Mathematical Competencies in the Digital Era: An Introduction

Uffe Thomas Jankvist, Eirini Geraniou, Mathilde Kjær Pedersen, Cecilie Carlsen Bach, and Rikke Maagaard Gregersen

1 Rationale Behind the Book

Mathematics education has been experiencing two rather distinct, yet related, 'paradigm shifts'. The first is to do with the massive introduction of digital technologies (DT) in the teaching and learning of the subject (e.g., Trouche et al., 2013); the second is to do with a shift from the traditional focusing on mastering of skills and knowledge to being concerned with the possession and development of *mathematical competencies* (e.g., Stacey, 2010; Stacey & Turner, 2015). This book focuses on the potential *interplay* between these two paradigm shifts by considering the connection of different theoretical perspectives, e.g., by drawing on the notion of 'networking of theories' (e.g., Bikner-Ahsbahs & Prediger, 2010; Prediger et al., 2008).

A somewhat recent Organisation for Economic Co-operation and Development (OECD, 2015) report states that merely adding 21st-century technologies to 20th-century teaching practices is rather likely to dilute the effectiveness of teaching than to inform and enhance it. This observation is in line with recent research in mathematics education; while studies from the 1980s and 1990s tend to point to the

U. T. Jankvist (✉) · M. K. Pedersen · C. C. Bach · R. M. Gregersen
Danish School of Education, Aarhus University, Copenhagen, Denmark
e-mail: utj@edu.au.dk

M. K. Pedersen
e-mail: mkp@edu.au.dk

C. C. Bach
e-mail: ccb@edu.au.dk

R. M. Gregersen
e-mail: rmg@edu.au.dk

E. Geraniou
UCL Institute of Education, UCL, London, UK
e-mail: e.geraniou@ucl.ac.uk

U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_1

positive effects of DT on teaching and learning, recent research often has a more balanced view on these effects (Drijvers, 2015; Weigand, 2014) or even point to negative effects (Jankvist et al., 2019). DT in the form of mathematical software—for example, Dynamic Geometry Systems (DGSs) or Computer Algebra Systems (CASs)—can today perform many of the mathematical tasks and processes that students traditionally are expected to do (e.g., reduction, equation solving, functional analysis, etc.). Newer research thus address the role of mathematical competencies in the digital era (e.g., Geraniou & Jankvist, 2019).

From an international perspective, Denmark offers *unique opportunities* for carrying out research on this interplay for two reasons. Firstly, Denmark is one of the few countries in the world that has gone 'all in' on introducing DT in its mathematics programmes of primary, secondary and upper secondary school. However, this has not been an altogether positive experience (Matematikkommissionen, 2016), since DT has also brought along new and unforeseen difficulties related to students' learning of mathematics (Jankvist & Misfeldt, 2015). Secondly, Denmark is one of the few countries that have competencies descriptions of its mathematics programmes all the way from primary school through upper secondary school to tertiary level, including teacher training—and all of these are based on the Danish so-called KOM framework (Niss & Højgaard, 2011, 2019; Niss & Jensen, 2002), which was also previously adopted by OECD for the Programme for International Student Assessment (PISA) (OECD, 2013). Niss and Højgaard (2011) define a *mathematical competency* as (an individual's) "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (p. 49). The KOM (competencies and the learning of mathematics) framework operates with eight distinct, yet mutually related, mathematical competencies, three second-order competencies referred to as types of overview and judgement and six didactico-pedagogical competencies for mathematics teachers.

Thus far, the discussions of using DT and developing mathematical competencies have often run on separate tracks, both within practice and within research. One reason for this may be due to many of the mathematics educational theories and results being framed within a setting of skills and knowledge rather than one of competency. New research surrounding KOM in the digital era does, however, address the potential of connecting the KOM framework with more established theoretical perspectives of mathematics education in situations involving DT. Recent research studies show that the KOM framework for example connects well with both Trouche (2005) and colleagues' instrumental approach as well as the theoretical perspective of semiotic mediations (Bartolini-Bussi & Mariotti, 2008) in relation to the use of DT as well as for example Vergnaud's (2009) theory of conceptual fields in relation to more cognitive elements of mathematical concept formation.

