



Parallel Simulations of Dynamic Interaction Between Train Pantographs and an Overhead Catenary Line

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Abstract. The paper describes a computational model and an original software system, UKS-Dynamic, for the analysis of dynamic interaction between train pantographs and an overhead catenary system at high-speed railway lines. The study focuses on the problem of reducing non-physical high-frequency oscillations arising in dynamic simulations due to the spatial discretization of flexible wires of the catenary. A number of model problems are solved both analytically and numerically, and the quality of high-frequency mode suppression is studied for several suppression techniques, including specific time integrators (beta-Newmark and generalized-alpha methods) and Rayleigh damping. The model is validated against the solutions of etalon problems given by the EN 50318:2018 standard for alternating and direct current catenary lines at a train speed of 320 km/h. Code parallelization employs the OpenMP library; the code profiling results are presented for both serial and parallel implementations.

Keywords: Overhead catenary line · Dynamic simulation · High-frequency oscillation suppression

1 Introduction

In the design of overhead contact lines for high-speed railway tracks, it is important to ensure a high quality of electric current collection. The analysis of the current collection quality requires a realistic simulation of the dynamic contact between train pantographs and contact wires. These contact interactions are influenced by the elastic waves propagating in the catenary system. At train speed values of 350–400 km/h, high-frequency oscillations can physically emerge in the catenary. On the other hand, the finite element models of catenaries also spawn non-physical high-frequency oscillation modes that occur due to the space discretization of continuous elastic wires. These non-physical high-frequency modes can be withdrawn from the solution using different suppression techniques, such as specific time integrators (beta-Newmark and generalized-alpha methods), Rayleigh damping and output filtering.

The goals of this study are: (a) to analyze dynamic processes occurring in the catenary system by solving a number of analytical model problems; (b) to test the quality of numerical time integrators on model problems; (c) to examine high-frequency suppression techniques for the degree of dissipation. After considering the model problems, we switch to real-life catenaries and demonstrate the validation of the presented computational model against the etalon problem solution from [1].

The UKS computational system for the design of railway overhead catenary lines (OHL) has been developed by the team of Universal Catenary Systems Co. and researchers from St. Petersburg Polytechnic University over the past twenty years [2, 3]. Computational models involved in OHL dynamics simulations are constantly upgraded in accordance with the international standards developed by the European Committee for Electrotechnical Standardization [1]. To the authors' knowledge, the described computational package is the only domestic professional software used at the industrial level and supporting all stages of the OCL design, from the initial scratch to the final technical documentation albums for railway construction and maintenance staff.

The functionality of the software system includes:

- UKS-Static module for the non-linear static finite element analyses of catenary lines, trusses and frames with large nodal displacements;
- UKS-Dynamic module for the linear dynamic finite element analyses of catenary lines, including near real-time simulations of contact interaction between pantographs and the catenary;
- statistical analysis of the simulation output (mean value, standard deviation calculation);
- spectral analysis of the simulation output;
- output signal filtering;
- GUI pre- and post-processing modules: integration with AutoCAD, production of design documentation, visualization of simulation results.

The presented computational system was successfully applied for designing overhead catenary lines at high-speed railway lines, including the Moscow—St. Petersburg 250 km/h line (currently in operation), the Moscow—St. Petersburg 400 km/h line (under construction) and the Moscow—Kazan 400 km/h line (design stage completed, construction suspended).

The rest of the paper is organized as follows: Sect. 2 describes the mathematical models employed in the design of catenary lines; in Sect. 3, model problems are stated; Sect. 4 describes the numerical time integration schemes employed in the analyses (namely, beta-Newmark and generalized alpha method families); Sect. 5 outlines the features of the numerical implementation; in Sect. 6, the simulation results are presented, and the quality of high-frequency suppression is analyzed both for model problems and one etalon problem from the EN 50318:2018 standard [1] for a train speed of 320 km/h; Sect. 7 describes the OpenMP parallelization of the code and presents the code profiling results.

2 Mathematical Models

2.1 Catenary Line Model

A schematic view of a section of the overhead catenary line is shown in Fig. 1: the messenger wire and the contact wire are supported by cantilevers, which are, in turn, mounted to supports. The contact wire is fixed by the cantilevers' steady arms and droppers mounted between the contact wire and the messenger wire.

