Mathematics Education in the Digital Era

Uffe Thomas Jankvist
Eirini Geraniou *Editors*

# Mathematical Competencies in the Digital Era

Springer

# Mathematics Education in the Digital Era

Volume 20

The Mathematics Education in the Digital Era (MEDE) series explores ways in which digital technologies support mathematics teaching and the learning of Net Gen'ers, paying attention also to educational debates. Each volume will address one specific issue in mathematics education (e.g., visual mathematics and cyber-learning; inclusive and community based e-learning; teaching in the digital era), in an attempt to explore fundamental assumptions about teaching and learning mathematics in the presence of digital technologies. This series aims to attract diverse readers including: researchers in mathematics education, mathematicians, cognitive scientists and computer scientists, graduate students in education, policy-makers, educational software developers, administrators and teachers-practitioners. Among other things, the high quality scientific work published in this series will address questions related to the suitability of pedagogies and digital technologies for new generations of mathematics students. The series will also provide readers with deeper insight into how innovative teaching and assessment practices emerge, make their way into the classroom, and shape the learning of young students who have grown up with technology. The series will also look at how to bridge theory and practice to enhance the different learning styles of today's students and turn their motivation and natural interest in technology into an additional support for meaningful mathematics learning. The series provides the opportunity for the dissemination of findings that address the effects of digital technologies on learning outcomes and their integration into effective teaching practices; the potential of mathematics educational software for the transformation of instruction and curricula; and the power of the e-learning of mathematics, as inclusive and community-based, yet personalized and hands-on.

**Submit your proposal:** Please contact the publishing editor at Springer: Marianna.Georgouli@springer.com

**Forthcoming titles:**

- Mathematics Education in the Age of Artificial Intelligence—Philipe R. Richard, M. Pilar Velez and Steven van Vaerenbergh (eds.)
- Mathematical Work in Educational Context—Alain Kuzniak, Elizabeth Montoya-Delgadillo and Philippe R. Richard (eds.)
- The Mathematics Teacher in the Digital Era: International Research on Professional Learning and Practice (2nd Edition)—Alison Clark-Wilson, Ornella Robutti and Nathalie Sinclair (eds.)
- Quantitative Reasoning in Mathematics and Science Education—Gülseren Karagöz Akar, İsmail Özgür Zembat, Selahattin Arslan and Patrick W. Thompson (eds.)
- The Evolution of Research on Teaching Mathematics—Agida Manizade, Nils Fredrik Buchholtz and Kim Beswick (eds.)—Open Access!

Uffe Thomas Jankvist · Eirini Geraniou
Editors

# Mathematical Competencies in the Digital Era

Springer

*Editors*
Uffe Thomas Jankvist
Danish School of Education
Aarhus University
Copenhagen, Denmark

Eirini Geraniou
Curriculum, Pedagogy and Assessment
UCL Institute of Education
London, UK

ISSN 2211-8136 ISSN 2211-8144 (electronic)
Mathematics Education in the Digital Era
ISBN 978-3-031-10140-3 ISBN 978-3-031-10141-0 (eBook)
https://doi.org/10.1007/978-3-031-10141-0



This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

*To all believers and non-believers in the power, value and impact of digital technologies for mathematics education and their influence regarding mathematical competencies.*

# Foreword

*Mathematical Competencies in the Digital Era*, edited by Uffe Thomas Jankvist and Eirini Geraniou, is a book in the series *Mathematics Education in the Digital Era*, edited by Dragana Martinovic and Viktor Freiman. The *Digital Era*, in this book, refers to two layers: mathematical digital competencies to be developed, but even with higher emphasis, the development of mathematical competencies with the support of digital tools.

The focus on mathematical competencies (also conceptualized as mathematical practices, in other contexts) has a strong tradition in Denmark, where Mogens Niss and colleagues not only developed a coherent framework, the mathematics competencies framework, the so-called KOM-framework. Denmark is an interesting curricular laboratory of international interest, because the KOM framework has been consequently established in the Danish curricula and from there influenced many other countries and regions and also the previous PISA frameworks. In recent years, digital technologies have been introduced to classrooms, but the editors and authors are driven by the main idea that more research is needed to use the digital technologies productively for supporting the development of mathematical competencies and mathematical digital competencies.

Thereby, the ambitious book presents design research studies from Denmark and other countries that develop learning opportunities with digital technologies to support the development of particular mathematical competencies and that investigate the initiated teaching–learning processes.

The design as well as the empirical investigations draw upon the KOM framework as well as upon different theoretical constructs and perspectives in mathematics education research that were particularly generated for situations involving digital technologies in the teaching and learning of mathematics. For example, the chapters refer to the instrumental approach, instrumental orchestration, semiotic mediations, the anthropological theory of the didactic, mathematics-related beliefs, programming and computational thinking, mathematical digital competencies and the theory of conceptual fields. In each of the chapters, the authors find other ways to combine the KOM framework with one or two other theoretical perspectives or constructs, and in each of the cases, different strategies are applied for networking the theories. In this way, the book documents highly interesting cases of *networking experiences* on four levels:

(1) On the *practical level*, two highly relevant challenges in classrooms practices are combined in many different ways, each time the development of one competency and the use of digital technologies, and this networked design work results in several promising instructional approaches that can enrich classroom practice.
(2) On the *theoretical and empirical level*, the design research requires coordinating, local integrating or synthesizing theoretical approaches to make sense of the investigated teaching–learning processes.
(3) On the *personal level*, these tasks have been conducted by networking of researchers from different countries: sustained exchange of ideas and collaborations between Scandinavian researchers familiar with the KOM framework and a number of international scholars have resulted in an impressive range of co-authored contributions in the book's 18 chapters.
(4) On the *methodological level*, the authors present intriguing ideas about comparing and contrasting theories in relation to the use of digital technologies and mathematical competencies and other constructs from the KOM framework. The editors and authors have carried out substantial networking practices to showcase how KOM's descriptions of mathematical competencies, etc., can be refined and deepened using other frameworks from both inside and outside of mathematics education research. This in itself is a significant contribution that offers some foundational work for further connections of theories moving towards a higher level of integration for the KOM framework's constructs in the digital era. The audience of this book should recognize the combined (networked) framework approach suggested in each chapter, as they provide specific foci for attention and a potential basis for an increasingly unified discourse of research and practice in mathematics education.

In my view, one of the strongest aspects of the work presented in this book is the comprehensive exposition of how additional frameworks augment and refine the KOM framework when digital technologies are in play. Its multi-layer work reveals a fascinating read for mathematics education researchers, graduate students and teacher educators worldwide, who are particularly interested in the digital technologies' influence on mathematical competencies as presented by different authors using various educational theories. I congratulate the editors and the authors for this insightful work!

I hope this compelling book will find many readers who enjoy diving into *Mathematical Competencies in the Digital Era*!

Dortmund, Germany Susanne Prediger

# Acknowledgements

A big thank you to all contributing authors, without whom this book would not have been possible. Also a warm thank you to doctoral students Cecilie Carlsen Bach, Mathilde Kjær Pedersen and Rikke Maagaard Gregersen who efficiently and effectively assisted in the many practical matters surrounding this book's coming into being.

# Contents

# Mathematical Competencies in the Digital Era: An Introduction

**Uffe Thomas Jankvist, Eirini Geraniou, Mathilde Kjær Pedersen, Cecilie Carlsen Bach, and Rikke Maagaard Gregersen**

## 1 Rationale Behind the Book

Mathematics education has been experiencing two rather distinct, yet related, 'paradigm shifts'. The first is to do with the massive introduction of digital technologies (DT) in the teaching and learning of the subject (e.g., Trouche et al., 2013); the second is to do with a shift from the traditional focusing on mastering of skills and knowledge to being concerned with the possession and development of *mathematical competencies* (e.g., Stacey, 2010; Stacey & Turner, 2015). This book focuses on the potential *interplay* between these two paradigm shifts by considering the connection of different theoretical perspectives, e.g., by drawing on the notion of 'networking of theories' (e.g., Bikner-Ahsbahs & Prediger, 2010; Prediger et al., 2008).

A somewhat recent Organisation for Economic Co-operation and Development (OECD, 2015) report states that merely adding 21st-century technologies to 20th-century teaching practices is rather likely to dilute the effectiveness of teaching than to inform and enhance it. This observation is in line with recent research in mathematics education; while studies from the 1980s and 1990s tend to point to the

U. T. Jankvist (✉) · M. K. Pedersen · C. C. Bach · R. M. Gregersen
Danish School of Education, Aarhus University, Copenhagen, Denmark
e-mail: utj@edu.au.dk

M. K. Pedersen
e-mail: mkp@edu.au.dk

C. C. Bach
e-mail: ccb@edu.au.dk

R. M. Gregersen
e-mail: rmg@edu.au.dk

E. Geraniou
UCL Institute of Education, UCL, London, UK
e-mail: e.geraniou@ucl.ac.uk



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_1

positive effects of DT on teaching and learning, recent research often has a more balanced view on these effects (Drijvers, 2015; Weigand, 2014) or even point to negative effects (Jankvist et al., 2019). DT in the form of mathematical software—for example, Dynamic Geometry Systems (DGSs) or Computer Algebra Systems (CASs)—can today perform many of the mathematical tasks and processes that students traditionally are expected to do (e.g., reduction, equation solving, functional analysis, etc.). Newer research thus address the role of mathematical competencies in the digital era (e.g., Geraniou & Jankvist, 2019).

From an international perspective, Denmark offers *unique opportunities* for carrying out research on this interplay for two reasons. Firstly, Denmark is one of the few countries in the world that has gone 'all in' on introducing DT in its mathematics programmes of primary, secondary and upper secondary school. However, this has not been an altogether positive experience (Matematikkommissionen, 2016), since DT has also brought along new and unforeseen difficulties related to students' learning of mathematics (Jankvist & Misfeldt, 2015). Secondly, Denmark is one of the few countries that have competencies descriptions of its mathematics programmes all the way from primary school through upper secondary school to tertiary level, including teacher training—and all of these are based on the Danish so-called KOM framework (Niss & Højgaard, 2011, 2019; Niss & Jensen, 2002), which was also previously adopted by OECD for the Programme for International Student Assessment (PISA) (OECD, 2013). Niss and Højgaard (2011) define a *mathematical competency* as (an individual's) "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (p. 49). The KOM (competencies and the learning of mathematics) framework operates with eight distinct, yet mutually related, mathematical competencies, three second-order competencies referred to as types of overview and judgement and six didactico-pedagogical competencies for mathematics teachers.

Thus far, the discussions of using DT and developing mathematical competencies have often run on separate tracks, both within practice and within research. One reason for this may be due to many of the mathematics educational theories and results being framed within a setting of skills and knowledge rather than one of competency. New research surrounding KOM in the digital era does, however, address the potential of connecting the KOM framework with more established theoretical perspectives of mathematics education in situations involving DT. Recent research studies show that the KOM framework for example connects well with both Trouche (2005) and colleagues' instrumental approach as well as the theoretical perspective of semiotic mediations (Bartolini-Bussi & Mariotti, 2008) in relation to the use of DT as well as for example Vergnaud's (2009) theory of conceptual fields in relation to more cognitive elements of mathematical concept formation.

The chapters of this book provide further illustration of such connections. Several of the empirical cases in the book stem from the Danish educational system. Still, the book also offers a broader international perspective by, on the one hand, drawing on cases from other countries, for example, Australia, China, Costa Rica, England and Sweden and, on the other hand, having a wide range of international researchers co-author several of the chapters involving cases from the Danish educational system.

## 2 The Structure of the Book

Besides this introduction, the chapters of the book fall into four parts, each part addressing different aspects of the KOM framework.

The first part, Setting the Scene, consists of two introductory chapters. These chapters address the KOM framework and the potentials of connecting it with other theoretical perspectives from mathematics education research, both from a general point of view and specifically in relation to the use of DT.

The second part, which is on the eight mathematical competencies, consists of eight chapters, one for each of the mathematical competencies described in the KOM framework. These are:

- *Mathematical thinking competency*—engaging in mathematical enquiry.
- *Mathematical problem handling competency*—posing and solving mathematical problems.
- *Mathematical modelling competency*—analysing and constructing mathematical models.
- *Mathematical reasoning competency*—assessing and producing justification of mathematical claims.
- *Mathematical representation competency*—dealing with different representations of mathematical entities.
- *Mathematical symbols and formalism competency*—handling mathematical symbols and formalisms.
- *Mathematical communication competency*—communicating in, with and about mathematics.
- *Mathematical aids and tools competency*—handling material aids and tools for mathematical activity.

The third part consists of three chapters, each addressing one of the KOM framework's three types of overview and judgement. According to Niss and Højgaard, (2011), overview and judgement are *insights* that "enable the person mastering them to have a set of views allowing him or her overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture" (p. 74). The three types of overview and judgement are:

- The actual application of mathematics within other disciplines and fields of practice.
- The historical development of mathematics, seen from internal as well as from sociocultural perspectives.
- The nature of mathematics as a discipline.

The fourth part broadens the scene by providing four chapters that are of general nature in relation to the KOM framework in the digital era. Among other things, these chapters address assessment of mathematical competencies, teachers' didactico-pedagogical competencies in situations involving DT, mathematical competencies in relation to computational thinking and programming in school.

## 3 Introduction to Setting the Scene

In "On the mathematical competencies framework and its potentials for connecting with other theoretical perspectives", Mogens Niss and Uffe Thomas Jankvist present and discuss the KOM framework in general as well as its potentials for connecting with other theoretical perspectives. As part of this discussion, the notions of 'theory', 'theoretical framework' and 'theoretical perspectives' are examined and discussed with the purpose of placing KOM in this landscape. The chapter also provides examples of connections between the KOM framework and other theoretical perspectives, one example in terms of mathematical competencies (the modelling competency), one example in terms of the three types of overview and judgement (the historical development of mathematics) and one example in terms of the didactico-mathematical competencies (uncovering learning). Based on their discussion of the potential of connecting aspects of the KOM framework with other theoretical constructs, the authors point out that at the current stage of development perhaps 'mutual fertilisation' of the entities under consideration should be the goal rather than actual 'networking' of these entities into new frameworks.

While the first chapter does not address the role of DT explicitly in relation to the KOM framework, the second chapter by Eirini Geraniou and Morten Misfeldt does exactly this. In their chapter, "The mathematical competencies framework and digital technologies", the two authors discuss the influence as well as the potential of DT in relation to the KOM framework's eight mathematical competencies, six didactico-pedagogical competencies for teachers and three types of overview and judgement. Influenced by the notion of mathematical digital competency (MDC) by Geraniou and Jankvist, (2019), through the use of examples (from England, Sweden, and the well-known international problem—'the four-color theorem'), they argue for the need to revisit the KOM framework and add a layer that represents the interplay between technology and competencies. However, not thinking of this as a meta-competency of the tools and aids competency, but rather allowing technology to take a more central role in each of the mathematical competencies described by the KOM framework. In Geraniou's and Misfeldt's eyes, MDCs should be at par with mathematical competencies rather than being a sub-competency.

## 4 Introduction to The Eight Mathematical Competencies

The second part presents the eight mathematical competencies of the KOM framework in the perspective of DT use. Each chapter in this part corresponds to one of the eight mathematical competencies. Since all eight competencies are mutually related, a chapter may involve other competencies than the one in focus. All chapters present cases illustrating the interplay between the competency in play and the use of DT. Most of the chapters involve DGS, but there are also examples of CAS and various programming tools.

In "Processes of mathematical thinking competency in interactions with a digital tool", Mathilde Kjær Pedersen discusses students' mathematical thinking competency in learning situations of differentiability involving TI-nspire, which involves both DGS and CAS. In parallel, an empirical case stemming from Danish upper secondary school is analysed from the lenses of the instrumental approach (e.g., Drijvers et al., 2013), conceptual fields (Vergnaud, 2009) and semiotic mediation (Bartolini-Bussi & Mariotti, 2008). The chapter discusses which aspects of the mathematical thinking competency can be identified in the students' interaction with the DT.

In their chapter "Mathematical competencies framework meets problem-solving research in mathematics education", Mario Sánchez Aguilar, Martha Leticia García Rodríguez and William Enrique Poveda Fernández focus on the problem handling competency. The authors connect the KOM framework with other notions from mathematics education research related to problem-solving, which together provide a suitable construct for analysing problem-solving processes. Through analyses of an empirical case stemming from a virtual course for prospective lower secondary school teachers in Costa Rica, a student (i.e., a prospective teacher) worked with Euclidean geometry using GeoGebra (DGS). The authors connect the problem handling competency and the aids and tools competency with Santos-Trigo and Camacho Machín's (2013) framework of using DT in problem-solving processes.

In "Mathematical modelling and digital tools—and how a merger can support students' learning", Britta Eyrich Jessen and Tinne Hoff Kjeldsen debate the modelling competency in relation to DT using two cases from upper secondary school in Denmark. Based on parallel analyses on the modelling cycle (Kjeldsen & Blomhøj, 2006) and the Media-Milieu dialectics from the anthropological theory of the didactic (ATD) (Chevallard, 2007), they compare and contrast the theoretical perspectives and their analyses. Such a networking approach is applicable for the authors to discuss how to merge DT and the modelling competency with the purpose of supporting students' learning.

In their chapter "Lower secondary students' reasoning competency in a digital environment: The case of instrumented justification", Rikke Maagaard Gregersen and Anna Baccaglini-Frank focus on the reasoning competency when students use

GeoGebra (DGS). Within the chapter, an analytic tool coordinating the technique-scheme duality from the instrumental approach (Drijvers et al., 2013) with Toulmin's (2003) argumentation model is developed. The authors utilise the tool for an analysis of two Danish lower secondary school students collaborating on a task concerning variable points.

In "Mathematical representation competency in the era of digital representations of mathematical objects", Ingi Heinesen Højsted and Maria Alessandra Mariotti address students' possession and development of the representation competency in a context of using DGS. The authors utilise two cases of Danish lower secondary school students collaborating on a task concerning geometry in GeoGebra, which are analysed and discussed with respect to the representation competency. They hypothesise that the representation competency in the context of DT is closely related to the complexity of the dynamic dependency of the mathematical representation itself.

Linda Marie Ahl and Ola Helenius display a case on programming, exemplified using Scratch. Within this chapter "New demands on the symbols and formalism competency in the digital era", Vergnaud's (2009) theory of conceptual fields and the KOM framework are coordinated to gain deeper understanding of the symbols and formalism competency. The two theoretical perspectives offer different grain sizes for analysing the programming situation.

In "Activating mathematical communication competency when using DGE—is it possible?", Cecilie Carlsen Bach and Angelika Bikner-Ahsbahs investigate mathematical communication competency through the concept of tool-based mathematical communication. Such a concept is developed through a coordinated analysis of cases of Danish lower secondary school students using a DGS template. The two theoretical perspectives, instrumentation profiles, part of the instrumental approach to mathematics education (Guin & Trouche, 1998) and two dialogical genres (O'Connell & Kowal, 2012), are chosen with respect to KOM.

Morten Misfeldt, Uffe Thomas Jankvist and Eirini Geraniou investigate the application of the aids and tools competency in relation to DT and new virtual manipulatives in "An embodied cognition view on the KOM framework's aids and tools competency in relation to digital technologies". Different examples are analysed using the instrumental approach (Guin & Trouche, 2002) and embodied instrumentation (Drijvers, 2019) to discuss students' aids and tools competency in situations involving DT. The chapter includes examples of tasks from Danish higher education involving CAS and DGS as well as non-empirical analyses of virtual manipulatives, located within online mathematical learning environments, such as Mathletics and TouchCount.

## 5 Introduction to The Three Types of Overview and Judgement

In "Mathematics in action: On the who, where and how of the constructions and use of mathematical models in society", Raimundo Elicer and Morten Blomhøj address the KOM framework's first type of overview and judgement related to the actual application of mathematics through a discussion of mathematical models in society. They take the KOM framework's critical stance to this and address the role of such models in the digital era by networking the notion of internal and external reflections regarding mathematical modelling and the instrumentation–instrumentalisation duality of the instrumental approach. Two teaching experiences from higher education in Denmark serve as illustrative cases in this respect.

Marianne Thomsen and Kathleen M. Clark consider the KOM framework's second type of overview and judgement on the historical development of mathematics, seen from internal as well as from sociocultural perspectives. In "Perspectives on embedding the historical development of mathematics in mathematical tasks", they do so in relation to how working with the interplay between original (historical) mathematical sources and DT can support students' development of this type of overview and judgement, and thereby reinforce the dialectical relation between the praxis and logos block relying amongst other theoretical constructs on the ATD. The authors describe and draw on two empirical examples from Denmark, one from a 7th-grade classroom and one from an in-service teacher course.

In their chapter "Facilitating teachers' reflections on the nature of mathematics through an online community", Maria Kirstine Østergaard and Dandan Sun address the third type of overview and judgement from a teacher's professional development perspective. Although teachers' overview and judgement concerning the nature of mathematics as a discipline must be regarded as an essential part of their overall mathematical competence, this is seldom the object of teachers' professional development. The authors apply theoretical constructs related to beliefs in combination with the KOM framework in order to investigate how an online teacher community can provide opportunities, both for gaining experience with the nature of mathematics as a discipline as well as for reflections on such. The empirical case stems from China, where the authors have monitored a Chinese teacher who participated in an online teacher education programme. The authors exemplify how DT can facilitate and motivate the participant teachers' reflections on their own existing beliefs, for instance, by making these more conscious and nuanced.

## 6 Introduction to Broadening the Scene

Ingi Heinesen Højsted, Eirini Geraniou and Uffe Thomas Jankvist investigate one Danish teacher's practice in a dynamic geometry teaching sequence aiming to support students' development of mathematical reasoning. In "Teachers' facilitation of students' mathematical reasoning in a dynamic geometry environment: An analysis through three lenses", the three lenses used are the KOM framework's description of (didactico-pedagogical) mathematical competencies for teaching, the theory of instrumental orchestration and the theoretical construct of justificational mediations. The authors argue that the use of these three theoretical constructs enabled them to capture different levels of analysis as a synthesising result, and therefore to widen the scope of the KOM framework by integrating aspects from the theory of instrumental orchestration and the justification mediations framework. In fact, the latter allowed for an exemplification of a teacher's chosen mediations in utilising both their mathematical competencies and their instrumental orchestrations in supporting students' mathematical learning, comprising the reasoning competency.

In "Mathematical competencies and programming: The Swedish case", Kajsa Bråting, Cecilia Kilhamn and Lennart Rolandsson present opportunities and challenges regarding the integration of programming in school mathematics, focusing on the case of Sweden. The authors discuss how using programming in mathematics teaching and learning can affect the development of students' mathematical competencies. In more detail, they present three examples. The first one showcases differences regarding syntax and semantics in programming and mathematics and how these affect learning, while the second and third examples draw attention to the teachers' views of how programming may enhance the traditional learning of school mathematics. In all three examples, different mathematical competencies are discussed.

In their chapter "Coordinating mathematical competencies and computational thinking practices from a networking of theories point of view", Andreas Tamborg and Kim André Stavenæs Refvik discuss the evolving role computational thinking plays in mathematics education and its potential with regards to mathematical competencies. They build on the taxonomy of Weintrop et al. (2016) for computational thinking practices along with the KOM framework to attempt a coordination of mathematical competencies and computational thinking practices from a networking of theories perspective, and articulate which mathematical competencies may support students with engaging in computational thinking practices. They exemplify their analysis with mathematical tasks used in Denmark and Sweden, but also in PISA assessments.

Finally, Ross Turner, David Tout and Jim Spithill present in "A rich view of mathematics education and assessment: Mathematical competencies" how the KOM framework has influenced the OECD's Programme for International Student Assessment (PISA) assessment of mathematical literacy and its reporting, but also how it can be applied to develop KOM-inspired PISA assessment items. The authors present an evidence-based case for making use of mathematical competencies and their assessment in the digital era.

## 7 A Platform for Further Discussion and Research

The present book can be seen as a first attempt to connect—or network—different theoretical perspectives in mathematics education with the KOM framework on mathematical competencies for situations that in some way or another involve DT. The types and variants of theoretical perspectives in play are numerous, stretching from more comprehensive theoretical perspectives such as Chevallard and colleagues' ATD to much smaller or local constructs, notions and distinctions. Some of these of course address aspects of DT use in an explicit manner, for example, the instrumental approach or that of semiotic mediations. Some connect the elements of the KOM framework, through a competency or two, with in-depth analyses of mathematical concept formation, for example, through Vergnaud's theory of conceptual fields. Some again go beyond the scope of the KOM framework's content, for instance, when drawing on the notion of teachers' mathematics-related beliefs in relation to overview and judgement. The involvement of different DT in the chapters mirrors the digital era of mathematics education through the many different possibilities of choosing DT for teaching, such as programming, CAS and DGS.

In the chapters on the eight mathematical competencies in Part II, the authors often utilise both theoretical perspectives of the use of DT, for most of the chapters the instrumental approach, and theoretical perspectives related to the specific competency. For example, theoretical perspectives on problem-solving when focusing on the problem handling competency, or theoretical perspectives on modelling when investigating the modelling competency. The same appears to be the case in Part III and Part IV. In Part III, authors draw on theoretical perspectives closely related to the specific types of overview and judgement, such as mathematical modelling or use of history of mathematics in the teaching and learning of mathematics, as well as the instrumental approach. For Part IV, authors connect aspects of KOM to the interest at issue in the chapter, such as instrumental orchestration and the six didactico-pedagogical competencies, or the eight mathematical competencies' relations to programming and computational thinking.

Another commonality throughout the chapters of the book is that several of these apply the networking strategies of 'coordinating' or 'combining', which both serve as means to understand an empirical phenomenon through triangulation of theoretical perspectives (Prediger et al., 2008). Furthermore, several chapters contain parallel analyses in order to secure both a focus on DT and on the mathematical competency in question.

Other chapters apply networking practices merely as reflections of the processes and of the selection of the theoretical perspectives in play (Prediger et al., 2008). In such cases, networking of theories may be identified as a kind of 'eclecticism' of theoretical perspectives on a well-informed basis (Køppe, 2008). In any of the different involvements of networking of theories, the actual connection of theories serves the purpose of shedding light on the processes of choosing and of using the theoretical perspectives in relation to the specific framework of KOM. Often, KOM acts as the broader lens, while the other theoretical perspectives serve the purposes of providing finer graded analyses.

Hence, the book offers a variety of different ways to link theoretical perspectives from mathematics education research with elements of the KOM framework in teaching and learning situations involving DT. The book in itself serves the purpose of obtaining "various forms of *mutual fertilisation* of the entities under consideration" as pointed out by Niss and Jankvist (2022, p. 35). In this light, the endeavours of the contributors of this book may be seen as "valuable for furthering the theoretical development of our field to engage in analysing, comparing and contrasting different constructs and frameworks in considerable detail in order to uncover their similarities and differences" (2022, p. 35); yet, in this book specifically in relation to mathematical competencies and use of DT. We cannot escape the influence of DT in the teaching and learning of mathematics, and hence also for each of the eight mathematical competencies, as pointed out by Geraniou and Misfeldt (2022), which too promotes the notion of mathematical digital competencies (Geraniou & Jankvist, 2019). All in all, *mathematical competencies in the digital era.*

**Ackowledgements** This book was partly prepared in the frame of project 8018-00062B under *Independent Research Fund Denmark*.

## References

Bartolini-Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed.), (pp. 746–783). Routledge. https://doi.org/10.4324/9780203930236

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Seeking new Frontiers* (pp. 483–506). Springer-Verlag. https://doi.org/10.1007/978-3-642-00742-2_46

Blomhøj, M., & Kjeldsen, T. H. (2006). Teaching mathematical modelling through project work—Experiences from an in-service course for upper secondary teachers. *ZDM Mathematics Education, 38*, 163–177. https://doi.org/10.1007/BF02655887

Chevallard, Y. (2007). Un concept en émergence: la dialectique des médias et des milieux. In G. Gueudet, & Y. Matheron (Eds.), *Actes du séminaire national de didactique des mathématiques* (pp. 344–366), ARDM et IREM de Paris 7.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 135–151). Springer Verlag. https://doi.org/10.1007/978-3-319-17187-6_8

Drijvers, P. (2019). Embodied instrumentation: Combining different views on using digital technology in mathematics education. In U. T. Jankvist, M. Van den Heuvel-Panhuizen & M. Veldhuis

(Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 8–28). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, *82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Geraniou, E., & Misfeldt, M. (2022). The mathematical competencies framework and digital technologies. In U. T. Jankvist, & E. Geraniou, (Eds.), *Mathematical competencies in the digital era* (pp. 39–60). Springer.

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: Necessity of instrumental orchestration. *ZDM Mathematics Education, 34*(5), 204–211. https://doi.org/10.1007/BF02655823

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics? *For the Learning of Mathematics, 35*(1), 15–20.

Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). What happens when CAS-procedures are objectified?—The case of "solve" and "desolve." *Educational Studies in Mathematics, 101*(1), 67–81. https://doi.org/10.1007/s10649-019-09888-5

Køppe, S. (2008). En moderat eklekticisme. [A moderate eclecticism.] *Psyke & Logos*, *29*(1), 15–35.

Matematikkommissionen (2016). *Matematikkommissionen – afrapportering*. Retrieved from: http://gymnasieskolen.dk/sites/default/files/matematikkommissionen_afrapportering.pdf

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jankvist, U. T. (2022). On the mathematical competencies framework and its potentials for connecting with other theoretical perspectives. In U. T. Jankvist, & E. Geraniou, (Eds.), *Mathematical competencies in the digital era* (pp. 15–38). Springer.

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

O'Connell, D. C., & Kowal, S. (2012). *Dialogical genres. Empractical and conversational listening and speaking.* Springer. https://doi.org/10.1007/978-1-4614-3529-7

OECD. (2013). *PISA 2012 assessment and analytical framework: Mathematics.* OECD Publishing. https://doi.org/10.1787/9789264190511-en

OECD. (2015). *Students, computers and learning—Making the connection.* OECD Publishing.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches—First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Santos-Trigo, M., & Camacho Machín, M. (2013). Framing the use of computational technology in problem solving approaches. *The Mathematics Enthusiast*, *10*(1–2), 279–302.

Stacey, K. (2010). Mathematical and scientific literacy around the world. *Journal of Science and Mathematics Education in Southeast Asia, 33*(1), 1–16.

Stacey, K., & Turner, R. (Eds.) (2015). *Assessing mathematical literacy—The PISA experience.* Springer. https://doi.org/10.1007/978-3-319-10121-7

Toulmin, S. E. (2003). *The uses of argument: Updated edition* (2nd ed.). Cambridge University Press.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). Springer. https://doi.org/10.1007/b101602

Trouche, L., Drijvers, P., Gueudet, G., & Sacristán, A. I. (2013). Technology driven developments and policy implications for mathematics education. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 753–789). Springer. https://doi.org/10.1007/978-1-4614-4684-2

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

Weigand, H.-G. (2014). Looking back and ahead—Didactical implications for the use of digital technologies in the next decade. *Teaching Mathematics and Its Applications, 33*(1), 3–15. https://doi.org/10.1093/teamat/hru006

Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal of Science Education and Technology, 25*(1), 127–147. https://doi.org/10.1007/s10956-015-9581-5

# Setting the Scene

# On the Mathematical Competencies Framework and Its Potentials for Connecting with Other Theoretical Perspectives

**Mogens Niss and Uffe Thomas Jankvist**

## 1 Introduction

The growing plethora of theories, theoretical constructs, frameworks and perspectives in mathematics education, many of which stand alone in more or less 'splendid isolation', has given rise to the need for finding ways to bring them together in some sort of fruitful interaction in order to better serve the theoretical interests of mathematics education research, especially as regards creating some order in the chaos of theoretical constructs and frameworks. Naturally, this may be done in numerous different ways, but the overarching term that has been used in the research community for this endeavour is 'networking' of theories, theoretical perspectives, or theoretical frameworks (e.g., Bikner-Ahsbahs & Prediger, 2014). This chapter concerns such 'networking' of theoretical perspectives with the Danish mathematical competencies framework, the so-called KOM framework (Niss & Højgaard, 2011, 2019), which first appeared in Danish some twenty years ago (Niss & Jensen, 2002).

Now, engaging in such a discussion surely requires that we first come to terms with the meaning(s) of 'theoretical framework' and 'theory' in the field of mathematics education research, and not the least with how the KOM framework may be placed in this landscape. So, the two next sections of the chapter are devoted to this endeavour. Following this, we provide three rich examples of the KOM framework's potential for connection with other theoretical perspectives, both inside and outside of mathematics education research. The first example concerns mathematical modelling, an area which, as a separate subdomain of mathematics education research, has

M. Niss
Department of Science and Environment, Roskilde University, Roskilde, Denmark
e-mail: mn@ruc.dk

U. T. Jankvist (✉)
Danish School of Education, Aarhus University, Copenhagen, Denmark
e-mail: utj@edu.au.dk



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_2

undergone its own theoretical development, intricately linked to the fostering and possession of mathematical modelling competency. Secondly, we discuss one of the KOM framework's three so-called types of overview and judgement—the type that concerns the historical development of mathematics, seen from internal as well as from socio-cultural perspectives—and its relation to the use of history in mathematics education. Thirdly, we address one of the KOM framework's six didactic and pedagogical teacher competencies, namely the competency of uncovering learning. These examples thus stem from different sub-domains of the KOM framework, the eight mathematical competencies, the three types of overview and judgement and the six didactic and pedagogical teacher competencies (see also Sect. 3). Finally, we relate the three examples to the ongoing discussion of networking of theoretical perspectives in mathematics education research and provide our suggestions related to potential connections with the KOM framework.[1]

## 2 What Are Theoretical Frameworks in Mathematics Education?

The notion of 'theoretical framework', as well as those of its closer relatives, 'theory', 'theoretical construct', 'theoretical perspective', 'theoretical approach' and the slightly more distant ones 'conceptual framework' and 'research framework', are highly significant in mathematics education research but at the same time notoriously ill-defined and extremely multifaceted. This implies that no sensible discussion involving members of this cluster of notions can be conducted without an attempt to provide at least a minimum of conceptual clarification of the notions invoked in the given context. Offering an outline of the most important of these notions is the aim of this section.

The fundamental notions are those of 'theory' and 'theoretical'. They are derived from the ancient Greek verb *theorein*, 'to consider', and the noun *theoria*, 'consideration' or 'speculation'. These terms suggest that, in the first place, 'theory', 'theoretical' and 'theorising' consist in thinking and reflection, rather than in actions dealing with material objects and in forming perceptions and conceptions of certain segments of the world rather than in interacting with it in empirical or practical terms. Needless to say, this neither prevents theorising from being inspired by dealing with the practical, material world nor from having consequences for interaction with that world. In writings concerning the notion of 'theory' the following three meanings, whether explicit or implicit, are prevalent:

[1] It should be noticed that this chapter, unlike the majority of other chapters in this book, does not specifically address the use of digital technology in relation to the KOM framework, since this matter is dealt with in the book's following chapter by Geraniou and Misfeldt (2022).

- A theory can be a *hypothesis* that cannot be, or has not yet been, substantiated;
- A theory can be a collection of *beliefs*, *rules* or *principles* that are meant to *guide action or behaviour*;
- A theory can be a more or less connected *edifice of claims* intended to explain or predict phenomena occurring within some domain.

In this chapter, we shall adhere to the third of these meanings. More precisely, we adopt the following *minimalist* definition—inspired by Niss (2007)—of the concept of theory:

A *theory* is a theory *of something*, i.e., it deals with certain sorts of *objects* and *phenomena* and includes terms for these. Its purpose is to produce *corroborated claims* about these objects and phenomena, typically in response to *questions* posed about them. These claims are generated by some *means*, on some *grounds*, involving some fundamental *methodology/ies*.

Thus, *specifying a theory* amounts to specifying (at least) the objects and phenomena covered by the constructs, the nature of the questions posed and of the claims produced, as well as the ways in which these claims are (supposed to be) obtained.

A more elaborate and exhaustive definition of theory, going beyond the minimalist one just stated, reads:

> A *theory* is an organised *network of concepts* (including, ideas, notions, distinctions and terms) and *claims* (oftentimes taken as answers to *questions*) about some extensive *domain of objects and phenomena*. The concepts of the theory are *linked in a hierarchy*, in which a subset of concepts, taken to be *basic*, is employed to form the other concepts in the hierarchy. The *claims* are either basic assumptions or axioms, taken as *fundamental*, or statements obtained from the fundamental claims by *formal or material derivation* (including reasoning). (Niss, 2007, p. 1308)

The concept of 'theory' is the most elaborate, complex and demanding one within the cluster of notions considered here. This is not the place to offer a more elaborate discussion of the notion of theory and of what can justifiably be perceived as a theory in mathematics education research. Suffice it to quote Kilpatrick (2010): "To say that something is a theory in mathematics education—rather than, say, an approach, theoretical framework, theoretical perspective, or model—is to make an exceedingly strong claim [...] I am happy to talk about theorising, adopting a theoretical stance, or employing a theoretical framework but I do not see extant theoretical constructions as warranting the label of theory." (p. 4). Although we would, in fact, grant the label 'theory' to some constructions in mathematics education—e.g., Brousseau's (1997) theory of didactical situations, or Chevellard's (2019) anthropological theory of the didactic (ATD), and the APOS theory developed by Dubinsky (1991)—we largely share Kilpatrick's position. At any rate, the number of theories in mathematics education is, at best, extremely small.

Even if the notion of 'theoretical construct' can, of course, refer to a component of a theory as just defined (like 'didactical contract' forms part of Brousseau's 'theory of didactical situations'), it does not have to. A 'theoretical construct' may well stand alone and in its most basic form be nothing more than a singular concept introduced by way of a definition. It may or may not rest on certain assumptions or hypotheses, and it may or may not involve certain claims. In other words, a 'theoretical construct' is a much less ambitious notion than 'theory', which does not mean that it is less important. As examples, we can think of the construct 'concept image' introduced by Vinner, Hershkowitz and Tall (see, e.g., Vinner & Hershkowitz, 1980; Tall & Vinner, 1981), the notion of 'mathematical beliefs' (see, e.g., Leder et al., 2002) or that of teachers' 'pedagogical content knowledge', PCK, (Shulman, 1987). Oftentimes a theoretical construct is introduced as part of a distinction. This is the case with 'concept image' in contradistinction to 'concept definition', 'mathematical beliefs' as distinct from 'mathematical knowledge and skills', and teachers' PCK as distinct from teachers' subject matter knowledge.

Some researchers consider a 'theoretical framework' to be a 'theory', which, together with other entities, is employed as a framework for conducting research or making research expositions, e.g., Cai et al. (2019): "… we use the term theoretical framework broadly […] to encompass the set of assumptions, theories, hypotheses and claims (as well as the relationships between them) that guide the researcher's thinking about the phenomenon being studied" (p. 219).

We, too, use the term 'theoretical framework' broadly as a framework guiding a research study but, in contrast to Cai et al. (2019), for us, such a framework is a collection of one or more theoretical constructs (as defined above), which frames—i.e., provides the foundation for—the conceptualisation, design or carrying out of the study, including its interpretations, analyses or inferences. The elements of a theoretical framework do not have to be linked so as to form a full-fledged theory. In fact, the framework does not even have to be coherent or exhaustive but may take the shape of *bricolage* (Cobb, 2007; Gravemeijer, 1994) of singular theoretical constructs. Since a 'theoretical framework' does not necessarily involve any theory, it is a vaguer and less tight notion than 'theory' but, for exactly that reason, much more prevalent in mathematics education research.

What, then, is a 'theoretical approach' to a research enterprise? We suggest that this simply means that the *problématique* of the research is (at least in parts) primarily dealt with by way of conceptual and theoretical, as opposed to empirical, means of investigation. This typically involves invoking or creating some theoretical constructs in the considerations. Finally, a 'theoretical perspective' consists in adopting one or more theoretical approaches as part of the treatment of the *problématique.*

There are different purposes of theories or, more modestly, theoretical frameworks in mathematics education, namely, to provide:

- *explanation* of observed facts or phenomena within the domain supposedly covered by the theory
- *prediction* of the (possible) occurrence of phenomena
- *guidance* for action or behaviour
- a structured *set of lenses* through which parts of the world can be investigated.

Theories may also serve as a safeguard against unscientific approaches and procedures or as protection against opponents who are sceptical or hostile to mathematics education research. There are basically two categories of theories (theoretical frameworks) with regard to mathematics education research. Theories *of* or *about mathematics education as an object* deal with aspects of the teaching and learning of mathematics, for example, focusing on functions or on mathematical reasoning, whereas theories *in* or *for mathematics education research* are used by researchers in underpinning, framing and carrying out their investigations. As examples, we mention Piaget's theory of schemes and the theory of statistical testing. We might add *meta-theories* (e.g., didactical engineering or the didactics of mathematics as a design science), which have mathematics education research as their object of study. However, we abstain from going further into such theories here.

A number of observations can be made concerning the nature, place and role of theory in mathematics education. Firstly, the very notion and concept of 'theory' is gaining increasing importance in mathematics education research. Yet, there are no well-established, unified and exhaustive theories *of/about* mathematics education as an object, neither *in/for* mathematics education research. The same is true of meta-theories about mathematics education research as an object. Instead, diversity prevails and does so to an increasing extent. Secondly, theories of/about and in/for mathematics education (research) used to be, and many still are, borrowed from other fields, for example: mathematics itself, including its history, epistemology and sociology; statistics and psychometrics; general philosophical theories (especially about epistemology); general psychological theories (especially about learning and cognition); neuroscience; general theories of education and pedagogy (especially about curriculum and teaching); linguistic theories (including semiotics); and theories from the social sciences. However, we are now witnessing a strong movement away from 'theory borrowing' only, towards 'theory building' (Lesh et al., 2014). Thirdly, our field is moving away from 'fights' between theoretical positions, towards *bricolage* (Cobb, 2007; Gravemeijer, 1994) and networking of theories (Bikner-Ahsbahs & Prediger, 2010; Prediger et al., 2008; Radford, 2008).

Theories/theoretical frameworks of/about mathematics education as an object and in/for mathematics education research have different target levels (or grain sizes), ranging from *local* theories with a rather specific focus on a particular topic or issue, to *medium-level* theories with a broader, yet far from universal, focus on a generic set of topics or issues across several domains. As already mentioned, there are not really any global or grand theories that purport to cover all aspects of mathematics education or mathematics education research. Identifying, characterising and analysing local and medium-level theories is a massive task that goes far beyond the scope of a book chapter. So, we shall confine ourselves, here, to briefly listing candidates for theories of the two kinds. Local theories (theoretical constructs or frameworks) include the 'concept definition'/'concept image' distinction (Tall & Vinner, 1981; Vinner & Hershkowitz, 1980) and the 'theory of *Grundvorstellungen*', i.e., fundamental mathematical ideas, (vom Hofe & Blum, 2016), both of which have a cognitive learning focus. Medium-level theories are the most prevalent ones and include Brousseau's (1997) 'theory of didactical situations', which deals with aspects of the relationship between teaching and learning mathematics' and Duval's (2006) 'semiotic registers', which has an epistemological learning focus but does not deal with teaching. The theoretical status of such theories and theory germs is highly diverse and often unclear. The majority are primarily conceptual frameworks, inspired by philosophical considerations. To the extent they have empirical components, these are predominantly interpretive. The roles in mathematics education research and development of these constructions are numerous. Sometimes a theory first and foremost offers terminology. Sometimes a theory offers an overarching perspective from which mathematics education research can be conducted or viewed. Sometimes a theory aims to organise sets of observations and interpretations. And sometimes a theory offers a methodology for carrying out research.

## 3 Placing the KOM Framework in the Landscape of Theoretical Frameworks

In order to be able to place the KOM framework in relation to the set of notions dealt with in the previous section, we begin by briefly outlining the key components of this framework (Niss & Højgaard, 2011, 2019; Niss & Jensen, 2002).

The fundamental idea behind the framework is to come to grips with what it means to do, or, more ambitiously, to master mathematics. In other words, what does it mean to possess mathematical competence? So, the primary focus is not to capture what it means to know mathematics but to capture what competent *enactment* of mathematics amounts to. A related fundamental idea is that characterising mathematical competence should be of an overarching and generic nature and hence should not depend on any particular mathematical content, nor refer to any particular education level. Thus, the KOM framework takes its point of departure in the following definition:

> *Mathematical competence* is someone's insightful readiness to act appropriately in response to all kinds of *mathematical* challenges pertaining to a given situation. (Niss & Højgaard, 2019, p. 12)

A key issue then is to identify the most important constituents in mathematical competence, or differently put, in the (successful) enactment of mathematics. The approach taken is to focus on what can be considered the core of such enactment, namely posing and answering questions within, about, and by means of mathematics. The ability to pose and answer such questions requires mastery of the language and tools of mathematics. Taken as a whole, being able to pose and answer mathematics-related questions and to master the language and tools of mathematics constitutes mathematical competence (not to be mixed up with competency). But what does that involve, more specifically?

In the KOM framework this is conceptualised in terms of eight mathematical competen*cies* (in plural), based on the following definition:

> *A mathematical competency* is someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations. (Niss & Højgaard, 2019, p. 14)

It remains to identify and describe what specific sorts of challenges are being referred to in this definition. Eight different competencies and corresponding challenges are identified in the framework (for further details, see Niss & Højgaard, 2019). The first four of these, attempt to flesh out what it *means to pose and answer questions* in and by means of mathematics (Niss & Højgaard, 2019, pp. 15–16).

- *Mathematical thinking competency—engaging in mathematical enquiry*: The ability to relate to and pose the *kinds* of generic questions that are characteristic of mathematics and relate to the *kinds* of answers that may be expected to such questions. The competency also includes the ability to relate to the varying scope of mathematical concepts and terms, and to different categories of mathematical statements as well as to the nature and roles of abstraction and generalisation.
- *Mathematical problem handling competency—posing and solving mathematical problems*: The ability to pose (including identify, specify and formulate) and to solve different kinds of mathematical problems. This involves the ability to devise and implement solution strategies. The competency also includes the ability to critically analyse and evaluate attempted problem solutions.
- *Mathematical modelling competency—analysing and constructing mathematical models*: The ability to construct mathematical models in order to deal with extra-mathematical questions, contexts and situations, as well as the ability to critically analyse and evaluate extant or proposed models.
- *Mathematical reasoning competency—assessing and producing justification of mathematical claims*: The ability to analyse or produce written or oral arguments (chains of statements linked by inferences) put forward to justify mathematical claims.

Posing and answering questions in and by means of mathematics requires the ability to *handle the language, constructs and tools of mathematics*, which is conceptualised in terms of the following four competencies (Niss & Højgaard, 2019, pp. 17–18):

- *Mathematical representation competency—dealing with different representations of mathematical entities*: The ability to interpret, translate, move between, choose amongst and employ various types of mathematical representations (verbal, material, symbolic, tabular, graphic, diagrammatic, visual), as well as the ability to relate to the uses, scopes and limitations of such representations.
- *Mathematical symbols and formalism competency—handling mathematical symbols and formalisms*: The ability to relate to and deal with mathematical symbols, symbolic expressions and transformations, as well as with the rules and formalisms that govern them. This ability has a receptive side—decoding, interpreting and assessing extant symbolic expressions and transformations—as

well as a constructive side—introducing and employing symbols and formalisms in dealing with mathematical contexts and situations.

- *Mathematical communication competency—communicating in, with and about mathematics*: The ability to engage in different kinds of written, oral, visual or gestural mathematical communication, either as a recipient and interpreter of others' mathematical communication or as an active, productive communicator.
- *Mathematical aids and tools competency—handling material aids and tools for mathematical activity:* The ability to deal with material aids and tools (physical objects or instruments, special papers or charts and digital technologies) designed to facilitate mathematical work. This includes critically analysing and reflectively employing such aids and tools in mathematical activity whilst paying attention to their affordances and limitations.

There are four important remarks that should be made on these competencies. Firstly, the competencies are of a cognitive nature, i.e., no affective or volitional components are involved in their definition. It is not because such components are considered insignificant, on the contrary, but they are conceptually and substantively different from the cognitive ones and hence should not be mixed up with them. Secondly, as is evident from the above descriptions, each competency is born with an inherent duality between a *receptive* facet and a *constructive* facet. The receptive facet is to do with an individual's ability to relate to and navigate with respect to extant considerations and processes. The constructive facet focuses on the individual's ability to independently activate the competency and constructively employ it in dealing with given contexts and situations. Thirdly, it is a crucial idea that all the competencies are mutually overlapping, yet clearly distinct from each other. This is depicted in the so-called *competency flower*, Fig. 1.

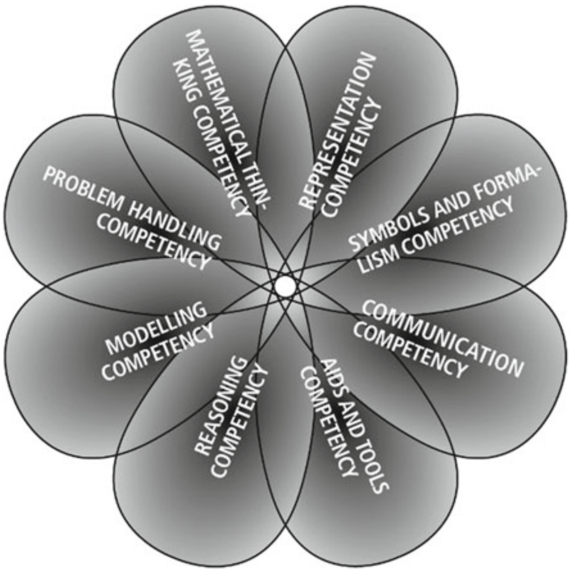



**Fig. 1** The mathematical competency flower (Niss & Højgaard, 2019, p. 19)

Fourthly, the definitions and descriptions of the competencies presented above do not involve particular mathematical content or topic areas, nor particular education levels. However, they do make sense across all such areas and levels. This is analogous to the usual characterisation of linguistic competence as consisting of four competencies (or two, if one prefers to pair them) (Gregersen et al., 2003): The ability to understand and relate to other people's oral speech, the ability to understand and relate to other people's written texts, the ability to express and make oneself understood orally, and the ability to express and make oneself understood in writing.

The KOM framework also conceptualises an individual's *possession of each competency* at a given point in time in terms of three (qualitative) dimensions: the degree of coverage, the radius of action and the technical level of possession. The *degree of coverage* specifies how large a part of the definitorial characteristics of the competency that individual masters. The *radius of action* concerns in how large a set of diverse contexts and situations the individual can activate the competency. Finally, the *technical level* specifies the set of mathematical concepts, methods, theories and results that the individual can activate when exercising the competency. The individual's current possession of the competency then is represented as an idealised three-dimensional box composed of the components just mentioned. *Progression* in possession of a competency then amounts to an increase in one or more of these dimensions and a decrease in none, which corresponds to some form of expansion of the box. Also, if just one of these dimensions takes the 'value zero' for some individual, the box collapses and hence that individual does not possess the competency at issue at the given point in time.

