Chapter 14
The New Math in Hungary: Tamás Varga's Complex Mathematics Education Reform

Katalin Gosztonyi

Abstract In Hungary, a reform movement called Complex Mathematics Education was led by Tamás Varga from 1963 to 1978. The Complex Mathematics Education reform is clearly inscribed in the context of the international reform movements of the New Math period, but it also bears specificities which can be traced back to a local, Hungarian tradition of mathematics and mathematics education: A heuristic epistemology of mathematics, and an approach to the teaching of mathematics which can be described with the terms "guided discovery." With his work on this reform curriculum and the related experimentations, Tamás Varga himself contributed also to the development of international research on mathematics education, especially in the domains of logic, combinatorics, and probability. In this chapter, we present a brief chronology of the reform; an analysis of its historical context and the different influences which shaped it; the characteristics of the curriculum; and the related resources and expected teaching practices. We illustrate with some examples how the reform combined lessons from the New Math movement with local, Hungarian influence.

Keywords Combinatorics · Complex Mathematics Education · Curriculum reform · Guided Discovery · Heuristic epistemology · Hungary · Logic · Mathematical discovery · New Math · Probability · Problem-solving · Resources · Series of problems · Teaching practices · Tamás Varga

Introduction[1]

In Hungary, a reform movement called Complex Mathematics Education was led by Tamás Varga from 1963 to 1978. In 1978, the reform was implemented in the frame of a general reform of national curricula. Varga's reform curriculum covered the compulsory education of the period, from grade 1 to grade 8 (6–14-year-old students). The Complex Mathematics Education reform is clearly inscribed in

[1] This chapter is based on my PhD thesis (Gosztonyi 2015) and on earlier publications, mainly (Gosztonyi 2016, 2020). Some paragraphs are taken from (Gosztonyi 2020).

K. Gosztonyi (✉)
Eötvös Loránd University of Budapest, Budapest, Hungary

Alfréd Rényi Institute of Mathematics, Budapest, Hungary

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the context of the international reform movements of the New Math[2] period: Varga referred to international experiences and explained how they inspired his work. The Hungarian reform shows many characteristics which were widespread in the period's other reforms. However, the Complex Mathematics Education reform also bears specificities which can be traced back to a local, Hungarian tradition of mathematics and mathematics education: A heuristic epistemology of mathematics, and an approach to the teaching of mathematics which can be described with the terms "guided discovery." With his work on this reform curriculum and the related experimentations, Tamás Varga himself contributed also to the development of international research on mathematics education, especially in the domains of logic, combinatorics, and probability.

In this chapter, we present a brief chronology of the reform; an analysis of its historical context and the different influences which shaped it; the characteristics of the curriculum; and the related resources and expected teaching practices. We illustrate with some examples how the reform combined lessons from the New Math movement with local, Hungarian influence.

A Brief Chronology of the Reform

The experimental project named "Complex Mathematics Education" was started by Tamás Varga in 1963, with two first-grade classes in a primary school in Budapest. The project was progressively extended to further educational levels and other schools in different parts of the country, and attained, in the early 1970s, about 100 classes from grade 1 to 8 (covering all levels of compulsory education) all over the country.

The project was directed by the National Pedagogical Institute (*Országos Pedagógiai Intézet*, OPI), in close collaboration with a group hosted in the Mathematics Research Institute, responsible for the development of the curricula for high school classes specialized in mathematics (under the direction of János Surányi). The closest associates of Varga in this project were Endréné Gádor, Sándor Pálfy, Istvánné Halmos, Eszter C. Neményi, and Julianna Szendrei (Szendrei 2007).

Varga's project was not the only experimentation concerning mathematics education in Hungary in this period. At the beginning of the 1970s, the project was chosen among several other projects by a ministerial committee, led by János Szendrei, as the basis of the new curriculum. A provisional and optional version of the curriculum was introduced in 1974, and the obligatory version in 1978, in the framework of a general curricular reform.

It is important to underline that Varga's team would have preferred a slower introduction of the reform, on a voluntary basis, in order to prepare more conveniently the participating teachers—but this was not supported by politics and was contradictory to the centralized direction of the educational system. The progressive introduction during a period of 4 years can thus be understood as a compromise.

Despite the important efforts of Varga's team concerning the production of resources (textbooks, teacher's guides, periodicals, etc.) and teacher education sessions (Varga 1975) the reform met significant resistance and debates. The curriculum and the related resources were modified in 1985. But despite the partial failure and the modifications, a significant continuity with Varga's curriculum can be observed in the following decades (Pálfalvi 2000).

[2] In Hungary, the international movement is mainly known by the name of "New Math." Therefore, I keep this expression as reference to the international movement, although other expressions (as "modern mathematics") are widespread, especially in Europe.

Tamás Varga, the Leader of the Reform

Tamás Varga (Kunszentmiklós, November 3, 1919–Budapest, November 1, 1987), the leader of the reform, was born as the second of seven children to a Calvinist priest, Tamás Vargha. The family provided a dynamic intellectual milieu: Several of his siblings become also well-known intellectuals, namely the novel-writer and journalist Domokos Varga and Balázs Vargha, a writer, historian of literature and anthropologist of games.[3] They all recognize the influence of their parents on their later work, for example, the logical and linguistic games played with their fathers (Balogh 2014; Dancs 2016). The three brothers were also in good contact with the Calvinist educator, linguist, and writer Sándor Karácsony—we will come back to his influence on Varga's work later.