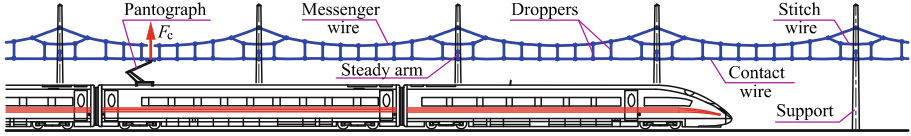


Fig. 1. Catenary line with supporting elements (supports, cantilevers and fixation arms)

In the presented model, catenary wires (contact and messenger wires and droppers) are simulated as ideally flexible threads using finite elements of “link” type with the account for unloading (folding) under a negative axial force. Supporting constructions (rotating cantilevers and fixation arms) are modeled as truss elements connected to catenary wires. Supporting constructions are included in the model to take into account their response (for example, at rotation) on the position and tension of catenary wires.

2.2 Non-linear Static Analysis

The wires of the catenary line are modeled as elastic, pre-tensioned, ideally flexible threads. At the first simulation stage, the static configuration of the catenary line under the action of gravity, pre-tension and static pressure from pantographs is determined. Static analysis takes into account the actual configuration of the catenary under the acting loads (“large displacements” analysis). The 3D displacement field of each wire in the catenary is described in the local coordinate system (which is attached to the reference configuration of the wire) by the following equations:

$$\begin{cases} H(u', v', w') \frac{d^2 v'}{dx'^2} + q'(x') = 0 \\ H(u', v', w') \frac{d^2 w'}{dx'^2} + p'(x') = 0 \\ ES \frac{d^2 u'}{dx'^2} + \tau'(x') = 0, \end{cases} \quad (1)$$

where x' is the axial coordinate in the local basis, $v(x')$, $w(x')$ are the transversal displacements, $u(x')$ is the axial displacement, ES is the axial stiffness of the wire, $H(x')$ is the axial force, $q'(x')$, $p'(x')$ are the transversal distributed loads, $\tau'(x')$ is the axial distributed load.

After the finite element discretization of 1 and the transformation of element matrices to the global coordinates, the resulting system of nonlinear algebraic equations is written as:

$$K(U)U = F_{\text{ext}}, \tag{2}$$

where U is the vector of nodal displacements measured from the reference (unloaded) configuration, $K(U)$ is the stiffness matrix and F_{ext} is the vector of external nodal loads.

System 2 is solved using the fixed-point iteration method with the relaxation factor $r \in (0; 1]$. The i^{th} iteration of the method is written as follows:

$$K(U_{i-1})\Delta U_i = r(F_{\text{ext}} - K(U_{i-1})U_{i-1}), \quad U_i = U_{i-1} + \Delta U_i.$$

2.3 Dynamic Analysis

In dynamic analysis, small oscillations of the wires around their static configuration are considered. In the local coordinate system, the oscillations of each wire are modeled with the following equations:

$$\begin{cases} m \frac{\partial^2 v'}{\partial t^2} + 2D \frac{\partial v'}{\partial t} - H_{\text{static}}(x') \frac{\partial^2 v'}{\partial x'^2} = q'(x', t') \\ m \frac{\partial^2 w'}{\partial t^2} + 2D \frac{\partial w'}{\partial t} - H_{\text{static}}(x') \frac{\partial^2 w'}{\partial x'^2} = p'(x', t') \\ m \frac{\partial^2 u'}{\partial t^2} + 2D \frac{\partial u'}{\partial t} - ES \frac{\partial^2 u'}{\partial x'^2} = \tau'(x', t), \end{cases} \tag{3}$$

where m is the mass per unit length, g is the gravity acceleration, $H_{\text{static}}(x)$ is the axial tension force obtained from static analysis, D is the viscous damping coefficient.

In the finite element formulation, a linear ODE system is solved:

$$M\ddot{U} + B\dot{U} + KU = F_{\text{ext}}(t). \tag{4}$$

Here M is the mass matrix, B is the damping matrix. The constant stiffness matrix K is computed in the reference configuration obtained from static analysis 2.

During time integration, the tensions in the droppers are checked; in the case of dropper unloading, the stiffness matrix is corrected, and the time step is repeated until the stiffness matrix is stabilized (see [2] for more details).

Supporting elements (such as fixation arms or cantilevers) are modeled as visco-elastic nodal supports at this stage.

