Chapter 17
The Influence of Royaumont on Mathematics Education in the USA

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Abstract The influence of the 1959 Royaumont Seminar on US mathematics education is described. Five periods are discussed: The 1960s New Math, the post-New Math of the 1970s and 1980s, the 1990s reform, the 2000s standards-based era, and the 2010s Common Core State Standards. The landmarks are initially described by tracing the teaching of geometry, but then other key Royaumont themes are developed: The use of set theory, logic, and mathematical structure; the use of problem solving and inquiry; and the role of research mathematicians. The views of three European mathematicians, Jean Dieudonné, René Thom, and Hans Freudenthal, whose writings in US publications stimulated important discussions, provide a second lens for examining the streams of thought that emerged. Examples from US mathematics education research, state policy documents, and instructional materials are given to illustrate the impact of the Royaumont meeting on the USA.

Keywords Burt Kaufman · Edward G. Begle · Euclidean geometry · Hans Freudenthal · Jean Dieudonné · Logic · Modern mathematics · New Math · Nicolas Bourbaki · Problem solving · Proof · René Thom · Rigor · Royaumont Seminar · Set theory · Spatial thinking · Standards · US mathematics education

Introduction

The story of post-Royaumont mathematics education in the USA involves a myriad of players in 50 states each of which has its own policies and instructional programs. It begins with roughly one decade of the New Math era involving a large number of the nations' school districts, but certainly shy of a majority. The ridicule of the New Math and its demise is well known (see, e.g., Kilpatrick 2012; Phillips 2015). But the influence of the Royaumont meeting is not simply the story of the New Math, for indeed, the New Math in the USA, in particular, the School Mathematics Study Group (SMSG), originated prior to Royaumont; it is the story of the streams of thought and reactions to the ideas expressed at Royaumont. So, although we will begin with the New Math, rather than discussing what has been detailed elsewhere, we will identify four basic themes from Royaumont to use as a

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framework for thinking about its influence in the USA. Without any doubt, the story of mathematics education in the USA post-1960 is vastly different from the prior 60 years.

The following Royaumont themes are considered: Concerns about the teaching of geometry; a demand for a “modern” curriculum including set theory, logic, axiomatic and deductive reasoning; the role of problem solving and inquiry in instruction; and the role of research mathematicians in this process. Because Dieudonné’s statement “Euclid must go!” attracted such attention, and because the geometry story is helpful to understanding views of rigor, axiomatics, language, spatial thinking, and problem solving, geometry is used to lay out certain guideposts. Following the New Math decade, we have an era of research alongside a basic skills emphasis (very roughly two decades 1970 to 1990), a decade of reform, a decade of standards-based instruction, and the past decade of the Common Core State Standards, the closest the USA has come to a national curriculum.

Emerging Ideas on Problem Solving and a “Modern Curriculum”

The story of the New Math of the 1960s in the USA is well documented (Kilpatrick 2012; Phillips 2015; Roberts 2015), so our focus here is on two fundamental issues raised in Royaumont:

1. The need for new approaches to develop “latent talents” and promote interest in mathematics.
2. The “modernization” of topics (set theory, replacement of classical Euclidean geometry by affine geometry, and deductive proof based on axiomatic characterizations of modern mathematical structures. This would include new textbooks along with teacher preparation guided by an increased role of research mathematicians (Moon 1986). Of the five US participants at Royaumont: Edward G. Begle, Howard Fehr, Robert Rourke, Marshall Stone, and Albert Tucker, Begle and Fehr would go on to lead curriculum projects, and both Stone and Tucker would be involved in subsequent meetings and debates, and Rourke would author articles and textbooks.

While at the Royaumont Seminar Marshall Stone from the USA offered the following thoughts about the development of new approaches:

> In truth, we find ourselves faced with an extremely urgent pedagogical problem. It is all too evident that in our primary schools we are failing to develop at all efficiently, or at all adequately, the latent mathematical talents and interests of the average child. (OEEC 1961, p. 22)

Stone’s remarks were rooted in concerns that surfaced repeatedly in the USA and would lead to many attempts at reform, and ultimately lead to multiple pedagogical approaches. Some would be based upon rigorous development of the content in texts and lectures, others under the umbrella of student-centered learning, including various notions of problem solving, inquiry-based learning, discovery learning, realistic mathematics education, and more.

The earliest use of inquiry in mathematics education in the USA was the work of Warren Colburn (1793–1833) who wrote arithmetic texts emphasizing student invention of computational procedures and mental arithmetic (Bidwell and Clason 1970; Colburn 1863). Colburn attributed his ideas to the Swiss educator Johann Pestalozzi (1726–1827) (Cohen 1999). About one hundred thousand copies of Colburn’s texts were sold in the 1850s. The preface to Colburn’s edition of *Arithmetic Upon the Inductive Method of Instruction* (Colburn 1824) described his ideas:

> Set a student to work on an addition problem without telling him what to do. He will discover what is to be done, and invent a way to do it. Let him perform several in his own way, and then suggest some method a little different from his, and nearer to the common method. If he readily comprehends it, he will be pleased with it, and adopt it. If he does not, his mind is not yet prepared for it, and should be allowed to continue his own way longer, and then it should be suggested again. (pp. 4–5)

In the 1930s, Louis P. Benezet expressed concern about students' abilities to understand their computations and offered ideas regarding the role of context. As a superintendent of public schools in New Hampshire, he felt students were asked to learn a great deal of arithmetic, including long division, for which they had no use and that the early introduction of arithmetic contributed to a "dulling of their reasoning abilities." For example, Benezet posed the following question to a class of elementary students: "If I can walk a hundred yards in a minute, how many miles can I walk in an hour, keeping the same speed?" Nearly all students responded 6000. If he beamed approval, the class was satisfied. But if he said that meant he could walk from New Hampshire to San Francisco and return in an hour, the students would laugh and look foolish (Benezet 1935). Like Marshall Stone at the Royaumont Seminar, Benezet was concerned about how students will behave and think in their later years. He claimed that if one paid little attention to instruction in arithmetic operations in the elementary grades, it could be later learned effectively in a much shorter period of time if a focus on thinking or problem solving had been the emphasis in earlier years.

Moving beyond concerns of children's acquisition of computational procedures and basic applications is the emphasis on problem solving. This pair of words would take on a variety of meanings as the post-Royaumont era unfolded. The first prominent discussion of problem solving in the USA is George Pólya's (1945) book, *How to Solve It*. Later Pólya would prepare additional volumes, *Mathematics and Plausible Reasoning* (Pólya 1954a, b) and *Mathematical Discovery* (Pólya 1962). Pólya provided a four-step platform for implementing problem solving: Understanding the problem, devising a plan for a solution, implementing the plan, and reviewing or "looking back" or checking over what has been done. It is these ideas that Pólya set forth and explored in his books. Thus, his approach is not deductive—that is to say, for him, mathematics was not a deductive science in which the learner started from some set of definitions and postulates and then proceeded to proofs using the rules of logic. In fact, the discussions of solutions at the end of his book explicitly cited the importance of inductive reasoning, much as Colburn had indicated over a century earlier. The influence was enormous, with a major impact on teachers of mathematics, mathematics teacher educators, and mathematicians. Many of the problems discussed in his texts would find their way into subsequent curricula over the next half-century (usually without citation; of course, many of the problems had been part of mathematicians' repertoire for a long time).

When Pólya's books came onto the scene in mathematics education, school mathematics was dominated by a focus on drills at the expense of emphasizing mathematical thinking or problem solving. Vogeli (1976) commented:

> In the eyes of many thoughtful members of the mathematical community, the picture of mathematics education in American high schools in 1950 was not a pretty one … in their opinion there was undue emphasis being placed on skills, and unnecessary preoccupation with the immediate usefulness of what was taught, and an unfortunate distortion of the students' ideas as to the nature of mathematics (Vogeli quoting Wooton 1965, p. 7).

It is in this context that the *New Math* was born and began to grow in the USA. During the period shortly after 1950, there were quite a number of projects centered on improving the mathematics curriculum in US schools (Becker 1967; Hayden 1981). The project that came to be the largest in the history of the country and whose goal was to modify the existing school curriculum was the School Mathematics Study Group (SMSG).

The "modern" curriculum emerged in the USA in February 1958 at a Chicago conference on research potential and training and at the mathematics meeting of the National Science Foundation that led to the formation of SMSG. According to Phillips (2015) a group of mathematicians including A. Adrian Albert, Edward G. Begle, Henri F. Bohnenblust, Paul Rosenbloom, Marshall H. Stone, and Raymond Wilder suggested at Chicago the focus be shifted from graduate education to primary and secondary schools. Shortly thereafter, Begle, a Professor of Mathematics at Yale University, became the Director of SMSG, and by summer he had an advisory board consisting of about 25 people and writing began. The National Science Foundation provided substantial funding to support the SMSG work (Krieghbaum and Rawson 1969; Wooton 1965) including teacher professional development, which SMSG regarded as critical to success.

