Chapter 13
Reforms Inspired by *Mathématique Moderne* in Poland, 1967–1980

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Abstract Efforts to modernize Polish mathematics education began in the first half of the twentieth century. After 1957, the moving spirit of the Polish reforms was Zofia Krygowska; a description of her role in Poland and in international fora is augmented with explicit quotations from her books and articles. In 1967, under a strong influence of the French and Belgian versions of New Math, a radical reform of Polish secondary mathematics education was introduced, followed by an equally radical reform of primary education. Unfortunately, the implementation of the latter was combined with a fundamental change of the whole 12-year schooling system to an unclear 10-year system. In 1980, when the reform reached grade 4, the Solidarity movement forced the government to abandon it; yet, some changes were irreversible. In 2008, the last remains of New Math disappeared from the Polish core curriculum. In the concluding part of this chapter, the unique phenomena of New Math reforms and their ideology are discussed.

Keywords Algebraic errors · Axiomatic geometry · CIEAEM · Curricula · Curriculum reforms · Educational ideology · Geometry textbooks · Georges Papy · Hans Freudenthal · ICMI · Jean Piaget · Mathematical structures · Mathematics teachers · New Math · Polish education · Primary mathematics education · Sets · Stefan Straszewicz · Zofia Krygowska

Early Polish Efforts to Modernize Mathematics Education[1]

When the Fourth International Congress of Mathematicians (held in Rome in 1908) resolved to establish the International Commission on Mathematical Instruction (ICMI)[2] and Felix Klein became its president, the state of Poland did not exist. Its territory was divided among the empires of Germany, Russia and Austro-Hungary; Polish students attended three different school systems. Yet, when ICMI invited educators from each member state to prepare a report on the practice of mathematics teaching in their country, Polish mathematicians managed to prepare their own report, which was published in

[1] In this chapter, we make use of the material gathered in Semadeni (2020).

[2] Until the 1950s mainly indicated with the French acronym CIEM (*Commission Internationale de l'Enseignement Mathématique*) or the German IMUK (*Internationale Mathematische Unterrichtskommission*). This history is described in Howson (1984), where the case of Poland before 1914 is mentioned.

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Figure 13.1 Stefan Straszewicz

Figure 13.2 Zofia Krygowska

the journal *L'Enseignement Mathématique* in July 1911 (Cercle mathématico-physique de Varsovie 1911). Officially, it was a separate part of the Russian report.

In independent Poland (1918–1939), Witold Wilkosz and some other mathematicians were engaged in improving secondary education. With Klein's ideas in mind, they discussed the question of logical precision in textbooks, and tried to keep balance between abstract mathematical structures and their concrete examples. Otto Nikodym (1930) argued that modern mathematics should be seen as a renaissance of Greek thought, incorporating an axiomatic approach and reasoning based on logic, while elementary mathematics education, seen from a modern viewpoint, was replete with errors. A report on the university preparation of Polish prospective mathematics teachers was submitted to ICMI at a meeting in Zurich in 1932. Among authors of textbooks for 11- to 17-year-olds there were top mathematicians: Wacław Sierpiński and Stefan Banach. Their textbooks were well written, in traditional style, with moderate scope, avoiding novelties.

After the 1939 German invasion, secondary and tertiary education systems were not allowed in Poland; only a part of the elementary and vocational school systems continued to operate, with practice-oriented reduced curricula. However, a huge system of underground Polish education was organized. It was an amazing network of clandestine classes, held in small groups of up to 5–10 students, hidden in people's homes or disguised as another activity allowed by the Nazis. It is estimated that over a million Polish children learned in covert primary schools, about a hundred thousand students secretly attended secondary schools, and ten thousand studied at the university level. Some 15% of teachers lost their lives during that period.

After 1945, contacts of Polish citizens with Western Europe were strictly limited by communist authorities. The 1956 political thaw made foreign trips possible, but it was still difficult to obtain a passport, while lack of hard currency was another obstacle.

For half a century, a leader in Polish mathematics education was Stefan Straszewicz (1889–1983) (Figure 13.1), a professor at Warsaw Technical University. His PhD from the University of Zurich in 1914 was supervised by Ernst Zermelo, the founder of axiomatic set theory. In 1932, Straszewicz became the Polish national representative to ICMI and continued in that role until 1972 after the reconstitution of ICMI in 1952; during 1963–1966, he was a vice-president of the Executive Committee of ICMI. From 1949 to 1969 he was the chairman of the Commission on Mathematics Curricula of the Ministry of Education. In 1950 he organized the Polish Mathematical Olympiad for secondary school students and for the next 20 years he was its chairman. From 1953 to 1957, he was the president of the Polish Mathematical Society.

However, it was Zofia Krygowska (Figure 13.2) who played the key role in reforms of mathematics education in Poland inspired by New Math; she was also very active internationally.

Krygowska's Role in Poland and in International Reform Debates

Zofia Krygowska (officially Anna Zofia Krygowska,[3] 1904–1988) attended primary and secondary schools in Zakopane in the Polish highlands. She was eager to read books, also in French and German, including original versions of classics like Goethe's *Faust* and *Der Zauberberg* [The Magic Mountain] by Thomas Mann. As a mathematics graduate from the Jagiellonian University in Cracow, she became a teacher in 1927 and taught in primary schools (from grade 1 on) and in secondary schools. Her passions were school and mountains. During the War she acted as a liaison (under the cover of a bookkeeper visiting sawmills) between the Polish Underground Teaching Organization and secret local groups in the highlands, helping them, supervising, organizing examinations, carrying textbooks and other materials in a backpack, and teaching. She often experienced a great fear; 40 years later she recalled it as a nightmare (Krygowska 1985).

