Chapter 10
Papy's Reform of Mathematics Education in Belgium: Development, Implementation, and Controversy

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Abstract The modern mathematics movement in Belgium is inextricably linked to Georges Papy, a flamboyant and uncompromising professor of algebra at the Free University of Brussels. From the late 1950s, Papy reshaped the content of secondary school mathematics by basing it upon the unifying themes of sets, relations, and algebraic structures. Meanwhile, he innovated the pedagogy of mathematics by functionally interweaving his rigorous discourse with multicolored arrow graphs, filmstrips as non-verbal proofs, and playful drawings, as manifested in his revolutionary textbook series *Mathématique Moderne*. From 1961, his Belgian Centre for Mathematics Pedagogy coordinated the various reform actions: Curriculum development, classroom experiments, and in-service teacher training. Although the Belgian mathematics education community was divided about Papy's agenda, *zeitgeist*, media propaganda, and political support made it possible for Papy to realize his reform almost entirely. After the generalized and compulsory introduction of modern mathematics in Belgian secondary schools in 1968–1969, the primary schools followed in the 1970s.

Keywords Belgian Centre for Mathematics Pedagogy · Belgian experiments · CIEAEM · Days of Arlon · Frédérique Lenger · Georges Papy · Implementation of reform · Kindergarten teachers · Léon Derwidué · *Mathématique Moderne* · Minicomputer · Modern mathematics · Structuralist approach · Teacher recycling · Teacher re-education · Teacher training · Willy Servais

Introduction

As a small country, Belgium played a pioneering role in the worldwide modern mathematics movement of the 1960s and early 1970s. Compared to most other countries, the Belgian reform movement started early (before the Royaumont Seminar, which was held in 1959), was quite radical, and survived for a long time (until the early 1980s). The movement was driven by the passionate reformer Georges Papy, president of the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM)/*International Commission for the Study and Improvement of Mathematics Teaching* during the 1960s, an often invited *expert* at international conferences, and the author of a groundbreaking textbook series *Mathématique Moderne*. Papy inspired reformers all over the world, but this in no way meant that his ideas were welcomed uncritically in

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other countries. Reformers outside Belgium—as well as some within Belgium—considered Papy's project unfeasible for most secondary school students and teachers. Nevertheless, because of their refreshing approach, his textbooks in particular were often recommended as "compulsory literature" for pre-service and in-service teachers.

Although Papy is recognized as the leading, if not uncontested, architect of the modern mathematics reform movement in Belgium, the path toward the reform was paved by others. As shown in Chap. 3 in this volume, discussions on the direction of a modernization process of mathematics education were held within the CIEAEM from the early 1950s onward. Several Belgians, including Lucien Delmotte, Louis Jeronnez, Frédérique Lenger, Paul Libois, and Willy Servais, actively participated in these discussions. When, in 1953, the *Société belge de Professeurs de Mathématiques* (SBPM) [Belgian Society of Mathematics Teachers] was founded by active CIEAEM members, the quality and possible improvement of secondary school curricula became an important concern, in particular at the Society's annual conferences and in its journal *Mathematica & Paedagogia* (De Bock and Vanpaemel 2019, Chap. 3). The debates within the Society culminated in the development in August 1958 of an experimental program for the teaching of modern mathematics by Frédérique Lenger and Willy Servais (Le programme B des écoles normales gardiennes 1958–1959), and in an experiment based on that program.

Toward Modern Mathematics at the Secondary Level

The First Experiment with Future Kindergarten Teachers

The first Belgian experiment with modern mathematics, based on the Lenger-Servais program, was run during the school year 1958–1959 in two schools for future kindergarten teachers in the French-speaking part of Belgium (one in Arlon and the other in Liège). The experiment was led by Frédérique Lenger and Madeleine Lepropre, and participants were 15–16-year old students who certainly did not belong to the top streams of education for mathematics ("Some of them did not hide their fundamental hostility toward mathematics … and the persons who teach it," Papy 1968, p. 27). The experimental course (3 h of mathematics per week) started with fundamental notions from set theory, related to the genesis of natural numbers, and from topology as a basis for the study of geometry (Lenger and Lepropre 1959). After introducing the notion of set, the course continued with set-theoretical topics such as relations of inclusion and equality of sets, the main operations on sets (intersection, union, and Cartesian product), and correspondences between sets (e.g., one-to-one correspondences). Examples were taken from students' experiences and from school life. The second part of the course included an arithmetic and a geometric track (treated in parallel in, respectively, one and 2 h per week). In arithmetic, operations with natural numbers and properties of these operations were discussed from a set-theoretical perspective, as well as their application in the decimal system. Geometry started with some intuitive topological notions (e.g., open and closed figures, interior and exterior of a closed curve) and culminated in a study of the basic plane figures. This study was primarily oriented to geometrical transformations and led to the concepts of symmetry, congruence, and similarity.

According to Lenger and Lepropre, the experiment was a success. As their main evidence, they referred to the encouraging lively and active response from students to the new material.

> Education was provided in these classes in an atmosphere of happiness. The hostility of the students toward mathematics had completely disappeared. We saw vividly that today's children are in resonance with the mathematics that is currently in use. (Papy 1968, p. 29)

If the experiment did prove one thing, Lenger and Lepropre maintained, it was that modern mathematics did not put off students, but on the contrary inspired them with a "taste for mathematics." The experimenters invited other teachers to share their experiences and to find still better ways to make the teaching methods more active.

After the first year of experimentation with modern mathematics, Lenger and Servais realized that they needed the assistance of an academic mathematician to help them with the mathematical problems that would arise with the design of new teaching programs for follow-up experiments. Georges Papy (1920–2011), a professor of algebra at the *Université libre de Bruxelles* [Free University of Brussels], was contacted (Papy 1968). Papy was a promising research mathematician who had not yet shown a strong interest in educational problems. However, he was not completely unknown among mathematics teachers. Papy had published in *Mathematica & Paedagogia* (a primarily mathematical article on the scalar product in which he argued that it would be advantageous to introduce some of such "unifying concepts" into secondary school mathematics; Papy 1954–1955), and he had also intervened in debates at the 1956 SBPM annual conference (with a position in favor of teaching concepts of modern algebra at the secondary level; Boigelot et al. 1956–1957). It is also worth mentioning that about that time Papy was assigned as secretary of a newly formed committee at his university that was to deal with the teaching of basic mathematics (*Le Soir,* April 27, 1958, pp. 1–2).

