{x_i} is a Dirac delta measure in $x_i \in \mathbb{R}^n$. We consider the ergodic control problem (normalized with respect to the number N of particles)

$$J_N(b_N, x_0) = \limsup_{T\to+\infty} \frac{1}{NT}\int_0^T \mathbb{E}_{x_0}\left[\frac{|b_N(X_t)|^2}{2} + \mathcal{V}_N(X_t)\right]\mathrm{d}t. \tag{33}$$

The corresponding (normalized) energy functional (analogous to the one defined in (27)) is

$$\begin{aligned}\mathcal{E}_N(\rho_N) &= \mathcal{E}_{K,N}(\rho_N) + \mathcal{E}_{P,N}(\rho_N)\\ &= \frac{1}{N}\left(\int_{\mathbb{R}^{nN}} \frac{|\nabla\rho_N|^2}{2\rho_N}\mathrm{d}x + \int_{\mathbb{R}^{nN}} \mathcal{V}_N(x)\rho_N(x)\mathrm{d}x\right),\end{aligned} \tag{34}$$

where ρ_N is a positive Lebesgue integrable function such that $\int_{\mathbb{R}^{nN}} \rho_N(x)\mathrm{d}x = 1$. We also consider the value function

$$\mathfrak{I}_N = \operatorname{ess\,sup}_{x_0\in\mathbb{R}^n}\left(\inf_{b_N\in C^1(\mathbb{R}^{nN},\mathbb{R}^{nN})} J_N(b_N, x_0)\right). \tag{35}$$

Remark 5.1 It is important to note that in the case where $\mathcal{V}$ is of the form (24) (i.e. for the mean-field BEC), from the definition of $\mathcal{V}_N$, we get:

$$\mathcal{V}_N(x_1, \ldots, x_N) = \sum_{k=1}^{N} V(x_k) + \frac{1}{N-1}\sum_{k,h=1}^{N} v_0(x_k - x_h),$$

which is exactly the total potential of the Hamiltonian H_N in (1), where v_N is the mean field scaling limit of v_0 (see Eq. (6) in the case $\beta = 0$).

In the setting described above we are able to prove two convergence results: the first one involves the invariant measure $\rho_{0,N}$ of the optimal controlled process $X_{N,t}$, the second concerns the law of the process $X_{N,t}$ on the whole path space $C^0([0,T], \mathbb{R}^{nN})$. In order to state the two convergence results we introduce the relative entropy between probability densities ρ_N, ρ'_N on $\mathbb{R}^{nN}$ resp. between probability measures $\mathbb{P}, \mathbb{Q}$ on the path space $C^0([0,T], \mathbb{R}^n)$ as

$$H_N(\rho_N|\rho'_N)) := \begin{cases} \int_{\mathbb{R}^{nN}} \log\left(\frac{\rho_N(x)}{\rho'_N(x)}\right) \rho_N(x)\mathrm{d}x & \text{if } \mathrm{supp}(\rho_N) \subset \mathrm{supp}(\rho'_N) \\ +\infty & \text{elsewhere} \end{cases},$$

$$\mathcal{H}_{C^0([0,T],\mathbb{R}^n)}(\mathbb{P}|\mathbb{Q}) := \int_\Omega \log\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(\omega)\right) \mathbb{P}(\mathrm{d}\omega).$$

Theorem 5.1 *Suppose that $\mathcal{V}$ satisfies hypotheses $\mathcal{V}$, $C\mathcal{V}$ and $Q\mathcal{V}$ and let $\rho_{0,N}$ and ρ_0 be the unique minimizers of the energies (34) and (27) respectively, then we have as $N \to +\infty$*

$$\frac{1}{N} H_N(\rho_{0,N}|\rho_0^{\otimes N}) \to 0. \tag{36}$$

Remark 5.2 It is important to note that if $\rho_{0,N}^{(k)}(x_1, \ldots, x_k)$ denotes the marginal of the measure $\rho_{0,N}(x_1, \ldots, x_N)$ with respect to the first k variables, then Theorem 5.1 implies $H_k(\rho_{0,N}^{(k)}|\rho_0^{\otimes k}) \to 0$ (see [4] for the details). By the Csiszar–Kullback inequality [39, 57], this implies that $\rho_{0,N}^{(k)}$ converges to $\rho_0^{\otimes k}$ in $L^1(\mathbb{R}^{nk})$ and it also means that the N-particle system (32) (when evaluated at the optimal control $b_N = \frac{1}{2}\nabla \log \rho_{N,0}$) is Kac and entropy chaotic (see, e.g., [53] for the definition of these properties).

Let $\mathbb{P}_{0,N}$ be the probability law of the process $X_{N,t}$, on the path space $C^0([0,T], \mathbb{R}^{nN})$, for the case where $b_N(x) = \frac{1}{2}\nabla \log(\rho_{0,N}(x))$, $x \in \mathbb{R}^{nN}$ is the optimal control and the law of the initial condition $X_{N,0}$ is the invariant (optimal) measure $\rho_{0,N}$. We denote by $\mathbb{P}_{0,N}^{(k)}$ the marginal of $\mathbb{P}_{0,N}$ on the path space of the first k particles $C^0([0,T], \mathbb{R}^{nk})$. Finally, let $\mathbb{P}_0$ be the law of the process X_t solution to Eq. (16), with the optimal control $b(x_1) = \frac{1}{2}\nabla \log(\rho_0(x_1))$, $x_1 \in \mathbb{R}^n$, and starting at the invariant measure ρ_0.