The basic philosophy of SMSG was that textbooks were the key, and the plan was to produce sample textbooks where the materials would go out of print shortly. The sample texts and the accompanying teacher commentaries were to be of such high quality that commercial publishing houses would quickly publish materials based upon the model. College and university professors of mathematics worked with high school teachers in producing the materials and by the mid-1960s, SMSG had produced, "tried out" and put into final form a set of sample texts for grades K through 12. In 1962, SMSG began a large-scale, long-term study of mathematics curricula involving more than one hundred thousand students in 40 states—this was the National Longitudinal Study of Mathematical Abilities (NLSMA) (Begle and Wilson 1970). Students were tested in grades 4 through 12 in the fall and spring of each of the 5 years of the study. Many educators believed this was the kind of study needed in order to study the effectiveness of new curricula, since many effects may not show up in short-term studies.

Consistent with Royaumont themes, the materials placed emphasis on structure and precision in the use of modern mathematical language (including set theory). The role of proof in mathematics was important in the materials, especially in grade 9 algebra, grade 10 geometry, and later grades. Students were involved in the curriculum through reading the materials and in general, by doing mathematics. But these characteristics led some mathematicians to oppose the SMSG efforts: Morris Kline was a prominent outspoken critic; George Pólya, Max Schiffer, Lipman Bers, and more than 60 US and Canadian mathematicians reacted by petitioning that the new materials should be more appropriate for all students, not just bright, mathematics-bound students; abstract mathematics should not be introduced too early to students; more emphasis on conjecturing should precede formal proof (Ahlfors et al. 1962); Max Beberman was opposed to "too much" high school mathematics being introduced at the lower grade levels; and more emphasis was needed on arithmetical operations (see Kilpatrick 2012).

The debate between the reform of SMSG and critics rose to a high level with pronouncements that the top-down approach of SMSG was inappropriate and that, even, the "New Math" was a failure. But school mathematics would never be the same as during the pre-SMSG days and most came to realize that change cannot be successful unless teachers were deeply involved (see Begle 1969). Indeed, there was and still is a view that everything in the classroom is mediated through the teacher. In the end, it became clear to some that more focus needed to be placed on basic skills, and what followed, in the USA was a "back to basics" movement fomented by the general public (Phillips 2015).

Post-Royaumont Geometry in the USA

Undoubtedly the most famous utterance at the Royaumont Seminar was Jean Dieudonné's "Euclid must go!" (OEEC 1961, p. 35). Dieudonné continued, "The result may perhaps be a bit startling" (p. 35) adding that "everything about triangles" (p. 35) can be omitted, referring to triangles as "artificial playthings" (p. 41). A "sharp controversy" (p. 46) was provoked by these statements. However, in order to understand the impact, it is necessary to move beyond Dieudonné's sensationalizing three-word proclamation and unpack the competing ideas more thoroughly.

Although he paid tribute to Euclid's accomplishments, there is no doubt that Dieudonné was completely serious about eliminating nearly all of classical Euclidean Geometry from pre-collegiate mathematics and replacing it with content he deemed essential for success in university mathematics. University mathematics was to be aligned with the foundations that mathematicians had developed during the first half of the twentieth century, including the approaches taken in the Bourbaki books (see Chap. 3 in this volume). Dieudonné's content summary gave as high school topics:

1. Matrices and determinants
2. Elementary calculus
3. Graphs of functions and curves in parametric form
4. Complex numbers
5. Polar coordinates

Perhaps more fundamental were Dieudonné's two guiding principles. The first was that before any formal theory could be developed, learners needed to acquire familiarity with the material on an experimental or semi-experimental basis with "constant appeal to intuition" (OEEC 1961, p. 39). Second, regarding logical inference, "it should always be presented with absolute honesty—that is without trying to hide gaps or flaws in the argument" (p. 39).

At the time of Royaumont in the USA there were two axiom systems in US geometry courses; those that retained the spirit of Euclid's original five axioms and those that sought to bring more "rigor" by incorporating Birkhoff's axioms. We discuss to what extent these would be sustained over the next 60 years, and if not, how they would be replaced and how Royaumont ideas influenced these new directions.

In 1963, the National Science Foundation sponsored a conference in Cambridge, Massachusetts that included 29 researchers, most of them affiliated with mathematics departments in major universities in the USA. In the foreword, Frances Kapel, of the US Commission of Education would write: "The step here taken by mathematicians is one that all scholars in all the disciplines must sooner or later attempt to take" (Goals for School Mathematics 1964, p. ix). The report's geometry was close to Dieudonné's vision. Geometry in grade 9 included the following: (a) Intuitive and synthetic geometry to the Pythagorean theorem, (b) coordinate descriptions of conics, (c) motions in Euclidean space, (d) linear algebra (including linear independence and vector spaces), (e) complex numbers, (f) trigonometry, (g) projective geometry and tensors. Grade 10 would expand on these topics, including the geometry and topology of the complex plane (neighborhoods, continuous functions, the fundamental theorem of algebra, and more) and linear algebra through to the Cayley-Hamilton theorem. The authors added: "The subject matter we are proposing can roughly be described by saying that a student who has worked through the full 13 years of mathematics in grade K to 12 should have a level of training comparable to three years of top-level of college training today" (p. 7).

A conference on geometry held in Carbondale, Illinois, was sponsored by the Comprehensive School Mathematics Program (CSMP) in 1970 (Figure 17.1). In the meeting report, CSMP Director

Figure 17.1 1970 CSMP meeting organizers Burt Kaufman (left) and Hans-Georg Steiner. (Image from an appreciation collection of letters provided by Terry Kaufman)

Burt Kaufman cited the 1963 Cambridge meeting as providing the impetus for the CSMP (Kaufman 1971). Among the participants, Friedrich Bachman spoke on *n*-gons and polyhedra, Harold S. M. Coxeter on inversive geometry, Arthur Engel on geometric activities in elementary School, Peter Hilton on topology in high school, Paul Kelley on topology and transformations in high school, Victor Klee on research problems in high school, and Howard Levi on geometric algebra. Hans Freudenthal also gave a presentation that took a different tack and is discussed in greater detail below.

The CSMP described their vision for a K–12 geometry curriculum (CSMP Staff 1971). In K–2 one has exploration of shape and in subsequent grades students learn about congruence and isometries of the plane (reflections and rotations). In middle grades the notions of parallel and perpendicular emerge along with related shape classification (squares, rhombi), and in three dimensions experiments leading to Euler's formula for polyhedra. Vectors were to be introduced in terms of translations. Similarity, area, volume, and angle would then emerge to complete the preparation for the secondary program. The authors asserted: "Every student will have experienced geometric activities commensurate with his ability and interest" (p. 283). At the secondary level, the proposed geometry "will develop only a fragment of geometry, but that fragment rigorously" (p. 284). For the core of this book, we have:

> Affine geometry in 2- and 3-space will be studied with considerable emphasis given to characterizing the finite affine planes. Hilbert's system will be mentioned, but Euclidean geometry will not formally be developed in Book 10 beyond ordered affine geometry. Rather, projective planes will be investigated, particularly finite projective planes. This will be an easy task, as the usual correspondence between affine and projective planes will be established. […] The concluding chapter of Book 10 will pick up the history of the parallel postulate and it should serve as an informal introduction to non-Euclidean geometry for any student who wishes to read further in that subject. (p. 284)

To use a US phrase, the CSMP 1970 secondary proposal would be the "last hurrah" given the Royaumont ideas expressed by Dieudonné to emerge in the USA.

According to the *Unified Modern Mathematics* text (Fehr and Fey 1969), transformation geometry would play a major role. Although the emphasis on isometries of the plane was not delineated in Dieudonné's Royaumont presentation, he did discuss "a deeper study of the groups of plane geometry" (OEEC 1961, p. 43) which we interpret as advocating such a position. Few of the topics discussed by the mathematicians in Carbondale would (or could) leave a mark on the secondary curriculum in the USA, with the exception being transformation geometry involving basic isometries of the plane. Rigid motions of the plane and dilations would enter and remain through today's Common Core era. If one attributes this emphasis to Dieudonné, then indeed in this small way he had a lasting impact.