Mathematical competence and the eight mathematical competencies that constitute mathematical competence all are to do with people's enactment of mathematics. However, the KOM framework also involves a characterisation of what it means to come to grips with the state and role of *mathematics as a discipline*, beyond the enactment of mathematics and the possession of mathematical competence. The framework identifies three kinds of *overview and judgement* concerning mathematics as a discipline (see Niss & Højgaard, 2019, pp. 24–25). The first of these is the *application of mathematics* as it actually takes place within other disciplines or fields of practice, in science and society. The application of mathematics is always brought about—explicitly or implicitly—by way of mathematical models and modelling. The second kind is the *historical development* of mathematics, seen from internal as well as from socio-cultural perspectives, with a focus on the driving forces and mechanisms behind this development. The third kind of overview and judgement concerns the *nature of mathematics* as a discipline, with an emphasis on the similarities and differences between mathematics and other fields and disciplines. These kinds of overview and judgement are all to do with posing and answering questions *about mathematics as a whole*, thus complementing the questions about the enactment of mathematics that are on the agenda in mathematical competence and competencies.

Let us summarise the constructs of the KOM framework. The framework attempts to capture and characterise what it means to be able to enact mathematics with some degree of mastery and what it means to come to grips with the state and role of mathematics as a discipline. This is done by first introducing the theoretical construct of mathematical competen*ce* and its main constituent constructs in terms of eight distinct, but mutually overlapping, mathematical competen*cies*, as well as three kinds of overview and judgement, which address the most significant aspects of mathematics as a discipline as it manifests itself in culture, science and society. The first four competencies concern posing and answering questions within and by means of mathematics, whereas the last four competencies deal with mastering mathematical language, constructs and tools. The competencies are figuratively organised as a flower with eight petals. Each competency has a dual nature as it consists of a receptive and a productive facet. The possession of a given competency is conceptualised in terms of three dimensions, which (are likely to) develop over time. The three types of overview and judgement are also to do with posing and answering questions, here, however, *about* mathematics, rather than within and by means of mathematics.

The constructs just mentioned form the primary concepts of the KOM framework. There are also lots of secondary concepts in the framework that are involved in spelling out and detailing the primary concepts.

The *fundamental claims* of the framework are that: (1) the enactment of mathematics can, in fact, be comprehensively captured—conceptually—in terms of mathematical competence and its constituent eight competencies (each of which has a dual nature) and their interrelations; (2) the competencies make sense in relation to any mathematical content and at any education level; (3) possession of a competency can be characterised in terms of three dimensions (degree of coverage, radius of action, technical level); and (4) the state and role of mathematics as a discipline can be characterised in terms of three kinds of overview and judgement (the actual application of mathematics, the historical development of mathematics, the specific nature of mathematics). These claims can be perceived as the axioms of the framework. The framework also contains a number of derived claims: Mathematical competence (and competencies) and overview and judgement concerning mathematics as a discipline are two independent constructs in the sense that none of them can be determined in terms of the other, even though there is, of course, a multitude of connections between them. The eight competencies have a non-empty intersection but are, nevertheless, clearly delineated and distinct. If we focus on the exercise of any one of the competencies the others can be involved as 'auxiliary troops' in various ways, depending on the context and situation at issue. Both the fundamental and the derived claims are obtained by analytic reflection on mathematical activity and practices, rooted in experiences of a wide variety of mathematical enactment, as well as by conceptual analyses based on the definitions of the constructs.

Against this background, we can now conclude that the KOM framework does not satisfy the criteria for a full-fledged theory as presented in Sect. 2. Rather it is a theoretico-analytic framework that perhaps may be perceived as a theory in the making, even though this would indeed be a long-term process. But a framework for what? For theoretical and empirical investigations of the mastery of mathematics in generic terms, as well as for overview and judgement concerning the state and role of mathematics as a discipline, both when it comes to individual human beings and to the explicit and implicit manifestation of these notions in curricula, teaching/learning materials, assessment and actual mathematics teaching. Such use is first and foremost of an *analytical* nature. However, the framework can also be used for *normative* purposes in the design of curricula, teaching/learning materials, modes and instruments of assessment, teaching/learning environments and for the orchestration and implementation of teaching/learning activities.

The KOM framework also contains a (largely *normative*) sub-framework for capturing the competencies of competent mathematics teachers. This sub-framework of course relates to the KOM framework, but in some respects, it can, in fact, stand alone.

A competent mathematics teacher can effectively foster the development of students' mathematical competencies. This of course requires that the teacher possesses these competencies him/herself, at least at a level corresponding to the education level at which he or she teaches. But much more than that is needed, namely didactico-pedagogical competencies specifically related to mathematics. The KOM framework identifies six such competencies.

- *Curriculum competency,* i.e., to analyse, relate to and implement existing mathematics curricula and syllabi and to construct (parts of) new ones.
- *Teaching competency*, i.e., to devise, plan, organise, orchestrate and carry out mathematics teaching, including creating a rich spectrum of teaching/learning situations; find, judge, select and create teaching materials; inspire and motivate students; discuss curricula and justify teaching/learning activities to students.
- *Uncovering learning competency*, i.e., to uncover, interpret and analyse students' learning of mathematics as well as their notions, beliefs and attitudes regarding mathematics. This includes identifying and charting the learning development of the individual students.
- *Assessment competency*, i.e., the ability to identify, characterise and assess students' learning outcomes and mathematical competencies, so as to assist and inform individual students and other relevant parties. This includes selecting, modifying, constructing, critically analysing and implementing a varied set of assessment modes and instruments to serve a variety of formative and summative purposes.
- *Collaboration competency*, i.e., the ability to collaborate with different kinds of colleagues within and outside mathematics, as well as with others (e.g., parents, authorities), about mathematics education and its conditions.
- *Professional development competency*, i.e., the ability to develop one's own competency as a mathematics teacher (in fact a meta-competency), including participating in and relating to activities of professional development, such as in-service courses, projects, conferences; reflecting upon one's own teaching and needs for development; keeping oneself updated on new developments and trends in research or practice.

Although these mathematics teacher competencies make sense and are relevant irrespective of which notion of mathematical mastery teachers (are supposed to) adopt, they are instantiated in a special way if that notion is based on mathematical competencies as in the KOM framework.

## 4 An Example Related to Mathematical Modelling

The last half-century has witnessed an ever-growing interest within mathematics education circles in the application of mathematics in a wide variety of extra-mathematical domains. A key feature of this development is the realisation that mathematical models form the crucial ingredient in the external application of mathematics. Hence, the construction of such models, also known as *mathematical modelling*, becomes highly significant. Initially, the interest in mathematical applications, models and modelling was primarily cultivated in circles preoccupied with tertiary education focusing on putting mathematics to use in different extra-mathematical domains such as engineering, physics, life sciences and economics. At an early stage, only relatively few mathematics educators working in or on general mathematics education paid substantive attention to mathematical applications, models and modelling. From the 1980's onwards, more and more mathematics educators at large became engaged in the teaching and learning of mathematical models and modelling as a domain of research and development. This is reflected in the step-wise establishment of an *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA, cf. ictma.net), which was accepted as an affiliated study group of the *International Commission on Mathematical Instruction* in 2003.

Within this community, various theoretical frameworks for research and development have been produced over the years, many of which are relatively independent of other frameworks in mathematics education, especially in the early stages of development in the field. Here, we focus on one such theoretical framework. Its core is the so-called *modelling cycle*, which consists of the conceptually necessary stages in the construction of any mathematical model, typically represented by a diagram (Niss & Blum, 2020). Different variants of the modelling cycle are in use in the field. They primarily differ in the extent to which they detail some stages of the modelling process more than others, but they are all derived from the same basic modelling cycle:

1. Structuring and analysing a situation belonging to some extra-mathematical domain.
2. Mathematising the situation, i.e., translating selected objects (and relations between them) and questions about them from the extra-mathematical situation into objects, relations and questions belonging to some chosen mathematical domain.
3. Answering (by way of mathematical considerations) the mathematical questions arising from the mathematised situation.
4. Translating the mathematical answers obtained in the mathematical domain back to extra-mathematical answers pertaining to the corresponding domain.
5. Validating the model outcomes and evaluating the model with respect to its quality and its relevance for the purpose for which it was constructed.
6. Modifying the model or constructing a new one in case the model is deemed deficient or unsatisfactory.

The modelling cycle is to be perceived as an analytic reconstruction of the elements necessarily present, explicitly or implicitly, in any modelling process. It is not meant to be a description of the itinerary that concrete modellers must or do follow when actually performing modelling. Around the establishment of the modelling cycle, the community has developed a rich terminology coined to capture and describe a plethora of aspects of mathematical modelling, both per se and in relation to mathematics teaching and learning. This terminology now forms part of the 'standard' framework for research on the teaching and learning of mathematical modelling. In such research (and development), the modelling cycle constitutes a reference point used to frame and underpin research studies, be they empirical or theoretical.

There is an intimate relationship between the framework focusing on the modelling cycle and the modelling competency of the KOM framework. As mentioned in Sect. 3, the modelling competency consists in the ability to construct mathematical models in order to deal with extra-mathematical questions, contexts and situations, as well as the ability to critically analyse and evaluate extant or proposed models. Attempts to detail the content of this competency oftentimes make use of the modelling cycle as outlined above. Some researchers (e.g., Maass, 2006) even speak of modelling competen*cies* in the plural for the ability to undertake the individual processes in the modelling cycle, whereas others (e.g., Galbraith & Stillman, 2006) prefer to speak of the ability to undertake a given process in the modelling cycle, e.g., mathematisation, as a *sub-competency.* This difference corresponds to a difference in conceptual approach amongst two factions of mathematical modelling researchers, those who perceive the notion of modelling competen*cy* as nothing but an aggregation of a bunch of competen*cies* and those who perceive modelling competency as a separate integral entity, in which a set of sub-competencies linked to the stages of the modelling cycle can be discerned.

This section exemplifies how the KOM framework can constructively communicate with a framework, the modelling cycle, designed to underpin research and development on the teaching and learning of mathematical modelling. The two frameworks mutually fertilise each other in that each framework provides additional perspectives and concretisation to the other. Perhaps this is not too surprising, since some researchers have been engaged in developing both frameworks.

## 5 An Example Related to Overview and Judgement Regarding the Historical Development of Mathematics

Niss and Højgaard (2019) explained the relationship between mathematical competencies and overview and judgement concerning mathematics as a discipline as follows:

> ... whilst the enactment of mathematics in a variety of challenging situations and the exercise of mathematical competencies do of course represent and draw upon crucial aspects of mathematics as a discipline, these aspects do not in and of themselves provide learners and practitioners with a structured and coherent knowledge about and image of mathematics as a discipline. (Niss & Højgaard, 2019, p. 24)

As mentioned, the KOM framework's three types of overview and judgement embrace essential features of mathematics as a discipline that are found crucial for mathematics' relationships with nature, society and culture, as part of the notion of mastery of mathematics. In this section, we address the second type: overview and judgement concerning the historical development of mathematics, seen from an internal as well as from a socio-cultural perspective.

In a similar way that mathematical modelling may serve both as a goal in and of itself and as a means for fostering mathematical understanding (Niss, 2009), so may the use of history of mathematics (Jankvist, 2009). For example, developing students' mathematical modelling competency as part of a mathematics programme may be a curricular aim in itself, whereas using modelling to support the teaching of mathematical concepts—e.g., modelling instantaneous velocity motivates and underpins the concept of the derivative—is an example of using modelling as a tool for something else. Similarly, the history of mathematics may be used as a cognitive, pedagogical, or motivational tool, but it may also be seen and used as a goal in and of itself. This should not be confused with the history of mathematics viewed as an independent topic (as taught *per se*), but as a mathematics education goal in the sense of making students aware that mathematics has *a* history and has developed as a discipline over millennia. And this is where the KOM framework's second type of overview and judgement enters the picture.

The focus of this type of overview and judgement is the fact that mathematics has developed in culturally and socially determined environments and is subject to external needs within other disciplines and fields of practice, as well as being subject to internal goals serving its own theoretical aims. This type of overview and judgement is exemplified by questions related to how mathematics has developed through the ages; what internal and/or external driving forces have motivated the development; what types of actors were involved; what the interplay was with other scientific fields; etc. Addressing such questions is of course closely related to research on history, including history of mathematics. Hence, in order to develop this type of overview and judgement with students, one may draw on—connect—theoretical constructs from both research in history and on history of mathematics, as well as, of course, constructs from mathematics education research. We exemplify this in the following.

One distinction from the field of history of mathematics that lends itself to the second type of overview and judgement is that between Whig and non-Whig historical writing. The notion of Whig history is due to the British historian Herbert Butterfield, who in 1931 defined this as a way of measuring the past in terms of the present (Butterfield, 1973). Differently put, "what one considers significant in history is precisely what leads to something deemed significant today" (Fried, 2001, p. 395). Kragh (1987) calls this type of history anachronistic and states that within such an historiography it is "considered legitimate, if not necessary, that the historian should 'intervene' in the past with the knowledge that he possesses by virtue of his placement later in time" (p. 89). Kjeldsen et al. (2004) state that a mathematician studying the history of his or her subject often is inclined to take such an approach and, hence, judge the contents of earlier mathematics by applying the standards of modern mathematics. Within the history of mathematics, Rowe (1996) refers to such historians as mathematical historians (as opposed to the cultural historians). Extreme examples of mathematical historians are members of the Bourbaki group, in particular, Jean Dieudonné and André Weil. For instance, in a discussion of why (and how) to do history of mathematics, Weil (1978) claimed that "it is impossible for us to analyse properly the contents of Book V and VII of Euclid's without the concept of group and even that of groups with operators, since the ratios of magnitudes are treated as a multiplicative group operating on the additive group of the magnitudes themselves" (p. 232). A (cultural) historian, on the other hand, will approach the history from a non-Whig point of view and will, thus, "typically look for differences in the mathematics of different times and in different locations and explore historical changes in mathematics, without using modern ideas as a yardstick" (Kjeldsen et al., 2004, p. 12). From the point of view of the second type of overview and judgement, which essentially aims to use history as a goal in mathematics education, it is clear that the sole use of the Whig approach to history cannot achieve this aim. This does not imply, of course, that such an approach has nothing to offer to mathematics education. A modern understanding of past mathematics can in fact shed light on the conceptual and structural features of past mathematics, as long as it is kept in mind that such an understanding is not identical to past mathematicians' understanding of what they were doing.

One way to pursue a non-Whig use of history in the mathematics classroom is by using primary historical source materials since these are not pre-digested by textbook authors and others —and are in fact open to students' own interpretation (Fried, 2001). However, although rewarding, both in terms of history as a goal and history as a tool, the use of primary sources is demanding for students as well as for teachers (e.g., Jahnke, 2000). Still, if willing to go 'the extra mile', students' learning outcome is indeed promising (e.g., Barnett et al., 2014; Jankvist, 2013; Kjeldsen & Blomhøj, 2012). Based on two experiences of using primary historical material, Kjeldsen et al. (2022) analysed the involved students' reflections with respect to how and in what sense the work using such material developed what they refer to as the students' (mathematico-)historical awareness—inspired by the Danish historian Jensen (2003). More precisely, they offer a categorisation of students' reflections with respect to: (1) reflections on what mathematicians do when producing mathematics, providing students with some insights into 'mathematics in the making'; (2) the extent to which the students look at the sources with an enlightenment purpose ('observer history') or from a forward-looking perspective ('action history'); and (3) effects of developing historical awareness for students' relationship with mathematics in their (future) lives. (To some extent, the first type concerns an anti-Whig perspective, whereas the second is more closely related to a Whig one.) This is to say that the notion of historical awareness may give some further 'flesh' to the second type of overview and judgement when it comes to using primary historical sources in the classroom. Hence, from a history of mathematics point of view, the second type of overview and judgement may be connected with the construct of distinguishing between Whig and non-Whig history.

From a mathematics education research point of view, students' overview and judgement of mathematics as a discipline consist of a complex combination of both knowledge—mathematical as well as historical—and beliefs about mathematics as a discipline (Jankvist, 2015). Now, *beliefs* are often defined as "lenses through which one looks when interpreting the world" (Philipp, 2007, p. 258). Beliefs are related to emotions and attitudes, yet they are of a cognitive and more stable nature. According to Green (1971), beliefs are organised in clusters, containing both central and peripheral beliefs. Beliefs within a cluster are interrelated. Hence, any change in beliefs may have consequences for the entire cluster, or for parts of it. Central beliefs can be psychologically important to their owner, which also partly explains why changing students' beliefs may be both difficult and time-consuming (Green, 1971). In a context of using history of mathematics as an essential means for developing upper secondary school students' overview and judgement regarding the historical development of mathematics, Jankvist (2015) argued that changes occur first in students' *views*. Here, we consider "views to be something less persistent than beliefs, but with the potential to develop into beliefs at a later point in time" (p. 53). Jankvist (2015) further defined students' *images of mathematics as a discipline* to be made up of their beliefs as well as of their views. 'Mathematics as a discipline' refers to the fact that students may possess beliefs about different aspects of what they have experienced as mathematics. Hence, students' images of mathematics as a discipline are closely related to their overview and judgement. The knowledge part of the latter may come into play if we (with Green, 1971) distinguish between beliefs that are evidentially held and beliefs that are non-evidentially held. Evidentially held beliefs are based on experiences or reason, whereas non-evidentially held beliefs stem from other sources of influence (e.g., teachers, parents, society) or are derived from already existing beliefs. Non-evidentially held beliefs are typically rather difficult to change through reason and may be thought of as convictions that cannot be argued against. In the study reported by Jankvist (2015), the students were given an opportunity to become aware of their own beliefs (about mathematics as a discipline) and then were confronted with evidence—in the form of concrete cases from the history of mathematics—to test their beliefs and views and possibly change them. Jankvist (2015) concluded: "Not until students have access to evidence—or counter-evidence—are they likely to criticise rationally, reason about and reflect upon their images of mathematics as a discipline and possibly accommodate and change them" (p. 55).

## 6 An Example Related to the Didactico-Pedagogical Competency of Uncovering Learning

Uncovering students' learning of mathematics is a complex and multifaceted undertaking which is part of any mathematics teacher's job; for example, when he or she is monitoring students' learning outcomes of his or her own teaching, is providing formative feedback to students and other relevant parties, or when summatively assessing students' learning, whether as part of high stakes testing or examinations or as part of marking students' achievements throughout a certain period.

The crux of this undertaking evidently lies in the conception of what mathematical learning *is* and what the constituents of such learning are supposed to be. Does mathematical learning consist in knowing and citing some set of singular facts, such as definitions, propositions and other results? In conceptually linking different mathematical entities across topics? In understanding mathematical ideas, principles and processes? In the ability to exert a set of procedural skills? In the ability to prove mathematical statements and reason mathematically in unrehearsed situations? In the ability to solve mathematical problems? In possessing (certain) mathematical competencies? Or in…? Thus, uncovering students' learning is necessarily filtered through the notion of mathematical learning that directly or indirectly is adopted by the 'uncoverer'. Depending on what specific notion is adopted, uncovering mathematical learning is related to the ways in which this notion is conceptualised in the theory and practice of mathematics education. As regards research on this theme, interplay with some of the multiple constructs and frameworks that deal with the notion of learning obviously becomes relevant or even necessary. For instance, if the notion focuses on, say, linking different mathematical entities across topics, frameworks dealing with the construction, interpretation and use of concept maps lend themselves to the research endeavour. If the notion focuses on, say, the ability to solve mathematical problems, it is evident that frameworks for mathematical problem solving (e.g., Schoenfeld, 2014) ought to be taken into consideration and if the notion rather focuses on students' mathematical competencies, the KOM framework's mathematical competencies naturally enter the stage. Other relevant constructs and frameworks attempt to characterise mathematical understanding (e.g., Freudenthal, 1991; Sierpinska, 1994) or mathematical reasoning and proof (e.g., Balacheff, 1980; Harel & Sowder, 2007) and so on and so forth.

One thing is to settle the issue of what constitutes mathematical learning. Once this is in place, the next issue is whether and how it is possible to uncover actual students' learning. What means are available for this purpose? Here, the means for uncovering students' mathematical learning have a non-empty intersection with the means for assessing students' learning outcomes, especially the more sophisticated modes and instruments that actually try to base assessment on insights into students' learning. This implies that links with the numerous constructs and frameworks for assessment in mathematics education (e.g., Niss, 1993; OECD, 2013; Stacey & Turner, 2015) might well deserve to be explored. Suitable means for uncovering (and eventually assessing) students' learning include: observation of classroom, group or individual student work; face-to-face interviews; student portfolios; written questionnaires; student essays; student projects—short term or long term; student-produced posters; student lectures; student panel discussions; oral exams; written exam papers on problem tasks; etc. Each of these kinds of instruments can shed light on some—but not on all—aspects of students' mathematical learning. In-depth uncovering of such learning needs an array of different modes and instruments.

Uncovering students' mathematical learning is but one of six mathematics teacher competencies in the KOM framework, as presented in Sect. 3. On a final and general note, it should be mentioned that this (sub-)framework can, of course, be compared, contrasted and connected with other frameworks for (mathematics) teacher capabilities such as Pedagogical Content Knowledge (Shulman, 1987) and Mathematical Knowledge for Teaching (Ball, 1991).

## 7 Potentials for Connecting and Networking with Other Theoretical Constructs

Ideas concerning 'networking' of theories or theoretical perspectives have now been around for a couple of decades (see, e.g., Kidron et al., 2018). But what does 'networking' mean in this context? The very term suggests a graph-theoretical metaphor in which a set of nodes (also called vertices)—which here are theoretical entities such as theoretical perspectives, constructs, frameworks and theories—are connected (linked) pairwise in some way or another, thus forming a graph or a network. Links are typically called edges. It does not have to be the case that links exist between any pair of nodes and links may very well be of different kinds. The question in our context is what kinds of links are involved in the networks at issue? Here, as suggested by Prediger et al. (2008), a link may consist in comparing two nodes, or of contrasting them. It may also consist in combining them, coordinating them, making a synthesis out of them, integrating them, or using one as a means for the other. A fundamental issue for linking two theoretical entities is whether these represent two different ways of dealing with the *same* object(s) or phenomena, or whether they deal with *different* objects or phenomena. This is closely related to the even more fundamental issue of what the purpose of linking them is. It does not make sense to link two theoretical entities unless there is a purpose of doing so. Another issue is if a network contains only two entities or whether more entities are involved. Yet another issue is whether there is a logical or substantive ordering of two entities, such that one presupposes the other.

Prediger et al. (2008) talk about *connecting strategies* with respect to theories. A connecting strategy presupposes the existence of an underlying purpose of connecting the theories involved, which the strategy is supposed to pursue. In their paper, Prediger et al. focus on one particular aspect of such strategies: the degree to which a given theory is integrated with the other theories considered in the enterprise. This gives rise to a one-dimensional scale, stretching from 'ignoring other theories' to 'unifying globally' (see Fig. 2).

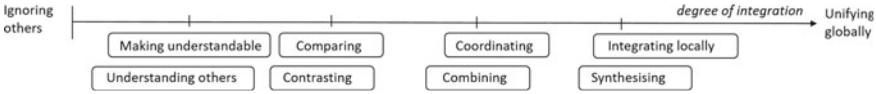



**Fig. 2** Connection strategies for networking of one theory with one or several other theories (based on Prediger et al., 2008, p. 170)

As regards the KOM framework, we gave three examples above of constructs that can be involved in networking with other constructs.

The modelling competency of the KOM framework was connected to the modelling cycle construct in the field of the didactics of mathematical modelling. One purpose of connecting these constructs is that one construct can shed light on and deepen the understanding of the other. Invoking the modelling competency in research on the modelling cycle makes it possible to focus on what it takes for someone to be able to perform the various steps and processes in the modelling cycle. Invoking the modelling cycle in research on the modelling competency provides us with a framework to scaffold a detailed understanding of the specifics of this competency in a variety of contexts. Thus, the two frameworks serve as mutual fertilisers for one another. The linking of the two constructs takes the form of *intertwining* them, i.e., putting them in direct productive contact with one another without losing the ability to clearly distinguish between them. In other words, it would be to go too far to consider the connection an integration of the two constructs.

Overview and judgement regarding the historical development of mathematics, another construct in the KOM framework, was connected to the distinction between Whig and non-Whig history, a general construct considered in history as a discipline, especially as regards the history of science, or more broadly the history of ideas. One purpose of this connection is that the Whig-non-Whig distinction can serve as one way, among others, to focus research on the historical development of mathematics as an element of mathematics education. Conversely, the construct of overview and judgement regarding the historical development of mathematics in an educational context can be one way of providing an exemplification of the nature and significance of the Whig-non-Whig distinction in research on the history of mathematics. The connection in play here is a combination of the two theoretical entities.

Similarly, the purpose of connecting overview and judgement regarding the historical development of mathematics with the construct of mathematics-related beliefs is for the two constructs to fertilise one another. Research on students' mathematics-related beliefs can be given a focus by invoking students' general views and understanding of the historical development of mathematics to (co-)shape their beliefs. And vice versa, research on students' overview and judgement regarding the historical development of mathematics can be enriched by implicating their mathematics-related beliefs. Here, the connection of the two constructs can be perceived as an example of coordinating two constructs.

When it comes to connecting the *uncovering of mathematical learning* competency with other constructs, the purpose depends on which aspects of mathematical learning are in focus. If students' development of mathematical competencies is in focus, the purpose of connection may be to consider methods for detecting and uncovering these competencies with given students, which would be an instance of combining two constructs. If instead, the learning aspect in focus is on uncovering students' ability to relate different mathematical concepts and propositions across topics, the purpose of linking this sort of uncovering with frameworks dealing with concept maps may well be to use concept maps as a *methodological means* to assist the uncovering of the ability under consideration here.

So far, we have concentrated on outlining how three specific constructs in the KOM framework can engage in networking with other selected constructs in mathematics education research. Due to the nature of the KOM framework and its overarching conceptualisation of mathematical mastery by means of eight mathematical competencies and three types of overview and judgement along with its six didactico-pedagogical mathematics teacher competencies, numerous other theoretical entities from mathematics education research naturally lend themselves to engage in networking with KOM constructs. In what follows, we briefly indicate a few other possibilities of connecting KOM constructs to other theoretical entities in mathematics education.

Since any of the mathematical competencies can only be exercised and developed in dealing with mathematical subject matter, the so-called '*Stoffdidaktik*', i.e., mathematics education research and development that addresses the structuring and sequencing of specific mathematical concepts, topics, methods, etc., which is especially cultivated in German-speaking communities, has a lot to offer. Examples of theoretical entities related to understanding that may be combined with elements of the KOM framework are Skemp's (1976) classical distinction between relational and instrumental understanding or Vergnaud's (2009) theory of conceptual fields (which also discusses mathematical competence explicitly).

If we focus on the different aspects of the eight competencies there certainly are possibilities for connecting these constructs with more local theoretical entities addressing aspects related to these. Regarding the mathematical reasoning competency, for example, Harel and Sowder's (2007) description of three overarching classes of so-called proof schemes, defined as that which, for a person or a community, constitutes ascertaining and persuading of the truth of some given statement, is an obvious candidate for networking. As final examples, we can link KOM's representation competency with Duval's (2006) framework of semiotic registers and representational shifts between them, whilst KOM's aids and tools competency can be linked with the so-called instrumental approach (Guin & Trouche, 1998), which describes the process of instrumental genesis, i.e., the process of becoming acquainted with an artefact, say a digital one and making it a personal instrument (e.g., Geraniou & Jankvist, 2019).

A multitude of other theoretical entities as candidates for networking with the KOM framework could be mentioned, but the ones listed suffice to suggest that this is likely to be a promising endeavour.

## 8 Towards Mutual Fertilisation

In the existing literature and in the previous sections we have encountered a variety of different ways to connect and network theoretical entities in mathematics education research. Connecting such entities should serve a purpose, and we have also encountered a variety of such purposes. When it comes to linking the KOM framework or some of its constructs with other theoretical entities, we suggest that at this stage of development, the general common purpose of such linking with a few exceptions is to obtain various forms of *mutual fertilisation* of the entities under consideration rather than to create new frameworks by synthesising or integrating them. However, it would be extremely valuable for furthering the theoretical development of our field to engage in analysing, comparing and contrasting different constructs and frameworks in considerable detail in order to uncover their similarities and differences. In view of its complexity, multifaceted nature and multiple ramifications, the KOM framework is a rich source for such an undertaking.

**Acknowledgements** This chapter is partly written under project 8018-00062B of *Independent Research Fund Denmark*.

The authors want to thank the reviewers of this chapter for several helpful comments and suggestions.

## References

Balacheff, N. (1980). Une étude, a l'aide de graphes, de démonstrations mathématiques formulées par des élèves. *Educational Studies in Mathematics, 11*(1), 91–111. https://doi.org/10.1007/BF00369161

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 1–48). JAI Press Inc.

Barnett, J. H., Lodder, J., & Pengelley, D. (2014). The pedagogy of primary historical sources in mathematics: Classroom practice meets theoretical frameworks. *Science & Education, 23*(1), 7–27. https://doi.org/10.1007/s11191-013-9618-1

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—an approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 483–506). Springer.

Bikner-Ahsbahs, A., & Prediger, S. (Eds.). (2014). *Networking of theories as a research practice in mathematics education.* Springer.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* In B. Nicolas, C. Martin, S. Rosamund, & W. Virgina (Eds.). Kluwer Academic Publishers.

Butterfield, H. (1973). *The Whig interpretation of history*. Penguin Books Ltd. (First published by Bell, 1931.)

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., Cirillo, M., Kramer, S. L., & Hiebert, J. (2019). Theoretical framing as justifying. *Journal for Research in Mathematics Education, 50*(3), 218–224. https://doi.org/10.5951/jresematheduc.50.3.0218

Chevellard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://www.jasme.jp/hjme/download/05_Yves%20Chevallard.pdf.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). Information Age Publishing.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–126). Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Freudenthal, H. (1991). *Revisiting mathematics education—China lectures*. Kluwer Academic Publishers.

Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science & Education, 10*(4), 391–408. https://doi.org/10.1023/A:1011205014608

Galbarith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *ZDM—Mathematics Education, 38*(2), 143–162. https://doi.org/10.1007/BF02655886

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Geraniou, E., & Misfeldt, M. (2022). The mathematical competencies framework and digital technologies. In U. T. Jankvist, & E. Geraniou, (Eds.), *Mathematical competencies in the digital era* (pp. 39–60). Springer.

Gravemeijer, K. (1994). Educational development and developmental research. *Journal for Research in Mathematics Education, 25*(5), 443–471. https://doi.org/10.2307/749485

Green, T. F. (1971). *The activities of teaching*. Mcgraw-Hill Book Company.

Gregersen, F., Elf, N. F., Lassen, S., Laursen, H. P., Lørring, L., Mortensen, F. H., & Voigt, K. (2003). *Fremtidens danskfag—en diskussion af danskfaglighed og et bud på dens fremtid* [The future of the subject Danish—A discussion of the characteristics of Danish as a subject now and in the future]. The Ministry of Education.

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*, 195–227. https://doi.org/10.1023/A:1009892720043

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Information Age Publishing.

Jahnke, H. N. (2000). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education, the ICMI study* (pp. 291–328). Kluwer Academic Publishers.

Jankvist, U. T. (2009). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261. https://doi.org/10.1007/s10649-008-9174-9

Jankvist, U. T. (2013). History, application and philosophy in mathematics education: HAPh—A use of primary sources. *Science & Education, 22*(3), 635–656. https://doi.org/10.1007/s11191-012-9470-8

Jankvist, U. T. (2015). Changing students' images of "mathematics as a discipline." *Journal of Mathematical Behavior, 38*, 41–56. https://doi.org/10.1016/j.jmathb.2015.02.002

Jensen, B. E. (2003). *Historie-livsverden og fag.* [History-life-world and subject.] Gyldendal.

Kidron, I., Bosch, M, Monaghan, J., & Palmér H. (2018). Theoretical perspectives and approaches in mathematics education research. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 254–268). Routledge—Taylor & Francis Group.

Kilpatrick, J. (2010). Preface to Part I. In B. Sriraman & L. English (Eds.), *Theories of mathematics education—Seeking new Frontiers* (pp. 3–5). Springer.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327–349. https://doi.org/10.1007/s10649-011-9352-z

Kjeldsen, T. H., Clark, K. M., Jankvist, U. T. (2022). Developing historical awareness through the use of primary sources in the teaching and learning of mathematics. In: C. Michelsen, A. Beckmann, V. Freiman, U. T. Jankvist, A. Savard (Eds.) *Mathematics and Its connections to the arts and sciences (MACAS): Vol. 19 Mathematics Education in the Digital Era* (pp. 45–68). Springer. https://doi.org/10.1007/978-3-031-10518-0_4

Kjeldsen, T. H., Pedersen, S. A., & Sonne-Hansen, L. M. (2004). Introduction. In T. H. Kjeldsen, S. A. Pedersen, & L. M. Sonne-Hansen (Eds.), *New trends in the history and philosophy of mathematics, Vol. 19 of studies in philosophy* (pp. 11–27). University Press of Southern Denmark.

Kragh, H. (1987). *An introduction to the historiography of science*. Cambridge University Press.

Leder, G. C., Pehkonen, E., & Torner, G. (Eds.) (2002). *Beliefs: A hidden variable in mathematics education*. Kluwer Academic Publishers.

Lesh, R., Sriraman, B., & English, L. (2014). Theories of learning mathematics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 615–623). Springer Science.

Maass. K. (2006). What are modelling competencies? *ZDM—Mathematics Education, 38*(2), 113–142. https://doi.org/10.1007/BF02655885

Niss, M. (2007). Reflections on the state of and trends in research on mathematics teaching and learning: From here to Utopia. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1293–1312). Information Age Publishing.

Niss, M. (2009). Perspectives on the balance between application and modelling and 'pure' mathematics in the teaching and learning of mathematics. In M. Menghini, F. Furinghetti, L. Giacardi, & F. Arzarello (Eds.), *The first century of the international commission on mathematical instruction (1908–2008), reflecting and shaping the world of mathematics education* (pp. 69–84). Instituto della Enciclopedia Italiana fondata da Giovanni Treccani.

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling*. Routledge.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Niss, M. (Ed.) (1993). *Cases of assessment in mathematics education—An ICMI study*. Springer Science.

OECD. (2013). *PISA 2012 assessment and analytical framework: Mathematics*. OECD Publishing. https://doi.org/10.1787/9789264190511-en

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Information Age Publishing.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches—First steps towards a conceptual framework. *ZDM—Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM—Mathematics Education, 40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Rowe, D. E. (1996). New trends and old images in the history of mathematics. In R. Calinger (Ed.), *Vita mathematica—Historical research and integration with teaching*, No. 40 in MAA notes (pp. 3–16). The Mathematical Association of America.

Schoenfeld, A. H. (2014). *Mathematical problem solving*. Elsevier Science.
Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review 57*(1), 1–22. https://doi.org/10.17763/haer.57.1.j463w79r56455411
Sierpinska, A. (1994). *Understanding in mathematics*. Studies in mathematics education series: 2. The Falmer Press.
Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.
Stacey, K., & Turner, R. (Eds.). (2015). *Assessing mathematical literacy—The PISA experience*. Springer.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619
Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727
Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *PME 4: Proceedings of the fourth international conference for the psychology of mathematics education* (pp. 177–184). University of California.
vom Hofe, R., & Blum, W. (2016). "Grundvorstellungen" as a category of subject-matter didactics. *Journal Für Mathematik-Didaktik, 37*, 225–254. https://doi.org/10.1007/s13138-016-0107-3
Weil, A. (1978). History of mathematics: Why and how. In O. Lehto (Ed.), *Proceedings of the international congress of mathematicians* (pp. 227–236). Academia Scientiarum Fennica, August 15–23, 1978.

# The Mathematical Competencies Framework and Digital Technologies

**Eirini Geraniou and Morten Misfeldt**

## 1 Introduction

There are rapid advances in technology and research have evidenced its great potential in mathematics education (e.g., Noss & Hoyles, 1996; Ruthven, 2011; Young, 2017). Technology-enhanced instruction with the support of well-designed digital tools has great benefits for students' mathematical learning and for addressing students' known difficulties with learning mathematics (e.g., Noss & Hoyles, 1996). Alongside the benefits and opportunities of technology in mathematics education, there are numerous challenges for both teachers and students that need to be addressed. Hence why the technology-enhanced mathematics education has not always met expectations (e.g., Drijvers et al., 2010). The teachers' perspectives and their abilities to develop a new repertoire of appropriate teaching practices for technology-rich classrooms play a crucial role in identifying the best strategies for supporting the successful integration of digital technologies (DT) in the mathematics classrooms (e.g., Bozkurt & Ruthven, 2017).

In recent decades, there have been a plethora of educational technologies offered to mathematics teachers, which can be helpful, but at the same time daunting. For example, Niss (2016) argued that DT can offer "marvels" and "disasters" to mathematics education. The outcomes of the use of DT very much depend on how it is perceived and used by students and/or teachers. As Niss (2016) wisely put

E. Geraniou (✉)
UCL Institute of Education, London, UK
e-mail: e.geraniou@ucl.ac.uk

M. Misfeldt
University of Copenhagen, Copenhagen, Denmark
e-mail: misfeldt@ind.ku.dk

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_3

it "The outcome crucially depends on the role and place of technology in the entire curriculum, […] and on the specific relationships that exist between that component and other curriculum components, as well as on the teacher's design and implementation of the teaching–learning environment and of the instructional sequences that (are supposed to) take place within this environment. Of course, the outcome also depends on the nature of the digital affordances offered by the ICT systems at issue and on their technical and pedagogico-didactic quality" (p. 248). Mathematics teachers' digital competencies are not sufficiently well-developed and, as a result, the provision of technology overwhelms teachers leading to sporadic, superficial and potentially less effective use of DT. Considering that teachers' competencies are positively correlated to teaching quality, which affects pupil outcomes (Kunter et al., 2013), then there is a need to consider mathematics teachers' mathematical competencies (Niss & Højgaard, 2019) as well as digital competencies (Ferrari, 2012; Hatlevik & Christophersen, 2013). Being influenced by Geraniou and Jankvist's (2019) work, which focussed on the interplay of students' mathematical competencies and digital competencies and introduced the term *mathematical digital competency*, we argue that there is a need to look into teachers' mathematical digital competencies too.

To conceptualize competence or in other words, the set of skills students need to have and teachers employ to teach, we draw on Niss and Højgaard's (2011, 2019) work regarding the KOM framework of mathematical competencies. This framework introduces the concept of mathematical competence, a number of mathematical competencies and their potential in mathematical teaching and learning (Niss & Højgaard, 2011; Niss & Jensen, 2002). Niss and Højgaard (2011) offered a description of mathematical competence stating that it "comprises having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role" (p. 49). In a more recent publication, as mentioned earlier, the same authors defined mathematical competence as "insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to situations" (Niss & Højgaard, 2019, p. 12).

In light of the KOM framework regarding students' mathematical competencies, Geraniou and Jankvist (2019) introduced Mathematical Digital Competency (MDC) in recognition of the need for learners to develop competence in using technology to solve mathematical problems. They proposed the possession of MDC if any of the following three characteristics are evident in students' actions:

- MDC1—"*Being able to engage in a techno-mathematical discourse*. […]
- MDC2—*Being aware of which digital tools to apply within different mathematical situations and context and being aware of the different tools' capabilities and limitations*. […]
- MDC3—*Being able to use digital technology reflectively in problem solving and when learning mathematics*" (Geraniou & Jankvist, 2019, p. 43).

It is equally important to support teachers' development of MDC to support their students' development of MDC. We understand teachers' mathematical digital competencies (MDCs) as the set of skills teachers need (or have) to select and implement technology in mathematics teaching and learning in productive ways.

In their 2019 paper, Niss and Højgaard focussed on "*What would an up-to-date conceptual account of the notions of mathematical competence and of mathematical competencies look like?*" (p. 9). In this chapter we add to that question the potential impact technology may have on the notions of mathematical competence. Of course, one of the KOM mathematical competencies, the aids and tools competency, refers to DT and in particular how DT are designed to represent and facilitate mathematical work (Niss & Højgaard, 2019, p. 18), but we argue that DT play a role in all the KOM mathematical competencies. In more detail, we look into how technology may influence mathematical competencies, as shared in the KOM framework, the students' eight mathematical competencies and meta-competencies and the teachers' six mathematical competencies. We present each of those three sets of competencies in the KOM framework and exemplify how technology impacts and influences them. We conclude by discussing the potential need of revising the KOM framework to include mathematical digital competencies to support the argument that technology inadvertently transforms mathematical competencies. The aim of this chapter is to exemplify the impact of digital technologies on students' and teachers' mathematical competencies and meta-competencies and therefore build a case for the need for expanding the KOM framework's notion of competencies, teacher competencies and meta-competencies (described as overview and judgement), in a way that goes beyond how MDC expands/augments the eight mathematical competencies.

## 2 Mathematical Competencies[1]

The KOM framework's main mathematical competence is split into eight distinct, yet related, competencies, which are divided into two groups. The first group comprises the first four competencies and refers to "*the ability to ask and answer questions in and with mathematics*" (Niss & Højgaard, 2011, p. 50). The second group comprises the last four competencies and refers to "*the ability to deal with mathematical language and tools*" (Niss & Højgaard, 2011, p. 50). Below we define each of the eight mathematical competencies referring to Niss and Højgaard's (2011, 2019) papers (Fig. 1).

[1] This section restates concepts that were introduced in chapter one of this volume, in order for the chapter to be readable on its own.

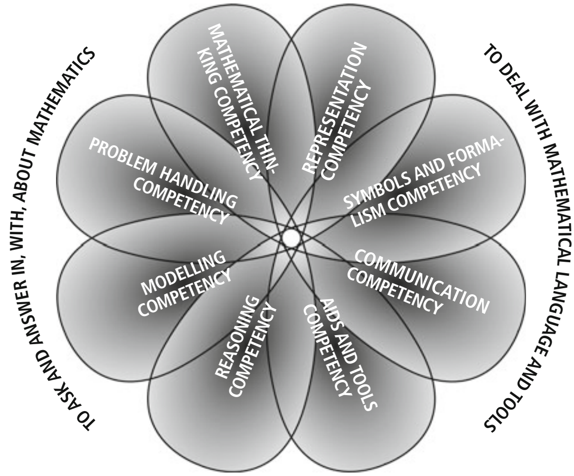



**Fig. 1** The eight overlapping mathematical competencies as diagrammatically presented by Niss and Højgaard (2019, p. 19) adapted from Niss and Jensen (2002) and referred to as the "KOM flower"

## *2.1 First Set: Posing and Answering Questions in and by Means of Mathematics*

1. The Mathematical Thinking competency is about being able to relate to
   a. "and pose the kinds of generic questions that are characteristic of mathematics and relate to the nature of answers that may be expected to such questions" […]
   b. "the varying scope, within different contexts, of a mathematical concept or term, as well as distinguishing between different types and roles of mathematical statements (including definitions, if–then claims, universal claims, existence claims, statements concerning singular cases and conjectures) and navigating with regard to the role of logical connectives and quantifiers in such statements, be they propositions or predicates". […]
   c. and propose " "abstractions" of concepts and theories and "generalization" of claims (including theorems and formulae) as processes in mathematical activity" (Niss & Højgaard, 2019, p. 15).

2. The Mathematical Problem handling competency is about being able to
   a. "pose (i.e., identify, delineate, specify and formulate) and to solve different kinds of mathematical problems within and across a variety of mathematical domains" […]
   b. "critically analyze and evaluate one's own and others' attempted problem solutions" […]

c. “devise and implement strategies to solve mathematical problems” (Niss & Højgaard, 2019, p. 15).

3. The Mathematical Modelling competency is about being able to

a. “construct […] mathematical models [...] to deal with extra-mathematical questions, contexts and situations”, […]
b. “critically analyze and evaluate existing or proposed models, whilst taking purposes, data, facts, features and properties of the extra-mathematical domain being modelled into account, are the core of this competency.” […]
c. relate to and navigate “within and across the key processes of the “modelling cycle” in its various manifestations (e.g., Blomhøj & Jensen, 2003; Blum & Leiß, 2007; Niss, 2010).” (Niss & Højgaard, 2019, p. 16).

4. The Mathematical Reasoning competency involves the ability to

a. “analyze or produce arguments (i.e., chains of statements linked by inferences) put forward in oral or written form to justify mathematical claims”
b. “both constructively provid[*e*] justification of mathematical claims and critically analys[*e*] and assess[] existing or proposed justification attempts. The competency deals with a wide spectrum of forms of justification, ranging from reviewing or providing examples (or counter-examples) over heuristics and local deduction to rigorous proof based on logical deduction from certain axioms” (Niss & Højgaard, 2019, p. 16).

## *2.2 Second Set: Handling the Language, Constructs and Tools of Mathematics*

1. The Mathematical Representation competency is about the ability to

a. “interpret as well as translate and move between a wide range of representations (e.g., verbal, material, symbolic, tabular, graphic, diagrammatic or visual) of mathematical objects, phenomena, relationships and processes”, […]
b. “reflectively choose and make use of one or several such representations in dealing with mathematical situations and tasks”, […]
c. relate “to the scopes and limitations—including strengths and weaknesses—of the representations involved in given settings.” (Niss & Højgaard, 2019, p. 17).

2. The Mathematical Symbols and Formalism competency is about the ability to

   a. "relate to and deal with mathematical symbols, symbolic expressions and transformations, as well as with the rules and theoretical frameworks (formalisms) that govern them, constitutes the key component of this competency" […]
   b. decode and interpret "instances of symbolic expressions and transformations, as well as formalisms, already present"
   c. introduce and employ "symbols and formalism in dealing with mathematical contexts and situations" (Niss & Højgaard, 2019, p. 17).

3. The Mathematical Communication competency is about an individual's ability to

   a. "engage in written, oral, visual or gestural mathematical communication, in different genres, styles and registers and at different levels of conceptual, theoretical and technical precision, either as an interpreter of others' communication or as an active, constructive communicator, constitutes the core of this competency" (Niss & Højgaard, 2019, p. 17).

4. The Mathematical Aids and Tools competency focuses on the ability to

   a. deal "with material aids and tools for mathematical activity, ranging from concrete physical objects and instruments, over specially designed papers and charts, to a wide spectrum of digital technologies designed to represent and facilitate mathematical work". […]
   b. "put such aids and tools to constructive use in mathematical work, as well as to critically relate to one's own and others' use of such aids and tools",
   c. pay "attention to the affordances and limitations of different mathematical aids and tools and choos[*e*] between them on that basis" (Niss & Højgaard, 2019, p. 18).

## *2.3 Exemplifying Technology Influence and Impact on Mathematical Competencies: Representation and Reasoning Competencies*

We present an example previously published by Jankvist and Geraniou (2021) to showcase the value, but also the influence and the impact of DT on students' mathematical competencies. We focus though on two mathematical competencies, the *representation competency* and the *reasoning competency*. In this example, two British students, Oscar (15 years old) and Alice (13 years old), worked as a pair on a task involving interpreting Euclid's Proposition 41 with the support of GeoGebra. Oscar was familiar with GeoGebra, whereas Alice had never used GeoGebra before this study.

At first, the two students were presented with Proposition 41 of the Euclid's Elements Book 1 (the Fitzpatrick translation into English, i.e., Fitzpatrick, 2008), μα΄.

Ἐὰν παραλληλόγραμμον τριγώνῳ βάσιν τε ἔχῃ τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾖ, διπλάσιόν ἐστί τὸ παραλληλόγραμμον τοῦ τριγώνου.

Proposition 41

If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.

**Fig. 2** Proposition 41's first sentence presented in both its original language Greek (left) and its translation in English (right) (as shared by Fitzpatrick, 2008)

without any diagrams so as to investigate their interpretation of the worded proposition (cf. Fig. 2), whilst being encouraged to use verbal comments and/or written diagrams (see Fig. 3). Both students seemed to interpret Proposition 41 correctly and

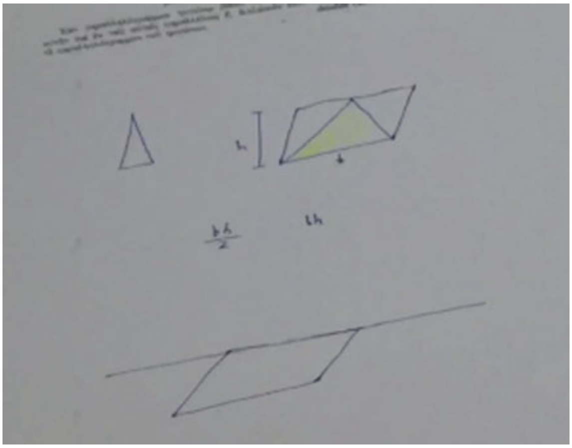
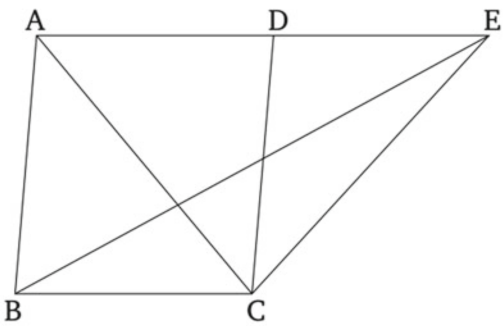


**Fig. 3** Students' pen-and-paper diagram for Proposition 41, as presented in Jankvist and Geraniou's (2021) paper (p. 9)

produced a diagram on paper to represent it (mathematical representation competency). Next, they were shown the full original description of this Proposition 41 as presented in Euclid's Elements Book 1 (Fitzpatrick, 2008) and here in Fig. 4 below and were asked to use GeoGebra to "create" the described situation. The objective was for students to discuss and articulate Proposition 41 in their own words and arguments so as to share their understanding of what the proposition states.

Students' ability to interpret what Proposition 41 stated and produce a diagram of their own that correctly represents the proposition, showcases their representation competency. They appreciated that their diagram on paper is a specific "static" case of a parallelogram and a triangle, but it was not clear if Alice in particular was able to recognize the generalizability of the proposition, whereas Oscar seemed to correctly interpret why their diagram was correct and matches what the proposition claims (Jankvist & Geraniou, 2021).




For let parallelogram $ABCD$ have the same base $BC$ as triangle $EBC$, and let it be between the same parallels, $BC$ and $AE$. I say that parallelogram $ABCD$ is double (the area) of triangle $BEC$.

