Processes of Mathematical Thinking Competency in Interactions with a Digital Tool

Mathilde Kjær Pedersen

1 Introduction

Around the turn of the millennium, the term "mathematical competence" entered the discussion of mastering mathematics as a more comprehensive concept than procedures, skills, knowledge and understanding (e.g., Niss & Højgaard, 2019; Stacey, 2010). Around the same time, the role of digital technologies significantly increased in the teaching and learning of mathematics (Trouche et al., 2013). Studies of the conceptualization of specific mathematical concepts in relation to competencies and the use of digital tools have been carried out (e.g., Kendal & Stacey, 2000; Weigand & Bichler, 2010). With the discussion of mathematical digital competencies, Geraniou and Jankvist (2019) address the requirement students face to draw on both mathematical and digital competencies in learning situations. To capture the students' mathematical competencies, Geraniou and Jankvist (2019) use the Danish mathematical competency framework, referred to as KOM, i.e., a framework describing what it means to master mathematics (Niss & Højgaard, 2019). Furthermore, to describe the interplay of mathematical and digital competencies, the authors apply the theoretical perspectives of Drijvers et al. (2013) instrumental genesis and Vergnuad's (2009) conceptual fields.

M. K. Pedersen (✉)
Danish School of Education, Arhus University, Copenhagen, Denmark
e-mail: mkp@edu.au.dk

U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_4

In their analysis, Geraniou and Jankvist (2019) address all the mathematical competencies in the KOM framework except the mathematical thinking competency. Without entering the discussion of mathematical digital competencies, I find it interesting to investigate the interplay between students' mathematical thinking competency and their use of digital technologies. However, since mathematical competencies cannot be developed or exercised without working with a mathematical subject matter, and since aspects of a mathematical competency appear differently depending on the level of mathematics education (Niss & Højgaard, 2019), it is necessary to specify the contextual setting of the specific mathematical thinking competency I will investigate.

In the subject of mathematics in upper secondary school, differential calculus plays an important role. Digital technologies, such as computer algebra systems (CAS) and dynamic geometry systems (DGS), can allow teachers to teach and students to study concepts of differentiability in new ways (Hohenwarter et al., 2008). However, incorporating digital tools into the learning of mathematical concepts can also lead to disasters, in which students objectify CAS procedures as mathematical objects (Jankvist et al., 2019). Niss (2016) argues that digital technologies may serve to enhance or replace mathematical competencies, depending on for what, when and how they are used. Contributing to the discussion of the interplay between mathematical competencies and the use of digital technologies, I address the following questions: (1) *Which processes of the mathematical thinking competency can be identified as part of students' work with instances of differentiability and non-differentiability?* (2) *How can these processes of the students' mathematical thinking competency interact with the students' use of a given digital tool?*

For this investigation, I present an empirical example of two students working with the concept of differentiability using both the dynamic graphic window and the CAS window of TI-*n*spire. Addressing the first question, I analyze the case through the lens of the mathematical thinking competency in order to identify the processes by which this particular competency appears in this case. When analyzing the students' mathematical competencies, the analyses can benefit from the use of other theoretical frameworks and constructs within mathematics education research (Jankvist & Niss, 2015). I also consider different perspectives on the use of digital technologies in mathematics education when analyzing the students' use of these tools. The research practice *Networking of Theories* offers different strategies for networking theoretical approaches of mathematics education research (Prediger et al., 2008). Thus, to investigate the second question, I aim to illustrate the interaction between the students' mathematical thinking competency and their use of a digital tool by applying the networking strategy *combining*, using key elements of the theoretical perspectives of instrumental genesis (Drijvers et al., 2013), conceptual fields (Vergnaud, 2009) and semiotic mediation (Bussi & Mariotti, 2008). In the following three sub-sections, I present the theoretical perspectives included in my analyses.

2 The Mathematical Thinking Competency of the KOM Framework

The KOM framework[1] is a set of descriptions for mastering mathematics across institutional levels and mathematical topics (Niss et al., 2016). In the KOM framework, possessing overall mathematical competence is defined as "someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations" (Niss & Højgaard, 2019, p. 12). In the framework, mathematical competence is divided into eight distinct but mutually linked mathematical competencies. A mathematical competency is defined in relation to a specific sort of mathematical challenge, in contrast to general mathematical competence, which includes a variety of mathematical challenges.

In the KOM framework, the mathematical thinking competency includes the processes of engaging in and reflecting upon mathematical inquiry (Niss & Højgaard, 2019). To be more specific, it involves "being able to relate to and pose the *kinds* of generic questions that are characteristic of mathematics and relate to the nature of answers that may be expected to such questions" (Niss & Højgaard, 2019, p. 15, italics in original). I term these processes of the competency the question–answer aspect.

In line with this aspect are the processes of

> distinguishing between different types and roles of mathematical statements (including definitions, if-then claims, universal claims, existence claims, statements concerning singular cases, and conjectures), and navigating with regard to the role of logical connectives and quantifiers in such statements, be they propositions or predicates (Niss & Højgaard, 2019, p. 15).

For instance, to possess elements of the question–answer aspect, one needs to know the differences between the mathematical claims. I term these processes the mathematical statements aspect.