Papy responded positively to Lenger and Servais' request for mathematical help in future experiments. However, Papy did not confine himself to providing technical advice, but immediately took charge of the project. In September 1959, he started his own experiment in the *École "Berkendael,"* a school for kindergarten teachers in Brussels, and expanded his experimental actions year after year. In this, *"Berkendael"* period, Lenger became Papy's partner, both professionally and personally (they married on October 1, 1960). From then on, Georges and Frédérique Papy (or Frédérique Papy-Lenger) would form a complementary team "driven by a shared vision and commitment that would guide the movement for more than a decade" (Noël 1993, p. 56). The experiments of the Papys would eventually lead to a generalized introduction of modern mathematics in Belgian secondary schools (starting in September 1968), and about a decade later, in Belgian primary schools.

A Ten-Year Experimental Trajectory at the Secondary Level

Papy's first classroom experiments were built on Lenger and Lepropre's work. During the 1959–1960 and 1960–1961 school years, Papy himself taught two experimental classes to future kindergarten teachers, 15–16-year-olds, in the *École "Berkendael."* It was Papy's first attempt at teaching of mathematics at the secondary level (to students who were not particularly gifted in mathematics). For a research mathematician, this must undoubtedly have been a culture shock. At first sight, Papy's *"Berkendael"* course (Papy 1960) looked like a tough university course for future mathematicians, rather than a textbook for future kindergarten teachers. Papy built up his discourse from sets and relations, concepts he illustrated with simple and varied examples from elementary mathematics and from daily life (some likely generated by the students themselves). However, the emphasis soon shifted from these "concrete" examples to the basic definitions and principles, the precise terminology, and the symbolic language of set theory which served as a thinking tool and unifying element throughout the whole course, in particular, for an introduction to geometry, arithmetic, and topology. Structures of order and equivalence were revealed and emphasized, and from the very beginning, Papy promoted rigor and abstraction. Also, logical-deductive reasoning and proving were essential ingredients of Papy's structuralist approach, activities for which the students could rely on the representational tools of Venn and arrow diagrams. Papy deliberately left little room for intuition (in its common sense) which evidently raised the course difficulty level.

However, Papy's structural and abstract view on mathematics and his tendency to detach mathematical entities from concrete, intuitive objects were embedded in a pedagogical approach that proved to be very effective (Randour 2003). He was a master in interacting with the students, bringing them

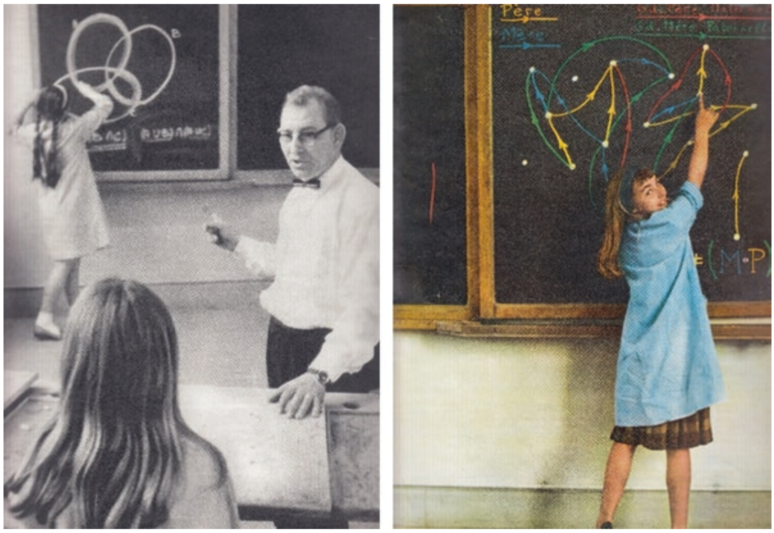

Figure 10.1 Papy experimenting with modern mathematics, early 1960s (left, at the blackboard a student proves the distributive law for union over intersection of sets; right, Christine Manet, one of his students, explains that the composition of relations is not commutative. (Hunebelle 1963)

step by step and without effort closer to correct mathematical conclusions. Furthermore, in his *"Berkendael"* experiments, Papy introduced a pedagogical innovation that would become his trademark, namely multicolored graphs and various other enlightening visualizations, all strongly appealing to the aesthetic and affective side of learning mathematics (Figure 10.1). Along with Papy's (and Lenger's) talents as teachers, this attractive and refreshing pedagogical approach undoubtedly contributed to the success of the experiment.

> Papy's merit is not so much in the content […] but in the teaching methods. Professor Papy and his wife show genuine talent. (Robert Baillieu, professor of mathematics at the Catholic University of Leuven, quoted by Stievenart 1968, pp. 18–19)

Apparently, if packaged in an appealing pedagogical approach, the abstract tenor of the mathematics in the spirit of Bourbaki (1939) was no obstacle for the future kindergarten teachers of *"Berkendael."* On the contrary, according to the teacher–mathematician in charge, they proved to be very receptive to this kind of advanced mathematics. "Papy judges the results fully satisfactory. […] The supreme logic of higher mathematics is directly assimilated by any moderately gifted mind" (De Latil 1960, p. 543).

Although Papy evaluated his experiment with 15–16-year-old future kindergarten teachers as successful, he considered it necessary to start the reform efforts from an earlier age. In May 1961, he published his *Suggestions pour un nouveau programme de mathématique dans la classe de sixième* [Suggestions for a new mathematics curriculum in the first year of secondary schools] (Papy 1961), based on his experiences in *"Berkendael."* In this curriculum, Papy proposed the theory of sets as the starting point for the teaching of mathematics from the age of 12.