For let $AC$ have been joined. So triangle $ABC$ is equal to triangle $EBC$. For it is on the same base, $BC$, as ($EBC$), and between the same parallels, $BC$ and $AE$ [Prop. 1.37]. But, parallelogram $ABCD$ is double (the area) of triangle $ABC$. For the diagonal $AC$ cuts the former in half [Prop. 1.34]. So parallelogram $ABCD$ is also double (the area) of triangle $EBC$.

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

**Fig. 4** Euclid's Proposition 41's proof as shared in the Elements book (Fitzpatrick, 2008, p. 41)

## *2.4 Exploration and Argumentation Using GeoGebra*

Alice and Oscar explored what Proposition 41 says by creating a construction in GeoGebra, similar to the one they had created earlier on paper (representation competency). They started off by creating a square, which is a special case of a parallelogram, as they both agreed. By creating a parallel line to the base, they were able to then choose a point (point E in Fig. 5a) on that parallel line to construct a triangle with the same base as the square.

As seen in Fig. 5, Alice experimented by dragging the top vertex of the triangle along the top parallel line, as prompted by Oscar. This process allowed Alice to trial numerous cases and offered her enough evidence to conjecture that any of these triangles has indeed the same area as triangle ABD. This experimental process and the creation of potentially infinitely many cases of Proposition 41 in GeoGebra (representation competency) would not have been possible on pen-and-paper. It allowed Alice, with the help of a knowledgeable other, Oscar, to reflect on their initial statement and potentially recognize why the area of any triangle ABE is half the area of

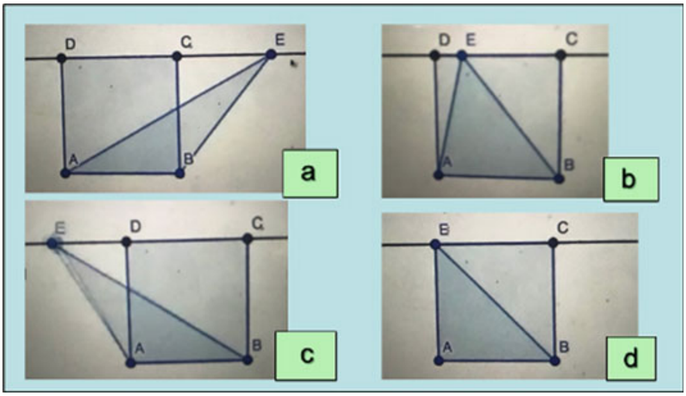



**Fig. 5** Students' GeoGebra constructions of Proposition 41, as presented in Jankvist and Geraniou's (2021) paper (p. 11)

the square ABCD (reasoning competency). In that sense, the DGS provokes conjectures and encourages students to engage in suitable justifications, involving either geometrical or algebraic reasoning.

Oscar relied on GeoGebra's features to argue that the base of both the square (which was the special case of a parallelogram in their construction) and the triangle are the same, claiming that they were constructed in that way. Following the same argumentation process, he claimed that the height will remain constant as the top line, going through points D, C and E (see Fig. 5), is parallel to the base. Later on in the interview, when challenged by the researcher, he shifted to pen-and-paper to prove the proposition 41 and said: "So, base times height is double the area of the triangle and we know that the area of the triangle is base times height divided by 2. [Writes on paper: bh = 2 × (bh/2)] […] and we can simplify this. Cancel these two out [referring to the 2's]" (Jankvist & Geraniou, 2021, p. 17—see Fig. 6).

In the case of Oscar, it seems that he was confident with his written proof of Proposition 41 and used GeoGebra to validate his thinking and his proof on paper (see Fig. 6). He was certainly influenced by their GeoGebra interactions, but relied on his prior mathematical knowledge, in particular, the area of a parallelogram or a square and the area of a triangle to prove the mathematical relationship as stated

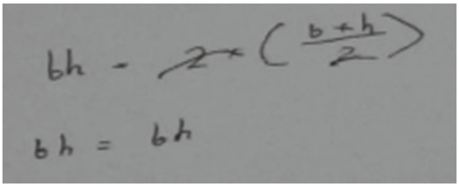


**Fig. 6** Oscar's proof on paper, as presented in Jankvist and Geraniou (2021, p. 18)

in Proposition 41. Oscar seems to possess a deductive proof scheme, as argued by Jankvist and Geraniou (2021).

Alice, on the other hand, questioned what GeoGebra offers in terms of proving a mathematical statement such as Proposition 41. Perhaps "seeing" that the proposition is true by creating a GeoGebra construction, which allows for multiple cases to be trialled, is enough to convince her.

> Oscar: So, explain to me why you think the triangle is half the parallelogram.
>
> Alice: Because it's between two parallel lines. And it's the same base and height.
>
> Oscar: Okay, so you just said the proposition again. You haven't really proved it.
>
> Alice: Because we've done two experiments and we proved the point? (dialogue presented in Jankvist & Geraniou, 2021, p. 17).

This last claim by Alice in the above dialogue may lead us to believe that there is a potential lack of understanding of what "proof" is (Dreyfus, 1999), whilst at the same time there is great "trust" in GeoGebra as a tool for exploring and justifying mathematical statements such as the one presented in Proposition 41. As Jankvist and Geraniou (2021) argued "we witness a classical case of a student, who jumps to general conclusions based on exploration via dragging as described by Mariotti (2006) and Mason (1991), which could be viewed as a development of an empirical proof scheme" (p. 15).

Of course, we do not argue that GeoGebra is to be used for offering a mathematically valid proof. We argue that technology, and in this case GeoGebra, is an additional tool in students' resources (or toolbox if we may say) that supports their mathematical work, but also enriches it. Students relied on their mathematical knowledge as well as their mathematical competencies and their knowledge of GeoGebra's features and functionalities and their own skills in using GeoGebra. Students' interactions with a tool, such as GeoGebra, enriched their experimentation with a mathematical problem (exploring Proposition 41), enabled them to "see" the mathematics and the mathematical relationships using dynamic representations compared to the static representation presented on Euclid's book of elements (mathematical representation competency) and argue about the mathematics using numerous cases to support their conjectures (mathematical reasoning competency). We can argue that GeoGebra "amplified" students' mathematical competencies and offered a technology-enriched mathematical exploration. Students' knowledge of and competencies in using GeoGebra in combination with their mathematical knowledge of the area of a triangle and a parallelogram and their mathematical representation and reasoning competencies, or in other words their mathematical digital competencies (Geraniou & Jankvist, 2019), led to a successful mathematical learning experience.

## 3 Meta-Competencies

In order to understand the broader nature of mathematics Niss and Højgaard have developed 2nd order competencies of *overview and judgements*. These "*include* *insights into essential features of mathematics as a discipline in [their] notion of mastery of mathematics"* (Niss & Højgaard, 2019 p. 24). They describe three types of overview and judgement:

1. **The actual application of mathematics within other disciplines and fields of practice**
   This first meta-competency is a matter of seeing where mathematics is in play in various practices in society, including in places where mathematics is hidden in technologies and procedures (Misfeldt & Jankvist, 2020). More specifically "Mathematics is widely used for extra-mathematical purposes in a large variety of everyday, occupational, societal, scholarly and scientific undertakings. This use is brought about by the explicit or implicit construction or utilization of mathematical models. Exactly which people are in fact using mathematics? When, and in what contexts and situations do they use it and for what purposes? In what ways do they use it, and what are the competencies they possess and activate for so doing?" (Niss & Højgaard, 2019, p. 24).
2. **The historical development of mathematics, seen from internal as well as from socio-cultural perspectives**
   This second meta-competency relates to knowing the history of mathematics and how the development of mathematics has happened in interlude with the rest of the society. The specific description is "Irrespective of which philosophical position one might take on the nature of the relationship between mathematics and reality, it is an undeniable fact that mathematics has developed as a discipline over numerous millennia. It is also a fact that mathematics was sometimes and in some respects developed in close interaction with external needs within other disciplines and fields of practice, and sometimes and in some respects in "splendid isolation" whilst pursuing its own internal goals and serving its own theoretical needs. What are the forces and mechanisms behind the historical development of mathematics in society and culture? In what respects and under what conditions and circumstances is the development of mathematics primarily influenced by internal forces, respectively by external forces?" (Niss & Højgaard, 2019, p. 24).
3. **The nature of mathematics as a discipline.**
   The last meta-competency considers the epistemological and ontological status of mathematics as a discipline. What properties are particular to mathematics and what properties are shared with other disciplines. It is described as follows: "As a scientific discipline mathematics shares some properties with other disciplines whereas several other significant properties are peculiar and specific to mathematics, in particular the ways in which mathematics obtains and justifies its results. What exactly are the properties that mathematics has in common with other fields and disciplines, what are the properties that are peculiar to mathematics, and what are the causes for the similarities as well as for the differences?" (Niss & Højgaard, 2019, p. 25).

### 3.1 Exemplifying Technology Influence and Impact on Mathematical Meta-Competencies: Proofs Using Computers

A good example of the impact of technology on mathematical meta-competencies is the discussion that the introduction of computers in proof work poses to the nature of mathematics as a discipline. One famous example of a computer proof is the proof of the four-colour theorem. The four-colour theorem is establishing the fact that all ordinary maps can be coloured with only four colours, in such a way so that there are no two adjacent regions with the same colour (see Fig. 7).

The proof relies entirely on a detailed investigation of thousands of cases that no human will be able to perform in a lifetime. Hence the proof in part consists of a computer program performing this investigation, as well as its output (Appel & Haken, 1977; Appel et al., 1977). Such an approach to proofs has led to a debate about to what extent it still can be argued that mathematics is a priori, since the computer's contribution to the evidence of a theorem is somehow equated with empirical knowledge. This is an ongoing debate in the philosophical literature on the nature of mathematics, where the question of whether the a priori nature of mathematics can be maintained when the discipline accepts sentences that can only be proved using a computer.

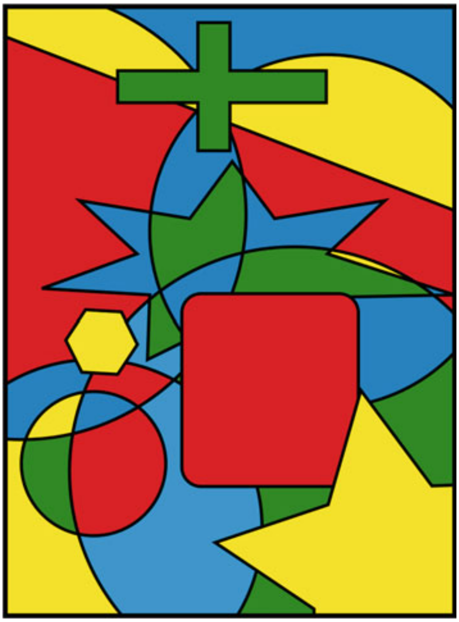


**Fig. 7** An illustration of the four-colour theorem from https://commons.wikimedia.org/wiki/File:FourColorMapEx.png

Some philosophers are suggesting that this type of example means that mathematics ceases to be a priori, because of the unsurveyable nature of a computer-assisted proof (CAP). This position was clearly stated by Tymoczko (1979), and as described by the second author in a previous publication.

> This line of argument was fleshed out by Tymoczko (1979) in connection to the computer-assisted proof of the four-colour theorem (4CT) that was published in 1977 (Appel & Haken, 1977; Appel et al., 1977). The proof is a prototypical CAP that combines a classical proof with a lemma where a large number of special cases are calculated using a computer. These calculations, however, are so numerous that the proof of the lemma ends up being "too long" to be read by any human mathematician in the sense that it cannot be read by a mathematician in a human lifetime (Tymoczko, 1979, 62). For this reason, Tymoczko categorizes the proof as unsurveyable; it cannot be read or understood in totality by any (human) mathematician. Thus, the published proof is a proof "where a key lemma is justified by an appeal to the results of certain computer runs or, as we might say, "by computer". This appeal to computers ... is ultimately a report on a successful experiment" (Tymoczko, 1979, 63). When mathematicians nevertheless see the proof as acceptable and convincing they do so because they have faith in the computer experiment. (Johansen & Misfeldt, 2017 p. 113)

Hence Tymoczko (1979) suggests that the influence of digital tools can lead to a mathematics that no longer is a priori—but much closer to empirical types of truth. On the contrary, there has been a recent push back from analytical philosophy, especially McEvoy (2008, 2013) who argues that mathematics continues to be a priori despite an active use of computer reasoning. The details of his argument is beyond the scope of this paper (see Johansen & Misfeldt, 2017 for an in-depth exploration of the problem), but both Tymoczko's hard claim about the changes in mathematical reasoning and the fact that this is a living debate in the philosophy of mathematics shows that there is a severe impact of technology on the nature of mathematics as a discipline.

One can of course ask what relevance this philosophical discussion has for mathematics teaching and learning practices? In the previous section, we saw how digital tools in the mathematical classroom influenced both reasoning and representation competencies for students. The fact that GeoGebra offers new ways to argue for mathematical results and explore mathematical phenomena is well established, but we suggest that the case of the four-colour problem, shows us that this change is not only a matter of new pedagogical possibilities in the classroom, it is also a manifestation of changes in the nature of the discipline. Hence the development of technology-supported proof schemes should perhaps be viewed as an obligation for a mathematics teaching that is up-to-date with the development of the discipline.

## 4 Teacher Competencies

The description of mathematics teacher competencies has two components; the first has to do with the pedagogical and didactic skills it takes to be a good mathematics teacher—expressed as six teacher competencies, and the second is a description of the eight mathematical competencies formulated in a way that highlights what aspects are needed to have in order to teach. In this part of the work, we focus on the six teacher competencies. The mathematical competencies of teachers are to a large degree redundant with the previous description of the eight competencies.

Niss and Højgaard (2011) stress the interaction between mathematical competencies and the competencies needed to be a mathematics teacher:

> Despite the fact that in the following chapters we describe these two components individually, it cannot be stressed enough that, in a good teacher, they are integrated in the sense that he or she can both apply competent mathematical points of view to every didactic or pedagogical problem, and relate to the didactic/pedagogical potential in the mathematical abilities and insights he or she possesses, as well as being able to bring these two components together in an integrated manner in teaching. (Niss & Højgaard, 2011, p. 84)

In the following we describe how the six teacher competencies are defined, we will use direct quotations from the original report (Niss & Højgaard, 2011).

## *4.1 Curriculum Competency—Being Able to Evaluate and Draw up Curricula*

The first teacher competency has to do with interpretation of learning standards and national curricula, and compare across various curriculum suggestions and participate in the development of curricula. Niss and Højgaard (2011) express this as:

> This competency comprises, on the one hand, being able to study, analyze and relate to every current or possible future framework curriculum for mathematics teaching at the relevant educational stage, and being able to evaluate these plans and their significance for one's actual teaching.
>
> On the other hand, it comprises being able to *draw up and implement* different types of curricula and course plans with different purposes and aims at different levels taking into consideration the overarching frameworks and terms which may exist, both under current conditions and those in the expected future. (Niss & Højgaard, 2011, p. 86)

## *4.2 Teaching Competency—Being Able to Think Out, Plan and Carry Out Teaching*

This competency has to do with planning, conducting and justifying teaching, as well as motivating and engaging with the students—it is a comprehensive and broad competency with many elements. Niss and Højgaard (2011) express this as:

> This competency comprises being able to, with overview and together with the students, think out, plan and carry out concrete teaching sequences with different purposes and aims.
>
> This involves the creation of an abundant spectrum of teaching and learning situations, including the planning and organization of activities for students and student groups with consideration being given to their characteristics and needs. It also covers the selection

> and presentation of tasks as well as the other assignments and challenges of the students' activities. In addition, it comprises being able to find, judge, select and produce different types of teaching means and material. Furthermore, the competency involves being able to justify and discuss with the students the content, form and perspectives of the teaching and being able to motivate and inspire students to become engaged in mathematical activities, as well as being able to create room for students' own initiatives in mathematics teaching. (Niss & Højgaard, 2011, p. 86)

### *4.3 Competency of Revealing Learning—Being Able to Reveal and Interpret Students' Learning*

The competency of revealing and interpreting student learning has to do with the teacher's ability to conduct cognitive and emotional empathy with their students' mathematical learning situation and use this to improve the learning situation for the students. Niss and Højgaard (2011) express this as:

> "This competency comprises being able to reveal and interpret students' actual mathematical learning and mastery of mathematical competencies as well as their conceptions, beliefs of and attitudes to mathematics and it includes being able to identify the development of these over time.
>
> Part of the competency is being able to get behind the facade of the ways in which the individual's mathematics learning, understanding and mastery is expressed in concrete situations, with the intention of understanding and interpreting the cognitive and affective background for these." (Niss & Højgaard, 2011, p. 87)

### *4.4 Assessment Competency—Being Able to Reveal, Evaluate and Characterize the Students' Mathematical Yield and Competencies*

The ability to evaluate and assess the students. It is related to the competency of revealing, but focussed on the assessment side (fairness, communication with the students) rather than on diagnosing the learning process. It has both a summative and formative side. Niss and Højgaard (2011) express this as:

> This competency comprises being able to select or construct, as well as utilize, a broad spectrum of forms and instruments to reveal and evaluate a student's or student group's mathematical yield and competencies, both during the course of teaching and at the end of it, and both in absolute and relative terms.
>
> Included in this is the ability to *critically relate to* the validity and extent of the conclusions reached via the use of the individual assessment instruments. This competency is a precondition for continuous assessment, i.e., assessment carried out during the course of teaching, and includes the ability to *characterize* the individual student's yield and competencies and the ability to be able to *communicate* with the student about the observations and interpretations

> made, and then *help* him or her to correct, improve or further develop his or her mathematical competencies. The same is true for final assessments, including examinations, even though the guidance in this situation is often of a different nature. (Niss & Højgaard, 2011, p. 87)

## *4.5 Cooperation Competency—Being Able to Cooperate with Colleagues and Others Regarding Teaching and Its Boundary Conditions*

Cooperation is a general professional competency, also needed by mathematics teachers. Niss and Højgaard (2011) express this as:

> This competency comprises, first of all, being able to cooperate with colleagues, both subject colleagues and colleagues in other subjects, about matters of significance to mathematics teaching. Included in this is the ability to bring the above-mentioned four competencies into play in mathematical, pedagogical and didactic cooperative projects and in discussions with different types of colleagues.
>
> Secondly, the competency includes the ability to cooperate with people beyond the staff room, e.g., the parents of students, administrative agencies, the authorities, etc. about the boundary conditions of teaching. (Niss & Højgaard, 2011, p. 88)

## *4.6 Professional Development Competency—Being Able to Develop one's Competency as a Mathematics Teacher*

Similar to the cooperation competency—successful mathematics teachers also need to conduct professional development competency. Niss and Højgaard (2011) express this as:

> This competency comprises being able to develop one's competency as a mathematics teacher. In other words, it is a kind of meta-competency.
>
> More concretely it involves being able to enter into and relate to activities that can serve the development of one's mathematical, didactic and pedagogical competency, taking into consideration changing conditions, circumstances and possibilities. This is about being able to reflect on one's teaching and discuss it with mathematics colleagues, being able to identify a developmental need, and being able to select or arrange as well as evaluate activities that can promote the desired development whether or not there is talk of external in-service training and further education courses, conferences or projects with colleagues and activities like, e.g., participation in study groups and research projects. It is also about keeping oneself up-to-date with the latest trends, new material and new literature in one's field, about benefiting from relevant research and development contributions, and maybe even about writing articles or books of a mathematical, didactic or pedagogical nature. (Niss & Højgaard, 2011, p. 88)

### *4.7 Exemplifying Technological Influence on Teacher Mathematical Competencies—Introducing Computational Thinking into Mathematics; Curriculum Competency and Professional Development Competency*

Recently many countries have been implementing computational thinking in compulsory education. This has led to a range of decisions and challenges (who should teach this in school and in relation to which topic). In many cases, mathematics teachers are being included as part of the solution to this challenge. One example of this is the Sweden case. In 2017, the Swedish government decided to include programming in the national mathematics curriculum from grades 1 through 12. This integration thus concerned all mathematics teachers, leading to a major implementation and training challenge. To address this the ministry of education initiated the development of a number of in-service training activities located at a digital portal, developed during "Boost for Mathematics" (a national in-service training program). Despite this initiative, Swedish mathematics teachers stated rather clearly that they did not feel ready to conduct teaching in programming when the change was initiated in 2018 (Misfeldt et al., 2019). Hence this change challenged Swedish teachers both in relation to their curriculum competency and in relation to their professional development competency. Clearly, the way that Sweden chose to include programming in the curriculum, has put significant pressure on teachers to learn new skills and take new objectives into their teaching. Currently, similar changes are happening globally (Bocconi et al., 2016; Kallia et al., 2021), and this will put pressure on the capacity of the educational systems and the teacher competencies in most of the world.

## 5 Mathematical Competencies Under the Influence of Digital Affordances

Without a doubt, technology appears widely in mathematics and mathematics education, in many mathematics teaching- and learning situations. Of course, the type of technology used and how it is used varies. Discussing mathematical competencies, and in particular, the KOM framework is more important than ever as it allows us to focus on students' mathematical development, as supported by their teachers, now that technology has taken a central role in the mathematics education community.

Digital technology has clearly changed the learning environment in many cases. The emerging division of labour between techniques involving digital technologies and more classical techniques has been investigated in numerous mathematics education research papers. As Turner points out (in this book), with a quote from Gravemeijer et al. (2017):

> "today, basically all mathematical operations that are taught in primary, secondary and tertiary education can be performed by computers and are performed by computers in the world outside school." [...and hence ...] we have to shift away from teaching competencies that compete with what computers can do and start focusing on competencies that complement computer capabilities. (Gravemeijer et al., 2017, p. 107)

This division of labour has for instance been investigated in relation to the use of Computer Algebra Systems, highlighting the problem of "blackboxing" and the related problematic tendency for the students' mathematical activities to move in a pragmatic direction (Artigue, 2010).

With reference to Niss's (2016) descriptions of the potential marvels and disasters that DT can bring to mathematics education, it is important to stress that we cannot expect DT to teach mathematics to children, instead we expect that "Digital technologies may serve to:

- enhance a wide variety of mathematical capacities and

- replace some mathematical competencies (Niss, 2016, p. 248).

In a similar fashion, DT does not in itself support the development of specific mathematical competencies, but it can be the case. And on a similar note, specific technologies can enhance mathematical representation, reasoning or any one of the other competencies. But DT can also replace certain mathematical competencies more or less entirely, such as when CAS systems are used for solving equations (see for example Jankvist & Misfeldt, 2015) and hence symbol and formalism competency and reasoning competency.

In the KOM framework, technology is discussed solely as part of the Tools and Aids competency. Even though we exemplified how technology impacts mathematical competencies focusing only on some of these competencies, we argue that technology appears in every mathematical competency and meta-competency. We are therefore proposing the addition of a "technology-enhanced" competence element in each mathematical competency. In our view, the KOM framework needs to be revisited and expanded by including a sub-competence within every mathematical competency that involves technology use, i.e., a technology thread of sub-competencies (see for example Fig. 8 for a potential diagrammatic representation of a technology-enriched KOM flower).

But let us now try and convince you by offering some further arguments about our proposition in light of our earlier discussions in this chapter. Technology has been developed to support mathematical operations (e.g., calculators) and explorations (e.g., graphic calculators, Desmos, GeoGebra, etc.) and both teachers and students are expected to develop certain skills to use such technology for mathematical teaching and learning.

In example 1, we discussed how computers and in particular a dynamic geometry software (GeoGebra) influence students' mathematical competencies, but also how students' mathematical competencies influence their explorations and interactions with GeoGebra. We are claiming that there are synergetic processes taking place in

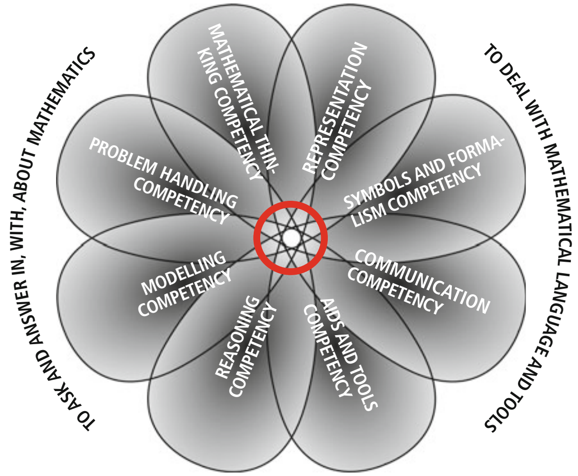



**Fig. 8** The KOM flower (Niss & Højgaard, 2019, p. 19) with the proposed "technology-enhanced" competence layer

such an educational context. Oscar and Alice relied on their mathematical knowledge of the area of a triangle and a parallelogram and their mathematical competencies to create a diagram and offer mathematically valid justifications for why Proposition 41 is true (MDC3). Alice seemed to rely on an empirical proof scheme, possibly triggered but certainly supported by GeoGebra, whereas Oscar seemed to rely on a deductive proof scheme and used GeoGebra to validate his conjectures and proof on paper. Undoubtedly, GeoGebra enabled both students to create a dynamic construction to explore the mathematical statement offered by Proposition 41 and trial numerous cases. Oscar was aware of GeoGebra's capabilities and limitations and supported Alice in producing the dynamic construction (MDC2).

Example 2 shows that the nature of mathematical argumentation and of what counts as mathematical truth is at stake when computers are used in mathematical proving activities. The example discusses the consequences of technology for the mathematical meta-competencies. However, there is a clear resemblance to example 1 and the focus on mathematical argumentation. One can say that "modern" mathematical reasoning processes that involve digital technology is exactly an example of the way that we as mathematics educators should refocus in the direction of competencies that *complement* rather than *resemble* what computers can do. This example shows two interesting points: (1) mathematical meta-competencies are affected by digital technology (as described in the example) and (2) there is a strong need to develop skills that complement what computers can do and allows students to maintain mathematical agency and curiosity.

Example 3 shows something about how rapid the changes are in relation to technology and mathematics teaching. Clearly, the work with programming and computational thinking is currently affecting many mathematics teachers worldwide and calls for them to enact curriculum and professional development competency. Of course, the current rush for implementing programming and technology education in compulsory education is a special situation, but viewed in a more longitudinal fashion, digital technologies have been a challenge to the status quo in teaching and learning of mathematics for many years. Just to mention two important changes. Computer algebra systems and the problems with blackboxing of algebraic reductions has given rise to a number of pedagogical problems over the last 20 years. Likewise, the increased use of personalized learning environments and portals, that are supporting teaching and learning processes, are requiring teachers to change their practices. These changes call for teachers that have flexible teacher competencies and are able to adapt to new technological situations without losing sight of the mathematical development of their students.

In all these three examples, we showcased and discussed the interconnections between mathematical competencies and technology use for educational purposes. We considered how technology impacts mathematical work in society and industry, how technology influences mathematics as a research discipline, how technology supports and potentially enriches mathematical work in the classroom, and finally how technology has become a common theme in discussions regarding educational policy and the mathematics curriculum. Our discussions call for some augmentation of the current three elements of MDCs described in the introductory section, to include a Mathematical Digital meta-Competence: *being able to discuss what is happening to the mathematics as a discipline when digital technologies come into the picture.*

We are furthermore suggesting an updated way to look at the interplay between technology and competencies. Not thinking of it as a meta-competency of the Tools and Aids competency, but allowing technology to take a more central role in each of the mathematical competencies described by the KOM framework (see Fig. 8 for a diagrammatic representation). We still strongly believe that the KOM framework and all mathematical competencies presented in that framework are crucial for students' and teachers' development regarding mathematics education. We put emphasis on the inevitable influence of technology and argue that mathematical digital competencies should be at par with mathematical competencies rather than being a sub-competency. Also, teacher competencies should evolve to allow for an effective management of the impact of technology in mathematics education.

Relating to the work of Niss (2016) and Niss and Højgaard (2019), we are arguing for a need to have an additional layer in the KOM framework that takes into account technology influence across the eight mathematical competencies, as well as in relation to teacher competencies and meta-competencies. In fact, in the digital age, it is difficult to "escape" the influence and impact of digital technologies in mathematics education. As Niss (2016) argues "knowledge of and insight into what is *variable* and what is *constant* contributes to making us sharper and wiser" (p. 240). So, is it safe to claim that the use of DT in mathematics education is more of a constant these days? We do not think we can unless we define what we mean by "use of DT". The use of powerpoint slides and perhaps an IWB is visible in most mathematics classrooms, but the use of GeoGebra, Desmos, excel and other mathematical digital tools is not always visible. To achieve any change in teacher's practices, we should definitely study teachers' practices and define the variables and constants. And we need to consider the influence of DT on mathematical competencies in detail and in relation to all aspects of mathematical competencies. As we have argued, DT influences all mathematical competencies as well as the meta-competencies and teacher competencies.

## References

Appel, K., & Haken, W. (1977). Every planar map is four colorable. Part 1: Discharging. *Illinois Journal of Mathematics, 21(3)*, 429–490.

Appel, K., Haken, W., & Koch, J. (1977). Every planar map is four colorable. Part II: Reducibility. *Illinois Journal of Mathematics, 21(3)*, 491–567.

Artigue, M. (2010). The future of teaching and learning mathematics with digital technologies. In C. Hoyles & J. -B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain. The 17th ICMI study* (pp. 463–475). Springer.

Blomhøj, M., & Højgaard Jensen, T. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching Mathematics and Its Applications, 22*(3), 123–139.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, engineering and economics* (pp. 222–231). Horwood.

Bocconi, S., Chioccariello, A., Dettori, G., Ferrari, A., & Engelhardt, K. (2016). *Developing computational thinking in compulsory education—Implications for policy and practice.* EUR 28295 EN. https://doi.org/10.2791/792158

Bozkurt, G., & Ruthven, K. (2017). Classroom-based professional expertise: A mathematics teacher's practice with technology. *Educational Studies in Mathematics, 94*(3), 309–328.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics, 38*(1–3), 85–109.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75(2)*, 213–234.

Ferrari, A. (2012). *Digital competence in practice: An analysis of frameworks. A technical report by the Joint Research Centre of the European Commission.* European Union.

Fitzpatrick, R. (2008). *Euclid's elements of geometry.* (http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf) [The Greek text of J. L. Heiberg (1883–1885) from Euclidis Elementa, edidit et Latine interpretatus est I. L. Heiberg, in aedibus B. G. Teubneri, 1883–1885 edited, and provided with a modern English translation, by Richard Fitzpatrick.]

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*, 29–45. https://doi.org/10.1007/s10649-019-09893-8

Gravemeijer, K., Stephan, M., Julie, C., Lin, F. L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education, 15*(1), 105–123.

Hatlevik, O. E., & Christophersen, K.-A. (2013). Digital competence at the beginning of upper secondary school: Identifying factors explaining digital inclusion. *Computers and Education, 63*, 240–247.

Jankvist, U. T., & Geraniou, E. (2021). "Whiteboxing" the content of a formal mathematical text in a dynamic geometry environment. *Digital Experience in Mathematics Education.* https://doi.org/10.1007/s40751-021-00088-6

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics? *For the Learning of Mathematics, 35*(1), 15–20.

Johansen, M. W., & Misfeldt, M. (2017). Computers as a source of a posteriori knowledge in mathematics. *International Studies in the Philosophy of Science, 30*(2), 111–127. https://doi.org/10.1080/02698595.2016.1265862

Kallia, M., van Borkulo, S., Drijvers, P., Barendsen, E., & Tolboom, J. (2021). Characterising computational thinking in mathematics education: A literature-informed Delphi study. *Research in Mathematics Education*. https://doi.org/10.1080/14794802.2020.1852104

Kunter, M., Klusmann, U., Baumert, J., Richter, D., Voss, T., & Hachfeld, A. (2013). Professional competence of teacher: Effects on instructional quality and student development. *Journal of Educational Psychology, 105*(3), 805–820.

Mariotti, M. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 173–204). Sense Publishers.

Mason, J. (1991). Questions about geometry. In D. Pimm & E. Love (Eds.), *Teaching and learning school mathematics: A reader* (pp. 77–90). Hodder and Stoughton.

McEvoy, M. (2008). The epistemological status of computer-assisted proofs. *Philosophia Mathematica, 16*, 374–387.

McEvoy, M. (2013). Experimental mathematics, computers and the a priori. *Synthese, 190*, 397–412.

Misfeldt, M., & Jankvist, U. T. (2020). Teknokritisk matematikundervisning: at åbne den skjulte matematik i demokratiets tjeneste. [Technocritical mathematics education: Opening the hidden mathematics in the service of democracy.] I C. Haas, & C. Mathiessen (red.), *Fagdidaktik og demokrati* (pp. 331–348). Samfundslitteratur.

Misfeldt, M., Szabo, A., & Helenius, O. (2019). Surveying teachers' conception of programming as a mathematics topic following the implementation of a new mathematics curriculum. In *The eleventh congress of the European Society for research in mathematics education (No. 13)*. Freudenthal Group; Freudenthal Institute; ERME.

Niss, M. (2010). Modelling a crucial aspect of students' mathematical modeling. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 43–59). Springer.

Niss, M. A. (2016). Mathematics standards and curricula under the influence of digital affordances: Different notions, meanings and roles in different parts of the world. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 239–250). Information Age Publishing.

Niss, M. & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002). Retrieved from: http://milne.ruc.dk/imfufatekster/pdf/485web_b.pdf

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*, 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings learning cultures and computers*. Dordrecht Kluwer.

Ruthven, K. (2011). Constituting digital tools and materials as classroom resources: The example of dynamic geometry. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 83–103). Springer.

Tymoczko, T. (1979). The Four-color problem and its philosophical significance. *Journal of Philosophy, 76*(2), 57–83.

Young, J. (2017). Technology-enhanced mathematics instruction: A second-order meta-analysis of 30 years of research. *Educational Research Review, 22*, 19–33.

# The Eight Mathematical Competencies

# Processes of Mathematical Thinking Competency in Interactions with a Digital Tool

**Mathilde Kjær Pedersen**

## 1 Introduction

Around the turn of the millennium, the term "mathematical competence" entered the discussion of mastering mathematics as a more comprehensive concept than procedures, skills, knowledge and understanding (e.g., Niss & Højgaard, 2019; Stacey, 2010). Around the same time, the role of digital technologies significantly increased in the teaching and learning of mathematics (Trouche et al., 2013). Studies of the conceptualization of specific mathematical concepts in relation to competencies and the use of digital tools have been carried out (e.g., Kendal & Stacey, 2000; Weigand & Bichler, 2010). With the discussion of mathematical digital competencies, Geraniou and Jankvist (2019) address the requirement students face to draw on both mathematical and digital competencies in learning situations. To capture the students' mathematical competencies, Geraniou and Jankvist (2019) use the Danish mathematical competency framework, referred to as KOM, i.e., a framework describing what it means to master mathematics (Niss & Højgaard, 2019). Furthermore, to describe the interplay of mathematical and digital competencies, the authors apply the theoretical perspectives of Drijvers et al. (2013) instrumental genesis and Vergnuad's (2009) conceptual fields.

M. K. Pedersen (✉)
Danish School of Education, Arhus University, Copenhagen, Denmark
e-mail: mkp@edu.au.dk

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_4

In their analysis, Geraniou and Jankvist (2019) address all the mathematical competencies in the KOM framework except the mathematical thinking competency. Without entering the discussion of mathematical digital competencies, I find it interesting to investigate the interplay between students' mathematical thinking competency and their use of digital technologies. However, since mathematical competencies cannot be developed or exercised without working with a mathematical subject matter, and since aspects of a mathematical competency appear differently depending on the level of mathematics education (Niss & Højgaard, 2019), it is necessary to specify the contextual setting of the specific mathematical thinking competency I will investigate.

In the subject of mathematics in upper secondary school, differential calculus plays an important role. Digital technologies, such as computer algebra systems (CAS) and dynamic geometry systems (DGS), can allow teachers to teach and students to study concepts of differentiability in new ways (Hohenwarter et al., 2008). However, incorporating digital tools into the learning of mathematical concepts can also lead to disasters, in which students objectify CAS procedures as mathematical objects (Jankvist et al., 2019). Niss (2016) argues that digital technologies may serve to enhance or replace mathematical competencies, depending on for what, when and how they are used. Contributing to the discussion of the interplay between mathematical competencies and the use of digital technologies, I address the following questions: (1) *Which processes of the mathematical thinking competency can be identified as part of students' work with instances of differentiability and non-differentiability?* (2) *How can these processes of the students' mathematical thinking competency interact with the students' use of a given digital tool?*

For this investigation, I present an empirical example of two students working with the concept of differentiability using both the dynamic graphic window and the CAS window of TI-*n*spire. Addressing the first question, I analyze the case through the lens of the mathematical thinking competency in order to identify the processes by which this particular competency appears in this case. When analyzing the students' mathematical competencies, the analyses can benefit from the use of other theoretical frameworks and constructs within mathematics education research (Jankvist & Niss, 2015). I also consider different perspectives on the use of digital technologies in mathematics education when analyzing the students' use of these tools. The research practice *Networking of Theories* offers different strategies for networking theoretical approaches of mathematics education research (Prediger et al., 2008). Thus, to investigate the second question, I aim to illustrate the interaction between the students' mathematical thinking competency and their use of a digital tool by applying the networking strategy *combining*, using key elements of the theoretical perspectives of instrumental genesis (Drijvers et al., 2013), conceptual fields (Vergnaud, 2009) and semiotic mediation (Bussi & Mariotti, 2008). In the following three sub-sections, I present the theoretical perspectives included in my analyses.

## 2 The Mathematical Thinking Competency of the KOM Framework

The KOM framework[1] is a set of descriptions for mastering mathematics across institutional levels and mathematical topics (Niss et al., 2016). In the KOM framework, possessing overall mathematical competence is defined as "someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations" (Niss & Højgaard, 2019, p. 12). In the framework, mathematical competence is divided into eight distinct but mutually linked mathematical competencies. A mathematical competency is defined in relation to a specific sort of mathematical challenge, in contrast to general mathematical competence, which includes a variety of mathematical challenges.

In the KOM framework, the mathematical thinking competency includes the processes of engaging in and reflecting upon mathematical inquiry (Niss & Højgaard, 2019). To be more specific, it involves "being able to relate to and pose the *kinds* of generic questions that are characteristic of mathematics and relate to the nature of answers that may be expected to such questions" (Niss & Højgaard, 2019, p. 15, italics in original). I term these processes of the competency the question–answer aspect.

In line with this aspect are the processes of

> distinguishing between different types and roles of mathematical statements (including definitions, if-then claims, universal claims, existence claims, statements concerning singular cases, and conjectures), and navigating with regard to the role of logical connectives and quantifiers in such statements, be they propositions or predicates (Niss & Højgaard, 2019, p. 15).

For instance, to possess elements of the question–answer aspect, one needs to know the differences between the mathematical claims. I term these processes the mathematical statements aspect.

Furthermore, the mathematical thinking competency includes the process of "relating to the varying scope, within different contexts, of a mathematical concept or term" (Niss & Højgaard, 2019, p. 15), which I term the scope of concept aspect. In relation to differentiability, this could include the meanings of differentiability for a given point, a given interval or a function as a whole. Moreover, it could include the different views of differentiability, from whether a function's graph is smooth to the $\varepsilon - \delta$ definition. Lastly, the competency involves "relating to and proposing "abstractions" of concepts and theories and "generalization" of claims (including theorems and formulae) as processes in mathematical activity" (Niss & Højgaard, 2019, p. 15, quotation marks in original). I term these processes the generalization–abstraction aspect. With the aspects of scope of concept and generalization–abstraction, the mathematical thinking competency concerns mathematical conceptualization.

[1] In this section, the mathematical thinking competency of the KOM framework is in focus, but for the chapter to be readable on its own, some of the concepts of the KOM framework are repeated from Chap. 2 (Niss & Jankvist, 2022) of this book.

## 3 Instrumental Genesis and Conceptual Fields

The perspective of instrumental genesis describes the complex process of transforming a tool into a useful mathematical instrument (Guin & Trouche, 1998); "useful" in this context refers to the tool's ability to help the user achieve an aim. Drijvers and colleagues describe instrumental genesis in terms of three dualities. The first duality is artefact-instrument, which distinguishes between the tool itself as a physical object (the artefact) and the tool as a psychological construct (the instrument) (Guin & Trouche, 1998). The second duality is instrumentation-instrumentalization, which concerns the direction of how the user interacts with the artefact. Instrumentation refers to how the artefact's configuration and features shape the user's way of thinking and doing. In contrast, instrumentalization refers to the user's way of thinking that directs the use of the artefact. (Drijvers et al., 2013) The third duality is technique-scheme, which, from a practical point of view, distinguishes between observable gestures (techniques) and unobservable cognitive structures that guide these techniques (schemes) (Drijvers et al., 2013).

The notion of scheme comes from Vergnaud's (2009) theory of conceptual fields. A conceptual field is a cognitive structure consisting of mathematical concepts and situations associated with each other, and "a scheme is *the invariant organization of activity for a certain class of situations*" (Vergnaud, 2009, p. 88, italics in original). In the development of mathematical knowledge, Vergnaud (2009) distinguishes between operational and predicative forms of knowledge. The operational form is the knowledge of doing, and the predicative form is the knowledge of articulation. Schemes are part of the operative form of knowledge, whereas language and symbols are part of the predicative form. Schemes consist of several aspects, one of which involves the two operational invariants: concepts-in-action and theorems-in-action. Concepts-in-action are the concepts we associate with and find relevant in the given situation. Theorems-in-action are propositions—considered true, but not necessarily articulated—stating which activities we can carry out with the concepts-in-action (Vergnaud, 2009).

Considering the concept of differentiability from the perspective of conceptual fields, differentiability builds on and connects to other concepts, such as linearity, slope, secant, tangent, limit and the derivative among others, all of which constitute their own conceptual fields. Furthermore, with the notions of concepts and theorems-in-action, a conceptual field of differentiability can include different ways of understanding differentiability. It is these relations and conceptualizations within the conceptual fields that I find particularly relevant for the mathematical thinking competency's scope of concept aspect (unfolded in the previous section). Furthermore, the dualities of instrumental genesis and the two-way interaction between user and tool can provide a deeper insight into the students' way of thinking and therefore their mathematical thinking competency.

## 4 Semiotic Mediation

The perspective of semiotic mediation focuses on the involvement of an artefact (understood in the same sense as in the perspective of instrumental genesis) as a tool of semiotic mediation in a mathematics teaching and learning setting (Bussi & Mariotti, 2008). A tool of semiotic mediation is an artefact the teacher intentionally uses to mediate specific mathematical content through a didactical sequence. The word "intentionally" is important here, as one can never be certain that the students using the artefact will infer the teacher's intended mathematical meanings (Bussi & Mariotti, 2008). This ultimately means that the artefact acquires a two-fold aim. It should be both an aid for the students to solve specific tasks and a tool of semiotic mediation related to specific mathematical knowledge (Bussi & Mariotti, 2008).

To account for this two-fold aim, Bussi and Mariotti (2008) distinguish between three categories of signs that indicate a student's progress from personal to mathematical meaning. The first category is artefact signs, which originate from the activities carried out with the artefact and are of personal meaning, based on experience. The second category is mathematical signs, which, in contrast, are signs of mathematical meaning related to the given mathematical content. The third category is pivot signs, which refer both to activities carried out with the artefact and to a mathematical domain. They function as a pivot in the progress from personal to mathematical meaning. Signs include different gestures, drawings and written and oral language (Bussi & Mariotti, 2008). With the framework of signs used in the a posteriori analysis of a teaching sequence, the process of semiotic mediation can bring forth aspects of students' making-meaning when they interact with an artefact.

## 5 Networking of Theories and the Roles of the Selected Theoretical Perspectives

Networking of theories is a research practice developed to make different theoretical and methodical perspectives in mathematics education research communicate with each other. It offers strategies for networking on a spectrum according to the degree of integration, from understanding others and making understandable to integrating locally and synthesizing (Prediger et al., 2008). Two of these strategies, combining and coordinating, are typically used to study an empirical phenomenon in more detail than could be achieved using only one perspective; but they are used in different ways. Combining is when two juxtaposed analyses use different theoretical lenses to capture different aspects of the same empirical phenomena. In contrast, coordinating is when a conceptual framework consists of well-fitting elements from different theoretical approaches (Bikner-Ahsbahs & Prediger, 2010). Therefore, coordinating requires that the cores of the theoretical approaches in question are more compatible. Hence, an important element for networking of theories is the focus on the core of a theoretical approach (Prediger et al., 2008).

Considering the core of the KOM framework, the authors behind the framework write in relation to the notion of mathematical competency:

> [t]he core of a mathematical competency is the enactment of mathematics in contexts and situations that present a certain kind of challenge. (Niss & Højgaard, 2019, p. 19)

The entire framework is a broad description of mathematics as a practice and is not anchored in a given theoretical perspective (Niss & Højgaard, 2019). From a network perspective, this may create difficulties, as one cannot go back to its origins to determine how compatible it is with the cores of the other theoretical approaches in question. However, it is possible to combine the mathematical thinking competency with parallel analyses using other lenses to elaborate the empirical phenomena of students exercising the mathematical thinking competency in interactions with a given digital tool.

As the KOM framework focuses on an individual's cognitive actions for doing and dealing with mathematics (Niss & Højgaard, 2019), the theoretical perspectives to help elaborate the mathematical thinking competency in interplay with the use of digital tools should also focus on the individual's cognition and actions. The theoretical perspectives of instrumental genesis (Drijvers et al., 2013) and conceptual fields (Vergnaud, 2009) have previously been proven suitable to describe the interplay between students' possession of mathematical competencies and the use of digital tools (Geraniou & Jankvist, 2019). Furthermore, I argue that the theoretical perspective of semiotic mediation offers a terminology to help us gain deeper insights into the students' meaning-making of a digital tool.

Instrumental genesis (Drijvers et al., 2013) and semiotic mediation (Bussi & Mariotti, 2008) are both based on the instrumental approach. The instrumental approach involves a Vygotskian perspective that emphasizes the use of instruments in learning processes but that also uses the Piagetian notion of scheme (Verillon & Rabardel, 1995). The notion of scheme is re-elaborated by Vergnaud (1996), who also draws on a Vygotskian perspective, arguing how notions from Piaget and Vygotsky complement each other. Thus, the cores of instrumental genesis (Drijvers et al., 2013), conceptual fields (Vergnaud, 2009) and semiotic mediation (Bussi & Mariotti, 2008) are compatible.

Since these theoretical approaches are compatible, they could potentially be used in a coordinated analysis of students' interactions with digital tools in a mathematics education setting. However, I have chosen to use these theoretical perspectives to conduct juxtaposed analyses using the networking strategy of combining to study the phenomenon of students exercising the mathematical thinking competency when working with digital tools. Using the mathematical thinking competency as a coarse-grained framework, I analyze the empirical case presented below to focus the attention on the students' processes of mathematical thinking competency for the subsequent finer-grained combined analyses.

In the following two sections, I account for the method of the empirical study and present a case in which I illustrate how the students exercise the mathematical thinking competency through their interaction with a digital tool.

# 6 Method and Selection of the Case

The case presented in this chapter is taken from a larger empirical study carried out in autumn 2020 in the classical stream of Danish upper secondary school, called STX. In the empirical study, 29 students participated in two lessons on differential calculus, each lesson lasting 90 minutes. The students collaborated in groups of two or three in order to capture their thinking through their mutual discussions. The students worked from a premade TI-*n*spire worksheet and wrote their answers in an appurtenant Word document, both of which were screencast recorded. The students were also recorded using their webcams so that their participation and any relevant hand gesticulations could be analyzed. The tasks on which the students worked represented the main exercises of the two lessons in the empirical study. All pairs/groups worked on these tasks for between 20 and 60 minutes, though their work was interrupted by various events, such as the first lesson ending, the teacher giving an introduction, having to engage in-class discussion, waiting for help or discussing topics unrelated to mathematics.

The video sequences of the students working on these tasks were first coded with the aspects of the mathematical thinking competency (described above) to identify relevant pieces of data. Analysis 1 below is an elaboration of this process for the given case. Based on this analysis, the case of Karen and Lily was selected, because the scope of concept aspect of the mathematical thinking competency was very clear in the initial coding, due to the students being persistent in their intuitive idea of differentiability (cf. Analysis 1). In order to elaborate on how the students interacted with and created meaning from the digital tool in relation to the scope of concept aspect of the mathematical thinking competency, the case was then analyzed using the perspectives of instrumental genesis and conceptual fields (Analysis 2) and semiotic mediation (Analysis 3).

# 7 Data: Exploring Differentiability Using Secant Lines

The case presents two students, Karen and Lily, working on a dynamic TI-*n*spire worksheet. This worksheet asks the students to investigate whether a given function $f$ (depicted by the blue graph in Fig. 1) is differentiable for given values, where $f$ is defined by

$$f(x) = \begin{cases} \frac{1}{15}x^3 - 1.2x^2 + 5.4x + 1.8, & -0.5 \leq x \leq 9 \\ 1.8, & 9 < x \leq 15 \end{cases}$$

With two sliders in the top right corner, the students can move $x_0$ on the $x$-axis and change the difference $\Delta x$. In this way, students can observe differentiability as a numerical approximation to the slope of the tangent line (Hohenwarter et al., 2008).

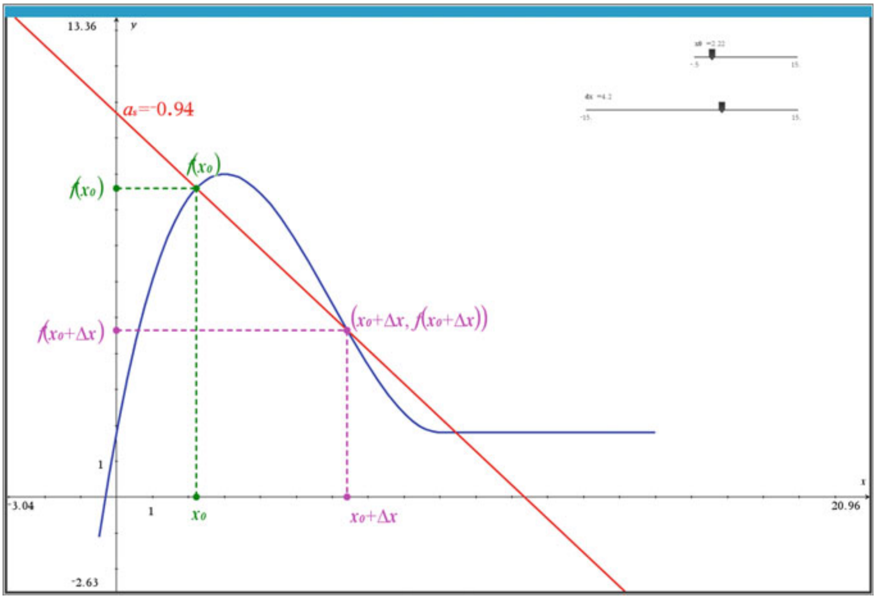



**Fig. 1** Snapshot from TI-nspire, the interactive representation of a function (the blue graph) and a changeable secant line (the red graph)

In the following dialogue, Karen and Lily work with the tool to investigate differentiability for $x_0 = 9$.