Furthermore, the mathematical thinking competency includes the process of "relating to the varying scope, within different contexts, of a mathematical concept or term" (Niss & Højgaard, 2019, p. 15), which I term the scope of concept aspect. In relation to differentiability, this could include the meanings of differentiability for a given point, a given interval or a function as a whole. Moreover, it could include the different views of differentiability, from whether a function's graph is smooth to the $\varepsilon-\delta$ definition. Lastly, the competency involves "relating to and proposing "abstractions" of concepts and theories and "generalization" of claims (including theorems and formulae) as processes in mathematical activity" (Niss & Højgaard, 2019, p. 15, quotation marks in original). I term these processes the generalization–abstraction aspect. With the aspects of scope of concept and generalization–abstraction, the mathematical thinking competency concerns mathematical conceptualization.

[1] In this section, the mathematical thinking competency of the KOM framework is in focus, but for the chapter to be readable on its own, some of the concepts of the KOM framework are repeated from Chap. 2 (Niss & Jankvist, 2022) of this book.

3 Instrumental Genesis and Conceptual Fields

The perspective of instrumental genesis describes the complex process of transforming a tool into a useful mathematical instrument (Guin & Trouche, 1998); "useful" in this context refers to the tool's ability to help the user achieve an aim. Drijvers and colleagues describe instrumental genesis in terms of three dualities. The first duality is artefact-instrument, which distinguishes between the tool itself as a physical object (the artefact) and the tool as a psychological construct (the instrument) (Guin & Trouche, 1998). The second duality is instrumentation-instrumentalization, which concerns the direction of how the user interacts with the artefact. Instrumentation refers to how the artefact's configuration and features shape the user's way of thinking and doing. In contrast, instrumentalization refers to the user's way of thinking that directs the use of the artefact. (Drijvers et al., 2013) The third duality is technique-scheme, which, from a practical point of view, distinguishes between observable gestures (techniques) and unobservable cognitive structures that guide these techniques (schemes) (Drijvers et al., 2013).

The notion of scheme comes from Vergnaud's (2009) theory of conceptual fields. A conceptual field is a cognitive structure consisting of mathematical concepts and situations associated with each other, and "a scheme is *the invariant organization of activity for a certain class of situations*" (Vergnaud, 2009, p. 88, italics in original). In the development of mathematical knowledge, Vergnaud (2009) distinguishes between operational and predicative forms of knowledge. The operational form is the knowledge of doing, and the predicative form is the knowledge of articulation. Schemes are part of the operative form of knowledge, whereas language and symbols are part of the predicative form. Schemes consist of several aspects, one of which involves the two operational invariants: concepts-in-action and theorems-in-action. Concepts-in-action are the concepts we associate with and find relevant in the given situation. Theorems-in-action are propositions—considered true, but not necessarily articulated—stating which activities we can carry out with the concepts-in-action (Vergnaud, 2009).

Considering the concept of differentiability from the perspective of conceptual fields, differentiability builds on and connects to other concepts, such as linearity, slope, secant, tangent, limit and the derivative among others, all of which constitute their own conceptual fields. Furthermore, with the notions of concepts and theorems-in-action, a conceptual field of differentiability can include different ways of understanding differentiability. It is these relations and conceptualizations within the conceptual fields that I find particularly relevant for the mathematical thinking competency's scope of concept aspect (unfolded in the previous section). Furthermore, the dualities of instrumental genesis and the two-way interaction between user and tool can provide a deeper insight into the students' way of thinking and therefore their mathematical thinking competency.

4 Semiotic Mediation

The perspective of semiotic mediation focuses on the involvement of an artefact (understood in the same sense as in the perspective of instrumental genesis) as a tool of semiotic mediation in a mathematics teaching and learning setting (Bussi & Mariotti, 2008). A tool of semiotic mediation is an artefact the teacher intentionally uses to mediate specific mathematical content through a didactical sequence. The word "intentionally" is important here, as one can never be certain that the students using the artefact will infer the teacher's intended mathematical meanings (Bussi & Mariotti, 2008). This ultimately means that the artefact acquires a two-fold aim. It should be both an aid for the students to solve specific tasks and a tool of semiotic mediation related to specific mathematical knowledge (Bussi & Mariotti, 2008).

To account for this two-fold aim, Bussi and Mariotti (2008) distinguish between three categories of signs that indicate a student's progress from personal to mathematical meaning. The first category is artefact signs, which originate from the activities carried out with the artefact and are of personal meaning, based on experience. The second category is mathematical signs, which, in contrast, are signs of mathematical meaning related to the given mathematical content. The third category is pivot signs, which refer both to activities carried out with the artefact and to a mathematical domain. They function as a pivot in the progress from personal to mathematical meaning. Signs include different gestures, drawings and written and oral language (Bussi & Mariotti, 2008). With the framework of signs used in the a posteriori analysis of a teaching sequence, the process of semiotic mediation can bring forth aspects of students' making-meaning when they interact with an artefact.