> While the space of Euclid could for a long time serve as the framework for a unified presentation of basic mathematics, it can no longer today, but its role can now be fulfilled by the *universe of sets*. Moreover, as it has been proved by experiments carried out in America, England, Russia, Poland and in our country, the teaching of the *basic notions of set theory* fascinates young students. It therefore seems sensible to propose that topic as *starting point* in secondary education. (Papy 1961, p. 21)

In addition to the language of sets and relations, Papy's curriculum proposal for the first year of secondary schools included the ring of integers, the binary and decimal numeration system, and an initiation to affine plane geometry.

Thanks to a positive recommendation by Henri Levarlet, general director of secondary education at the Ministry of Education, Papy's curriculum proposal was approved with an experimental status and permission was granted to continue and expand the experimental trajectory (Papy 1968). Based on his experimental curriculum, Papy started a new experiment from September 1961 in the first years of 12 schools (representing some 30 classes) for general secondary education (12–13-year-olds) (and from then on gradually in the two subsequent years). For the extension of the experiment to the second and the third years (13–15-year-olds), Papy developed a new experimental curriculum (Papy 1962), including, for the second year, the ordered field of real numbers and the real vector plane and, for the third year, the Euclidean vector plane and some elements of "classical" algebra (the equation of a straight line, the square root, functions of one real variable, polynomials, and the solution of systems of linear equations). Soon the experiment was expanded to several dozens of schools all over Belgium. Papy's experiments at the secondary school level resulted in his revolutionary textbook series entitled *Mathématique Moderne* (1963–1967), which we will discuss in the next section.

In order to coordinate the experimental trajectory and related initiatives, Papy had founded on May 24, 1961, the *Centre Belge de Pédagogie de la Mathématique* (CBPM) [Belgian Centre for Mathematics Pedagogy] of which he also became the chairman. It brought together a number of reform enthusiasts, both from universities and from secondary education. The Centre's goal was formulated in its *Articles* as "the study, the improvement and the reform of mathematics teaching. In particular, it will contribute to the promotion, the development and the diffusion of the teaching of modern mathematics" (Papy and Holvoet 1968, p. 133). This goal was realized by the development of experimental curricula, new textbooks, and teachers' courses, the organization of large-scale actions of teacher re-education (which will be discussed in a separate section), and by continuing the experimental trajectory. From 1968 onward, the CBPM also published its own journal *Nico*, a clear reference to Nicolas Bourbaki.

During the 1964–1965 school year, when the students who started the experiment in 1961 arrived in their fourth year of secondary school, the experiments were extended to the upper grades (15–18-year-olds), first in the scientific study streams and in 1967 also in the "non-scientific" streams. Core themes for the upper secondary level were linear algebra, mathematical analysis (founded on topology), and higher arithmetic (Papy 1968). According to Holvoet (1971), statistics and probability theory were also included (but we have not found any trace of that in the documents of the CBPM from the 1960s). Unfortunately, a detailed experimental curriculum was published only for the fourth year in the scientific streams, (CBPM 1966). This experimental curriculum was a mixture of new elements (combinatorial analysis, whole-number arithmetic), traditional elements (such as, for example, approximate calculations and quadratic equations), and repetitions and extensions of subjects that already had been taught in previous phases of the experiment (real vector spaces and the Euclidean vector plane). As the generalized introduction of modern mathematics in the first years of secondary schools approached, and the pressure was mounting to start with the first year, the completion of the experimental efforts for the upper classes seemed to have lost its direction and vigour.

> While the programs for the lower grades (12–15-year-olds) had been extensively tested in experimental classes, it has, unfortunately, not been the case for those for the upper grades. Only one of the classes involved in the experiment in 1961 received, under Frédérique Papy's direction, an experimental program throughout the six years of secondary school. That is not much to draw conclusions from. (Noël 1993, pp. 59–60)

Mathématique Moderne

In 1963, Papy started with the publication of *Mathématique Moderne* (in collaboration with his wife Frédérique), a textbook series revolutionary in content and layout, based on his previous classroom experimentation and intended for the teaching of modern mathematics to students from 12 to 18

Figure 10.2 Covers of Papy's *Mathématique Moderne*

(Figure 10.2). The series clearly shows how Papy reshaped mathematics education, both in terms of content and didactics.

In the first volume of the series (Papy 1963), Papy introduced the language of sets and relations, represented by Venn and arrow diagrams, respectively. These diagrams were intensively used for concept development, reasoning, and proof. The algebra of sets receives ample attention, not only because of its intrinsic value and interesting "new" applications but also because this algebra differed in several aspects from the usual "algebra of numbers," and thus could contribute to a better understanding of the latter. The symmetric difference of sets provided the first example of a group structure. In a geometric track, the plane was introduced as "an infinite set of points" and straight lines were introduced as subsets of the plane, whose mutual positions were explained with Venn diagrams. Papy paid considerable attention to proof and logical-deductive reasoning. A few initial propositions were selected as axioms, from which some simple and intuitively clear properties of parallelism and perpendicularity were proved. According to Papy, self-evident properties are particularly suited for learning to reason correctly and for understanding the essence of proof.

Papy also included some basic topological notions—he differentiated between an open and a closed "disk" and a circle (which only includes the "perimeter"). To visualize these notions, the red-green "traffic-light" convention (for parts that were ex/included) was introduced. New concepts, such as relations of order and equivalence, were commonly introduced with simple and familiar situations that encouraged the student "to take an active part in building the mathematical edifice" (p. vi). The geometric track also included an introduction to transformation and vector geometry. Translations or vectors were defined as classes of equipollent couples of points, the set of which forms a group under composition. In this section, Papy introduced the didactical tool of proof by film fragments: A sequence of suggestive images, from which a line of thought could be seen, was presented and students were asked to add justifications.