Theorem 5.2 *Under hypotheses $\mathcal{V}$, $C\mathcal{V}$ and $Q\mathcal{V}$ we have that for any $k \in \mathbb{N}$*

$$\lim_{N\uparrow+\infty} \mathcal{H}_{C^0([0,T],\mathbb{R}^{nk})}(\mathbb{P}_{0,N}^{(k)}|\mathbb{P}_0^{\otimes k}) = 0. \tag{37}$$

Proof ***(Idea of the Proof of Theorem 5.2)*** One first proves that the "value function" $\frac{1}{N}\mathfrak{I}_N$ of the N-particle system converges, as $N \to +\infty$, to $\mathfrak{I}$, i.e. the "value function" of the limit problem given by (16) and (17). A stronger result holds: the kinetic part of the energy of the N-particle system converges to the kinetic energy of the limit problem, namely

$$\lim_{N\to+\infty} \frac{1}{N} \int_{\mathbb{R}^{nN}} \frac{|\nabla \rho_{0,N}(x_1,\dots,x_N)|^2}{2\rho_{0,N}(x_1,\dots,x_N)} \mathrm{d}x_1 \dots \mathrm{d}x_N = \int_{\mathbb{R}^n} \frac{|\nabla \rho_0(x_1)|^2}{2\rho_0(x_1)} \mathrm{d}x_1, \tag{38}$$

(see [4, Theorem 5.1]). Furthermore, if $b_N(x) = \frac{1}{2}\nabla \log(\rho_{0,N}(x))$ and $b(x_1) = \frac{1}{2}\nabla \log(\rho_0(x_1))$, by Girsanov's theorem we get that

$$\frac{1}{N}\mathcal{H}_{C^0([0,T],\mathbb{R}^{nN})}(\mathbb{P}_{0,N}|\mathbb{P}_0^{\otimes N}) = \mathbb{E}_{\mathbb{P}_{0,N}}[|b_N^1(X_{N,t}) - b(X_{N,t}^1)|^2],$$

(with the upper index 1 indicating the first $\mathbb{R}^n$ component of the corresponding vector in $\mathbb{R}^{nN}$). Using Eq. (29) we get

$$\begin{aligned}\frac{1}{N}\mathbb{E}_{\mathbb{P}_{0,N}}[|b_N^1(X_s) - b(X_s^1)|^2] &= \int_{\mathbb{R}^{nN}} \frac{|\nabla_1 \varphi_{0,N}(x)|^2}{2}\mathrm{d}x - \mu_0 \\ &+ \int_{\mathbb{R}^{nN}} 2\left(\mathcal{V}(x^1,\rho_0) - \int_{\mathbb{R}^n} \partial_\mu \mathcal{V}(y,x^1,\rho_0)\rho(y)\mathrm{d}y\right)\varphi_{0,N}^2(x)\mathrm{d}x, \end{aligned} \tag{39}$$

where $\varphi_{0,N} = \sqrt{\rho_{0,N}}$. Exploiting the explicit formula (30) for μ_0, the convergence of $\frac{1}{N}\mathfrak{I}_N$ to $\mathfrak{I}$, for $N \to +\infty$, and the limit (38) we obtain that $\lim_{N\to+\infty} \frac{1}{N}\mathcal{H}_{C^0([0,T],\mathbb{R}^{nN})}(\mathbb{P}_{0,N}|\mathbb{P}_0^{\otimes N}) = 0$. Finally, by the inequality

$$\mathcal{H}_{C^0([0,T],\mathbb{R}^{nk})}(\mathbb{P}_{0,N}^{(k)}|\mathbb{P}_0^{\otimes k}) \le \frac{k}{N}\mathcal{H}_{C^0([0,T],\mathbb{R}^{nN})}(\mathbb{P}_{0,N}|\mathbb{P}_0^{\otimes N}), k = 1,\dots,N,$$

see [4], we get the thesis. □

6 The Case of the Dirac Delta Potential

In this section we propose to the reader a potential $\mathcal{V}$ of the following form

$$\mathcal{V}_\delta(x,\mu) = V_0(x) + g\delta_x * \mu, \tag{40}$$

where V_0 is a regular positive function growing at infinity ("trapping potential"), δ_x is the Dirac delta centered at $x \in \mathbb{R}^n$, $g \in \mathbb{R}_+$ is a strictly positive constant, $*$ stands for convolution, and μ is a probability measure. The potential $\mathcal{V}_\delta$ does not satisfy the regularity Hypotheses $\mathcal{V}$(i) and $\mathcal{V}$(iii). On the other hand it satisfies Hypothesis $\mathcal{V}$(ii) and C$\mathcal{V}$, and (whenever the Gâteaux derivative is well defined) we have $\partial^2_\mu(\tilde{\mathcal{V}}_\delta) = 2\delta_{x-y}$, where $\tilde{\mathcal{V}}_\delta = \int_{\mathbb{R}^n} \mathcal{V}(x,\mu)\mu(\mathrm{d}x)$, which is a positive definite distribution.