Tamás Varga married the psychologist Ágnes Binét, who was a student of Jean Piaget, and a close collaborator of Ferenc Mérei, a particularly influential Hungarian psychologist in the second half of the twentieth century, who specialized in child psychology.[4] These relationships contributed to the psychological background of Varga's work.

Varga, after his graduation from the University of Sciences of Budapest (the later Eötvös Loránd University) as a mathematics and physics teacher, was awarded a scholarship to spend 18 months at *Scuola Normale Superiore* in Pisa, Italy (Szendrei 2007). He started to teach mathematics at the secondary school of Kunszentmiklós in 1945, but very early, in 1947 he was invited to the Educational Ministry, and later to the Pedagogical Institute to work on new curricula and textbooks. From 1951, he worked in mathematics teacher education at the Eötvös Loránd University. From 1955 on, he also regularly taught mathematics in high school for one class at a time (Pálfalvi 2019).

The most important stimulation to start experimentations aiming for the renewal of mathematics curricula came from a series of lectures given by Zoltán Pál Dienes in Budapest in 1960, and a UNESCO symposium on mathematics education organized in Budapest in 1962 (Hungarian National Commission for UNESCO 1963; Pálfalvi 2019). Varga was charged, with Willy Servais, to edit a book based on this conference (Servais and Varga 1971).[5] After a short experimentation in 1961, Varga started the "Complex Mathematics Education" project in 1963 (see below).

In 1967, Varga was moved to the National Pedagogical Institute where he led the team responsible for the development of the reform project. Varga defended his doctoral thesis based on the reform project in 1975 (Figure 14.1), and he also obtained several national prizes in recognition of his work.

From the second half of the 1960s, Varga developed an international reputation within the emerging mathematics education research community. He published and was regularly invited both to the countries of the "Eastern" and the "Western" Blocs, including Poland, the USSR, France, Italy, Canada, and the USA among others. He was particularly recognized for his work in teaching logic, combinatorics, and probability.[6] He also became responsible for various international organizations: He was, for example, a member of the editorial board of *Educational Studies in Mathematics* and Vice-Chair of the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM)/International Commission for the Study and Improvement of Mathematics Teaching (Szendrei 2007).

[3] Tamás and Domokos abandoned the "h" from the spelling of their names.

[4] See, for example, their common work (Mérei and Binét 1970).

[5] About the relationship of Varga and Servais, see De Bock (2020).

[6] See, for example his publication on the teaching of these domains in *Educational Studies in Mathematics* (Varga 1972).

Figure 14.1 Varga (right) with János Surányi (left) and Ervin Fried (center) at Varga's thesis defense in 1975. (Courtesy of Mária Halmos; published with the authorization of Varga's family)

The Political and Institutional Context

In the period in question, Hungary was a socialist country, under the influence of the USSR. However, the reform started after an important political turn. In the 1950s, the hardest period of the dictatorship led to the revolution in 1956 and the following retribution, but a consolidation began in 1962. The 1960s and 1970s were the periods of softer authoritarianism with restricted oppression, some liberalization of the communist system and some opening toward the Western world (Romsics 1999). Although we do not have proof, it is likely that the possibility to organize an international UNESCO conference in Hungary in 1962 and to start experiments inspired by this conference are related to this political turn.

The frames of the educational system in which this reform arrived were established after 1946. Compulsory education was provided by the 8-grade single-structure "basic schools," comprising elementary (grades 1–4) and lower secondary (grades 5–8) education. Upper secondary education was provided by general and vocational secondary schools. During the 1950s and 1960s, regulation of the educational system was extremely centralized, with detailed curricular instructions. Soviet influence and the communist ideology were quite apparent in instructions as well as in teaching materials in this period. From the late 1960s however, a slow liberalization occurred (Báthory 2001): The influence of ideology was pushed into the background, pedagogical and psychological considerations were taken into account, and differentiation, as well as teachers' autonomy and liberty, were emphasized. This turn plays a crucial role in the preparation of the 1978 reform, and—as we will see below—Varga's project can be considered a pioneer in this sense.

Thus, contrarily to some other reforms of the New Math era which fitted in the frame of a unification process of the educational system, like the creation of the "*collège unique*" in France (d'Enfert and Kahn 2011), the Hungarian movement emerged from a unified, centralized system and fitted into a liberalization process. The impact of this context is perceptible on many aspects of Varga's reform, even in less evident characteristics such as the role of mathematical language: While the abovementioned French "mathématiques modernes" reform insisted on the unifying role of formal mathematical language, Varga's complex mathematics education reform emphasized the importance of working with the diversity of students' personal expressions.

The Hungarian Reform in the Context of the International New Math Movement

The international New Math movement is often considered as being developed in the context of the Cold War's scientific and technological competition. Thus, it would be an obvious hypothesis that the New Math was a Western movement, without relevant contributions from the "Eastern Bloc" or with two parallel movements in the two "blocs." However, the Hungarian reform is a good example illustrating that this was not the case. Varga always declared being influenced by the New Math; from the 1960s, he actively participated in the work of different international organizations of the movement like the UNESCO and the CIEAEM,[7] he was invited to and published in the US, Canada, France, and Italy, among others. In his doctoral thesis (Varga 1975) as well as in his colleagues' memories according to interview data, Eastern influence was much less important in his work—although he also published in several Eastern Bloc countries; his only important partner from these countries was Anna Krygovska, the leader of the Polish reform—also an active and recognized contributor to the international New Math movement (Figure 14.2).