Between 1963 and the mid-1970s, the Secondary School Mathematics Curriculum Improvement Study (SSMCIS), led by Howard F. Fehr of Columbia University Teachers College, prepared texts titled *Unified Modern Mathematics*. Fehr had chaired Section II at the Royaumont Seminar and was a principal editor of *New Thinking in School Mathematics* (OEEC 1961), so it is no surprise that SSMCIS echoed Royaumont themes, although also notable was the inclusion of computer science topics. SSMCIS would target the top 10 to 15 percent of students and reportedly some 25 000 students used the material. Fehr and Fey (1969) described an integrated curriculum (geometry across all 4 years). One finds in Course I: Transformations of the plane, segments, angles, isometries; in Course II: Groups, axiomatic affine geometry, coordinate geometry, transformations of the plane: isometries, length, area, volume; and in Course III: Introductions to matrices, algebra of matrices, affine space geometry, circular functions, vector geometry. So, we find Dieudonné's idea of including affine geometry a clear objective, together with algebraic topics essential to setting up the affine approach, but in a program only for a select few top students.

During the 1960s and 1970s, the axiomatic development of triangle congruence and similarity theorems to which Dieudonné had objected continued to dominate US geometry texts, where one typically found two-column proofs (statements on the left, reasons on the right). Many were structurally close to Euclid's text (e.g., Jurgenson et al. 1969) while some used Birkhoff's axioms, assuming a ruler placement postulate to embed the properties of the real numbers into measurement (School

Mathematics Study Group 1963). Additional topics which appeared by the mid-1970s included chapters on set theory and logic, coordinate geometry, and area and volume (Wood 1975). Outside of SSMCIS texts, one does not find significant incorporation of transformation geometry (e.g., hints of Klein's Erlangen program).

A pair of articles in the *American Scientist* by René Thom (1971) and Jean Dieudonné (1973) offers more clarity on the points of view that were emerging. That the *American Scientist* chose to engage these two stellar researchers is indicative of tensions that had emerged. Regarding Euclid, Thom stated: "Geometry is a natural and possibly irreplaceable intermediary between ordinary language and mathematical formalism." He later added, "There is hardly any doubt that, from a psychological and, for the writer, ontological point of view, the geometric continuum is the primordial entity" (p. 698). Referring to criticisms that had been made of Euclid for its lack of rigor, Thom continued "All this explains why the reproaches of inconsistency directed at Euclidean geometry are irrelevant; they do not touch the validity of local intuitive reasoning. […] Euclidean geometry is the first example of the transcription of a two- or three-dimensional spatial procedure into the one-dimensional language" (p. 698). Dieudonné responded with a defense of linear algebra, in particular, "geometric algebra" alluding to the book title of Emil Artin (1957) by stating: "Once the basic theorems of Euclidean geometry have been established by linear algebra (without coordinates, of course!), there is nothing to prevent the bright student from tackling the classical problems on triangles or conics if he is so minded. […] The basic principle of modern mathematics is to achieve a complete fusion between 'geometric' and 'algebraic' ideas, opposing 'geometry' to 'algebra' as Thom does is simply meaningless" (p. 19). The arguments that would play out in the USA were not simply a matter of no axioms versus axioms or which axioms to use, but in fact, it was more fundamental: Should the teaching of geometry be centered on a spatial domain (Thom's primordial entity) or instead on the fusion of algebra and geometry (according to Dieudonné).

Hans Freudenthal's discussions at the 1970 Carbondale Conference (Freudenthal 1971) offered a different perspective on the debates emerging in the USA. About Dieudonné's algebraic approach, Freudenthal opined, "The geometry allowed by linear algebra is an utterly dreary product" (p. 425) and about the axiomatic approach, "People today believe that geometry failed because it was not deductive enough; to my opinion it failed because its deductivity could not be reinvented by the learner but only imposed" (p. 418). Instead, citing work of Tatiana Ehrenfest and also the van Hieles, he added: "It sounds old fashioned to claim that geometry should be related to physical space" (p. 418) and he devoted the latter half of his article to discussion of examples of what in the USA would be referred to as spatial thinking, explaining also the importance of working in three dimensions. About high school students he stated: "Their spatial imagination had been killed by too much and too exclusive plane geometry" (p. 421). Freudenthal offered a view directly linked to his experiences as a research mathematician: "I espouse the philosophy of teaching mathematics related to reality. […] We mathematicians retain the mathematics we learned, because it is our business. People usually forget what is not related to the world in which they live. For most people mathematics cannot be an aim in itself; if they have learned it in an unrelated way, they will never be able to use it" (p. 420).

The writings of Dieudonné, Thom, and Freudenthal illustrated what, 15 years after Royaumont, would become three streams of thinking about geometry education in the USA. Those emphasizing algebraic (coordinate and linear algebra methods), those affirming classical Euclidean results, and those more focused specifically on spatial reasoning across a wider age span. We would not claim that these three mathematicians created these streams, but much of the US geometry activity in the next five decades can be understood through the ebb and flow between them. There would be substantial overlap and crossing between these streams and other layers of the pedagogical debates; for example, the role of problem solving as a basis for instructional design, and debates over the notion of "rigor."

The secondary geometry materials from the 1970s to the mid-1980s were reasonably stable. Except for differing "tastes" among axiom systems, roughly speaking what Dieudonné despised and Thom advocated was pretty well entrenched. However, during the 1970s and 1980s, research in geometry education in the USA began to dig deeply into spatial reasoning across all ages, as discussed in a survey article by Clements and Battista which appeared in Grouws (1992). Albeit loosely, this body of work was aligned with the third stream articulated by Freudenthal. This work would influence curriculum development in the 1990s, particularly in K–6 and the National Science Foundation (NSF)-funded curriculum projects (Clements 2003; TERC 1994).

Research at the time on proof looked at van Hiele levels and their implications. Shaughnessy and Burger (1985) discussed the spatial thinking necessary for successful deduction in geometry. Clements and Battista (1992) reported a "conflict" of views between using van Hiele levels versus a Piagetian perspective citing Driscoll's (1983) assertion that students need to be formal operational thinkers to completely understand and construct proofs. In their closing comments on proof, Clements and Battista (1992) offered an extremely pessimistic view, "However, our analysis of students' proof-making abilities reveals a far more devastating finding: Students are deficient in their ability to establish truth in geometry, and indeed, all of mathematics" (p. 442). The US mathematics educators' dilemma three decades post-Royaumont was that students were not learning proof and there was no consensus as to why. To some extent, this set the stage for events of the 1990s where axiomatic geometry would see a reduced emphasis, but not at all for Dieudonné's reasons.

Policy in the USA is recorded in state standards and frameworks. The 1985 California Framework called for "An adequate set of postulates to support proof of geometric theorems" (California Department of Education 1985 p. 40). Citing the NCTM Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics 1989), the 1992 California Mathematics Framework listed topics for increased attention and decreased attention (California Department of Education 1992): For *increased attention*: Integration across topics at all grade levels; coordinate and transformation approaches; development of short sequences of theorems; deductive arguments expressed orally and in sentence or paragraph form; computer-based explorations; three-dimensional geometry; realistic applications and modeling; and with *decreased attention* to Euclidean geometry as a complete axiomatic system; proofs of incidence or betweenness theorems; geometry from a synthetic view point; two-column proofs; inscribed and circumscribed polygons; theorems for circles involving segment ratios. Analytic geometry was listed as a separate course. The 1989 NCTM Standards, would include Standards 7: Geometry from a Synthetic Perspective, and Standards 8: Geometry from an Algebraic Perspective. This offered space for proofs as developed in classical Euclidian geometry, but the reference to an algebraic perspective meant the use of Cartesian coordinates, not the linear algebra approach to the affine plane which had been advocated by Dieudonné. Perhaps one finds some affirmation of Thom's views, but by this point, the three streams discussed above *had merged into a US blend*.

To get a sense of the new directions, four federally funded secondary projects from the 1990s attracted considerable attention and are still commercially available today (with charts showing alignment with the Common Core State Standards): *Core-Plus Mathematics Project* (CPMP 2015), the *Interactive Mathematics Program* (IMP; Fendel et al. 2015), the *University of Chicago School Mathematics Project* (UCSMP 2007), and *Core Connections* (formerly *CPM Mathematics*; Sallee et al. 2016). Initially, both Core-Plus and IMP were integrated programs, while CPM and UCSMP followed the traditional US algebra 1-geometry-algebra 2 sequence. While UCSMP's geometry has a structure closest to a deductive system, emphases on a deductive system that approximates the style of US texts from the previous decade were not part of the other three. Spatial reasoning was used to capture the reasons behind important results, for example, the IMP showed a proof of the Pythagorean theorem using an expansion of the well-known diagram previously used by SMSG.