5 Networking of Theories and the Roles of the Selected Theoretical Perspectives

Networking of theories is a research practice developed to make different theoretical and methodical perspectives in mathematics education research communicate with each other. It offers strategies for networking on a spectrum according to the degree of integration, from understanding others and making understandable to integrating locally and synthesizing (Prediger et al., 2008). Two of these strategies, combining and coordinating, are typically used to study an empirical phenomenon in more detail than could be achieved using only one perspective; but they are used in different ways. Combining is when two juxtaposed analyses use different theoretical lenses to capture different aspects of the same empirical phenomena. In contrast, coordinating is when a conceptual framework consists of well-fitting elements from different theoretical approaches (Bikner-Ahsbahs & Prediger, 2010). Therefore, coordinating requires that the cores of the theoretical approaches in question are more compatible. Hence, an important element for networking of theories is the focus on the core of a theoretical approach (Prediger et al., 2008).

Considering the core of the KOM framework, the authors behind the framework write in relation to the notion of mathematical competency:

> [t]he core of a mathematical competency is the enactment of mathematics in contexts and situations that present a certain kind of challenge. (Niss & Højgaard, 2019, p. 19)

The entire framework is a broad description of mathematics as a practice and is not anchored in a given theoretical perspective (Niss & Højgaard, 2019). From a network perspective, this may create difficulties, as one cannot go back to its origins to determine how compatible it is with the cores of the other theoretical approaches in question. However, it is possible to combine the mathematical thinking competency with parallel analyses using other lenses to elaborate the empirical phenomena of students exercising the mathematical thinking competency in interactions with a given digital tool.

As the KOM framework focuses on an individual's cognitive actions for doing and dealing with mathematics (Niss & Højgaard, 2019), the theoretical perspectives to help elaborate the mathematical thinking competency in interplay with the use of digital tools should also focus on the individual's cognition and actions. The theoretical perspectives of instrumental genesis (Drijvers et al., 2013) and conceptual fields (Vergnaud, 2009) have previously been proven suitable to describe the interplay between students' possession of mathematical competencies and the use of digital tools (Geraniou & Jankvist, 2019). Furthermore, I argue that the theoretical perspective of semiotic mediation offers a terminology to help us gain deeper insights into the students' meaning-making of a digital tool.

Instrumental genesis (Drijvers et al., 2013) and semiotic mediation (Bussi & Mariotti, 2008) are both based on the instrumental approach. The instrumental approach involves a Vygotskian perspective that emphasizes the use of instruments in learning processes but that also uses the Piagetian notion of scheme (Verillon & Rabardel, 1995). The notion of scheme is re-elaborated by Vergnaud (1996), who also draws on a Vygotskian perspective, arguing how notions from Piaget and Vygotsky complement each other. Thus, the cores of instrumental genesis (Drijvers et al., 2013), conceptual fields (Vergnaud, 2009) and semiotic mediation (Bussi & Mariotti, 2008) are compatible.

Since these theoretical approaches are compatible, they could potentially be used in a coordinated analysis of students' interactions with digital tools in a mathematics education setting. However, I have chosen to use these theoretical perspectives to conduct juxtaposed analyses using the networking strategy of combining to study the phenomenon of students exercising the mathematical thinking competency when working with digital tools. Using the mathematical thinking competency as a coarse-grained framework, I analyze the empirical case presented below to focus the attention on the students' processes of mathematical thinking competency for the subsequent finer-grained combined analyses.

In the following two sections, I account for the method of the empirical study and present a case in which I illustrate how the students exercise the mathematical thinking competency through their interaction with a digital tool.

6 Method and Selection of the Case

The case presented in this chapter is taken from a larger empirical study carried out in autumn 2020 in the classical stream of Danish upper secondary school, called STX. In the empirical study, 29 students participated in two lessons on differential calculus, each lesson lasting 90 minutes. The students collaborated in groups of two or three in order to capture their thinking through their mutual discussions. The students worked from a premade TI-*n*spire worksheet and wrote their answers in an appurtenant Word document, both of which were screencast recorded. The students were also recorded using their webcams so that their participation and any relevant hand gesticulations could be analyzed. The tasks on which the students worked represented the main exercises of the two lessons in the empirical study. All pairs/groups worked on these tasks for between 20 and 60 minutes, though their work was interrupted by various events, such as the first lesson ending, the teacher giving an introduction, having to engage in-class discussion, waiting for help or discussing topics unrelated to mathematics.

The video sequences of the students working on these tasks were first coded with the aspects of the mathematical thinking competency (described above) to identify relevant pieces of data. Analysis 1 below is an elaboration of this process for the given case. Based on this analysis, the case of Karen and Lily was selected, because the scope of concept aspect of the mathematical thinking competency was very clear in the initial coding, due to the students being persistent in their intuitive idea of differentiability (cf. Analysis 1). In order to elaborate on how the students interacted with and created meaning from the digital tool in relation to the scope of concept aspect of the mathematical thinking competency, the case was then analyzed using the perspectives of instrumental genesis and conceptual fields (Analysis 2) and semiotic mediation (Analysis 3).

7 Data: Exploring Differentiability Using Secant Lines

The case presents two students, Karen and Lily, working on a dynamic TI-*n*spire worksheet. This worksheet asks the students to investigate whether a given function f (depicted by the blue graph in Fig. 1) is differentiable for given values, where f is defined by

$$f(x) = \begin{cases} \frac{1}{15}x^3 - 1.2x^2 + 5.4x + 1.8, & -0.5 \leq x \leq 9 \\ 1.8, & 9 < x \leq 15 \end{cases}$$

With two sliders in the top right corner, the students can move x_0 on the x-axis and change the difference Δx. In this way, students can observe differentiability as a numerical approximation to the slope of the tangent line (Hohenwarter et al., 2008).