01. Lily: What does it mean for it to be differentiable? … When it is differentiable… we should read about it at some point. That, when it has such a tip, then it is not differentiable.
02. Karen: Yes, it is when it is curved. You should be able to walk from $a$ to $b$ and such [she grabs her book and reads out sections of the text]. Continuity. Called continuity when its graph is connected… from $a$… [she moves on to the paragraph on differentiability] and here, it should be without corners.[2]
03. Lily: [Looks at the graph in Fig. 1] But I guess there are no corners in this one… But, when it is differentiable, then it is without corners, so there cannot be such a tip on it. … Because then it could slope differently-ish like this [she holds one of her hands in different directions].
04. Karen: But this one [the graph in Fig. 1] is soft.
05. Lily: Yes, there are no corners.

Karen and Lily cannot work out how the information from the book can help them investigate differentiability using the tool, and they ask for help. Through guidance, Karen sets $x_0 = 9$ and controls the slider for $\Delta x$. First, she does it for negative $\Delta x$, but, because the slider jumps in small intervals, she ends up typing $\Delta x = -0.1$, for

[2] I translate the Danish word "knæk" as "corners", to illustrate a graph having one or more sharp bends but still being connected.

which the secant slope is 0. Afterward, she types $\Delta x = 0.1$, for which the secant slope is also 0.

06. Mathilde: There [the secant slope] is also 0. So, now it approaches 0 from right and left, so it then approaches the same value. In this case, 0.
07. Karen: Ah, so it is opposite, kind of like a mirror-ish?
08. Mathilde: Yes, let's say we worked with something that tended to 0 from the one side, but
09. Karen: something different … then it would not be differentiable. Ok. For example, if it has a corner.

The group moves on to investigate differentiability for $x_0 = 1$. Karen types $\Delta x = -0.1$, for which the secant slope is 3.3, and then she types $\Delta x = 0.1$, for which the slope is 3.1.

10. Karen: Then, I guess, it is not differentiable. But, that does not make any sense.
11. Lily: No, I don't think it does. But, well, I guess it approaches the same, but it is not quite the same. Is it just because there is a little bit of difference in the slope. Well, they both approach 3 [she mumbles something]… but does it have to be the exact same? What if we take 0.01?
12. Karen: I think so. Let us just try. [She types in $\Delta x = 0.01$, and the secant slope is 3.19.]
13. Lily: So it does… it does move closer to. It does approach…
14. Karen: But it is just not. I think they have to be right on the opposite side. … I do not feel, it would make any sense if it is not [differentiable].
15. Lily: Yes, because there are no corners, or jumps or anything.

From this, Karen and Lily presume differentiability, but, to be sure, they ask their teacher, who suggests they go even closer. Setting $\Delta x = \pm 0.0001$, they get the secant slopes to 3.2 on both sides and confirm their presumption. The group moves on to another function, where they have to investigate differentiability for $x_0 = 10$. For $\Delta x = 0.1$, the secant slope is $-0.1$, and for $\Delta x$ to $-0.1$, the secant slope is 0.

16. Karen: Oops, there it is not. Should we try more 0's, or what?

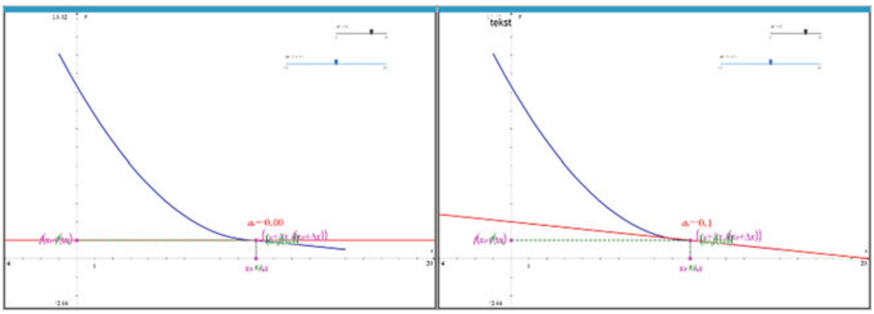


**Fig. 2** Non-differentiable function at $x = 10$. The secant slope is 0 for $\Delta x = 0.00001$ (left), and the secant slope is $-0.1$ for $\Delta x = -0.00001$ (right)

Det vil sige at alle er differentiable, fordi den er kontinuert samt uden knæk.
Hældning for $x_0 = 1$ er 3.2
Hældning for $x_0 = 3$ er 0.0
Hældning for $x_0 = 9$ er 0.0

**Fig. 3** Estimated slopes for the given $x$-values. The text translates as "That means that they are all differentiable, because it is continuous and without corners. Slope for $x_0 = 1$ is 3.2 Slope for $x_0 = 3$ is 0.0 Slope for $x_0 = 9$ is 0.0"

For $\Delta x = \pm 0.00001$ the slope is still 0 (Fig. 2 left), respectively $-0.1$ (Fig. 2 right).

Karen and Lily seem to see a little corner around $x = 10$, and they try with $\Delta x = \pm 0.000001$. The secant slope is still 0 for positive $\Delta x$ and $-0.1$ for negative $\Delta x$, and they conclude non-differentiability. Afterward, they get guidelines to calculate the derivative with TI-*n*spire CAS. Karen types in the given command for the first function with the value of $x_0 = 1$, to which TI-*n*spire CAS gives the result 3.2.

17. Karen: 3.2, ay! Ah, that was it… we did not calculate it. Should we then just [calculate it]?

They go back to the graphic window again and estimate their values for the limit of the secant slope for $\Delta x$ approaching 0 (Fig. 3).

Before using CAS on the last example, the students go back to the graphic investigation and write out their arguments for their conclusion of non-differentiability for $x_0 = 10$ (Fig. 4).

18. Karen: $x$ is 10, and there we got that it was not differentiable in ours.
19. Lily: Oh, yes. Should it not say undefined then?
20. Karen: I think so.

Karen types in the command to which the output is "undef" (the last calculation in Fig. 5).

21. Lily: Great. It is so nice when it works.
22. Karen: Great, yeah. It is the greatest when it actually works, and you finally … well, when you do not get it. It is really like ups and downs.

*Vi fandt ud af, at $x_0 = 2$ er differentiabel (differentiabelkvotient = -1.33), men at $x_0 =$ 10 ikke er differentiabel*
*Vi fandt ud af at hældningen for sekanten ved $x_0 = 10$ er forskellig, når man kom fra en positiv retning -0.1, men når man kommer fra en negativ retning, er hældningen 0*

**Fig. 4** The students' argumentation for differentiability and non-differentiability. The text translates as "We found out that $x_0 = 2$ is differentiable (differential quotient $= -1.33$), but that $x_0 = 10$ is not differentiable. We found out that the slope of the secant for $x_0 = 10$ is unequal when you come from a positive direction $-0.1$, but, when you come from a negative direction, the slope is 0"

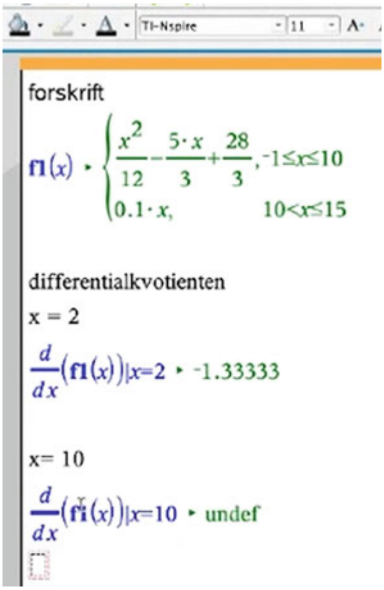



**Fig. 5** Calculations of the derivative for specific values in TI-nspire CAS

## 8 Analysis 1: Exercised Processes of the Mathematical Thinking Competency

To identify which processes of the mathematical thinking competency the students exercise, I analyze the case through the lens of the four aspects of the mathematical thinking competency. At first, Lily asks the question of what differentiability means (Line 01), which illustrates processes of the question–answer aspect. She recognizes the task as a mathematical question and relates this to the nature of the expected answer to such a task. This prompts her to search for a definition or explanation of differentiability, so they know which results they can interpret as "yes, it is" or "no, it is not differentiable". The students do not exercise the process of posing or relating to generic mathematical questions, which are also parts of the question–answer aspect.

In the students' search for an answer, relying on the explanation of differentiability as a function whose graph has no corners (Line 02), the two students exercise processes of the scope of concept aspect. For instance, Karen connects her understanding of differentiability as a graph with no corners to the actions carried out with the tool (Line 09). Here, she relates the scope of differentiability when simply looking at the graph of the function to investigate the limit of the secant slopes graphically.

Before starting their investigations, they intuitively consider the graph as being soft with no corners (Line 03–05). Hence, they presume that the function is differentiable. When they are to determine if the function is differentiable for $x_0 = 1$, they calculate the secant slopes to be 3.3, respectively, 3.1 for $\Delta x = \pm 0.1$. This makes Karen conclude non-differentiability, despite it looks like it has no corners (Line 10). Lily tries to relate the two understandings of differentiability by interpreting the dynamic output of the tool, adding more 0s to $\Delta x$ (Line 09–15). This relation is also seen in the instance of non-differentiability (Line 16). Throughout the actions with the tool, the students' intuitive view of differentiability as a graph with no corners guides the students' extension of the scope of the concept of differentiability.

When calculating the derivatives with CAS, they relate these results to their work with the tool (Line 17–20). Hereby, they expand their understanding of an answer of differentiability to include a result in the form of a number, or the output "undef" in the case of non-differentiability (Line 19–20). Hence, they return to the question–answer aspect of the mathematical thinking competency.

This analysis illustrates that the work with instances of differentiability and non-differentiability makes Karen and Lily exercise some processes of the question–answer aspect and some of the scope of concept aspect, but no processes of the mathematical statements or the generalization–abstraction aspect. Moreover, the students' expressions in Line 18–19 also illustrate how the students find coherence between the answers and the varying scopes. In the following analyses, I elaborate on how these processes are exercised during the students' work.

## 9 Analysis 2: Beginning Instrumental Genesis

In this section, I analyze the case using the perspectives of instrumental genesis and conceptual fields. In this case, the TI-*n*spire worksheet is the artefact at issue. Karen and Lily's starting point is their concept-in-action "differentiability is shown by a graph with no corners" (Line 01–02). Thereby, they have an intuitive idea of the function being differentiable (Line 03–05). This understanding becomes part of their predicative form of knowledge, but with no connection to any operative form of knowledge. As they have no theorem-in-action to draw on, they cannot infer how to act with the artefact in relation to their predicative form of knowledge and they ask for help. This illustrates the difficulties of beginning the instrumental genesis when they have no initial schemes to rely on.

The instrumentation of the artefact leads Karen to a theorem-in-action, saying that, for a function to be differentiable at a given point, the secant slope should be the same value for both negative and positive $\Delta x$, close to the given point. This worked for the first instance ($x_0 = 9$) with $\Delta x = \pm 0.1$. Therefore, Karen and Lily use this scheme to investigate differentiability by copying the technique of setting $\Delta x = \pm 0.1$ for the next point of interest ($x_0 = 1$). As this gives them two different secant slopes, respectively, for $\Delta x = \pm 0.1$, they get confused (Line 10–15). This instrumentation leads Karen to conclude non-differentiability, even though this conclusion does not match her initial concept-in-action of differentiability being a curved graph with no corners.

Because of the mismatch between the tool-induced theorem-in-action and the book-induced concept-in-action, Lily is not sure non-differentiability is the answer.

Although she cannot infer the exact limit, her scheme builds on the secant slope getting closer to something, which is observed by her technique of adding more 0s to $\Delta x$. In Lily's case, the duality of instrumentation-instrumentalization is essential. On the one hand, it is Lily's way of thinking of "approaching" (Line 11) that directs the use of the slider for $\Delta x$. On the other hand, the configuration of the tool and the constraints of the slider have encouraged her to think this way.

With $\Delta x = \pm 0.0001$, they conclude differentiability for $x_0 = 1$, which matches the concept-in-action of "differentiability-as-no-corners". This process illustrates how Lily's actions make Karen adjust her thinking of differentiability. For Karen, the two secant slopes still have to be equal, but now in the sense of $\Delta x$ close enough to 0. Both Lily's and Karen's schemes are confirmed by the instance of non-differentiability, where the two secant slopes differ from each other, keeping their respective values, regardless of how close $\Delta x$ is to 0 (Line 16, Fig. 2 left and right).

In the end, the CAS calculations of the derivatives also confirm their schemes, building on the instrumentation of the dynamic environment. Getting the exact value of the derivative puts the dynamic investigations into perspective and helps them develop a predicative form of knowledge in the form of arguments for differentiability and non-differentiability, respectively (Figs. 3 and 4). Also in this situation, the instance of non-differentiability functions as a confirmation, when Lily predicts the output to be "undefined" (Line 19), and, in this way, thinks with the tool.

This analysis illustrates how the students relate to the varying scope of differentiability from simply looking at the graph to estimating the limits of the secant slopes by relating their concept-in-action of "differentiability-as-no-corners" with their developed theorems-in-action from working with the tool. Through the development of schemes for using the dynamic worksheet as an instrument for determining differentiability, the students' two views on differentiability approach a conceptualization of differentiability with a similar scope. With the students' development of predicative knowledge, they expand their view on an expected answer of differentiability to include whether or not they can determine a limit. Therefore, this process also calls for them to exercise the question–answer aspect of the mathematical thinking competency.

To explore how the students obtain sense and meaning out of their interaction with the tool, I will now analyze the case from the perspective of semiotic mediation.

## 10 Analysis 3: Signs of Semiotic Mediation

Like above in the analysis from the perspective of instrumental genesis, the artefact is the specific dynamic TI-*n*spire worksheet with the two sliders. From this perspective, we consider it a tool of semiotic mediation in relation to the specific tasks of investigating differentiability and the mathematical content of differentiability.

Summing up their initial work with the artefact, Karen's question "Ah, so it is opposite, kind of like a mirror-ish?" (Line 07) can be seen as an artefact sign of how she understands the secant slopes' behavior in relation to differentiability. She connects this to the situation of non-differentiability and a graph with corners (Line 09). Thus, the terms "corner" and "no-corner" become pivot signs that hinge the behavior of the secant slopes to the concept of differentiability-as-no-corners. This allows her to speak of differentiability on a more general level than just related to the specific tasks.

In the next task for $x_0 = 1$, where the secant slopes do not "mirror" around the given value, Karen concludes "Then, I guess, it is not differentiable. But, that does not make any sense." This illustrates a discrepancy between her personal sense of differentiability within the artefact and her differentiability-as-no-corners understanding. Lily, on the other hand, remains focused on the dynamic of the artefact, expressed by artefact signs like "it approaches…" (Line 12) and "it does move closer to…" (Line 14). With confirmation from their teacher, this leads to an agreement between the results in the artefact and the differentiability-as-no-corners understanding for both students, as they can see the graph has no corner at the given point. For Lily, the movements of the artefact confirm her understanding of "approaching", and for Karen, the secant slopes do "mirror" around the given point for small enough $\Delta x$.

In the last task, Karen and Lily's written answers show their move from artefact signs to mathematical signs, with the pivot sign "slope" hinging the observed in the artefact with the derivative obtained by CAS (Figs. 3 and 4). During the exercises in the dynamic artefact, Karen does not pay attention to the values of the limit but to whether the slopes are equal on both sides of the given $x$-value. Lily seems aware of the limit in some sense, expressed by artefact signs of "approaching". When they return to the graphic investigations after calculating the first derivative, the exact values become their argumentation for whether the function is differentiable. In Fig. 3, they use the pivot signs of "corners" and "slope", whereas in Fig. 4, they use the mathematical sign "derivative". This indicates a move from the artefact and toward a more generalized concept of differentiability. However, the episode also illustrates that knowing the CAS command simplifies determining differentiability to whether the output is a number or the undefined-respond.

This analysis illustrates that as part of the students' meaning-making of their actions with and responses from the artefact they exercise the question–answer aspect of the mathematical thinking competency. Like in analysis 2, analysis 3 shows the importance of the individual tasks in relation to each other, which makes the students exercise the scope of concept aspect. Moreover, the attention to artefact signs and mathematical signs in analysis 3 indicates an initial, yet important, process of the generalization–abstraction aspect of the mathematical thinking competency, which analysis 1 did not capture.

## 11 Discussion and Conclusion

By viewing the same data through different lenses, the three analyses above illustrate processes of students' mathematical thinking competency in interactions with the use of the TI-*n*spire worksheet.

Analysis 1 uncovers which aspects of the mathematical thinking competency are exercised in the given case and which are not. The apparent processes of the mathematical thinking competency that the students exercise are of the question–answer and the scope of concept aspects. It is through the students' work with multiple instances of differentiability and non-differentiability that the students get the opportunities to relate to the scope of the concept in different contexts and develop their conception of differentiability, as well as to relate to the expected answer for determining differentiability.

Analyses 2 and 3 illustrate how these processes of mathematical thinking competency interact with the students' use of the TI-*n*spire dynamic template and CAS. First, the case illustrates that the question–answer aspect is part of the instrumentation of instrumental genesis. An expectation of the interplay between the mathematical thinking competency and the use of digital tools could be that relating to an expected answer would influence the instrumentalization. Having an idea of what kind of answer to be looking for may influence how to use the tool. However, Analysis 2 of the case illustrates that relating to the expected answer develops through the instrumentation and the development of schemes. Not until then, the question–answer aspect is exercised in the instrumentalization aspect as well. Thus, in this case, instrumentation and to some extent instrumentalization interact with the exercise of the question–answer aspect.

Second, the scope of concept aspect is exercised in the duality of instrumentation and instrumentalization. The students bring new operational knowledge including new-developed schemes into each new task, which develops both the instrumental genesis and the signs of the semiotic mediation as well as the scope of the concept of differentiability. Hereby, it seems that the students work on the individual tasks and relate these to each other, but do not generalize over the different instances. Nevertheless, illustrating the development from artefact signs to pivot and initial mathematical signs, Analysis 3 indicates that the students approach the conditions for the limit to exist and for the function to be differential at a more general level. This shows that the students' work with the tool includes an initial process of the generalization–abstraction aspect toward a more generalized concept of differentiability—an aspect of the mathematical thinking competency not obvious from the perspective of the mathematical thinking competency alone.

Using the networking strategy of combining, the three analyses can be said to have separate foci. The first analysis (with the mathematical thinking competency as a coarse-grained framework) helped navigate the following analyses using the other selected theoretical perspectives. As juxtaposed analyses, each theoretical perspective adds to a networked understanding of students exercising the mathematical thinking competency while interacting with the given TI-*n*spire worksheet.

The focus on enactment in the KOM framework fits well with the focus on interaction with the tool in the perspectives of instrumental genesis (Drijvers et al., 2013) and semiotic mediation (Bussi & Mariotti, 2008) as well as with the focus on the operational form of knowledge and the notion of scheme in the perspective of conceptual fields (Vergnaud, 2009). This indicates compatibility between the KOM framework and the theoretical perspectives applied in this study, thus, a potential for using the networking strategy coordinating. Hence, the theoretical perspectives can be pieced together as a conceptual framework to study the mathematical thinking competency in interaction with the use of a given digital technology.

The three juxtaposed analyses also illustrate which processes of the mathematical thinking competencies the students do not exercise. First of all, the involved tasks do not initiate the mathematical statements aspect or the part of the generalization–abstraction aspect involving the awareness of generalization and abstraction as mathematical activities. Nevertheless, it could be expected that the students would try to generalize more explicitly over the different instances of differentiability. Furthermore, having the view on differentiability-as-no-corners, the students could have asked why discontinuous functions or functions whose graphs have sharp corners are not differentiable and in this way add to the question–answer aspect as well as to the scope of concept aspect. Yet the results do indicate how single aspects of the mathematical thinking competency can be exercised in interaction with an explorative worksheet, like the TI-*n*spire worksheet presented in this chapter. Using an explorative environment can help students investigate both positive and negative instances of given mathematical concepts, processes or relations. However, it is important that the tasks supporting the students' use of the tool guide the students' explorations, for example, by clearly stating the interesting aspects and asking them to observe specific elements of the explorative environment. Finally, the tasks should explicitly ask the students to question the concept in different contexts, for instance, how the given concept is connected to other parts of the specific topic or how the given concept could be defined in other mathematical topics.

**Acknowledgements** Acknowledgements to Ola Helenius, Linda Ahl and Andreas Tamborg for valuable comments on earlier versions of this chapter.

Supported by Independent Research Fund Denmark [Grant no. 8018-00062B].

## References

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 483–506). Springer. https://doi.org/10.1007/978-3-642-00742-2_46

Bussi, M. G. B., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom. In L. D. English, M. B. Bussi, G. A. Jones, R. A. Lesh, B. Sriraman, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 746–783). Routledge. https://doi.org/10.4324/9780203930236.ch28

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses. *Educational Studies in Mathematics*, *82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Hohenwarter, M., Hohenwarter, J., Kreis, Y., & Lavicza, S. (2008, July 6–13). Teaching and calculus with free dynamic mathematics software GeoGebra [Paper presentation]. In *The 11th International Congress on Mathematical Education* (ICME 11), Monterey, Mexico.

Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). What happens when CAS procedures are objectified?—The case of "solve" and "desolve." *Educational Studies in Mathematics, 101*(1), 67–81. https://doi.org/10.1007/s10649-019-09888-5

Jankvist, U. T., & Niss, M. (2015). A framework for designing a research-based "maths counsellor" teacher programme. *Educational Studies in Mathematics, 90*(3), 259–284. https://doi.org/10.1007/s10649-015-9629-8

Kendal, M., & Stacey, K. (2000). Acquiring the concept of derivative: Teaching and learning with multiple representations and CAS. In T. Nakahara, & M. Koyama (Eds.), *Proceeding of the 24th conference of the international group for the psychology of mathematics education (PME 24)* (Vol. 3, pp. 127–134).

Niss, M. (2016). Mathematics standards and curricula under the influence of digital affordances: Different notions, meanings and roles in different parts of the world. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 239–250). Information Age Publishing.

Niss, M., Bruder, R., Planas, N., Turner, R., & Villa-Ochoa, J. A. (2016). Survey team on: Conceptualisation of the role of competencies, knowing and knowledge in mathematics education research. *ZDM Mathematics Education, 48*(5), 611–632. https://doi.org/10.1007/s11858-016-0799-3

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jankvist, U. T. (2022). On the mathematical competencies framework and its potentials for connecting with other theoretical perspectives. In U. T. Jankvist, & E. Geraniou (Eds.), *Mathematical competencies in the digital era* (pp. 15–38). Springer. https://doi.org/10.1007/978-3-031-10141-0_2

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Stacey, K. (2010). Mathematical and scientific literacy around the world. *Journal of Science and Mathematics Education in Southeast Asia, 33*, 1–16.

Trouche, L., Drijvers, P., Gueudet, G., & Sacristán, A. I. (2013). Technology driven developments and policy implications for mathematics education. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 753–789). Springer.

Vergnaud, G. (1996). Some of Piaget's fundamental ideas concerning didactics. *Prospects, 26*(1), 183–194. https://doi.org/10.1007/BF02195617

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101. https://doi.org/10.1007/BF03172796

Weigand, H.-G., & Bichler, E. (2010). Towards a competence model for the use of symbolic calculators in mathematics lessons: The case of functions. *ZDM Mathematics Education, 42*(7), 697–713. https://doi.org/10.1007/s11858-010-0274-5

# Mathematical Competencies Framework Meets Problem-Solving Research in Mathematics Education

**Mario Sánchez Aguilar, Martha Leticia García Rodríguez, and William Enrique Poveda Fernández**

## 1 Introduction

Nowadays, the mathematical instruction of many students around the world is immersed in technological environments supported by the use of collaborative digital platforms, telecommunication software, and other digital technologies. The digital tools to which students are exposed in these instructional environments modify their ways of solving and proposing mathematical problems, as well as the way they express and communicate such problems and their solutions. The transformation of the processes of solving and posing mathematical problems in these instructional environments is so profound that the isolated use of theoretical frameworks may prove insufficient to fully grasp the emergence and development of the strategies implemented by the students in these kinds of technological environments.

This chapter illustrates how the mathematical competencies framework (Niss & Højgaard, 2011, 2019) can function as an organizing framework for studying mathematical problem-solving processes supported by the use of digital tools. When used in combination with other compatible theoretical notions—as happens with some notions from research on mathematical problem solving—the mathematical competencies framework offers an enhanced analytical capability.

M. S. Aguilar (✉) · M. L. G. Rodríguez
Programa de Matemática Educativa, Instituto Politécnico Nacional, CICATA Legaria, Mexico city, Mexico
e-mail: mosanchez@ipn.mx

M. L. G. Rodríguez
e-mail: mlgarcia@ipn.mx

W. E. P. Fernández
Departamento de Educación Matemática, Escuela de Matemática, Universidad de Costa Rica, San José, Costa Rica
e-mail: william.poveda@ucr.ac.cr

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_5

Particularly, this chapter illustrates the potential of the networking of theories (Prediger et al., 2008) through a case study related to the mathematical work of a preservice mathematics teacher. A theoretical analysis of the problem-solving process of this future teacher when dealing with a geometrical problem aided by digital tools is presented as illustration. This chapter contributes not only to illustrating the potential of networking of theories as a research practice, but also to positioning the mathematical competencies framework as an additional theoretical tool available to researchers interested in investigating people's mathematical problem-solving processes.

Before presenting the case study and its analysis, the following sections introduce the reader to a set of theoretical notions that are necessary to follow the analysis. It begins with an outline of the mathematical competencies framework and later introduces the notion of networking of theories in mathematics education. Finally, some theoretical developments from problem-solving research in mathematics education are introduced.

## 2 Mathematical Competencies: An Overarching Framework

The mathematical competencies framework is derived from a Danish educational project called *Competencies and Mathematical Learning* (The KOM project), first reported by Niss and Jensen (2002). One of the driving forces behind this project was "to create valid and reliable forms of assessment of a person's mastery of mathematical competencies" (Niss & Højgaard, 2011, p. 8). Thus, the framework of mathematical competencies offers an overarching conceptualization of what it means for a person to be "mathematically competent".

At the heart of this conceptual framework is the notion of *mathematical competence*, defined as "someone's insightful readiness to act appropriately in response to all kinds of *mathematical* challenges pertaining to given situations (Niss & Højgaard, 2019, p. 12). Moreover, there are constituent components of mathematical competence called *mathematical competencies* defined as "someone's insightful readiness to act appropriately in response to a *specific sort* of mathematical *challenge* in given situations" (Niss & Højgaard, 2019, p. 14). According to this framework, the mathematical competencies call "for 'specific kinds of activation' of mathematics in order to answer questions, solve problems, understand phenomena, relationships or mechanisms, or to take a stance or make a decision" (Niss & Højgaard, 2019, p. 14).

Eight mathematical competencies[1] constitute the notion of mathematical competence. Although usually presented separately, these competencies may overlap and interact, depending on the situation and context. The eight competencies can be grouped into two categories (Niss & Højgaard, 2019). The first is the category of competencies for posing and answering questions in and by means of mathematics:

- Mathematical problem-handling competency;
- Mathematical reasoning competency;
- Mathematical modeling competency;
- Mathematical thinking competency.

The second is the category of competencies for handling the language, constructs, and tools of mathematics:

- Mathematical representation competency;
- Mathematical symbols and formalism competency;
- Mathematical communication competency;
- Mathematical aids and tools competency.

The case study that is analyzed in this chapter shows an evident intertwining between the mathematical problem-handling competency and the mathematical aids and tools competency. However, this does not mean that other mathematical competencies are not involved in the case study—they are also activated but to a lesser degree. Due to their importance in the case study analyzed, we will describe these two competencies in more detail below. A more comprehensive account of all eight mathematical competencies can be found in Niss and Højgaard (2011).

### *2.1 Problem-Handling Competency*

This competency relates to the capacity to pose and solve mathematical problems within and across a variety of mathematical domains. But what is meant by a mathematical *problem*? According to Niss and Højgaard (2011, p. 55):

> A (formulated) mathematical problem is a particular type of mathematical question, namely one where mathematical investigation is necessary to solve it. In a way, questions that can be answered by means of a (few) specific routine operations also fall under this definition of "problem". The types of questions that can be answered by activating routine skills are not included in the definition of mathematical problems in this context. The notion of a "mathematical problem" is therefore not absolute, but relative to the person faced with the problem. What may be a routine task for one person may be a problem for someone else and vice versa.

[1] There are small differences between the names of the mathematical competencies as presented in Niss and Højgaard (2011) and Niss and Højgaard (2019). In this chapter, we use abbreviated versions of the competencies' names as presented in Niss and Højgaard (2019).

As noted by Niss and Højgaard (2019, p. 15), "a key aspect of this competency is the ability to devise and implement strategies to solve mathematical problems". Niss and Højgaard (2011, p. 55) characterize this competency as follows:

> This competency partly involves being able to put forward, i.e., detect, formulate, delimitate and specify different kinds of mathematical problems, "pure" as well as "applied", "open" as well as "closed", and partly being able to solve such mathematical problems in their already formulated form, whether posed by oneself or by others, and, if necessary or desirable, in different ways.

## *2.2 Mathematical Aids and Tools Competency*

In this competency, reference is made to the ability to put into productive use material aids and tools as part of the mathematical activity. The material aids and tools considered in this competency may vary: blocks, rulers, abacuses, calculators, etc. As noted by Niss and Højgaard (2011), this competency involves having knowledge of the existence and properties of relevant tools used in mathematics, but also having an insight into their possibilities and limitations. It is important to note how Niss and Højgaard's (2011) considerations about mathematical aids and tools are applicable to a wide range of digital tools such as software, mobile devices, digital platforms, etc.

## *2.3 Facets and Dimensions of a Mathematical Competency*

All mathematical competencies are manifested either in a receptive facet or in a constructive facet. According to Niss and Højgaard (2019), in the receptive facet of a competency, the individual manifests the ability to "relate to and navigate with respect to considerations and processes which have already been introduced (typically by others) into a given context or situation" (p. 19), while in the constructive facet "the focus is on the individual's ability to independently invoke and activate the competency to put it to use for constructive purposes in given contexts and situations" (p. 19).

Moreover, individuals' mathematical competence manifests in different contexts and situations but never in its full range; that is, it is not possible to exhaustively and completely possess a mathematical competency. For this reason, there are three dimensions that allow definition and characterization of the degree of possession of a competency by an individual: degree of coverage, radius of action, and technical level. The three dimensions are defined as follows:

> The degree of coverage of a competency is the extent to which all the aspects that define and characterize the competency form part of that individual's possession of the competency. … The radius of action represents the range and variety of different contexts and situations in which the individual can successfully activate the competency. … [The technical level]

> denotes the level and degree of sophistication of the mathematical concepts, results, theories and methods which the individual can bring to bear when exercising the competency. (Niss & Højgaard, 2019, p. 21)

As mentioned earlier, the mathematical competencies framework offers an overarching conceptualization of what it means for a person to be mathematically competent. Since it includes competencies related to the use of tools and the handling of mathematical problems, it is a theoretical approach that allows framing the study of people solving and proposing mathematical problems when they are supported by the use of digital tools. In particular, the mathematical competencies framework can readily be used as an organizing framework that could be networked with compatible theoretical notions from the area of problem solving for analyzing the complex reality of students' mathematical work on collaborative digital platforms. Later on, it is illustrated how the networking of these theoretical notions can allow a fine-grained analysis of the ways people handle mathematical problems within technological settings. But before that, the notion of networking of theories in mathematics education is introduced in more detail.

## 3 Networking of Theories in Mathematics Education

The networking of theories arises as a way to take advantage of the wealth and variety of existing theoretical approaches in the field of mathematics education research. The diversity of theoretical lenses is considered a rich resource for grasping the complex reality of educational phenomena in mathematics instruction (Prediger et al., 2008).

The networking of theories can be interpreted as a research practice consisting of a set of methods and strategies to connect theoretical approaches. This connection can have several purposes such as understanding others and making one's own theories understandable, finding similarities and differences between theoretical approaches, understanding an empirical phenomenon or a piece of data, and developing a new piece of synthesized or integrated theory (Prediger et al., 2008).

Within the range of networking strategies, there are two focused on providing deeper insights into an empirical phenomenon or a piece of data, namely *combining* and *coordinating*. According to Prediger and Bikner-Ahsbahs (2014), these strategies are mostly used for a networked understanding of an empirical phenomenon or a piece of data. Here comes into play the idea of triangulation since "combining and coordinating means looking at the same phenomenon from different theoretical perspectives as a method for deepening insights into the phenomenon" (pp. 119–120).

The distinction between the strategies combining and coordinating depends on the degree of integration of theory elements with respect to their compatibility. While combining theoretical approaches does not necessitate the complete compatibility of the theoretical approaches under consideration, the coordination of theoretical approaches produces a conceptual framework with elements of different theories but with compatible cores (Prediger & Bikner-Ahsbahs, 2014; Prediger et al., 2008).

Taking these ideas together, it could be claimed that the method for implementing a coordinating strategy consists of, first, verifying the compatibility of the theoretical approaches to be coordinated and, second, conducting an analysis in which the same phenomenon is analyzed from different theoretical perspectives in order to deepen our understanding of the phenomenon. To verify theoretical compatibility, it is necessary to confirm that the particular approaches or notions that are intended to be coordinated share epistemological assumptions, and that their key premises and definitions are shared or at least do not contradict each other.

In the analysis presented later in this chapter, the coordinating strategy is used to analyze an empirical case. Such an analysis serves as an illustration of the ideas presented in this section. The following section presents an overview of some theoretical notions from problem-solving research, which will be used in coordination with the mathematical competencies framework.

## 4 Some Theoretical Developments from Problem-Solving Research in Mathematics Education

Problem solving is a well-established research area in the field of mathematics education. Its main purpose is to understand and relate the processes involved in formulating and solving problems and use such understanding to promote students' development of mathematical knowledge and problem-solving competencies (Santos-Trigo, 2020).

A fundamental notion in mathematical problem solving is that of a *problem*. Schoenfeld (1985) refers to the notion of problem as follows:

> The difficulty with defining the term *problem* is that problem solving is relative. The same tasks that call for significant efforts from some students may well be routine exercises for others, and answering them may just be a matter of recall for a given mathematician. Thus being a "problem" is not a property inherent in a mathematical task. … The word *problem* is used here in this relative sense, as a task that is difficult for the individual that is trying to solve it. (p. 74)

Another key notion in the area of mathematical problem solving is *heuristics*. An early definition of this notion can be found in Polya (1962/1981): "I wish to call *heuristics* the study that the present work attempts, the study of means and methods of problem solving." (p. x, emphasis in the original). Nevertheless, a heuristic can also be understood as a "generic rule that often helps in solving a range of non-routine problems. … Heuristics are an important aspect of mathematical problem solving, especially if we refer to them as the capabilities for mathematical reasoning that enable insightful problem solving" (Mousoulides & Sriraman, 2020, p. 331).

There are other theoretical developments emanating from mathematical problem solving. In particular, conceptual frameworks to characterize learners' progress and success in problem-solving activities have been developed in this research area. One of the first models to characterize the process of solving mathematical problems was proposed by Polya (1945/1957). Polya's model consists of four phases: understanding the problem, devising a plan, carrying out the plan, and looking back. Another widely used framework is the one proposed by Schoenfeld (1985). This framework helps to explain students' problem-solving behaviors in terms of four dimensions: the use of basic mathematical resources or knowledge base, the use of cognitive or heuristic strategies, the use of metacognitive or self-monitoring and control strategies, and students' beliefs about mathematics and problem solving.

Schoenfeld's (1985) work has served as inspiration and as a starting point for other theoretical developments within the mathematical problem-solving research area. Of particular relevance for the analysis presented in this chapter are the theoretical models that consider the role of digital tools—particularly the use of dynamic geometry systems (DGS)—in the process of mathematical problem solving (e.g., Jacinto & Carreira, 2017; Santos-Trigo & Camacho Machín, 2013). In the next section, a theoretical framework that considers the use of DGS into problem-solving processes is introduced. Notions of this framework are later coordinated with notions of the mathematical competencies framework for the analysis of a case study.

## *4.1 A Framework for the Systematic Use of Technology in Mathematical Problem Solving*

The use of digital tools has a profound effect on problem-solving processes. According to Santos-Trigo (2020, p. 690), digital tools provide "new opportunities for teachers and students to represent, explore, and solve mathematical problems and to extend mathematical discussions beyond formal settings". But what happens when subjects use systematically computational tools to make sense of the problem statement, represent, explore, and solve problems? This is a central question in the work of Santos-Trigo and Camacho Machín (2013), where they propose a framework to characterize ways of reasoning that emerge as a result of using digital tools when trying to solve a mathematical problem.

The framework proposed by Santos-Trigo and Camacho Machín (2013) stems from the systematic observation of prospective and in-service mathematics teachers solving mathematical problems with the aid of digital tools—mainly with the support of DGS—and takes into consideration previous theoretical models for mathematical problem solving. These researchers acknowledge that digital tools such as DGS introduce powerful heuristics to the problem-solving process such as dragging objects and finding loci of particular objects. The framework conceptualizes into the following four episodes the process of solving mathematical problems with the support of digital tools.

**The Comprehension Episode**. As in Polya's model (1945/1957), the comprehension episode refers to the process of understanding the problem statement by the problem-solver. Making sense of the problem statement is a fundamental step in any problem-solving approach. However, the comprehension episode considers the use of digital tools as a means of representing and exploring the problem with the intention of fully understanding it: "the use of the tool demands that the problem-solver thinks of the statement in terms of mathematical properties to use the proper software commands to represent and explore the problem" (Santos-Trigo & Camacho Machín, 2013, p. 287).

**The Problem Exploration Episode**. Once the problem statement has been understood, the problem-solver is in a position to begin the exploration of the problem with the help of digital tools. This exploration can serve the solver to broaden the perspectives from which the problem can be analyzed, but it can also be useful to formulate conjectures related to the problem that could later be confirmed or refuted.

**The Searching for Multiple Approaches Episode**. This is a continuation of the previous episode. This means that in this episode the problem is approached by the solver from different perspectives (e.g., analytical, geometric) and using different resources. This can favor the contrast of strengths and limitations associated with each approach. Also, it is in this episode where the problem-solver has the opportunity to test their initial conjectures on the problem.

**The Integration Episode**. This is an episode in which the different approaches and solutions to the problem are collectively presented and compared. This collective process favors the comparison of the different approaches, the identification of the scope of the proposed solutions, and the formulation of new problems.

## 4.2 *Coordination of Notions from Problem-Solving Research with the Mathematical Competencies Framework*

As mentioned before, the coordinating networking strategy is mostly used for a networked understanding of an empirical phenomenon or a piece of data. Furthermore, the coordination strategy must be implemented between theoretical approaches with compatible cores. So here, it is relevant to ask, are the mathematical competencies framework and the aforementioned theoretical framework coming from mathematical problem-solving research compatible?

We think that the answer to the previous question is: yes, they are. First, a shared premise between the mathematical competencies framework and mathematical problem-solving research is to consider the ability to solve and propose mathematical problems as a fundamental part of mathematical understanding. Second, the fundamental notion of *problem* as relative to the person attempting to solve it is also shared. Third, the mathematical competencies framework considers the use of tools and aids—including digital tools—in the mathematical activity, as is also considered in the framework proposed by Santos-Trigo and Camacho Machín (2013). Fourth, the ability to pose conjectures—which can be fundamental to a problem-solving process—is considered part of the *mathematical thinking competency* that involves being able to relate to and pose the kinds of generic questions that are characteristic of mathematics—including definitions, if–then claims, conjectures, etc. (see Niss & Højgaard, 2019, p. 15).

The potential of this networking of theories will be illustrated with the analysis of a case study. It consists of the problem-solving process of a geometric problem with the help of a DGS, carried out by a prospective mathematics teacher. In the following sections, the geometric problem addressed by the prospective mathematics teacher is presented, and the context in which this problem was implemented is briefly explained.

## 5 A Virtual Course on Euclidean Geometry for Prospective Mathematics Teachers

The case study analyzed in this chapter consists of the solving process of a geometric problem by a future mathematics teacher identified by the pseudonym "Anna". This case study was taken from a virtual course on Euclidean geometry, which is part of a training program for prospective Costa Rican lower secondary school teachers offered by the University of Costa Rica. This course is intended to promote future teachers' understanding of geometry as an axiomatic theory and develop certain mathematical abilities. In particular, it is expected to strengthen their ability to solve geometric problems with the help of a DGS.

The course lasted 12 weeks and was taught during the spring of 2020. The course was taught mainly asynchronously using the Moodle platform and using a Zoom room for synchronous meetings. Twenty-three Costa Rican future teachers in an age range of 19–22 years participated in the course. This is a course where future mathematics teachers study Euclidean geometric concepts, but also learn how to represent and manipulate them with the DGS GeoGebra. Additionally, the course emphasizes the exploration and solving processes of geometric problems using GeoGebra.

The teacher in charge of this course is the third author of this chapter. He has a Ph.D. in mathematics education and possesses a broad knowledge of mathematical problem solving. His role was to introduce the topics of the course through the Moodle platform, to design mathematical problems for the students, and to give them feedback on their solution processes. In turn, the students went through the materials corresponding to the topics introduced by the teacher on the Moodle platform. They also solved geometric problems and sent their solutions through the Moodle platform in PDF, Word, or GeoGebra formats. This last type of format is used for data files developed with the GeoGebra software; through these files, students could show the teacher their GeoGebra-based explorations and solutions to the geometric problems, including text explaining the followed procedure step by step.

### *5.1 A Geometric Problem and Some Prerequisites to Tackle It*

The illustrative case study presented in this chapter is based on the solving process of an open-ended geometric problem that was presented to students during week 10 of the course. This geometric problem (known as Varignon's theorem) was selected from an advanced stage of the course because it was expected that at this time the students would be more capable of solving geometric problems and handling DGS.

Before the problem was presented to the students, they were introduced to the following topics related to the theme "parallelism and parallelograms". These topics are part of the mathematical prerequisites considered necessary to address the problem:

- Definition of parallel lines.
- Two theorems for parallel lines:
  - *Theorem A*. If two lines in a plane are parallel to a third line, then they are parallel to each other.
  - *Theorem B*. The line segment formed by joining the midpoints of two sides of a triangle is parallel to the third side of the triangle.
- Properties of the angles that are formed when cutting two parallel lines with a secant line.
- GeoGebra construction of a line that is parallel to another line.
- Definition of quadrilateral, parallelogram, rhombus, rectangle, and square.

The problem was presented to the students through the following statement:

*Given any quadrilateral $ABCD$ and a quadrilateral $EFGH$ whose vertices are the midpoints of the sides of $ABCD$, what properties does $EFGH$ have?*

In addition, the problem statement was accompanied by a GeoGebra file that contained a dynamic representation of the problem (see Fig. 1). The instructions for the students were: (1) use the provided dynamic representation to explore the problem, (2) propose a solution to the problem, and (3) formulate new problems from the original problem and propose solutions for them.

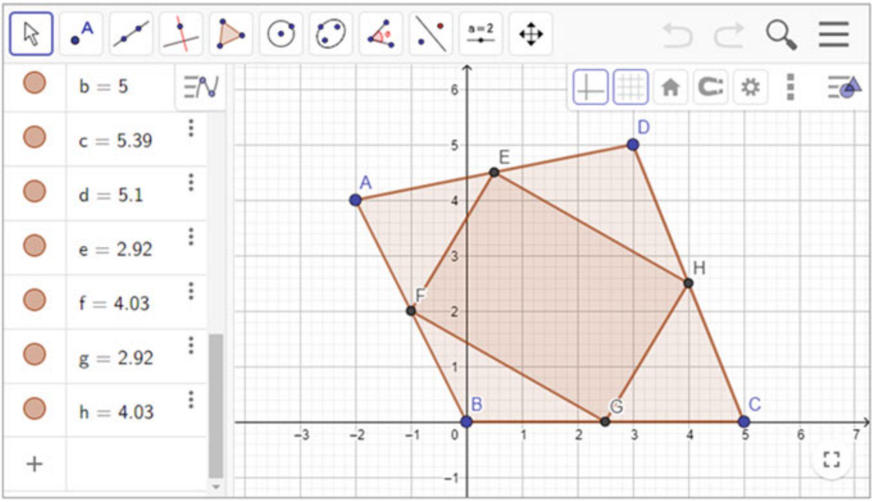



**Fig. 1** Representation of the problem provided to the students. The file allows the user to manipulate the represented figures by dragging their vertices

In the next section, we present Anna's solving process of this geometric problem. Following, we analyze this solving process with the use of a conceptual framework generated by the coordination of theoretical notions from the mathematical competencies framework and mathematical problem-solving research.

### 5.2 Anna's Solving Process of the Geometric Problem

Anna is an 18-year-old prospective mathematics teacher enrolled in the previously described course. Her case was selected because Anna's reports of the proposed solutions were explicit and detailed. Seven reports that she submitted over a two-week period were reviewed. These reports allowed the reconstruction of Anna's exploration and solution process of the geometric problem stated above. The reader may find it useful to use as a reference the representation of the problem located at https://www.geogebra.org/m/mkvps626 to follow the description of Anna's solution process.

### 5.3 Anna Formulates and Proves Conjecture 1

Anna declares having used the dynamic representation of the problem provided by the teacher to drag around the points $A$, $B$, $C$, and $D$ and formulates the following conjecture:

> *Conjecture 1: "Regardless of the position of points A, B, C and D, the quadrilateral $EFGH$ seems to be a parallelogram".*

To explore this conjecture, Anna uses the *Relation* tool in the GeoGebra system, which provides information about the possible relationship between two selected objects. Thus, Anna uses this tool to compare the opposite sides of quadrilateral $EFGH$ and confirms that they are congruent and parallel (see Fig. 2).

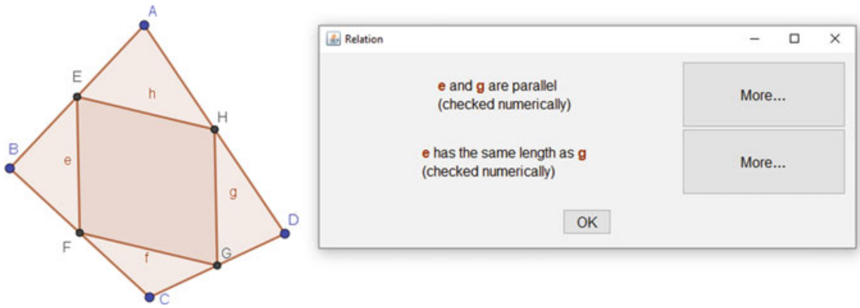



**Fig. 2** Excerpt from Anna's report illustrating how she used the Relation tool in the GeoGebra system to verify whether EFGH is a parallelogram

Then, Anna presents in her report a proof of conjecture 1. The following is a translation and transcription of that proof:

Given $\square ABCD$, $E$, $F$, $G$, and $H$ are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. $\overline{AC}$ and $\overline{BD}$ are its diagonals (see Fig. 3).

Conjecture to be proved: $\square EFGH$ is parallelogram.

| | |
|---|---|
| (1)$\triangle\ ABD$ | By definition of triangle |
| (2)$\triangle\ CBD$ | By definition of triangle |
| (3) $\overline{BD} \parallel \overline{EH}$ | By theorem B |
| (4) $\overline{BD} \parallel \overline{FG}$ | By theorem B |
| (5) $\overline{EH} \parallel \overline{FG}$ | By theorem A |
| (6) $\triangle\ BCA$ | By definition of triangle |
| (7) $\triangle\ DCA$ | By definition of triangle |
| (8) $\overline{CA} \parallel \overline{FE}$ | By theorem B |
| (9) $\overline{CA} \parallel \overline{GH}$ | By theorem B |
| (10) $\overline{FE} \parallel \overline{GH}$ | By theorem A |

$\therefore \square EFGH$ is a parallelogram by definition of parallelogram (see 5 and 10)

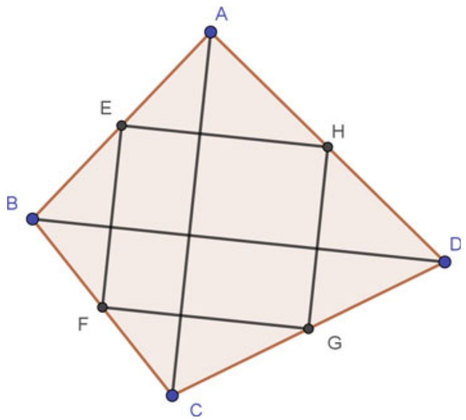



**Fig. 3** Image included by Anna in her report to illustrate her proof that □EFGH is a parallelogram

## 5.4 *Anna Formulates Conjectures 2 and 3 and Proposes Three Problems*

Anna also reports having explored the relationship between the areas of the quadrilaterals $ABCD$ and $EFGH$. One of her reports includes a screenshot of the GeoGebra system where we can observe that Anna added a grid to the plane and dynamically measured and displayed the areas of both quadrilaterals as dynamic text (see Fig. 4).

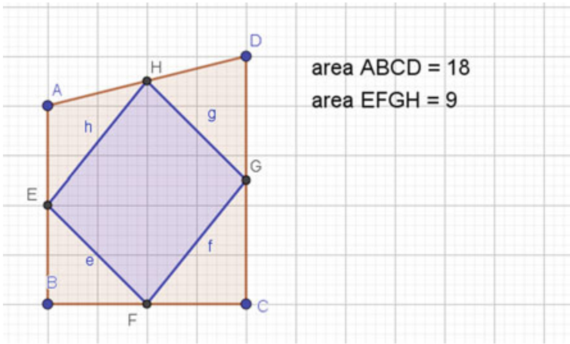



**Fig. 4** Excerpt from Anna's report showing her use of the grid on the plane and the simultaneous and dynamic measurement of the areas of the quadrilaterals *ABCD* and *EFGH*

Anna then formulates a second conjecture related to the relation of the areas of the quadrilaterals:

*Conjecture 2: "The area of $EFGH$ is half the area of $ABCD$".*

Anna did not provide a formal proof of this conjecture. Rather, she explains that she dragged the vertices of $ABCD$ and observed that in some cases the parallelogram $EFGH$ looks like a rhombus, a rectangle, or a square. She verified this observation by using particular cases of the quadrilateral $ABCD$ and using the grid as a reference. Thus, Anna proposes the following three problems:

- *Problem 1.* Is there a possibility that $EFGH$ is rhombus? What conditions must be met for this to happen?
- *Problem 2.* Is there a possibility that $EFGH$ is rectangle? What conditions must be met for this to happen?
- *Problem 3.* Is there a possibility that $EFGH$ is square? What conditions must be met for this to happen?