Here we do not consider the problem of proving that the optimal control ergodic problem has a unique optimal control (i.e. we do not prove here the equivalent of Theorem 4.1 for the potential (40)). We suppose that there exists a family $\mathcal{C}_{V_0} \subset C^1(\mathbb{R}^n,\mathbb{R}^n)$ of vector fields b on $\mathbb{R}^n$ (in general we expect that it can depend on the trapping potential V_0 in (40)) such that

$$\inf_{b\in\mathcal{C}_{V_0}} \left(\limsup_{T\to+\infty} \frac{1}{T} \left(\int_0^T \mathbb{E}_{x_0} \left[\frac{|b(X_t)|^2}{2} + V_0(X_t) + g\rho_{x_0,\alpha,t}(X_t) \right] \mathrm{d}t \right) \right) = \mathbb{E}_{X_0\sim\rho_0(x)\mathrm{d}x} \left[\frac{|\nabla\rho_0(X_t)|^2}{4\rho_0^2(X_t)} + V_0(X_t) + g\rho_0(X_t) \right], \tag{41}$$

where $\rho_{x_0,b,t}$ is the probability density of the law of the solution to the SDE (16) starting at $x_0 \in \mathbb{R}^n$ evaluated at time t, and ρ_0 is the density of the probability distribution minimizing the functional

$$\mathcal{E}_\delta(\rho) = \mathcal{E}_K(\rho) + \mathcal{E}_{\delta,P}(\rho) = \int_{\mathbb{R}^n} \left(\frac{|\nabla\rho(x)|^2}{4\rho(x)} + V_0(x)\rho(x) + g(\rho(x))^2 \right) \mathrm{d}x. \tag{42}$$

In other words we suppose that in the set $\mathcal{C}_\rho$ (introduced in (26)) the optimal control for the problem (16) with cost functional (17) and potential $\mathcal{V}_\delta$ (see [16] for an alternative derivation of a stochastic process associated with the above cost functional) exists and it is given by $b = \frac{\nabla\rho_0}{2\rho_0}$. What we want to consider here is an N-particle problem converging to the solution of the optimal control ergodic problem just described (namely we are looking for an analogous of Theorem 5.2 for the case where $\mathcal{V}$ is given by $\mathcal{V}_\delta$ in (40)).

In general, since $\mathcal{V}_\delta$ is not well-defined for positive measures μ that are not absolutely continuous measures, let us then consider an approximating potential of the form

$$\mathcal{V}_{\delta,N}(x,\mu) = V_0(x) + \int_{\mathbb{R}^n} v_N(x-y)\mu(\mathrm{d}y),$$

where $v_N : \mathbb{R}^n \to \mathbb{R}$ is a sequence of positive functions converging in the sense of distributions to a Dirac delta δ_0 when $N \to \infty$. Let us choose a specific sequence of the following form

$$v_N(\mathbf{r}) = \frac{N^{3\beta}}{N-1} v_0(N^\beta \mathbf{r}), \qquad x \in \mathbb{R}^n \tag{43}$$

for $\beta > 0$, where v_0 is a positive smooth radially symmetric function with compact support (as in formula (6)). We take the N-particle approximation having the control $b_N(x_1, \ldots, x_N)$ given by the logarithmic derivative of $\rho_{0,N}$, that is the minimal probability density of the energy functional $\mathcal{E}_\delta$ associated with $\mathcal{V}_{\delta,N}$, namely

$$\begin{aligned}\mathcal{E}_{\delta,N}(\rho) &= \mathcal{E}_{K,N}(\rho) + \mathcal{E}_{\delta,P,N}(\rho)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left(\int_{\mathbb{R}^{Nn}}\left(\frac{|\nabla_i \rho|^2}{4\rho} + V_0(x_i)\rho\right)\mathrm{d}x\right.\\ &\qquad\left. + \frac{1}{N-1}\sum_{j=1,\ldots,N,\, j\neq i}\int_{\mathbb{R}^{Nn}} v_N(x_i - x_j)\rho \mathrm{d}x\right).\end{aligned}$$

In the rest of the paper we show how the results on Bose–Einstein condensation (mainly for $n = 3$, see, e.g., [61–65, 69, 75, 82]) can be used to study the convergence of the N-particles approximation of the control problem with potential (40). For this reason hereafter we shall limit our discussion to the case $n = 3$.

6.1 The Intermediate Scaling Limit

The case $0 < \beta < 1$, where β is the parameter used in the rescaling (43), which is known as intermediate scaling limit, is very similar to the regular case ($\beta = 0$) that we discussed in Sect. 5. Indeed, in this case we can prove the following theorem.

Theorem 6.1 *Under the previous hypotheses and notations, if* $0 < \beta < 1$ *we have, as* $N \to +\infty$*, the convergence statements* $\mathcal{E}_{\delta,N}(\rho_{0,N}) \to \mathcal{E}_\delta(\rho_0)$*,* $\mathcal{E}_{\delta,P,N}(\rho_{0,N}) \to \mathcal{E}_{\delta,P}(\rho_0)$ *and* $\rho^{(1)}_{0,N} \to \rho_0$ *(where the last convergence is in the weak* L^1 *sense) with the constant* $g = \int_{\mathbb{R}^3} v_0(x)\mathrm{d}x$ *(where* $g \in \mathbb{R}_+$ *is the constant appearing in Eqs.* (40) *and* (41)*).*

Proof The proof of the theorem can be found in [62] for $0 \leq \beta < \frac{1}{3}$ (for any n and a more general class of potentials v_0 than the one considered here) and in [5] for $0 \leq \beta < 1$ (for $n = 3$ and a positive-definite interaction potential v_0). See also [83]. □

Theorem 6.1 is the analogue of the results mentioned in the proof of Theorem 5.2 in this context. Thanks to Theorem 6.1 we can repeat the reasoning performed in the proof of Theorem 5.2, obtaining:

Theorem 6.2 *Under the previous hypotheses and notations, if* $0 < \beta < 1$ *we have that the law* $\mathbb{P}^{(k)}_{0,N}$ *of the first* k *particles satisfying the system* (32)*, with* $\mathcal{V}$ *replaced* *by $\mathcal{V}_\delta$, converges in total variation on the path space $C^0([0,T],\mathbb{R}^{3k})$ to $\mathbb{P}_0^{\otimes k}$ (where $\mathbb{P}_0$ is the law on $C^0([0,T],\mathbb{R}^3)$ of the system (16) associated with (40)).*

Proof The proof can be found in [5]. □

6.2 The Gross–Pitaevskii Scaling Limit

The case $\beta = 1$ is completely different from the previous ones. The main difference between the cases $0 < \beta < 1$ and $\beta = 1$ is that in this latter case the value function convergence result of Theorem 6.1 does not hold.

