Projected AQIF Parallel Algorithm for Solving EHL Line and Point Contact Problems: Parallel Computing

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Abstract A novel parallel approach is developed for solving EHL line and point contact problems. The main motivation of algorithm comes from solving a discrete variational inequality problems on parallel computer by introducing a novel solver named as projected alternate quadrant interlocking factorization (PAQIF). The PAQIF has the property that when complementarity system

$$\begin{aligned} L_0 x &\geq b, \\ x &\geq 0, \\ x(L_0 x - b) &= 0 \end{aligned}$$

is banded with semibandwidth β_v, the space generated by $e_i., e_{n-i}; 1 \leq i \leq \beta_v$ is invariant under the transformation W^{-1}. Hence PAQIF is combined with partitioned scheme that renders a divide and conquer algorithm for solution of the banded linear complementarity system. The idea is extended to EHL problems by developing suitable preconditioner in the form of banded matrix.

1 Introduction

In a wide range of lubricated industrial devices studied, due to varying partial differential equations (PDEs) behaviour in Reynold's equation in the model (known as Elasto-hydrodynamic lubrication (EHL) see for examples [1, 2, 4]), depicting the pressure distribution and film thickness gap having considerable amount of difficulty

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when the numerical simulation is done on serial computer. A very fine mesh is essential to capture inherited physics behind the model which generates a large memory requirement and computational complexity during the computation. Such a challenge can be compromised if discretized Reynold's equation is approximated in the form of a banded linear system during fix point iteration. Such banded linear systems often give rise to very large narrow banded linear systems which can be dense or sparse within the band. As result it is essential to develop robust parallel algorithms to meet the memory requirement and reduce the computational complexity by sharing the load on parallel computers. We discuss a novel parallel approach known as projected alternate quadrant interlocking factorization (PAQIF) to tackle the above mentioned extremities.

2 The Mathematical Model Problem

The mathematical formulation of the EHL problem consists of the set of nonlinear PDEs in the form of inequalities (see [1, 2] and [4] for more details) described as

$$\begin{aligned} \frac{\partial(\rho H)}{\partial x} - \frac{\partial}{\partial x}\Big(\epsilon\frac{\partial p}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\epsilon\frac{\partial p}{\partial y}\Big) &\geq 0 \quad \forall x, y \in \Omega \\ p(x, y) &\geq 0 \quad \forall x, y \in \Omega, \\ p(x, y)\Big[\frac{\partial(\rho H)}{\partial x} - \frac{\partial}{\partial x}\Big(\epsilon\frac{\partial p}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\epsilon\frac{\partial p}{\partial y}\Big)\Big] &= 0 \quad \forall x, y \in \Omega, \\ p(x, y) &= 0 \quad \forall x, y \in \partial\Omega. \end{aligned} \tag{1}$$

The elastic regime of the film thickness gap H between two contacting surfaces is governed by

$$H(p) = H_0 + \frac{x^2 + y^2}{2} + \frac{2}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{p(x', y')dx'dy'}{\sqrt{(x - x')^2 + (y - y')^2}}. \tag{2}$$

The dimensionless force balance equation are defined as follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x', y')dx'dy' = \frac{3\pi}{2}, \tag{3}$$

Here term ϵ is defined as

$$\epsilon = \frac{\rho H^3}{\eta\lambda},$$

where dimensionless viscosity η is defined as dimensionless density ρ and speed parameter λ

2.1 The PAQIF Algorithm

We consider the linear complementarity problem (LCP) define as below

$$\begin{aligned} LU(x) &\geq f(x), \quad x \in \Omega, \\ U(x) &\geq 0, \quad x \in \Omega, \\ U(x)^T.[LU(x) - f(x)] &= 0, \quad x \in \Omega, \\ U(x) &= g(x), \quad x \in \partial\Omega. \end{aligned} \tag{4}$$

Now we subdivide the LCP into r blocks linear sub-complementarity problem (LSCP) each of size n along the main diagonal such that $N = nr$, where r is the number of processors available. From Eq. (4), LSCP is expressed as

$$\begin{aligned} L_-^{(m)}U^{(m-1)}(x) + L_0^{(m)}U^{(m)}(x) + L_+^{(m)}U^{(m+1)}(x) \geq f^{(m)}(x), \quad m = 1, 2, \ldots, r \\ U^{(m)}(x) \geq 0 \\ U^{(m)}(x)^T.(L_-^{(m)}U^{(m-1)}(x) + L_0^{(m)}U^{(m)}(x) + L_+^{(m)}U^{(m+1)}(x) - f^{(m)}(x)) = 0, \end{aligned} \tag{5}$$