Elementary and middle-school curricula in the 1990s "reform" curricula also contained significant spatial reasoning illustrating the impact of the prior decades' research. One K–5 program would emphasize spatial thinking, *Investigations in Number, Data and Space* (TERC 1994) and a University of Wisconsin-Freudenthal Institute 6–8 program *Mathematics in Context* (Wisconsin Center for Education Research and Freudenthal Institute 1997) included spatial models in number and proportional reasoning that would reemerge with the implementation of the Common Core State Standards. To be sure, many of the texts available in the 1980s remained prominent (and constituted the majority in use), but the "reform" materials' geometry introduced a Freudenthal-style spatial stream to K–8.

In the mathematical community, there were many objections based upon a sense of a loss of rigor sacrificed to different pedagogy, and by the mid-1990s the "math wars" erupted (Becker and Jacob 2000a, b; Haimo and Milgram 2000; Schoenfeld 2004). Referring to what happened three decades earlier those unhappy with the reform would label the 1990s' curriculum as the "new new math."

The National Council of Teachers of Mathematics (NCTM) undertook the most ambitious collaborative project in developing standards that has ever occurred in the USA, *Principles and Standards for School Mathematics* (PSSM) (2000). Joan Ferrini-Mundy led a writing team that vetted drafts to a variety of "affiliated response groups," including pure and applied mathematicians via their professional organizations. This document gave the best indication of the thinking and compromises reached among US mathematics education stakeholders four decades after Royaumont. In PSSM 2000 Geometry we find:

Instructional programs from pre-kindergarten through grade 12 should enable each and every student to:

- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- Apply transformations and use symmetry to analyze mathematical situations
- Use visualization, spatial reasoning, and geometric modeling to solve problems (NCTM 2000, p. 41)

Looking at the grade bands (K–2, 3–5, 6–8, 9–12), in particular high school, we see that the document invited everyone to the table, including a fourth contingent that may be more concerned with "mathematical correctness" rather than with detailing specific content. In secondary we find no mention of a deductive system based on axioms, with the closest discussion in grades 6–8 to "create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship" (p. 42). In grades 9–12 there was "establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others" (p. 42). Although the algebra standards included a heading "Represent and analyze mathematical situations and structures using algebraic symbols" (p. 37) there was nothing resembling affine geometry (although there were some references to functions that underlay calculus ideas). In grades 6–8 we had, "examine the congruence, similarity, and line or rotational symmetry of objects using transformations" (p. 43) which is expanded in grades 9–12 to "understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices; use various representations to help understand the effects of simple transformations and their compositions." There is substantive spatial reasoning throughout K–8 and then in grades 9–12 we find, "use geometric models to gain insights into, and answer questions in other areas of mathematics and other areas of interest such as art and architecture" (p. 41).

So, what do we conclude about post-Royaumont geometry in the USA at the end of the 20th century? We have the US blend of the three streams. After several decades of rather testy disagreement, the mathematics education leadership decided in 2000 to bring all views to the classroom and let them attain a balance. Moreover, consistent with principles embedded in US educational thinking at the time, these standards were intended for *all* students, not just college preparatory students. Similar consensus-building efforts attempted around the same time ignored geometry. For example, the National Academies produced a volume, *Adding it Up* (National Research Council 2001), and this 454-page research overview had a mere 7½ pages devoted to geometry, most of which dealt with measurement. The volume would have little noticeable impact.

The NCTM did produce a volume of research *A Research Companion to Principles and Standards for School Mathematics* (Kilpatrick, Martin, & Schifter, 2003). Douglas H. Clements (2003) authored a chapter on teaching and learning geometry that cited research from the 1980s which again offered a pessimistic view of high school students' proof-writing abilities. He wrote, "Through the grades, the curriculum tends to name more geometric objects but not require deeper levels of analysis (Fuys, Geddes, & Tischler, 1988)" (p. 151), and maintained that the demands of spatial thinking embedded in the PSSM provided a better alternative. Reminiscent of Freudenthal Clements (2003) offered the following stark assessment of the state of geometry in the USA prior to PSSM:

> According to a former president of NCTM, geometry is the "forgotten strand" of mathematics (Lappan 1999) … Most current curricular and teaching practices [in Geometry] are, simply, abominable. They promote little learning or conceptual change. They often do more harm than good. They leave students unprepared for further study of geometry and the many other mathematics topics and subjects that depend on geometric knowledge. We need to do better; research provides support for the NCTM Standards and specific guidelines for teaching and learning to aid such and effort. (p. 171)

Would the new US blend bring geometry out of the doldrums over the next two decades? If it did, it would be a significant, although indirect, post-Royaumont consequence.

During 1995–2010, 35–50 years after Royaumont, geometry instruction in the USA, like much of the rest of instruction in K–12, became increasingly governed by the emergence of state standards. Each of the 50 states created and adopted content standards during the latter 1990s to the early 2000s. The 1997 California Mathematics Standards (California Department of Education 1997) were clear about the necessity of proof within a deductive system. The first two geometry standards were: "1. Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning" and "2. Students write geometric proofs, including proofs by contradiction" (p. 42). Dieudonné's despised triangle topics were nailed down in Standards 4 and 5: "4. Students prove basic theorems involving congruence and similarity" and "5. Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles" (p. 42). There were further Standards involving coordinate geometry, basic trigonometry, and rigid motions of the plane. California's standards were largely written by Stanford University mathematicians at the request of the State Board of Education (Schoenfeld 2004) and in general, one could say that Dieudonné's ideas were lost almost entirely, Thom's vision remained vaguely in place, and the Freudenthal stream nearly absent (including in K–8). The content had not shifted from the 1980s, but one can see an increased demand for attention to be given to mathematicians' notion of rigor and proof.

There was significant variation in the standards with some states including more experiential geometry and spatial thinking. However, we know of none that set aside a thorough treatment of triangle congruence and similarity as Dieudonné had advocated, albeit we do find right-triangle trigonometry and coordinate methods that many educators in the USA view as providing key pre-calculus preparation. The impact of this standards movement cannot be overstated. Teachers and teacher educators would come to use the word "standards" and the phrase "standards-based instruction" regularly, and the alignment of state standards and state-wide assessment would be an expectation of federal legislation *No Child Left Behind* (NCLB), written at the urging of President George W. Bush in collaboration with Senator Ted Kennedy. As a consequence, what was taught was more closely aligned to the standards because that was what was tested (and not just what the textbook companies produced).

In June 2010, the Common Core State Standards, sponsored by the National Governors Association and the Council of Chief State School Officers, would be released in Mathematics and Language Arts. Encouraged by President Obama's Administration's criteria for receipt of federal *Race to the Top* funds, 45 States and the District of Columbia would, within a short period of time, adopt the CCSSM (2010) (with minor variations allowed). Confrey and Krupa (2012) put it this way:

> The progress made in creating and refining national standards for mathematics is undeniable. In less than twenty years, we have progressed as a nation from no standards, to multiple sets of standards, to a voluntary set (CCSSM), each created with clear attention to salient features affecting students' learning. (p. 6)

The CCSSM contain eight *Standards of Mathematical Practice*, grade-level content descriptions for each grade K–8, and High School content descriptions for each of Number and Quantity, Algebra, Functions, Modeling, Geometry, Statistics and Probability. Although the developers of the CCSSM were explicit that their standards did not specify pedagogy, as of this writing, the CCSSM still plays a major role in determining the content and the approaches used. Moreover, some states have released open-source materials, for example, Engage New York (2020), that have become widely referenced and adopted, leading to greater uniformity of content across the nation since prior to the 1960s.

In High School Geometry, one does not find a major departure from the PSSM in terms of content. In K–6, however, there is a substantial change. The spatial thinking that was a major part of the research beginning in the 1970s and embedded in PSSM is lost in CCSSM. K–6 Geometry is mostly social knowledge of properties of figures, and measurement formulae (e.g., limiting shape decomposition to demonstrations of fractional areas) with an introduction to coordinates in grade 5. Although some topics allow time for play in K–3, which includes, "Reason with Shapes and their Attributes," by grades 4–6 the topics either involve measurement or classification of shapes. By and large, we find essentially none of the spatial objects and problems that Freudenthal articulated in 1970, nor those highlighted in the subsequent US research. In this regard, the transition in Geometry from the PSSM to the CCSSM is a huge loss to those following the stream articulated by Freudenthal.

In middle and high school, transformation geometry (isometries and dilations) is used to set up ideas of congruence and similarity. There are standards expecting students to "prove theorems" including results involving congruence, similarity, parallels, and circles, but no set of axioms or assumptions is specified. Introductory trigonometry is embedded in the right-triangle section as well as a considerable amount of coordinate geometry, geometric constructions as well as area and volume. It is a very full package of topics, much more than was expected in the 1960s texts. The programs outlined by Dieudonné and Thom are distant. If asked to place CCSSM geometry into post-Royaumont categories, the answers might best be "none of the above" or "tiny bits and pieces of each."