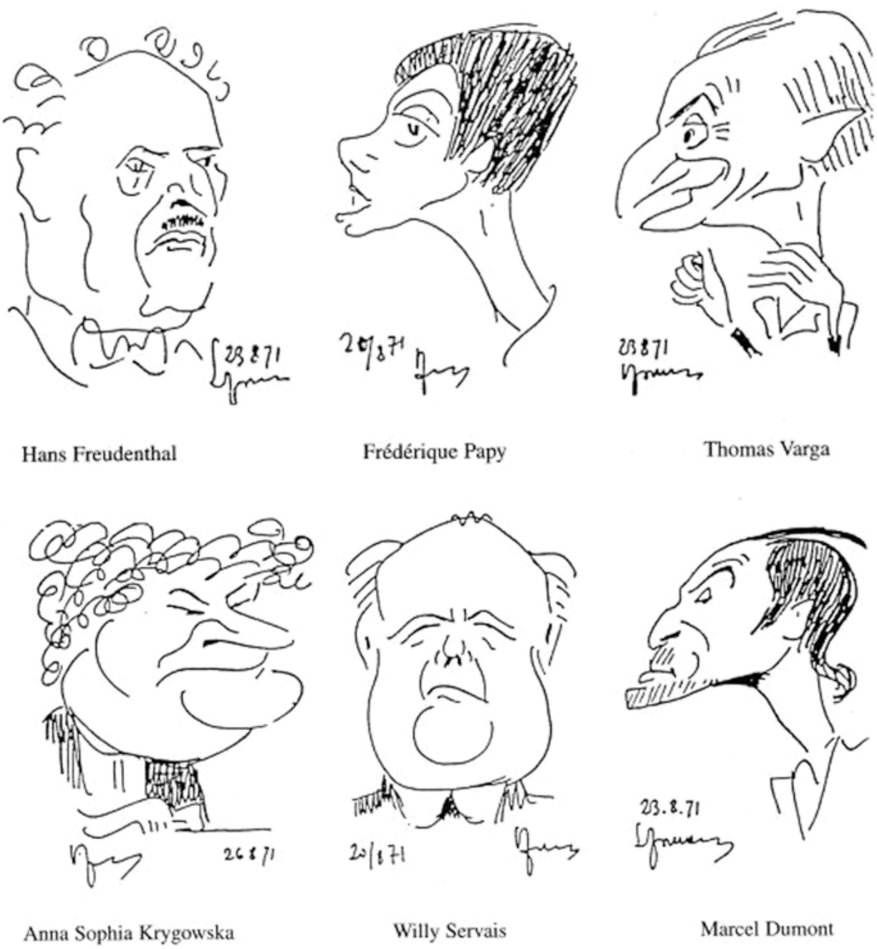


Figure 14.2 Varga among other eminent actors of the New Math era. (Caricatures by Leon Jeśmanowicz, CIEAEM meeting, Poland 1971)

[7] See, for example, the hommage to Varga on the site of the CIEAEM: http://www.cieaem.org/?q=system/files/varga.pdf

The impact of the New Math movement can be observed in the Hungarian reform: The introduction of a coherent subject named "mathematics" instead of "arithmetic and measurement"; new mathematical domains introduced in early ages like sets or logic; the reference to Piaget's psychology and Dienes's mathematical games; the important role of manipulative tools, etc. However, Varga was critical of some aspects of the New Math reforms, especially the excessive emphasis on mathematical formalism—as we will see below.

Pedagogical and Psychological Background: A Complex Situation

The pedagogical and psychological background to the reform was quite complicated. Jean Piaget's influence was obviously important, but there were other significant influences on Varga's thinking. His wife, Ágnes Binét was a psychologist and worked with the most influential Hungarian psychologist of the period, Ferenc Mérei. Varga refers to some Soviet pedagogues too, but only a few times in his politically relevant writings: So, it is difficult to know if these are real or only politically motivated references. However, Lev Vygotsky is almost missing from his references, although Varga's conception shows some similarities with Vygotsky's socio-constructivist ideas. The socio-constructivist approach, as well as the importance of visual intuition in the learning process, was also inspired by the views of Sándor Karácsony, who was a Calvinist educator and philosopher, who had contact with most of the mathematicians mentioned in the section below (Máté 2006). According to Varga's colleagues and family, Karácsony had a great influence on him—but he was not referred to in Varga's writings, again because of political reasons. In summary, pedagogical and psychological influences seem to be quite complex and their more detailed identification would need further research.

A "Heuristic" Epistemology of Mathematics

In the reform movements of the New Math period, questions of mathematical epistemology come to the fore. Renewing mathematics education meant not only the introduction of new, "modern" topics into the curriculum, but also a "modern" view of mathematics, and the intention to present it as a coherent domain. But, while internationally the Bourbaki group had a decisive influence on the mathematical epistemology of the period's reforms, emphasizing mathematical structures, modern formal language, and a coherent axiomatic construction of mathematics based on set theory, one of the most important specificities of Varga's reform was the impact of a "heuristic" epistemology of mathematics, developed in the Hungarian mathematical community of the twentieth century (Gosztonyi 2016).

This tradition was developed partly in the teaching of young mathematical talents[8] and went back at least to the beginning of the twentieth century. Varga himself had intensive personal contact with several mathematicians representative of this tradition (László Kalmár, Rózsa Péter, Alfréd Rényi, and János Surányi, among others) since the 1940s; and they all supported, more or less actively, Varga's later reform movement. These mathematicians, together with well-known thinkers like George Pólya and Imre Lakatos, represent a quite coherent epistemology of mathematics, which is closely related to questions of mathematics education and published mostly in texts popularizing mathematics and in lectures about mathematics education.