Theorem 6.3 *Under the previous hypotheses and notations, if $\beta = 1$ we have that, as $N \to \infty$ $\mathcal{E}_{\delta,N}(\rho_{0,N}) \to \mathcal{E}_\delta(\rho_0)$ and $\rho^{(1)}_{0,N} \to \rho_0$ (where the latter convergence is in the weak sense in L^1) for $g = 4\pi a$ (where $g \in \mathbb{R}_+$ is the constant appearing in Eqs. (40) and (41), and $a > 0$ is the scattering length of the interaction potential v_0 (see [64])). Furthermore, putting $\hat{s} = \frac{1}{g}\int_{\mathbb{R}^3} \frac{|\nabla\rho_0|^2}{\rho_0}\mathrm{d}x \in (0,1)$ we have, as $N \to +\infty$:*

$$\mathcal{E}_{K,N}(\rho_{0,N}) \to \mathcal{E}_{\delta,K}(\rho_0) + g\hat{s}\int_{\mathbb{R}^3} \rho_0^2(x)\mathrm{d}x.$$

Proof The proof of the first part of the theorem is a well-known result proven in [63, 65, 75]. The second part is proved in [65]. □

In this case we cannot repeat the reasoning of Theorem 5.2 since we are not able to prove that the relative entropy $\mathcal{H}(\mathbb{P}^{(k)}_{0,N}|\mathbb{P}_0^{\otimes k})$ converges to 0 (in fact we do not know whether the relative entropy converges to 0 or to another value). On the other hand it is possible to prove a weaker result for $\beta = 1$, we have namely:

Theorem 6.4 *Under the previous hypotheses and notations, if $\beta = 1$ we have that the law $\mathbb{P}^{(k)}_{0,N}$ converges weakly on the path space $C^0([0,T],\mathbb{R}^{3k})$ to $\mathbb{P}_0^{\otimes k}$.*

Proof The proof can be found in [6]. □

Remark 6.1 In [72] a different kind of convergence is proven and in [86] a transition to chaos result for the particle system related to the control problem is obtained.

7 Future Research Lines

We plan to extend our stochastic approach to BEC in three different directions. First it would be interesting to face the more difficult Gross–Pitaevskii scaling limit ($\beta = 1$) with a similar optimal control approach. Since this scaling limit gives rise to a singular action functional we could try to extend stochastic mechanics to this non-linear singular Schrödinger model on one hand and obtain the solution to a singular optimal control problem. Since our ergodic control problem can be looked upon as being of McKean–Vlasov type, both the drift of the SDE and the potential depending on the probability density of the invariant measure, that is, on the law of the stochastic process, we hope to be able to prove the complete BEC (in the sense of Definition 2.1) and its justification by taking advantage of the advanced stochastic techniques developed in connection with the well-studied McKean–Vlasov optimal control problem (see, e.g., [21, 33, 58], and also [19, 85]). Finally, a big effort would be needed to extend our stochastic approach to the general time-dependent setting. This is not a direct consequence of the ground state case, even in the mathematical physics approach. Indeed, the proof of BEC with a time-dependent wave function requires different techniques (see, e.g., [1, 20, 27, 28, 43]).