> The assimilation of a proof involves several stages which we should try to keep separate. The first step is for the student to understand the film so that he can explain it in informal language. Next he must be able to reconstruct the argument himself. After this comes the stage where more formal justifications are required. Only after all this do we turn our attention to the proper setting out of the proof. (p. viii)

Regarding algebra, Papy first anchored students' pre-knowledge about numbers and their operations in a set-theoretical framework. Natural numbers were defined as cardinal numbers of finite sets and the addition and multiplication of such numbers were related to, respectively, the union and Cartesian product of sets. The positional notation of numbers was revisited by studying the binary system. To introduce integers and their addition, Papy proposed a combat game with red and blue counters, representing oppositely signed numbers that "kill" each other when coming in the same compartment.

Properties of the operations with integers were strongly emphasized and led to the discovery of a group and ring structure. The first volume of *Mathématique Moderne* concluded with a chapter on (abstract) groups, bringing together and systematizing several "concrete" examples from the previous chapters.

In the second volume of the series (Papy 1965), the field of real numbers was constructed, in a mathematically rigorous way. Papy's starting point was a process of binary graduation of a straight line. By inserting the axioms of Archimedes and continuity, he established a one-to-one correspondence between the points on a line and the set of numbers, represented by terminating or non-terminating binaries, at a certain moment called "real numbers." Then, the order and additive structure of the points (vectors) on that line were transferred to the set of real numbers. For the multiplicative structure, Papy first defined the multiplication of real numbers by means of homotheties (= homothetic mappings) and then deduced the basic properties of multiplication from the corresponding properties of the composition of homotheties. The ordered field of real numbers showed up as the ultimate reward. The rational numbers were defined after the real numbers and their structure appeared to be an ordered subfield of that of the real numbers. The real vector plane served as an example of the general concept of vector space and as a basis for affine analytic plane geometry.

In *Euclid Now* (Papy 1967a)—the third volume in the series—the axiomatic-deductive building up of plane geometry was continued and finally resulted in a contemporary vector-based exposition of Euclidean (metric) geometry for 14–15-year-old students.

> Euclid's *Elements* exposed the basic mathematics of his time, about 300 years before J.-C. The monumental work of Nicolas Bourbaki presents, at the highest level, the basic mathematics of today. The "MMs" [= Papy's textbook series] want to expose the Elements of today's basic mathematics for adolescents … and people of any age and schooling who wish to initiate themselves in the mathematics of our time. (p. vii)

Transformations and groups which were generated by these transformations played a key role in Papy's construction of (Euclidean) geometry. Isometries were defined via the composition of a finite number of (perpendicular) line reflections. The different types—translations, rotations, reflections, and glide reflections—and their possible compositions received considerable attention. Colorful classification schemes based on Venn diagrams were presented and group structures were highlighted. Each time a group was discovered, it provoked an *Aha-Erlebnis*: When a student recognized a known abstract structure in a new setting, it was hoped that he or she might be able to apply all previously learned knowledge and skills about this structure to that setting.

> Over the past half century, mathematics has switched from the artisanal stage to the industrial stage. The machine tools of our factories made it possible to save human muscular effort. The great structures of contemporary mathematics allow to save the human mind. (p. vii)

Transformation approaches were also promoted as an alternative to traditional methods in school geometry as it was claimed that such approaches were much more intuitive and universal.

> The outdated artisanal technique based on congruence of triangles must be abandoned in favor of translations, rotations, and reflections, which are much more intuitive and whose scope goes far beyond the framework of elementary geometry alone. (p. ix)

Once the group of isometries was established, the fundamental concepts of Euclidean (metric) geometry could be introduced. The distance between a pair of points and the length of a line segment were defined by means of isometries (and from then on, isometries gained their etymological meaning of "length preserving transformations"). Definitions of the norm of a vector and the scalar product of two vectors followed. The "natural structure" for Euclidean geometry—a vector space equipped with a scalar product—was thereby created. Classical results, such as the Pythagorean theorem could be proved easily within that structure.

> Certain statements, once fundamental, are reduced to the rank of simple corollaries. That they now stop cluttering up the memory of our students. If necessary, they would be able to retrieve these results by routine use of one of the machine tools of modern mathematics. (p. ix)

The fourth volume in the series remained unpublished. The fifth volume (Papy 1966) presented an introduction to combinatorial analysis and higher arithmetic, based on the theory and language of sets and relations. In the sixth volume (Papy 1967b), Papy first retraced, in brief, the laborious path from the original "intuitive" (synthetic) axioms of geometry to the establishment of a Euclidean vector plane structure, the path that the students had followed from the age of 12 to 15. This summary was intended to prepare these students for the second step which Papy described as a *psychological reversal*: The structure of a Euclidean vector plane was taken as a new and "unique" starting axiom for the further development of plane geometry. This approach opened the way for the future study of higher-dimensional Euclidean spaces, in particular for building up solid geometry. At the end of the book complex numbers were introduced as direct similarities. By relying on the structure of the latter and isomorphism, it was proved that complex numbers form a field, extending the field of real numbers.

Volumes of *Mathématique Moderne* were translated into several languages, including Danish, Dutch, English, German, Italian, Japanese, Romanian, and Spanish. To the best of our knowledge, these translations served mainly as a source of information for teachers and all those who participated in the reform debate in other countries ("You can accept or reject Papy's choices for mathematics education, but in any case you can't ignore them," Campedelli and Giannarelli 1972, p. v). However, it is difficult to overestimate the impact of Papy and his *Mathématique Moderne* on the international mathematics education scene during the 1960s (De Bock and Vanpaemel 2019, Chap. 6). Papy acted as an uncompromising modern mathematics ambassador at major international conferences of that period and defended, with verve and authority, his views on the modernization of mathematics teaching. Already at the 1963 OECD conference in Athens, Papy presented an extended sneak preview of the mathematical content and methodological approach of the first two volumes of *Mathématique Moderne* (Papy 1964). Papy's design of teaching modern mathematics was well received by the other OECD experts:

> The example given by […] Mr. Papy were stimulating as to what can be accomplished by a proper blend of modern mathematical ideas with very conscious psychological methods of presentation. When students are directed toward the *discovery of mathematical patterns* and the self-construction of mathematical entities (such as the real numbers), motivation and permanency of learning are greatly enhanced. (OECD 1964, p. 296)

Large-Scale Recycling of Teachers: The Days of Arlon

The content, approach, and results of the experiments of Papy and his team were largely disseminated among Belgian mathematics teachers. Several initiatives were taken by different actors, but the *Journées d'Arlon* [Days of Arlon] undoubtedly had the greatest impact. In July 1959, instigated by Lenger, the SBPM organized, in the Belgian city of Arlon, an intensive teacher training course on set theory and the principles of topology (Papy 1959). The course, which lasted 3 days and was attended by about 150 participants, was the first edition of the Arlon days, a series of annual in-service teacher training courses—at that time usually called "recycling courses"—aimed at introducing Belgian teachers to modern mathematics, both in terms of content and didactics.