Anna begins the exploration of the problems that she proposed herself, particularly problem 1. She reports having dragged the vertices of the rectangle $ABCD$, using as a reference the grid in the plane, and writes that in some cases a rhombus is formed. She also reports having drawn the diagonals of the rectangle $EFGH$ and measuring the angle that they form when they intersect to verify that it is a right angle. She concludes her exploration with the following conjecture:

*Conjecture 3: "For $EFGH$ to be rhombus, its diagonals must be perpendicular and bisect. This happens if $ABCD$ is square or rectangle".*

Anna's solving process continues and leads to the formulation of a conjecture about the characteristics of $EFGH$ when $ABCD$ is a non-convex quadrilateral. However, due to space limitations, we will leave the description of the process up to this point, and we will continue with its theoretical analysis in the next section.

## 6 A Networked Analysis of Anna's Solving Process

The analysis of Anna's solving process of the geometrical problem is started by applying the mathematical competencies framework while trying to identify the potentialities and limitations that this framework offers for the analysis of a solution process. We complement the analysis by using one of the episodes considered in the framework proposed by Santos-Trigo and Camacho Machín (2013).

### *6.1 Mathematical Competencies Framework as an Overarching Framework for Analyzing Anna's Problem-Solving Process*

If we analyze Anna's solving process through the lens of the mathematical competencies framework, it is possible to identify what other mathematical competencies come into play in her solving process in addition to the mathematical problem-handling competency. For instance, it is evident that Anna uses the mathematical aids and tools competency which manifests itself through her confident use of the GeoGebra system, for example, when she uses the relation tool to get information about the possible relationship between two selected objects. Likewise, it could be argued that Anna brings into play the mathematical thinking competency due to her apparent understanding of the meaning of a conjecture and a proof. Also, the mathematical representation competency is activated in Anna's handling and exploration of the representations of geometric objects in GeoGebra, for example, when she drags around the vertices of a parallelogram to formulate conjectures. All these mathematical competencies are manifested by Anna in a constructive facet.

Continuing with this analysis, it could be further argued that other competencies are activated during Anna's solution process—albeit perhaps with a lesser degree of coverage because only some aspects that define and characterize the competencies are put into play. These include, for instance, the mathematical symbols and formalism competency and the mathematical communication competency due to the mathematical symbols and formal rules involved in developing and expressing her proof of Conjecture 1. Probably also the mathematical modeling competency should be considered since she explores geometrical models and their features.

Furthermore, the mathematical competencies framework allows us to characterize in more detail the mathematical competencies activated in Anna's problem-solving process through the dimensions of a competency degree of coverage, radius of action, and technical level. As an illustrative case, we consider Anna's mathematical problem-handling competency. It is difficult to express precisely the magnitude of these dimensions, because they are qualitative in nature (Niss & Højgaard, 2019, p. 22) and the mathematical competencies framework does not provide a reference point or any tool that facilitates the partial ordering of a competency possession. In the case of Anna's mathematical problem-handling competency, it is fair to claim that it has a broad degree of coverage because she demonstrates the ability to identify potential solutions to the proposed problem, but she also shows the capability to propose new geometrical problems by herself. The radius of action of the competency manifested through this case study is more limited since her problem tackling competency only manifests itself in a geometric context—although this does not imply that Anna cannot solve mathematical problems in other contexts and situations. Finally, the technical level that Anna's competency implies is not basic since it involves the knowledge and application of definitions and properties of various geometric objects, as well as theorems associated with the properties of those objects. A similar analysis can be conducted for the other mathematical competencies involved in Anna's problem-solving process.

As we have illustrated, the mathematical competencies framework can help formulate a clear description of the competencies that an individual activates when solving a mathematical problem aided by digital tools. However, by coordinating this framework with notions from mathematical problem solving research, it is possible to unpack more details about Anna's activation of those various competencies. The next section illustrates how a theoretical coordination like this one is useful in exploring these elements at a finer grain size.

### *6.2 Producing a More Fine-Grained Analysis Through the Coordination of Theoretical Notions*

Research on mathematical problem solving has developed theoretical notions related to the trajectories that individuals usually follow when trying to solve mathematical problems with the help of digital tools. The coordination of these notions with the mathematical competencies framework can produce a more fine-grained analysis of Anna's mathematical problem-handling competency.

For instance, the framework proposed by Santos-Trigo and Camacho Machín (2013) indicates that the problem-solving process supported with DGS initially goes through comprehension and exploration episodes. We assume that there is an understanding of the problem on the part of Anna that is confirmed in her way of exploring and conjecturing about it. However, if we zoom in and focus the analysis on the problem exploration episode—that is, the moments in which the individual conducts the exploration of the problems with the aid of digital tools—we will be able to appreciate smaller components of problem-handling competency in action. An example is the heuristics that the individual puts into play in the exploration process. In the case of Anna, it is possible to see that the heuristics of dragging vertices and measuring line segments and angles were essential for Anna to be able to identify patterns and formulate conjectures. The application of these two heuristics together with the use of the grid in the plane was fundamental to verify the proposed conjectures. In turn, these conjectures were the raw material for Anna's suggested solutions and posed problems.

Zooming in on the problem-solving episodes and paying attention to the heuristics that the problem-solver enacts in such episodes allows us to more clearly appreciate the way in which the different mathematical competencies are manifested and intertwined. For example, when Anna is exploring Conjecture 1, she uses the relation tool in the GeoGebra system which provides information about the possible relationship between two selected objects. The heuristic of comparing two geometric objects through the relation tool is a manifestation of Anna's mathematical aids and tools competency, particularly of her knowledge of the possibilities of the GeoGebra system and her ability to put that knowledge into productive use. However, Anna also manifests the ability to interpret the outcomes generated by the relation tool. That is, by confirming with the help of the DGS that the opposite sides of the quadrilateral $EFGH$ are parallel and congruent (see Fig. 2), Anna concludes—and later proves—that $EFGH$ is a parallelogram. However, we argue that Anna would not be able to interpret the outcomes of the relation tool without having some knowledge about the properties of parallelograms. In other words, in order to interpret the relation tool outcomes, Anna must also activate her mathematical representation competency, in particular, her knowledge about representations of geometric objects.

## 7 Conclusion

The mathematical competencies framework is an overarching framework that allows for identifying the different competencies—and dimensions of these competencies—that an individual brings into play by being involved in some type of mathematical activity. In the particular case of solving mathematical problems, the mathematical competencies framework allows us to identify what other competences individuals put into play when trying to solve mathematical problems, beyond the somewhat obvious mathematical problem-handling competency. We have argued and illustrated that the mathematical competencies framework can function as a platform compatible with theoretical developments of mathematical problem-solving research, which in turn allows us to deepen our understanding of mathematical problem-solving processes through a fine-grained analysis of its evolution and development.

We hope that the modest theoretical exercise that we present in this chapter will inspire other mathematical problem-solving specialists to explore the possibilities and potentialities of the networking of theories for the analysis of the problem-solving process.

## References

Jacinto, H., & Carreira, S. (2017). Mathematical problem solving with technology: The techno-mathematical fluency of a student-with-GeoGebra. *International Journal of Science and Mathematics Education, 15*(6), 1115–1136. https://doi.org/10.1007/s10763-016-9728-8

Mousoulides, N., & Sriraman, B. (2020). Heuristics in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 331–333). Springer. https://doi.org/10.1007/978-3-030-15789-0_172

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Polya. (1957). *How to solve it: A new aspect of mathematical method* (2nd ed.). Anchor Books (Original work published 1945).

Polya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined ed.). Wiley (Original work published 1962).

Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 117–125). Springer. https://doi.org/10.1007/978-3-319-05389-9_8

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM–The International Journal on Mathematics Education*, *40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Santos-Trigo, M. (2020). Problem-solving in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 686–693). Springer. https://doi.org/10.1007/978-3-030-15789-0_129

Santos-Trigo, M., & Camacho Machín, M. (2013). Framing the use of computational technology in problem solving approaches. *The Mathematics Enthusiast, 10*(1–2), 279–302.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press.

# Mathematical Modelling and Digital Tools—And How a Merger Can Support Students' Learning

**Britta Eyrich Jessen and Tinne Hoff Kjeldsen**

## 1 Introduction

Mathematical modelling is as most parts of mathematics education increasingly affected by the use of digital technologies. According to Siller and Greefrath (2010), when merging modelling and technology more advanced calculations and realistic problems can be addressed, potentially serving both pedagogical, psychological, cultural and pragmatic aims for the teaching of modelling. However, in a subsequent study they identify the non-impact of the use of technologies on students' development of modelling competency when engaged with traditional tasks. They conclude that the technological tool "cannot simply be seen as a facilitator of learning mathematical modelling, at least not if the tasks are not changed". (Greefrath et al., 2018, p. 243). This means, if we wish to capture the potential pedagogical, psychological, cultural and pragmatic aims of including technologies in modelling, we need to understand the potential roles played by digital tools and how they affect the modelling activities in order to design modelling activities supporting students' learning. In this chapter, we contribute to this by analysing two case studies designed with the aim of supporting students' development of modelling competency and learning content knowledge while explicitly drawing on digital tools. We discuss how the digital resources affect the modelling activities and learning outcomes by pursuing the following research question:

B. E. Jessen (✉)
Department of Science Education, University of Copenhagen, Copenhagen, Denmark
e-mail: britta.jessen@ind.ku.dk

T. H. Kjeldsen
Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark
e-mail: thk@math.ku.dk



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_6

How can the merger of mathematical modelling and digital resources support students' learning of mathematical knowledge together with development of their modelling and mathematical digital competencies?

Furthermore, we will discuss potentials and pitfalls for students' learning of mathematics when merging modelling and digital resources.

Mathematical modelling covers a number of different theoretical constructs (e.g., see Barquero & Jessen, 2020; Niss & Blum, 2020), however in this chapter we delimit ourselves to address mathematical modelling from perspectives explicitly related to modelling competency. According to Niss and Højgaard (2019) "Mathematical competence is someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations" (Niss & Højgaard, 2019, p. 12), which again should hold true for the modelling competency defined as "being able to construct […] mathematical models, as well as to critically analyse and evaluate existing or proposed models, whilst taking purposes, data, facts, features and properties of the extra-mathematical domain being modelled into account", (Niss & Højgaard, 2019, p. 16). Niss and Blum (2020, p. 13) present several theoretical constructs that reflect the relation between mathematics and the real world, which draw on the notion of sub-competencies. Here we employ the modelling cycle described by Blomhøj and Jensen (2007), depicted in Fig. 1, where the sub-competencies are indicated by the arrows (a)–(f) covering: formulation of task based on the perceived reality, systematisation, mathematisation into a mathematical system, mathematical analysis, interpretation, or evaluation against the mathematical system before validating the model against the perceived reality. Blomhøj and Jensen (2007) stress that these sub-competencies cannot stand alone, which is why

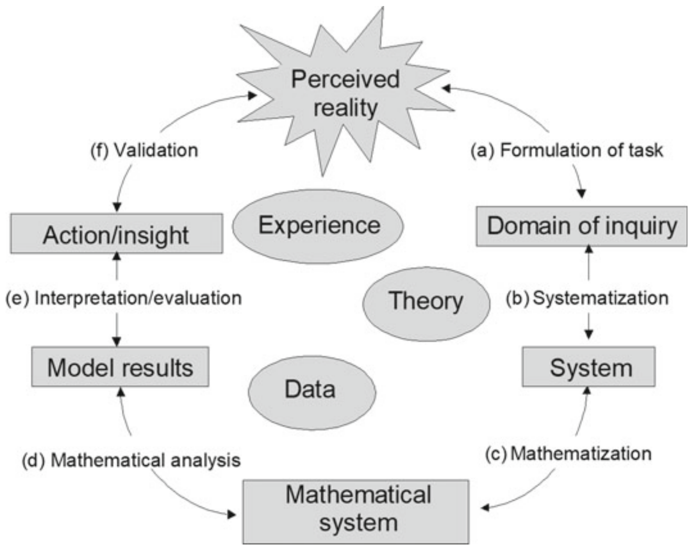



**Fig. 1** The six step modelling cycle (Blomhøj & Kjeldsen, 2006) adapted from Blomhøj and Jensen (2007)

students' experiences, theory and data available, must be taken into consideration regarding what constitutes modelling competency. These are relevant aspects for our case studies.

To analyse the learning potentials and their realisations with respect to modelling competency and mathematical digital competency (Geraniou & Jankvist, 2019) we need a theoretical tool that allows us to tune in on the role played by the digital tools, which is not captured by the modelling cycle in Fig. 1, nor in the versions suggested by Siller and Greefrath (2010). For this, we use the media-milieu dialectics from the Anthropological Theory of the Didactic (ATD). Our analysis is based on strategies for networking theories in terms of 'coordinating and combining' as described by Prediger et al. (2008, p. 172). In the following, we present the two cases before introducing the media-milieu dialectics and mathematical digital competency. That is, we follow the methodology coined parallel analysis by Prediger and colleagues (2008), by first presenting the phenomenon to be studied followed by a short presentation of the theoretical construct employed for the analysis and applied to the phenomenon before changing to the next theoretical construct, see also Bikner-Ahsbahs et al. (2014) and Kidron et al. (2014). By comparing and contrasting the findings provided through the parallel analysis from the three perspectives, we discuss the potentials of merging digital tools and modelling. Thus, from the lenses offered by the media-milieu dialectics, modelling competency and mathematical digital competency we seek to narrow the gap of knowledge concerning potential roles played by digital tools in modelling processes.

## 2 First Case Study—Pirates of the Caribbean

Our first case study is a teaching activity developed by Danish upper secondary mathematics teachers as part of an in-service course on how to teach inquiry-based in mathematics as required by reformed curricula. The didactical notions taught were transposed elements of task design from the perspective of ATD (Jessen & Rasmussen, 2020). The course was 7 sessions where small groups of teachers from different schools, worked and developed activities together in ways adopting elements of Japanese lesson study to ensure the continuous shared development and evaluation of teaching practices (Jessen, 2019). The case study is an activity on the introduction of vectors in grade 11. The problem posed by the teachers were the following:

> $Q_0$: You are a captain in the Golden Days of piracy in the Caribbean and you are to guide your ship from Havana to Santo Domingo (see the attached map). Your crew covers 'landlubbers', 'treasure hunters' and sailors. They only answer to directions formulated as: 'go 20 miles south, then 30 miles southeast (SE) and then 100 miles north west (NW)'. A while ago you made the distance from Aruba to Montserrat in 3 days and you expect to travel by the same average speed this time.
>
> What orders would you give your crew and when do you expect to arrive? (Jessen et al. 2021)

The students had two hours for solving the problem in groups. They were asked to share their preliminary solutions twice during the two hours. The teachers had prepared a GeoGebra file, with an ancient map of the Caribbean Sea imported, where the mentioned cities were marked as fixed points. Furthermore, the teachers had created a function called 'vector from beginning point', assuring vectors to appear between the two points defining them. Finally, the students were told where in the textbook they could find information about vectors and handed out a picture of a compass rose. There is no obvious optimal sailing route from Havana at the north coast of Cuba to Santo Domingo in the Dominican Republic placed at the south coast of Hispaniola. Therefore, the problem invites the students to inquire and explain their choices.

From the planning group attending the professional development course, the usual teacher of the class taught the lesson. The rest of the planning group observed the teaching, took notes and video recorded the sharing sessions. All student groups had to present, which they did by connecting their computers to the projector showing their work in GeoGebra. The episodes below are based on the video recordings of the students' sharing sessions.

During the first plenum presentation, most groups have been working on question $Q_0$, determining the optimal sailing route by placing points and connecting those with line segments. This is related to $Q_0$, but different from it. Several groups were content with these initial routes, but did encounter challenges, when calculating the length of the route and how to translate the line segments into instructions for the crew. Moreover, the line segments were of different length, why to sum them was perceived cumbersome by the students. To figure out what instructions to give the crew, the students tried to measure angles between the line segments and the $x$-axis in the coordinate system. They did not succeed. Some groups decided to search for

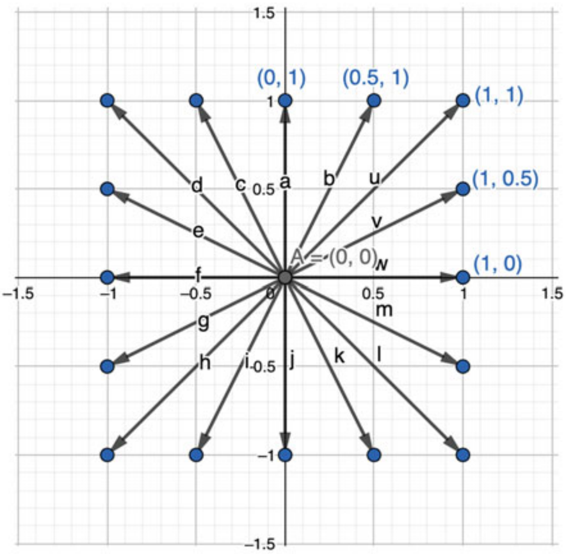



**Fig. 2** Students' initial compass rose in GeoGebra

the notion of vector by 'googling'. This gave them short videos showing how to work with vectors in GeoGebra. They learned the syntax for writing vectors in the input field. This gave them vectors starting from (0,0) in the coordinate system. Then they created the compass rose shown in Fig. 2. The group had created the optimal route by copying and dragging 'unit vectors' from this compass rose into a long sequence in the map in GeoGebra. The first student presenting this work remarked that it made it easier to calculate the length of the route. They argued how the length of each compass rose vector, $\overrightarrow{u_i}$, had to be added the number of times they appeared in the route. Then the route could be described as:

$$c_1 \cdot \overrightarrow{u_1} + c_2 \cdot \overrightarrow{u_2} + \cdots$$

where $c_i \in \mathbb{N}$ denotes the number of times vector $\overrightarrow{u_i}$ is used in the optimal route. After the group had presented this, the teacher asked a specific group:

> T: "Didn't you encounter the notion of 'unit vector' when reading parts of the suggested pages in the textbook? Did you notice what defines a unit vector?"

The students were quiet for a while, mumbling, then one student replied:

> $S_1$: "we are not sure what you are asking for, but at least their length is 1".

> T: "Yes, does this hold true for those unit vectors?" [points to the whiteboard showing the students' version of Fig. 2].

Students from several groups are mumbling and another student answers:

> $S_2$: "No, all the vectors need to end at the [circumference of the] unit circle, the angles are not all correct there" [pointing at the figure].

In the second plenum session, most groups had continued working on creating optimal routes using the notion of vector. Some groups referred to what they have read in the suggested pages of the textbook, but other groups had used internet searches. They found the Cartesian coordinate representation and the geometric representation of magnitude and direction and improved their compass roses in GeoGebra, e.g., defining the north east unit vector as:

$$\overrightarrow{u_{NE}} = \begin{pmatrix} \cos(45°) \\ \sin(45°) \end{pmatrix}$$

One group found that instead of connecting the same vector several times in a row, they could prolong it as they called it, by multiplying the unit vector with the number of vectors needed. Another group added that in fact they could multiply the vector with any real number. Thus, scalar multiplication was introduced.

Eventually, the students needed to find the length of the route. Some groups argued that since the unit vectors had length one, $\left|\overrightarrow{u_i}\right| = 1$, the length of the route could be described as:

$$c_1 \cdot \left|\overrightarrow{u_1}\right| + c_2 \cdot \left|\overrightarrow{u_2}\right| + \cdots = c_1 + c_2 + \cdots$$

Another group decided to create their entire route as one vector

$$\overrightarrow{u_{\text{total}}} = k_1 \cdot \overrightarrow{u_1} + k_2 \cdot \overrightarrow{u_2} + \cdots$$

Then they wrote the syntax in the input field computing the length of the entire route:

$$\left|\overrightarrow{u_{\text{total}}}\right|$$

During their presentation, it was shown within their shared GeoGebra file how the length of the actual sailing route was equal to the direct line between the two cities, though it went across land. Initially the group did not find this strange, but after discussing why some groups got significantly longer routes, the students discovered that

$$c_1 \cdot \left|\overrightarrow{u_1}\right| + c_2 \cdot \left|\overrightarrow{u_2}\right| + \cdots \neq \left|c_1 \cdot \overrightarrow{u_1} + c_2 \cdot \overrightarrow{u_2} + \cdots\right|$$

The lesson ended by the teacher asking the groups to write their answer to $Q_0$ in a document and hand it in together with their GeoGebra worksheets. The students left the class wondering how multiplication works for vectors.

## 3 The Second Case Study—Anaesthesia

The second case study is a project on how to build models for administering drugs during surgery. It is part of a larger group of projects that is developed for a two semester first year university course on mathematical modelling of dynamical systems of which every student worked with six projects in groups of 4–6 students. The course was designed to teach calculus through a modelling and problem oriented approach to develop (parts of) the students' modelling competency. The anaesthesia project is also discussed by Blomhøj and Jensen (2003), who analyse how the formulation of the modelling problem and the supervision through dialogues with each group support students' development of modelling competency with special focus on the sub-processes (c), (d) and (e) in Fig. 1. Here we focus on the role of MatLab as an essential digital tool in the modelling projects. Below we present just enough of the students' work with the anaesthesia modelling problem in order to analyse the potentials of the project and the role of MatLab. This will naturally have some similarities with the analysis by Blomhøj and Jensen (2003) regarding modelling competency. However, our analysis of the role played by the digital tool, MatLab, goes beyond the scope of their paper.

Each modelling project is driven by a problem guiding the students' work and taking point of departure in a data set. Each project comes with an introduction to the context of the problem (called 'Background') and a collection of 'hints' of which some can be seen as sub-problems guiding the students' independent work (Blomhøj et al., 2008c). Each group of students documented their work in a report. The reports were required to contain a description in prose of the system modelled, a diagram (if feasible) illustrating the dynamics, the delimitation of the system and considerations hereof. Data should be presented, described and reflected upon, a mathematical representation of the system (equations, graphs or similar) should be described, along with discussions of the parameters of the model including possibilities of estimating values for them. Numerical and analytical analyses, interpretations and discussions of model results with respect to the original problem, and finally, a concluding discussion of the model(s)' status and applicability (Blomhøj et al., 2008c, p. 3). The requirements for the report and the formulation of the problem of the project with hints were designed in accordance with the modelling cycle and the mathematical themes covered to ensure students' development of modelling competency and learning of mathematical knowledge within calculus.

The problem of the anaesthesia project was: How to dose anaesthetics during surgery? The problem was based on models that are parts of a training simulator, which has been used at Herlev Hospital in Denmark by medical staff in charge of anaesthetics (Fig. 3).

**17 Anæstesi**

Hvordan skal anæstesistoffer doseres under en operation?

Uddybning af problemstillingen: Hvordan kan man opnå kendskab til koncentrationen af anæstesistoffer i vævet (hvor stofferne virker) på baggrund af koncentrationen af stofferne i blodet (hvor de kan måles), og kan denne viden benyttes til at bestemme en doseringsstrategi, der holder koncentrationen af anæstesistoffet konstant i vævet under en operation?

**Baggrund**

En lang række anæstetika (bedøvelsesstoffer), der anvendes til næstesi (narkose), indsprøjtes direkte i blodbanen eller indåndes og føres derefter gennem lungerne ind i blodbanen. Anæstesistofferne transporteres derefter med blodet rundt i kroppen, hvor de dels opløses i fedtvævet, dels bindes til proteiner i de enkelte organer. Stoffet udskilles enten direkte gennem udåndingsluft, sved og urin, eller det nedbrydes til andre stoffer, der kan udskilles. Formålet med indsprøjtning af anæstesistoffer er primært at påvirke hjernen, og vores hovedinteresse er derfor at

| Tid [timer] | Koncentration [$\mu$-g/l] |
|---|---|
| 0.0 | 846 |
| 0.1 | 591 |
| 0.2 | 468 |
| 0.3 | 385 |
| 0.4 | 374 |
| 0.5 | 325 |
| 1.0 | 229 |
| 1.5 | 183 |
| 2.0 | 166 |
| 2.5 | 132 |
| 3.0 | 106 |
| 3.5 | 83 |
| 4.0 | 70 |
| 4.5 | 58 |

**Fig. 3** A copy of the first page of the anaesthesia project and the data. Every project description begins with a box containing a question, here "How to dose anaesthetics during surgery?". The data table shows the concentration of the drug pancuronium in the bloodstream at different times after the injection of 4 mg drug at time $t = 0$. (After Hull, 1979). (Blomhøj et al., 2008c, pp. 57–58)

The students worked with the project during three weeks along with regular classes of the course (2.5 h twice a week). Two sessions were reserved for students' group work where they worked together, discussed their project with other groups working on the same problem and used the teacher as a supervisor for discussions and help.

Guided by the first set of 'hints'/sub-problems, the students.

- constructed a curve from data of the logarithm of the concentration of drug in the bloodstream as a function of time (see Fig. 4),
- drew a two-compartment diagram from which they derived a system of differential equations for the concentration of drug in the bloodstream and the tissue respectively (see Fig. 5),
- derived expressions for the eigenvalues, $\alpha$ and $\beta$, in terms of the parameters $a_1, a_2, a_3$, and tried to convince themselves that the eigenvalues are none-equal, negative real numbers.

The students used their existing knowledge, their modelling competency developed through their previous projects in the course (the anaesthesia project is in the group of the 5th projects), MatLab and their textbook to produce the plot in Fig. 4 (and a similar plot with linear approximations for the time intervals [0, 0.5] and [0.5, 4.5], respectively) and an expression for the eigenvalues in terms of the parameters in the model. Dealing with algebraic manipulations, estimating upper bounds for expressions and rewriting them as expressions they know for certain are positive or negative, are not standard procedures for them.

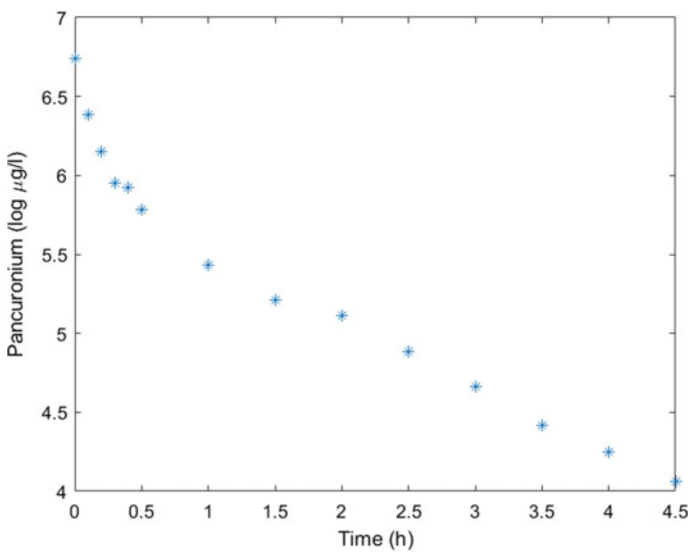
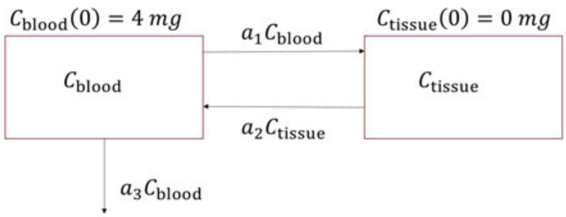



**Fig. 4** Plot of data from the table in Fig. 3. Time in hours from the injection is on the first axis and the logarithm of the concentration in µ-gr/l is on the second axis




$$C'_{blood}(t) = -(a_1 + a_3)C_{blood}(t) + a_2 C_{tissue}(t); \quad C_{blood}(0) = 4\,mg$$

$$C'_{tissue}(t) = a_1 C_{blood}(t) - a_2 C_{tissue}(t); \quad C_{tissue}(0) = 0\,mg$$

**Fig. 5** Compartment diagram and system of differential equations of the dynamics of the flow of drug between the bloodstream and the tissue

In the second set of 'hints', the students were first asked to solve the system of differential equations analytically. Many students turned to MatLab, as they had done in their previous projects, to use MatLab to solve the system and draw graphs of the solutions. These students did not initially reflect upon the fact that their MatLab procedure requires numerical values for the parameters, which they did not have at this point, and hence the syntax failed. Thus, MatLab 'forced' the students to return to the analytic work for deriving solutions with symbols for the eigenvalues and the parameters. Secondly, the students were asked to estimate the eigenvalues and the parameter $a_2$ such that the logarithm to the function describing the concentration of pancuronium in the bloodstream fits the given data. This required the students to combine their knowledge of the analytic solution as a sum of two exponential functions, the sign of the eigenvalues and their knowledge of straight lines in semi-log plots to judge and take actions on how to estimate the eigenvalues. Their existing knowledge included reading semi-log plots and estimate parameters for exponential functions through linear regression performed in MatLab, which was elaborated by their work with the semi-log plot (Fig. 4), which does not resemble a straight line.

For this, they received two further hints: to consider what the curve looks like for later time values, and show the following relations between the eigenvalues and the parameters:

$$a_2 a_3 = \alpha\beta \text{ and } a_1 + a_2 + a_3 = -(\alpha + \beta)$$

Focusing on the last part of the plot in Fig. 4, several groups usually turned to MatLab to perform linear regression and received an exponential function as the solution. However, this does not fit the events around the time when the drug initially was injected. Gradually, during dialogue with the teacher and fellow students (see also Blomhøj & Jensen, 2003), these groups realised, that the initial curve looks similar, and through linear regression they gained another exponential function. This led the groups to raise several questions: How can it be that the analytic solution seems to be the sum of two exponential functions for the entire period? How can the solution be one exponential function in one time interval and another exponential function in another interval? Can one of those be equal to zero? Some groups graphed both functions in the same MatLab window, discovering when one and the other is dominant. The students used the estimates for the eigenvalues together with the relations between the eigenvalues and the parameters to derive estimates for $a_1$ and $a_3$. Lacking only one parameter ($a_2$), the students could now use their MatLab syntax to solve the systems of differential equations by altering the value of $a_{2.}$. For each value of $a_{2.}$, they plotted the corresponding solution, and compared with the plot of data until they reached a sufficiently good fit with data. This qualitative approach of finding the best fit made the students uneasy, not trusting their own judgement for estimating the parameter.

During the construction of the anaesthesia model, the students realised the necessity of algebraic skills, mathematical representation and reasoning competencies in order to be able to use mathematical modelling. With estimates for all parameters, they used MatLab to plot the functions representing the concentrations of drug in the bloodstream and in the tissue (Fig. 6). Thus, they had developed a 'mathematical laboratory' for monitoring the amount of drug. They used this laboratory to, finally, answering the original question of how to dose pancuronium during surgery. They extended their compartment model by implementing a drop, adjusting their mathematical model accordingly and used MatLab to experiment with various plans for continuous intravenous dosing to ensure that the amount of drug in the tissue stays between the thresholds of risking the patient wakes up and the critical limit where the drug becomes dangerous. They used MatLab to validate their plan for administering a steady state of drug in the tissue for a two hours surgery, as shown in Fig. 6.

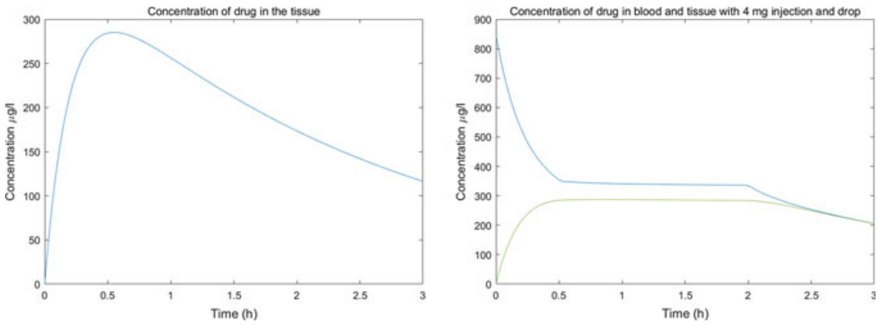



**Fig. 6** Solutions to the system of differential equations solved by MatLab. The left side shows the plot of the concentration of drug in the bloodstream with an injection of 4 mg drug at time $t = 0$. The right side shows the plots of the concentration of drug in the bloodstream (blue) and in the tissue (green) for the extended model with a drop implemented in the compartment model, where a steady state is reached and maintained for 2 hours

## 4 Media-Milieu Dialectic Through the Herbartian Schema

In our analysis we are drawing on the notion of media-milieu dialectic from ATD (Chevallard, 2007), since this allows us to analyse conditions and constraints for strategies pursued by students (Kidron et al., 2014). In our cases, this means how the design and the digital resources further or delimit students' work with the modelling problems at stake. When discussing the interplay between media and milieu, we employ the Herbartian schema to depict the dynamics between the two and when one resource changes role from media to milieu and vice versa. The Herbartian schema is different from cognitive schemes, though the students' engagement with the dialectic between media and milieu might lead to development of their cognitive schemes. The Herbartian schema describes the dynamics of a didactic system, $S$, which consists of a group of learners $X$, studying the question $Q$ under guidance of $Y$ (Chevallard, 2007). $X$ can be a whole class of students, a smaller group of students or a single student, $s$. Similarly, $Y$ can be a group of supervisors, a single teacher, $y$, or $Y = \emptyset$, in case of self-study processes. The system consisting of $X$, $Y$ and $Q$ brings into being (denoted by ➥) a 'personal answer', $A^{\heartsuit}$, to $Q$. We depict the didactic system as:

$$S(X; Y; Q) \text{➥} A^{\heartsuit}$$

$A^{\heartsuit}$ is personal since it is not to be found in textbooks, webpages etc., but is the result of a joined process of $X$ and $Y$ interacting with the milieu, $M$, which Chevallard (2008) defines as "a fuzzy and changing set of didactic 'tools' of different kinds that $X$, acting under the supervision of $Y$, has to bring together (➡)" (Chevallard, 2008, p. 2). The process is depicted in the developed Herbartian schema:

$$S(X; Y; Q \text{➡} M) \text{➥} A^{\heartsuit}$$

The milieu,$M$, consists of existing answers, $A_i^{\diamond}$, in terms of students' previously developed knowledge (the rhombus indicates that these are equal or similar to official or shared knowledge within the specific domain), works,$W_j$, drawn upon, which can be textbooks, webpages and all kinds of resources produced to disseminate knowledge, but also videos and newspaper articles etc. and data, $D_k$, which can be generated through experiments, be collected from databases or provided as part of the problem. The data can be both quantitative and qualitative (Chevallard, 2019):

$$M = \left\{A_1^{\diamond}, A_2^{\diamond}, \ldots, A_l^{\diamond}, W_1, W_2, \ldots, W_m, D_1, D_2 \ldots, D_n\right\}$$

All elements of the milieu are considered media which "designate here any representation system of a part of the natural or social world addressed to a certain audience" (Chevallard, 2007, p. 1, our translation), brought together as being relevant for the study of $Q$. Hence the didactic milieu of the Herbartian schema covers media supporting and nurturing the study of $Q$ and "can include an a-didactic milieu in the sense given by TDS, that is, a system of objects acting as a fragment of 'nature' for Q, able to produce objective feedback about its possible answers" (Kidron et al, 2014, p. 158, where TDS is short for Theory of Didactic Situations (Brousseau, 1997)). Thus, in the modelling context the validation takes place against the perceived reality, why media representing this must be part of the milieu. The dialectic of the media and the milieu can be described as drawing on knowledge from media using it to search for new knowledge to be tested in the milieu and converted to media. Media and milieu refer to the functioning of an object in a learning process. That is, digital tools play the role of media, when used for more pragmatic purposes as providing answers to be studied or to be used when developing $A^{\heartsuit}$. When digital tools are used for exploring notions or validating answers, we consider them as milieu for the study of $Q$. Similarly, the teachers or student peers can both provide answers to be studied (being media) or they can validate answers provided by other students (representing the milieu).

### 4.1 The Dialectics and Piracy

The didactical system describing plenum sessions from the first case study consists of

$$S(X_1; X_2; X_3; X_4; X_5; y; Q_0) \rightarrow A_1^{\heartsuit}$$

where $X_i$ represents each of the student groups, guided by the single teacher, $y$, studying the question of guiding a ship around the Caribbean Sea, $Q_0$. During the first sharing session the class derives a preliminary answer denoted $A_1^{\heartsuit}$. The idea was to construct the route as line segments or by (some unit) vectors added together. The initial work of each group can be described by the system:

$$S(X_i; y; Q_0) \rightarrow A_{X_{i,1}}^{\heartsuit}$$

The answers shared during plenum sessions vary but guide the study towards the development of $A_1^{\heartsuit}$, reflecting different group answers. The milieu brought together by the groups and the teacher can be described as:

$$M = \left\{ A_1^{\diamond}, A_2^{\diamond}, A_3^{\diamond}, A_4^{\diamond}, A_5^{\diamond}, \ldots W_1, W_2, W_3, \ldots D_1, D_2, Q_0, Q_0{'} \right\}$$

The data is the map in GeoGebra, $D_1$, and the time spent on the previous trip, $D_2$. The students drew on existing answers such as how to construct fixed and movable points, line segments, measuring length and angles in GeoGebra, which is represented by $A_1^{\diamond}$, $A_2^{\diamond}$, $A_3^{\diamond}$, $A_4^{\diamond}$, $A_5^{\diamond}$... The teacher brought into the milieu the GeoGebra function 'vectors from beginning point', specific pages of the textbook and the compass rose, $W_1$, $W_2$, $W_3$. Initially the students chose to answer the question $Q_0{}'$: What is the optimal route? Their answers were validated against the milieu containing the original question, $Q_0$, requiring the instructions to be understandable for sailors. Here their initial answer failed, which created the need for the students to study the notion of vector and exploring the milieu leading to searches for further media. We see from the video recording that they refer to resources found as web searches. We denote these as '…' in the Herbartian schema, not knowing them exactly. Thus, we can conclude that the nature of the question and students' existing knowledge about GeoGebra and geometry made it possible for the students to engage with the problem, formulate initial hypotheses of an answer and created the need to study further media, driving the modelling process.

Some students experimented with the notion of vectors, some by constructing the compass rose in GeoGebra to build the route. GeoGebra became the milieu where students have decomposed the knowledge studied from media, reconstructing this as answers for $Q_0$. Thus, students gained the ability to draw vectors digitally, apply them for preliminary solutions for the modelling problem at stake. Further, we observed how the students became media for each other. Some groups got inspired by the compass rose in GeoGebra as to secure instructions comprehensible for sailors. The idea presented in Fig. 2 is an answer, $A_{X_{i,1}}{}^{\blacktriangledown}$, for the presenting group, but becomes a work, $W^{\diamond}_{X_{i,n}}$, to be studied by other groups. During the following dialogue initiated by the teacher regarding unit vectors, the answering group validated the compass rose against their newly gained knowledge on unit vectors and existing knowledge on the unit circle, correcting their initial perception of the notion. All became works to be studied by other groups. Hence, the plenum session captured episodes where the digital tool both acts as media and as milieu. Such processes were also observed regarding the students' presentations, which means that the students continuously go forth and back between decomposing and reconstructing answers or engage in the dialectic between media and milieu.

Similar dynamics are found in the second session, where several groups used the idea of unit vectors, which led to discussions of sums of vectors leading to multiplication with scalar, calculation of length of vectors and the sum of those. The constructions of elaborated answers are based on further media studied (in the textbook, web searches etc.) employing new notions and syntax in the milieu offered by the map in GeoGebra. GeoGebra plays the role as milieu when demonstrating the 'paradox' that: $\overrightarrow{u_{\text{total}}} = k_1 \cdot \overrightarrow{u_1} + k_2 \cdot \overrightarrow{u_2} + \cdots$ does not entail $\left|\overrightarrow{u_{\text{total}}}\right| = k_1 \cdot \left|\overrightarrow{u_1}\right| + k_2 \cdot \left|\overrightarrow{u_2}\right| + \cdots$. This validation led the students to study the media on properties of the sum and length of vectors and how to calculate those. Thus, media invited for explorations within the milieu, validating the answer, which led to further study of media. Thus, what is seen from this analysis is the fuzzy nature of mathematical modelling when modelling the optimal sailing route and how the digital tools delimit their first answers, when students believe it to be sufficient to use their existing skills within GeoGebra and later, the option of adding vectors providing them with a different measure, than what they asked for. However, the GeoGebra combined with the specific phrasing of $Q_0$, the works proposed by the teacher, students' autonomous web searches drove the process of testing, adjusting and evaluating continuously improved models.

### 4.2 Dialectics and Anaesthetics

In our second case, we have, during the two project sessions, didactical systems consisting of a group of students working under the guidance of the teacher—and occasionally by other groups, but a significant amount of self-study occur. We depict this as:

$$S(X_i; y; Q) \Rightarrow A_{X_{i,1}}{}^{\heartsuit} \text{ or } S(X_i; \emptyset; Q) \Rightarrow A_{X_{i,1}}{}^{\heartsuit}$$

The milieu consists of:

$$M = \left\{A_1^{\diamond}, A_2^{\diamond}, A_3^{\diamond}, \ldots, W_1, W_2, W_3 \ldots, \ldots D_1, Q_0, Q_1, Q_2, \ldots\right\}$$

Here we denote the problem formulation with sub-problems as $Q_0$, $Q_1$, $Q_2$, .... The problem came with a data set of the decay of the concentration of drug in the bloodstream as a function of time, $D_1$. The problem was formulated in relation to what had been taught previously, why students' existing knowledge in terms of existing answers $A_1^{\diamond}, A_2^{\diamond}, A_3^{\diamond}, \ldots$ are considered the natural resource for answering the problem, $Q_0$. If not all answers are well enough established it is suggested that students restudy the course material. There is a reference to a paper by Hull (1979) retrieved by some groups where other groups used general web searches, represented as $W_1$, $W_2$, $W_3$... It varies between groups how much the milieu is developed when engaged with the project.

The first set of sub-problems, $Q_1 - Q_3$, draws on previous answers developed by the students $A_1^{\diamond}, A_2^{\diamond}, A_3^{\diamond}, \ldots$ concerning the plot of data, the construction of the compartment diagram and the system of differential equations, leading to the derivation of expressions of the eigenvalues in terms of the parameters in the model. During their work with the characterisation of the eigenvalues, students' previously gained knowledge and the textbook acted as milieu, where some of them consulted their upper secondary textbooks as media. The gained knowledge constituting the first answers became media to be restudied for the second set of sub-problems. Thus, we see an emphasis put on revisiting the content knowledge when building elements of the model through answering the sub-questions and less experimentation and evaluation against the perceived realty, which is mainly mathematical.

In the second set of sub-problems, some groups wanted to use MatLab for solving the systems of differential equations (i.e., as media), though not knowing how to type the equations, which made them realise the need for further analysis. Thus, MatLab prompted the students for reconstruction of their mathematical knowledge before allowing MatLab to provide the answers. In this episode, MatLab functioned both as media and as milieu. When fitting the curves to estimate the value of $a_2$, the students used MatLab for explorations, why MatLab functioned as the milieu for exploration and validation. In the final part of the project where the students answered a concrete version of the original problem of how to administer the drug during an operation, MatLab again functioned as milieu—both when the students experimented with how to implement a drop in the model, and when they eventually used MatLab for validating their dosing plan. Thus, by explicitly identifying when a tool as MatLab function as media or milieu in the students' process of study, we can see how the students' knowledge about needed and relevant input for MatLab and potential answers initially delimits the modelling process. However, most students demonstrate the maturity to revisit their mathematical work to match relevant input for MatLab. Knowledge about these dynamics is relevant to know to design the scaffolding of the modelling activity providing the students with sufficient openness to demonstrate autonomy and still being able to engage with the problem, learning both mathematics and developing modelling competency.

## 5 Mathematical Digital Competency and Modelling

Mathematical digital competency is a recent construct that builds on the competence framework by Niss and Højgaard (2019). The competence framework covers in total eight competencies, where modelling is one of them. Another is the aids and tools competency covering the ability to know of and how to use various aids and tools when engaged in mathematical activity and critically reflect upon when to use those aids and tools (Niss & Højgaard, 2019, p. 18). This also links to other descriptions of digital competence (Ferrari, 2012). However, Geraniou and Jankvist (2019) claim that a definition must take into account how the processes of using digital tools transform those tools into mathematical instruments affecting students' cognitive schemes referring to Vergnaud (2009) and support students' learning of mathematical concepts as it has been argued by Artigue (2002) and Guin and Trouche (1999).

When digital tools become instruments rather than artefacts, situations emerge where "a tool can shape and affect a student's thinking and actions" (Geraniou & Jankvist, 2019, p. 36), though this is often the result of a long process where the tool initially have had a pragmatic value as e.g., advanced calculator, but gradually becomes an environment for explorations and construction of new knowledge gaining an epistemic value. The latter is to be strived for, as Artigue (2010) have argued that the pragmatic purpose of digital tools can have little or negative impact on students' learning outcomes. Based on relations between theoretical constructs regarding digital tools and learning processes within mathematics, Geraniou and Jankvist propose the following definition of mathematical digital competency as:

> Being able to engage in a techno-mathematical discourse. […] Being aware of which digital tools to apply within different mathematical situations and contexts, and being aware of the different tools' capabilities and limitations […] Being able to use digital technology

reflectively in problem solving and when learning mathematics. (Geraniou & Jankvist, 2019, p. 43)

This definition of mathematical digital competency captures more nuances of what it means insightfully to employ digital tools when engaged in mathematical activities including modelling processes and prompts the idea of mathematical digital competency shaping mathematical thinking. This indicates that digital technologies shape the modelling processes more than what is captured by the work of Siller and Greefrath (2010), though still depending on the nature of the modelling problem posed.

## *5.1 Competencies and Piracy*

Revisiting the first case study from the perspective of modelling, the students are eventually able to model the sailing route using vectors to provide sound instructions for the crew. The route is constructed based on the 'perceived reality', represented by the data $D_1$ and $D_2$. Though the situation is not real, it nurtures students' willingness to systematise and mathematise the problem. The domain of inquiry is delimited by $D_1$, the sketching environment and the requirement of using vectors. The students' experience with GeoGebra and the geometric notions $A_1 - A_5$, made the students reformulate the task handed out to $Q'$, ignoring theory on vectors. The mathematisation using the $A_1 - A_5$, proved insufficient when validating the route against the perceived reality of manoeuvring ships centuries ago. This prompted the students to study theory of vectors, $W_1, W_2, \ldots$, re-mathematising the problem. The insights gained from evaluating the second modelling of the route, using the compass rose of Fig. 2, led the students to revisit the theory in terms of unit vector and unit circle. After a new mathematical analysis, the groups modelled routes answering the problem. From this we see that students demonstrated capacity and willingness to engage in all aspects of mathematical modelling furthering their modelling competency.

From the perspective of mathematical digital competency, we consider both GeoGebra and the web searches as tools employed by the students. The tools function both as media and works to be studied, but also as the milieu against which knowledge is decomposed and reconstructed as answers in terms of suggested routes. Throughout the two lessons, students mainly worked in the digital environment using the discourse of the program, though linked to the discourse of geometry and in particular vectors. They discussed advantages of Cartesian definition, but also magnitude and direction linking this to the unit circle. This demonstrates their familiarity with the program and their ability to engage in techno-mathematical discourse, which is further developed through the notion of vectors. The tool certainly shaped their initial approach to the problem, but they also demonstrated reflective use of the tool and ability to search for further knowledge through web searches, improving their capabilities within GeoGebra. The experimentation carried out in the program indicated limitations with respect to their understanding of the syntax of finding length of vectors and a limited understanding of the notion itself. They partly resolved this by sharing their work with the class realising their mistake. This demonstrates how the milieu, in terms of other groups and the digital tool, shaped the thinking of the students, which we consider an illustration of students being "able to use digital technology reflectively in problem-solving and when learning mathematics" (Geraniou & Jankvist, 2019, p. 43). In particular, the analysis through media-milieu dialectic emphasises how this emerge.

### *5.2 Competencies and Anaesthesia*

In the second case, we have a modelling problem, which design is based on the modelling cycle (Blomhøj & Jensen, 2007). The students were guided by the 'hints' or sub-problems to overcome the challenges of the mathematisation, though freely employing the mathematics required to characterise the eigenvalues and deduce $a_1$ and $a_3$ in terms of $a_2$ after determining the eigenvalues, interpreting back and forth between graphical and algebraic representations of the model as described above. When estimating $a_2$, the students fitted their model against the data from reality provided by the teacher. In the last part of the project, the students experimented freely with the model, extending it by adding a drop or implementing a time schedule for giving a new dosis. Thus, we can argue that they engaged in all phases of the modelling cycle, except (a) 'the formulation of task', and (maybe) (b) 'systematisation', as argued by Blomhøj and Jensen (2003), since these are provided in the description of 'background' in the problem formulation of the project, see Fig. 3, furthering their mathematical knowledge and modelling competency.

With respect to the mathematical digital competency, not all students were able to engage in a techno- mathematical discourse at the outset of the project work, and initially some of them were not able to use MatLab reflexively. They seemed to perceive MatLab as this black box providing them with answers. However, by revisiting the sub-problems, students realised the need for reformulating the problem, solving parts of it with pen, paper and algebraic reasoning in order to develop the problem into a problem that fits MatLab syntax, which allowed them to get the answers needed for the development of their model of the dosing problem. From student reports and the media-milieu dialectics analysis, we see that MatLab moved from having a pragmatic value to an epistemic value in this specific context. The modelling activity and sub-problem supported the dialectic between media and milieu, in which MatLab played a significant role when solving the system of differential equations. The 'laboratory' the students created in MatLab for the dosing problem created a mathematical 'microscope' to 'look' into the tissue (of the brain) without having direct access. This illustrated the power of mathematical modelling for the students in a very direct manner. The students did neither discover nor explore new mathematical notions, as in the vector project. However, the students gained security concerning MatLab as an essential tool for modelling processes and the project developed their mathematical digital competency as awareness of when to use the tool, what is needed before the tool can be used and for what. The case of anaesthetics both developed elements of modelling competency and mathematical digital competency.