Let us close with a look back to the basic problems underlying the study in this article, in order to insert them into a "future research line" prospective. Since the Enlightenment, much of the development of mathematics has been influenced by problems that arose in connection with the investigation of nature, in particular physical phenomena. Especially in the last century and into the present one, the description and interpretations of quantum phenomena have played an important role (for instance, in relation to classical deterministic and stochastic dynamical systems, among other examples). For the description of quantum phenomena ideas and methods coming from the theory of infinite-dimensional spaces, and operators on them, play a central role on the "abstract level" (accompanied by a more "concrete level", like the Schrödinger equation of non-relativistic quantum mechanics). But this is certainly not the whole story, as is seen from the early steps of quantum mechanics itself, where other areas of mathematics entered and got enriched in one way or the other, e.g., the representation theory of Lie groups (to express transformation properties of observables and conservation laws), see, e.g., [66]. Variational principles also played a founding role, inasmuch as quantum mechanics can be seen as a deformation of classical mechanics, and reciprocally (see, e.g., [89]), and the most important variational principles have originated in classical mechanics (see, e.g., [2, 22] and references therein). These influences are certainly present in the genesis of R. Feynman's reformulation of quantum mechanics in terms of the famous heuristic "Feynman path integral", that became quite important both in physics and mathematics inspired by physics. We recall that Feynman's original approach consisted in describing the quantum mechanical evolution by an "integral kernel" of the form $e^{\frac{i}{\hbar}S(\gamma)}$ (i being the imaginary unit, $\hbar$ Plank's constant and $S(\gamma)$ the action functional, i.e., a time integral of a Lagrangian, for a path γ in a space of continuous paths). A heuristic variational principle would then permit us to get classical orbits from integrals involving $e^{\frac{i}{\hbar}S(\gamma)}$ expressing, for example, the solutions of Schrödinger equation, in the "semiclassical limit" (where $\hbar$ is considered to be very small). This general programme has found some mathematical realization in single cases, e.g. non-relativistic quantum mechanics in flat space (see, e.g., [11, 15, 68]) and also in some more geometric settings and for quantum fields (see, e.g, [10, 60]). But the mathematical and physical potentiality is much richer, even at the non-relativistic level, see the deep discussions of this issue in the work of J.-C. Zambrini (e.g. [88, 89]). Another aspect of mathematics developed in connection with quantum mechanics is stochastic analysis. It has its origins in the 1923 work of N. Wiener on Brownian motion, where the heat semigroup kernel plays a role similar to the above Feynman kernel. It yields the solutions of the heat equation, the parabolic analogue of the Schrödinger equation (but Wiener himself was also interested in quantum mechanics, see [67]). The "Wiener path integral" and its transformations play an important role in stochastic differential equations, invented by mathematicians like S. Bernstein (1932) and K. Itô (1948). It is emblematic that the same K. Itô who founded the probabilistic Itô calculus also gave a first approach to the Feynman path integral ([54, 55]; for further developments, see, e.g., the references [10, 11, 68] cited above). As we mentioned in Sect. 3, Nelson's stochastic mechanics is a probabilistic approach to quantum mechanics; Euclidean methods in quantum field theory (also strongly influenced by Nelson as a tool for the construction of relativistic, hence hyperbolic, thus Feynman's type) quantum fields are also ways to bring together Feynman's methods and probabilistic methods; there is a further approach, put forward by J.-C. Zambrini, and called by him "Euclidean stochastic mechanics", that exports to the world of probability structures that are somewhat hidden inside Feynman's hyperbolic formalism. Zambrini actually took much inspiration from Schrödinger's work, based on a time symmetric view of the heat equation (rather than Schrödinger's equation, see, e.g., [89]). Our point is to observe that there is an immense amount of work to be done in mathematics to better understand all these interwoven and fascinating structures.

The relations between Nelson's stochastic mechanics and variational principles in the study of certain quantum mechanical problems ([51, 87] mentioned in Sect. 3) have also generated a lot of interest in the study of the "Schrödinger probabilistic problem". Here a lot of activity has recently been developed, e.g., in [17, 35, 46, 59, 74, 88]. In this line there are also connections with probabilistic and analytic works on optimal transport (see, e.g., [36, 59, 70]) that deserve much further attention. The world of systems of many quantum particles, and their limits (see [19, 85]), proper of quantum statistical mechanics, is another area of application where such methods should be very useful. This also would imply applications to other areas of science like biology, mathematical finance and game theory (see for instance [33, 34, 41]).

A final comment concerns the developments of similar constructions and connections in the world of fields (and strings) instead of particles. This can be seen as an infinite-dimensional extension of the work we just discussed in relation to non-relativistic quantum theory, as the path γ at a fixed time t in the above Feynman approach, instead of taking values in a finite-dimensional space, would take values in infinite-dimensional spaces. Here the constraints of invariance with respect to transformation groups, imposed by the Poincaré invariance of relativistic quantum fields, causes in addition worse local singularities for the paths (much stronger than the non-differentiability of the Wiener paths), and forces regularizations and limits much more involved and challenging than those involved in the non-relativistic world. The Euclidean methods of constructive quantum field theory based on a construction at imaginary time ("heat equation world") followed by an analytic continuation of the relevant correlation functions to real time ("Schrödinger equation world") have been useful for the construction and the study of models in space-time dimensions up to 3 (see, e.g., [48, 77, 84], for similar methods for path integrals in quantum statistical mechanics see, e.g., [12]). Recently new constructive methods based on a singular stochastic partial differential equation (of the Parisi–Wu stochastic quantization type) initiated by Hairer and Gubinelli–Imkeller–Perkowski have been developed, see for instance [14, 49, 50, 52]. Also here there are relations to variational principles [18] and elliptic methods [3]. However a fully fledged transposition to the field case of the methods related to stochastic mechanics and the corresponding variational methods has still to be elaborated.

Another aspect that might be useful to examine more closely is that in a certain limit (like the non-relativistic one starting from relativistic models with polynomial interactions) the Hilbert space becomes a direct sum of spaces with a fixed number N of particles with point interactions, similar to the non-relativistic models used in Sects. 2–6 for deriving asymptotically, for $N \to +\infty$, the Gross–Pitaevskii equation. The N-scaling used there for $\beta = 1$ is a prototype of the quantum field renormalization procedures and has been used to give a meaning to the Hamiltonian (1) in the case where v_0 is a Dirac delta distribution, see references [7, 9, 42, 47] (where interesting connections with the Efimov and Thomas atomic physics effects are discussed). It also gives a meaning to the non-relativistic limit of the mentioned quantum theoretical models at least in space time dimension 2 (see the references in [7]). Similar methods might also be helpful in dimension 3 and 4, but more work has definitely to be done.

In conclusion, we mentioned some open problems involving processes with finite-dimensional resp. infinite-dimensional state space, and both using infinite-dimensional analysis: there is indeed plenty of room for new developments out there, and many more beautiful flowers to be found!

Acknowledgements The first named author thanks the DFG for the financial support via the grant AL 214/50-1 "Invariant measures for SPDEs and Asymptotics". The first and second named authors are also funded by the DFG under Germany's Excellence Strategy—GZ 2047/1, project-id 390685813.