The EN 50318:2018 standard [1] does not require using any specific structural damping model; the standard requires that the damping of the overhead contact line is adjusted to a non-dimensional damping rate (ratio of damping vs. critical damping) of 0.1% to 0.15% for the overhead contact line. The standard recommends using Rayleigh damping in the discretized (finite element) model:

$$B = \alpha M + \beta K, \quad \alpha = 1.25 \cdot 10^2 \text{ s}^{-1}, \quad \beta = 10^{-4} \text{ s}. \tag{5}$$

The Rayleigh damping model [4] was originally proposed to mimic internal damping of materials in structures. However, it does not agree with nature

experiments [5]: material damping does not tend to depend on the frequency [5]. Nevertheless, the Rayleigh damping model can be useful for suppressing artificial high-frequency oscillations arising in the numerical solution due to the distortion of the frequency characteristics of a continuous elastic body after spatial discretization.

In the Rayleigh model, the damping ratio ζ depends on the mode frequency ω as follows [4]:

$$\zeta = \frac{n}{\omega} = \frac{1}{2} \left(\frac{\alpha}{\omega} + \beta\omega \right), \tag{6}$$

where n is the decay ratio. The first natural frequency of the catenary line is typically 1 Hz. The substitution of $\omega = 2\pi \times 1$ Hz and 5 into 6 gives the damping ratio $\zeta = 0.0013$.

2.4 Pantograph Model. Contact Interaction Between Pantographs and the Contact Wire

Pantographs are modeled as discrete systems containing two or three lumped masses connected with elastic springs, dry friction elements and viscous dampers as shown in Fig. 2.

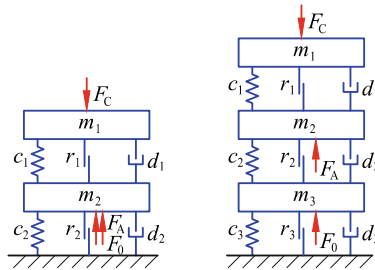


Fig. 2. Two-mass and three-mass pantograph models

In Fig. 2, the vectors F_A and F_0 are the aerodynamic lift force and static push force, correspondingly, and the vector F_C is the force of contact interaction between the contact wire and the pantograph.

Contact interaction between the pantograph and the contact wire is modeled with the penalty method [6]: a restoring force proportional to the value of the mutual penetration of the pantograph and the CW is applied to the contact wire in order to eliminate the penetration of the contacting parts. More details on applying the method to the UKS-dynamic model can be found in [2].

3 Model Problems Setup

In the model problems, small transversal oscillations of an ideally flexible pre-tensioned wire are modeled using Eqs. 3–4. In problems 1 and 2, the oscillations

are free (the only load is gravity). In the 3rd problem, the oscillations are driven by the moving force of the constant magnitude. The pretension value and all characteristics of the wire (see Table 1) represent a real-life contact wire. The length of the wire $L = 50$ m is a typical railway span length.

Table 1. Parameters of the model problems

Parameter	Value
Span length L	50 m
Pretension T	20 kN
Mass per unit length m	1 kg/m
Viscous damping coefficient $2D/m$	0.010185342 s ⁻¹
Disturbing force F in problem 1	10 N
Impact impulse S in problem 2	25 kg · m/s
Moving force F magnitude in problem 3	110 N
Moving force velocity V in problem 3	135 m/s

Due to the lack of space, we omit the expressions for analytical and semi-analytical solutions of the model problems; however, the curves representing these solutions are presented in the Results section.

3.1 Problems 1,2: Free Oscillations Excited Statically and Dynamically (by Impact Interaction)

In problem 1, free oscillations of the wire are excited in a static way: at $x = L/2$, a lumped force is applied (quasi-statically) and then released at $t = 0$ s. In problem 2, free oscillations are excited by impact interaction: at $x = L/2$, the impulse $s = M \cdot V_0$ is applied at $t = 0$ s.

3.2 Problem 3: Oscillations Driven by a Constant Push Force Moving Along the Span

In problem 3, driven oscillations of the wire are excited by the vertical force F moving with the constant speed V along the span. In this case, the critical (resonance) speed of the load V_{cr} equals to the speed of the wave propagation c along the span: $V_{cr} = c$, where $c = \sqrt{\frac{T}{m}} = 141$ m/s.

4 Numerical Time-Integration Methods

The time-integration schemes implemented and tested in the current study include Newmark-beta schemes [8] and the generalized alpha-method [9]. All used methods are absolutely stable, implicit and have the 2nd order of precision except for the 1st order Newmark-beta scheme with $\beta = 0.3025$, $\gamma = 0.6$.

Designed for Hamiltonian systems, the methods significantly differ in computational complexity, internal dissipation and wave dispersion. We performed a comparative study of these methods to estimate the quality of numerical solutions in catenary line dynamics simulations.