For each partition r, Eq. (5) can be reformulated as

$$\begin{aligned} L_0^{(m)}U^{(m)}(x) \geq f^{(m)}(x) - \begin{bmatrix} L_-^{(m)}U_L^{(m-1)}(x) \\ 0 \\ . \\ . \\ 0 \\ L_+^{(m)}U_F^{(m+1)}(x) \end{bmatrix}_{n\times 1} := f^{*(m)}(x), \quad m = 1, \ldots, r \\ U^{(m)}(x) \geq 0 \\ U^{(m)}(x)^T.(L_0^{(m)}U^{(m)}(x) - f^{*(m)}(x)) = 0, \end{aligned} \tag{6}$$

where $U_L^{(m-1)}(x)$ and $U_F^{(m+1)}(x)$ are $\beta_v \times 1$ vectors picked up from the last and first β_v components of the solution vector $U^{(m-1)}(x)$ and $U^{(m+1)}(x)$, respectively. Now we decouple the LSCP in Eq. (6) for parallel processors. Note that in Eq. (6) $f^{*(m)}(x)$ differs from $f^{(m)}(x)$ only in its first β_v and last β_v components. In order to factorize $L_0^{(m)}$ into $W_0^{(m)} Z_0^{(m)}$, we consider the space generated by

$$\text{Span}_{1 \leq i \leq \beta_v} \{e_i, e_{n-i+1}\}$$

is invariant under the matrix $W_0^{(m)}$ (and so for ${W^{(m)}}_0^{-1}$), where

$$e_j := (0, 0, \ldots, 0, 1_{j^{th} term}, 0, \ldots, 0).$$

Let $[L_0^{(m)}]_{n \times n}$ (say $n = 2s$), $[W_0]_{n \times n}$ and $[Z_0]_{n \times n}$ matrices such that $L_0 = W_0 Z_0$, The above factorization can be proved that the method is stable for nonsingular diagonally dominant. Over all method is now outlined in brief as follows (see [3, 4] in details):

Step 1: For $m = 1, 2, \ldots, r$ factorize in parallel

$$L_0^{(m)} = W_0^{(m)} Z_0^{(m)}$$

Step 2: For $m = 1, 2, \ldots, r$ compute $Y^{(m)}$ in parallel

$$W_0^{(m)} Y^{(m)} = F^{(m)}$$

Step 3: For $m = 1, 2, \ldots, r$ get inverse of $2\beta_v \times 2\beta_v$ matrix obtained by collecting first β_v and last β_v rows and columns of $W_0^{(m)}$ in parallel.
Step 4: Solve the reduced system from the subsystem by collecting first β_v and last β_v equations from each block. Then form normal equations, Solve system for $U_F^{(m)}$ and $U_L^{(m)}$, $m = 1, 2, \ldots, r$.
Step 5: Project $U_F^{(m)}$ and $U_L^{(m)}$, $m = 1, 2, \ldots, r$ into convex set K, where

$$K = \{p \in U : p \geq 0\}.$$

Step 6: For $m = 1, 2, \ldots, r$ solve $U_M^{(m)}$ in parallel.
Step 7: Project $U_M^{(m)}$, $m = 1, 2, \ldots, r$ into convex set K.

3 Numerical Results

We discretize the EHL model problem defined in Eqn (1) using finite difference method (see for example [4]). The domain decomposition method is used here for solving problem on parallel computers. We have used PAQIF algorithm during the fix point inner iteration process of the the computation. The speedup performance and efficiency plot of PAQIF algorithm is shown for varying grid points in Figs. 1 and 2 respectively. The converged pressure profile and gap plot are shown in Figs. 3

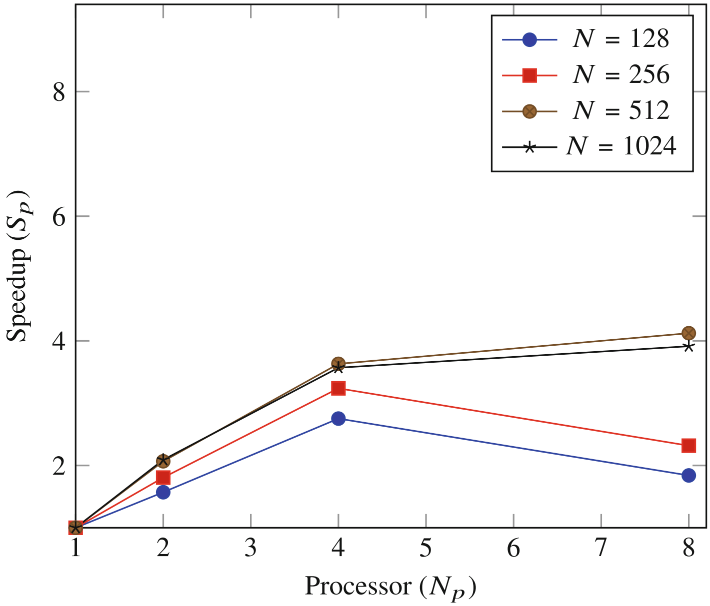

Fig. 1 Speedup plot for the cases $N = 128, 256, 512, 1024$, where bandwidth of matrix $\beta_v = 2$