The chapters of this book provide further illustration of such connections. Several of the empirical cases in the book stem from the Danish educational system. Still, the book also offers a broader international perspective by, on the one hand, drawing on cases from other countries, for example, Australia, China, Costa Rica, England and Sweden and, on the other hand, having a wide range of international researchers co-author several of the chapters involving cases from the Danish educational system.

2 The Structure of the Book

Besides this introduction, the chapters of the book fall into four parts, each part addressing different aspects of the KOM framework.

The first part, Setting the Scene, consists of two introductory chapters. These chapters address the KOM framework and the potentials of connecting it with other theoretical perspectives from mathematics education research, both from a general point of view and specifically in relation to the use of DT.

The second part, which is on the eight mathematical competencies, consists of eight chapters, one for each of the mathematical competencies described in the KOM framework. These are:

- *Mathematical thinking competency*—engaging in mathematical enquiry.
- *Mathematical problem handling competency*—posing and solving mathematical problems.
- *Mathematical modelling competency*—analysing and constructing mathematical models.
- *Mathematical reasoning competency*—assessing and producing justification of mathematical claims.
- *Mathematical representation competency*—dealing with different representations of mathematical entities.
- *Mathematical symbols and formalism competency*—handling mathematical symbols and formalisms.
- *Mathematical communication competency*—communicating in, with and about mathematics.
- *Mathematical aids and tools competency*—handling material aids and tools for mathematical activity.

The third part consists of three chapters, each addressing one of the KOM framework's three types of overview and judgement. According to Niss and Højgaard, (2011), overview and judgement are *insights* that "enable the person mastering them to have a set of views allowing him or her overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture" (p. 74). The three types of overview and judgement are:

- The actual application of mathematics within other disciplines and fields of practice.
- The historical development of mathematics, seen from internal as well as from sociocultural perspectives.
- The nature of mathematics as a discipline.

The fourth part broadens the scene by providing four chapters that are of general nature in relation to the KOM framework in the digital era. Among other things, these chapters address assessment of mathematical competencies, teachers' didactico-pedagogical competencies in situations involving DT, mathematical competencies in relation to computational thinking and programming in school.

3 Introduction to Setting the Scene

In "On the mathematical competencies framework and its potentials for connecting with other theoretical perspectives", Mogens Niss and Uffe Thomas Jankvist present and discuss the KOM framework in general as well as its potentials for connecting with other theoretical perspectives. As part of this discussion, the notions of 'theory', 'theoretical framework' and 'theoretical perspectives' are examined and discussed with the purpose of placing KOM in this landscape. The chapter also provides examples of connections between the KOM framework and other theoretical perspectives, one example in terms of mathematical competencies (the modelling competency), one example in terms of the three types of overview and judgement (the historical development of mathematics) and one example in terms of the didactico-mathematical competencies (uncovering learning). Based on their discussion of the potential of connecting aspects of the KOM framework with other theoretical constructs, the authors point out that at the current stage of development perhaps 'mutual fertilisation' of the entities under consideration should be the goal rather than actual 'networking' of these entities into new frameworks.

While the first chapter does not address the role of DT explicitly in relation to the KOM framework, the second chapter by Eirini Geraniou and Morten Misfeldt does exactly this. In their chapter, "The mathematical competencies framework and digital technologies", the two authors discuss the influence as well as the potential of DT in relation to the KOM framework's eight mathematical competencies, six didactico-pedagogical competencies for teachers and three types of overview and judgement. Influenced by the notion of mathematical digital competency (MDC) by Geraniou and Jankvist, (2019), through the use of examples (from England, Sweden, and the well-known international problem—'the four-color theorem'), they argue for the need to revisit the KOM framework and add a layer that represents the interplay between technology and competencies. However, not thinking of this as a meta-competency of the tools and aids competency, but rather allowing technology to take a more central role in each of the mathematical competencies described by the KOM framework. In Geraniou's and Misfeldt's eyes, MDCs should be at par with mathematical competencies rather than being a sub-competency.