The generalized alpha-method for numerical time integration [9] was proposed for suppressing non-physical high-frequency oscillations occurring in numerical solutions due to spatial discretization. The method is described with the following formulas:

$$\begin{cases} (1 - \alpha_m)MA_{n+1} + \alpha_mMA_n = \alpha_fF_n + (1 - \alpha_f)F_{n+1} \\ \frac{U_{n+1} - U_n}{\tau} = V_n + \tau(\frac{1}{2} - \beta)A_n + \tau\beta A_{n+1} \\ \frac{V_{n+1} - V_n}{\tau} = (1 - \gamma)A_n + \gamma A_{n+1}, \end{cases} \quad (7)$$

where $n + 1$ is the current time layer number, τ is the time integration step, U , V , A are the arrays of nodal displacements, velocities and accelerations, correspondingly, and α_m , α_f , β , γ are the parameters of the method. The parameters α_m , α_f were calculated depending on the desired dissipation of high frequencies using the formula proposed in [9]:

$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2, \quad \gamma = \frac{1}{2} + \alpha_f - \alpha_m, \quad (8)$$

where $\rho_\infty = \lim_{\omega \rightarrow \infty} \rho$, ρ is the spectral radius of the transfer matrix of the method.

The generalized alpha methods defined by formulae 7–8 are absolutely stable and have the second order of precision [9]. The values $\rho_\infty = 0$ and $\rho_\infty = 0$ correspond to the total and zero dissipation of the method at high frequencies, correspondingly. Imposing $\alpha_m = \alpha_f = 0$ in 7 produces the family of beta-Newton methods. In this study, the absolutely stable implicit first order beta-Newton scheme with $\beta = 0.3025$, $\gamma = 0.6$ and the second order trapezoidal rule with $\beta = 0.25$, $\gamma = 0.5$ were tested. In all time integration schemes, the time step was chosen so that the Courant number [10] equaled 0.5. The maximal finite element length varied between 10 cm and 25 cm.

5 Software Implementation

The software system basically consists of two modules, UKS-Static and UKS-Dynamic, aimed for the static and dynamic analyses of mechanical interactions between train pantographs and an overhead catenary line. The computational core of the system is implemented in Fortran 2018 and parallelized with the OpenMP library [11]. The core employs the classical mathematical libraries LAPACK and BLAS [12] for solving systems of linear algebraic equations. The system is closely integrated with the AutoCAD software [13]: the AutoCAD system is used for the pre- and post-processing of simulations, as well as for the production of design documentation. Integration with AutoCAD is performed

via AutoLISP scripts. The post-processing utilities of the system also employ the GTK library for creating animations.

The workflow of the UKS-Dynamic computational module is presented in Fig. 3, with the external libraries (BLAS/LAPACK and GTK) shown as blue rectangles. The main submodules of UKS-Dynamic are listed below:

- Dat, DataFileRes—submodules defining initial data: basic global constants, variables and data structures;
- ConvCalcVal—converts data types and performs geometric calculations;
- ToSurface—produces the graphical output using the GTK Cairo library;
- OutDataRes—sets scale factors for graphs, colors for suspension elements and graphs;
- ToAutoCAD—translates graphical data into AutoCAD;
- MfCacl—simulates interaction between pantographs and the catenary line;
- Pantograph2, Pantograph3—calculate two-mass and three-mass pantograph configurations;
- WithoutPantograph—simulates the dynamics of the catenary line in the absence of pantographs;
- WithPantograph—simulates the dynamics of interaction between pantographs and the catenary line within one tensioning section;
- Overlap—simulates the dynamics of interaction between pantographs and the catenary line within two tensioning sections, taking into account the overlap zone;
- LibFS—low-level C language library for file I/O, interacts with a Windows API;
- FSWrappers, FortranFSWrappers—Fortran wrappers for C functions and interfaces for lower-level functions from FSWrappers and LibFS modules, correspondingly;
- Drawing15b, Drawing15bACAD—export catenary line views and data graphs to PNG files and AutoCAD;
- UKSDynamic—program entry point.

6 Results

6.1 Model Problems: Free Oscillations

In model problems simulations, the finite element length was 25 cm, and the time step value was 1 ms. In Fig. 4 and 5, the analytical solutions are shown by red lines. The green lines correspond to the generalized alpha-method with $\rho_\infty = 0.1$, the blue and violet lines correspond to the beta-Newmark method with $\beta = 0.3025$, $\gamma = 0.6$ and the beta-Newmark trapezoidal rule $\beta = 0.25$, $\gamma = 0.5$, correspondingly.