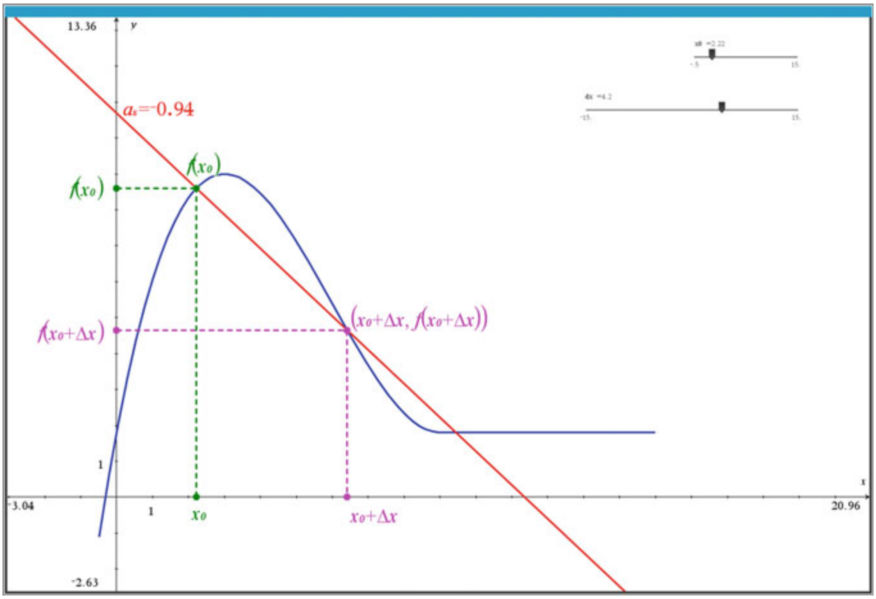

Fig. 1 Snapshot from TI-nspire, the interactive representation of a function (the blue graph) and a changeable secant line (the red graph)

In the following dialogue, Karen and Lily work with the tool to investigate differentiability for $x_0 = 9$.

01. Lily: What does it mean for it to be differentiable? … When it is differentiable… we should read about it at some point. That, when it has such a tip, then it is not differentiable.
02. Karen: Yes, it is when it is curved. You should be able to walk from a to b and such [she grabs her book and reads out sections of the text]. Continuity. Called continuity when its graph is connected… from a… [she moves on to the paragraph on differentiability] and here, it should be without corners.[2]
03. Lily: [Looks at the graph in Fig. 1] But I guess there are no corners in this one… But, when it is differentiable, then it is without corners, so there cannot be such a tip on it. … Because then it could slope differently-ish like this [she holds one of her hands in different directions].
04. Karen: But this one [the graph in Fig. 1] is soft.
05. Lily: Yes, there are no corners.

Karen and Lily cannot work out how the information from the book can help them investigate differentiability using the tool, and they ask for help. Through guidance, Karen sets $x_0 = 9$ and controls the slider for Δx. First, she does it for negative Δx, but, because the slider jumps in small intervals, she ends up typing $\Delta x = -0.1$, for

[2] I translate the Danish word “knæk” as “corners”, to illustrate a graph having one or more sharp bends but still being connected.

which the secant slope is 0. Afterward, she types $\Delta x = 0.1$, for which the secant slope is also 0.

06. Mathilde: There [the secant slope] is also 0. So, now it approaches 0 from right and left, so it then approaches the same value. In this case, 0.
07. Karen: Ah, so it is opposite, kind of like a mirror-ish?
08. Mathilde: Yes, let's say we worked with something that tended to 0 from the one side, but
09. Karen: something different … then it would not be differentiable. Ok. For example, if it has a corner.

The group moves on to investigate differentiability for $x_0 = 1$. Karen types $\Delta x = -0.1$, for which the secant slope is 3.3, and then she types $\Delta x = 0.1$, for which the slope is 3.1.

10. Karen: Then, I guess, it is not differentiable. But, that does not make any sense.
11. Lily: No, I don't think it does. But, well, I guess it approaches the same, but it is not quite the same. Is it just because there is a little bit of difference in the slope. Well, they both approach 3 [she mumbles something]… but does it have to be the exact same? What if we take 0.01?
12. Karen: I think so. Let us just try. [She types in $\Delta x = 0.01$, and the secant slope is 3.19.]
13. Lily: So it does… it does move closer to. It does approach…
14. Karen: But it is just not. I think they have to be right on the opposite side. … I do not feel, it would make any sense if it is not [differentiable].
15. Lily: Yes, because there are no corners, or jumps or anything.

From this, Karen and Lily presume differentiability, but, to be sure, they ask their teacher, who suggests they go even closer. Setting $\Delta x = \pm 0.0001$, they get the secant slopes to 3.2 on both sides and confirm their presumption. The group moves on to another function, where they have to investigate differentiability for $x_0 = 10$. For $\Delta x = 0.1$, the secant slope is -0.1, and for Δx to -0.1, the secant slope is 0.