This "heuristic" approach presents mathematics as a constantly developing creation of the human mind; this development is guided by a series of problems. According to them, the source of mathematics is intuition and experience; mathematical activity is basically dialogical, and teaching mathemat-

[8] Nowadays its most important representative is Lajos Pósa. See http://agondolkodasorome.hu/en/

ics can be seen as a joint activity of the students and of the teacher, where the teacher acts as an aid in students' rediscovery of mathematics. Teachers discourage excessive formalism, seeing formal language also as a result of development. They describe mathematics as a creative activity close to playing and to the arts (Gosztonyi 2016).

The "Complexity" of Varga's Reform

As we can see from the paragraphs above, the influences shaping the Hungarian reform were multiple and quite complex. Its name, "Complex Mathematics Education" characterizes several other aspects of the reform: First, the complexity of its ambitions, transforming the content and the structure of the curriculum, the teaching practices, the teaching materials, the resources for students and teachers, and also teacher education. Second, the complexity of the mathematical content, covering a diversity of domains and rich intersections between these domains. And finally, its pedagogical complexity—the reform emphasized the importance of rich teaching practices, using a variety of tools and teaching methods.

Varga's Curriculum

As we mentioned above, like other reformers of the New Math period, Varga aimed to integrate new topics in mathematics education and to present mathematics as a coherent science, with the curriculum organized in accordance with modern mathematics. But Varga interpreted "new topics" as well as the internal "coherence" of the curriculum in his own way, integrating considerations of the reforms in other countries with the traditions of Hungarian mathematics.

The introduction of new topics involved basing concepts on sets and relations, with a strengthened role of algebra—as in a number of reforms of the same period in other nations. But, for Varga, it also meant introducing logic, combinatorics, probability, or algorithmic thinking into primary and lower secondary school education, which were the research domains of the abovementioned Hungarian mathematicians supporting the reform.

Instead of a strict axiomatic hierarchy of mathematical domains, the internal coherence of the curriculum was to be ensured by the parallel, spiral presentation of five big domains, all being present throughout the whole curriculum, with frequent and various internal connections being made between them: (a) sets and logic; (b) arithmetic and algebra; (c) relations, functions, and series; (d) geometry and measure; and (e) combinatorics, probability, and statistics.

Another significant characteristic of Varga's curriculum was its flexible structure: "Suggested" and "compulsory" topics were distinguished from "requirements." As he explained: "Many concepts and skills not appearing as requirements in the school year where they are first mentioned in the syllabus, get enrolled to them in subsequent years when they are supposed to become ripened" (Halmos and Varga 1978, p. 231).

This organization offered important liberty to teachers, allowed for differentiation among students, provided a rich and varied experimental basis for the progressive generalization and abstraction of mathematical notions, and supported a learning process based on mathematical discovery by which elements of mathematical knowledge could emerge as tools during problem-solving situations.

Textbooks, Teachers' Guides, and Expected Teaching Practices

Beyond the reform of the curriculum, Varga's team sought to implement important changes in teaching practices, especially those giving place to students' problem-solving activities, mathematical games, and the use of various manipulative tools—Dienes' blocks or Cuisenaire rods for example. Varga was aware that the reform implied important changes from teachers, concerning their mathematical knowledge, their teaching practices and their beliefs both concerning mathematics and student learning. Therefore, they had to be prepared accordingly (Varga 1975). While in the first years of the experimentations, the teachers' preparation was assured by regular personal contact with the reform leaders, later, in the period of large-scale introduction that was no longer possible. Teacher-education sessions were organized and teacher-education videos were prepared, but the main source of information for teachers were written resources.

In this period in Hungary, only one collection of textbooks and teacher's handbooks was available, prepared by the same persons who had been responsible for preparing the curriculum. For the primary schools, like the case in other countries in the New Math period, worksheets were available, but these were meant to be used only as partial resources for various activities. Official teacher's guides served as the main resources for teachers. For middle school, textbooks and worksheets were provided, with (much less detailed) teacher's guides. The analysis of these resources, their explicit content, and their organization contributes also to an understanding of Varga's conceptions of desirable teaching practices (Gosztonyi 2015).

The primary-school teacher's guides followed a special structure: Their main part contained quasi-continuous text mixing examples of tasks with mathematical, didactical and pedagogical commentaries. They were organized in thematic chapters, following the abovementioned five "big" domains of the curriculum. The tasks and small problems were described with several possible variations, and suggestions for inventing new tasks which could generate shorter or longer series of tasks and activities (Gosztonyi 2017). The guides also gave ideas for their realization: They often described possible student reactions (based on the experimentations) and advised teachers on how to deal with them.

After this main part, the books presented a *possible* syllabus for the year, emphasizing that it was only an example, and encouraging teachers to elaborate their own teaching progression for the year. In fact, the task of following the offered syllabus was not easy because the main domains were treated in parallel, most of the lessons often contained activities from several domains which were located in different thematic chapters of the book. And since the thematic chapters were not very structured and contained many internal and implicit references, teachers had to know them quite well in order to make efficient use of them.

Concerning middle school textbooks, one unusual characteristic was the way they introduced new knowledge: They presented fictional dialogues of students discovering new knowledge while they discussed a small series of mathematical problems. The teacher's guide encouraged teachers to provoke similar discussions in classrooms.

Many of the tasks proposed in these resources could be described as "rich tasks" in several senses. Firstly, and similar to other emerging didactical theories of the period,[9] they described problem situations allowing students to make sense of mathematical notions; required an active contribution to the construction of mathematical knowledge. These tasks were also rich in the sense that many of them connected several mathematical domains. Finally, they are rich by being open to multiple solution strategies.