Figure 4 shows the simulation results of problem 1 (free oscillations of the wire excited by a static force) for a viscous damping model (with the diagonal damping matrix B) and a Rayleigh damping model. The displacement dynamics

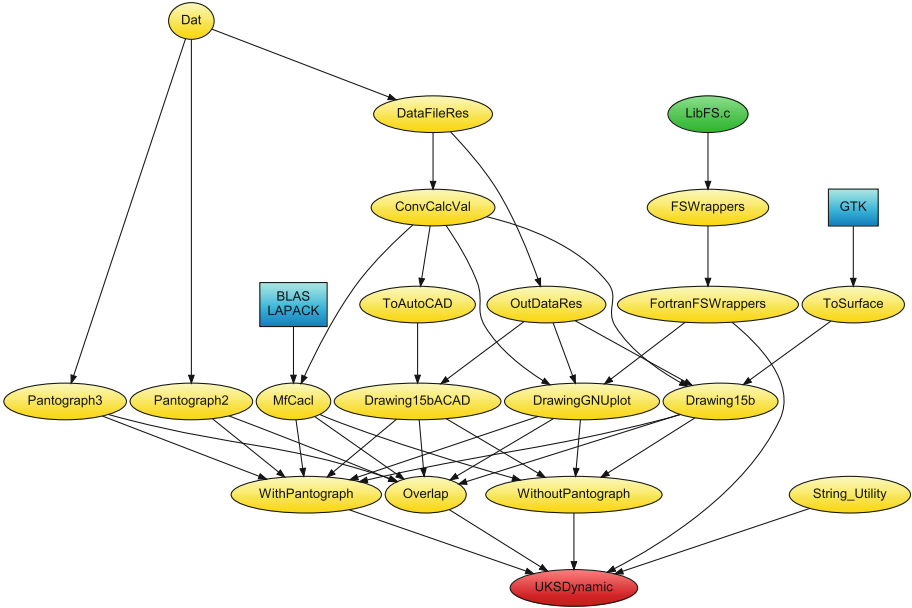


Fig. 3. Workflow of the UKS-Dynamic computational module

is well mimicked by all methods, independently of the damping model (Fig. 4a, c). In the absence of Rayleigh damping, the velocity dynamics graph (Fig. 4b) shows intensive non-physical oscillations for all methods except beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$. The latter has a very strong internal dissipation (which is expressed in a higher oscillation decay compared to the analytical solution). Since this beta-Newmark scheme has only the first order of precision, the accuracy of the solution should be controlled with a sufficiently small time step.

Together with Rayleigh damping (Fig. 4c, d), all schemes show satisfactory results, although beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$ tends to overdamp the solution.

Figure 5 shows the simulation results of problem 2 (free oscillations of the wire excited by impact interaction). In the absence of Rayleigh damping, the displacements (Fig. 5a) oscillate heavily in all methods except beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$. In Fig. 5b, the wave shape is shown at the time moment $t = 0.1$ s (the wave crosses the span in about 0.3 s); the shape is distorted in all time integration schemes (however, it should be noted, that all non-physical oscillations and distortions shown in Fig. 4 and 5 tend to decay with a decrease in the finite element size).

Together with Rayleigh damping (Fig. 4 and 5c, d), all schemes show satisfactory results, with some overdamping in beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$.