The Engage New York (2020) geometry materials in grade 10 offer a picture of what is happening as a result of CCSSM. There are five modules in the course. The first two deal with rigid motions and dilations to provide approaches to congruence, similarity, and right-triangle trigonometry. The third deals with 3-dimensions and volume. The fourth develops coordinates and a modeling problem. The final module is concerned with circles, properties of triangles and quadrilaterals related to parallel lines and intersections, and closes with Ptolemy's Theorem on cyclic quadrilaterals. Students are expected to experiment, describe their geometric reasoning, and prepare two-column proofs using given theorems. The materials faithfully cover the CCSSM Geometry standards, but add almost no topics in addition to the minimum needed to cover the CCSSM list.

As noted by Stanic and Kilpatrick (1992), societal pressures in the USA surrounding college and university preparation have greatly influenced the directions of the curriculum. The University of California (UC) and the California State University are the only systems in the USA that review the course content of high schools to affirm or deny a course meeting their systems' admission requirements. After California adopted the CCSSM for its state standards, the UC Board of Admissions and Relations with Schools (BOARS) changed its mathematics discussion to say, "the Common Core State Standards in Mathematics offers a starting point for developing courses." At the same time, some Engineering-focused academies in California high schools had students skip geometry so that they could study calculus earlier, thereby fulfilling a version of Dieudonné's argument that junior high geometry was sufficient to succeed at the university—the schools were also under local pressure to have their students pass the Advanced Placement Calculus examination. Engineering faculty from the nine University of California undergraduate campuses urged BOARS to specify that the 3 years in mathematics preparation had to include geometry forcing these high schools to retreat.

Post-Royaumont Sets, Logic, and Structure

While Dieudonné's statement "Euclid must go!" attracted great attention, as far as the general public in the USA was concerned the New Math came to be synonymous with funny things like sets and base seven arithmetic. At the time of Royaumont the SMSG program was focused on the secondary level, but by early 1961 the SMSG leadership had turned its attention to the elementary grades (Phillips 2015). Earlier, SMSG had sponsored meetings with experts in psychology and listened to the ideas of Jerome Bruner, the Harvard psychologist who spoke about Jean Piaget's research (Kilpatrick 1964; Phillips 2015) including the importance of mathematical structure and part-whole relations. The result would be textbooks in the USA that introduced sets and one-to-one correspondence with the addition of single digit numbers represented by unions of sets in primary grades.

The approaches of SMSG and CSMP programs in K–6 aligned with those articulated at the 1963 Cambridge Conference, which built upon Royaumont and where according to Weaver (1964), "The notions of function and set are to be used throughout; of course, set theory and formal logic should not be emphasized as such, but the child should be able to build his early mathematical experience into his habitual language" (p. 208). In grades 4–6 in addition to arithmetic computation, SMSG students studied intersections and unions of sets, representing numbers in bases other than ten, and examined sequences of numbers for patterns (that was the second author's personal experience in grades 4–6). Similarly, with the CSMP, for which there would be many editions published between 1969 and 1982, students would see the Boolean combinations of sets in K–5 (including Venn diagrams) through a strand *The Languages of Strings and Arrows*, where the Teachers' Guide stated, "Two fundamental modes of thought for understanding the world around us are the classification of objects into sets and the study of relationships among objects" (CSMP 1976). In upper elementary school, the student projects and games involved factors and multiples relationships, and functional relationships created by strings of operations to illuminate the structure of the number system.

In SMSG, middle-school students studied modular arithmetic and the structure of the Klein 4-group through the set of symmetries of a non-square rectangle (Phillips 2015). The study of structure was not limited to side explorations, for example, the distributive law provided the basis for the multiplicative structure of negative integers. CSMP used its Languages of Strings and Arrows to develop modular arithmetic, probability, functions, and other topics where the relationships were encoded by graphs and trees.

Across high school topics, set theory would be used by SMSG to give definitions. To the ire of some applied scientists (Diamond 1962), SMSG defined a function to be a set of ordered pairs for which each input from the domain was assigned a unique output in the range, instead of emphasizing the dependence of one variable on another as would be more suggestive in a practical context. SSMCIS would in fact do this; the 1969 edition of their ninth-grade text (SSMCIS 1969) would use the terminology of mappings and assignments for functions, while at the same time stating the criteria for a (functional) mapping between two sets including domain, range, and image. The early SMSG author Dolciani and her collaborators on a geometry text (see Jurgenson et al. 1969) included an introductory chapter on set theory as well as logic involving "if-then" statements. SMSG director Begle pushed, mostly unsuccessfully, for textbook publishers to incorporate the SMSG approaches; however, as noted by Phillips (2015), "Dolciani's *Modern Algebra* and Moise and Down's *Geometry* were, in fact, the only two books ever considered to be suitable replacements for the SMSG." Discussion of set theory would largely disappear from US texts by the mid-1970s. By 1984 the updated versions of Dolciani's texts would not have these sections, although one finds intersection, union, and member ($\cap$, $\cup$, $\in$) in the symbols list at the end of the text. Kilpatrick (2012) noted, "Basing an introduction to algebra on axioms, definitions, and theorems proved unwieldy for teachers and pupils to handle … On the other hand, many of the ideas brought into school mathematics by the new math have remained. For example, textbooks still refer to sets of numbers and sets of points" (p. 569).

Given the centrality of set theory to communication among mathematicians, and given the Royaumont imperative to include set theory, it is surprising to find little research on the teaching of these topics. Of the 29 chapters in the NCTM *Handbook of Research on Mathematics Teaching and Learning* (Grouws 1992), none would have substantive discussions of the teaching of set theory or logic. While mentioning Piaget's work, Karen Fuson's (1992) chapter on "number addition and subtraction" parsed out children's strategies and language surrounding numbers documented by numerous researchers, but there was no discussion of the role of informal set theory.

The discussion of functions in Carolyn Kieran's (1992) chapter on algebra examined the impact of the structural (as a set of ordered pairs) definition. Recalling Freudenthal's view that functions are a dependency between variables that are distinguished as dependent and independent, Kieran pointed out the prominence of the structural definition in the curriculum and its ill effects, citing Even (1988) who found that as a result, prospective teachers had a limited idea of what a function was. Moreover, since students had ample experience with functional dependence relations in prior mathematics and science classes, Kieran cited research showing that students could successfully use a process conception of function and at the same time be baffled with the structural definition—findings consistent with research in collegiate education (Dubinsky et al. 1994). As the function concept for university mathematics is so critical, and given that Royaumont mathematicians advocated set theory and structural approaches, from the tenor of the Kieran article we might conclude that the impact of Royaumont in this regard had been nothing short of disastrous.

Subsequent compilations of research in the USA would not offer insight into the consequences of the direct teaching of set theory or logic. *Adding it Up* (National Research Council 2001) did not even have "set" in its index, and it did not provide a substantive discussion of their role in mathematics teaching; the only mention of "function" was in the context of using technology to solve equations. However, by no means were mathematics teachers and education researchers in the USA ignoring proper mathematical language and reasoning. In the *Research Companion to Principles and Standards for School Mathematics* (Kilpatrick et al. 2003), there was a chapter on "communication and language" by Magdalene Lampert and Paul Cobb and a chapter on "reasoning and proof" by Erna Yackel and Gila Hanna. Lampert and Cobb (2003) noted a shift between "those who emphasize the acquisition metaphor and those who emphasize the participation metaphor," very different views of teaching and learning, and indicated why there was little interest in examining explicit teaching of set theory and logic. Reasoning and Proof was one of the five process standards in the NCTM Principles and Standards, but there was no emphasis on the language of sets and axiomatic foundations. Yackel and Hanna (2003) cited mathematicians William Thurston and Yuri Manin who expressed the view that the role of proof is to "advance human understanding" and "to make them wiser" (p. 228). The chapter noted that in the 1990s there had been more than 100 articles devoted to the teaching of proof, and the references to research mathematicians' beliefs indicated that the cross current of ideas between the mathematics and mathematics education research communities had continued after Royaumont.

Like the 1992 Handbook, the NCTM's *Second Handbook of Research on Mathematics Teaching and Learning* (Lester 2007) did not offer any discussion of the role of set theory in instruction. There were two chapters on algebra—early algebra and middle school through college algebra, but aside from mentioning algebraic properties (commutative, associative, etc.), there was little discussion of axioms, the structure of number systems, or algebraic objects in the sense of Royaumont. There was a chapter on the teaching of proof in which Guershon Harel and Larry Sowder (2007) asked, "Does explicit teaching of logic work?" They respond, "Studies of the effects of an explicit attention to logic have not, however, indicated that there is then a pay-off in proof-writing" (p. 835). After a detailed discussion of research on proof, Harel and Sowder's chapter ended noting that although students demonstrate abilities with empirical reasoning and providing examples, neither they nor their teachers have a solid grasp of the purpose of, or the ability to construct proof.