After the War, she worked at the Teachers Training Centre in Cracow. In 1952, she received a PhD in mathematics from the Jagiellonian University; her thesis was on the limits of precision in teaching elementary geometry. She became a lecturer at the College of Teacher Training in Cracow, which later evolved into what is now the Pedagogical University of Cracow.[4] She worked on many issues, such as educating teachers of mathematics, developing didactics of mathematics to make it a scientific discipline, making use of concepts of developmental psychology, and discussing reforms to the mathematics curriculum. In 1958, her school resolved to organize a Chair of Methodology of

[3] In her books and articles published in Poland, she always wrote only her middle name "Zofia"; she also celebrated her name day as Zofia. When she was abroad, Anna Zofia was used according to what was written in her passport.

[4] For many years it was named *Wyższa Szkoła Pedagogiczna*, literally "Higher School of Education," corresponding to the German term *Pädagogische Hochschule*. In 1999 the school became *Akademia Pedagogiczna* and in 2008 it became a university.

Mathematics Teaching, later renamed as Chair of Didactics of Mathematics. This was the first such chair in Poland. It was part of the Faculty of Mathematics and Physics, not of the Faculty of Education (Nowecki 1992).

In 1956, Straszewicz and Krygowska were members of the official delegation of the Polish Ministry of Education to the UNESCO International Conference in Geneva. The conference included a part devoted to mathematics (regarded then as the most conservative of all school subjects). Jean Piaget participated in it and expressed his view that from a psychological viewpoint, the fundamental structures of mathematics (i.e., algebraic structures, order structures, and topological structures) could and should be a basis of a new organization of school curricula (Krygowska 1957).

A significant result of that conference was that Krygowska opened up contacts with mathematicians in Western Europe (Félix 1986). She was a good friend of many of them, in particular Willy Servais, Hans Freudenthal, Hans-Georg Steiner, and Tamás Varga.

In 1957, she joined the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM)/International Commission for the Study and Improvement of Mathematics Teaching. It was established in the years 1950–1952 as an independent body and consisted of enthusiastic people.[5] In the first period (till 1960) the Commission was dominated by three personalities: Its president (the mathematician Gustave Choquet, who stressed the role of fundamental structures of modern mathematics in education), its vice-president (the psychologist Jean Piaget, who opened new perspectives for research on logico-mathematical constructions of the mind), and its secretary (the educationist Caleb Gattegno, who worked on connecting the above ideas with a didactical context) (see also Chap. 3 in this volume).

Piaget, Choquet, and André Lichnerowicz influenced Krygowska's structuralist approach to secondary school mathematics. She wrote:

> The modest content of school curricula gives many opportunities to reveal to students fundamental algebraic structures, analogies and isomorphisms of various fragments of school mathematics which are isolated in students' minds; this can be done in a way which is natural, simple and accessible to them. It makes possible to break down rigid barriers dividing particular branches of elementary mathematics and to show students that there are concepts which intervene—in particularly interesting ways—in completely different areas. One can easily make a general concept of an operation available to students, show examples of some "unusual" operations, non-commutative or non-associative. The school subject matter is pervaded by the concepts of operation, inverse operation, group, field, equivalence relation, order relations, transformations, invariants, and isomorphisms. Neglecting these concepts is completely unnatural, is an unnecessary curtain separating the student's thought from contemporary mathematical thought. (Krygowska 1959, p. 18)

Krygowska was influenced by Piaget's idea of relations between fundamental structures of modern mathematics and structures of cognitive development. The first operations used by children, derived from a general coordination of their actions on objects, are precisely divided into three wide categories. One of them consists of those operations where reversibility is obtained in a way akin to algebraic structures; the second—those where reversibility is obtained by reciprocity (as in structures of order, of seriation); and the third—those related to neighborhoods and continuity, that is, topological structures, genetically prior to metric structures. He compared this with Bourbaki's analogical concept of three mother structures (Piaget 1970).

In the second period 1960–1970, CIEAEM was dominated by its president Georges Papy (Figure 13.3), a professor of algebra at the University of Brussels, Belgium (see also Chap. 10 in this volume). His goals were as follows: a curriculum with a strong component of algebraic structures, modernization of geometry teaching, and modernization of primary education (Castelnuovo 1981).

The 14th *rencontre* [meeting] of CIEAEM was held in Cracow in August 1960. It was heavily influenced by Papy and his idea of *mathématique de base* ["base mathematics"]. He stressed that in

[5] The legal setting of CIEAEM should be contrasted with that of ICMI, which officially is a commission of the International Mathematical Union (IMU). Members of ICMI are states (one of them is Poland), while members of CIEAEM are people.

Figure 13.3 Papy, presented as the "pope of modern mathematics." (Caricature by Leon Jeśmanowicz, 1971)

each epoch of the history of mathematics there were certain concepts, relations between them, theorems and rules forming the "base mathematics"; in modern mathematics, this fundamental role is played by sets and relations. Papy claimed that it should be much easier to teach these fundamental ideas when the learners were not yet educated and their brains were fresh, "without bad habits." One day he made an impressive show with children aged 8–12 years. Schools were closed, and children were found in a summer play center and brought to the conference venue. The theme of those activities was relations (a university subject). Papy spoke French and his words were *ad hoc* translated to Polish for the children. Children were first asked to represent themselves on the blackboard, in some very simple way. They talked about how to do this and drew small marks ×, augmented later with numbers chosen by themselves. As one of the next steps, the children were asked to make marks for their sisters and brothers; when they were asked to indicate these relationships, they drew arrows from children to their sisters and other arrows to brothers. After that, Papy drew a separate graph representing some other children and arrows showing sisters and brothers; he then asked what additional arrows must be there for sure (e.g., in the case of a brother of a brother, this meant the transitivity of the relation). Teachers who watched this were astonished: The children were able to use simple graphical schemes and argued correctly (Korczowski 1961; Krygowska 1961).