The richness manifested itself in the collections of tasks too: The diversity of the problems' contexts, the diversity of manipulative tools, from ordinary objects of the classroom to specially designed tools like the Cuisenaire rods, the logicality of Dienes's tasks; and the diversity of representational

[9] Like that of Guy Brousseau and Hans Freudenthal.

tools. One of the aims of this diversity was the support of a progressive abstraction process based on a variety of experiences, and the other was to present mathematics as a playful and pleasant activity.

These elements not only contributed to the didactical richness of the reform but also to the complexity of the work expected from teachers: Important mathematical and pedagogical competencies and a high level of autonomy were necessary to implement the reform's conception successfully.

An Example: Probability in Varga's Curriculum

We illustrate the principles and characteristics described above with an example from the teaching of a specific domain: The teaching of probability.

The Reasons to Teach Probability

Varga was particularly recognized by the emerging international mathematics education community for his work in the teaching of logic, combinatorics, and probability. He introduced these domains into the reform curriculum systematically, from the first grade on. Several of his papers discussed the teaching of these domains, especially concerning lower grades (Varga 1967, 1970, 1972, 1982). Here we summarize his views on the introduction of probability at the primary and middle school level.

Probability is part of the 5th group of domains of the reform curriculum, "combinatorics, statistic, and probability." These domains are taught in close connection to each other, but their connections to other domains are also emphasized, contributing to the diversity of domains and the dialectic network between them, one of the important principles of Varga's curriculum. Beyond combinatorics and statistics, probability is connected for example to logic (notions of certain, possible, impossible), and offers a domain of application for fractions, decimal numbers, and percentages, helping to make sense of rational numbers and their different forms of representation (Varga 1972, 1982).

Beyond this aspect, Varga underlined several arguments for the teaching and the early introduction of these domains, which are in close connection with the underlying epistemological principles discussed above (Varga 1967, 1982). These domains can be introduced without important theoretical background knowledge and starting with games and manipulative experiences, students can be familiarized very early with processes of mathematical abstraction through the development of these domains. They allow the introduction of a diversity of representations, such as tables and tree diagrams. They offer opportunities to practice estimation, something which Varga considered to be important in the learning of mathematics.

> My own view is that estimating, guessing, predicting, mentally representing the future and expressing our opinion about it is a human ability which should play a greater role in education than it does now. All these activities […] get kids personally involved in learning. (Varga 1982, p. 30)

Estimation had a general educational value according to Varga. He discussed potential counter-arguments in this sense:

> Reasons must be strong, maybe not unrelated to sentences from "a child should not have an opinion" to "a child should have not will." If this is a correct conjecture, then the issue is a more general one about education, not necessarily school education or school math in particular. (Varga 1982, p. 30)

Finally, probability was also an important domain, according to Varga, to represent another, less-known nature of mathematics, the mathematics of the *uncertain*.

> Perhaps, the main reason for an early introduction of this branch of mathematics is that it is essentially different from other parts of mathematics. […]

> Mathematics […] does not only concern itself with certainties which can be expressed by exact statements but also with uncertain things of a chance character. This sounds strange, because it might be thought that if something is uncertain we can only state uncertain things about it. On the one hand we have everyday life, full of chance events, full of "perhaps" and "probable." On the other side there is mathematics with its perfect circles and triangles, with its exact statements which unfortunately are not a great deal of use to describe the real world. How could one somehow mix up these two sides? (Varga 1967, p. 2)
>
> […] The words "probably," or "about" or "estimate" which we have used here hardly express for us much more than uncertainty. This is what separates our method of reasoning from the more usual forms of mathematics. The strength of the calculus of probability lies in the fact that we can give such words an exact meaning of a quantitative character. (Varga 1967, p. 4)

These considerations illustrate well the impact of Hungarian mathematicians on Varga's reform (Gosztonyi 2015, 2016). Research on probability was developed in Hungary by Alfréd Rényi, one of the mathematicians supporting the reform. The arguments above make a strong echo with the ones Rényi formulated in his book, *Letters on Probability,* popularizing probability (Rényi 1972). This impact is also apparent in the way Varga constructed the probability curricula that we will present below. But echo can be found with Pólya's (1954) arguments on the importance of plausible reasoning or with Kalmár's (1967) and Lakatos' (1976) conceptions of the "quasi-empirical" nature of mathematics.

The Probability Curriculum

In a manuscript of a paper from the early 1980s[10] Varga (1980) explained the logic of the construction of the probability curriculum. He insisted on the importance of an early introduction to the domain, saying that it is easier to form the thinking of students at a young age. He referred on this subject to the work of Jean Piaget and Bärbel Inhelder, *La genèse de l'idée de hazard chez l'enfant* (Piaget and Inhelder 1951). The curriculum was constructed by the following steps:

1. Activities to distinguish certain, possible, and impossible events
2. Activities to construct the notion of more and less probable
3. Notion of equiprobable events, based on symmetry arguments: Comparison of events on the bases of the number of equiprobable elementary events
4. Recognizing analogies and equivalences between experiments and events
5. Establishing relationships between relative frequency and probability
6. Calculating probability on a universe of events
7. Conditional probability

The first four points cover basically the curriculum of the first three grades and build only on natural numbers. From the fifth point, fractions and decimal numbers come into play: Probability appears as a domain of application of fractions and decimal numbers and contributes to give sense to the notion of fractions. The relation between probability and arithmetic is more of a dialectic than hierarchic nature.