## 6 Discussion and Concluding Remarks

Returning to our research question concerning how the merger of mathematical modelling and digital resources can support students' development of mathematical knowledge and their modelling and digital competencies, including the discussion of potentials and pitfalls for students' learning of mathematics in this context, we see potentials in both cases. Both problems are designed such that digital tools are required in the solution process. In the vector project, GeoGebra is chosen explicitly as the milieu capturing the perceived reality. When students engage with the problem, GeoGebra also becomes media, providing answers to be studied, which means further evaluation, systematisation and mathematisation. All elements allowing modelling to be a driver for learning mathematical knowledge. In the anaesthetics project, MatLab is required implicitly, since the design of the problem and the reality modelled lead to expressions, systems of differential equations and numerical analyses impossible to complete by pen and paper. MatLab kept changing role between functioning as milieu or media. Thus, we might argue that these modelling problems, in contrast to the study presented by Greefrath and colleagues (2018), take the digital tool into consideration and allow the presence of the digital tools to frame the problems. The digital tool becomes more than an advanced calculator to which certain mathematical work can be outsourced but shapes the thinking and the modelling process of the students. In this sense, the digital tools further the students' learning by nurturing their experimentation through constructions of preliminary models, which are validated against reality using the tools. Thus, pedagogical aims of inquiry combined with a digitalisation of education are gained in those projects.

However, in both cases we also see elements of students demonstrating a naïve trust in the digital tools as the strategy for solving advanced problems. At university level, we see this when students go directly to MatLab before considering if and how to state the problem in terms of what can be handled by MatLab. In the case of piracy, students trust any answer provided by GeoGebra as correct, despite the graphing showing the absurdity of the answer, when calculating the length of the route. Here the mistake was corrected by validation during plenum session, which was an explicit part of the design of the vector project. In the anaesthetia project, the sub-problems were designed to guide the students realising the need for analytic 'pen and paper' work in combination with MatLab, and dialogue with fellow students and the teacher was an essential part for realising the learning goals (for this last part, see also Blomhøj & Jensen, 2003). Those episodes indicate that the productive merger between modelling and digital tools requires 'scaffolding' in the design for how to overcome unproductive use of the digital tools. When this is done deliberately, we can argue that the merger of modelling and digital tools potentially can further students' mathematical modelling and digital competencies and nurture the learning of mathematical knowledge.

**Acknowledgements** Lundbeck Foundation Project R284-2017-2997.

## References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274. https://doi.org/10.1023/A:1022103903080

Artigue, M. (2010). The future of teaching and learning mathematics with digital technologies. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics education and technology-rethinking the terrain. The 17th ICMI study* (pp. 463–475). Springer. https://doi.org/10.1007/978-1-4419-0146-0_23

Barquero, B., & Jessen, B. E. (2020). Impact of theoretical perspectives on the design of mathematical modelling tasks. *Avances de Investigacion en Educacion Matematica, 17*, 98–113. https://doi.org/10.35763/AIEM.V0I17.317

Bikner-Ahsbahs, A., Prediger, S., Artigue, M., Arzarello, F., Bosch, M., Dreyfus, T., … & Schäfer, I. (2014). Starting points for dealing with the diversity of theories. In A. Bikner-AhsBahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 3–12). Springer. https://doi.org/10.1007/978-3-319-05389-9_1

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching Mathematics and Its Applications, 22*(3), 123–139. https://doi.org/10.1093/teamat/22.3.123

Blomhøj, M., & Jensen, T. H. (2007). What's all the fuss about competencies? In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 45–56). Springer. https://doi.org/10.1007/978-0-387-29822-1_3

Blomhøj, M., & Kjeldsen, T. H. (2006). Teaching mathematical modelling through project work—Experiences from an in-service course for upper secondary teachers. *ZDM Mathematics Education, 38*(2), 163–177. https://doi.org/10.1007/BF02655887

Blomhøj, M., Kjeldsen, T. H., & Ottesen, J. (2008). *Problemstillinger til miniprojekter i matematisk modellering*. August 2008. Nat.Bas, RUC, Roskilde University.

Brousseau, G. (1997). Theory of didactical situations in mathematics. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, (Eds.), & Trans.). Kluwer.

Chevallard, Y. (2007). Un concept en émergence: la dialectique des médias et des milieux. In *Actes du séminaire national de didactique des mathématiques* (pp. 344–366).

Chevallard, Y. (2008). Afterthoughts on a seeming didactical paradox. In *International conference "Efficacité & Équité en Éducation"*, Rennes, France. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Yves_Chevallard_-_On_Inquiry-Based_Instruction.pdf

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114.

Ferrari, A. (2012). *Digital competence in practice: An analysis of frameworks.* A technical report by the Joint Research Centre of the European Commission. European Union. https://doi.org/10.2791/82116

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Greefrath, G., Hertleif, C., & Siller, H. S. (2018). Mathematical modelling with digital tools—A quantitative study on mathematising with dynamic geometry software. *ZDM Mathematics Education, 50*, 233–244. https://doi.org/10.1007/s11858-018-0924-6

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Hull, C. J. (1979). Pharmakokinetics and pharmacodynamics. *British Journal of Anaesthesia, 51*, 579–594. https://doi.org/10.1093/bja/51.7.579

Jessen, B. E. (2019). Teachers learning to design and implement mathematical modelling activities through collaboration. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education* (pp. 1182–1189). Freudenthal Group & Freudenthal Institute, Utrecht University.

Jessen, B. E., & Rasmussen, K. (2020). What knowledge do in-service teachers need to create SRP's? *Educaçao Matemática Pesquissa*, *22*(4), 680–693. https://doi.org/10.23925/1983-3156.2020v22i4p680-693

Jessen, B. E., Borby, J., & Kroman, J. (2021). *Definitioner og regneregler for vektorer*. Ressource for profesional development: Nationalt Center for Udviklingen af Matematikundervisningen. Retrieved from: https://matematikdidaktik.dk/temaer/undersoegelsesbaseret-matematikundervisning-i-gymnasiet/definitioner-og-regneregler-for-vektorer/

Kidron, I., Artigue, M., Bosch, M., Dreyfus, T., & Haspekian, M. (2014). Context, milieu, and media-milieus dialectic: A case study on networking of AiC, TDS, and ATD. In A. Bikner-AhsBahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 153–177). Springer. https://doi.org/10.1007/978-3-319-05389-9_10

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies of Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling*. Routledge.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Siller, H.-St., & Greefrath, G. (2010). Mathematical modelling in class regarding to technology. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the sixth congress of the European Society for Research in Mathematics Education* (pp. 2136–2145). Institut National de Recherche Pédagogique.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

# Lower Secondary Students' Reasoning Competency in a Digital Environment: The Case of Instrumented Justification

**Rikke Maagaard Gregersen and Anna Baccaglini-Frank**

## 1 How Does the Use of Digital Technology Influence Students' Mathematical Reasoning Competency?

The work presented in this chapter is part of a broader research problem stemming from the following Danish education context that, however, is arguably an important matter in other countries. In Denmark, the Mathematical Competencies framework (KOM) (Niss & Højgaard, 2019) highly influences the curricular goals (UVM, 2019). KOM defines a mathematical competency as "…someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 6). In all, there are eight distinct yet interrelated competencies; here, we will be focusing in particular on the *reasoning competency*.

Although KOM, which was developed at the start of the century, acknowledges digital technology in mathematical practices, it does not account for the prevalence and the role that digital technologies now play in mathematics programs at all educational levels. In Demark, GeoGebra is the primary dynamic geometry environment (DGE) used early on for mathematics teaching (Højsted, 2020b). Indeed, DGEs are considered to support students' mathematical reasoning competency (e.g., Højsted, 2020a). For example, they can support students in connecting mathematical theory with empirical explorations or identifying geometrical invariants as key properties of geometrical figures and relationships (e.g., Højsted, 2020c; Leung et al., 2013; Sinclair & Robutti, 2013).

R. M. Gregersen (✉)
Danish School of Education, Aarhus University, Aarhus, Denmark
e-mail: rmg@edu.au.dk

A. Baccaglini-Frank
Department of Mathematics, University of Pisa, Pisa, Italy
e-mail: anna.baccaglinifrank@unipi.it



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_7

Our work stems from the need to deepen digital technology aspects of KOM's competencies descriptions, as Geraniou and Jankvist (2019, 2020) advocated at a practical and theoretical level. Specifically, we intend to contribute to partially bridging this research gap by offering a theoretical tool to analyze how a digital interactive environment like GeoGebra can contribute to lower secondary school students' reasoning competency in a mathematical domain at the crossroads between algebra and geometry.

This contribution is also quite relevant from an internationally broader perspective. Indeed, in addition to being a DGE, GeoGebra features an "algebra view" with the symbolic representations of items that appear in the graphic view. This is a feature shared by computer algebra systems (CAS) in general that has been studied especially in the context of functions and related concepts in calculus (e.g., Artigue, 2002; Drijvers et al., 2013; Lagrange, 2010, 2014; Takači et al., 2015). However, the potential of dynamic geometry and algebra environments is yet to be fully unveiled (Hohenwarter & Jones, 2007), especially at the lower secondary school level.

In the following paragraphs of this section, we will clarify what is intended in the KOM framework by *reasoning competency* and how we intend to approach it. Then we will provide an overview of our conceptual framework, explaining how we adopted each construct, connecting it with others, to reach the theoretical tool that we designed by putting it in relation to Toulmin's argumentation model (from now on Toulmin's model) and the Theory of Instrumental Genesis. We will then use the tool designed to study students' argumentation processes in an interactive digital environment; specifically, we analyze excerpts from two students' efforts at solving a task in GeoGebra in which the objects in play are described algebraically and graphically. Finally, we will discuss our findings, leading to the notion of *instrumented justification* to frame the process captured by the analytic tool.

### *1.1 Reasoning Competency in the KOM Framework*

The reasoning competency includes the ability:

- to produce oral or written arguments (i.e., chains of statements linked by inferences) and to justify mathematical claims;
- to critically analyze and assess existing or proposed claims and justification attempts.

So the competency explicitly considers *justification*, hinting at various forms of justification, ranging from reviewing or providing examples to rigorous proof (Niss & Højgaard, 2011, 2019). Niss and Højgaard (2019) also note that reasoning goes beyond justifying theorems and formulae, extending to the justification of any mathematical conclusion obtained through mathematical methods or inference.

This way of describing reasoning competency—especially the first ability presented—resonates highly with research on *argumentation*. Indeed, we situate our work within this discourse, and we use "argumentation" to refer to all processes aimed at producing and validating mathematical claims.

Many studies have shown that students can struggle with both identifying the relevant properties and structuring a mathematical argument (e.g., Duval, 2007). Moreover, when engaging in argumentation, students might rely on authorities such as standard formulas, teacher's statements, or technology instead of their mathematical knowledge (Harel & Sowder, 2007; Lithner, 2008). Argumentation commonly aims to change the epistemic value of a mathematical claim (Duval, 2007). Consistently with Jeannotte and Kieran (2017), we consider justification a specific type of argumentation process "… that, by searching for data, warrant, and backing, allows for modifying the epistemic value of a narrative." (Jeannotte & Kieran, 2017, p. 12).

### *1.2 Designing an Analytical Tool from a Complex Theoretical Panorama*

Our work is situated in a rather complex conceptual framework that we need to clarify, explaining mutual relationships between the theoretical approaches and the theoretical constructs we use. As discussed above, the broad framework within which we situate this work in the KOM is a quite general framework organizing the main competencies needed to become a proficient mathematician. However, it lacks detail for students' uses of digital technology. The compatibility of KOM with a theory designed specifically to analyze students' use of digital technology has already been explored by Geraniou and Jankvist (2019). Using the same theories, we take a step into further analytic detail to gain insight into students' reasoning processes, specifically justification seen as a particular process of argumentation supported by digital technology. To do this, we use Toulmin's model, designed to capture the structure of argumentations and adapt it to the context of a digital interactive environments using the *scheme-technique* duality from the Theory of Instrumental Genesis (TIG). We do this with the intention to understand the empirical phenomenon of students' justification processes in a digital environment.

The analytical tool we introduce here is re-elaborated from the one presented in Gregersen and Baccaglini-Frank (2020). The TIG describes how an artifact such as GeoGebra can become an instrument for an individual who engages in solving a task (Rabardel & Bourmaud, 2003). Moreover, Drijvers et al. (2013) have elaborated within the TIG three dualistic processes; we consider the *scheme-technique* duality. Techniques are considered the "visible part" of doing that relies on the "invisible part", the solver's schemes, that direct and organize techniques. Moreover, schemes contain concepts and rules which regulate actions. This duality assumes that part of the scheme can be inferred from observing actions.

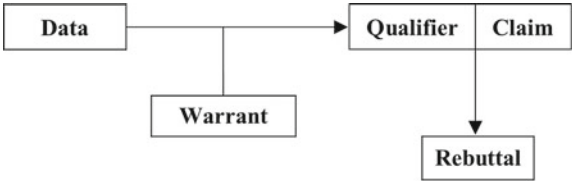



**Fig. 1** Basic elements in Toulmin's argumentation model (Toulmin, 2003)

Traditionally, the TIG has been applied to gain insight into students' learning processes solving specific mathematical tasks using a digital environment, for example, finding the solutions of an equation with CAS (e.g., Artigue, 2002; Jupri et al., 2016). In our case, students will be using GeoGebra to solve mathematical tasks, but they might also be arguing in favor of or against certain claims arising in their solution process, and we are interested in capturing this. The scheme-technique duality alone is not sufficient, as we want to gain insight into students' justification processes, as particular argumentation processes, so key structural aspects should not get lost.

We, therefore, use Toulmin's model (Toulmin, 2003) to keep track of these processes. In Toulmin's model, the *claim* is a statement of the speaker, uttered with a certain indication of likelihood called the *qualifier*. This is supported by *data* that is facts and *warrants* that are inference rules which connect the data to the claim. Finally, the *rebuttal* denotes conditions for or limits of the claim. Figure 1 depicts such a model, as introduced by Toulmin (2003).

Commonly for Toulmin's model, the unit of analysis is a finalized argument restricted to a single sentence. However, a key aspect of the justification processes we aim at capturing is the change of the qualifier of a claim, possibly leading to the rejection or restatement of the original claim. Therefore, our units of analysis consist of students' actions (including utterances and gestures, both technology-mediated and not) between their first utterance of a claim and a restatement of the claim, that we call *re-claim,* involving a change in the qualifier. The qualifier can then be inferred from the student's actions; for example, a statement can be uttered with hesitation, or if a student continues to search for data, we can infer that the student is not yet convinced that the claim is true. The qualifier can change from "possible" to "more possible", "less possible", "true", or "false". To change the claim's qualifier, the students argue in favor or against the initial claim as they generate *data* that constitute factual evidence. Figure 2 shows a generic diagram of our adapted Toulmin's model, our new analytical tool: in the top right corner, noted in gray, is the first uttered claim along with a qualifier; below is the re-claim, with a new qualifier.

A second feature of our analytical tool is that a *technique* frame appears next to the data. This is because the main source of data, as students attempt to justify claims in a digital interactive environment like GeoGebra, is the effect of their use of techniques (as described in the TIG). The invisible schemes direct and organize actions with or on the data, but they also contain conceptual elements and rules that regulate actions (Drijvers et al., 2013). Such rules can be seen in the model as *warrants,* which are inference rules that connect the data to the claim.

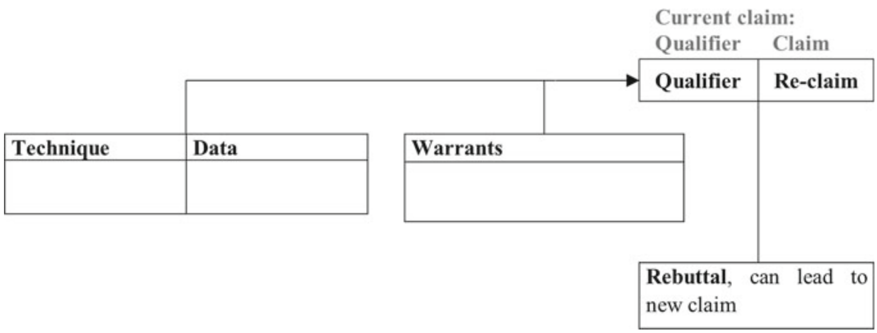



**Fig. 2** Adaptation of Toulmin's model: an analytical tool for students' justification processes in a digital interactive environment

In short, we look at actions as warrants connecting data to claims: we make inferences on the students' (usually implicit) warrants, through their verbal utterances and visible actions, to gain insight into their justification processes and, more in general, into their reasoning competency. For example, since warrants consist of inference rules connecting data to the claim, they direct how data is interpreted as evidence (Baccaglini-Frank, 2019). Making such inferences about students' warrants, thanks to the structural setup provided by the tool, seems to be completely coherent with Toulmin's (2003) statement about warrants being potentially implicit.

A justification process can be made up of various justification sub-processes (Pedemonte, 2007). Each of the sub-processes constituting units of analysis can be analyzed using the tool in Fig. 2. Since a justification sub-processes can build on previous sub-processes, for the same student or pair of students (see the following section), analyses of successive sub-processes through the analytical tool may contain long lists of data and warrants. For this chapter, we do not graphically link successive justification sub-processes, but we take into account previous units by recalling relevant data and warrants previously generated and used in the new unit analyzed.

## 2 Method

In the case introduced below, the task that the students are solving is taken from a sequence of tasks designed by the first author. It stems from her doctoral work, with the general aim to explore the potentials of basic tools in the algebra view of GeoGebra (e.g., typed-in expressions, sliders, variable points) concerning students' justification processes. The sequence of tasks was assigned to students in pairs in three 7th grade classrooms in two 90-min sessions. All students had prior experience using GeoGebra. 17 pairs agreed to be recorded as they worked on the tasks, capturing their screens, faces, and voices to make more accurate inferences, especially about the qualifiers of their claims. The transcriptions in this chapter have been translated from Danish to English.

The students were asked to work in pairs so that through the interaction with a peer, we could gain more insight into the students' justification processes. This approach is common in the use of Toulmin's model in mathematics education research (eg., Fukawa-Connelly & Silverman, 2015; Knipping, 2008; Pedemonte, 2007). Acknowledging that Toulmin's model originally only takes into account a single individual's argumentation, we keep track of discrepancies between each student's position with respect to their warrants by labeling warrants that seem to be held by one (and not the other) student. If students seem to hold the same warrant, we do not label it.

## *2.1 Task Design*

In the task we consider in this chapter, students are given the points A = (1, $s$), and B = ($s$, 1) (see Fig. 3) and asked to construct a point C, dependent on $s$, so that C and A move in parallel [directions]. Then they are asked: *Can C = B? If so, when?*

The algebra view and its tools are accessible, but the toolbar is restricted to the cursor, the line construction tool, the parallel line construction tool, and the perpendicular line construction tool (see Fig. 3). This design choice was made to ensure that the students used the tools accessible in the algebra view. Previous tasks introduce the trace tool to create a trace mark of dynamic points dependent on the variable by dragging the slider. The slider can also be animated to make the variable change "on its own". This was not introduced, but it was used by some students, including those presented in the case here. The default range of a variable represented

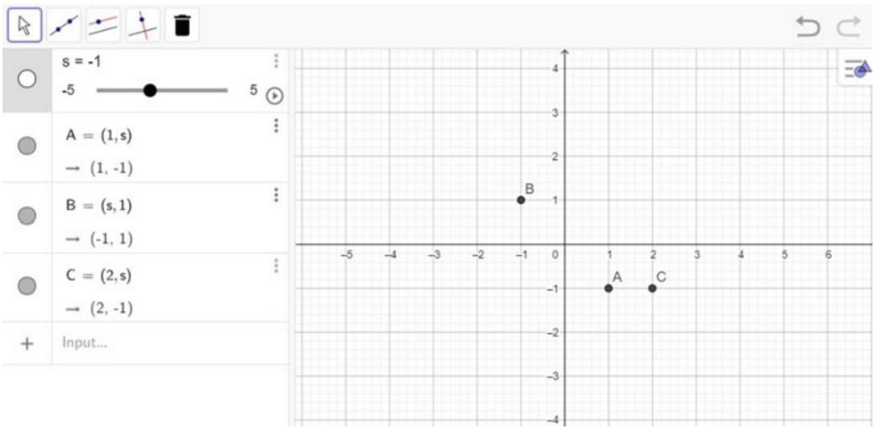



**Fig. 3** GeoGebra setting of the task. To the left is the algebra view. From top: a slider for the variable $s$, points A, B, and a possible construction of C, all dependent on s and displayed in the graphics view on the right with the coordinate plane

by a slider in GeoGebra is [−5, 5]; it can be changed, but for this task, it was left to this default setting.

We now provide a preliminary analysis of the task concerning possibilities it offers for the students' justification processes. For our students' age group, we only expected justifications less formal than proof.

The mathematical concepts at play are variable points on the coordinate plane, equality, parallelism, and intersection of lines (or segments). How these concepts are represented in GeoGebra's graphic view increases the complexity. Although the task refers to "parallel movement", students who see this mathematically as points belonging to parallel lines may have a deeper insight into how to solve the task. We will clarify how the mathematical concepts can come into play in the solution of the task.

The lines, and therefore parallelism and intersection, are indirectly represented. One of the coordinates of each point is defined by a variable, so the points move on the plane as the variable is changed. The coordinates of each point do not just refer to a single point on the plane but to a set of points restricted by the limits of the variable ([(−5), 5] as set by default), describing a segment that is either parallel or perpendicular to the $x$-axis. These segments can be represented by activating the trace of the variable point describing it when the slider is dragged. If the slider is animated (i.e., it moves automatically), it provides the opportunity of focusing on the movement of the dependent objects, in this case, points A, B, and C. With the coordinates given, points A and B have the same coordinates when $s = 1$, but B and C cannot be equal, that is, they cannot occupy the same place on the screen at the same time. However, the position and movement of A, B, and C can be altered by changing the expression containing $s$ of their coordinates, either by adding a term or changing the coefficient. The latter also changes the length (and hence the set of points) described by the trace mark.

Depending on the students' knowledge of generality, they may approach the task "Can C = B? If so, when?" in different ways. If they only consider the *specific point* C that they construct, C = B only if their point C intersects with B in a single point, with fixed coordinates, that hence need to be identical for both B and C. If, instead, the students consider C as a *set of points on a specific line parallel to the trace of A*, for example, $x = 2$, the point of the intersection of the traces left by C and B identifies a possible equality. If the students consider C as the *set of all points on any line parallel to the trace of A*, the answer could be a general expression such as "C = B, if C = $(d, \frac{1}{d}s)$" or C = $(d, s - (d - 1))$, when $s = d$. Of course, this is beyond our expectations for the students in this study.

## 2.2 Presentation of the Case

We selected episodes from the work of two 14-year-old girls, Em and Isa, because they were one of the two pairs of students in the larger study who answered that it is possible to have B = C. Figure 4 presents the GeoGebra applet, with the points and

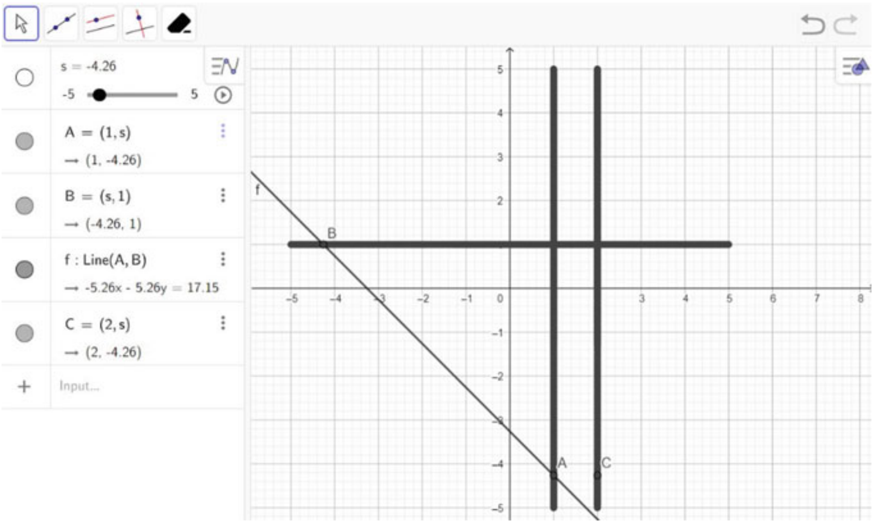



**Fig. 4** Screenshot of Em and Isa's screen showing trace marks from A, B, and C. Replication: https://www.geogebra.org/m/khjnewpm

line constructed as they started the task. It displays: points A = (1, *s*) and B = (*s*, 1), along with point C = (2, *s*), constructed by the students, and a line connecting B and A, which is unrelated to the task at hand. The trace is active for all three points, and the animations are turned on for the slider of *s*.

## 3 Analysis of the Students' Justification Process

In all excerpts, Isa is controlling the computer. The first author acted as a teacher-interviewer in the classroom along with the regular math teacher.

In the transcription, "what is done" and authors' notes are enclosed in square brackets. The transcript is presented in three excerpts that capture the main justification sub-processes of their general justification process. After each excerpt, we provide its analysis through our analytical tool. We label reappearing warrants in bold: the label **WE** concerns Warrants of Equality of points, **WP** concerns position and behavior of points, and **WT** concerns Traces.

## Excerpt 1: justification sub-process 1

| | | |
|---|---|---|
| 1 | Isa | Okay then, can C be equal to B? |
| 2 | Both | [Observe GeoGebra until line 10]. |
| 3 | Em | Collide, collide, collide—no |
| 4 | Isa | No, they cannot be equal to each other. Because these [points at A then B with cursor], they can be equal to each other. |
| 5 | Isa | But C can't. |
| 6 | Em | They will never collide. |
| 7 | Isa | That is because C, C is too slow. |
| 8 | Em | It is too far away. |
| 9 | Isa | Yes. |
| 10 | Em | It is too far away. |
| 11 | Isa | At least the C we have made cannot. |
| 12 | Isa | [Types answer: "no it cannot"]. |
| 13 | Isa | Okay, justify your answer [Reads from written assignment]. |
| 14 | Em | But there will probably be some that can if they are further away (Fig. 5). |

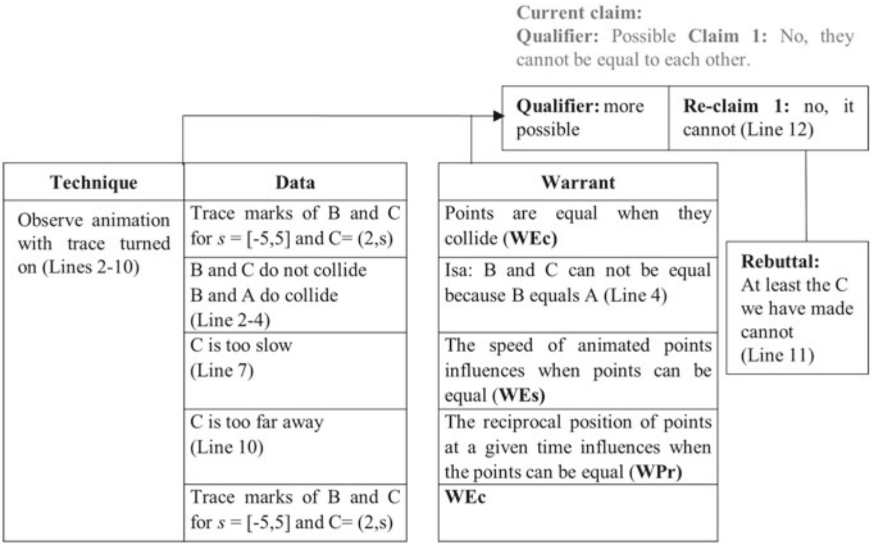



**Fig. 5** Justification sub-process 1 through the lens of the analytical

Notice that the rebuttal leads to an opposing claim (line 14): "But there will probably be some that can" with the qualifier *possible*. This opposing claim becomes the students' Claim 2.

### Excerpt 2: justification sub-process 2

Em and Isa gesture how the points move on the screen. Then they go back to observing the screen.

| | | |
|---|---|---|
| 15 | Isa | Okay… Ehm… Wait a minute. If we do like this. [Stops animation, edit C to (2, 2*s*), and start the animation]. |
| 16 | Both | [Observe the animation]. |
| 17 | Isa | Okay, so no. Why does it not? [Clicks the back arrow and C returns to C = (2, *s*)]. |
| 18 | Both | [Observe the animation]. |
| 19 | Isa | Hmm, let's see. |
| 20 | Em | They will never collide. |
| 21 | Isa | They will never, ehm, it can never be C equal to B because C is too slow. |
| 22 | Em | Because C moves parallel to A. |
| 23 | Isa | Yes |
| 24 | Em | And when A and B collide, then C is next to it. All the time, so it will never be able to get there, unless… |
| 25 | Isa | What? |
| 26 | Isa | No not unless. It just can't (Fig. 6). |

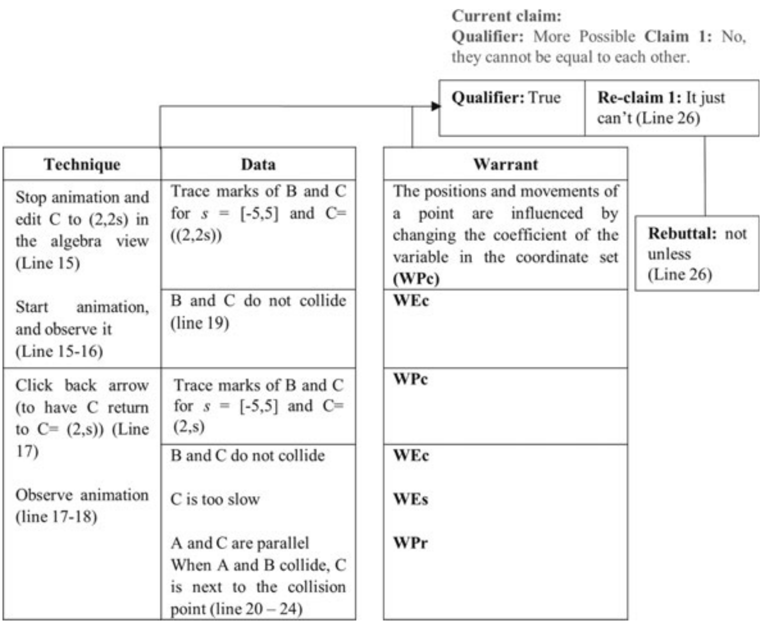



**Fig. 6** Justification sub-process 2 through the lens of the analytical tool

Since claims 1 and 2 stated in excerpt 1 are opposing, this change in the qualifier for claim 1 also inflicts change in the qualifier for claim 2, which goes from *possible* to *false*.

**Excerpt 3 (after an intervention of the researcher): justification sub-process 3**

At this point, the first author intervenes, prompting the students to continue searching for a possible "collision", implying that she disagrees with their re-claim 1. She then guides them to stop the animation and consider the position of C. She suggests: "… try and move it [the slider] so that B collides with the trace of C", and she suggests exploring the *x*- or *y*-value for C. Then excerpt 3 follows:

| | | |
|---|---|---|
| 27 | Isa | So, let's see. Look. |
| 28 | Em | Wait a minute. Must it [point B] collide with A at the same time? |
| 29 | Isa | No not at the same time, it [point C] just needs to be parallel with A. |
| 30 | Em | Okay, okay. |
| 31 | Isa | And the lines don't need to have the same length. |
| 32 | Em | Yes. |
| 33 | Isa | [Clicks to edit the *y*-value of C—this also makes the trace disappear in the graphics view]. |
| 34 | Em | Can we do like this, and then we need to move C down there [points at (0, 2)]. |
| 35 | Isa | Yes, but how do we do that? |
| 36 | Em | Right, now it [point C] starts at two, so it starts there. Can we get it to start further down? [points at (2, 2) then (0, 2)]. |
| 37 | Isa | Oh yeah, it starts here [points at (2, 2)]. |
| 38 | Em | Can you get it to start at minus one? |
| 39 | Isa | [Edits C from $(2, s)$ to $(-1, s)$. Starts animation.] |
| 40 | Both | [Observe the screen]. |
| 41 | Em | Wait, they might collide. A still collides, so no. |
| 42 | Isa | No, but we need to change… |
| 43 | Isa | [Stops animation]. |
| 44 | Em | So we get it to start a little further down, then it might do like this. [Gestures on the screen how C and B approach (1, 1) to collide. C from 4th quadrant and B from 2nd quadrant]. And A then, something, it will be before. |
| 45 | Isa | Yes, but it is $s$ we have to change. |
| 46 | Em | Is it $s$ we have to change then? |
| 47 | Isa | Yes, can we do like this then? |
| 48 | Isa | [Edits C to $C = (2, 0.5s)$, Starts animation]. |
| 49 | Both | [Observe screen]. |
| 50 | Em | It [point C] is still moving parallel with A, Isa. |
| 51 | Isa | Yes, it is supposed to do that. |
| 52 | Em | B and A still collide at the same time! |
| 53 | Isa | Yes, but C is a little behind, C is half the time behind al-ways, okay. [Collision of C and B happens for $s = 2$]. |

54 Em So, they can do it!

55 Isa Yes!

56 Em Oh, so it was just a little too fast (Fig. 7).

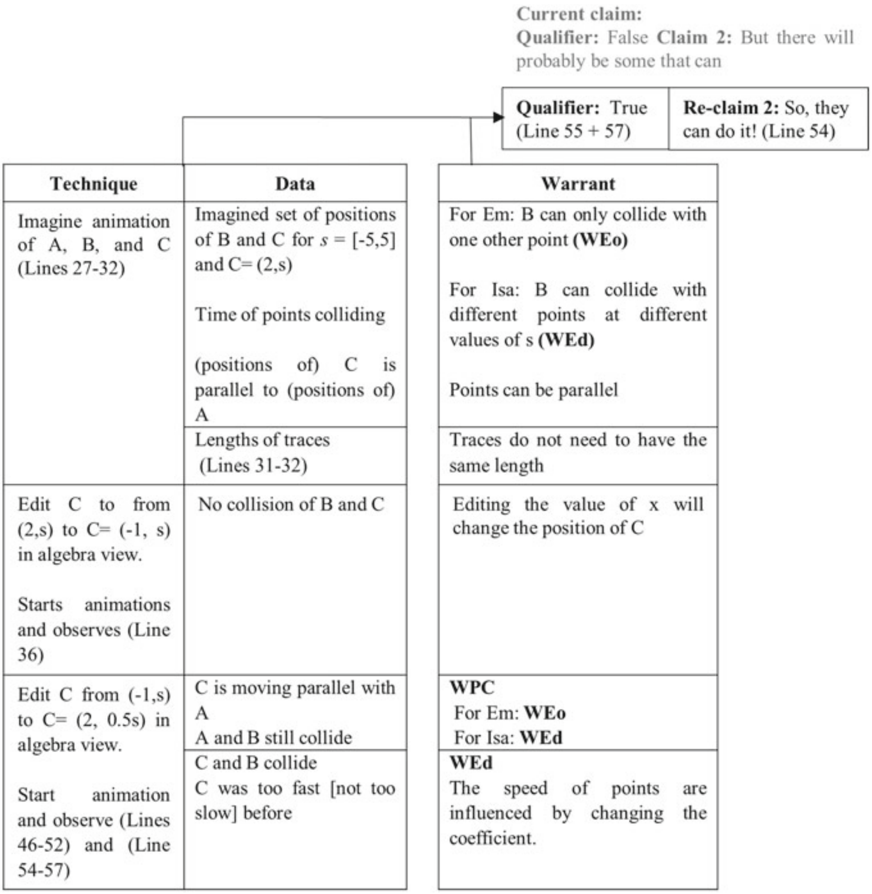



**Fig. 7** Justification sub-process 3 through the lens of the analytical tool

Again, as claims 1 and 2 are opposing, the change in the qualifier for claim 2 inflicts a change in the qualifier for claim 1, shifting it from *true* to *false*.

# 4 Results and Discussion

In this section, we discuss findings from the analyses of Em and Isa's overall justification process. Specifically, we discuss the two main claims that the students put forth, focusing on their mutual relationship given by the changes in their qualifiers. Then, we give an overview of the students' instrumented techniques and warrants used to generate and interpret data and discuss the insight we gain on the students' reasoning competency.

## 4.1 Claims and Qualifier Change

Em and Isa argue for two opposing claims:

Claim 1: B and C cannot be equal.
Claim 2: B and C can be equal.

As the two claims are opposing, an increase in the qualifier of Claim 1 toward *true* leads to an implicit decrease in the qualifier of Claim 2 toward *false* and vice versa. This is illustrated in Table 1.

**Table 1** The changes of qualifiers of the two claims from the initial claim throughout the three justification sub-processes

| | Initial claim | Sub-process 1 | Sub-process 2 | Sub-process 3 |
|---|---|---|---|---|
| Claim 1 | Possible | More possible | True | False |
| Implicit change | | | ↓ | ↑ |
| Claim 2 | | Possible | False | True |

Arrows indicate the implicit decrease in the qualifier of one claim in relation to the increase of the other

For Isa and Em, claims 1 and 2 correspond to two possible responses to the question in the task. Precisely what spurs Claim 2 about the possible equality of B and C is unclear. Neither student refers explicitly to the trace marks left by C and B or to their intersection. Claim 2 seems to be related to the rebuttal in line 11: "At least the C we have made cannot", suggesting that the students extend the constructed point C to all points on the trajectory $x = 2$. The students seem to be seeking phenomenological evidence of a "collision" between B and C to further convince themselves of Claim 2. As they fail to produce a collision in the second justification sub-process, the qualifier of Claim 1 shifts to true, while Claim 2 shifts to false. After the researcher's intervention, suggesting that the "collision" is possible, the students persist and eventually produce an example of C = B. The students seem to view this as evidence confirming Claim 2 and leading to the rejection of Claim 1.

### *4.2 Instrumented Techniques, Data, and Warrants*

The students use the following two techniques ($T_n$):

$T_1$: start the animation of the slider in the algebra view, then observe the screen where the trace is turned on for all points;
$T_2$: edit the coordinates of a point in the algebra view, then use $T_1$.

$T_2$ can be further subdivided into:

$T_{2C}$: edit coefficient for a variable.
$T_{2T}$: edit a term in the constant coordinate.

In Excerpt 1, $T_1$ generates the data that leads to Claim 1 (line 3). The students describe the points as moving and the equality of points as "collision" rather than as an intersection of lines. For point C, they seem to perceive the situation differently: Isa talks about C as "too slow" (lines 7 and 21), from which we infer the warrant "the speed of animated points influences when points can be equal". On the other hand, Em describes C in relation to other points: "too far away" (line 8 and 10) "C is next to it" (line24) ("it" refers to where A and B collide). We infer the warrant here to be: "the reciprocal positions of points at a given time influence when the points can be equal". This suggests that animating the slider can give the impression of points moving along parallel and perpendicular trajectories, as discussed in the preliminary analysis of the task.

$T_{2C}$ is used twice and $T_{2T}$ once to produce a collision of C and B, as evidence of Claim 2. Both techniques are used in trial and error strategies. $T_{2T}$ is first used in Excerpt 2, but it does not produce the collision. Without discussing the data generated, the students return to the original description of C. Em suggests that "because C is parallel to A" (line 22) and as "A collides with B", Claim 2 will never be possible. We infer this justification process to rely on the warrant **WEo**, "B can only collide with one other point", that feasibly emerges in Excerpt 1 from Isa's words: "No, they cannot be equal to each other. Because these [points at A then B with cursor], they can be equal to each other" (line 4). **WEo** seems to be used again twice in Excerpt 3, but not by Isa (line 29). Em seems to value this (mathematically false) warrant (lines 50 and 52); it is only when she sees the collision of B and C that she abandons **WEo**.

Conceivably, without the author's intervention, the students would have settled with Claim 1. However, such an intervention spurs the students to continue searching for evidence for Claim 2. Indeed, in Excerpt 3, Em suggests to "move it [C] further down" (lines 34 and 36), but neither student knows how to accomplish this. Em tries to use $T_{2T}$ to do this, but as she changes the $x$-value, point C moves in an unexpected (for her) way, horizontally instead of vertically. By enacting $T_{2C}$, Em is able to edit the coefficient of $s$ to 0.5, relying on the warrant **WPc**, "The positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set". This results in the desired collision that the students interpret as evidence strongly supporting Claim 2.

We note that the students never seem to consider the trace alone as evidence in support of Claim 2, even though the first author had given a strong hint in this direction: only one warrant seems to refer to the trace. Moreover, the students seem to refer to specific lines or trajectories (though lines are never mentioned explicitly) through the names of the points moving on them. This leads them to speak of A and C as "being parallel". However, eventually, Em refers to the *movement of A and C* as being parallel (line 50). We see this as a small step toward the distinction between "colliding points" and "intersecting lines", which we see as key in the students' potential progress in this mathematical domain.

# 5 Discussion

In this section, we start by discussing the specific situation of Isa and Em in relation to the reasoning competency; then, we reflect upon what is gained by the analytical tool we designed and used and on the theoretical implications of the coordination of the TIG and Toulmin' model.

## *5.1 How Does Isa and Em's Use of Digital Technology Influence Their Reasoning Competency?*

Isa and Em engage in justification by using GeoGebra to generate, explore, and interpret data. As mentioned in the case presentation, most pairs of students did not argue for possible equality of points B and C since they only considered the point C constructed initially, without thinking about tweaking its coordinates. On the other hand, Em and Isa seem to reach a conception of C as the set of points on the trajectory $x = 2$. We see this as an essential step in Em and Isa's mathematical reasoning that allowed them to make significant advances in their exploration and reasoning.

Isa and Em's data generation is limited by the techniques they implement, primarily $T_{2C}$ and $T_1$, which do not include adding a term to the expression containing the variable. Whether the data they generate is interpreted as evidence for or against a claim relies heavily on their warrants. While Em relies very much on the warrant **WEo** (B can only collide with one other point), Isa interprets the data through the warrant **WEd** (B can collide with different points at different values of $s$). Such warrants lead to interpretations of the data as that constitute primarily phenomenological evidence (Baccaglini-Frank, 2019) of their claims. This is also the case for warrants **WEs** (the speed of animated points influences when points can be equal) and **WPr** (the reciprocal position of points at a given time influences when the points can be equal). However, the warrant **WEd**, and more so the warrant **WPc** (the positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set), start to establish relationships between the dynamic points and the algebraic expressions.

We believe that interpreting the movements and positions of points in relation to the expressions in the coordinate sets in the algebra view could have helped activate (or construct) more profound mathematical knowledge.

## *5.2 Gaining Insights into Argumentation Processes in a Digital Environment*

The adaption of Toulmin's model to the context of a digital interactive environment through the use of the scheme-technique duality has led to an analytical tool that sheds light on how the duality can play out in justification processes in a digital interactive environment. Our analytical tool provides the techniques as ways of generating data, inferring warrants thanks to the schemes-technique duality, and it shows how data is interpreted as evidence for a certain claim.

In particular, the tool provides structure to the observed justification processes, organizing visible elements and allowing us to make inferences about the implicit warrants and the qualifier. Indeed, through the inferred warrants, we interpret the visible parts of the argumentations and their relations that provide insight into the students' more general reasoning competency. Since a warrant is an explicit hypothesis about students' conceptions (and misconceptions) relative to the mathematical concepts they are grappling with, the students' warrants are what allow them to interpret feedback from the digital environment as evidence for their claims. For example, the warrants **WPc** (the positions and movements of a point are influenced by changing the coefficient of the variable in the coordinate set) and **WEs** (the speed of animated points influences when points can be equal) reflect the students' conceptualizations of point C, which they seem to see as a "generalized" point relative to the value of $s$ and to the expression in the coordinate set. Such warrants allow the students to interpret the generated data as different sets of C. Through such warrants, obtaining evidence for a claim becomes a matter of generating data that "represents" the claim.

Further, the student's development and exploration of techniques empower them to generate further data in their justification processes. Techniques that involve both the graphical view and the algebraic view might further activate their reasoning competency, as we noted earlier.

### 5.3 *Theoretical Implications of Adapting Toulmin's Model Through the Scheme-Technique Duality*

We start by noting that the two theoretical approaches we consider are not symmetrical. The TIG draws on developmental psychology studied by Gerald Vergnaud and partly on cognitive ergonomics (Artigue & Trouche, 2021; Rabardel & Bourmaud, 2003). Later the construct of *technique* from the Anthropological Theory of the Didactic was adopted and reinterpreted within the TIG (Artigue & Trouche, 2021), leading to the development of the scheme-technique duality (Drijvers et al., 2013). On the other hand, Toulmin's model is not a theory but an analytical model. This makes it more flexible and applicable across sciences (Toulmin, 2003). It was originally positioned in the discipline of law (Toulmin, 2003), although its use in mathematics education has been extensive.

Despite the asymmetry outlined above, we see the conceptualization of *knowledge* as the main linkage between the scheme-technique duality and Toulmin's model. In Toulmin's model, data is observable factual knowledge that can imply more implicit knowledge in the form of warrants. In the duality, knowledge appears in terms of schemes that are partly visible, thanks to the related techniques (Drijvers et al., 2013). There is a parallel distinction between the observable and the implicit that has allowed us to link warrants to the notion of schemes and data (and its generation) techniques. Hence, we can see techniques as "windows" onto the students' knowledge about the objects at play (warrants), as they generate, notice, and interpret observable facts (data) as evidence of their claims in a justification process.

In our effort to understand the specific phenomenon of student justification processes in a digital environment, we also found ourselves in need of adapting the units of analysis. This led to the conception of sub-processes of justification within a greater process. Moreover, the sub-processes that correspond to the units of analysis capture the transition from a "claim" to a "re-claim", which is structurally different from Toulmin's original model. Indeed, rather than a static, finished argument, our sub-processes capture the formation of arguments aimed at changing the qualifier of a claim or reformulating the claim itself as a "re-claim". Such adaptation of the unit of analysis makes Toulmin's model more compliant with the scheme-technique duality.

The adaption of Toulmin's model to the context of argumentation in a digital interactive environment seems to provide a significant tool for particular types of argumentation processes that we refer to as *instrumented justification*. We conclude with the proposal of a definition for such a process.

*Instrumented justification is a process through which a student modifies the qualifier of one (or more related) claim(s) using techniques in a digital environment to generate and search for data and warrants constituting evidence for such claim(s).*

## 6 Concluding Remarks

In this chapter, we set out to contribute to a deepening of digital technology aspects of KOM's (Niss & Højgaard, 2019) reasoning competency. We have approached this by developing an analytical tool by adapting Toulmin's argumentation model through the scheme-technique duality from the Theory of Instrumental Genesis to define and capture students' processes of *instrumented justification*. The tool has provided us with a lens through which to gain insight into how the students' use of a digital environment is intertwined with their justification processes and hence with their reasoning competency. In a digital environment like GeoGebra, the students' interpretations of the objects represented are key in how the students consider them as evidence for a claim.

The theoretical developments presented in this chapter should, in our future research, be put under further scrutiny to consider how they align with other aspects of the TIG and the KOM; for example, how processes of instrumental genesis in the context of instrumented justification unfold, as well as other interesting aspects. To do this, we see potential in using a networking approach (Prediger et al., 2008) that conceives this first attempt of ours as a form of coordination between the TIG and Toulmin's model, where Radford (2008) suggests a comparison of principles, methodology, and paradigmatic questions to consider the compatibility between the coordinated elements.

Finally, from a mathematical teaching/learning point of view, the case of Em and Isa that we investigated in the chapter revealed a tension between what they referred to as "colliding points" and a yet implicit notion of intersecting trajectories. Initially, the students argued that $B = C$ was not possible as the "collision" did not occur. To overcome this interpretation, we conjecture that it is necessary for the students to reach a generalized conception of C, as *any* point on *any* vertical line instead of the specific point (e.g., $C = (2, s)$) that moves in a certain way along the vertical line. Such a generalization would entail overcoming the specific dynamic behavior of point C and conceiving its dynamism in a more general way. We see this as closely related to a broader issue of dynamism and temporality of mathematical objects, as discussed, for example, by Sinclair et al. (2009). The fine-grained analyses obtained through our analytical tool suggest that awareness of students (mostly implicit) warrants used in instrumented justification processes, and thus related to specific techniques carried out within the digital environment, can provide precious insights into their mathematical reasoning competence.

**Acknowledgements** Supported by Independent Research Fund Denmark [Grant no. 8018-00062B].

## References


Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), Article 245. https://doi.org/10.1023/A:1022103903080

Artigue, M., & Trouche, L. (2021). Revisiting the french didactic tradition through technological lenses. *Mathematics, 9*(6), Article 629. https://www.mdpi.com/2227-7390/9/6/629

Baccaglini-Frank, A. (2019). Dragging, instrumented abduction and evidence, in processes of conjecture generation in a dynamic geometry environment. *ZDM, 51*(5), 779–791. https://doi.org/10.1007/s11858-019-01046-8

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses. *Educational Studies in Mathematics, 82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in school. From history, epistemology and cognition to classroom practice* (pp. 135–161). Brill. https://doi.org/10.1163/9789087901691_009

Fukawa-Connelly, T., & Silverman, J. (2015). The development of mathematical argumentation in an unmoderated, asynchronous multi-user dynamic geometry environment. *Contemporary Issues in Technology and Teacher Education, 15*(4), 445–488.

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency". *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Geraniou, E., & Jankvist, U. T. (2020). "Mathematical digital competencies for teaching" from a networking of theories perspective. In R. B. B. Barzel, L. Göbel, M. Pohl, H. Ruchniewicz, F. Schacht, & D. Thurm (Eds.), *Proceedings of the 14th international conference on technology in mathematics teaching—ICTMT 14 Essen, Germany*, 22nd to 25th of July 2019 (pp. 368–375). https://doi.org/10.17185/duepublico/48820

Gregersen, R. M., & Baccaglini-Frank, A. (2020). Developing an analytical tool of the processes of justificational mediation. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H.-G. Weigand (Eds.), *Proceedings of the tenth ERME topic conference (ETC 10) on mathematics education in the digital age (MEDA), 16–18 Septe 2020 in Linz, Austria* (pp. 451–458). https://hal.archives-ouvertes.fr/hal-02932218

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), *Second handbook of research on mathematics education, Information Age Pub Inc., Greenwich.*

Hohenwarter, M., & Jones, K. (2007). Ways of linking geometry and algebra: The case of GeoGebra. BSRLM Geometry Working Group. *Day Conference British Society for Research into Learning Mathematics, 27*(3), 126–131. https://bsrlm.org.uk/publications/proceedings-of-day-conference/

Højsted, I. H. (2020a). Guidelines for utilizing affordances of dynamic geometry environments to support development of reasoning competency. *Nordic Studies in Mathematics Education, 25*(2), 71–98.