For a long period of time Papy's ideas influenced Krygowska's thinking about mathematics education. For her, three ideas were particularly important: (a) stressing the role of fundamental algebraic structures; (b) starting with *mathématique de base* very early; and (c) advocating Venn diagrams and arrow graphs as non-verbal language representing two main concepts of this "base mathematics." She believed in the explanatory power of Venn diagrams for children and recommended using them even in the case where the closed curves symbolized intersecting straight lines (Krygowska 1977b). Her structuralist attitude was extended to primary grades:

> The structure of the ring Z of integers is the main subject in those experimental classes [devised by Krygowska]; the study of it is the core of a topic rich in problems of algebra, arithmetic, geometry … Examples of finite rings facilitate an approach to a general idea of an operation (example of a preconcept). (Krygowska 1971b, p. 103)

Henryk Moroz, in his thesis which was supervised by Krygowska, presented this idea in the following way:

> In the project of a new mathematics curriculum for grades I–IV, worked out by Z. Krygowska and the present author, the central theme is the structure of the ring of integers. Results of an experimental study irrefutably show that the structure of the ring of integers is for the pupil of age 7–11 definitely more intelligible, more interesting and more instructive than the structure of the set of non-negative rational numbers. (Moroz 1972, p. 36)

Even the Cuisenaire rods, teaching tools providing the child with visual, muscular, and tactile images, were seen in structural terms:

> In order to stress the role of clear scientific conception played in the conception of activity-based teaching, let us mention another approach to elements of arithmetic, namely that based on the Cuisenaire material. Theoretically, we start with an algebraic structure of the arithmetic of non-negative integers, that is, a semigroup. Children solve various problems. (Krygowska 1977a, p. 97)
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> Using numbers in color, the pupil of grade 1 in the preparatory period, before introducing numbers, solves various problems, and doing that he/she becomes aware of the structure of a semigroup. In the set of numbers in color, the child discovers an equivalence relation: Blocks of the same color are equally long. (Moroz 1986, p. 33)

In those times, many advanced mathematical ideas were identified by observers in children's activities. However, later the attitude changed: "There is a long distance to cover between the spontaneous, unconscious use of structures and their becoming conscious" (Piaget and Garcia 1989, p. 25).

In 1964, at a conference of Polish mathematics educators, a decisive idea of didactics of mathematics as a separate multidisciplinary academic subject was formulated by Krygowska. She presented inspiring visions and her arguments sounded convincing (Nowecki 1990a). At that conference Zdzisław Opial, an outstanding mathematician from the Jagiellonian University, said:

> A compromise between mathematics of the second part of the 20th century and that of the 19th century is impossible. Consequently, a compromise between modern mathematics and the present school mathematics is impossible. … links between school and university mathematics were broken many years ago and it is necessary to fix them and to raise the school mathematics back to the height of contemporary mathematics. (quoted by Nowecki 1984, p. 23)
>
> In contemporary mathematics there are numerous simple elements that organize this difficult subject, elements common to all branches of mathematics, and these elements should be transferred to the school. They should be used widely, so that school mathematics becomes a reasonable introduction to the activities of the contemporary man. One has to do this necessarily, even if this is impossible! (Opial 1966, p. 7)

These radical statements expressed the feelings of many mathematicians at that time, and reflect the atmosphere which preceded the 1967 reform (see the next section).

In 1965, Krygowska spent two months at the UNESCO center in Paris as an editor of the first volume of *Tendances Nouvelles de l'Enseignement des Mathématiques/*New Trends in Mathematics Teaching (published in 1966). She also contributed to volumes II (1970), III (1972), and IV (1979) of the *Tendances*.

At the 16th International Congress of Mathematicians (ICM) held in Nice (1970), Krygowska delivered an invited address on problems of modern teacher education in the section *Histoire et Enseignement* (Krygowska 1971a). She also actively participated in two other ICMs: Moscow (1966) and Warsaw (1983). At ICME-1, the First International Congress on Mathematical Education held in 1969 in Lyon (France), Krygowska delivered a plenary lecture *Le texte Mathématique dans l'Enseignement* [Mathematical text in the teaching]; in particular, she explained in detail how to help a reader understand Bourbaki's introductory statements initiating a series of definitions leading to that of a scheme S of construction of an echelon on n terms from a theory stronger than set theory (Bourbaki 1957). This definition started with the words: "A scheme of construction of echelons is a sequence $c_1, c_2, \ldots, c_n$ of pairs of natural numbers $c_i = (a_i, b_i)$ satisfying the following conditions ..." (Krygowska 1969, p. 361).

At ICME-2 (1972) in Exeter (UK), she worked in the section on *The Professional Training of Mathematics Teachers*. At ICME-3 (1976) in Karlsruhe (Germany), she was a member of the Program Committee and was the reporter in Section A2 *Mathematics Education at Upper Primary and Junior High School Level* (see Figure 13.4). She also lectured at numerous conferences in various countries and visited many groups working on problems of mathematics education (Ciosek 2005; Siwek 2004).

In 1970 Papy resigned as the president of CIEAEM and Krygowska became his successor. She remained in that position during 1970–1975; after 1975 she was an honorary president. Her disciple, Stefan Turnau was the president of CIEAEM during 1981–1982 (Gellert et al. 2015). In August 1971 she organized the 23rd *rencontre* of CIEAEM in Cracow, a meeting which considered questions related to the teaching of logic in schools.