16. Karen: Oops, there it is not. Should we try more 0's, or what?

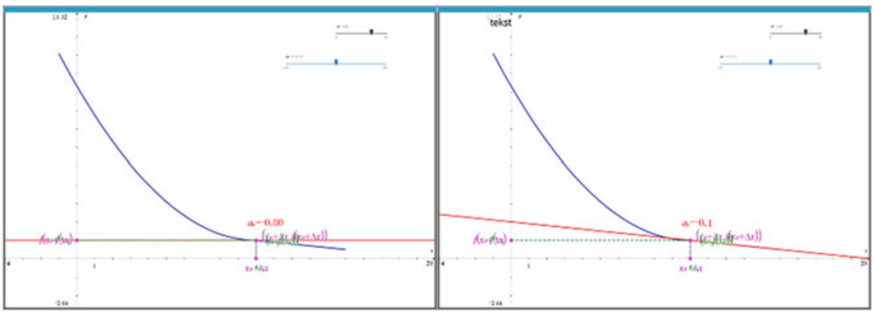

Fig. 2 Non-differentiable function at $x = 10$. The secant slope is 0 for $\Delta x = 0.00001$ (left), and the secant slope is -0.1 for $\Delta x = -0.00001$ (right)

Det vil sige at alle er differentiable, fordi den er kontinuert samt uden knæk.
Hældning for $x_0 = 1$ er 3.2
Hældning for $x_0 = 3$ er 0.0
Hældning for $x_0 = 9$ er 0.0

Fig. 3 Estimated slopes for the given x-values. The text translates as "That means that they are all differentiable, because it is continuous and without corners. Slope for $x_0 = 1$ is 3.2 Slope for $x_0 = 3$ is 0.0 Slope for $x_0 = 9$ is 0.0"

For $\Delta x = \pm 0.00001$ the slope is still 0 (Fig. 2 left), respectively -0.1 (Fig. 2 right).

Karen and Lily seem to see a little corner around $x = 10$, and they try with $\Delta x = \pm 0.000001$. The secant slope is still 0 for positive Δx and -0.1 for negative Δx, and they conclude non-differentiability. Afterward, they get guidelines to calculate the derivative with TI-*n*spire CAS. Karen types in the given command for the first function with the value of $x_0 = 1$, to which TI-*n*spire CAS gives the result 3.2.

17. Karen: 3.2, ay! Ah, that was it… we did not calculate it. Should we then just [calculate it]?

They go back to the graphic window again and estimate their values for the limit of the secant slope for Δx approaching 0 (Fig. 3).

Before using CAS on the last example, the students go back to the graphic investigation and write out their arguments for their conclusion of non-differentiability for $x_0 = 10$ (Fig. 4).

18. Karen: x is 10, and there we got that it was not differentiable in ours.
19. Lily: Oh, yes. Should it not say undefined then?
20. Karen: I think so.

Karen types in the command to which the output is "undef" (the last calculation in Fig. 5).

21. Lily: Great. It is so nice when it works.
22. Karen: Great, yeah. It is the greatest when it actually works, and you finally … well, when you do not get it. It is really like ups and downs.

Vi fandt ud af, at $x_0 = 2$ er differentiabel (differentiabelkvotient = -1.33), men at $x_0 =$ 10 ikke er differentiabel
Vi fandt ud af at hældningen for sekanten ved $x_0 = 10$ er forskellig, når man kom fra en positiv retning -0.1, men når man kommer fra en negativ retning, er hældningen 0

Fig. 4 The students' argumentation for differentiability and non-differentiability. The text translates as "We found out that $x_0 = 2$ is differentiable (differential quotient $= -1.33$), but that $x_0 = 10$ is not differentiable. We found out that the slope of the secant for $x_0 = 10$ is unequal when you come from a positive direction -0.1, but, when you come from a negative direction, the slope is 0"

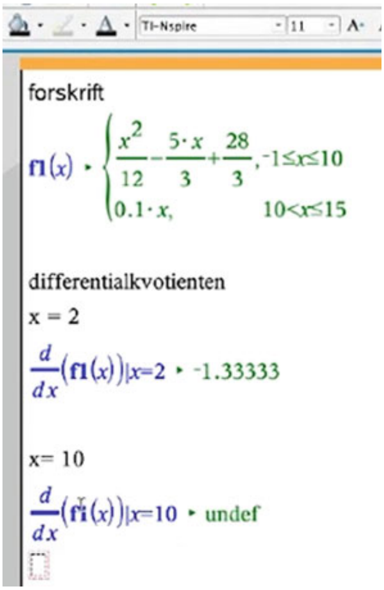

Fig. 5 Calculations of the derivative for specific values in TI-nspire CAS

8 Analysis 1: Exercised Processes of the Mathematical Thinking Competency

To identify which processes of the mathematical thinking competency the students exercise, I analyze the case through the lens of the four aspects of the mathematical thinking competency. At first, Lily asks the question of what differentiability means (Line 01), which illustrates processes of the question–answer aspect. She recognizes the task as a mathematical question and relates this to the nature of the expected answer to such a task. This prompts her to search for a definition or explanation of differentiability, so they know which results they can interpret as "yes, it is" or "no, it is not differentiable". The students do not exercise the process of posing or relating to generic mathematical questions, which are also parts of the question–answer aspect.