Højsted, I. H. (2020b). Teachers reporting on dynamic geometry utilization related to reasoning competency in Danish lower secondary school. *Digital Experiences in Mathematics Education, 6*(1), 91–105. https://doi.org/10.1007/s40751-020-00059-3

Højsted, I. H. (2020c). A "toolbox puzzle" approach to bridge the gap between conjectures and proof in dynamic geometry. In E. F. A. Donevska-Todorova, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H. -G. Weigang (Eds.), *The tenth ERME topic conference (ETC 10) on mathematics education in the digital age (MEDA)*. https://hal.archives-ouvertes.fr/hal-02932218

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–16. https://doi.org/10.1007/s10649-017-9761-8

Jupri, A., Drijvers, P., & van den Heuvel-Panhuizen, M. (2016). An instrumentation theory view on students' use of an applet for algebraic substitution. *International Journal of Technology in Mathematics Education*, *23*(2), 63–80.

Knipping, C. (2008). A method for revealing structures of argumentations in classroom proving processes. *ZDM Mathematics Education, 40*(3), 427. https://doi.org/10.1007/s11858-008-0095-y

Lagrange, J.-B. (2010). Teaching and learning about functions at upper secondary level: Designing and experimenting the software environment Casyopée. *International Journal of Mathematical Education in Science and Technology, 41*(2), 243–255. https://doi.org/10.1080/00207390903372395

Lagrange, J.-B. (2014). New representational infrastructures: Broadening the focus on functions. *Teaching Mathematics and Its Applications: An International Journal of the IMA, 33*(3), 179–192. https://doi.org/10.1093/teamat/hru018

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460. https://doi.org/10.1007/s10649-013-9492-4

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*(3), 255–276. https://doi.org/10.1007/s10649-007-9104-2

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66*(1), 23–41. https://doi.org/10.1007/s10649-006-9057-x

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*(5), 665–691. https://doi.org/10.1016/s0953-5438(03)00058-4

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM Mathematics Education, 40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM Mathematics Education*, *41*, 441–452. https://doi.org/10.1007/s11858-009-0180-x

Sinclair, N., & Robutti, O. (2013). Technology and the role of proof: The case of dynamic geometry. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 571–596). Springer, New York. https://doi.org/10.1007/978-1-4614-4684-2_19

Takači, D., Stankov, G., & Milanovic, I. (2015). Efficiency of learning environment using GeoGebra when calculus contents are learned in collaborative groups. *Computers & Education, 82*, 421–431. https://doi.org/10.1016/j.compedu.2014.12.002

Toulmin, S. E. (2003). *The uses of argument* (2nd ed.). Cambridge University Press. (Updated Edition).

UVM. (2019). *Fællesmål for faget matematik. [Common objectives for the subject of mathematics].* https://www.emu.dk/grundskole/matematik/faelles-mal

# Mathematical Representation Competency in the Era of Digital Representations of Mathematical Objects

**Ingi Heinesen Højsted and Maria Alessandra Mariotti**

## 1 Introduction

Any mathematical activity involves the use of representations of mathematical objects. The necessity of representations emerges from the fact that mathematical objects are abstract ideas that are not perceptually available except through semiotic representations (Duval, 2006). Ability in dealing with representations is therefore an important constituent of mathematical competency. In the KOM framework, representation competency concerns being able to manage representations, that obviously includes treating and converting within and between different representations (Niss & Højgaard, 2011, 2019), but also any other ability concerning the meaningful and effective use of semiotic representations in accomplishing mathematical tasks.

Historically, different ways of representing mathematical objects have evolved as new technologies become available. In mathematics education research, the nature and role of representations in mathematics learning and teaching has received ample attention, not least since the introduction of digital tools (Balacheff & Kaput, 1996; Goldin & Janvier, 1998a, 1998b; Janvier, 1987; Kaput, 1998; Morgan et al., 2009; Vergnaud, 1998). Once the use of digital tools in mathematical activities became customary, new types of representation systems were established and in this context new abilities become necessary to exploit the new potentialities offered. From a didactic point of view, an important issue arises concerning which new abilities

I. H. Højsted (✉)
Faculty of Education, University of the Faroe Islands, Tórshavn, Faroe Islands
e-mail: ingih@setur.fo

M. A. Mariotti
Department of Information Engineering and Mathematics Science, University of Siena, Siena, Italy

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_8

that new digital representation systems require and how such abilities can be developed at school in order to support or perhaps not impede students' development of representation competency.

The aim of this chapter is to contribute to outlining what the possession of representation competency constitutes in the context of digital representations of mathematical objects and propose criteria for such competency, also discussing implications for educational practice to exploit the new representation systems. We claim that geometric knowledge plays a fundamental role in interpreting graphical representations of mathematical objects (to be elaborated later), and, therefore, we focus on the case of representations in geometry and, in particular, on digital representations of geometrical objects in a dynamic geometry environment (DGE). The following research question guides our effort: *What characterizes development and possession of representation competency in the context of digital representations of mathematical objects?*

After the introduction of pertinent conceptual frameworks and constructs in the next section, we analyze empirical examples of students working in DGE. Based on the specific cases, we then discuss a more general development of representation competency in the context of digital representations of mathematical objects. From a networking of theories perspective, we then consider the relationship between the KOM framework and other external theoretical constructs employed in our discussion.

## 2 Conceptual Frameworks and Constructs

In the following three subsections, we elaborate on representation competency from the KOM framework; dynamic representations of geometrical objects in DGE; and then we outline the fundamental role of geometric knowledge in interpreting graphical representations of mathematical objects.

### *2.1 Representation Competency in the KOM Framework*

Mathematical representation competency concerns, in basic terms, an individual's ability to deal with different representations of mathematical objects, phenomena, problems or situations.

It comprises being able to interpret different kinds of mathematical representations as well as being able to translate and move between an extensive range of such mathematical representations (e.g., verbal, symbolic, graphic). This includes awareness not only of the connections between different forms of representations of the same object, but also awareness of the "scopes and limitations—including strengths and weaknesses—of the representations involved in given settings" (Niss & Højgaard, 2019, p. 17). On the basis of such awareness, the competency includes being able to select and utilize particular representations in order to deal with certain tasks or contexts.

As with each of the other competencies in the KOM-framework, the representation competency comprises a productive side, which refers to oneself being able to carry out the constituents of representation competency. The competency also comprises an investigative side, which pertains to the ability to follow, comprehend, analyze, and assess another subject's mobilization of representation competency (Niss & Højgaard, 2011).

Elaborating on the affordances of digital technologies in relation to the development of mathematical competencies, Niss (2016) considers that digital technologies may

> help generate student experiences of mathematics-laden processes and phenomena that might be difficult to obtain by other means; create platforms and spaces for exploration in which mathematical entities can be investigated through manipulation and variation; produce static and dynamic images of objects, phenomena, and processes that are otherwise difficult to capture and grasp; create connections between different representations of a given mathematical entity [...] (p. 248)

A digital technology that comprises affordances, which may support several of the aspects outlined above, are dynamic geometry environments.

### *2.2 Representations of Mathematical Objects—from Paper and Pencil to DGE*

Since the introduction of DGE in the mid 1980s (Oldknow, 1997), much has been written about the potentialities and complexities emerging in the move from paper and pencil geometry to dynamic geometry.

A DGE can be considered as a microworld that mimics a theoretical system, usually Euclidean geometry (Balacheff & Kaput, 1996; Healy & Hoyles, 2002). This type of environment offers a new kind of dynamic representation of geometrical objects that are produced as a result of construction tools, which induce certain geometrical properties, chosen by the user (Laborde, 2005). Subsequently, using the dragging tool, the user may manipulate the figure while its constructed properties are preserved. The dependency relations between the elements of the figure that were induced in the construction process are locked in a hierarchy of dependencies, which decide the behavior of the object during dragging (Hölzl et al., 1994). As stated by Laborde (2005), a crucial feature of dynamic representations is

> their quasi-independence of the user once they have been created: when the user drags one element of the diagram, it is modified according to the geometry of its constructions rather than the wishes of the user. This is not the case in paper-and-pencil diagrams […] (p. 161).

The possibility of dragging a figure while its constructed properties are preserved, has been widely recognized as an affordance that may support development of mathematical reasoning, generalization and the development of conjectures (e.g., Arzarello et al., 2002; Baccaglini & Mariotti, 2010; Healy & Hoyles, 2002; Laborde, 2002, 2005; Leung, 2015). According to Laborde (2005), the relationship between perceptual and theoretical aspects of a diagram is favored in dynamic representations in a DGE, because the behavior of the new kind of representation during dragging is controlled by Euclidean theory.

However, there are at least two layers of complexity involved in competency in interpreting these dynamic representations: firstly, it involves being able to interpret dynamic dependency, i.e., awareness of dependency relations between the objects' movements through dragging. Studies have shown that students' competency in interpreting the dynamic dependency relations inherent in the dynamic representations is not immediate. For example, Talmon and Yerushalmy (2004) found that junior high and graduate students predicted a dynamic behavior that was "contrary to the behavior that would be expected based on the order of construction" (p. 114). Secondly, it involves being able to perform a mathematical/geometrical interpretation of dynamic dependency in terms of logical dependency between geometrical properties. This ability is not immediate for students, either. For example, the classroom experiments reported by Arzarello et al. (2002) showed "that the software itself does not grant the transition from empirical to generic objects, from perceptive to theoretical level" (p. 71). Efficacy in managing the geometrical interpretation of dynamic phenomena in a DGE, that is of dynamic representations, can be considered a highly demanding competence concerning a new type of representation of mathematical objects.

### *2.3 The Fundamental Role of Geometrical Knowledge in Dealing with Representations*

As outlined above, we intend to consider the competencies that are required to treat representations of geometric objects, in particular, we will consider representations in a DGE. This is a somewhat specific focus, however, in our view, it may provide a fundamental contribution. As a matter of fact, any contribution about the needs in terms of competencies required to manage graphical representation of geometrical objects—e.g., to be productive in problem-solving or conceptualizing geometrical objects—can give us enlightenment on what kind of competencies are needed for interpreting and using effectively graphical representations of any mathematical object.

Let us consider the paradigmatic example of the graphical representation of a function in a Cartesian System, commonly called a *graph*. Actually, we have at least three different representations for functions: tables, formulas, and graphs. As far as the graph is concerned, it is a set of points, and a geometrical object; thus, any drawing of it, is not a mere pictorial representation, it is the representation of such a geometrical object. The way we view, conceptualize, and analyze the graph of a function is therefore affected by competencies that we develop for interpreting the representation of geometrical objects, competencies that can be reinvested in the elaboration of the graphic representation of the function, even though a function is not a geometrical object per se.

> Geometric intuition plays an important role beyond geometry itself: we mention only its fundamental role in mathematics. Starting with the concept of continuity, which is based on the intuition of the continuity of a line, one recalls the presentation of functions by means of curves, the complex plane, etc. (Alexandrov, 1994, p. 366)

Hence, sharing Alexandrov's claim, we consider geometry to be a particular fundamental type of knowledge, because it constitutes the basis to elaborate on graphical representations of mathematical objects, though not specifically geometrical objects. A mathematician may consider the graphical representation of a function as more immediate, more intuitive (Fischbein, 1999). However, this is not true simply because it is an image, rather because such an image can benefit of an expert's interpretation as the representation of a geometrical object. As a matter of fact, the refined interpretation that is at the base of such intuition is not available for a novice, at least not in the same way that it is for the mathematician, because the mathematician has already developed specific high demanding representation competencies, i.e., he or she possesses somewhat of a "mathematical eye" (Mariotti & Baccaglini-Frank, 2018).

In the following we present examples illustrating different developmental stages of representation competency in the context of interpreting representations of geometrical objects in a DGE. At the same time, the examples will show how specific tasks can foster students' awareness of geometrical interpretation of dynamic images.

## 3 Method and Context

We present two case studies of Danish 8th grade students (age 13–14) working together in pairs using GeoGebra on a shared computer to solve tasks from a worksheet. The students had basic knowledge of the software layout and tools for construction from previous GeoGebra experience. In the selection of data, we chose, based on the teacher's assessment, a high achieving pair, Sif and Ole, and an average/low achieving pair, Dan and Jan, in order to show examples of students at different developmental stages in relation to possessing representation competency in the context of interpreting representations of geometrical objects in a DGE.

The students are working on a particular type of task, denominated “dependency tasks” (Højsted & Mariotti, 2020), which are characterized by a twofold aim: firstly, the aim is to support students in becoming aware of dynamic dependency, which is dependency between objects’ movements during dragging in a DGE. Such awareness is needed in order to interpret dynamical representations of geometrical objects, and therefore, we argue, it is a constituent of representation competency. The first aim is fundamental and a prerequisite for a second aim, which is to support students in interpreting dynamic dependency geometrically as logical dependency between geometrical properties.

Each task starts with a construction part, asking the students to perform a particular construction using certain GeoGebra tools. The tasks then follow the task design heuristic of Predict-Observe-Explain (White & Gunstone, 2014, pp. 44–65), in which the students are required first to predict what will happen to certain elements of the construction during dragging, then to observe what actually happens, and finally to explain discrepancies between prediction and observation. According to the objective of a dependency task, the three related requests are expected to foster students’ awareness of the geometrical interpretation of what happens on the screen: the possible discrepancy between prediction and observation can lead students to come back to the construction process in the attempt to resolve differences, and in so doing gaining awareness of dependency relations. The formulations of the three specific tasks are presented in the next section together with the analysis of the examples.

Data was collected in the form of video, screencast, and written products. The data is analyzed, in the form of transcripts of verbal utterances as well as gestures that were captured on video, in order to find evidence of students’ awareness of the abovementioned aim. That is, do they grasp dynamic dependency, and are they able to interpret the dynamic dependency geometrically? Hence, our analytical approach comprises investigating, to what extent the students have developed an interpretative frame that enables them to predict and/or describe the result of a dragging action in terms of the dynamic dependencies that are embedded in the construction, and secondly, if the students are able to explain their perceptual experience on the basis of geometrical properties.

## 4 Presentation of Data and Ensuing Analysis

We first present and analyze data collected from the high achieving pair of students, Sif and Ole, as they work on tasks 1–3, followed by medium-low achieving students, Dan and Jan, working on the same tasks. The two cases will highlight different possession and development of representation competency.

## Sif and Ole—task 1

In task 1, the students are required to construct two points, A and B, and the midpoint C, using GeoGebra´s midpoint tool. We follow the events as Sif and Ole reach task 1d, in which they have to predict:

| | | |
|---|---|---|
| 95 | Sif: | [*reading task formulation 1d*] What do you think happens to the other points when you drag point C? Guess and justify first. |
| 96 | Ole: | It’s all moving together. |
| 97 | Sif: | Then everything moves because C must be in the middle. Then they will move in relation to C? [*the tone indicates a question and she looks at Ole*] |
| 98 | Ole: | I think so. |
| 99 | Sif: | Then one could imagine that it was just a line moving around (Fig. 1) |

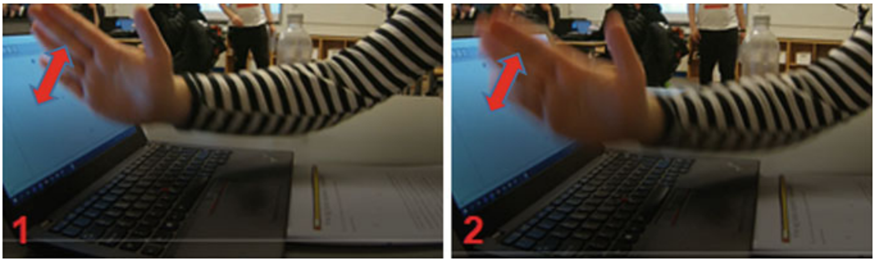



**Fig. 1** Sif moves her hand to represent a line going through the three points. The movement indicates that it remains parallel to the initial line

| | | |
|---|---|---|
| 100 | Ole: | Yes exactly, in parallel. |
| 101 | Sif: | Okay, so we just say that everything will move relative to point C. [*writing*] |
| 103 | Sif: | Yes, because it must be in the middle in relation to C. [*Sif tries to drag point C*] |
| 104 | Ole: | Oh! |
| 105 | Sif: | One cannot move C. [*Sif writes down*] |
| 109 | Sif: | Ehhh, and why can’t you? … |

The students expect the dynamic representation to move as a rigid structure (line 96–103). Their prediction is grounded in a geometrical interpretation, meaning that they are justifying their prediction by referring to the geometrical properties, which they induced in the construction process (line 97, 101, 103). While Sif says that the structure will move around (line 99), her gesture (Fig. 1) suggests that the movement of the line is restricted to an orthogonal translation, remaining parallel to the initial position. Ole understands her suggestion and makes it explicit (line 100). Apparently, the students expect dynamic dependency relationships in GeoGebra to be non-hierarchical. In the literature, the discernment between locked and free objects, that is independent and dependent objects, is sometimes referred to as parent/child relationship (e.g., Talmon & Yerushalmy, 2004). In GeoGebra, it is not possible to drag dependent objects (children) that are derived from independent objects (parents), and since the midpoint C is derived from points A and B, it is a locked object that cannot be dragged.[1] The students' ability to interpret this aspect of dynamic representations is yet to be developed, i.e., this part of the representation competency is still not matured, however, apparently the conflict between prediction and control spurs the students' interest (line 104–109).

The teacher was in the vicinity and chose to intervene at this point. However, his intervention did not support the students in grasping the phenomena they observed. Rather, the teacher shifted their attention from the non-draggable midpoint to the draggable points A and B, explaining what they already knew and giving an authoritarian argument (see also Chap. Teachers' Facilitation of Students' Mathematical Reasoning in a Dynamic Geometry Environment: An Analysis Through Three Lenses of this book) (Højsted & Mariotti, 2020). Without resolving the cognitive conflict, the students moved on to the next task.

**Sif and Ole—task 2**

Sif has constructed line f through points A and B as requested in task 2a.

| | | |
|---|---|---|
| 153 | Sif: | (*Reading task formulation 2b*) What do you think happens with line *f* when you drag point A or B? First guess and justify your guess to your mate. |
| 156 | Sif: | Ok, if you drag A or B, then… then the line changes. So it changes its slope, does it not? |
| 157 | Ole: | Yes |
| 158 | Sif: | It doesn't change length because that line is |
| 159 | Sif/Ole: | (*speak at the same time*) Infinite |
| 174 | Sif: | But it does not make a difference to the line how far there is between the points |
| 180 | Sif: | (*reading task formulation 2c*) What is the relationship between the line and points A and B? |
| 185 | Sif: | (*writing the answer and talking out loud*) A and B determine the slope. |

Sif and Ole are able to interpret the dynamic representation, showing signs of awareness of the dependency relation between the points and the line, and they are

[1] Other DGE, however, do allow dragging the dependent points, e.g., Geometer Sketchpad (Mackrell, 2011). Being able to interpret dynamic representations (development of representation competency) is therefore a non-trivial task—the behavior of the dynamic representations during dragging is grounded not only in geometry, but also in software design choices.

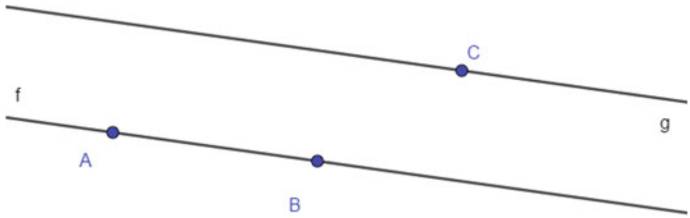



**Fig. 2** Reconstruction of Sif´s construction

therefore able to predict the movement of the line when dragging the points (line 156–157, 160, 185). They also show awareness of the infinite length of the line (158–159). They provide a geometrical interpretation referring to the geometrical properties of the objects they observe: the points, the line, its slope, and length. Since there is only a line, their idea of slope is not geometrically consistent, however, it seems that the students are implicitly considering the spontaneous reference to the horizontal direction, as if the line was embedded in a horizontal/vertical reference system, which they may have previously experienced in GeoGebra. This conceptualization belongs to the projection of a spontaneous reference frame coming from our experience of the physical world, where the horizontal laying/vertical direction constitutes the privileged organizing frame. We expect the students to be framing the image and interpreting it according to this frame.

### Sif and Ole—task 3

According to the instruction in task 3a, Sif has constructed line f through points A and B and the parallel line g through point C, using the parallel line tool (Fig. 2).

| | | |
|---|---|---|
| 254 | Sif: | (*reading task 3b formulation aloud*) What do you think happens with line *g* when you drag points A or B? Guess first and justify your guess to your mate |
| 259 | Sif: | (*writing and talking*) It will move, depending on line *f*'s (movement/position? she does not finish her sentence) |
| 260 | Sif: | (*reading*) Examine afterwards what happened. |

(*Sif drags point B, Ole is talking to another student, he has lost concentration completely*)

| | | |
|---|---|---|
| 268 | Sif: | Oh, this is so cool. So, the gap between the lines, it is not the same, but they keep laying… OLE, concentrate! |
| 270 | Ole: | Sorry |
| 271 | Sif: | The space between the lines is not the same, but they remain parallel |
| 273 | Sif: | Satisfying. |
| 274 | Sif: | We agree that they remain like that |
| 278 | Sif: | In fact, I thought they might keep the same length between them, but that didn't happen. |

It seems Sif expected line g to remain parallel and maintain the same distance from line f. Perhaps the usual expression of the definition of parallelism has influenced her prediction, hence she is surprised. The dependency relation is explicitly stated "it will move, depending on line f´s" (line 259) although the sentence was not finished. She is surprised and seems also intrigued when she experiences that the lines remain parallel, yet the distance is not maintained (line 268–278).

| | | |
|---|---|---|
| 308 | Sif: | (*reading 3d*) What is the relation between line *f* and line *g*? |
| 309 | Sif: | They are parallel. Can we just write that? |
| 310 | Ole: | Yes |
| 311 | Sif: | But it is line *f* that determines. I mean, it decides, it remains just as it is. Ok |

Sif is able to geometrically interpret the dynamic representation, highlighting the parallelism between the two lines as the invariant property. In addition, Sif´s final utterance "it is line f that determines" "it decides, it remains just as it is" (line 311) indicates her awareness of the hierarchical dependency relation between lines f (independent object) and g (dependent object), and it is a sign of her emerging awareness of the hierarchical dependencies that are embedded in dynamic DGE constructions.

### Jan and Dan—task 1

Dan and Jan constructed two points, A and B, and the midpoint C, using GeoGebra´s midpoint tool. They are working on task 1.d and about to make a guess about what happens if they drag midpoint C:

| | | |
|---|---|---|
| 33 | Jan: | I think that when you move point C, then they move *[likely means A and B]* |
| 35 | Jan: | so if we move it here, and then drag it *[talking about point C]*, then they move like this *(moving hand up and down, see* Fig. 3*)* |

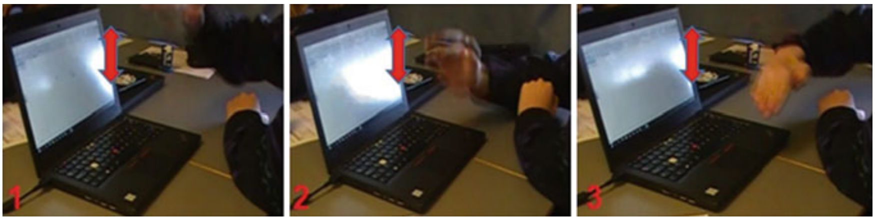



**Fig. 3** Jan moving his hand up and down as if it were a line going through points A, B, and C

| | | |
|---|---|---|
| 36 | Dan: | move along in parallel |
| 37 | Jan: | hm, is it called parallel?… |
| 38 | Jan: | Yes, parallel to |
| 39 | Dan: | Horizontally |
| 40 | Jan: | Like if you have a stick… and lift it in the middle (Fig. 4) |


**Fig. 4** Jan simulates holding the middle of a stick and moving it up and down

| | | |
|---|---|---|
| 42 | Dan: | and in parallel |
| 43 | Dan: | The sides *[the extreme of the stick]* move along |
| 44 | Jan: | Exactly |
| 48 | Jan: | Let's try *(Tries to drag C)* |
| 49 | Jan: | Nothing is happening… It should. *(They are puzzled)* |
| 52 | Jan: | *(asks some other students)* Did something happen for you with C? *(does not get a response)* |
| 53 | Dan: | Nothing happened (*writing and talking*) |
| 54 | Jan: | You are not supposed to use this one? *(clicks on "Move graphics view" command, which can be used to move the viewing screen around the plane),* |
| 55 | Dan: | no, that one *(indicating the "Move" command, which is used to drag elements)* |
| 56 | Jan: | …What's next? *(they move on to the next task)* |

Jan has difficulty in describing the movement he expects (line 33–44), however, it seems his interpretation of the dynamic representation is that the construction will move like a rigid system. They use the image of a stick (line 40), unveiling that they expect the construction to be constrained. Also, they use the geometrical term "parallel" (line 36, 37, 38, 42), though the notion of parallelism is not an appropriate mathematical interpretation of the dynamic representation; however, it is consistent with Danish everyday language use of the term to describe something happening at the same time. Their interpretation is not grounded in geometry, that is, they are not reflecting about or addressing the geometrical properties of the construction, which they induced. Rather, their interpretation is grounded in everyday life experiences of physical objects, for instance, lifting the middle of a stick results in the whole stick being elevated. When the construction does not behave as expected, they are puzzled, ask their peers, and look for an answer in the software, before writing that nothing happened, and moving on to the next task (line 52–56).

**Jan and Dan—task 2**

Jan uses the line tool and clicks twice on the screen, thereby constructing line f through points A and B. They are now about to make a prediction.

| | | |
|---|---|---|
| 111 | Jan: | Ok, (*reading the task formulation 2b*) what do you think happens with line f |
| 112 | Dan: | (*continuing the reading*) when you drag points A or B |
| 113 | Jan: | ehh (*gesturing with his hands, it is not clear what he means*) |
| 115 | Jan: | ehh, it moves (*gestures with his hand from side to side in a curving movement, like tracing a semicircle*) (Fig. 5) |

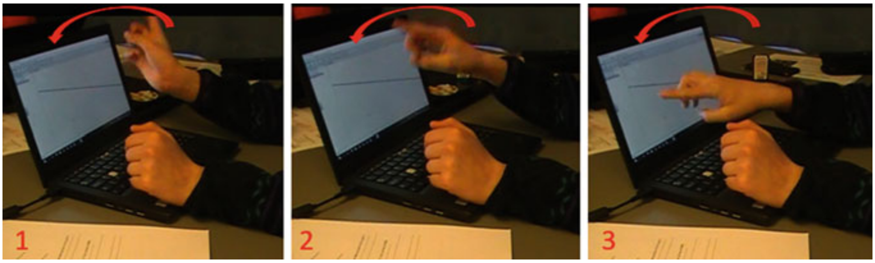



**Fig. 5** Jan "draws" a semicircle with his index finger

| | | |
|---|---|---|
| 117 | Jan: | I think A will be the midpoint for B and vice versa |
| 118 | Dan: | What? (*getting ready to write*) Ok, A becomes the midpoint for B? |
| 119 | Jan: | no A (*marks point A on screen with his finger*)… (*stops for 3* s) [*seems to be thinking, choosing his words*] |
| 120 | Jan: | When you drag A, B becomes the midpoint and when you drag B, A becomes the midpoint. |
| (*Jan makes a circular gesture with his finger on the screen, now drawing a full circle*) | | |
| 126 | Jan: | Yes! Then I think it [*point A*] will move, (*does a circular motion*), with a radius of that which is between them [*in danish "med en radius på det der er mellem dem" indicating radius AB*]. In a circle you can move them around |
| 128 | Dan: | What? |

(continued)

(continued)

| | | |
|---|---|---|
| 129 | Jan: | I don't know how [*does not seem sure how to formulate his thoughts*] |
| 132 | Dan: | (*Tuts, gesturing that he doesn't understand/is frustrated)* Write. (*hands over pencil and textbook to Jan*) |
| 137 | Jan: | So if we drag A (*marks point A on screen with finger*) then it should very well be able to go all the way around B with a radius of what is in between them. |
| (*Jan moves point A around*) | | |
| 140 | Jan: | Ok it was maybe… (*seems insecure about his previous answer*) |
| 141 | Dan: | No… let's answer the question |
| 145 | Jan: | B becomes the midpoint of A |
| 156 | Jan: | (*reading task formulation 2c)* What is the relation |
| 157 | Dan: | (*continuing the reading*) between the line and points A and B? (Fig. 6) |


**Fig. 6** Reconstruction of Jan's construction

| | | |
|---|---|---|
| 158 | Jan: | Eh … that … that the relation … when you drag one of the points … you can move it around the other point (*gesturing with his hand in a circular motion*) |
| 160 | Dan: | around what? |
| 161 | Jan: | They are always on a parallel line |

In Jan´s prediction (lines 115–137) he seemingly projects onto the construction dependency properties that are not there. He seems to expect that the points A and B are movable, but that the length AB will be preserved, hence that A can be dragged on the circle with center B and radius AB and conversely, B can be dragged on the circle with center A and radius AB.

The students mention several times that if one point is dragged, the other becomes its midpoint. It may be because they interpret it as the center of a circle with radius AB. However, later when they have found out that the length AB is not preserved (line 140), they still talk of point B becoming the midpoint of point A (line 145). Hence, in their interpretation of the dynamic representation, they are neither capable of correctly predicting the dependency between objects' movements during dragging, nor able to interpret geometrically, using the geometrical terms correctly to describe what they saw during dragging. In particular, the use of the term "midpoint" does not seem to correspond to the correct geometrical notion, rather to a more general meaning as "something that stays in the middle", which might also refer to the center of a circle. The term "parallel" is used frequently by this pair of students, here and in the previous task, with different meanings that are not mathematically consistent; in this case it seems that "parallel" indicates that the line is straight, or perhaps just that some properties are maintained during dragging, perhaps the term "parallel" is intended to express invariance. Even if they use geometrical terms "A, B, line, midpoint, parallel, radius", they do not produce a geometrical interpretation. They do not grasp the geometrical relationship between the points and the line: the geometrical terms are used to describe what they imagine or what they observe, but not consistently with the geometrical properties of the constructed figure.

**Jan and Dan—task 3**

Jan has constructed line f through points A and B and the parallel line g through point C, using the parallel line tool. They are now about to make a prediction.

| | | |
|---|---|---|
| 255 | Jan: | (*reading task formulation 3b*) What do you think happens with line *g* when you drag points A or B? Guess first and justify your guess to your mate. |
| 258 | Jan: | When you drag points A and B… eh, it runs parallel to it (*gestures with his hand as a line, moving back and forward*). |
| 260 | Dan: | It runs parallel to it. |
| 265 | Dan: | (*Dan laughs*) What are you saying? It just runs in parallel? |
| 268 | Jan: | Yes, If I drag B.. down here for example, (*gesturing with his hands, two lines which remain parallel*) then it stays parallel (Fig. 7) |

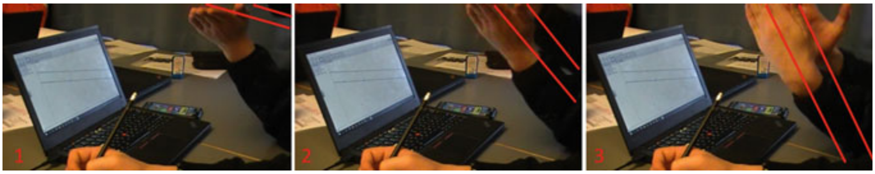


**Fig. 7** Jan gestures two lines remaining parallel. The added red lines show the direction of Jan´s hands

| | | |
|---|---|---|
| 270 | Dan: | It stays parallel… So try and do it |
| (*Jan drags point B*) | | |
| 271 | Jan: | You see, it's always parallel. |
| 273 | Jan: | wooooooo… yes |
| 310 | Jan: | (*reading 3d*) What is the relationship between … |
| 311 | Jan: | It is that, no matter what, no matter what, they are parallel (*gesturing with his hands two parallel lines*) (*Dan writing*) |

(continued)

(continued)

| | | |
|---|---|---|
| 313 | Dan: | No matter what… (*he forgot what Jan said, and does not understand himself*) |
| 314 | Jan: | Can't you remember that? |
| 315 | Dan: | No matter what, they are parallel (*Writing*). Now we're done (*they shake hands*) |

Jan is to some extent becoming aware how the software works. In his interpretation of the dynamic representation, he seemingly projects dependency properties onto the construction and expects that they are maintained during dragging (line 268). While he has predicted incorrectly in tasks 1 and 2, he predicts correctly in task 3, and is satisfied with his achievement (line 273). The expression "runs in parallel" (line 260) is used to describe the motion when the two lines remain parallel during dragging. He does not justify his prediction referring to dependency relations or geometrical properties induced in the construction process; hence we cannot claim that he has reached a geometrical interpretation. However, his words and his gestures witness the emergence of awareness of hierarchical dependencies. This is not the case for Dan. Though the workload is shared, Jan is the one who tries to understand and solve the tasks, while Dan just writes the answers. Dan finds the tasks difficult and does not seem to understand much of what is going on (line 132, 265).

## 5 Discussion

We see in task 1d, that both pairs of students do not immediately expect the relations between elements to be hierarchical. In the case of Sif and Ole, the conflict between what they predicted and what they experienced in task 1d, may be seen as the starting point of a chain of events, where they act, observe, and become aware of the hierarchical nature of dependencies in a DGE. In task 3, we see Sif, in particular, indicating her awareness of the hierarchical dependency relation between lines f and g. In general, Sif and Ole´s ability to interpret the dynamic representation geometrically is demonstrated in ascending moves (Arzarello et al., 2002), from their perceptual experience to geometrical properties, i.e., they explain what they observe on the basis of geometrical properties. We view these as signs of development of representation competency, and such a competency seems to become more consolidated and evolve further during the three tasks, till it encompasses hierarchical dependency relations in DGE. The case indicates that dependency tasks can support students in interpreting dynamic figures, both at a basic level, where the students´ can interpret dynamic dependency, and at a more advanced one, that they can interpret dynamic dependency geometrically.

Dan is not actively participating in trying to solve the tasks and seems to have no comprehension of the functioning of the program or of the geometry that is embedded in it. Instead, Jan projects properties onto the construction, which are not there (expected A to be the midpoint of B and vice versa). We could argue that Jan is just starting to develop an interpretive frame; however, even if he is trying to interpret what he observes and uses geometrical terms, his interpretation does not seem to be anchored in geometry. This case shows that interpreting dynamic representations is a non-trivial task, and that even if the DGE task design is an important constituent of supporting this awareness, the task design and the technology that is exploited is, in and of itself, not sufficient. To support this pair of students, it would be important that the teacher intervenes to make students aware of this geometrical interpretation, as pointed out by Arzarello et al. (2002) "the teacher plays a very important role in students' approach to theoretical thinking. Technology itself cannot bring about an educational change" (p. 71).

The cases that we presented show examples related to the use of a specific digital technology, a DGE, with the aim of characterizing what the development and possession of representation competency constitutes in the context of graphical dynamic geometrical representations of geometric objects. However, we consider this to be only the first step in developing representation competency, in accordance with our main claim about the fundamental role of geometrical knowledge in dealing with representations. The second step involves dealing with graphical dynamic geometric representations of any mathematical object.

If we attempt to generalize our example to graphical representations of any mathematical object, which is the research aim that we set out to explore, then this ability resonates with what Dreyfus (1994) coined as "visual reasoning", which describes the ability to reason analytically about visual images, in "visually-based analytical thought processes" (p. 109). Sif and Ole are able to reason analytically about the dynamic figure, referring to geometric properties in their prediction and to explain any observed phenomenon appearing on the screen. According to Dreyfus, visual reasoning can be conducive in learning mathematics, however, it "is based on expertise—it will be unhelpful if not impossible for the uninitiated" (p. 116). This is the case with Dan and Jan, who clearly need the intervention of the teacher.

While our cases refer to geometric objects, other studies deal with what may typically be thought of as non-geometrical mathematical objects that are represented geometrically, e.g., if we return to Dreyfus' (1994) paper, he refers to students' successful development of "visual reasoning" in Artigue's (1989, as cited in Dreyfus, 1994) study, in which the students work exclusively with graphical representations, studying differential equations "based on reasoning with functions that are not given explicitly by a formula" (Dreyfus, 1994, p. 117). The students successfully manage to "infer graphical information about curves from graphical information about their derivatives" (p. 117).

## 6 Networking Representation Competency and Other Theoretical Constructs

The KOM frameworks description of representation competency plays an important theoretical role in this chapter; however, we have also adopted theoretical constructs external to the KOM framework.

Reflecting on the functioning of these constructs, we can recognize their importance in relation to different aspects of our analysis. Some of the constructs relate to the specific context of DGE and its potentialities, e.g., dynamic dependency and dependency tasks (Højsted & Mariotti, 2020) or the relationship between perceptual and theoretical aspects of a diagram (Arzarello et al., 2002; Laborde, 2005). Other constructs are used to elaborate on the fundamental role of geometrical knowledge in dealing with representations, e.g., geometric intuition (Alexandrov, 1994), while some contribute to a fine-grained analysis of what it means to possess representation competency in the context of digital representations of mathematical objects, e.g., the possession of a "mathematical eye" (Mariotti & Baccaglini-Frank, 2018) or abilities in visual reasoning (Dreyfus, 1994). Additionally, we can add our own theoretical contribution comprising the hypothesis of two steps in the representation competency in relation to digital representations of mathematical objects. The first step concerns the ability to interpret and use representations of dynamic geometrical objects, which we argue involves two layers of complexity, firstly, being able to interpret dynamic dependency, and secondly, being able to interpret dynamic dependency geometrically, i.e., as a logical dependency between geometrical properties. The second step concerns the ability to interpret and use representations of non-geometrical objects through a geometrical interpretation.

From a networking of theories point of view, the theoretical constructs used seem conducive to elaborate on the broad description of representation competency that is provided in the KOM framework, in order to fit the specific context of digital representations of mathematical objects.

## 7 Conclusion

Our contribution focuses on the general issue of representation and in particular on the ability needed for using and interpreting representations of mathematical objects, which is a part of mathematical representation competency. We set out to identify what characterizes development and possession of representation competency in the specific context of digital representations of mathematical objects. We hypothesize two steps of this competency, a more basic one, which is ability to interpret and use dynamic representations of geometrical objects. We have argued that this first step comprises at least two layers of complexity, (1) being able to interpret dynamic dependency, (2) being able to interpret dynamic dependency geometrically. The identification of the first step is supported by our empirical examples, in which Sif and

Ole show signs of possessing this aspect of representation competency to some extent, and we found that working on the specific dependency tasks seems to support further development: an emerging awareness of hierarchical dependency relations appears. This pair of students is able to use and interpret dynamic geometrical representation of geometrical objects. This first step can be considered as a bridge toward achieving the second step, which is the ability to interpret geometrically the use of graphic representations for non-geometrical objects.

## References


Alexandrov, A. D. (1994). Geometry as an element of culture. In D. Robitaille, D. H. Wheeler, & C. Kieran (Eds.), *Selected lectures from the seventh international congress on mathematical education* (pp. 365–368). Les Presses de l'Université Laval.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt Für Didaktik Der Mathematik, 34*(3), 66–72. https://doi.org/10.1007/BF02655708

Balacheff, N. & Kaput, J. J. (1996). Computer-Based Learning Environments in Mathematics. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, C. Laborde (Eds.) *International handbook of mathematics education. Kluwer international handbooks of education, vol. 4.* (pp. 469–501). Springer. https://doi.org/10.1007/978-94-009-1465-0_14

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining-dragging model. *International Journal of Computers for Mathematical Learning, 15*(3), 225–253. https://doi.org/10.1007/s10758-010-9169-3

Duval, R. (2006). A cognitive analysis of problems of comprehension in the learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Dreyfus, T. (1994). Imagery and reasoning in mathematics and mathematics education. In D. Robitaille, D. Wheeler, & C. Kieran (Eds.), *Selected lectures from the 7th international congress on mathematical education* (pp. 107–122). Les presses de l'université Laval.

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics, 38*(1), 11–50. https://doi.org/10.1023/A:1003488222875

Goldin, G. A., & Janvier, C. (1998a). Representations and the psychology of mathematics education: Part I. *The Journal of Mathematical Behavior, 17*(1), 1–134.

Goldin, G. A., & Janvier, C. (1998b). Representations and the psychology of mathematics education: Part II. *The Journal of Mathematical Behavior, 17*(2), 135–301.

Healy, L., & Hoyles, C. (2002). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning, 6*, 235–256. https://doi.org/10.1023/A:1013305627916

Hölzl, R., Healy, L., Hoyles, C., & Noss, R. (1994). Geometrical relationships and dependencies in Cabri. *Micromath, 10*(3), 8–11.

Højsted, I. H., & Mariotti, M. A. (2020). *Analysing signs emerging from students' work on a designed dependency task in dynamic geometry.* Unpublished manuscript.

Janvier, C. (Ed.). (1987). *Problems of representation in the teaching and learning of mathematics.* Lawrence Erlbaum Associates.

Kaput, J. J. (1998). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *The Journal of Mathematical Behavior, 17*(2), 265–281. https://doi.org/10.1016/S0364-0213(99)80062-7

Laborde, C. (2002). Integration of technology in the design of geometry tasks with cabri-geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317. https://doi.org/10.1023/A:1013309728825

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero (Eds.) *Meaning in mathematics education. Mathematics education library (Vol. 37).* Springer

Leung, A. (2015). Discernment and reasoning in dynamic geometry environments. In S. Cho (Ed.), *Selected regular lectures from the 12th Iiternational congress on mathematical education* (pp. 451–469). Springer.

Mackrell, K. (2011). Design decisions in interactive geometry software. *ZDM, 43*(3), 373–387. https://doi.org/10.1007/s11858-011-0327-4

Mariotti, M.A., & Baccaglini-Frank, A. (2018). Developing the mathematical eye through problem-solving in a dynamic geometry environment. In N. Amado, S. Carreira & K. Jones (Eds.) *Broadening the scope of research on mathematical problem solving. Research in mathematics education.* Springer. https://doi.org/10.1007/978-3-319-99861-9_7

Morgan, C., Mariotti, M. A., & Maffei, L. (2009). Representation in computational environments: Epistemological and social distance. *International Journal of Computers for Mathematical Learning, 14*(3), 241–263. https://doi.org/10.1007/s10758-009-9156-8

Niss, M. A. (2016). Mathematics standards and curricula under the influence of digital affordances: Different notions, meanings and roles in different parts of the World. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 239–250). Information Age Publishing. Research in Mathematics Education.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Oldknow, A. (1997). Dynamic geometry software—A powerful tool for teaching mathematics, not just geometry! In W. Fraunholz (Ed.), *Proceedings of 3rd international conference on technology in mathematics teaching*. University of Koblenz.

Talmon, V., & Yerushalmy, M. (2004). Understanding dynamic behavior: Parent-child relations in DGE. *Educational Studies in Mathematics, 57*(1), 91–119. https://doi.org/10.1023/B:EDUC.0000047052.57084.d8

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior, 17*(2), 167–181.

White, R. T., & Gunstone, R. F. (2014). *Probing understanding*. Routledge.

# New Demands on the Symbols and Formalism Competency in the Digital Era

**Linda Marie Ahl and Ola Helenius**

## 1 Introduction

When Russell and Whitehead published the first edition of *Principia Mathematica* in 1910–1913, the interest in the roles of symbols and formalism in mathematical reasoning was greatly revived. Russell and Whitehead were unsatisfied with how elementary mathematics like whole numbers relied on intuitive reasoning. They thought that unless mathematics is put on completely formal grounds, the ugly head of self-reference might always lurk in the background and allow false statements to be proved which would be mathematical blasphemy. To rectify the situation, they developed a finite symbol system where every admissible inference was explicitly described. This allowed elementary arithmetic to be developed without any reference to real-world inferences whatsoever (Russell & Whitehead, 1973). The core of their contribution is an extreme use of mathematical symbol systems and formalism.

The story of Principia Mathematica serves as an important backdrop because it displays that advanced mathematical reasoning is possible in purely symbolic systems. But when we learn mathematics, we have initially almost no access to the symbol systems we end up using. How we initially understand the symbol systems has to be grounded in the experiences we have access to, and becoming competent in reasoning with the help of symbol systems has to be learned. But what does it mean to master mathematical symbol systems and formalism and how does the developmental progression look? One way of speaking about this is in terms of mathematical competence: "Mathematical competence is someone's insightful readiness to act

L. M. Ahl (✉)
Uppsala University, Uppsala, Sweden
e-mail: linda.ahl@edu.uu.se

O. Helenius
University of Gothenburg, Gothenburg, Sweden
e-mail: ola.helenius@ncm.gu.se

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_9

appropriately in response to all kinds of mathematical challenges pertaining to given situations" (Niss & Højgaard, 2019, p. 12). Within this view, a major constituent of mathematical competence is the *competency of handling mathematical symbolism and formalism* (Niss & Jensen, 2002). The framework for mathematical competence (hereafter the KOM framework) may be used for designing curricula on any level of mathematics education (Niss & Højgaard, 2019). However, to be useful in teaching practice, the competency description requires thoughtfully designed activities, based on an idea of progress in students' knowledge. Since no description of such developmental progress is covered by the KOM framework, we need additional theory to put the competencies to work.

In this chapter, we will argue that while the competence framework from Niss and Jensen provides a powerful way of viewing mathematical symbolism and formalism, the framework lacks features needed to explain how a learning progression involves becoming a competent user of mathematical symbolism and formalism might look. We will also argue that an elaborate understanding of the symbolism and formalism competency is particularly important in the digital era. Our aim is therefore threefold: first, to argue for the possibility to network the competence framework with theories from Vergnaud as well as with an elaborated view on how symbol systems are constructed; second, to argue for a way of thinking about progress in conceptual knowledge based on an epistemological shift from meaning-making in situations and iconic representations to meaning-making in mathematical symbol system; third, to exemplify how the ways of working associated to the digital era put new demands on the symbols and formalism competency.

The outline of the rest of this chapter is as follows. First, we will elaborate theoretically on symbol systems. We will then move on to present a theoretical account of how symbol systems can emerge from intuitively accessible forms of mathematical reasoning. We develop this theoretical account by building on work by Vergnaud which we refine by merging it with the symbol system theory. We will then go on to briefly introduce the framework for mathematical competence (hereafter the KOM framework) developed by Niss and Jensen and describe why it makes sense to network the KOM framework with the Vergnaud-based work. Thereafter, we carry out this networking to elaborate on the symbols and formalism competency from a learning progression point of view. As the title promises, we will also problematize new demands on the symbols and formalism competency in the digital era. Therefore, we examine an example from elementary programming in a block-based programming environment and explain how it challenges the relationship between symbolism and intuition in elementary school mathematics. We close with a conclusion concerning new demands on the symbols and formalism competency in the digital era, namely that the same class of concrete situations can be conceptualized and formalized in competing ways in block programming code versus mathematics.

## 2 The Nature of Symbol Systems

A symbol can be any kind of mark, inscription, gesture, sound, etc., that is understood to stand for something else, like an object, relationship, or idea. In semiotics, the totality of *something standing for something else* is called a sign. A sign is composed of a signifier that stands for or denotes a signified. Signs, in general, can have natural causes like thunder is a sign for the presence of lightning or a particular form of buzzing sound can be a sign of a signified mosquito (Saussure, 1974). Signs, in general, can also have an iconic relationship between the signified and the signifier, like how ☺ can denote a smiling human face. The concept of a symbol, in most traditions, is taken to mean a token of human construction that has no causal relationship with what it stands for in opposition to signs where the signifier has an iconic relationship to the signified (Brandon & Hornstein, 1986). If the physical token IIII appears in a discussion about quantities, you will have a good chance of figuring out what it might mean even if you did not see the physical token before. The token IIII is iconic relative to its meaning as signifying a quantity. We can, for example, directly manipulate the token IIII and obtain I and III, signifying how the quantity four can be constructed from one and three. There is no corresponding way of manipulating the token 4 to obtain any information on the quantity it represents because it is not iconic. If the token 4 appears, you have no chance of drawing inferences about its quantitative meaning unless someone told you what the token is supposed to stand for or used the token 4 in some communicative circumstance where its meaning could be inferred. The form of the token 4 has no relation to its meaning except through discursive association.

So, while a symbol is characterized by being a token with no physical connection to what it is set to signify, in symbol *systems,* physical manipulation of the placement of symbols in relation to other symbols regularly shifts the meaning in what is symbolized in specified ways. We will exemplify after a formal description of how a symbol system can be understood. The description we will use comes from Harnad (1990), and according to him and the ones he builds on, a symbol system is:

1. a set of arbitrary physical tokens that are
2. manipulated on the basis of "explicit rules" that are
3. likewise physical tokens and strings of tokens. The rule-governed symbol-token manipulation is based
4. purely on the shape of the symbol tokens (not their "meaning"), i.e., it is purely syntactic and consists of
5. "rulefully combining" and recombining symbol tokens. There are
6. primitive atomic symbol tokens and
7. composite symbol-token strings. The entire system and all its parts—the atomic tokens, the composite tokens, the syntactic manipulations both actual and possible and the rules—are all
8. "semantically interpretable": The syntax can be systematically assigned a meaning (e.g., as standing for objects, as describing states of affairs) (Harnad, 1990, p. 336).

Mathematics is full of symbol systems satisfying these rules. Most well-known is probably the base ten position system for expressing numbers. The standard way of expressing definite integrals, the standard way of expressing function with the notion $f(x)$, coordinate systems, and the associated $(x, y)$ notation to denote points in the plane are other examples. Mathematical symbol systems are regularly intertwined. For example, in the notational system for functions, we write $f(10)$ to express the value of the function $f$ at $x = 10$. We then use the base ten positions system. Symbol systems can also build on each other. For example, the decimal system recycles the base ten position system. It adds only the decimal point symbol and extends the place value evaluation system to include negative powers of 10.

Let us look a little closer at how whole number arithmetic is extended to rational number arithmetic through a symbol system extension (Mendelson, 2008). Again, it is done by adding only one single atomic symbol, $\div$ or /. Whole number arithmetic is symbolized with the ten digits, $+$, $-$, $=$, and $\cdot$. We then introduce / by saying that $a/b$ should have the same role in the symbol system as a symbol $c$ for which $a = b \cdot c$ holds. Note that this is a rule expressed without /, that is, in the symbol system we already had. We can then go on to set up the substitution rule $a/b = c/d$ if $a \cdot d = b \cdot c$ and the transformation rules $(a/b) \cdot (c/d) = (a \cdot c)/(b \cdot d)$ and $(a/b) + (c/d) = (a \cdot d + c \cdot b)/(b \cdot d)$. After checking a set of conditions that ensure that the syntax of our new extended symbol system is well defined (something that takes a few pages in a standard book in abstract algebra or elementary number theory and also includes taking special care of 0 and 1) we can now operate in our new symbol system. Effectively, we added only one atomic symbol token and some manipulation rules for it that all fell back on the existing rules of the previous system. After doing this, we can formulate a relatively short list of conditions for which sentences are allowed (e.g., $a/(b + c)$ is allowed, but $a/{+}b\cdot$ is not allowed) and which substitution and transformation rules exist. Then, we can operate with $a/b$. We should note that we did not follow the definition of the symbol system above fully, as we did not express all rules in terms of tokens with no meanings as point 4 states. Going down that route would be to go full Principia Mathematica and since it takes Russell and Whitehead 200 pages to reach the conclusion that $1 + 1 = 2$, such a demonstration would not fit this chapter.