Figure 13.4 Hans-Georg Steiner, Zofia Krygowska, and Hans Freudenthal at a meeting in Oberwolfach (Germany) preparing for ICME-3 on December 8, 1975. (Photo collection H.-G. Steiner)

During the period 1963–1972, Krygowska was a member of the Main Council of Higher Education, an advisory board of the Minister of Education in Poland. She succeeded in convincing authorities of the need to grant PhDs in academic disciplines (mathematics, biology, etc.), including cases where these concentrated on educational problems of disciplines.

Krygowska actively promoted her vision of mathematics education as a field of research (see Nowecki 1984). She maintained that first one had to establish a theoretical and methodological basis of the field, emerging from mathematics itself, psychology, pedagogy, sociology, philosophy, and methodology of research, and also to work out a new conception of elementary mathematics reflecting the present stage of the development of mathematics, overwhelming the historic, anachronic structure of school curricula. Second, it was necessary to set up a suitable legal and organizational framework for educating teachers and to specify the psychological and pedagogical conditions needed for implementing it. Third, modernization of the subject content, teaching methods and activities needed to be carried out by the teacher, teaching resources, working out theoretical concepts, curricula and experimental verifications of them. She also stressed that didactics of mathematics was an independent scientific discipline, albeit interdisciplinary and still at the beginning of its development. Moreover, in this field, one would never have "absolute truths," independent of the time, country and of general culture which forms the minds independently of teaching (Félix 1986).

Krygowska created a vivid center of research on mathematics education in Cracow, and many foreigners came to join the activities. During the period 1968–1986, she was the principal supervisor of 26 PhDs in mathematics granted for research theses which were basically in the field of mathematics education (Turnau 1983). In 1972, she became professor emeritus, but her activity did not diminish at all. The list of her publications is in Nowecki (1990b).

Her great success was the creation in 1982 of a research journal *Dydaktyka Matematyki* (now *Didactica Mathematicae*, with articles in English), published by the Polish Mathematical Society. She also served on editorial boards of several other journals: *Educational Studies in Mathematics* (1968–1978), *Recherches en Didactique des Mathématiques* (1980–1988), the Polish journal *Wiadomości Matematyczne* (1963–1988), and the Belgian *Nico* (1969–1975).

In 1977, she received a honoris causa doctorate from her school and a honorary membership of the Polish Mathematical Society. Hans Freudenthal praised her with words: "Among many people carrying on research in mathematics education only a few represent suitable level of mathematical and pedagogical quality. Madame Krygowska is one of them, and perhaps the most important and most active, and the only one who has created a school with outstanding achievements. Cracow's school of educational research is internationally recognized" (quoted by Nowecki 1984, p. 9). In 1984 many people came to Cracow for a conference celebrating Krygowska's 80th birthday, including Josette Adda, Emma Castelnuovo, Hans Freudenthal, Claude Gaulin, Georges Glaeser, Colette Laborde, and Tamás Varga. Hans-Georg Steiner sent a paper to be read.

The 1967 Change in the Mathematics Curriculum for Secondary Schools in Poland

In 1962, the Polish government decided to change the whole structure of general education from a 7 + 4 system to 8 + 4. During the period 1963–1971, the new curriculum was successively *implemented* in grades 5–8 and then in the secondary school grades 1–4.

The Polish Mathematical Society organized debates on the anticipated changes. Official documents had been worked out by the ministerial commission on mathematical curricula chaired by Straszewicz (with Krygowska, Opial, Leon Jeśmanowicz, and other mathematicians as members). Straszewicz was in favor of modernizing Polish curricula, but regarded New Math as too radical.

The curriculum and textbooks for grades 5–8 were then modified but remained rather traditional. In the accompanying textbooks for grades 7–8, an intuitive concept of a set was used occasionally, for example as the set of numbers satisfying the inequality $2x - 5 < 8$. In grade 8, the notion of a function F from a set X to a set Y, with symbols such as $y = F(x)$, was introduced (Straszewicz 1966). In sharp contrast, the curriculum for the secondary school was heavily influenced by the spirit of *Mathématique Moderne*.[6] In 1967, two new textbooks—*Algebra* and separately *Geometry*—were published for grade 1 of the lyceum. Novelties in the algebra textbook were rather moderate (although it was an introduction to calculus rather than an algebra course), but Krygowska's geometry strongly stirred mathematicians and teachers.

Krygowska's textbook developed the conception of the so-called global deduction in geometry, based on a fixed axiom system (contrasted with local deduction, i.e., without spelling out axioms). However, the word "axiom" did not appear in the book; they were called "basic properties." In her approach, two other systems were also assumed: Naïve set theory and the real number system. A geometric figure was defined as an arbitrary set of points of the plane.

The general concept of a metric space was introduced and explained with two examples. The first example was the set of real numbers with the distance $|a-b|$; the second was a set A consisting of a species of trees, a set B of b species of trees, and the distance between them defined as the quotient of the numbers $a + b - 2w$ and $a + b - w$, where w is the number of species common to A and B.[7] A totally ordered set was also defined.

In a condensed form, omitting several auxiliary definitions, the ten axioms I–X may be summarized as follows (Krygowska and Maroszkowa 1967; Semadeni et al. 1970).

Property I. A straight line is a geometric figure consisting of infinitely many points. Each point of the plane belongs to infinitely many straight lines. For point A, the symbol (A) denotes the pencil of straight lines defined as the set of all straight lines passing through A.

[6] The main inspiration came from France, Belgium, and Germany. The influence of British and US reforms was marginal.

[7] This idea of becomes clear if the reader realizes that $a + b - 2w$ is the cardinal number of the symmetric difference $(A \setminus B) \cup (B \setminus A)$ while $a + b - w$ is the cardinal number of the of the union $A \cup B$.