In the students' search for an answer, relying on the explanation of differentiability as a function whose graph has no corners (Line 02), the two students exercise processes of the scope of concept aspect. For instance, Karen connects her understanding of differentiability as a graph with no corners to the actions carried out with the tool (Line 09). Here, she relates the scope of differentiability when simply looking at the graph of the function to investigate the limit of the secant slopes graphically.

Before starting their investigations, they intuitively consider the graph as being soft with no corners (Line 03–05). Hence, they presume that the function is differentiable. When they are to determine if the function is differentiable for $x_0 = 1$, they calculate the secant slopes to be 3.3, respectively, 3.1 for $\Delta x = \pm 0.1$. This makes Karen conclude non-differentiability, despite it looks like it has no corners (Line 10). Lily tries to relate the two understandings of differentiability by interpreting the dynamic output of the tool, adding more 0s to Δx (Line 09–15). This relation is also seen in the instance of non-differentiability (Line 16). Throughout the actions with the tool, the students' intuitive view of differentiability as a graph with no corners guides the students' extension of the scope of the concept of differentiability.

When calculating the derivatives with CAS, they relate these results to their work with the tool (Line 17–20). Hereby, they expand their understanding of an answer of differentiability to include a result in the form of a number, or the output "undef" in the case of non-differentiability (Line 19–20). Hence, they return to the question–answer aspect of the mathematical thinking competency.

This analysis illustrates that the work with instances of differentiability and non-differentiability makes Karen and Lily exercise some processes of the question–answer aspect and some of the scope of concept aspect, but no processes of the mathematical statements or the generalization–abstraction aspect. Moreover, the students' expressions in Line 18–19 also illustrate how the students find coherence between the answers and the varying scopes. In the following analyses, I elaborate on how these processes are exercised during the students' work.

9 Analysis 2: Beginning Instrumental Genesis

In this section, I analyze the case using the perspectives of instrumental genesis and conceptual fields. In this case, the TI-*n*spire worksheet is the artefact at issue. Karen and Lily's starting point is their concept-in-action "differentiability is shown by a graph with no corners" (Line 01–02). Thereby, they have an intuitive idea of the function being differentiable (Line 03–05). This understanding becomes part of their predicative form of knowledge, but with no connection to any operative form of knowledge. As they have no theorem-in-action to draw on, they cannot infer how to act with the artefact in relation to their predicative form of knowledge and they ask for help. This illustrates the difficulties of beginning the instrumental genesis when they have no initial schemes to rely on.

The instrumentation of the artefact leads Karen to a theorem-in-action, saying that, for a function to be differentiable at a given point, the secant slope should be the same value for both negative and positive Δx, close to the given point. This worked for the first instance ($x_0 = 9$) with $\Delta x = \pm 0.1$. Therefore, Karen and Lily use this scheme to investigate differentiability by copying the technique of setting $\Delta x = \pm 0.1$ for the next point of interest ($x_0 = 1$). As this gives them two different secant slopes, respectively, for $\Delta x = \pm 0.1$, they get confused (Line 10–15). This instrumentation leads Karen to conclude non-differentiability, even though this conclusion does not match her initial concept-in-action of differentiability being a curved graph with no corners.

Because of the mismatch between the tool-induced theorem-in-action and the book-induced concept-in-action, Lily is not sure non-differentiability is the answer.

Although she cannot infer the exact limit, her scheme builds on the secant slope getting closer to something, which is observed by her technique of adding more 0s to Δx. In Lily's case, the duality of instrumentation-instrumentalization is essential. On the one hand, it is Lily's way of thinking of "approaching" (Line 11) that directs the use of the slider for Δx. On the other hand, the configuration of the tool and the constraints of the slider have encouraged her to think this way.

With $\Delta x = \pm 0.0001$, they conclude differentiability for $x_0 = 1$, which matches the concept-in-action of "differentiability-as-no-corners". This process illustrates how Lily's actions make Karen adjust her thinking of differentiability. For Karen, the two secant slopes still have to be equal, but now in the sense of Δx close enough to 0. Both Lily's and Karen's schemes are confirmed by the instance of non-differentiability, where the two secant slopes differ from each other, keeping their respective values, regardless of how close Δx is to 0 (Line 16, Fig. 2 left and right).

In the end, the CAS calculations of the derivatives also confirm their schemes, building on the instrumentation of the dynamic environment. Getting the exact value of the derivative puts the dynamic investigations into perspective and helps them develop a predicative form of knowledge in the form of arguments for differentiability and non-differentiability, respectively (Figs. 3 and 4). Also in this situation, the instance of non-differentiability functions as a confirmation, when Lily predicts the output to be "undefined" (Line 19), and, in this way, thinks with the tool.

This analysis illustrates how the students relate to the varying scope of differentiability from simply looking at the graph to estimating the limits of the secant slopes by relating their concept-in-action of "differentiability-as-no-corners" with their developed theorems-in-action from working with the tool. Through the development of schemes for using the dynamic worksheet as an instrument for determining differentiability, the students' two views on differentiability approach a conceptualization of differentiability with a similar scope. With the students' development of predicative knowledge, they expand their view on an expected answer of differentiability to include whether or not they can determine a limit. Therefore, this process also calls for them to exercise the question–answer aspect of the mathematical thinking competency.