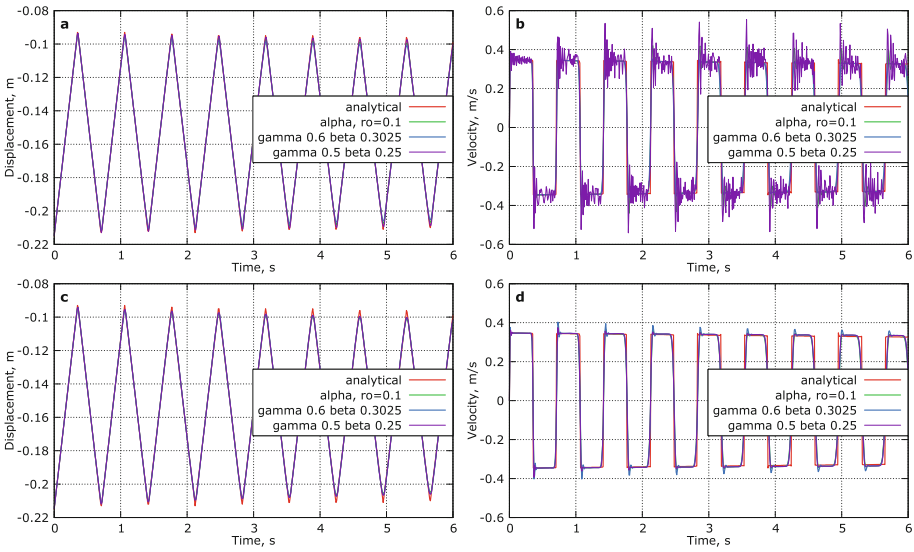


Fig. 4. Model problem 1: free oscillations of the wire excited by a lumped force. Displacement and velocity in the middle of the span, viscous damping (a, b) and Rayleigh damping (c, d)

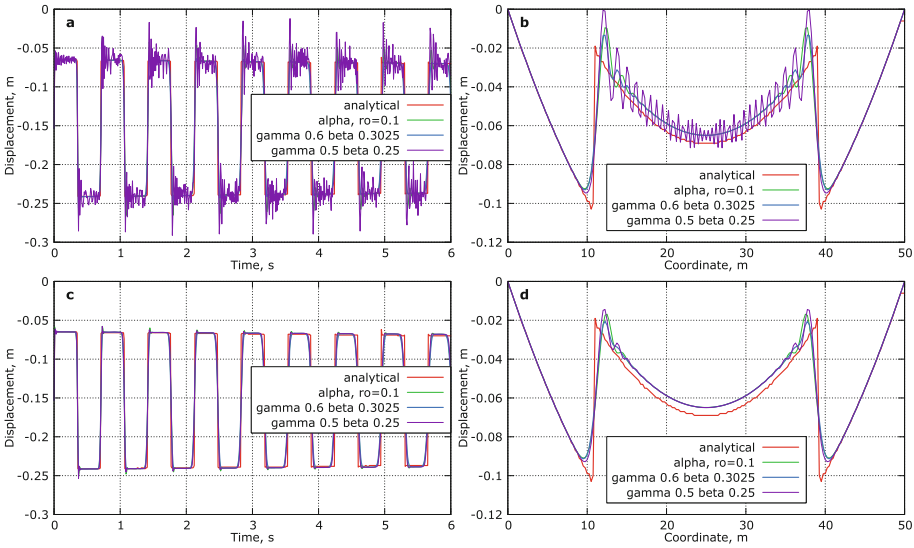


Fig. 5. Model problem 2: free oscillations of the wire excited by impact interaction. Displacements in the mid-span and wave shape at $t = 0.1$ s, viscous damping (a, b) and Rayleigh damping (c, d)

6.2 Model Problem: Driven Oscillations

In the problem of driven oscillations, only the subcritical load velocity value (135 m/s) is considered; supercritical train speeds are prohibited by design standards. According to the analytical solution, the critical load speed equals to the wave propagation speed, which is 141 m/s for the considered wire. When the load speed is under the critical value, the maximal displacement occurs in the point of force application. The distribution of displacements at the moment when the load is at the middle of the span is shown in Fig. 6. The beta-Newmark scheme $\beta = 0.3025$, $\gamma = 0.6$ shows the smoothest results.

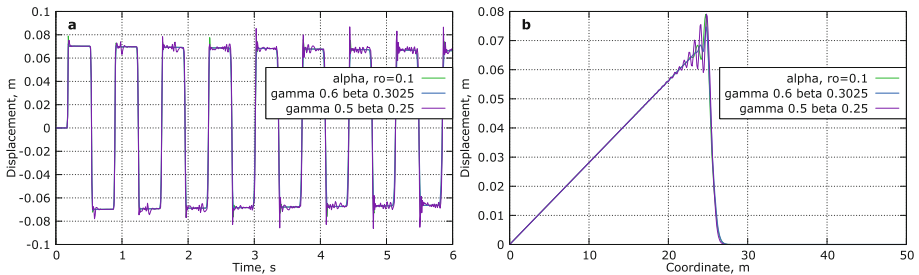


Fig. 6. Model problem 3: driven oscillations, load speed 135 m/s (486 km/h). Wave shape at the moment when the force is in the middle of the span