Property II. For any two distinct points *A* and *B*, there is one and only one straight line passing through them, denoted as pr.*AB*. This was complemented with a symbolic form, displayed in a special box:

$$A \neq B \Rightarrow (A) \cap (B) = \{\text{pr.}AB\}.$$

The students had to decode the meaning of these symbols as well as the implication: If points *A* and *B* are different, then the intersection of the pencils (*A*) and (*B*) is the set consisting of a single element, namely the straight line passing through *A* and *B*. Such formal symbolic versions of various statements, using the special system of symbols introduced in this textbook, appeared several times.

Property III. In each pencil of straight lines, there is exactly one straight line parallel to a given straight line.

Definition: We say that we have determined a distance in a set *Z* if to each pair of elements of *Z* we have assigned a non-negative number, called the distance of these elements, such that the distance from *a* to *b* is zero if and only if $a = b$; for any elements, *a, b*, the distance from *a* to *b* is equal to the distance from *b* to *a*; for any elements, *a, b, c*, the distance from *a* to *c* is not greater than the sum of distances from *a* to *b* and from *b* to *c*.

Property IV. In the plane, there is a distance that assigns to each pair *X, Y* of points a number denoted by *XY* such that:

(a) Points *A, B, C* are colinear if and only if $AB = AC + CB$ or $AB = AC - CB$,
(b) Points *A, B, C* are not colinear if and only if $|AC - CB| < AB < AC + CB$.

Property V. On each straight line, there are exactly two mutually inverse total orders such that a point *B* lies between points *A, C* if and only if *A, B, C* are distinct points and $AC = AB + BC$.

Property VI. On each half-line, there is a unique point such that its distance to the origin is equal to a given non-negative number. The family of all straight lines parallel to a straight line *a* is denoted as (*a*) and is called the direction of the line.

Property VII. Any parallel projection preserves the natural order of any straight line, whose direction is different from the direction of the projection.

A point is called a boundary point of a figure if in each neighborhood of that point there are points of the figure and points not belonging to it. The boundary of a figure is the set of all its boundary points. An interior point of a figure is a point having a neighborhood contained in it. An exterior point of a figure is a point having a neighborhood disjoint with it.

Property VIII. Any line segment whose one end is an interior point of a figure and the other end is an exterior point of it has common points with the boundary of the figure. Any line segment whose one end is an interior point of a disk and the other end is an exterior point of it has exactly one common point with the boundary of the disk. The same is assumed if the word "segment" is replaced by "circular arc." Several definitions were then formulated: A convex figure, a bounded figure, an unbounded figure, a polygonal chain, a simple polygonal chain, and a simple closed polygonal chain.

Property IX. Any simple closed polygonal chain divides the plane into two connected figures, one bounded and the other unbounded.[8]

Property X. For any two points of the plane, there is a non-identical isometry of the plane such that these points are fixed points of the transformation.

A section of the textbook concerned vectors (defined as ordered pairs of points). This was continued in grade 2, where the notion of the scalar product of two vectors was introduced and used in a proof of the Pythagorean theorem (Krygowska 1968b).

[8] This axiom is a version of the celebrated Jordan curve theorem in topology.

Students faced serious difficulties while attempting to learn from Krygowska's textbook. One of the obstacles was the concept of a pencil of straight lines, defined as the set of all straight lines passing through a given point or as the set of all lines parallel to a given line. It plays a significant role in Properties II, III, and VII, for example in the implication $A \neq B \Rightarrow (A)\cap(B) = \{\text{pr.}AB\}$. Students could intuitively grasp the idea of a single figure as a set of points (a circle, a straight line), but a pencil was a much more advanced concept: It is a set whose elements are also sets.

Many people regard sets not in the distributive sense, as it is assumed in mathematics, but in the collective sense (called also "mereological"), typical of common thinking. Whereas set theory is founded on the membership relation between sets and their elements, mereology is based on a very general concept of part-whole.[9] Collective set theory is characterized by its principle: If *A* is an element of *B* and *B* is an element of *C*, then *A* is an element of *C*. A simple example of a collective approach has the USA regarded as a set of states; states are elements of the USA. Each state is (collectively) regarded as the set of its counties and counties are elements of the state. Consequently, by the above rule, counties are also elements of the USA. One may ask a simple question which helps to distinguish the distributive approach from the collective one: Is the set of US counties contained in the set of US states? Collectively, the answer is affirmative; distributively—of course, negative (let us note that in set theory a subset cannot have more elements than a set).

One of the reasons why students found it hard to comprehend Krygowska's pencils of straight lines was that they intuitively interpreted the pencils in a collective sense. Thus, for them, points were elements of lines; lines were elements of a pencil, consequently, points were elements of the pencil. Moreover, each point of the plane belongs to a line of the pencil, and therefore—according to this line of thinking—it also belongs to the pencil. Thus, for such a student, the pencil was much the same as the whole plane. Moreover, students regarded the intersection of two lines—collectively—as just their common point *A*, while—in the distributive interpretation—the common part of two lines is a set, a singleton$\{A\}$ and not the point *A*, as $\{A\} \neq A$. The situation was further complicated by the special system of symbols introduced in the textbook: A pencil of lines passing through *A* was denoted as (A), the direction of a line *a* was denoted as (a), while A^m symbolized the orthogonal projection of *A* on the line *m*.

Krygowska's textbook was a subject of many heated discussions; one of them was organized by the Polish Mathematical Society (Semadeni et al. 1970). For several years, students were buying the textbook at the beginning of the school year, but it was not used in the classroom; teachers preferred to use some collections of problems.