To explore how the students obtain sense and meaning out of their interaction with the tool, I will now analyze the case from the perspective of semiotic mediation.

10 Analysis 3: Signs of Semiotic Mediation

Like above in the analysis from the perspective of instrumental genesis, the artefact is the specific dynamic TI-*n*spire worksheet with the two sliders. From this perspective, we consider it a tool of semiotic mediation in relation to the specific tasks of investigating differentiability and the mathematical content of differentiability.

Summing up their initial work with the artefact, Karen's question "Ah, so it is opposite, kind of like a mirror-ish?" (Line 07) can be seen as an artefact sign of how she understands the secant slopes' behavior in relation to differentiability. She connects this to the situation of non-differentiability and a graph with corners (Line 09). Thus, the terms "corner" and "no-corner" become pivot signs that hinge the behavior of the secant slopes to the concept of differentiability-as-no-corners. This allows her to speak of differentiability on a more general level than just related to the specific tasks.

In the next task for $x_0 = 1$, where the secant slopes do not "mirror" around the given value, Karen concludes "Then, I guess, it is not differentiable. But, that does not make any sense." This illustrates a discrepancy between her personal sense of differentiability within the artefact and her differentiability-as-no-corners understanding. Lily, on the other hand, remains focused on the dynamic of the artefact, expressed by artefact signs like "it approaches…" (Line 12) and "it does move closer to…" (Line 14). With confirmation from their teacher, this leads to an agreement between the results in the artefact and the differentiability-as-no-corners understanding for both students, as they can see the graph has no corner at the given point. For Lily, the movements of the artefact confirm her understanding of "approaching", and for Karen, the secant slopes do "mirror" around the given point for small enough Δx.

In the last task, Karen and Lily's written answers show their move from artefact signs to mathematical signs, with the pivot sign "slope" hinging the observed in the artefact with the derivative obtained by CAS (Figs. 3 and 4). During the exercises in the dynamic artefact, Karen does not pay attention to the values of the limit but to whether the slopes are equal on both sides of the given x-value. Lily seems aware of the limit in some sense, expressed by artefact signs of "approaching". When they return to the graphic investigations after calculating the first derivative, the exact values become their argumentation for whether the function is differentiable. In Fig. 3, they use the pivot signs of "corners" and "slope", whereas in Fig. 4, they use the mathematical sign "derivative". This indicates a move from the artefact and toward a more generalized concept of differentiability. However, the episode also illustrates that knowing the CAS command simplifies determining differentiability to whether the output is a number or the undefined-respond.

This analysis illustrates that as part of the students' meaning-making of their actions with and responses from the artefact they exercise the question–answer aspect of the mathematical thinking competency. Like in analysis 2, analysis 3 shows the importance of the individual tasks in relation to each other, which makes the students exercise the scope of concept aspect. Moreover, the attention to artefact signs and mathematical signs in analysis 3 indicates an initial, yet important, process of the generalization–abstraction aspect of the mathematical thinking competency, which analysis 1 did not capture.

11 Discussion and Conclusion

By viewing the same data through different lenses, the three analyses above illustrate processes of students' mathematical thinking competency in interactions with the use of the TI-*n*spire worksheet.

Analysis 1 uncovers which aspects of the mathematical thinking competency are exercised in the given case and which are not. The apparent processes of the mathematical thinking competency that the students exercise are of the question–answer and the scope of concept aspects. It is through the students' work with multiple instances of differentiability and non-differentiability that the students get the opportunities to relate to the scope of the concept in different contexts and develop their conception of differentiability, as well as to relate to the expected answer for determining differentiability.

Analyses 2 and 3 illustrate how these processes of mathematical thinking competency interact with the students' use of the TI-*n*spire dynamic template and CAS. First, the case illustrates that the question–answer aspect is part of the instrumentation of instrumental genesis. An expectation of the interplay between the mathematical thinking competency and the use of digital tools could be that relating to an expected answer would influence the instrumentalization. Having an idea of what kind of answer to be looking for may influence how to use the tool. However, Analysis 2 of the case illustrates that relating to the expected answer develops through the instrumentation and the development of schemes. Not until then, the question–answer aspect is exercised in the instrumentalization aspect as well. Thus, in this case, instrumentation and to some extent instrumentalization interact with the exercise of the question–answer aspect.

Second, the scope of concept aspect is exercised in the duality of instrumentation and instrumentalization. The students bring new operational knowledge including new-developed schemes into each new task, which develops both the instrumental genesis and the signs of the semiotic mediation as well as the scope of the concept of differentiability. Hereby, it seems that the students work on the individual tasks and relate these to each other, but do not generalize over the different instances. Nevertheless, illustrating the development from artefact signs to pivot and initial mathematical signs, Analysis 3 indicates that the students approach the conditions for the limit to exist and for the function to be differential at a more general level. This shows that the students' work with the tool includes an initial process of the generalization–abstraction aspect toward a more generalized concept of differentiability—an aspect of the mathematical thinking competency not obvious from the perspective of the mathematical thinking competency alone.