The second edition of the Arlon days (1960), organized by the Belgian Ministry of National Education and Culture, was devoted to the study of relations and graphs—two core ingredients of Papy's *"Berkendael"* course which was distributed among the participants. From 1961 to 1968, the CBPM took charge of the practical organization of the Arlon days, in collaboration with the Ministry which supported the initiative financially and morally. Already in a 1962 circular, Victor Larock, socialist Minister of Education of Belgium at that time, drew teachers' attention to the Arlon days and to other in-service training courses. Their purpose, he said, was

> to expand their knowledge of modern mathematics, to reflect on the educational problems raised by the teaching of contemporary mathematics, to convince themselves of their extreme importance and to collaborate for finding solutions. (Larock 1962, p. 8)

Henri Janne, the successor of Larock and also a French-speaking socialist and a former rector of the *Université Libre de Bruxelles*, pledged his full support for the reform activities of his political "friend" Papy.[1] In his opening address to the Arlon days of 1964, of which he had accepted the presidency, Janne formulated his endorsements in the following way:

> I would like to congratulate the promoters, and especially pay tribute to the effort that has been done for many years by Professor Papy who, in addition to his activity as a scientific creator, has made himself an apostle—the word will not shock him—of the current reform. His effort has an incontestable international influence, and I believe, is currently undisputed. … The CBPM … did an extraordinary effort to spread the new mathematics and a pedagogy of its teaching. (Janne 1964, pp. 9–10)

For this and other more intrinsic reasons, the Arlon days, held each year at the beginning of the summer holydays, became more and more successful, with about 600 participants attending at its peak. Most of those who attended were Belgian mathematics teachers, but there were also some university professors, inspectors, political officials, and foreign guests. From the third to the tenth edition the following themes were programmed: Groups (1961), vector spaces (1962), exterior algebra and determinants (1963), new paths in the teaching of analysis (1964), the Euclidean vector plane (1965), the teaching of mathematical analysis in the second year of the scientific stream (1966), the teaching of integral calculus in the first year of the scientific stream (1967), and the position of calculation in a modern teaching of mathematics (1968). The insights and materials of the Arlon days were further disseminated by working groups which were coordinated by the CBPM and locally led by benevolent instructors. These working groups, more than 20 in total, were active in all main Belgian cities and reached yearly up to 3000 teachers (Holvoet 1968). In relative terms, however, this was still a very small minority of the Belgian teaching staff for mathematics (Adé 1973–1974).

The days of Arlon had great impact, even outside Belgium. In an interview from the 1980s, Piet Vredenduin, a prominent mathematics educator from the Netherlands who participated to the courses as a foreign guest, looked back:

> I have learned a lot in Belgium. They were ahead of us. Every year Papy organized a weeklong course on modern subjects in Arlon. These were excellent. (Goffree 1985, p. 163)

For Belgian mathematics teachers the Arlon days were not just one of the many in-service training courses focusing on new mathematics and its didactics. The days have been described as an exciting experience, connecting many people who felt themselves being part of a big and ambitious project, across the linguistic and ideological boundaries which were still strongly present in Belgian society. At that time (and still now) most schools in Belgium belong to one of two mighty educational networks, one representing the "free" (usually Catholic) schools, and the other, uniting the publicly-run schools. Of course, mobilizing actors of these two educational networks and of the two main linguistic Belgian communities (the French- and the Dutch-speaking) had a strategic-political dimension—Papy needed all these actors' and their organizations' support for his reform to succeed—but as Vanhamme (1991) testified, the unifying power of mathematics was also one of Papy's profound convictions.

Implementation and Controversy

During the mid-1960s the reform movement was in a winning mood. In the school year 1963–1964, a working group of the CBPM had developed an improved version of the experimental programs from 1961 (and 1962) for the lower secondary level (CBPM 1964), the structure of which was of course very similar to that of Papy's *Mathématique Moderne*. At that time, the original programs for respectively the first, second, and third years were already run in about 100, 20, and 5 secondary school classes, respectively. In his opening lecture to the sixth edition of the Arlon days (1964),

[1] Between 1963 and 1964, Papy himself had been a member of the Belgian Senate for the Socialist Party.

Minister Janne had officially allowed the improved modern mathematics program as an alternative for the traditional mathematics curriculum for 12–15-year-olds:

> This program has the great advantage of being fully taught and of taking into account the experiments that have already been realized … In view of the quality of this working group and of the evidence provided by the previous experiments, *I decided to authorize this program, of course on an experimental and optional basis, as early as the next school year* [1964–1965]. (Janne 1964, p. 11)

The decision to make the optional program compulsory in the first years of secondary education from September 1, 1968, was made by Janne as Minister of Education in an outgoing government and announced in a circular of May 14, 1965, which also urged teachers to prepare (Janne 1965).