4 Introduction to The Eight Mathematical Competencies

The second part presents the eight mathematical competencies of the KOM framework in the perspective of DT use. Each chapter in this part corresponds to one of the eight mathematical competencies. Since all eight competencies are mutually related, a chapter may involve other competencies than the one in focus. All chapters present cases illustrating the interplay between the competency in play and the use of DT. Most of the chapters involve DGS, but there are also examples of CAS and various programming tools.

In "Processes of mathematical thinking competency in interactions with a digital tool", Mathilde Kjær Pedersen discusses students' mathematical thinking competency in learning situations of differentiability involving TI-nspire, which involves both DGS and CAS. In parallel, an empirical case stemming from Danish upper secondary school is analysed from the lenses of the instrumental approach (e.g., Drijvers et al., 2013), conceptual fields (Vergnaud, 2009) and semiotic mediation (Bartolini-Bussi & Mariotti, 2008). The chapter discusses which aspects of the mathematical thinking competency can be identified in the students' interaction with the DT.

In their chapter "Mathematical competencies framework meets problem-solving research in mathematics education", Mario Sánchez Aguilar, Martha Leticia García Rodríguez and William Enrique Poveda Fernández focus on the problem handling competency. The authors connect the KOM framework with other notions from mathematics education research related to problem-solving, which together provide a suitable construct for analysing problem-solving processes. Through analyses of an empirical case stemming from a virtual course for prospective lower secondary school teachers in Costa Rica, a student (i.e., a prospective teacher) worked with Euclidean geometry using GeoGebra (DGS). The authors connect the problem handling competency and the aids and tools competency with Santos-Trigo and Camacho Machín's (2013) framework of using DT in problem-solving processes.

In "Mathematical modelling and digital tools—and how a merger can support students' learning", Britta Eyrich Jessen and Tinne Hoff Kjeldsen debate the modelling competency in relation to DT using two cases from upper secondary school in Denmark. Based on parallel analyses on the modelling cycle (Kjeldsen & Blomhøj, 2006) and the Media-Milieu dialectics from the anthropological theory of the didactic (ATD) (Chevallard, 2007), they compare and contrast the theoretical perspectives and their analyses. Such a networking approach is applicable for the authors to discuss how to merge DT and the modelling competency with the purpose of supporting students' learning.

In their chapter "Lower secondary students' reasoning competency in a digital environment: The case of instrumented justification", Rikke Maagaard Gregersen and Anna Baccaglini-Frank focus on the reasoning competency when students use

GeoGebra (DGS). Within the chapter, an analytic tool coordinating the technique-scheme duality from the instrumental approach (Drijvers et al., 2013) with Toulmin's (2003) argumentation model is developed. The authors utilise the tool for an analysis of two Danish lower secondary school students collaborating on a task concerning variable points.

In "Mathematical representation competency in the era of digital representations of mathematical objects", Ingi Heinesen Højsted and Maria Alessandra Mariotti address students' possession and development of the representation competency in a context of using DGS. The authors utilise two cases of Danish lower secondary school students collaborating on a task concerning geometry in GeoGebra, which are analysed and discussed with respect to the representation competency. They hypothesise that the representation competency in the context of DT is closely related to the complexity of the dynamic dependency of the mathematical representation itself.

Linda Marie Ahl and Ola Helenius display a case on programming, exemplified using Scratch. Within this chapter "New demands on the symbols and formalism competency in the digital era", Vergnaud's (2009) theory of conceptual fields and the KOM framework are coordinated to gain deeper understanding of the symbols and formalism competency. The two theoretical perspectives offer different grain sizes for analysing the programming situation.

In "Activating mathematical communication competency when using DGE—is it possible?", Cecilie Carlsen Bach and Angelika Bikner-Ahsbahs investigate mathematical communication competency through the concept of tool-based mathematical communication. Such a concept is developed through a coordinated analysis of cases of Danish lower secondary school students using a DGS template. The two theoretical perspectives, instrumentation profiles, part of the instrumental approach to mathematics education (Guin & Trouche, 1998) and two dialogical genres (O'Connell & Kowal, 2012), are chosen with respect to KOM.