Changes in the Polish Mathematics Curriculum for the Early Grades During the Period 1970–1980

The 1962 reform left primary grades 1–4 unchanged and Krygowska started to work out her own project for a new curriculum for these grades. Its Polish version was published (see Krygowska 1968a) and a similar French version also appeared (see Krygowska 1971b). In the latter paper, she presented a clear picture of her intended content of mathematics. The list of topics began with a remark that it

[9] Questions of wholes and parts were dealt with by philosophers since Plato, in particular by Edmund Husserl in 1901. A systematic theory, very technical and difficult, was worked out by Stanisław Leśniewski (1929), who had coined the term *mereology* in 1927 (inspired by the Greek word μέρος, *méros*, part). In this system there is no empty set, an element x is not distinguished from the singleton $\{x\}$, and the element relation is transitive. A comprehensive non-technical survey of mereology is given by Gruszczyński and Pietruszczak (2010). In the didactical context, the difference between distributive and collective concepts of a set and related students' difficulties are discussed in Bryll and Sochacki (1997).

was intended for each class, not only for experimental classes. Below we reproduce that part of it for grade 1 (7-year-olds) and for grade 2.

Grade 1 (5 lessons of 45 minutes per week). Use of structured material (the number is not "visible," but manipulations are a qualitative preparation to arithmetic).

1. Sets, subsets. Arrangement of elements of a set. Preconcept of the order. Logical games with the material; and, or, not. Transformation of a set into a set; graph. Equipotent sets. Introduction to algebraic operations on the material; pre-concept of inverse operation, of operation compatible with an order (for instance, with Cuisenaire rods).

2. Natural numbers from 0 to 20 (complete study), from 0 to 1000 (without looking for perfection). Solving equations of the type $a + x = b$, $x - a = b$, $a - x = b$, and applying them to small problems.

Addition and subtraction are introduced simultaneously (the same situation, the same thinking, expressed in different ways). The same applies to multiplication and division (in a case where one can multiply, one can also divide). Introducing the fraction as an operator composed of multiplication and a division.[10] Introducing parentheses, the order of operations, a^n as an abbreviation of $a \times a \times ... \times a$. At the end of the school year, if it is possible, children are introduced to base two and three (e.g., with the material of Dienes).

Grade 2 (6 lessons of 45 minutes per week).

1. Introducing symbols of set theory: $\in$, $\notin$, $\subset$, $\not\subset$, $\cap$, $\cup$, $\setminus$, $\emptyset$. Logical games (material of Dienes), subject much appreciated; while practicing one proposes negating conjunctions and disjunctions of propositions.

2. Arithmetic of natural numbers. Algorithms of operations in base two, three, and ten. Converting from one base to another. Equations of the type $ax + b = c$, $a + x < b$, $ax > b$. Examples of functions in N, empirical functions (with examples taken from geography) and their various presentations (bar graphs, pie charts, etc.).

3. Applications to simple combinatorial problems (without formulas); the number of permutations; the number of shortest paths on a square lattice (e.g., on geoboards). During this, intuitive initiation to recurrent reasoning: Discover the rule! (Krygowska 1971b, pp. 103–106; the experimental curriculum was also described in Krygowska and Moroz 1970).

Krygowska's project on the curriculum for primary grades was not accepted by the ministerial commission. Straszewicz organized a subcommission consisting of educators dealing with primary education, including Zofia Cydzik, and the mathematician Zbigniew Semadeni. The subcommission prepared a project of a curriculum for grades 1–4. After discussions and alterations, Straszewicz submitted it as an official project to the Minister of Education in 1970. It was tested in selected schools

[10] The idea of presenting fractions as "operators" originated in France, but was uncritically accepted and advocated by Krygowska, who tried to introduce it to the compulsory national curriculum. Traditionally first examples of fractions had been shown to children based on activities such as cutting a circle or a rectangle into equal (i.e., congruent) parts. However, the general dogma that from the beginning mathematics should be presented in terms of set theory and based on the structure of the ring of integers was not compatible with fractions based on properties of geometric figures and activities such as paper cutting. A fraction, for example, 3/4, was then interpreted as a composition of two functions: multiplication by 3 followed by dividing by 4 (or dividing by 4 followed by multiplication by 3). The domain of such function 3/4 was the set of integers divisible by 4 (so that the values were integers). With this approach, however, addition of fractions, say, 3/4 (function defined on the set of numbers divisible by 4) and 1/5 (function defined on the set of numbers divisible by 5) lead to a function defined on the set of numbers divisible by 20. Thus, an operator was an equivalence class of composite functions on certain classes of integers. Krygowska wanted examples of fractions as operators already in grade 1 of primary education.

and published (in the journal *Matematyka*, *3* in 1971). Its text is reproduced as an annex to Krygowska (1971b).

The content of the ministerial curriculum was then regarded as a compromise between that of Krygowska and those submitted by other educators.[11] As a first step to the anticipated reform, new textbooks were published for grade 1 (one by Ewa Puchalska and Marek Ryger, and another by Cydzik) and for grade 2 (by Moroz).

The government sponsored a huge program of preparation for teachers, including 3 years of 30-min lectures with mini-films (prepared by Krygowska, Semadeni, and others) which were broadcast once a week on Channel 2 of the Polish National TV, augmented by discussions on Polish Radio. The texts of the TV presentations were successively published as annexes to a biweekly journal of the Ministry of Education, and their reworked, extended, and colorfully illustrated versions were published in four volumes (Semadeni 1981, 1984, 1986, 1988). A major part (134 pages) of Volume 1 was written by Alina Szemińska, the co-author with Piaget of the groundbreaking *La Genèse du Nombre Chez l'Enfant*, on the origin of the child's concept of number (Piaget and Szemińska 1941).