References

1. R. Adami, F. Golse and A. Teta. Rigorous derivation of the cubic NLS in dimension one. *Journal of Statistical Physics* **127**, no. 6, 1193–1220 (2007).
2. S. Albeverio, A.B. Cruzeiro and D. Holm, editors. *Stochastic geometric mechanics*. Volume 202 of Springer Proceedings in Mathematics & Statistics. Papers from the Research Semester "Geometric Mechanics—Variational and Stochastic Methods" held at the Centre Interfacultaire Bernoulli (CIB), Ecole Polytechnique Fédérale de Lausanne, Lausanne, January–June, 2015. Springer, Cham (2017).

3. S. Albeverio, F. De Vecchi and M. Gubinelli. The elliptic stochastic quantization of some two dimensional Euclidean QFTs. In: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 57, pp. 2372–2414, Institut Henri Poincaré (2021).
4. S. Albeverio, F. De Vecchi, A. Romano and S. Ugolini. Mean-field limit for a class of stochastic ergodic control problems. *SIAM J. Control Optim.* **60**(1), 479–504 (2022).
5. S. Albeverio, F.C. De Vecchi, A. Romano and S. Ugolini. Strong Kac's chaos in the mean-field Bose–Einstein condensation. *Stoch. Dyn.* **20**(5):2050031, 21 (2020).
6. S. Albeverio, F.C. De Vecchi and S. Ugolini. Entropy chaos and Bose–Einstein condensation. *J. Stat. Phys.* **168**(3), 483–507 (2017).
7. S. Albeverio and R. Figari. Quantum fields and point interactions. *Rend. Mat. Appl. (7)* **39**(2), 161–180 (2018).
8. S. Albeverio, H. Föllmer, L. Gross and E. Nelson. *Mathematical physics at Saint-Flour. Probability at Saint-Flour.* Reprints of selected lectures from the Summer Schools. Springer, Heidelberg (2012).
9. S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden. *Solvable models in quantum mechanics*. Springer Science & Business Media (2012).
10. S. Albeverio, A. Hahn and A.N. Sengupta. Rigorous Feynman path integrals, with applications to quantum theory, gauge fields, and topological invariants. In: *Stochastic analysis and mathematical physics (SAMP/ANESTOC 2002)*, pp. 1–60, World Sci. Publ., River Edge, NJ (2004).
11. S. Albeverio, R. Høegh-Krohn and S. Mazzucchi. *Mathematical theory of Feynman path integrals. An introduction.* Volume 523 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, second edition (2008).
12. S. Albeverio, Y. Kondratiev, Y. Kozitsky and M. Roeckner, The statistical mechanics of quantum lattice systems. A path integral approach, *Eur. Math. Soc.* (2009)
13. S. Albeverio and P. Kurasov. *Singular perturbations of differential operators. Solvable Schrödinger type operators.* Volume 271 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (2000).
14. S. Albeverio and S. Kusuoka. The invariant measure and the flow associated to the Φ^4_3-quantum field model. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **20**(4), 1359–1427 (2020).
15. S. Albeverio and S. Mazzucchi. Path integral: mathematical aspects. *Scholarpedia* **6**(1), 8832 (2011).
16. S. Albeverio and S. Ugolini. A Doob h-transform of the Gross–Pitaevskii Hamiltonian. *J. Stat. Phys.* **161**(2), 486–508 (2015).
17. J. Backhoff, G. Conforti, I. Gentil and C. Léonard. The mean field Schrödinger problem: ergodic behavior, entropy estimates and functional inequalities. *Probab. Theory Related Fields* **178**(1–2), 475–530 (2020).
18. N. Barashkov and M. Gubinelli. A variational method for Φ^4_3. *Duke Math. J.* **169**(17), 3339–3415 (2020).
19. C. Bardos, L. Erdős, F. Golse, N. Mauser and H.-T. Yau. Derivation of the Schrödinger–Poisson equation from the quantum N-body problem. *C. R. Math. Acad. Sci. Paris* **334**(6), 515–520 (2002).
20. C. Bardos, F. Golse and N. Mauser. Weak coupling limit of the N-particle Schrödinger equation. *Methods and Applications of Analysis* **7**(2), 275–294 (2000).
21. E. Bayraktar, A. Cosso, and H. Pham. Randomized dynamic programming principle and Feynman–Kac representation for optimal control of McKean–Vlasov dynamics. *Trans. Amer. Math. Soc.* **370**(3), 2115–2160 (2018).
22. P. Blanchard and E. Brüning. *Mathematical methods in physics. Distributions, Hilbert space operators, variational methods, and applications in quantum physics.* Volume 69 of Progress in Mathematical Physics. Birkhäuser/Springer, Cham, second edition (2015).
23. P. Blanchard, P. Combe and W. Zheng. *Mathematical and physical aspects of stochastic mechanics*. Lecture Notes in Physics Volume 281. Springer (1987).
24. C. Boccato, C. Brennecke, S. Cenatiempo and B. Schlein. Complete Bose–Einstein condensation in the Gross–Pitaevskii regime. *Comm. Math. Phys.* **359**(3), 975–1026 (2018).