6.3 Validation of the Model Against the Etalon Problem from EN 50318:2018

The described computational model was successfully validated against etalon problems and experimental data for real existing overhead contact line sections of high speed railway lines (Annex A, B of EN 50318:2018, [1]). Figure 7 presents simulation results for the etalon problem of a catenary line containing a messenger wire, one contact wire, two pantographs located at a 200 m distance (the three lumped masses model of the pantograph is used according to [1]). The messenger wire is connected to fixed points via spring-damping elements. The contact wire is connected to fixed points via supporting elements, i.e. steady arms. The finite element model contains 22 spans, 10 of them are reference spans (according to [1]). Pantographs start moving at the beginning of the section. Ten referent spans are located in the middle of the section, between supports 7 and 17. The time integration step is 0.5 ms. The maximal finite element length is 0.1 m. The contact stiffness in the penalty method is 50 000 N/m, in accordance with the recommendations of EN 50318:2002. The sampling 200 Hz (the sampling interval is 5 ms) is decoupled from the time integration step. The output signal is filtered by band filters with bandwidths 0–20 Hz, 0–5 Hz and 5–20 Hz according to [1].

The initial configuration of the catenary and pantographs is shown in Fig. 7a. The dynamics of the vertical elevations of two contacts along the train trajectory is presented in Fig. 7b. Figure 7c, d shows the statistical distribution of the contact force values. The variation of the contact force along the track after filtering with a bandwidth of 0–20 Hz is presented in Fig. 7e, f. The simulated parameters, which are the most important for the current collection quality, are listed in the tables in Fig. 7.

All simulation results for the etalon model fit into the reference ranges given in [1].

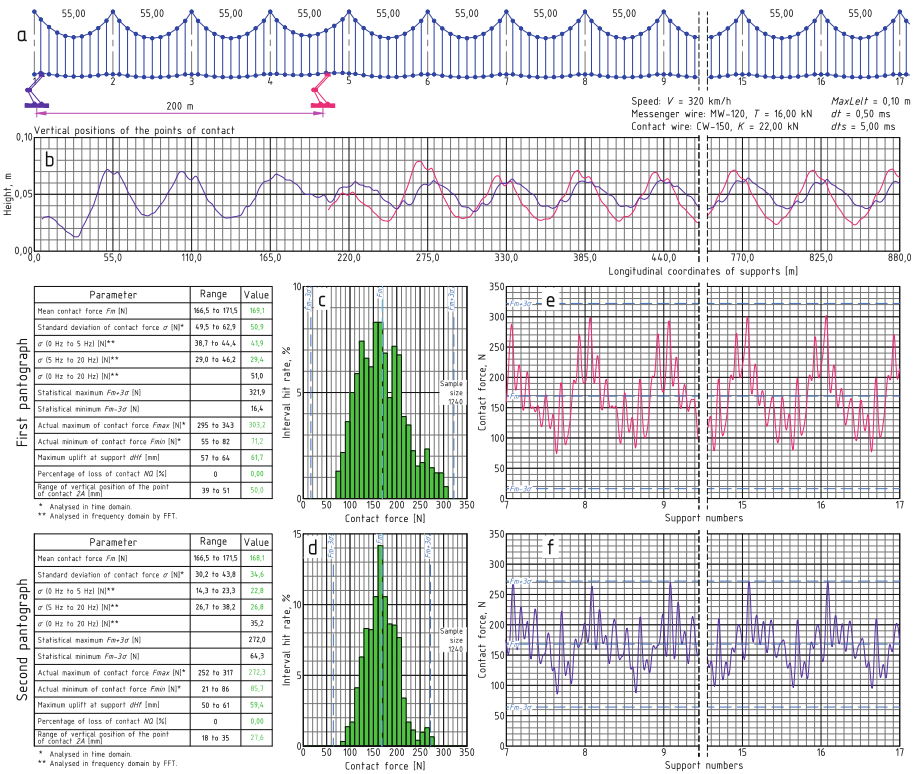


Fig. 7. Etalon problem solution, train speed of 320 km/h: (a) initial configuration of the catenary and pantographs, (b) vertical positions of two contact points along the train trajectory, (c, d) statistical distribution of the contact force values, (e, f) variation of the contact force along the track

7 OpenMP Parallelization and Code Performance

To identify the bottlenecks of the UKS-Dynamics program, the profiling of the serial version was performed using the gprof profiler. The profiling showed that the main bottleneck was the matrix multiplication procedure: it took about 54% of the elapsed time. In the second place, it was the linear algebraic equations system solution in the LAPACK dpbtrs package (19%), wherein the factorization itself did not take up significant resources (taking into account the fact that it is taken out of the time integration loop). Parallelization was performed using the OpenMP library. The loops containing matrix multiplications and the procedures for simulation output filtering were parallelized. Additionally, a number of loops were rewritten to enable automatic parallelization by the compiler.