In 1985, California Mathematics Framework (California Department of Education 1985) made no mention of set theory as a discipline to be taught. But "logic" remained as a strand and the document included recognition of the logical terms "all, some, and, or, if-then, not" as well as general discussions of mathematical reasoning, including the difference between inductive and deductive reasoning. Logic would be replaced by "logic and language" in California's 1992 *Framework* (California Department of Education 1992), but without any specificity. It mostly gave general advice such as "Proofs should make difficult things clear, not make clear things difficult." The 1989 NCTM *Standards* had Standards 2: Mathematics as communication and 3: Mathematics as reasoning, but again, there would be no references to set theory or emphasis on the structure of mathematical objects as envisioned by Dieudonné. The discussion of proof was largely limited to activities such as make and test conjectures, formulate counterexamples, follow logical arguments, judge the validity of arguments, construct simple valid arguments, aspects where students had been noted to be successful by Harel and Sowder (Lester 2007).

We know of no state standards from the late 1990s movement that had explicit sections dealing with set theory or logic. Mathematical reasoning might be a strand, but formal deductive proof would not be a major emphasis outside of the context of geometry, and even then, not regularly. The *NCTM Principles and Standards* (National Council of Teachers of Mathematics 2000) would include "reasoning and proof" as well as "communication" as standards topics, and the importance of proof is strongly stated, but not the axiomatic development of mathematics. In the CCSSM (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010), neither set theory nor logic are topics for focused discussion. And while the process standards include Standards 2: Reason abstractly and quantitatively, and 3: Construct viable arguments and critique the reasoning of others, one does not find deductive arguments based on a system of axioms as a priority. So, although the document achieved considerable consensus it offers little change since the mid-1970s concerning set theory or deductive proof. The choice of language in Standard 7 was interesting—"Look for and make use of structure," which is reminiscent of Royaumont language. But the meanings are different. With Royaumont, structure referred to mathematical objects (number systems, groups of symmetries, etc.), while the CCSSM standard refers to developing a student's habit of mind when examining an expression and how the pieces of an expression might interact as part of the whole.

We ask where this leaves higher education, particularly for STEM majors for whom the Royaumont participants advocated such a dramatic change? We mention two areas of activity: the calculus reform and the proliferation of courses on discrete math, set theory, and introduction to proof. During the 1990s, the NSF funded a number of calculus projects, and while the courses offered new pedagogical approaches and many integrated technologies, they were also deeply concerned about the treatment of function. Collegiate mathematics educators were dealing head on with a fundamental problem that the Royaumont conferees had hoped to prevent, namely, *too many University students had inadequate understanding of perhaps the most important mathematical construction: Functions.* In 1992, the Mathematical Association of America (MAA) published a volume of research on the topic (Dubinsky and Harel 1992) and most NSF-funded courses developed functions from multiple perspectives: symbolic definitions, numerical tables, graphs, and at times via algorithms. To what extent this rectified the failures noted earlier is hard to say, but it certainly placed discussion of collegiate understanding of function front and center.

Although in part due to the expanded interest in computer science, courses in discrete mathematics or "introduction to proof" proliferated beginning in the 1980s. One reviewer of an introduction to proof text (Seldon 2013) claimed to have counted at least 25 texts on the market in 2009. That these courses include a substantial amount of set theory is not a surprise given that it had vanished from the K–12 curriculum. Typically, these courses also reteach the function concept as well as basic proof approaches such as proof by contradiction and induction. Again, US collegiate educators were dealing with topics the Royaumont conferees had hoped they would not. At many institutions, the course is a roadblock for mathematics majors to enter advanced courses.

Post-Royaumont Thinking About Problem Solving

Although work on problem solving in the USA predates 1959, its use in instruction was given impetus by Royaumont and the New Math. As noted by De Bock and Zwaneveld (2020) an important subgroup at Royaumont pressed for the inclusion of applications and modeling in the curriculum, and this fitted naturally with increasing interest in didactics and the emerging Realistic Mathematics Education (RME) movement with its view of problem solving as the act of mathematizing, where they quote Freudenthal (1968):

> The problem is not what kind of mathematics, but how mathematics has to be taught. In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. (p. 7)

Examining the literature on mathematics teaching and learning over the years, one finds that problem solving has been used with multiple meanings that range from solving rote exercises to the activities of professional mathematicians. In time, the discussion of problem solving would converge on the mental actions of an individual (at times a group) dealing with a question that included features they had not previously experienced. Understanding how a problem solver acts in a particular situation would become a central research topic and Alan Schoenfeld's (1992) chapter for the NCTM *Handbook of Research on Mathematics Teaching and Learning* summarized research on issues of problem solving, metacognition, and sense-making, citing171 references, most from the previous two decades.

Previously Schoenfeld had written an article with a provocative title, *When Good Teaching Leads to Bad Results: The Disasters of "Well Taught" Mathematics Courses* (Schoenfeld 1988), in which he outlined four "beliefs" that students acquire in traditional classes, the fourth of which, amplifying Doyle (1988) states "One succeeds in school by performing the tasks, to the letter, as described by the teachers. Corollary: Learning is an incidental by-product to 'getting the work done'" (p. 151). Schoenfeld's subsequent research examined in detail students' problem-solving strategies, where in contrast to Pólya's ideal approaches that one might try to teach, he introduced a dose of reality as to what is really going on when students tackle problems. Echoing Freudenthal, according to Schoenfeld (1992) learning to think mathematically means "developing a mathematical point of view," "valuing the processes of mathematization and abstraction" (p. 335), and a natural bent to do these things.

When posing a problem for instructional use many questions were discussed by US educators during the 1990s: How close is it to a routine exercise, would an inductive method be appropriate, what is the role of the context of the question (e.g., to make it familiar or to provide a tool for solutions), what representations might emerge, will the learner acquire the skills that previously were taught by drill, how open-ended is the question and what does "open ended" mean (multiple approaches, multiples solutions, just initiating a direction)? Then, as materials proliferated, questions of an instructional trajectory began to be considered: How tightly ordered should the materials be or is there a more fluid landscape that learners might explore, could everything be handled through inquiry-based learning or does time need to be set aside for "practice," are there multiple entry points to assure that the needs of a diversity of learners are met through "differentiated" instruction?

The National Council of Teachers of Mathematics (NCTM) prioritized the theme "problem solving" following the "back to basics" movement of the 1970s. In quick succession, the NCTM published *An Agenda for Action* (NCTM 1980) and its 1980 *Yearbook* (Krulik 1980), and they were followed by *Everybody Counts* of the National Research Council (1989), NCTM's *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), and *Principles and Standards for School Mathematics* (NCTM, 2000). In each problem solving was presented as an important goal for teaching school mathematics and, indeed, it is one of NCTM's process goals for school mathematics. The problem-solving standard of NCTM states:

> Instructional programs from prekindergarten through grade 12 should enable all students to
> - Build new mathematical knowledge through problem solving

- Solve problems that arise in mathematics and other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving. (p. 402)

In *Everybody Counts* (National Research Council 1989)—as also reflected in the *Standards* (NCTM, 1989, 2000) and *Reshaping School Mathematics* (National Research Council 1990)—we find:

> Mathematics is a living subject that seeks to understand patterns which permeate both the world around us and the mind within us. Although the language of mathematics is based on rules that must be learned, it is important that students move beyond rules to be able to express things in the language of mathematics. This transformation suggests change in both curricular content and instructional style. It involves renewed effort to focus on: Searching for solutions, not just memorizing procedures; exploring patterns, not just learning formulas; formulating conjectures, not just doing exercises. (p. 13)

By the mid-1990s, the notion of problem solving would become so central to the US post-Royaumont reforms that leaders would argue problem solving should become the "basis for reform in curriculum and instruction." In an article whose title bears that phrase, Hiebert et al. (1996) would write in their abstract:

> We argue that reform in curriculum and instruction should be based on allowing students to problematize the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems. In mathematics, this principle fits under the umbrella of problem solving, but our interpretation is different from many problem-solving approaches. (p.12)

Studies of the implementation of problem solving in the classroom would emerge. They reveal a simple, but critical post-Royaumont idea, that silly as it sounds today had it been better articulated at that time it may have led to different outcomes. Namely, the dynamics of a classroom are complex and approaches developed in isolation may not play out as intended in a dynamic classroom. Magdalene Lampert (1990, 2001) described the myriad of issues that underlie teacher decision-making; from students' affect, engagement, and entry points, to organizing student-led discussions so that their ideas will be honored and move the class forward, not to mention external pressures the teacher faces. In short, the USA was learning that using problem solving as a basis for instruction, however, it is construed, is a complex process that requires extensive professional development to accomplish.