Morten Misfeldt, Uffe Thomas Jankvist and Eirini Geraniou investigate the application of the aids and tools competency in relation to DT and new virtual manipulatives in "An embodied cognition view on the KOM framework's aids and tools competency in relation to digital technologies". Different examples are analysed using the instrumental approach (Guin & Trouche, 2002) and embodied instrumentation (Drijvers, 2019) to discuss students' aids and tools competency in situations involving DT. The chapter includes examples of tasks from Danish higher education involving CAS and DGS as well as non-empirical analyses of virtual manipulatives, located within online mathematical learning environments, such as Mathletics and TouchCount.

5 Introduction to The Three Types of Overview and Judgement

In "Mathematics in action: On the who, where and how of the constructions and use of mathematical models in society", Raimundo Elicer and Morten Blomhøj address the KOM framework's first type of overview and judgement related to the actual application of mathematics through a discussion of mathematical models in society. They take the KOM framework's critical stance to this and address the role of such models in the digital era by networking the notion of internal and external reflections regarding mathematical modelling and the instrumentation–instrumentalisation duality of the instrumental approach. Two teaching experiences from higher education in Denmark serve as illustrative cases in this respect.

Marianne Thomsen and Kathleen M. Clark consider the KOM framework's second type of overview and judgement on the historical development of mathematics, seen from internal as well as from sociocultural perspectives. In "Perspectives on embedding the historical development of mathematics in mathematical tasks", they do so in relation to how working with the interplay between original (historical) mathematical sources and DT can support students' development of this type of overview and judgement, and thereby reinforce the dialectical relation between the praxis and logos block relying amongst other theoretical constructs on the ATD. The authors describe and draw on two empirical examples from Denmark, one from a 7th-grade classroom and one from an in-service teacher course.

In their chapter "Facilitating teachers' reflections on the nature of mathematics through an online community", Maria Kirstine Østergaard and Dandan Sun address the third type of overview and judgement from a teacher's professional development perspective. Although teachers' overview and judgement concerning the nature of mathematics as a discipline must be regarded as an essential part of their overall mathematical competence, this is seldom the object of teachers' professional development. The authors apply theoretical constructs related to beliefs in combination with the KOM framework in order to investigate how an online teacher community can provide opportunities, both for gaining experience with the nature of mathematics as a discipline as well as for reflections on such. The empirical case stems from China, where the authors have monitored a Chinese teacher who participated in an online teacher education programme. The authors exemplify how DT can facilitate and motivate the participant teachers' reflections on their own existing beliefs, for instance, by making these more conscious and nuanced.

6 Introduction to Broadening the Scene

Ingi Heinesen Højsted, Eirini Geraniou and Uffe Thomas Jankvist investigate one Danish teacher's practice in a dynamic geometry teaching sequence aiming to support students' development of mathematical reasoning. In "Teachers' facilitation of students' mathematical reasoning in a dynamic geometry environment: An analysis through three lenses", the three lenses used are the KOM framework's description of (didactico-pedagogical) mathematical competencies for teaching, the theory of instrumental orchestration and the theoretical construct of justificational mediations. The authors argue that the use of these three theoretical constructs enabled them to capture different levels of analysis as a synthesising result, and therefore to widen the scope of the KOM framework by integrating aspects from the theory of instrumental orchestration and the justification mediations framework. In fact, the latter allowed for an exemplification of a teacher's chosen mediations in utilising both their mathematical competencies and their instrumental orchestrations in supporting students' mathematical learning, comprising the reasoning competency.

In "Mathematical competencies and programming: The Swedish case", Kajsa Bråting, Cecilia Kilhamn and Lennart Rolandsson present opportunities and challenges regarding the integration of programming in school mathematics, focusing on the case of Sweden. The authors discuss how using programming in mathematics teaching and learning can affect the development of students' mathematical competencies. In more detail, they present three examples. The first one showcases differences regarding syntax and semantics in programming and mathematics and how these affect learning, while the second and third examples draw attention to the teachers' views of how programming may enhance the traditional learning of school mathematics. In all three examples, different mathematical competencies are discussed.

In their chapter "Coordinating mathematical competencies and computational thinking practices from a networking of theories point of view", Andreas Tamborg and Kim André Stavenæs Refvik discuss the evolving role computational thinking plays in mathematics education and its potential with regards to mathematical competencies. They build on the taxonomy of Weintrop et al. (2016) for computational thinking practices along with the KOM framework to attempt a coordination of mathematical competencies and computational thinking practices from a networking of theories perspective, and articulate which mathematical competencies may support students with engaging in computational thinking practices. They exemplify their analysis with mathematical tasks used in Denmark and Sweden, but also in PISA assessments.