This summary shows a slow, gradual progression of probabilistic notions. The first grades do not actually deal with probability calculus: The aim here is to familiarize students with a probabilistic way of thinking. Mathematical abstraction is developed progressively, based on experiences.

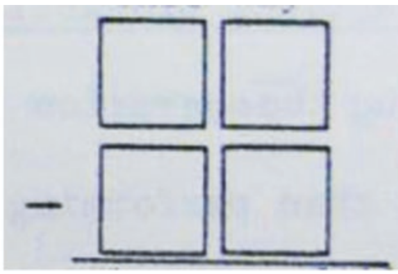
Analysis of specific tasks from Varga's papers and from textbooks (Gosztonyi 2015) shows that this progression builds on a dialectic of two different approaches to probability: Frequentist and classical, Laplacian approaches. Experimentations with various probabilistic games, the observation and

[10] We only dispose the manuscript without a precise date. It was certainly written some years after the official introduction of the new curriculum, as it refers to related experiences.

analysis of frequencies help to gain experience and develop students' understanding of probability. Nevertheless, frequentist techniques of probability calculus are not developed in detail and serve for estimation but not for justification and proof.[11] Estimations and hypotheses are justified following a classical approach to probability, based on theoretical models and symmetry arguments, gaining progressively more and more place at higher grades, and in close connection with the learning of combinatorics.

Examples of Activities

1. **Subtraction by Rolling a Dice** The following example, "Subtraction by rolling a dice," represents several important characteristics of Varga's approach, and gives also a good example of the apparition of probabilistic thinking in the early years of the curriculum (Figure 14.3).



The goal is to make the difference as great as possible. They can fill the boxes in any order, but only with random numbers produced by rolling dice. After a number is given by the dice, children write it into one of the boxes which is still empty and cannot change it. After filling the third box they have no choice for the fourth random number. Those with the largest difference get three points, those with the second largest get two, those with the next get one and the others none. Then they start again.

Figure 14.3 New topics for the elementary school math curriculum. (Varga 1982, p. 28)

This game was one of the activities allowing students to discover the notion of probability, while they exercised themselves in making subtractions and they acquired a deeper understanding of the operation's properties. The competition, while a pleasant activity for pupils, motivated them to develop strategies and offered an experimental basis for comparing different strategies.

Students, combining experiments and logical argumentation, would have the opportunity to observe that there are more and less efficient strategies, but none of them would provide a certainty for winning. They would however not make any formal calculation of probabilities at this stage (this corresponds to phase 2 of the probability curriculum). Varga mentioned that a more formal study of these strategies was possible at higher levels:

[11] Contrarily to the contemporary experimentation of Brousseau, realized at the early 1970s (see, e.g., Brousseau et al. 2001).

Their intuitions lead them to different strategies which are then tested at further games. At somewhat higher level, still in the elementary school, the strategies can be formulated more precisely, incorporated into computer programs, and tested more precisely in a great number of trials. At a much higher level calculating the expected values related to specific strategies and finding the best strategy may be the subject of a research for undergraduates. (Varga 1982, p. 29)

2. **A Game with Three Disks** In another paper, Varga presented a somewhat similar problem situation with a more detailed discussion of the process of exploring different strategies. The situation was the following:

Three apparently identical disks are put into a box. However, one of them has a cross on each side, one is blank on both sides, and one has a cross on one side and nothing on the other.

I draw a disk at random and show you one of its faces at random. This latter is as essential as the first. I would fake the whole situation by looking at the disk after drawing it and deciding myself which face to show you, if the faces are different. Try to guess the other face. We repeat this experiment several times. You can never be sure. Still say each time what you think most likely. (Varga 1970, p. 424)

Face shown	X	O	O	X	X	X	O	O	X	X	Strategy used :
Guess for the other face	X	O	X	O	X	O	X	O	X	O	X and O
Other face	X	X	O	X	X	O	X	O	X	X	alternately
Hits	✓			✓	✓	✓	✓	✓			

Table 1 : Recording a set of ten games (0 = blank face)

A	X, 0, X, 0 ...										
B	0, X, 0, X ...										
C	X, X, 0,0, X,X, 0,0, etc.										
D	X always										
E	0 always										
F	Random										
G	Other face last time										
H	Opposite of that										
I	Face shown last time										
J	Opposite of that										
K	Face shown this time										
L	Opposite of that										
M	X until I win, change when I lose, keep when I win ...										

Table 2 : Recording the number of hits in consecutive sets for various strategies.

Figure 14.4 Two of the three tables associated to the game with three disks. (Varga 1970, p. 426)

The exploration of the situation happens in several phases, a table is associated with each of them (see Figure 14.4). In the first phase (Table 1), students play 10 games and note their results. This is again a competition: They look for the students with the most hits. Then the class discusses the different strategies. They compare their efficiency, each student taking charge of one of the strategies, and note down the number of hits obtained with it after each set of ten games (see Table 2).