A variety of views of problem solving emerged with varying levels of implementation. To illustrate the possibilities, we discuss the influence of Japanese style *open-ended problem solving*, and investigations based on ideas of Realistic Mathematics Education (RME) from the Freudenthal Institute where context is used to develop big ideas, representations, and tools. These and other approaches arose as US mathematics educators labored to put the 1960s New Math and 1990s Math Wars behind them and to develop both the skills the public demands and the understandings mathematicians and the mathematical education communities deem necessary without compromising the other. While most parties to the debates agreed that "balance" is the answer, solving this elusive equation remained the most significant obstacle to the improvements enunciated at the Royaumont meeting.

In the 1990s, interest in Japanese mathematics education would expand in the USA, after the dissemination of comparative research (case studies) conducted in the *Third International Mathematics and Science Study* (TIMSS) (Stigler et al. 1999) and publications dealing with Japanese lesson study (e.g., Hiebert and Stigler 1999; Takahashi 2016, 2021). Research mathematicians were involved in evaluating the mathematical content and student thinking in the TIMSS lesson study research (Hiebert et al. 1999; Manaster 1998) invigorating interest across that community. In 1997, the California Standards Commission specifically asked staff to check the alignment of proposed standards against the Japan mathematics standards, and interest in Japanese textbooks grew (Bishop 1999).

Japanese thinking regarding problem solving came to the USA and became more and more known (see, e.g., Becker 1992). The book *The Open-Ended Approach: A New Proposal for Teaching Mathematics* (Becker & Shimada, 2007) reported on Japanese developmental research concerned with evaluating higher-order thinking using open-ended problems as a theme. Research in problem solving in school mathematics was carried out beginning in the early 1970s in Japan at the same time as it became a focus in the USA. Joint US-Japan research on problem solving included a bi-national seminar on problem solving (Becker 1992). The research showed that lessons based on solving "open-ended" problems have a rich potential for improving teaching and learning, and as a result subsequent studies focused on teaching using open problems in classrooms (Hashimoto and Becker 1999).

The Japanese approaches included problems that have exactly one correct answer, but many different ways to get it (the process is open), problems that have several of many correct answers (the end products are open), and those with students formulating their own problems (the ways of formulating problems are open). This approach to problem solving and research was close to that in the NCTM (1989, 2000) Standards documents. The research was reported in detail in Becker (1992), Becker and Shimada (1997), and Hashimoto and Becker (1999).

Example: (Sugiyama 1989) There are three parks A, B, C in which many boys and girls are playing. The areas of the parks A, B, C are, respectively, 500 square meters, 500 square meters, and 300 square meters. The number of boys and girls playing in these parks is, respectively, 40, 30, and 30. Which park is most crowded?

A	B	C
500	500	300
40	30	30

There are many ways to get the answer. Then ask the students what threads run through all the solution methods. Students will see that in each case one variable is held constant and the other examined. This is a general solving strategy in this and many other cases.

Extended: Suppose there is a park D with an area of 520 square meters and 47 boys and girls, is park D more crowded than C or A?

Questions that arise include are decimal fractions appropriate to use? Which solution do you think is best? Why?

In the USA, similar problems appeared where students used a ratio table as well, for example, in comparative pricing problems (Jacob and Fosnot 2007; Wisconsin Center et al. 2013a). Other Japanese open-ended problems circulated widely in the USA had students in elementary grades investigating arithmetic sequences (see, e.g., Hashimoto 1987). Similar problems can be found in SMSG in the 1960s and reappeared in the 1990s reform materials where middle-school students studied figures and used spatial reasoning to develop the number patterns that emerge, usually arithmetic sequences but at times geometric sequences as well (Wisconsin Center et al. 2013b).

The Freudenthal Institute of the Netherlands's work on Realistic Mathematics Education (RME) used an understanding of the role of context that was more closely linked to learners' mathematical reasoning than was typically the case in the USA. This work has been incorporated by US researchers and educators, often in collaboration with Freudental researchers: the NSF-funded middle school curricula; *Mathematics in Context* (Wisconsin Center for Education Research and Freudenthal Institute 1997); *Math in the City* at City College of New York; and the *Freudenthal Institute West* at the University of Colorado, Boulder. These projects considered two aspects (among many) relevant to the discussion here, the role of context and the design of instructional trajectories that linked problems and inquiry into a curricular unit. We give an example from the work at *Math in the City* to illustrate features of the role of context.

Example: The teacher tells fourth grade students about the water and juice machines in the teachers' lounge and asks two questions.

In the teachers' lounge we have a water machine and a juice machine. I talked to the person who fills them and learned they each contain 156 bottles. That got me thinking. I wonder how many six-packs the machine will hold?

When he was filling the juice machine I saw inside and there were six columns that the bottles are stacked in, one for each of the six flavors of juice in the machine. (The teacher shows students a picture of an array with six

Figure 17.2 Use of the representation of the context as a tool for thinking about the link between two forms of division

columns.) So, I am now wondering how many bottles of each flavor fit in the juice machine when it is full? (Natale and Fosnot 2007)

The context is realistic in that it is familiar (the students have seen soda machines before) and so the student can imagine the details. But the problems are not "trivialized word problems" where computation is randomly matched to a situation (as is often the case in US textbooks). The context was carefully chosen to involve an array, which is the model of multiplication that the students will also use for division. The model becomes a tool for the students' computations where the authors note that models go through three stages of development: model of the situation, model of students' strategies, and model as a tool for thinking (Gravemeijer 1999).

Both forms of division are posed simultaneously as the main goal of the problem is for students to use the array model to see why they give the same result. Most students at this age will work on the two problems separately and the focus of the class discussion of their work is to find out why the two answers are the same. Some students will draw pictures or build a model out of cubes to see that the six-packs could be arranged along the rows of the juice machine, where each row would have the six juice flavors in the juice machine (Figure 17.2).

In the introductory discussion the emergent strategies, big ideas, and models in the unit are described. In this problem students are using strategies that require unitizing chunks of bottles (how many sixes) in a context where the array model is natural. The next stage of the developmental trajectory includes a context where students think about boxes of ten six-packs, so that in a repeated subtraction process to determine the number of sixes in 156, the more efficient removing 60 at a time, rather than 6 at a time emerges. Ultimately, an efficient computational algorithm for division emerges, but again, the original context and the array is a representational intermediary that remains available for the students thinking about division.

Post Royaumont Research Mathematicians in the USA and K–12

The Royaumont meeting included a balance of research mathematicians, secondary school teachers of mathematics, and representatives from education ministries or outstanding educators (OEEC 1961) (We use the terminology "research mathematicians" in this discussion for those whose academic careers began in pure or applied mathematics even though many would likely prefer to simply be called a "mathematician"). Moreover, the Royaumont report sees the training of teachers as critical, and "thus university professors of mathematics should devote a part of their professional time to the important task of teacher education" (Moon 1986, p. 43). But the Royaumont balance would not play out evenly over time in the USA, in fact it would experience swings. The original SMSG secondary team to some extent would reflect a balance, but as SMSG expanded research mathematicians working at the elementary level would play a dominant role. The CSMP and SSMCIS development teams were better balanced than others but again were largely guided by research mathematicians.

At the time of Royaumont research mathematicians in the USA "were not supposed to care about school mathematics" (Phillips 2015, p. 48). Richard Brauer, speaking as President of the American Mathematical Society (AMS), said that K–12 curriculum development was "not Society business." In 1961, Edward Begle would take a position in the education department at Stanford University where according to Phillips (2015) "his request for a joint appointment in the math department was rebuffed, apparently to his great dismay" (p. 81). In 1969, Begle would plea for educational research, "I trust that by now we have convinced those who are concerned with the improvement of mathematics education that we are faced with many serious problems … The progress toward solution of these problems can only come from careful empirical research" (Begle 1969, p. 242). During 1970–1990, US educational research activity would expand enormously and appears to have sidelined research mathematicians, with educational researchers playing a prominent role, for example, the overwhelming majority of authors in the NCTM handbooks (Grouws 1992; Lester 2007) were education researchers. This could have spelled the end of their involvement, but attitudes would gradually change within the mathematical community, and collaboration with educational researchers would become more valued and no longer considered an oddity.

Many US research mathematicians first come into contact with K–12 issues while teaching courses for prospective teachers. Some would then write texts for such courses or became involved in teacher professional development as part of an NSF- or state-funded curriculum project. Math circles were organized where research mathematicians facilitated problem-solving discussions with students or teachers, with an eye toward increasing interest in the more formal tools of mathematics. Others would play leading roles in mathematics competitions: junior high and high school MAA competitions and the USA and International Mathematics Olympiads. The agendas of meetings of the AMS and MAA show that over the past 40 years the number of K–16 education-related sessions has dramatically increased and participants represented the full spectrum of opinions regarding the "debates." These activities represent a level of engagement that was not seen in the pre-Royaumont era.