Unfortunately, this reform was heavily disturbed by the decision of the Polish government to implement a radical structural reform of the whole educational system. The 12-year schooling system was replaced by a 10-year system, modelled on the Soviet system of general education with a 10-year curriculum. The previous eight-year primary and two- to five-year general or vocational secondary schools were replaced by a uniform and mandatory 10-year program, called "secondary education for all." Official propaganda claimed that improved undergraduate teacher training, better methods, gathering children from rural areas into big schools, and avoiding unnecessary repetitions of the material, would enable the students to achieve in 10 years what previously needed 12 years.

The Minister of Education failed to obtain a parliamentary approval and yet decided to start the reform in 1977 as a *fait accompli*. A syllabus for grades 1–10 was announced (including, for example, "calculus for all"). The subject matter was squeezed from 12 years to 10, and primary education from 4 years to 3. Grade 4 of elementary education was declared to be the first grade of secondary education. In a new textbook prepared for grade 5, students were introduced to real numbers, including the irrationality of $\sqrt{2}$.

In 1980, when the reform reached grade 4, the Solidarity movement forced the government to abandon the reform. An indefinite postponing of it was announced on television in November 1980.[12] Nonetheless, general changes in primary grades turned out to be irreversible, whereas corrections of the mathematics curricula in grades 4–8 lasted until the end of the communist rule in Poland in 1989. In 2008 the last remains of formal logic and sets disappeared from the Polish core curriculum.[13]

Closing Reflections

At ICME-2 in Exeter (1972), René Thom opened his plenary address with the following words:

> The future historian of mathematics will not fail to be amazed by the extent of the movement of the 1960s known as Modern Mathematics. … Only dogmatic spirits (and they are not lacking among "modernists") can believe

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[11] However, some years later it became clear that there was too much material and it had to be reduced. A thorough analysis of the ways of introducing elements of New Math to early grades and their long-term adverse consequences is given in Gruszczyk-Kolczyńska (2017).

[12] Some weeks later, on December 13, 1981, General Wojciech Jaruzelski announced the introduction of martial law in Poland, suspending people's constitutional rights, curfew, telephone connections cut, special permissions for intercity travelling, no classes at universities for several months.

[13] The history of the reforms of Polish mathematics curricula in the period 1963–1990 is outlined in Molęda and Piesyk (1993).

> that there is in these questions a truth capable of being logically established and before which one needs must bow. … "Modern Mathematics" has a very complex origin and composition. (Thom 1973, p. 130)

The unique phenomenon of the New Math movement has been the subject of many heated discussions and thorough analyses.[14] A particularly intriguing question is: How was it possible that so many competent mathematicians in so many countries eagerly promoted educational ideas which included elements not compatible with common sense?

Among the many faces of those reforms, one may list important novelties, like the Cuisenaire rods. The idea of getting rid of obsolete fragments from curricula seemed reasonable. Proposals of modernizing curricula were supported with quotations from Piagetian theory of cognitive development. The idea that modern mathematics is based on set theory and on formal structures was promoted by outstanding leaders (e.g., Dieudonné, Choquet, Lichnerowicz) and also by those who benefited as authors and publishers of new educational materials. Sharply critical opinions of Thom, Jean Leray, and some other top mathematicians were not heard.

A new educational paradigm emerged: The right approach to school mathematics should be based on sets, basic structures of mathematics, and precisely defined concepts. At the peak of reform enthusiasm, a particularly harmful slogan was heard: Forget what you had earlier learned at school!

The reformers took into account the qualifications of teachers but neglected their susceptibility to change. It was believed that after a certain initial period teachers would master the new content, whereas the question of methods of teaching was passed over. Some outstanding educators devised such activities that they themselves might be able to implement, but they ignored the fact that it might not be possible for most teachers to do the same.

However, plausible arguments do not explain the scope of the reforms and the radicalism of the proposals.[15] Let us look for other hints, related to the significant question: What is didactics of mathematics? According to a popular view, it is a study of how to teach mathematics properly, a study of the skill, and a study of curricula. However, didactics of mathematics is also a domain of scientific research.[16] One of the energetic promoters of this idea was Krygowska. Determining its present scope continues to be a subject of various studies (Sierpińska and Kilpatrick 1998).

A key hint to the unique phenomenon of New Math seems to be an interpretation suggested by Anna Sierpińska in her plenary address that opened ICME-8 in Sevilla in 1996. She found that programs of research and action in mathematical education, that of New Math and later, evolved on three planes: "The plane of ideology, the plane of theory, and the plane of didactic action" (Sierpińska 1996, p. 22). The term ideology has political connotations, and may also be interpreted as, for example, the ideology of postmodernism or, more generally, as "a set of beliefs or philosophies attributed to a person or group of persons, especially as held for reasons that are not purely epistemic, in which practical elements are as prominent as theoretical ones" or as "a coherent system of ideas that rely on a few basic assumptions about reality that may or may not have any factual basis" ("Ideology" n.d.).[17]

Thus, if New Math is thought of as an ideology, then certain features of the reforms are easier to comprehend. The ideology of New Math was characterized by a strong faith in the rightness of the basic postulates, an aspiration to cover all of mathematics education, overwhelming enthusiasm, disregarding facts not fitting the assumed vision of mathematics education, and labelling of those who

[14] Hans Freudenthal was one of the first mathematicians who critically described main features of the new trends and pinpointed dangers involved in them (Freudenthal 1963, further developed in Freudenthal 1979).

[15] David Pimm, discussing various proposals of reforms, used the words "monomaniacal enthusiasm" (Sierpińska 1996, p. 23).

[16] As a symbolic date, one may assume 1893, the year of establishing the Chair in Mathematics Education at the University of Göttingen.