25. V.I. Bogachev, N.V. Krylov, M. Röckner and S.V. Shaposhnikov. *Fokker–Planck–Kolmogorov equations*. Volume 207 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI (2015).
26. C. Brennecke, P. Thành Nam, M. Napiórkowski and B. Schlein. Fluctuations of N-particle quantum dynamics around the nonlinear Schrödinger equation. In: *Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire*, volume 36, pp. 1201–1235, Elsevier (2019).
27. C. Brennecke and B. Schlein. Gross–Pitaevskii dynamics for Bose–Einstein condensates. *Anal. PDE* **12**(6), 1513–1596 (2019).
28. M. Caporaletti, A. Deuchert and B. Schlein. Dynamics of mean-field bosons at positive temperature. *arXiv preprint* arXiv:2203.17204 (2022).
29. E. Carlen. Stochastic mechanics: a look back and a look ahead. In: *Diffusion, quantum theory, and radically elementary mathematics*, volume 47 of Math. Notes, pp. 117–139, Princeton Univ. Press, Princeton, NJ (2006).
30. E.A. Carlen. Conservative diffusions. *Comm. Math. Phys.* **94**(3), 293–315 (1984).
31. E.A. Carlen. Existence and sample path properties of the diffusions in Nelson's stochastic mechanics. In: *Stochastic processes–mathematics and physics (Bielefeld, 1984)*, volume 1158 of Lecture Notes in Math., pp. 25–51, Springer, Berlin (1986).
32. E.A. Carlen. Progress and problems in stochastic mechanics. In: *Stochastic methods in mathematics and physics (Karpacz, 1988)*, pp. 3–31, World Sci. Publ., Teaneck, NJ (1989).
33. R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I. Mean field FBSDEs, control, and games.* Volume 83 of Probability Theory and Stochastic Modelling. Springer, Cham (2018).
34. R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. II. Mean field games with common noise and master equations.* Volume 84 of Probability Theory and Stochastic Modelling. Springer, Cham (2018).
35. P. Cattiaux, G. Conforti, I. Gentil and C. Léonard. Time reversal of diffusion processes under a finite entropy condition. *arXiv preprint* arXiv:2104.07708 (2021).
36. Y. Chen, T. Georgiou and M. Pavon. On the relation between optimal transport and Schrödinger bridges: a stochastic control viewpoint. *J. Optim. Theory Appl.* **169**(2), 671–691 (2016).
37. E.A. Cornell and C.E. Wieman. Bose–Einstein condensation in a dilute gas: the first 70 years and some recent experiments (Nobel lecture). *Chemphyschem* **3**(6), 476–493 (2002).
38. A.B. Cruzeiro, L. Wu, and J.C. Zambrini. Bernstein processes associated with a Markov process. In: *Stochastic Analysis and Mathematical Physics*, pp. 41–72, Springer (2000).
39. I. Csiszár. Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.* **2**, 299–318 (1967).
40. N. Cufaro Petroni and L. Morato. Entangled states in stochastic mechanics. *J. Phys. A* **33**(33), 5833–5848 (2000).
41. P. Dai Pra. Stochastic mean-field dynamics and applications to life sciences. In: *Stochastic dynamics out of equilibrium*, volume 282 of Springer Proc. Math. Stat., pp. 3–27, Springer, Cham (2019).
42. G.F. Dell'Antonio, R. Figari and A. Teta. Hamiltonians for systems of N particles interacting through point interactions. *Ann. Inst. H. Poincaré Phys. Théor.* **60**(3), 253–290 (1994).
43. L. Erdős, B. Schlein and H.-T. Yau. Rigorous derivation of the Gross–Pitaevskii equation. *Physical review letters* **98**(4), 040404 (2007).
44. G. Fabbri, F. Gozzi and A. Swiech. *Stochastic optimal control in infinite dimension. Dynamic programming and HJB equations*. With a contribution by Marco Fuhrman and Gianmario Tessitore. Volume 82 of Probability Theory and Stochastic Modelling, Springer, Cham (2017).
45. W.H. Fleming and H.M. Soner. Controlled Markov processes and viscosity solutions. Stochastic Modelling and Applied Probability, Volume 25. Springer Science & Business Media (2006).
46. H. Föllmer. Random fields and diffusion processes. In: *École d' Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, pp. 101–203, Springer, Berlin (1988).
47. M. Gallone and A. Michelangeli. Self-adjoint extension schemes and modern applications to quantum Hamiltonians. *arXiv preprint* arXiv:2201.10205 (2022).