The computational time spent on solving the etalon problem from EN 50318:2018 for a train speed of 320 km/h with a finite element length of 0.1 m and a time step of 0.5 ms is shown in Fig. 9. The results are obtained at the optimization level of the compiler -O2 on an Intel Core i5-3450 3.5 GHz computer with two physical and four logical cores. A relatively modest speed-up is due to the usage of an algorithmically sequential Holeyky solver for the SLAE (system of linear algebraic equations) solution. However, iterative SLAE solvers are less preferable here due to small problem sizes ($\approx 10^4 - 10^5$ degrees of freedom) (Fig. 8).

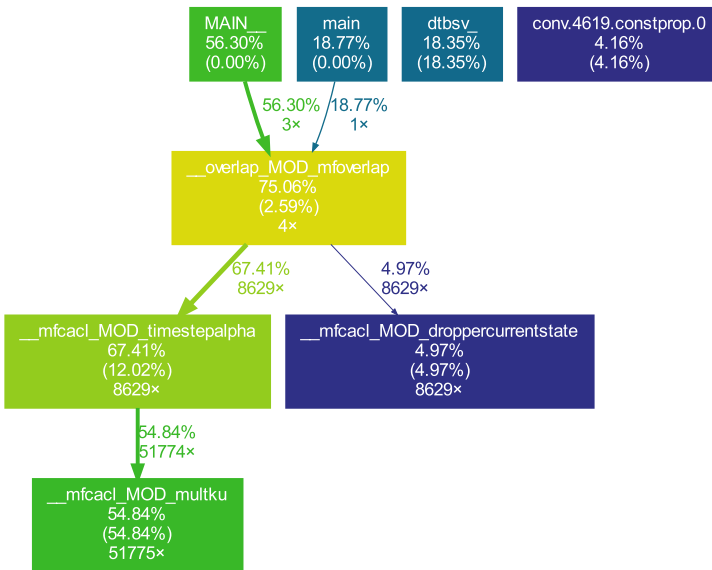


Fig. 8. Serial mode: code profiling results

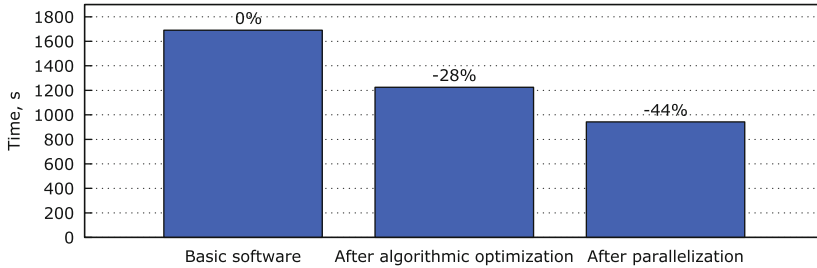


Fig. 9. Computational time spent on the etalon problem solution in serial and parallel modes on an Intel Core i5-3450 3.5 GHz standalone computer

8 Conclusions

Several techniques of high-frequency oscillation suppression were tested and applied to the problem of modeling the dynamics of interaction between train pantographs and a catenary line. The techniques include three special time integration schemes (generalized-alpha with $\rho_\infty = 0.1$, beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$ and beta-Newmark trapezoidal rule $\beta = 0.25$, $\gamma = 0.5$) and Rayleigh damping. The simulation results indicate that Rayleigh damping alone is not sufficient in high-frequency suppression. Qualitatively, beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$ is the best high-frequency suppression scheme in all computational tests, despite the first order of precision. Quantitatively, beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$ in combination with Rayleigh damping tends to overdamp the problem due to the lower accuracy of the scheme; appropriate time stepping should be used. Another advantage of this scheme is lower computational costs compared to generalized-alpha methods. The trapezoidal rule produced intensive non-physical oscillations and cannot be recommended for wire dynamics modeling.

For the etalon problem solution, beta-Newmark $\beta = 0.3025$, $\gamma = 0.6$ was chosen as the fastest and most robust integration scheme. All simulation results for the etalon model fit into the reference ranges given in [1]. The serial optimization and OpenMP parallelization of the program were performed; the computational time spent on the etalon problem solution was reduced by 44%. Further work on improving the performance of the program will be continued.

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