Finally, Ross Turner, David Tout and Jim Spithill present in "A rich view of mathematics education and assessment: Mathematical competencies" how the KOM framework has influenced the OECD's Programme for International Student Assessment (PISA) assessment of mathematical literacy and its reporting, but also how it can be applied to develop KOM-inspired PISA assessment items. The authors present an evidence-based case for making use of mathematical competencies and their assessment in the digital era.

7 A Platform for Further Discussion and Research

The present book can be seen as a first attempt to connect—or network—different theoretical perspectives in mathematics education with the KOM framework on mathematical competencies for situations that in some way or another involve DT. The types and variants of theoretical perspectives in play are numerous, stretching from more comprehensive theoretical perspectives such as Chevallard and colleagues' ATD to much smaller or local constructs, notions and distinctions. Some of these of course address aspects of DT use in an explicit manner, for example, the instrumental approach or that of semiotic mediations. Some connect the elements of the KOM framework, through a competency or two, with in-depth analyses of mathematical concept formation, for example, through Vergnaud's theory of conceptual fields. Some again go beyond the scope of the KOM framework's content, for instance, when drawing on the notion of teachers' mathematics-related beliefs in relation to overview and judgement. The involvement of different DT in the chapters mirrors the digital era of mathematics education through the many different possibilities of choosing DT for teaching, such as programming, CAS and DGS.

In the chapters on the eight mathematical competencies in Part II, the authors often utilise both theoretical perspectives of the use of DT, for most of the chapters the instrumental approach, and theoretical perspectives related to the specific competency. For example, theoretical perspectives on problem-solving when focusing on the problem handling competency, or theoretical perspectives on modelling when investigating the modelling competency. The same appears to be the case in Part III and Part IV. In Part III, authors draw on theoretical perspectives closely related to the specific types of overview and judgement, such as mathematical modelling or use of history of mathematics in the teaching and learning of mathematics, as well as the instrumental approach. For Part IV, authors connect aspects of KOM to the interest at issue in the chapter, such as instrumental orchestration and the six didactico-pedagogical competencies, or the eight mathematical competencies' relations to programming and computational thinking.

Another commonality throughout the chapters of the book is that several of these apply the networking strategies of 'coordinating' or 'combining', which both serve as means to understand an empirical phenomenon through triangulation of theoretical perspectives (Prediger et al., 2008). Furthermore, several chapters contain parallel analyses in order to secure both a focus on DT and on the mathematical competency in question.

Other chapters apply networking practices merely as reflections of the processes and of the selection of the theoretical perspectives in play (Prediger et al., 2008). In such cases, networking of theories may be identified as a kind of 'eclecticism' of theoretical perspectives on a well-informed basis (Køppe, 2008). In any of the different involvements of networking of theories, the actual connection of theories serves the purpose of shedding light on the processes of choosing and of using the theoretical perspectives in relation to the specific framework of KOM. Often, KOM acts as the broader lens, while the other theoretical perspectives serve the purposes of providing finer graded analyses.

Hence, the book offers a variety of different ways to link theoretical perspectives from mathematics education research with elements of the KOM framework in teaching and learning situations involving DT. The book in itself serves the purpose of obtaining "various forms of *mutual fertilisation* of the entities under consideration" as pointed out by Niss and Jankvist (2022, p. 35). In this light, the endeavours of the contributors of this book may be seen as "valuable for furthering the theoretical development of our field to engage in analysing, comparing and contrasting different constructs and frameworks in considerable detail in order to uncover their similarities and differences" (2022, p. 35); yet, in this book specifically in relation to mathematical competencies and use of DT. We cannot escape the influence of DT in the teaching and learning of mathematics, and hence also for each of the eight mathematical competencies, as pointed out by Geraniou and Misfeldt (2022), which too promotes the notion of mathematical digital competencies (Geraniou & Jankvist, 2019). All in all, *mathematical competencies in the digital era.*

Ackowledgements This book was partly prepared in the frame of project 8018-00062B under *Independent Research Fund Denmark*.