48. J. Glimm and A. Jaffe. *Quantum physics. A functional integral point of view.* Springer-Verlag, New York-Berlin (1981).
49. M. Gubinelli and M. Hofmanová. A PDE construction of the Euclidean φ_3^4 quantum field theory. *Comm. Math. Phys.* **384**(1), 1–75 (2021).
50. M. Gubinelli, P. Imkeller and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi* 3:e6, 75 (2015).
51. F. Guerra and L.M. Morato. Quantization of dynamical systems and stochastic control theory. *Phys. Rev. D (3)* **27**(8), 1774–1786 (1983).
52. M. Hairer. A theory of regularity structures. *Invent. Math.* **198**(2), 269–504 (2014).
53. M. Hauray and S. Mischler. On Kac's chaos and related problems. *J. Funct. Anal.* **266**(10), 6055–6157 (2014).
54. K. Itô. Wiener integral and Feynman integral. In: *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability*, volume 4, pp. 227–238, Univ. of California Press (1961).
55. K. Itô. Generalized uniform complex measures in the Hilbertian metric space with their application to the Feynman integral. In: *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1*, pp. 145–161, Univ. California Press, Berkeley, Calif. (1967).
56. W. Ketterle and N.J. Van Druten. Evaporative cooling of trapped atoms. In: *Advances in atomic, molecular, and optical physics*, volume 37, pages 181–236, Elsevier (1996).
57. S. Kullback. A lower bound for discrimination information in terms of variation (Corresp.). *IEEE transactions on Information Theory* **13**(1), 126–127 (1967).
58. D. Lacker. Limit theory for controlled McKean–Vlasov dynamics. *SIAM J. Control Optim.* **55**(3), 1641–1672 (2017).
59. C. Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Contin. Dyn. Syst.* **34**(4), 1533–1574 (2014).
60. T. Lévy and A. Sengupta. Four chapters on low-dimensional gauge theories. In: *Stochastic geometric mechanics*, volume 202 of Springer Proc. Math. Stat., pp. 115–167, Springer, Cham (2017).
61. M. Lewin. Mean-field limit of Bose systems: rigorous results. In: Proceedings of the international Congress Assoc. Math. Phys. *arXiv preprint* arXiv:1510.04407 (2015).
62. M. Lewin, P. Nam and N. Rougerie. The mean-field approximation and the non-linear Schrödinger functional for trapped Bose gases. *Transactions of the American Mathematical Society* **368**(9), 6131–6157 (2016).
63. E.H. Lieb and R. Seiringer. Proof of Bose–Einstein condensation for dilute trapped gases. *Physical review letters* **88**(17), 170409 (2002).
64. E.H. Lieb, R. Seiringer, J.P. Solovej and J. Yngvason. *The mathematics of the Bose gas and its condensation*, volume 34 of Oberwolfach Seminars. Birkhäuser Verlag, Basel (2005).
65. E.H. Lieb, R. Seiringer and J. Yngvason. Bosons in a trap: A rigorous derivation of the Gross–Pitaevskii energy functional. In: *The Stability of Matter: From Atoms to Stars*, pp. 685–697, Springer (2001).
66. G. Mackey. *Mathematical foundations of quantum mechanics*. With a foreword by A.S. Wightman. Reprint of the 1963 original. Dover Publications, Inc., Mineola, NY (2004).
67. P.R. Masani. Norbert Wiener and the future of cybernetics. In: *Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994)*, volume 52 of Proc. Sympos. Appl. Math., pp. 473–503, Amer. Math. Soc., Providence, RI (1997).
68. S. Mazzucchi. *Mathematical Feynman path integrals and their applications*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2009).
69. A. Michelangeli. *Bose–Einstein condensation: analysis of problems and rigorous results*. PhD thesis, SISSA (2007).
70. T. Mikami. *Stochastic optimal transportation: stochastic control with fixed marginals*. Springer Nature (2021).
71. A. Millet, D. Nualart and M. Sanz. Integration by parts and time reversal for diffusion processes. *Ann. Probab.* **17**(1), 208–238 (1989).

72. L.M. Morato and S. Ugolini. Stochastic description of a Bose–Einstein condensate. *Ann. Henri Poincaré* **12**(8), 1601–1612 (2011).
73. L.M. Morato and S. Ugolini. Localization of relative entropy in Bose–Einstein condensation of trapped interacting bosons. In: *Seminar on Stochastic Analysis, Random Fields and Applications VII*, volume 67 of Progr. Probab., pp. 197–210, Birkhäuser/Springer, Basel (2013).
74. M. Nagasawa. Stochastic variational principle of Schrödinger processes. In: *Seminar on Stochastic Processes, 1989 (San Diego, CA, 1989)*, volume 18 of Progr. Probab., pp. 165–175, Birkhäuser Boston, Boston, MA (1990).
75. P.T. Nam, N. Rougerie and R. Seiringer. Ground states of large bosonic systems: the Gross–Pitaevskii limit revisited. Analysis & PDE **9**(2), 459–485 (2016).
76. E. Nelson. *Dynamical theories of Brownian motion*, volume 3. Princeton university press (1967).
77. E. Nelson. The free Markoff field. *J. Functional Analysis* **12**, 211–227 (1973).
78. E. Nelson. *Quantum fluctuations*. Princeton University Press (1985).
79. E. Nelson. Stochastic mechanics and random fields. In: *École d' Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, volume 1362 of Lecture Notes in Math., pp. 427–450, Springer, Berlin (1988).
80. L. Pitaevskii and S. Stringari. *Bose–Einstein condensation and superfluidity*. International series of monographs on physics, volume 164. Oxford University Press, 2016.
81. M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1978).
82. N. Rougerie. De Finetti theorems, mean-field limits and Bose–Einstein condensation. LMV Lecture Notes. *arXiv preprint* arXiv:1506.05263 (2015).
83. N. Rougerie. Scaling limits of bosonic ground states, from many-body to non-linear Schrödinger. *EMS Surveys in Mathematical Sciences* **7**(2), 253–408 (2021).
84. B. Simon. *The* $P(\varphi)_2$ *Euclidean (quantum) field theory*. Princeton Series in Physics. Princeton University Press, Princeton, N.J. (1974).
85. H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.* **52**(3), 569–615 (1980).
86. S. Ugolini. Bose–Einstein condensation: a transition to chaos result. *Commun. Stoch. Anal.* **6**(4), 565–587 (2012).
87. K. Yasue. Stochastic calculus of variations. *J. Functional Analysis* **41**(3), 327–340 (1981).
88. J.-C. Zambrini. Variational processes and stochastic versions of mechanics. *J. Math. Phys.* **27**(9), 2307–2330 (1986).
89. J.-C. Zambrini. The research program of stochastic deformation (with a view toward geometric mechanics). In: *Stochastic analysis: a series of lectures*, volume 68 of Progr. Probab., pp. 359–393, Birkhäuser/Springer, Basel (2015).