However, the success story of Papy and his CBPM during the 1960s did not mean that all members of the Belgian mathematics education community were in favor of the ongoing reform. Opposition came, for example, from *Mathématique et Technique* (MATEC) [Mathematics and Technique], an organization of mathematics teachers in technical schools, who deplored the loss of geometrical representations and the emphasis on logic and abstract concepts. Papy's curriculum proposals isolated mathematics from other courses such as technical drawing, for which understanding of spatial forms and representations was required. At the academic level, the opposition to Papy's reform was led by Léon Derwidué (1914–1971), a professor at the Faculty of Engineering of the University of Mons. Derwidué rejected what he believed to be a one-sided emphasis on the axiomatic, logical, and structural aspects of mathematics, and pleaded for a renewal that took into account the needs of the engineers and other users of mathematics, whose advice had been disregarded in the reforms (Derwidué 1962). Moreover, Derwidué did not see any good argument for teaching set theory to young children, a theory that "serves first and foremost to provide a logical, solid, and precise basis for reasoning, a role that can only be appreciated by sufficiently advanced minds" (Derwidué 1962, p. 6). He doubted whether this theory—as well as other modern mathematics content—could be used for the exposition of many classical topics whose knowledge he still considered essential (from the point of view of the users of mathematics). And even for students who would later devote themselves to pure mathematics, Derwidué was not convinced that learning mathematics on the basis of a Bourbaki-style presentation provided a good starting point:

> Moreover, is it not appropriate to start the mathematical education of every teenager, even those predestined to the purest mathematics, with the useful and concrete aspect, which, in its further development, will naturally reveal problems for which rigorous treatment appears necessary? (Derwidué 1962, p. 10)

Opponents of the reform, however, had little chance in the mid-1960s. Papy, backed by a loyal and powerful base in politics and academia, succeeded in vigorously defending "his" modern mathematics, both in academic and more popular forums, and did not hesitate to ridicule his opponents' views. Smet and Vannecke (2002) described how Papy, at a symposium under the slogan "Ahead with the reform," organized on December 1, 1966, in the Brussels Palace of Congresses, in front of 1700 participants, vociferously denounced traditional mathematics as obsolete and worthless. Some compared Papy's discourse with that expected at a political meeting, but he did not convince everyone, even though his experiments and other actions received ample attention in newspapers and popular magazines of the time. By labelling his approach as "mathématique moderne," he already condemned any opponent as being outdated or even reactionary. It also reduced the debate to a dilemma, as formulated by Papy in one of his famous one-liners: "la mathématique de Papy ou les mathématiques de papa?" [Papy's mathematics or daddy's mathematics?][2] (Mawhin 2004). Papy divided the Belgian (and parts of the international) mathematics education community: He left no one neutral—he created dedicated followers as well as fierce opponents—"Papy'ists" and "anti-Papy'ists" (Colot 1969).

[2] In French "la mathématique," singular, refers to the unified (modern) mathematics, in contrast to "les mathématiques," plural, referring to (traditional) mathematics as an umbrella term for several subdisciplines with little or no interrelationships.

When the due date of the reform approached, a drastic change in the political climate seemed to reverse Papy's chances. In March 1966, a new government was formed and the liberal Frans Grootjans became Minister of National Education. Papy's (socialist) political family, to which also the aforementioned Ministers Larock and Janne belonged, did not participate in this new government. Grootjans then had to take the ultimate decision about the mandatory introduction of modern mathematics, as announced by Janne in 1965. Soon, some doubts about the wisdom of going ahead with the decision arose. In a short communication "Modern mathematics for a time not in education" (*De Standaard,* December 16, 1966, p. 8), Grootjans took a reserved position. The Minister was probably influenced by a group of representatives from the Faculties of Science and Engineering. The Faculties of Science, which organized the studies in (higher) mathematics, were divided on the issue of modern mathematics in secondary schools—Brussels and Leuven were in favor, Ghent and Liège were against—but the Faculties of Engineering were also involved, as "users of mathematics," and they were unanimously against. Not surprisingly, the CBPM immediately expressed concern about a possible suspension of Janne's earlier commitment. On January 4, 1967, Grootjans clarified his position:

> You can be assured that I will not take any decision without being informed by all the authorities responsible for the teaching and application of mathematics. Only after I will be in possession of all advice on this matter, will I announce my position without prejudice. (*Mathematica & Paedagogia, 31,* 1967, p. 76)

To obtain the desired advice, Grootjans installed two national study commissions: A University Commission and a Commission for Secondary Education. The University Commission, installed on March 20, 1967, firstly had to advise the Minister about the necessity to change the current mathematics curriculum of the scientific streams to meet the needs of the university, higher education, and trade and industry (and was thus composed of representatives of these sectors,[3] but complemented with two CBPM members including Frédérique Papy). Secondly, this commission had to enumerate in detail the crucial mathematical knowledge that should be provided by the secondary level, without developing a specific curriculum. Despite strong disagreements and tensions—the delegation from the University of Ghent withdrew its cooperation after the first meeting—the University Commission reached a compromise in favor of reform on September 12, 1967 (Feusels 1979). Although the Commission's recommendations and proposals had a marked modern mathematics signature, essentially reflecting the views of Papy and his team and not those of the University of Liège, some positions were more moderate (Commission Universitaire 1967). For example, a number of topics that Papy had considered outdated or useless, such as common plane and solid figures and their properties, relationships between sides and angles in a right-angled triangle, trigonometric formulas, spherical triangles, combinatorial analysis with and without repetition, estimation of numerical expressions, and error propagation, were nevertheless recognized as essential. The Commission for Secondary Education, consisting of inspectors and informed teachers at the secondary level, was then asked to elaborate in detail the final curriculum, taking into account not only the University Commission's advice, but also the traditional curriculum, and the results of the experiment with the optional (experimental) curriculum.

The Commission for Secondary Education soon agreed on a new curriculum for the first year of the secondary school. Although Georges Papy was not personally involved in its preparation, the Commission generally followed his view—both in terms of content and method—and thus, the new curriculum strongly resembled the CBPM's experimental curriculum (CBPM 1964). It consisted of six main sections: Sets, relations, natural numbers, integers, geometry, and "acquisitions to maintain." Only in the last section, in which some basic arithmetical and geometrical knowledge and skills from the primary school were repeated, applied, and expanded in a more or less intuitive way, was there any concession to the proponents of the classical approach. On April 11, 1968, a ministerial decision con-

[3] The universities appointed their own delegation, each consisting of one representative of the Faculty of Science and one of the Faculty of Engineering Science.

firmed the generalization of this modern curriculum, from September, 1968, in the first year of the general divisions of all secondary schools run by the state (Ministerie van Nationale Opvoeding/Ministère de l'Éducation Nationale 1968). The Catholic network implemented the reform in general secondary education at the same moment, but with a slightly different program (Nationaal Verbond van het Katholiek Middelbaar Onderwijs 1968). Technical schools, which had not previously experimented with modern mathematics, started 1 year later in a similar vein (Ministère de l'Éducation Nationale 1969).