Mathematics researchers would participate in the PSSM 2000 Affiliated Response Groups through their professional organizations (the AMS, MAA, and other groups). Many participated in standards and framework discussions in their states during the two decades beginning 1990s. William McCallum, a number theorist by training, would play a key role as a lead author of the Common Core State Standards and has played a large role in disseminating information about the CCSSM.

The research mathematics community has not been shy to offer opinions in public and professional forums. An open letter, *On the Mathematics Curriculum of the High School* (Ahlfors et al. 1962) signed by 65 research mathematicians offered a different point of view of "modern mathematics," explicitly recognizing in its opening sentences that the role of mathematicians engaging K–12 had changed, "The mathematicians of this country now have a more favorable climate in which to develop and gain acceptance of improvements in mathematics education" (p. 189). Research mathematicians would play substantive roles in the math wars of the 1990s (Becker and Jacob 2000a, b; Schoenfeld 2004). Some would write isolated reflective articles on teaching and learning that would become widely cited—for example, an article by William Thurston (of Princeton) offered insights about K–12 learning that linked that process to features of a PhD education where he described the phenomena he called "compression" (Thurston 1990). Hassler Whitney (Institute of Advanced Study) was inspired by the work of Louis P. Benezet and Roger Howe (Yale), who wrote about Liping Ma's work (Ma 1999). He wrote a number of articles dealing with the issue of arithmetic and children's understanding of computation in order to promote a middle ground in the skills versus understanding debates (Howe 1999, 2010, 2014; Whitney 1973, 1985, 1987). Others would blend their professional approaches with advocacy for equity and university access. For example, the Algebra Project of Robert Moses (Moses and Cobb 2001) advocated access to quality mathematics as a civil right. Moses, following a master's degree in logic (actually in a philosophy department), would devote years to civil rights work in Mississippi for which he would receive a MacArthur genius award (1982–1987), whose resources he used to found the Algebra project, and who since 2004 has served on the Education Advisory Committee of the Mathematical Sciences Research Institute.

A smaller number of US research mathematicians would shift gears and devote their attention almost entirely to mathematics education. Alan Schoenfeld would jump from topology to the Graduate School of Education at UC Berkeley and become known for his academic research in problem solving while expanding his research to cognitive issues and teacher education. Schoenfeld would also take on leadership roles in California mathematics education and nationally with projects with the NCTM and as a president of the American Educational Research Association (AERA). Hyman Bass, an AMS Cole Prize in Algebra recipient and National Academy of Science member, would retire from Columbia University and establish a second multi-decade career in mathematics education at the University of Michigan in collaboration with Deborah Ball making contributions to elementary school children's cognition, as well as mathematical knowledge for teaching (MKT) (Ball and Bass 2004). He would be a consistent voice about teaching and learning in the mathematics research community and would serve as President of the AMS and the International Commission on Mathematical Instruction (Bass 2005). In a different direction, geometer Hung-Hsi Wu would become vocal in the 1990s about his concerns about the state of mathematics education in the US and then devote two decades to working with and writing extensively about teacher preparation, recently publishing three volumes on the subject with the AMS (e.g., Wu 2020). Echoing Dieudonné's second principle, Wu would write that his books "give an exposition of the mathematics of grades 9–12 that is simultaneously mathematically correct and grade-level appropriate. The volumes are consistent with the Common Core State Standards for Mathematics and aim at presenting the mathematics of K–12 as a totally transparent subject" (back cover).

These examples offer responses to Royaumont which illustrate the roles of research mathematicians today in the USA. Their work on problem solving and learner's cognitive development responds to Dieudonné's first principle that learners need to acquire familiarity with mathematics on an experimental basis with "constant appeal to intuition" (OEEC 1961, p. 39). They also respond to his second principle regarding logical inference, that "it should always be presented with absolute honesty—that is without trying to hide gaps or flaws in the argument" (p. 39). And while our examples would likely have ample areas of disagreement among themselves (and the Royaumont participants), they illustrate an enormous level of dedication to K–12 among research mathematicians reflective of the post-Royaumont era that might not have matured in the same way had the USA bypassed "modern mathematics."

Concluding Remarks

We have considered post-Royaumont thinking in the USA across five time periods spanning 1960–2020 and considered the evolution of geometry and spatial thinking; formal mathematical language and structure in the curriculum, as well as problem solving and its relationship to "skills"; and the contributions of research mathematicians. In the first post-Royaumont decade we found three streams of thought that would evolve and used the writings of three mathematicians to characterize the starting points for these views: Jean Dieudonné who would articulate the Royaumont advocacy for using the rigor of the Bourbaki-style presentation; René Thom who would advocate equal rigor but with a different ontology regarding mathematical intuition; and Hans Freudenthal, who believed that for learners to best succeed they should mathematize their own reality, and that this need not conform to the day-to-day approaches of the mathematical research community.

The three streams of thought would ebb and flow over six decades and at times blend. During the 1960s, versions of axiomatic Euclidean geometry prevailed, including treatments with New Math emphases on proof, and Dieudonné's request for affine geometry would appear in the SSMCIS materials. Set theory and mathematical reasoning problems would appear in the New Math materials. The research mathematics community would play a large role in textbook development, both in the New Math and in more traditional high school texts as well. During the 1970s through the mid-1980s, most geometry instruction would retain axiomatic approaches similar to the pre-1960s texts, and that is roughly aligned with Thom's views. The language of sets and the vocabulary surrounding earlier New Math emphases on structure (e.g., algebraic laws) would migrate into traditional texts, but not, as a subject of study, leaving Dieudonné's views from Royaumont in the dust. An emphasis on computational skills would prevail in the post-New Math years. During this period research in mathematics education would expand and investigate spatial thinking across developmental levels in detail mirroring the views reflected in Freudenthal's 1970 Carbondale talk, but with fewer research mathematicians involved.

Starting in the late 1980s through the 1990s, building upon the prior decades' research and a growing literature on problem solving, a new "reform" would emerge K–16, where the content underlying the prior decades' curriculum was "problematized." A US blend would evolve in geometry, where younger students would be expected to engage in spatial experiments, then a bit later use diagrams and spatial reasoning to explain new results given known results, and then finally to develop proof in a variety of ways (ranging from little to fairly formal two-column proofs). Like the prior period, direct emphases on set theory, logic, and mathematical structure were largely absent although the language would be retained and an emphasis on mathematical reasoning would appear in the NCTM and state frameworks. The controversy and responses to these changes would re-engage the mathematical research community as the USA transitioned to the standards-based era of the late 1990s.

"Standards" would then be critical to the curriculum over the next two decades. During the 2000s, there were 50 state standards each with different notions of "balance" that the parties involved could squabble about. Then, in a dramatic shift leaving behind two centuries of local control, during 2010–2020 the nation's education would become aligned with the Common Core State Standards, the closest the USA has ever had to a national curriculum.

Although we have parsed out the transitions since the 1990s, it is not inappropriate to consider the residue of Royaumont and the three streams that emerged. Approaches using transformational geometry across K–12 became firmly embedded, a clear and lasting Royaumont impact. Beyond this, the US geometry blend would leave none of the post-Royaumont viewpoints as winners. The NCTM's (2000) *Standards* would outline a broad swath of topics and approaches in geometry as a "college preparatory" base curriculum for all students. In secondary geometry, the CCSSM, would, roughly speaking, follow this blend and include some proof to extend transformation and coordinate approaches. But the CCSSM's stark omission of spatial thinking in K–8, which consists mostly of rote tasks and vocabulary, would certainly have left Freudenthal heartbroken.

The NCTM *Standards* (2000) would emphasize reasoning and proof, but not formal mathematical structure and language (excepting some vocabulary), and this would not change during the standards eras. The CCSSM would mention proof in high school geometry, but its eight standards for mathematical practice that should guide instruction and curriculum development, including those labeled "Reason, abstractly and quantitatively," and "Attend to precision," one will not find the word "proof." Both Dieudonné and Thom, whose views of proper collegiate preparation were so different, would have been equally dismayed. This was not the case a decade earlier. We would by no means claim that mathematicians' notions of rigor and proof are not embedded in some CCSSM-aligned materials since many research mathematicians remain deeply engaged. So, although the Royaumont meeting stimulated serious introspection and changes in the USA over 60 years, in terms of the tenor set by the USA's movement toward national standards, outside of a handful of topics, with the CCSSM the last vestiges of Royaumont thinking seem to have been obliterated.

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