References

Bartolini-Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed.), (pp. 746–783). Routledge. https://doi.org/10.4324/9780203930236

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Seeking new Frontiers* (pp. 483–506). Springer-Verlag. https://doi.org/10.1007/978-3-642-00742-2_46

Blomhøj, M., & Kjeldsen, T. H. (2006). Teaching mathematical modelling through project work—Experiences from an in-service course for upper secondary teachers. *ZDM Mathematics Education, 38*, 163–177. https://doi.org/10.1007/BF02655887

Chevallard, Y. (2007). Un concept en émergence: la dialectique des médias et des milieux. In G. Gueudet, & Y. Matheron (Eds.), *Actes du séminaire national de didactique des mathématiques* (pp. 344–366), ARDM et IREM de Paris 7.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 135–151). Springer Verlag. https://doi.org/10.1007/978-3-319-17187-6_8

Drijvers, P. (2019). Embodied instrumentation: Combining different views on using digital technology in mathematics education. In U. T. Jankvist, M. Van den Heuvel-Panhuizen & M. Veldhuis

(Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 8–28). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, *82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Geraniou, E., & Misfeldt, M. (2022). The mathematical competencies framework and digital technologies. In U. T. Jankvist, & E. Geraniou, (Eds.), *Mathematical competencies in the digital era* (pp. 39–60). Springer.

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: Necessity of instrumental orchestration. *ZDM Mathematics Education, 34*(5), 204–211. https://doi.org/10.1007/BF02655823

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics? *For the Learning of Mathematics, 35*(1), 15–20.

Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). What happens when CAS-procedures are objectified?—The case of "solve" and "desolve." *Educational Studies in Mathematics, 101*(1), 67–81. https://doi.org/10.1007/s10649-019-09888-5

Køppe, S. (2008). En moderat eklecticisme. [A moderate eclecticism.] *Psyke & Logos*, *29*(1), 15–35.

Matematikkommissionen (2016). *Matematikkommissionen – afrapportering*. Retrieved from: http://gymnasieskolen.dk/sites/default/files/matematikkommissionen_afrapportering.pdf

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jankvist, U. T. (2022). On the mathematical competencies framework and its potentials for connecting with other theoretical perspectives. In U. T. Jankvist, & E. Geraniou, (Eds.), *Mathematical competencies in the digital era* (pp. 15–38). Springer.

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

O'Connell, D. C., & Kowal, S. (2012). *Dialogical genres. Empractical and conversational listening and speaking.* Springer. https://doi.org/10.1007/978-1-4614-3529-7

OECD. (2013). *PISA 2012 assessment and analytical framework: Mathematics.* OECD Publishing. https://doi.org/10.1787/9789264190511-en

OECD. (2015). *Students, computers and learning—Making the connection.* OECD Publishing.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches—First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Santos-Trigo, M., & Camacho Machín, M. (2013). Framing the use of computational technology in problem solving approaches. *The Mathematics Enthusiast*, *10*(1–2), 279–302.

Stacey, K. (2010). Mathematical and scientific literacy around the world. *Journal of Science and Mathematics Education in Southeast Asia, 33*(1), 1–16.

Stacey, K., & Turner, R. (Eds.) (2015). *Assessing mathematical literacy—The PISA experience.* Springer. https://doi.org/10.1007/978-3-319-10121-7

Toulmin, S. E. (2003). *The uses of argument: Updated edition* (2nd ed.). Cambridge University Press.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). Springer. https://doi.org/10.1007/b101602

Trouche, L., Drijvers, P., Gueudet, G., & Sacristán, A. I. (2013). Technology driven developments and policy implications for mathematics education. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 753–789). Springer. https://doi.org/10.1007/978-1-4614-4684-2

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

Weigand, H.-G. (2014). Looking back and ahead—Didactical implications for the use of digital technologies in the next decade. *Teaching Mathematics and Its Applications, 33*(1), 3–15. https://doi.org/10.1093/teamat/hru006

Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal of Science Education and Technology, 25*(1), 127–147. https://doi.org/10.1007/s10956-015-9581-5