Modern Mathematics in Belgian Primary Schools

A Reform Prepared in Various Experiments

In September 1967 the CBPM started the *experiment Frédérique*, which was aimed at preparing the reform at the primary level (Papy 1970, 1971). The experiment started with a class of 6–7-year-olds in the primary section of the *École "Berkendael."* Frédérique set out the general objective as follows:

> In the attempt to renew the teaching of mathematics at the primary level that I have undertaken since September 1967, one of my main objectives is to build, with the help of children, a house of mathematics. … For the student, the unitary structure offers security and comfort, essential elements of a climate that favors the development of intelligence and knowledge in mathematics. (Papy 1971, p. 160)

More specifically, the experiment aimed at introducing children to the relational world of modern mathematics, as well as initiating them, progressively, to calculation techniques in "sets of ever richer types of numbers" (Papy 1970, p. 95). Tools to realize these goals were, in addition to Venn diagrams and arrow graphs, Cuisenaire rods, Dienes logiblocs, and the *minicomputer*.

The minicomputer was not a computer or calculator, but a new teaching aid developed by Papy. It was a two-dimensional abacus with plates that were subdivided into four square sections, each colored according to the coding system of the Cuisenaire rods (Figure 10.3). In these plates, numbers were represented in a binary way by counters that could be played up and down, corresponding to the operations of doubling and halving (Papy 1969). Although the minicomputer was primarily based on the binary number representation, it also could be used for base 10 representation of numbers (by putting different plates, each representing a digit, next to each other).

> The method we used to introduce the 6-year old child to mechanical or mental numerical computation uses the decisive advantages of the binary over any other positional numeral system, while taking into account the decimal context in which we are housed. (Papy 1969, p. 333)

In the early 1970s, Frédérique published annotated accounts of her experimental classes in a four-volume book series entitled *Les Enfants et la Mathématique* [Children and Mathematics] (Frédérique 1970–1976). The series' style—with many colorful figures intended to elicit a mathematical idea or line of thought—resembled that of Papy's *Mathématique Moderne*, but the impact was much smaller partly because Frédérique's approach was almost not reproducible by "ordinary" teachers ("Frédérique's great didactic talents enable her to achieve results with young children that will be unattainable for many," Vredenduin 1975–1976, p. 165). Nevertheless, some modern mathematics enthusiast circles outside Belgium showed interest—such as, for example, Burt Kaufman's team in the USA which developed the Comprehensive School Mathematics Program and appointed Frédérique as *director of research*, a role that she fulfilled between 1973 and 1978 (Braunfeld 1973).

In September 1968, a second and larger-scale modern mathematics experiment started— namely the Waterloo experiment, conducted in the preparatory section of the *Athénée Royal de Waterloo* and led by Louis Jeronnez (Jeronnez and Lejeune 1972a, b).

Figure 10.3 Children aged 6 at work with Papy's minicomputer. (International Visual Aid Center, Brussels, late 1960s)

> The fundamental goal of our Waterloo experiment is to promote education that can better shape students' thinking, that promotes the spirit of personal research, that encourages children to think rather than to master techniques. (Jeronnez and Lejeune 1972a, p. 69)

Although the experiment was mainly focused on mathematical reflection, a lot of attention was paid to numbers, operations, and arithmetical skills that were deliberately trained. In this respect the Waterloo experiment distinguished itself from Frédérique's approach. Students' arithmetical skills were developed and individually supported through the manipulation of the Cuisenaire rods, which played a central role in the Waterloo experiment. In both Frédérique's and in Jeronnez's experiments, attention was given to the discovery of mathematical structures at an early stage. The Waterloo experiment was highly regarded in the French-speaking Community of Belgium: To Papy's dismay, the entire in-service training of primary school teachers in state-run schools was entrusted to the Waterloo group (Papy 1979).

The experimental efforts in the French-speaking Community of Belgium resulted in new curricula in the early 1970s. The main components were logic, sets and operations on sets, relations, structures, numbers, operations on numbers and their properties, measurement, an introduction to geometry, and word problems. The general aim was to find a balance between mathematical reasoning and the development of arithmetical skills in a renewed framework of sets and relations (Jeronnez and Lejeune 1972a). In Flanders, the Dutch-speaking Community of Belgium, a series of comparable experiments led to a reform of the mathematics curricula in the second half of the 1970s. Criticism of modern mathematics in primary education, and of its implementation, did not surface until the 1980s (e.g. Feys 1982).

Modern Mathematics in Daily Primary School Practice

The compulsory introduction of modern mathematics in primary schools in the 1970s thoroughly reshaped both the content and the didactics of the discipline at that level. This impressive operation was accompanied by the publication of new textbooks and the organization of various kinds of in-service training courses for teachers and even for parents. We discuss some major changes in primary school practice resulting from the reform. Sets and relations became the main ingredients of the new approach to mathematics education, not only as learning objectives in their own right but especially also as a means to frame most of the "traditional" mathematical contents (which the children were still expected to learn).

Knowledge of numbers and arithmetic preserved their importance, but attention shifted from being able to calculate quickly and accurately, and to perform standard procedures (such as the "rule of three") to insight into number systems, operations, and structures. To promote thinking and understanding, numerical situations and operations were often represented with different tools (Venn diagrams, arrow graphs, Cuisenaire rods). To gain a better understanding of the decimal system, some addition and subtraction problems in systems with number bases other than 10 were proposed. Word problems received less attention (but were still included in the curricula). Most reformers looked down on "applications of shop, garden, and kitchen mathematics" (Barbry 1974, p. 121). To visualize problems from everyday life, children had to use the appropriate tools of modern mathematics (such as schemes based on arrow or Venn diagrams). Children were not encouraged to make their own informal visualizations of a problem situation.