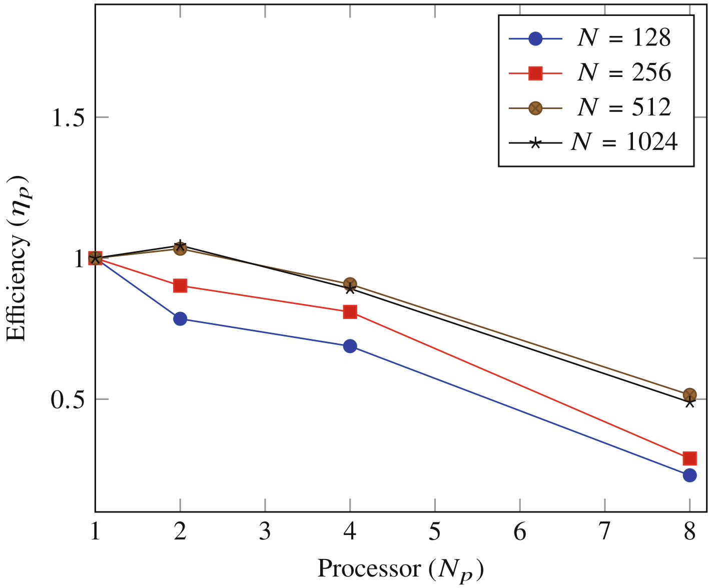

Fig. 2 Efficiency plot for the cases $N = 128, 256, 512, 1024$, where bandwidth of matrix $\beta_v = 2$

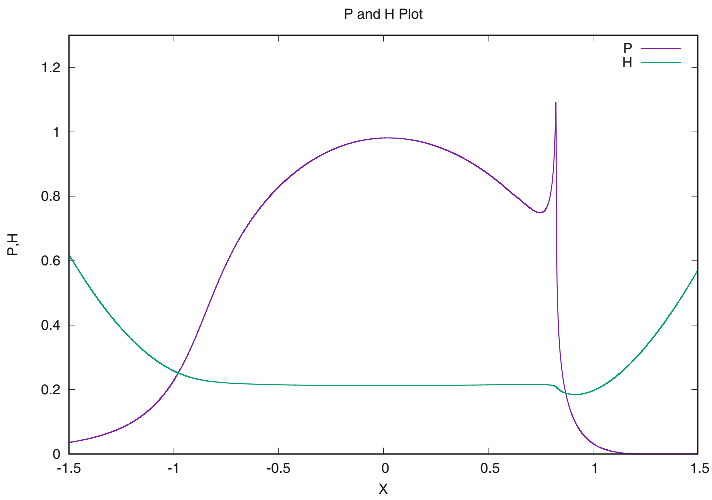

Fig. 3 EHL line contact, see [4]

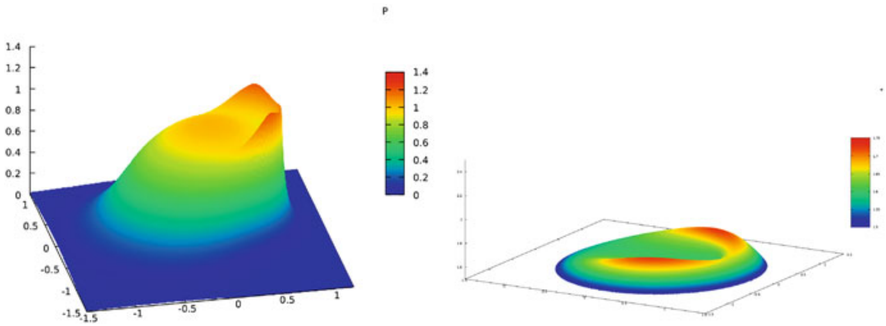

Fig. 4 Pressure P plot and 2-D Gap H plot for $M = 20$, $L = 10$, see [4]

and 4, respectively. We have performed all numerical computation on Dell Tower precision machine having processor specification Intel(R) Core(TM) i7-6700 CPU @ 3.40GHz.

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