Supported by their observations, students try to explain why one strategy is more efficient than another:

> Which strategy has proved to be the most successful in the long run? Who would explain why?
>
> That's right. If I mix the disks thoroughly, then about twice out of three occasions I draw a disk which is the same on the other face, and only once the one which is different on the other side. That is the reason why K turns out to be so good a strategy and L so poor. None of the other strategies listed above can raise the chances of winning above ½ or sink it below ½. This is true even if you have had sometimes spectacular successes with them. These are as little excluded as failures with K. But both are relatively rare. (Varga 1970, pp. 424–425)

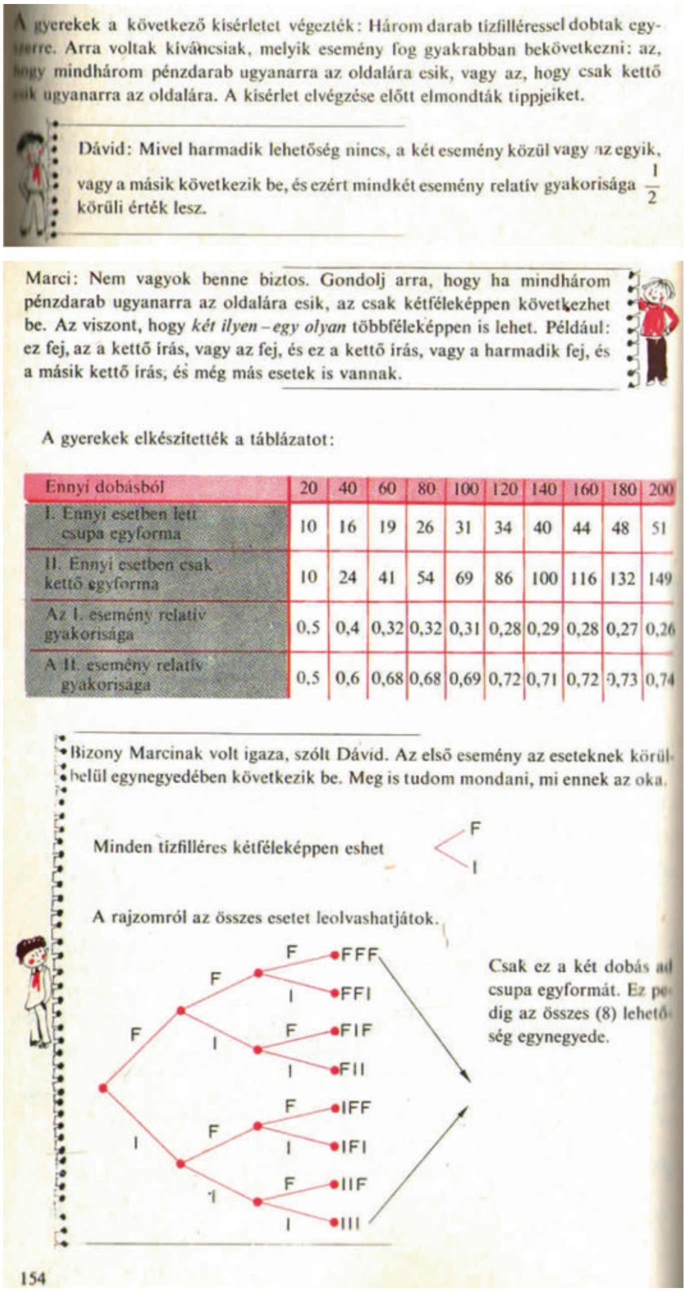
We can see here how Varga connected the frequentist and the classical approach to probability. Experiments served as a basis for concrete experiences on the problem before looking for logical argumentations based on classical probabilistic models.

It is also interesting to observe Varga's writing style here, representing a dialogic practice of teaching, characteristic of the *guided discovery* approach: We see the teacher's voice in a dialogue with students while he/she asks questions and responds to students reformulating the students' words.

3. **Tossing Three Coins—An Example From Varga's Textbooks** A similar process can be seen in the fifth-grade textbook developed for Varga's reform (Figure 14.5). The students toss three coins at the same time and want to observe which of the following events will happen more often: The three coins fall on the same side or one of them falls on a different side than the other two. First, they try to guess and argue about their guesses, but they do not reach an agreement. Then they do experiments and represent frequencies and relative frequencies in a table. This experimentation helps to confirm which guess was right, and based on this experience, students come up with a more grounded (classical probabilistic) argument. We can observe that a graphical representation supported the proof offered by one of the students.

Conclusions on the Case of Teaching Probability

The example from the probability curriculum illustrates many characteristics of Varga's approach. We see the introduction of a new, modern mathematical domain (although not the most typical one of the New Math era) from early grades on. It was introduced in a progressive, spiral way, starting with concrete, manipulative activities, which encouraged the active participation of students in the construction of mathematical knowledge; and abstraction was progressively developed based on these experiences. We can see how students' natural curiosity was emphasized through playful activities related to the learning of probability. We can observe the important role accorded to various representations supporting the abstraction process. We can also see that Varga's activities connected probability to various other mathematical domains, including the learning of basic operations, fractions, decimal numbers, and the domains of logic and combinatorics.



A gyerekek a következő kisérletet végezték: Három darab tízfilléressel dobtak egyszerre. Arra voltak kiváncsiak, melyik esemény fog gyakrabban bekövetkezni: az, hogy mindhárom pénzdarab ugyanarra az oldalára esik, vagy az, hogy csak kettő esik ugyanarra az oldalára. A kisérlet elvégzése előtt elmondták tippjeiket.

Dávid: Mivel harmadik lehetőség nincs, a két esemény közül vagy az egyik, vagy a másik következik be, és ezért mindkét esemény relatív gyakorisága $\frac{1}{2}$ körüli érték lesz.

Marci: Nem vagyok benne biztos. Gondolj arra, hogy ha mindhárom pénzdarab ugyanarra az oldalára esik, az csak kétféleképpen következhet be. Az viszont, hogy *két ilyen – egy olyan* többféleképpen is lehet. Például: ez fej, az a kettő írás, vagy az fej, és ez a kettő írás, vagy a harmadik fej, és a másik kettő írás, és még más esetek is vannak.