Probably the most radical change took place in the teaching of geometry. The plane, represented by the symbol Π, became an "infinite set of points" and lines and geometrical figures became "subsets of Π." In particular, the hierarchical order of the different plane figures was considered to be essential. Relations, such as "all rectangles are parallelograms," were highlighted and visualized in the language of sets. In the "exploration of space"—a curriculum component for 10–12-year-olds—this trend was continued with the classification of polyhedra according to diverse criteria. Solving applied problems about geometrical figures was considered less important. Besides, the correct use of an unequivocal terminology and symbol use was considered to be of utmost importance. Therefore, inaccuracies from the pre-modern mathematics programs were eliminated. For instance, a clear distinction was made between a "circle" and a "disk." A circle only referred to the border of the plane figure, and thus its area was no longer πr^2 but 0. The course in geometry also provided an introduction to transformation geometry. New topics, such as "reflection through an axis" and "axes of symmetry," had to prepare children to an extensive study of transformation geometry at the secondary level.

From the age of 10 onward, children were introduced to what was called *logical thinking*. In this special part of the mathematics course, they were expected to learn to use correctly the connectives "and" and "or" ("and/or") and their negation by the logical operator "not." In a next phase, children were also trained in the correct use of expressions such as "at least," "at most," "not all," "only if," "if and only if" and so on. Dienes logiblocs (a set of objects with restricted and well-defined features: rectangle, triangle, or disk; yellow, blue, or red; small or large, and thick or thin), with which all kinds of sorting and classification activities ("logical games") were devised, were a popular teaching material for promoting logical thinking:

> These blocks are used systematically all year round. This material is just fantastic. The little children can play with them as much as they want and structure little by little. The child gains a lot of experiences because he or she constantly discovers new aspects. It is exactly that self-discovery aspect that I have learned to appreciate in this new approach. (Mogensen 1970–1971, p. 241)

Although the Belgian curricula for primary mathematics were seriously affected by modern mathematics, it is unclear how drastically day-to-day teaching practices for mathematics were actually affected by it. It is apparent that computational and measurement techniques as well as word problem solving—key parts of the "old curriculum"—were not dropped by primary school teachers during the period of modern mathematics (especially not in Flanders and in the Catholic network where the influence of Papy and his collaborators tended to be less strong) (see, e.g. Verschaffel 2004). These skills were still considered important, although it was less evident that they needed to be integrated into the philosophy of modern mathematics.

Discussion

In the late 1950s, the Belgian modern mathematics movement found its leader in the strong personality of Georges Papy, professor of algebra at the Brussels University. Papy designed and carried out audacious experiments, developed new curricula and teaching materials, and engaged teachers through large-scale in-service education programs. Papy's actions were coordinated by the Belgian Centre for Mathematics Pedagogy, which had been founded in 1961, and received ample attention in the international mathematics education community. With the founding of his Centre, Papy was about 10 years ahead of similar institutes that were created in other Western European countries, such as France (IREMs), Germany (IDM), and the Netherlands (IOWO). However, although "study" of mathematics teaching was explicitly mentioned in the Centre's *Articles*, fundamental psychologically oriented research was not Papy's trademark: Papy wanted to move quickly to improve the teaching of (modern) mathematics.

In 1963 Papy published the first volume of the groundbreaking textbook series *Mathématique Moderne*, based on his experimental trajectory and intended for the teaching of modern mathematics to 12–18-year-olds. Inspired by the work of Bourbaki, Papy reshaped the content of secondary school mathematics by basing it on the unifying themes of sets, relations, and algebraic structures. Meanwhile, he proposed an innovative pedagogy using multicolored arrow graphs, playful drawings, and non-verbal proofs by means of film strips. In contrast to other influential textbooks of the time, such as those produced by the School Mathematics Study Group in the USA or the School Mathematics Project in England, Papy's textbooks were only used for teaching in experimental classes. When from 1968 to 1969 onward modern mathematics was made compulsory in Belgian secondary schools, the official programs were different and less ambitious than those developed by Papy and his team. Likely, this is a main reason why the series remained incomplete; in particular, it did not provide the necessary material for teaching modern mathematics at the upper secondary level. The series produced by Papy served as a major source of inspiration, both in terms of content and style, for mathematics educators and textbook developers during the 1960s and early 1970s, the period in which the modern mathematics reform was prepared and implemented in several countries.

The implementation of modern mathematics at the secondary level was preceded by a process of about 10 years of experimentation. The experiments, which basically examined whether a new subject, a particular curriculum, or a specific approach was feasible at a certain age with certain students, were always considered successful by those in charge. There have never been thorough, comparative evaluations of the extent to which specific educational goals were met. The final introduction of the reform at the secondary level did not happen in a serene atmosphere; in primary education, the controversy was less strong or even non-existent. For more than 20 years, modern mathematics was the dominant paradigm for the teaching and learning of mathematics in Belgium. Proper notations and symbols, the use of the right jargon, and theory development received increased attention, barriers between mathematical subdomains were largely eliminated, and geometry education was redirected toward transformation and vector geometry.

During the 1960s and 1970s, the vast majority of teachers and educators in Belgium expressed little or no criticism of modern mathematics. Although not necessarily inclined to reform, most remained silent; critics could not count on much support anyway. In the 1980s, partly influenced by international developments, criticism swelled, both with respect to the Bourbaki ideology for teaching mathematics and with respect to the way modern mathematics was implemented in Belgian schools. The criticisms of the early 1980s sounded loudest and sharpest at the primary level (e.g. Feys 1982), where modern mathematics was introduced last and probably least thoughtfully. It paved the way for the collapse of modern mathematics in Belgium, both in primary and secondary education, for a "reform of the reform," and for the emergence of new visions on teaching and learning mathematics in the 1990s.

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