[17] In the context of the mathematics curriculum, Richard Noss (1994) described ideology as "the body of ideas through which we see and with which we construct reality" (p. 2); this is related to his views that mathematics is a social construction.

questioned the reforms as traditionalists (or worse). A basic assumption of this ideology was a firm conviction that mathematics—something *not* divided into algebra, geometry, etc., but founded on set theory, axiomatic systems, and basic structures—was a definite, ultimate product of historical development. Such an attitude appears in the emotional declarations of Opial and Krygowska, quoted above.

Moreover, Krygowska, influenced by Papy, was convinced of the necessity of shaping the children's minds very early, according to the new paradigm. This explains the radicalism of the content of the curriculum forced by her in primary education. However, this does not explain why an incredible amount of content was squeezed into her intended curriculum for 7- to 9-year-old children, in particular, why she introduced exponentiation a^n as an abbreviation of $a \times a \times \ldots \times a$ in the very first grade of primary school, as well as parentheses, the order of operations and fractions as composite functions!

One of the outstanding features of Polish reforms worked out during the period 1960–1970 was an attempt to have a uniform approach to the whole of school mathematics.

> One has to think of the primary mathematics teaching in the perspective of the present structure of mathematics; to develop from the beginning those categories of mathematical thinking which will be used in the sequel, so that the next level of mathematics teaching should not be separated from the previous one by a threshold too difficult to pass for an average student. … On each level one should teach mathematics in its proper language. Its symbolic component facilitates learning, but one has to be aware of certain dangers. (Krygowska 1971b, p. 101, and p. 103)

A specific principle was: When the teacher introduces a new arithmetic operation, at the same time the inverse operation should also be introduced. Another principle was formulated in the context of upper secondary school: When the students learn some type of equation, they should also be shown corresponding types of inequalities; this was later applied to middle secondary school curricula, and finally solving inequalities was included in grade 2 of primary school (Krygowska 1971b).

Often, when there was a conflict between a general principle and a specific situation, priority was given to the principle—like, for example, when the principle was that geometric figures should be rigorously treated as sets, then that was done, even when it was incomprehensible to students (Krygowska and Maroszkowa 1967). When the principle was that arithmetic in grades 1–3 should be based on the structure of the ring Z of integers, fractions were introduced as functions on subsets of Z. When the principle was that children should meet basic concepts of modern mathematics very early, a lot of material was packed into syllabuses for grades 1 and 2 (in Krygowska's curriculum, reproduced above).

Krygowska paid much attention to questions about formulating and communicating mathematical ideas by the students, to the clarity and correctness of what they say or write. In several publications, she dealt with common algebraic errors, such as $(a + b)^2 = a^2 + b^2$, and described them in terms of correct/incorrect formalism.[18]

> The consequences of not taking the due care that the students' understanding of the structure of algebraic expressions be always correct and clear, are known: Glaring errors in algebraic transformations, "degenerate" formalism, which manifests itself in thoughtless, "slapdash" manipulation of symbols, is something totally different from correct formalism, which also consists in manipulating symbols, but in accordance to strictly applied rules. (Krygowska 1977b, p. 101)

Many examples of "degenerate formalism" were described in Ćwik (1984). This label, used by Krygowska and her followers, was unfortunate for two reasons. One is that the word "degenerate" connotes either becoming worse, lower in quality (but those students had never been good in algebraic transformations) or a low standard of behavior, depraved. Research by Agnieszka Demby has vividly shown another reason why such a label is not adequate. Such errors are a common feature in the initial period of algebraic experience of most students in grades 7 and 8. Moreover, she distinguished

[18] Technically, the student acted as if the function $f(x) = x^2$ were linear.

between (a) haphazard, ad hoc, non-consistent quasi-rules of certain groups of students, and (b) incorrect but consistent procedures of many other students in her study. Errors of either group looked much the same, but systematic inquiry showed essential differences in their thinking. Most students from the latter group made progress and later qualified for a higher level, whereas students at the quasi-rule level were likely to remain at a low level (Demby 1997). In Poland, the term "degenerate formalism" is still associated with Krygowska.

Krygowska regarded sets as a basis for mathematics education at both the primary and secondary levels. Later, however, she used to say that although sets themselves are not so important, they do provide a very effective language in geometry. In this context, a particularly interesting story was related by Papy. The students in a class were told to draw a square on paper and draw lines cutting the square into four identical parts. While the class was active with this task, one boy, particularly good in mathematics, did not follow the group, sat and thought. The intrigued teacher asked him and was astonished by the answer: "Only we two in this room know that such a cut is not possible." He explained that a point on the cutting line may go to one part only and parts will not be identical (Papy 1971).

Papy told this story as an argument for introducing sets early, already in primary education. In fact, just the opposite should be inferred: A consistent use of sets in elementary geometry may pose unexpected difficulties. Precise set-theoretical language fails to describe adequately a most simple Euclidean operation: Dividing a square into four congruent squares! Indeed, the point at the center can be assigned to only one of the four squares.[19] In the language of sets one cannot even deal with Euclid's Proposition 10: To cut a given finite straight-line in half; indeed, the middle point of the given segment can be assigned to only one of the halves[20]. Thus, if one accepts that the whole of mathematics must be expressed in terms of set theory only, the famous Dieudonné's statement "*A bas Euclide!*" ["Down with Euclid!"][21] is a definite consequence.

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[19] One can do this in the language of set theory using equivalence classes (e.g., modulo the ideal of sets of measure zero).

[20] A similar argument: The middle point should be split into two halves, was raised by Greek philosopher, Pyrrhonian skeptic Sextus Empiricus (second/third century AD, in his *Adversus Mathematicos*).

[21] Expressed in 1959 at a seminar in Royaumont.

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