A gyerekek elkészítették a táblázatot:

Ennyi dobásból	20	40	60	80	100	120	140	160	180	200
I. Ennyi esetben lett csupa egyforma	10	16	19	26	31	34	40	44	48	51
II. Ennyi esetben csak kettő egyforma	10	24	41	54	69	86	100	116	132	149
Az I. esemény relatív gyakorisága	0,5	0,4	0,32	0,32	0,31	0,28	0,29	0,28	0,27	0,26
A II. esemény relatív gyakorisága	0,5	0,6	0,68	0,68	0,69	0,72	0,71	0,72	0,73	0,74

Bizony Marcinak volt igaza, szólt Dávid. Az első esemény az eseteknek körülbelül egynegyedében következik be. Meg is tudom mondani, mi ennek az oka.

Minden tízfilléres kétféleképpen eshet

A rajzomról az összes esetet leolvashatjátok.

Csak ez a két dobás ad csupa egyformát. Ez pedig az összes (8) lehetőség egynegyede.

154

Figure 14.5 A probability problem from the fifth-grade textbook. (Eglesz et al. 1979, pp. 153–154)

The Impact and the Reception of the Reform

Varga's experimental project was accompanied by a multidimensional psychological impact analysis which led to the conclusion that the experimentation had been successful in various senses (Klein 1980). The students acquired the same mathematical knowledge that they would have in ordinary classrooms, yet in this case, they became involved with new elements of the reform curriculum, while among other things, their capacities of learning, their creativity, and their attitude toward mathematics improved.

That was also the subjective opinion of Varga's colleagues.[12] According to them, in the first period of experimentations, when the new approach was disseminated on a voluntary basis and by personal contact with the leaders of the experimentation, the dissemination was generally successful, and both teachers and students were satisfied. Problems started to arise in the 1970s, after that the project became the official basis of the prospective new curriculum. At that time the number of participating classes grew exponentially, teachers were obliged to participate and personal contact between the teachers and the leaders of the innovation was no longer possible. These conditions generated failures and resistance.

During this later period, and similar to many other reforms of the New Math era, Varga's Complex Mathematics Education reform provoked vivid public debate, and finally an important correction in 1985. Varga's former colleagues interpreted this as a failure, and they considered the obligatory introduction of the reform as the main reason for its rejection. According to them, Varga's approach should have been disseminated progressively in a bottom-up process, as it happened during the (generally successful) experimentation period—but this kind of slow diffusion was not politically supported. While a narrow circle of teachers (mostly colleagues of Varga and their disciples) continued to follow the *guided discovery* approach with success, the majority of Hungarian teachers did not really adopt it or integrated only partial elements of it into their practice.

Despite the negative reactions, Varga's work remains influential in Hungarian mathematics education, even today: It serves as a crucial reference for contemporary mathematics curricula, teacher education, and for the development of certain resources.[13]

Varga's work also exerted influence internationally. Beyond the impact on the teaching of logic, probability and combinatorics, his reform project had a significant impact in the Italian context in the 1970s (Bartolini Bussi 2020), and a Varga-Neményi Association, adopting Varga's methods and resources to Finnish primary school education has existed in Finland since 2005.[14]

Conclusion and Discussion

The Complex Mathematics Education project is an example of a reform inscribed in the international reform movements of the New Math era, significantly inspired by other reforms, conceptions, and endeavors of the period, but it is also shaped by local traditions and approaches.

The different elements of the reform, the curriculum, the task design, the resources, and the indications about expected teaching practices were conceived according to a coherent conception, which we named the *guided discovery* approach. This conception is partly related to some international trends of the New Math period (as the introduction of "modern mathematics" into the curricula, students' active participation in the construction of mathematical knowledge, the use of manipulative tools, etc.). At the same time, it corresponds to the abovementioned "heuristic" epistemology of mathematics, represented by Hungarian mathematicians, with problem-solving and mathematical discovery becoming major foci in the implemented curricula. It was supported by the flexible structure of the curriculum, the parallel, dialectic presentation of different mathematical domains, the use of various material tools and representations, the construction of long-term teaching processes in the form of a series of problems, and a dialogic guiding of the class. These elements allowed students to advance at their own pace, to have enough time and occasion to gain various experiences and to follow a slow abstraction process through progressive generalization.

[12] Based on interviews with M. Halmos, Cs. Kovács, E. Csahóczi, E. Neményi, L. Pálmay, and E. Deák made in 2013.

[13] Several curent textbook series can be traced back to the textbooks of Varga's team, partly with the same authors.

[14] https://varganemenyi.fi/

Many of the characteristics of Varga's *guided discovery* approach connect to other didactical approaches emerging in the 1970s. For example, Varga himself underlined the proximity with Freudenthal's ideas, although they developed their conceptions independently from each other (Varga 1975). Varga's conceptions, however, show some specificity—for example, in his sophisticated construction of the curriculum or in the idea of constructing teaching trajectories in the form of a series of problems. In this sense, Varga's work—not very well known in the current international mathematics education research community—has the potential to contribute to enrich ongoing international discussions on teaching mathematics through problem-solving approaches and through mathematical inquiry.

An ongoing research project supported by the Hungarian Academy of Sciences aims to revisit Varga's reform, contribute to the theoretical description of his approach, make it more available on a national and an international level, and connect it to other current didactical approaches. We also aim to adapt Varga's oeuvre in current contexts, with all the challenges of modern education; to develop extensions of this work for kindergarten and for high-school students; and to contribute to its dissemination by developing new ways to support teachers' work.

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