Using the networking strategy of combining, the three analyses can be said to have separate foci. The first analysis (with the mathematical thinking competency as a coarse-grained framework) helped navigate the following analyses using the other selected theoretical perspectives. As juxtaposed analyses, each theoretical perspective adds to a networked understanding of students exercising the mathematical thinking competency while interacting with the given TI-*n*spire worksheet.

The focus on enactment in the KOM framework fits well with the focus on interaction with the tool in the perspectives of instrumental genesis (Drijvers et al., 2013) and semiotic mediation (Bussi & Mariotti, 2008) as well as with the focus on the operational form of knowledge and the notion of scheme in the perspective of conceptual fields (Vergnaud, 2009). This indicates compatibility between the KOM framework and the theoretical perspectives applied in this study, thus, a potential for using the networking strategy coordinating. Hence, the theoretical perspectives can be pieced together as a conceptual framework to study the mathematical thinking competency in interaction with the use of a given digital technology.

The three juxtaposed analyses also illustrate which processes of the mathematical thinking competencies the students do not exercise. First of all, the involved tasks do not initiate the mathematical statements aspect or the part of the generalization–abstraction aspect involving the awareness of generalization and abstraction as mathematical activities. Nevertheless, it could be expected that the students would try to generalize more explicitly over the different instances of differentiability. Furthermore, having the view on differentiability-as-no-corners, the students could have asked why discontinuous functions or functions whose graphs have sharp corners are not differentiable and in this way add to the question–answer aspect as well as to the scope of concept aspect. Yet the results do indicate how single aspects of the mathematical thinking competency can be exercised in interaction with an explorative worksheet, like the TI-*n*spire worksheet presented in this chapter. Using an explorative environment can help students investigate both positive and negative instances of given mathematical concepts, processes or relations. However, it is important that the tasks supporting the students' use of the tool guide the students' explorations, for example, by clearly stating the interesting aspects and asking them to observe specific elements of the explorative environment. Finally, the tasks should explicitly ask the students to question the concept in different contexts, for instance, how the given concept is connected to other parts of the specific topic or how the given concept could be defined in other mathematical topics.

Acknowledgements Acknowledgements to Ola Helenius, Linda Ahl and Andreas Tamborg for valuable comments on earlier versions of this chapter.

Supported by Independent Research Fund Denmark [Grant no. 8018-00062B].

References

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 483–506). Springer. https://doi.org/10.1007/978-3-642-00742-2_46

Bussi, M. G. B., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom. In L. D. English, M. B. Bussi, G. A. Jones, R. A. Lesh, B. Sriraman, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 746–783). Routledge. https://doi.org/10.4324/9780203930236.ch28

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses. *Educational Studies in Mathematics*, *82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of “mathematical digital competency.” *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Hohenwarter, M., Hohenwarter, J., Kreis, Y., & Lavicza, S. (2008, July 6–13). Teaching and calculus with free dynamic mathematics software GeoGebra [Paper presentation]. In *The 11th International Congress on Mathematical Education* (ICME 11), Monterey, Mexico.

Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). What happens when CAS procedures are objectified?—The case of “solve” and “desolve.” *Educational Studies in Mathematics, 101*(1), 67–81. https://doi.org/10.1007/s10649-019-09888-5

Jankvist, U. T., & Niss, M. (2015). A framework for designing a research-based “maths counsellor” teacher programme. *Educational Studies in Mathematics, 90*(3), 259–284. https://doi.org/10.1007/s10649-015-9629-8

Kendal, M., & Stacey, K. (2000). Acquiring the concept of derivative: Teaching and learning with multiple representations and CAS. In T. Nakahara, & M. Koyama (Eds.), *Proceeding of the 24th conference of the international group for the psychology of mathematics education (PME 24)* (Vol. 3, pp. 127–134).

Niss, M. (2016). Mathematics standards and curricula under the influence of digital affordances: Different notions, meanings and roles in different parts of the world. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 239–250). Information Age Publishing.

Niss, M., Bruder, R., Planas, N., Turner, R., & Villa-Ochoa, J. A. (2016). Survey team on: Conceptualisation of the role of competencies, knowing and knowledge in mathematics education research. *ZDM Mathematics Education, 48*(5), 611–632. https://doi.org/10.1007/s11858-016-0799-3

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jankvist, U. T. (2022). On the mathematical competencies framework and its potentials for connecting with other theoretical perspectives. In U. T. Jankvist, & E. Geraniou (Eds.), *Mathematical competencies in the digital era* (pp. 15–38). Springer. https://doi.org/10.1007/978-3-031-10141-0_2

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Stacey, K. (2010). Mathematical and scientific literacy around the world. *Journal of Science and Mathematics Education in Southeast Asia, 33*, 1–16.

Trouche, L., Drijvers, P., Gueudet, G., & Sacristán, A. I. (2013). Technology driven developments and policy implications for mathematics education. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 753–789). Springer.

Vergnaud, G. (1996). Some of Piaget’s fundamental ideas concerning didactics. *Prospects, 26*(1), 183–194. https://doi.org/10.1007/BF02195617

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101. https://doi.org/10.1007/BF03172796

Weigand, H.-G., & Bichler, E. (2010). Towards a competence model for the use of symbolic calculators in mathematics lessons: The case of functions. *ZDM Mathematics Education, 42*(7), 697–713. https://doi.org/10.1007/s11858-010-0274-5