By defining the rules of the new system, like above, we get a new system that satisfies points 1–7 in the symbol system definition, mostly by inheritance from the previous system. In particular, the symbol / was introduced with no external meaning being assigned to it (fulfilling point 4). But what about point 8, the requirement of the new system to be semantically interpretable? One interpretation would be that since we already had additive inverses for all numbers in the whole number system, we would perhaps like to also have multiplicative inverses. The symbols $a/b$ give us exactly that because by using the rules we introduced we can show that for any $a$ (except 0) $a \cdot 1/a = a/a = 1$, and any $a/b$ ($a$, $b$ not 0) $a/b \cdot b/a = 1$, so the new elements have inverses too. So, we have created a new object where we can both add, subtract, multiply, and divide freely; a *field* in mathematical terms, more specifically the field of rational numbers.

In school mathematics, however, the symbol *a*/*b* is not introduced in the above way at all. Instead, *a*/*b* is used to denote several different but related situations, like equal sharing, equal grouping, and part-whole relationships, i.e., division and fractions. The manipulation rules are rather derived from the situations the symbol is set to signify than from the rules of additive and multiplicative arithmetic. Why will the manipulation rules derived from these different situations match up and why will they match up with the symbolically derived manipulation rules? The truth is that they will not match up *exactly* and a principal difficulty in acquiring competence in dealing with mathematical symbols and formalism comes down to dealing with this mismatch. This manifests the complicated relationship between mathematical meaning-making and mathematical symbolism. We will introduce theories from Vergnaud to explain this relationship further.

## 3 The Symbol Grounding Problem

How symbols get their meaning is a problem in philosophy and cognitive science alike: The symbol grounding problem (Harnad, 1990). To date, there is no conclusive theory for how abstract symbols acquire meaning in the mind. Even in the case of numerical cognition, there are competing theories (Leibovich & Ansari, 2016). But for the acquisition of school mathematics, we can give a pretty good description of how symbols usually acquire their meaning.

We base our reasoning on previous work (Ahl & Helenius, 2021a) based on theories by Vergnaud (1998, 2009). Vergnaud's theorization is characterized by an interest in the roles enunciation and symbolization play in conceptual development. Vergnaud pays great attention to the knowledge that is not yet symbolized and cannot be expressed other than in action. Building on Piaget, Vergnaud calls knowledge in action *operative knowledge* and knowledge that can be expressed in words and sentences or other symbolic forms for *predicative knowledge*. To theorize operative knowledge, he uses Piaget's concept of schemes, *the invariant organization of behavior for a certain class of situations* (Vergnaud, 1998). We will not go further into scheme theory in this chapter, but invariants and classes of situations are fundamental for our theorization. An invariant is something that stays the same as the situation varies within its class. For example, situations where you have two sets of discrete objects and decide to view them as one set, form a class. The "putting together" idea is invariant across such situations and can form a basis for the mathematical operation addition.

Armed with the idea of invariants and classes of situations and their interdependence, Vergnaud views a concept as a triplet $C = C(S, I, R)$, the set of situations, $S$, where the concept is meaningful, the invariants, $I$, and the representations, $R$, that can be used to represent invariants and situations (Vergnaud, 1997). The three elements should be thought of as intertwined in how concepts are psychologically structured and used. From an analytical point of view, it makes sense to separate them though. The invariants that form the core of a concept will on one hand manifest their meaning in situations and on the other hand in representations. We categorize representations in two forms, namely iconic representations and non-iconic representations which are almost always elements of symbol systems. It is important to note that situations and iconic representations can be grouped together in the sense that meaning-making can be based on direct experience, while the meaning of symbol systems can only be picked up from discursive inferences. From a symbol grounding perspective, the meaning of a symbol can either be grounded in relations in the symbol system, it can come from invariants in a situation, or invariants in iconic representations (Ahl & Helenius, 2021b). For example, the concept of a simple closed continuous curve in two-dimensional space can come from the situation of a string tied into a loop on the floor not crossing itself. In a coordinate system on paper, we can draw curves not intersecting themselves and let them end where they started and have an iconic representation of a simple closed continuous curve. Formally, we can define a simple closed curve in the plane by saying that it is an injective continuous map $\Phi : S^1 \to \mathbb{R}^2$ from the circle to the plane. Similarly, subtraction could be based on the classes of situations where you have some number of things and take some away. It can be based on the iconic images in Fig. 1 or $a - b$ can be defined by relations in symbol systems: as the number $c$ such that $a = b + c$.


**Fig. 1** Iconic representation of difference and hence subtraction constructed from the idea of two parts making up a whole. The whole is represented by the top bar, and the bottom row represents two parts making up the whole, with one missing—the difference. Three variants are displayed. A representation building on unit squares that you can count, a representation where the magnitudes are represented by the lengths or the bars and the numbers on them, and a generalized model where only the numbers on the parts give information on the quantities involved

Vergnaud emphasizes that schemes and their invariants can never be faithfully represented in symbol systems or by any other semiotic means (Vergnaud, 1998). We will exemplify. The symbol system *a*/*b* and its composition rules and additive manipulation rules are typically introduced by using two similar but nonidentical iconic part-whole representations. In Fig. 2, most would agree that the iconic representations in column A all represent 3/4 since three out of four items are colored. The form of these items is not considered to impact the meaning, as long as all forms are the same. And most would agree that the iconic representation in column C represents 3/2. One whole and one half, so together three halves. But what about column B? Depending on how we choose to look at it, it can be 3/4 or 3/2. This is disturbing since the elements in column B are just rigid transformations of either the bottom elements in column A or the elements in column C. So, there can be no rigid rule that maps either the representations of type A or the representations of type C onto the symbol system *a*/*b*.

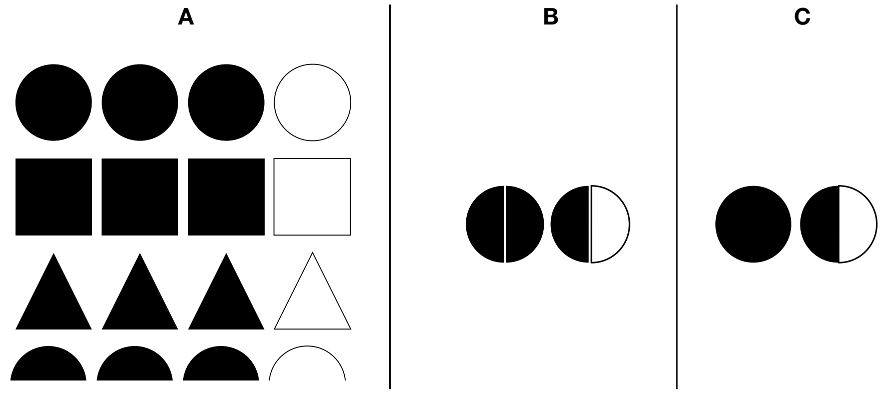



**Fig. 2** 3/4 in column A, 3/2 in column C, and both 3/4 or 3/2 in column B

While we often may want to believe that symbol system manipulation and composition rules are some kinds of crystallization of situations or iconic representations, it is rarely if ever so simple. Duval remarks that "a mark cannot function as a sign outside of the semiotic system in which its meaning takes on value in opposition to other signs within that system" (Duval, 2006, p. 110). But, as long as concepts are based on iconic representations, the signs can very well take on a meaning which is in opposition with other signs. The figure in column B, seen as a sign signifying some *a*/*b*, can represent both 3/4 and 3/2. Duval continues and claims that all mathematical symbol systems are based on rules of representation formation (Duval, 2006). This is true in the sense that within the symbol system itself, there must be clear evaluation and manipulation rules, but when it comes to what a symbol system represents in terms of different iconic representations or situations, the rules are not necessarily clear. The same symbol from the same system can have several different and partly conflicting representations in the form of invariants in situations or iconic representations. We have in previous work expressed this by saying that mathematical symbols are conceptually polysemic. They represent several but related meanings (Ahl & Helenius, 2021a, 2021b).

We also claim that in conceptual development, concepts are often first introduced by observing invariants in situations and iconic representations (Ahl & Helenius, 2021a). The symbols initially work as labels, and the symbol system manipulation rules are derived from invariants in the situations or iconic representations. But at some point, an epistemological shift has to occur. The main meaning-making has to be considered to be in the symbol systems and the relation they entail. Mathematical symbol system rules are never allowed to include contradictions and therefore act as an arbiter concerning which invariants in associated situations and iconic representations should count as valid. By acting on the arbiter's judgment, more fine-grained rules for interpreting situations and iconic representations must be invented to reinstall some elements of uniqueness. In the part-whole case displayed in Fig. 2, we can observe that the idea of parts and wholes will not suffice to sort out the contradictions.

But if we introduce the idea of a unit, we are on solid grounds again. If we decide the unit is one circle, column B represents 3/2. If we decide the unit is a half-circle, column B represents 3/4. Hence, the contradiction in symbol system representation led us to refine the invariants, making them on one hand more precise but on the other hand less intuitive and more reliant on refined explanations.

## 4 Networking KOM and the Symbol Grounding Theory into a Coordinated Conceptual Framework

The Danish KOM project (Niss, 2003; Niss & Jensen, 2002) is a prominent way of trying to explicate what knowing in mathematics might mean. The main tenet in this view of mathematical knowledge is that mathematical competence in general, that is, the ability to involve mathematics to deal with a range of situations or contexts, can be subdivided into overlapping but distinct mathematical competencies and that mathematical competence, in general, is constituted by the set of such competencies. We will argue why it is possible to coordinate the KOM framework with the Vergnaud-based theories we have presented above. We will use theory as a structured set of lenses (Niss, 2007) through which the phenomena of handling mathematical symbols and formalism may be investigated. By networking theoretical constructs from the KOM framework (Niss & Jensen, 2002) with theoretical constructs from Vergnaud's theories (e.g., Vergnaud, 1998, 2009), we aim to gain a deeper insight into the phenomena in contrast to what could be observed and explained with one theory alone. The way we network is through coordinating constructs. The possibility to coordinate theoretical constructs into a conceptual framework for analysis hinges on three crucial factors. First, the theories in the conceptual frameworks should be built by well-fitting elements (Prediger et al., 2008). Second, the philosophy that serves as the point of departure for observing the processes and objects in the respective frameworks should align. Third, the theories should differ somewhat in their focus, for example, by addressing different grain sizes.

The KOM framework and Vergnaud's theories have well-fitting elements for analyzing the symbols and formalism competency. Both the KOM framework and Vergnaud's theories, "The theory of conceptual fields" and "A comprehensive theory of representation", address student's learning, with a special focus on students' actions in situations. "Competence is someone's insightful readiness to act appropriately in response to the challenges of given situations." (Niss & Højgaard, 2019, p. 12) In a comprehensive theory of representations (Vergnaud, 1998), students' schemes, the invariant organization of behavior for a certain class of situations, are at the core for analyzing knowledge. The operational invariants, concepts-in-action, and theorems-in-action are the main concepts in the analysis which signals Vergnaud's analytical interest in knowledge in operative form. Similarly, "the competencies can be used as an analytic means for describing and characterizing the state of affairs concerning the competencies pursued (or not pursued) in a given segment of mathematics education." (Niss & Højgaard, 2019, p. 25) Further, Vergnaud theorizes and highlights the connections between students' actions in the operational form of knowledge and the predicative form of knowledge and to make that analysis he uses the ubiquitous operational invariants, the role of enunciation of invariants, and how they are constituted in action as well as in symbolic form. As we will show in our analysis this fits very well with the constitution of the symbol and formalism competency from the KOM framework.

The philosophy that serves as the point of departure for observing the processes and objects align, since both the competencies, the conceptual field, and the theory of representation are cognitive. While competence, in general, can be viewed as a continuum from cognitive traits to actual observed behavior (Blömeke et al., 2015), the KOM framework departures from a cognitive point of view in the characterization of what it means to be mathematically competent (Niss & Højgaard, 2019). Vergnaud draws mainly on Piaget in his theory building, but with impact from Vygotsky concerning the pivotal role of language. For Vergnaud, knowledge is adaptation in the Piagetian conceptualization. Development is understood as growth in student's conceptual fields by acquiring additional effective schemes to deal with certain classes of situations (Vergnaud, 2009). Schemes are certainly cognitive in nature. Thus, both theoretical perspectives we are networking with are concerned with knowledge in action and view that knowledge as a cognitive trait. This makes them philosophically compatible.

The theories differ in their grain sizes. While Niss and Højgaard's (2019) competency description describes the demands on students handling mathematical symbols and formalisms competency, without further explicating how this process is carried out, Vergnaud's theoretical constructs give us a lens to detail students' cognitive process when putting the competency to work. On a coarse grain size, the KOM framework gives us a lens for what to direct our attention to when analyzing the symbols and formalism competency. On a fine grain size, Vergnaud gives us theoretical concepts for analyzing students' actions with particular mathematical objects.

## 5 Symbolism and Formalism as a Mathematical Competency

One of the eight distinct competencies in the KOM framework is the mathematical symbol and formalisms competency. In the KOM framework, each competency has a receptive side, typically dealing with interpretation and understanding of mathematics that is already present in some circumstances, and a constructive side, typically dealing with employing or using some mathematics to deal with in some situation. We will spend this section on decomposing the symbols and formalism competency and will pay particular attention to the receptive and constructive sides. We use the description of the competency from a recent update by Niss and Højgaard:

> The ability to relate to and deal with mathematical symbols, symbolic expressions and transformations, as well as with the rules and theoretical frameworks (formalisms) that govern them, constitutes the key component of this competency. On the receptive side, this competency is to do with decoding and interpreting instances of symbolic expressions and transformations, as well as formalisms, already present, whereas the constructive side focuses on introducing and employing symbols and formalism in dealing with mathematical contexts and situations. (Niss & Højgaard, 2019, p. 19)

To be able to analyze what cognitive difficulties might be involved when developing the symbolism and formalism competency, we will decompose it into a schematic diagram (Fig. 3). The decomposition partly relies on analyzing the verbal description of the competency cited above and partly relies on coordinating this description with the theories based on Vergnaud's work and the symbol system theory we have introduced.

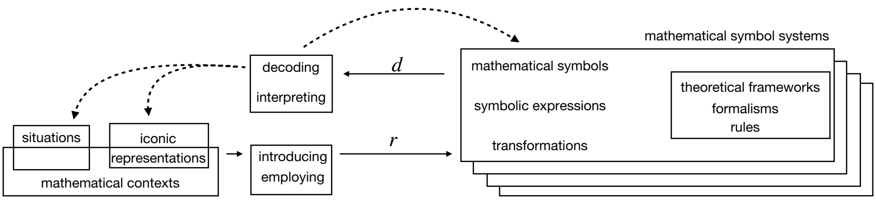



**Fig. 3** Mathematical symbolism and formalism competency decomposed

According to our description of symbol systems, we integrate the symbols and rules part of the competency description into the concept of a symbol system and display it in the box to the right. The multiplicity of such boxes indicates that several such systems might overlap and be in play at once. The arrow *d* indicates relating to the symbol systems by decoding or interpreting them. A question not dealt with in the KOM framework is what the symbols should be decoded into or interpreted as. In line with our previous reasoning, we claim that the target of the decoding/interpreting can essentially be of three types, indicated by dotted lines in Fig. 3: situations, iconic representations, and (possibly different) symbol systems. We also note that what is labeled as mathematical contexts and situations in the KOM framework largely overlaps with what we have dealt with as situations and iconic representations which explains why we overlap these boxes in our schematic description. The arrow *c* represents when elements in some mathematical contexts (situation or iconic representation) are understood by employing or introducing symbol system elements to denote or label them.

In the KOM description of the symbols and formalisms competency, it is hard to sense what progress in this competency might mean. In general, the KOM report handles progress in terms of three dimensions: the degree of coverage that concerns how much of the competency the person masters; the radius of action that covers the different mathematical subfields where the person can apply the competency; and the technical level concerning the complexity level within some field at which the person can act out the competency (Niss & Jensen, 2002). While these dimensions can help in sorting out to what degree a person possesses a competency, they say practically nothing on possible developmental progressions. Vergnaud's theory of conceptual fields, on the other hand, is specifically developed to understand progression on a medium and long-term basis, including, as we have seen, the role of symbolism. For example, it follows that the operative form of knowledge typically precedes the predicative form. The meaning of some symbolism is typically first grounded in invariants in classes of situations and iconic representations. That means that in Fig. 3 the arrow $c$ initially has a role of enunciating invariants and using invariants from multiplicities of situations and iconic representations to motivate symbol system manipulation and interpretation rules when a new conceptual field with new notations is encountered. At this point, the decoding/interpretation (arrow $d$) functions as inverses of the enunciation rules. But later, as we have argued, the multiplicities of symbol system rules will overrule certain invariants in the situations and iconic representation and how the icons and situations should be mathematically interpreted. So, the understanding of the relevant invariants in situations and iconic representations has to be refined.

In summary, the formulation of the symbolism and formalism competency together with the three dimensions of evaluation gives a good basis for understanding what the competency is about and for evaluating it. By coordinating the competency with the Vergnaud derived theories, we also gain access to a compatible theory of the development of the competency. The key point is the complicated relationship between situations/iconic representations and symbol systems and their interchanging roles for acting as the grounding factor. We will now move on to apply these insights by analyzing how the introduction of digital tools might affect the development of the symbolism and formalism competency.

## 6 Challenges in the Digital Era

In the KOM framework, the handling of digital tools is handled by the *aids and tools competency*. In this section, we show that the inclusion of digital tools into the mathematics classroom can also have profound effects on the symbol grounding problem and hence the development of the symbol and formalism competency. On one hand, the operation of the tools themselves may require specific formalism, but more interesting for us is that digital tools might also give access to different situations and iconical representations that may, or may not, support the creation of some established mathematical meaning (Jankvist & Misfeldt, 2015; Jankvist et al., 2019).

In this section, we will explore how the inclusion of digital tools may influence the symbol grounding problem and hence the symbolism and formalism competency. Our exploration will be guided by a classic geometrical task and how it might be solved by creating a program in the programming environment Scratch. The task is given in Fig. 4.

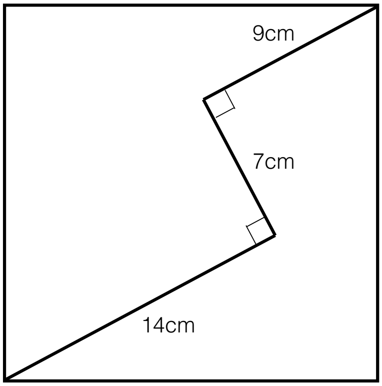



**Fig. 4** What is the length of the side of the square?

Without digital tools, a standard solution in geometry could be to realize that we only need to know the length of the diagonal since the length of the side can then be obtained by dividing by $\sqrt{2}$. By drawing the diagonal and using the properties of similar triangles, you can then set up some equations involving ratios and the Pythagorean theorem and eventually find the length of the diagonal. Such a solution would incorporate formalism in labeling sides in triangles, setting up relationships, solving equations, and so on, but not formalism related to coordinate geometry or vector geometry.

What can a solution in Scratch entail? We will discuss this question by analyzing a situation that Scratch code codifies. How could you walk the solution to the problem on a field in real life? Like this: Mark out any point where you happen to stand. In any direction, you happen to have your nose pointing, walk 14 steps, turn 90° left, walk

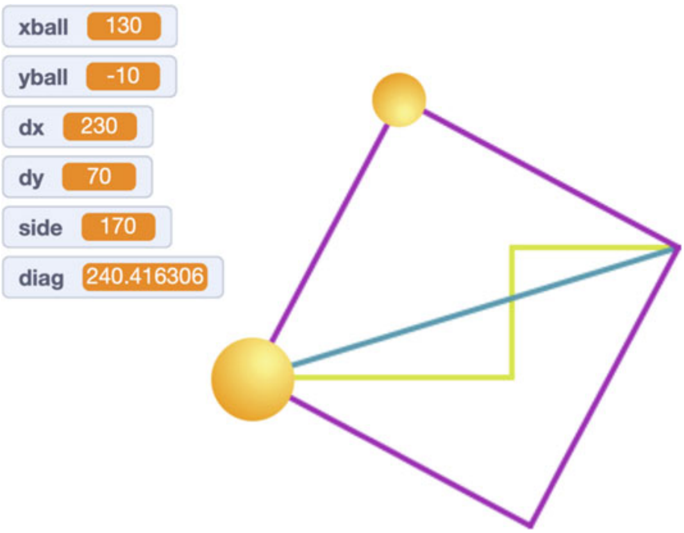



**Fig. 5** Result of running a Scratch program

7 steps, turn 90° right, walk 9 steps. Turn toward your original mark and count the steps back. You now know the length of the diagonal and can calculate the side length as before. Let us call what we just described a walking and turning (wt) situation. A central part of the Scratch block programming environment is a codification of the wt-situation type by commands like move? steps and turn? degrees. A virtual object, a so-called sprite, can be controlled on the Scratch screen by such commands and can also have its path drawn. Appendix shows a Scratch program where our walked solution is operationalized. The result is in Fig. 5.

Let us now analyze wt-situation, screen drawings, Scratch code, and corresponding mathematics in terms of situations, iconic representations, and formalism. First, the wt-situation is operationalized in Scratch formalism and produces Scratch screen iconic figures from the coded movements. Potentially, the wt-situation and related code could be related to vector geometry formalism, which is, however, not normally a part of elementary schooling. Some of the Scratch code deals with coordinates in terms of $x$ and $y$ values, and the Scratch screen has an underlying coordinate structure. This can on one hand be seen as a code version of mathematical coordinate formalism, albeit in an implicit way since our Scratch screen does not include any visible coordinate axes. On the other hand, potentially, both the mathematical coordinate formalism and the Scratch coordinate formalism could be grounded in an everyday situation, based on coordinate style naming of locations on a field. The finished drawing on the Scratch screen could also be seen as an iconic version of standard Euclidean style geometrical formalism. There is, however, no real way of creating a drawn square (or any other figure) except going through coordinate or movement Scratch code formalism. Therefore, exploring shapes through Scratch would have to go through introducing coordinate or movement style formalism, either in Scratch code or in Scratch code as well as in mathematical formalism. In Fig. 6, we have schematized all these connections. Solid lines represent connections that we consider manifested in typical lessons with where Scratch is used to draw figures by

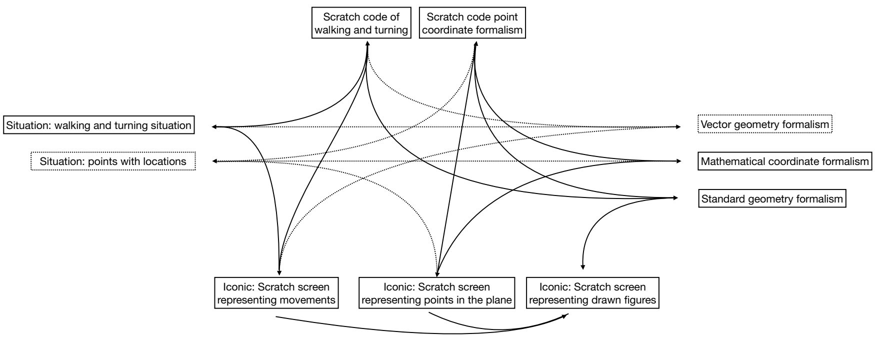



**Fig. 6** Points and movements situations, mathematical and Scratch formalisms; scratch screen iconic imagery; all interacting

letting a sprite move on the screen. Connections that we consider unusual for elementary school, but that potentially could be included, like for example vector geometry, are represented by dotted lines. Note in particular that not all possible connections have lines. For example, there is no connection between Scratch screen movements and standard geometry formalism, because such geometry cannot represent movement.

With this example, we want to illustrate that the introduction of the programming environment does not only introduce a new type of formalism. The programming environment also introduced new situations and iconic representations. For similar situations, different invariants might be chosen for operationalization in coding versus in mathematics. For example, in Scratch when a sprite represents some positions, it does not only represent the coordinates but also a direction, but neither in vector geometry nor coordinate geometry can a point have a direction. It should also be noted that in standard school geometry, and the figure in Fig. 4 is treated with no reference to either coordinates or vector geometry. The figure produced through Scratch is the same, but cannot be produced *without* the code versions of coordinates and vectors. Moreover, the complexities that are the results of different situations and iconic representations grounding the same symbol system might be multiplied because the programming introduces even more situations and iconic representations that may need to be considered. In a purely mathematical setting, we claimed that the symbol system should eventually serve as an arbiter and clear up how invariants need to be interpreted. But this role might be challenged when program code formalism is introduced in a way that operationalizes invariants in situations differently than the mathematical formalism does, like in the case of a "point" having direction. While all new connections might create new ways of introducing, explaining, and applying mathematical matters, the didactical challenges should not be underestimated.

In terms of the symbols and formalism competency, new questions arise. Should the competency include employing mathematical formalism to denote elements inside a computer program or from the product of a computer program? Should it include interpreting mathematical symbolism in terms of computer code or in terms of iconic representations, or situations that are related to the program code? This is far from the same problem dealt with by the aids and tools competency, where the focus is on using, for example, programming to deal with mathematical matters. In the case we have presented, the programming environment introduces a formalism world of its own as well as new types of situations and iconic representations to be dealt with.

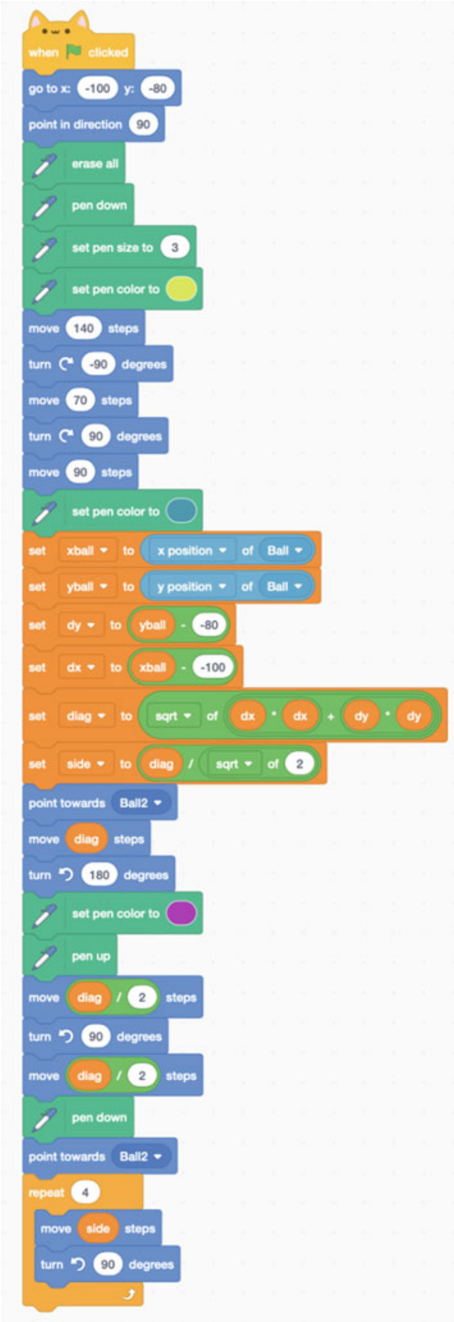
## 7 Conclusion

We have shown that the symbolism and formalism competency is deeply related to issues of mathematical conceptualization and meaning-making. From the developmental, Vergnaud-based theoretical perspective we employed, the symbolism and formalism competency will have different roles depending on how mature a concept is within an individual because the degree to which meaning-making resides in symbol system relations varies. By networking the symbolism and formalism competency with the Vergnaud-based theory, we could decompose the competency. On one hand, we got a more comprehensive and holistic description of how concepts like symbolic expressions, transformations, theoretical frameworks, and formalism used to describe the formal side of the competency stick together. These concepts all have a place under the symbol system theory umbrella. On the other hand, we could also describe how the concepts of decoding symbols and employing symbols used in the description of the symbolism and formalism competency can be interpreted and show that there are three distinct types of targets for such decoding, namely situations, iconic representations, and (possibly other) symbol systems. These results in some sense mimic the analysis made by Duval (2006) who discusses transformations within and between registers and characterizes registers as either monofunctional or multifunctional and as discursive or non-discursive. Symbol system, in the sense we have used the term, is mostly covered in the category of monofunctional discursive representations and iconic representations could be either mono- or multifunctional non-discursive representations in Duval's categorization. Situations, as we have used the concept, are not really represented in Duval's works. While some of our analysis has counterparts in Duval's work, we think the categorization into situations, iconic representations, and symbol systems are probably easier to think about for teachers and hence more applicable for discussions of teaching, in particular in relation to the symbolism and formalism competency and how it can be taught.

In the second part of the chapter, we analyzed new demands on the symbolism and formalism competency raised by the inclusion of digital tools, in particular block programming. Our analysis indicates that the symbolism and formalism competency must be developed alongside a progression where the meaning of concepts shifts from residing in situations and iconic representations to residing in relations in symbol systems. If one believes this analysis, then it is evident that the inclusion of any new experiential arena that brings in both new formalism, new iconic representations, and new ways of coding situations, alongside established mathematical iconic representations and formalism, will place higher demands on the symbolism and formalism competency. The analysis we provided gives a rather precise description of the nature of the challenge, namely that established mathematical objects (like a square) might come about through situations coded with new (programming) formalism that has unclear or complicated relationships with established school mathematics. Typically, when analyzing relationships between programming and mathematical formalism, the focus is on different meanings given to a particular symbol or concept, like the equal sign or the concept of a function (Partanen & Tolvanen, 2019). Our analysis shows that the challenges go much deeper.

To conclude, the KOM framework and in particular the identification and description of a symbolism and formalism competency is a powerful tool for thinking about one of the most important aspects of mathematical knowledge. Coordinating the competency with the more fine-grained theories based on Vergnaud's work gives access to understanding of the complications involved when developing the symbolism and formalism competency. This is why the networking of the KOM framework with the Vergnaud-based theories is a worthwhile effort.

## Appendix



when clicked
go to x: -100 y: -80
point in direction 90
erase all
pen down
set pen size to 3
set pen color to
move 140 steps
turn -90 degrees
move 70 steps
turn 90 degrees
move 90 steps
set pen color to
set xball to x position of Ball
set yball to y position of Ball
set dy to yball - -80
set dx to xball - -100
set diag to sqrt of dx * dx + dy * dy
set side to diag / sqrt of 2
point towards Ball2
move diag steps
turn 180 degrees
set pen color to
pen up
move diag / 2 steps
turn 90 degrees
move diag / 2 steps
pen down
point towards Ball2
repeat 4
move side steps
turn 90 degrees

## References

Ahl, L. M., & Helenius, O. (2021a). A framework for analyzing progress in concept knowledge in mathematics textbooks. In G. A. Nortvedt, N. A. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future. Proceedings of norma 20. The ninth nordic conference on Mathematics Education*, Oslo, 2021 (pp. 1–8). SMDF.

Ahl, L. M., & Helenius, O. (2021b). Polysemy and the role of representations for progress in concept knowledge. In Y. Liljekvist & J. Häggström (Eds.), *Sustainable mathematics education in a digitalized world. Proceedings of MADIF 12* (pp. 101–110). SMDF.

Blömeke, S., Gustafsson, J.-E., & Shavelson, R. J. (2015). Beyond dichotomies. *Zeitschrift Für Psychologie, 223*(1), 3–13. https://doi.org/10.1027/2151-2604/a000194

Brandon, R. N., & Hornstein, N. (1986). From icons to symbols: Some speculations on the origins of language. *Biology and Philosophy, 1*(2), 169–189. https://doi.org/10.1007/bf00142900

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Gödel, K. (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte Für Mathematik Und Physik, 38*(1), 173–198. https://doi.org/10.1007/bf01700692

Harnad, S. (1990). The symbol grounding problem. *Physica D: Nonlinear Phenomena, 42*, 335–346. https://doi.org/10.1016/0167-2789(90)90087-6

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics? *For the Learning of Mathematics, 35*(1), 15–20.

Jankvist, U. T., Misfeldt, M., & Sánchez Aguilar, M. (2019). What happens when CAS procedures are objectified?—The case of "solve" and "desolve." *Educational Studies in Mathematics, 101*(1), 67–81. https://doi.org/10.1007/s10649-019-09888-5

Leibovich, T., & Ansari, D. (2016). The symbol-grounding problem in numerical cognition: A review of theory, evidence, and outstanding questions. *Canadian Journal of Experimental Psychology/revue Canadienne de Psychologie Expérimentale, 70*(1), 12–23. https://doi.org/10.1037/cep0000070

Mendelson, E. (2008). *Number systems and the foundations of analysis*. Dover Publications, Inc.

Niss, M. A. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagatsis, & S. Papastavridis (Eds.), *3rd mediterranean conference on mathematical education - Athens, Hellas 3–4–5 January 2003* (pp. 116–124). Hellenic Mathematical Society.

Niss, M. A. (2007). The concept and role of theory in mathematics education: Plenary presentation (invited). In C. Bergsten, B. Grevholm, H. Skrømskag Måsøval, & F. Rønning (Eds.), *Relating practice and research in mathematics education: Proceedings of NORMA 05, fourth nordic conference on mathematics education* (pp. 97–110). TAPIR Akademisk Forlag.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Partanen, A.-M., & Tolvanen, P. (2019). Developing a frame for analysing different meanings of the concept of variable mediated by tasks in elementary-school mathematics textbooks. *Nordic Studies in Mathematics Education, 24*(3–4), 59–79.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Russell, B., & Whitehead, A. (1973). *Principia Mathematica* [PM] (Vol. I, 1910, Vol. II, 1912, Vol. III, 1913, Vol. I, 1925, Vols. II & III, 1927). Paperback Edition to* 56. Cambridge UP.

Saussure, F. de. (1974). *Course in general linguistics*. Fontana.

Vergnaud, G. (1997). The nature of mathematical concepts. In P. Bryant, & T. Nunes (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 5–28). Psychology Press/Erlbaum (UK) Taylor & Francis.

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior, 17*(2), 167–181. https://doi.org/10.1016/S0364-0213(99)80057-3

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

# Activating Mathematical Communication Competency When Using DGE—Is It Possible?

**Cecilie Carlsen Bach and Angelika Bikner-Ahsbahs**

## 1 Introduction

When students work collaboratively on mathematical issues using digital tools, their communication and handling of the tool develop in an intertwined manner. More specifically, students may use new words when using Computer Algebra Systems (CAS) (e.g., Schacht, 2015), or they may describe mathematical objects dynamically when using Dynamic Geometry Environments (DGE) (e.g., Jones, 2000). However, the use of a digital tool may also limit mathematical communication in favour of more pragmatic talking, which serves to conduct the practical activity rather than serving any communicative purposes. Jungwirth (2006) has called this kind of communication *empractical talk* in her investigation of students' use of CAS. These results indicate that the relationship between using a digital tool and communicating mathematically is not straightforward. As mathematical communication is highly relevant for learning mathematics (Morgan et al., 2014), there is a need to untangle its relation to using digital tools and more specifically to reason out its contribution to developing mathematical communication competency.

In previous case studies, we showed that mathematical communication and the use of DGE are interrelated: A mechanical-random DGE use was related to empractical communication, suppressing to put mathematical communication competency into practice, while a theoretical-resourceful DGE use dovetailed nicely with more

C. C. Bach (✉)
Danish School of Education, Aarhus University, Copenhagen, Denmark
e-mail: ccb@edu.au.dk

A. Bikner-Ahsbahs
Faculty 3—Mathematics and Computer Science, Bremen University, Bremen, Germany
e-mail: bikner@math.uni-bremen.de

Department of Language, Literature, Mathematics and Interpreting, Western Norway University of Applied Sciences, Bergen, Norway



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_10

dialogical communication, showing a high level of mathematical communication competency (Bach & Bikner-Ahsbahs, 2020). This chapter builds on these case studies and aims to characterise mathematical communication when using DGE in tool-based mathematical communication and to describe how it relates to the activation of students' communication competency initiated by a cognitively demanding task situation (Klieme et al., 2009) for mathematical communication within a digitally enhanced environment. Activation is not passive but entails acting related to the concept of competency. Given that the students use DGE, we ask:

RQ1: How is mathematical communication related to the use of DGE?
RQ2: How does the use of a digital environment activate mathematical communication competency?

To answer these research questions, we take a networking of theories approach, where we coordinate two theoretical perspectives: (1) mathematical communication and (2) instrumentation of using a digital tool. First, we embed mathematical communication competency in a background theory of communication that distinguishes between two communication genres—mathematical conversation and empractical communication. We then account for new cases of student pairs solving the same task as in the first trial of the case study with DGE and investigate them based on the two theoretical approaches. Finally, we add in-depth theorising of all the results by considering our approach as a case of theory networking.

## 2 Theoretical Framework

Research on the networking of theories has revealed a landscape of networking strategies used to relate theories and to further develop these relations. In this paper, we conduct case studies, for which we use the strategy of coordinating, which is "looking at the same phenomenon from different theoretical perspectives" (Prediger & Bikner-Ahsbahs, 2014, p. 119). It is used when "a conceptual framework (in the sense of Eisenhart, 1991) is built by fitting together elements from different theories for making sense of an empirical phenomenon" (Prediger & Bikner-Ahsbahs, 2014, p. 120) with the aim of theorising the phenomenon, which in our case is tool-based mathematical communication.

Research in networking of theories requires clarifying the term 'theory'. To this end, we draw on the notion of theory introduced by Radford (2008). Thus, a theory ($P$, $M$, $Q$) "(…) can be seen as a way of producing understandings and ways of action based on: a system, $P$, of basic principles (…) a methodology, $M$, (…) and a set, $Q$, of paradigmatic research questions (…)" (p. 320). The methodology $M$ justifies and encompasses ways of collecting data and methods related to $P$. $Q$ considers typical schemes of questions relevant for $P$ and $M$. By coordinating the two theories involved in our case studies, we can define a linking principle and a linking methodology, allowing us to reach the strategy of *integrating locally* (see Sabena et al., 2014). Thus, we can further theorise the phenomenon under investigation.

## 2.1 Mathematical Communication

The background theory of mathematical communication is placed in psycholinguistics based on the work of O'Connell and Kowal (2012), who point to the cognitive nature of linguistic resources used in a social setting of speaking and listening. Verbal communication is defined as an activity that "brings persons together somehow by means of spoken discourse" (p. 10), and whereas "discourse emphasises a corpus, communication emphasises social engagement" (p. 11). In this paper, the discourse is mathematical. When communicating, at least two individuals are participating and two concepts, *listener* and *speaker,* identify the participants' roles. Listening is different from hearing: Hearing is passive, whereas listening is active and includes validating the speaker's meanings and participation by "stance, gesture, gaze, nods and other signs of active engagement, including brief comments, which nonetheless leave the listener in listener's mode" (p. 6).

Within verbal communication, O'Connell and Kowal (2012) distinguish between two communication genres, *conversation* and *empractical communication.* Participating in a conversation is an important element in everyday life and demands that the participants are active, acting as both listener and speaker and showing qualities of common responsibility, such as *turn-taking*, *equal participation*, *open-endedness* and *verbal integrity.* When taking turns to speak, each utterance is related to the previous utterance of the others or oneself, where no hierarchical relation is permanently established. Open-endedness entails readiness to listen (also to oneself) and to move the communication forward. Verbal integrity entails that "one means what one says and says what one means. And one listens not only with a receptive heart but with an honest effort to review fairly what one receives from a speaker" (p. 21). Furthermore, "[p]revarication, trifling, prejudice, dogmatism, stubbornness (…) all these perspectives on the part of either listeners or speakers preclude verbal integrity" (p. 22). *Empractical communication* (cf. Bühler, 1934/1990) is less coherent than a conversation. It is a way of communicating where non-linguistic, practical activities dominate. This means that the talking is related to and based on the practical activity and its context, accompanied by many breaks (O'Connell & Kowal, 2012). For example, one of the participants may have a source of power, such as using the mouse when only one computer is available, thus causing a lack of turn-taking. In case a child uses the compass for the first time to draw a circle, she may think aloud "there, look …" indicating where to fix the centre, but it is not understandable outside the context.

We will refer to communication as *mathematical* when the discourse concerns mathematics, hence involving mathematical representations, concepts, objects or activities (Niss & Højgaard, 2011). We can distinguish between two genres—mathematical conversation and empractical mathematical communication. *Empractical mathematical communication* is subordinated to a practical activity related to mathematics, like drawing a circle. In this case, the communication focuses on the practicality of the activity and the mathematical content is addressed via this practicality rather than verbally. For example, students may use the word *solve* when using

CAS (Jungwirth, 2006), or when using DGE, students' explanations are influenced by dynamic features (Jones, 2000). *Mathematical conversation*, however, concerns communication in, with and about mathematics, to which the qualities of a conversation apply, including turn-taking, open-endedness and verbal mathematical integrity, verbal mathematical integrity meaning that "one means what one says and says what one means" (O'Connell & Kowal, 2012, p. 21) with respect to one's mathematical knowledge.

#### 2.1.1 Mathematical Communication Competency

Mathematical communication competency concerns being able to *express* oneself mathematically using mathematical terms and words, as well as *understanding* and *interpreting* other people's mathematical expressions (Niss & Højgaard, 2011). As we focus on oral communication between two students in this paper, the aim is for the students to have a dialogue showing mathematical conversation qualities, as a conversation involves the ability to interpret each other's mathematical expressions (listening) and to express oneself mathematically (speaking) (O'Connell & Kowal, 2012). To show mathematical communication competency, the communication must be of a mathematical nature, meaning that it entails the use of different mathematical words as well as words that are relevant in a mathematically related context. In a situation involving functions, these could be *equation*, *graph*, *slope* and *variables.*

Mathematical representations are important in mathematical communication. When students show mathematical communication competency, they deal with different mathematical media for representing mathematical objects (i.e., visual, written, oral and gestural) (Niss & Højgaard, 2019). In terms of functions, it could be tracing a graph, building an equation by a description in natural language or creating a table.

### *2.2 Instrumental Genesis and Instrumentation Profiles*

Instrumental genesis concerns the process of a student becoming able to use a digital tool in a given task in which the tool transforms from being an *artefact* to an individual *instrument* for a class of situations (Guin & Trouche, 1998). In this process, *instrumentation* means that a tool affects the student's actions and learning through building of cognitive schemes, thereby determining how the student is able to utilise the digital tool in connection with a task (Artigue, 2002).

Instrumentation processes vary depending on the students' mathematical understanding. Observing students using CAS, Guin and Trouche (1998) defined five work methods characterising students' instrumented actions, also called *instrumentation profiles*. The profiles involve a variety of instrumentation processes due to their complexity and re-organisation of activity, including students' understanding of mathematical objects, their knowledge about the tools and their ability to choose how to solve a task. We describe the profiles based on three different ways of approaching tools: (1) *Information tools*: theoretical knowledge, the ability to communicate with peers/teachers and to use a tool or paper and pencil. (2) *Understanding tools*: the ability to use the available information tools and to do semantic interpretations (i.e., decoding and translating between the involved representations), comparisons and logical inferences. (3) *Commando processes*: the ability to choose understanding tools.

The instrumentation profiles (see Guin & Trouche, 1998, pp. 214–216) are:

- Random: Students experience difficulties using either the digital tool or paper and pencil. Work is identified as copy-and-paste or trial-and-error strategies and there is no verification of work.
- Mechanical: Students are limited to using the digital tool, mostly doing simple calculations or investigations and avoiding mathematical reasoning. If the students reason, they tend to refer to the tool. Commando processes are rather weak.
- Resourceful: Students use a combination of several methods with paper and pencil, digital tools and theory. Commando processes are average. Often students compare and combine results from different resources and their way of investigating varies.
- Rational: Students primarily use paper and pencil. Students' command processes are strong, and they rely on mathematical inference when reasoning.
- Theoretical: Students show mathematical understanding and can therefore use their knowledge systematically. Their understanding of tools is good as well as their ability to use the tools. They verify their results.

## 2.3 Previous Results

Bach and Bikner-Ahsbahs (2020) found two relationships between students' communication and their way of using GeoGebra. The relationship is depicted in Fig. 1.

Our studies showed that empractical mathematical communication and mechanical-random instrumentation profiles appeared when students were immersed in the practical activity of using GeoGebra. Activating and practising mathematical communication competency in empractical talk seemed critical where listening was scarce, and speaking was reduced. This was different in mathematical conversation.[1] Mathematical conversation appeared in parallel with a theoretical-resourceful instrumentation profile, where students showed mathematical communication competency.

[1] Bach and Bikner-Ahsbahs (2020) labelled such communication *participatory communication.*

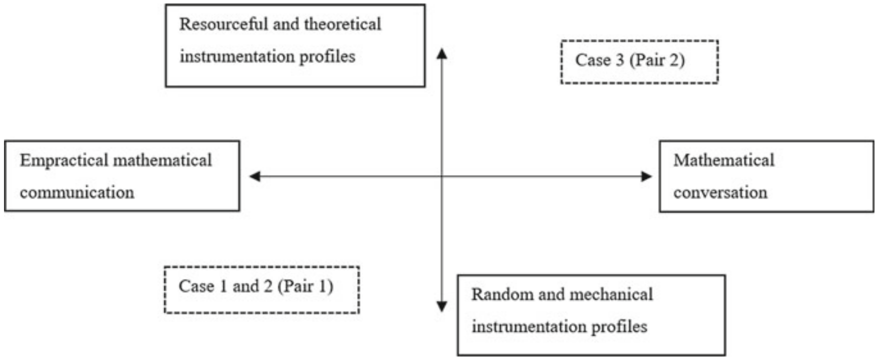



**Fig. 1** Illustration of the relationships between mathematical communication and instrumentation profiles based on the results presented by Bach and Bikner-Ahsbahs (2020)

## 3 Methodology

In the first trial of the case study, 9th-grade students (aged 14–16) from the same class worked in pairs on a task concerning functions as covariation (Bach & Bikner-Ahsbahs, 2020). In this chapter, we present data from the second trial collected in three 8th-grade classrooms (aged 13–15), where the task from the first trial was adjusted to the 8th-grade. We present examples from data collected in the second trial of the case study. We analyse three episodes from two perspectives: (1) mathematical communication competency shown in the communication genres and (2) instrumentation profiles in the way the tools are used. We then relate both perspectives in each case and finally merge all results to theorise them as a case of networking. Thus, we follow an argumentation thread of theory building rather than theory testing.

### *3.1 The Task*

Our task aims at cognitively activating mathematical communication competency and the use of GeoGebra. The design utilises ideas from Johnson and McClintock (2018) concerning functions as *covariation.* It demands to understand dependent and independent variables and how these change together related to the dynamic co-change (a term used by Schou & Bikner-Ahsbahs, 2022) of a rectangle and a function graph in the geometric window. The task is split into two parts with students working in pairs with a computer each. In this chapter, we focus on one part: *The Rectangle*. The task consists of eight subtasks involving a GeoGebra template (Fig. 2). Figure 2 shows a rectangle *ABCD* inserted in the coordinate system. The width of *AD* is fixed to be 3, whereas the length of *AB* varies. Point *A* can be dragged vertically. When dragging *A*, point *P*'s coordinates change as these are defined by the height *AB* (*P*'s *x*-coordinate) and the area of *ABCD* (*P*'s *y*-coordinate).

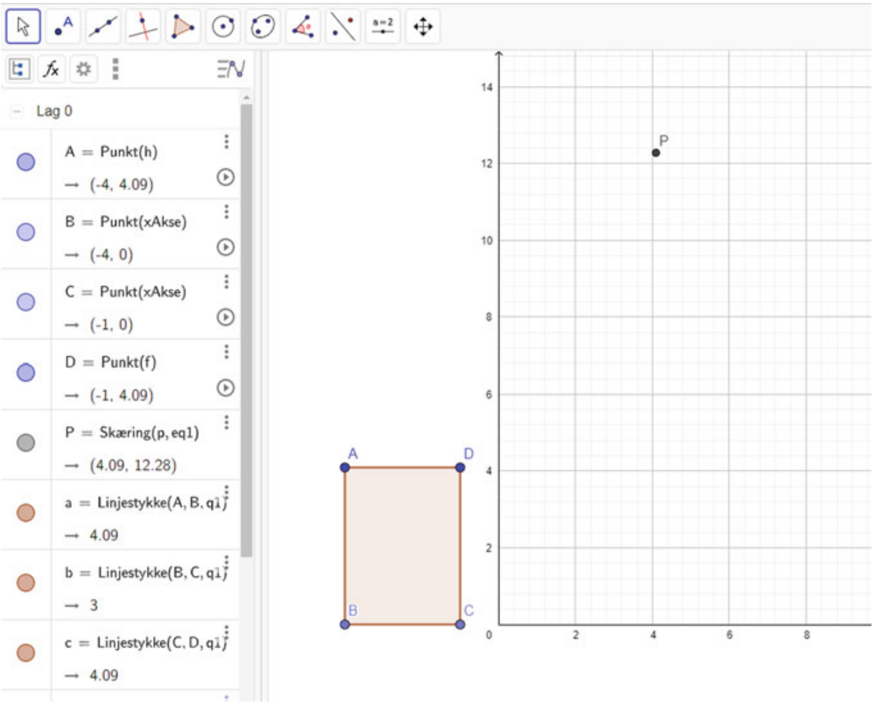



**Fig. 2** The GeoGebra template (https://www.geogebra.org; inspired by Johnson & McClintock, 2018)

First, the students explore the relationship between the rectangle and point $P$. Then, they individually fill in a predefined table concerning the relation between the height and the area of the rectangle (Table 1). Afterwards, the students model the equation of the functional relationship ($y = 3x$) and draw its graphical representation in GeoGebra as point $P$ moves on. Finally, they clarify the dependent and the independent variables.

**Table 1** The relation between height and area

| Height of AB (cm) | 1 | 3 | 4 | 7 | 10 |
|---|---|---|---|---|---|
| Area of the rectangle, ABCD | | | | | |

## *3.2 Data Collection*

Data was collected from three classes in October–November 2020.[2] The students were introduced to linear functions and GeoGebra in a minor course (three weeks)

[2] This was during the COVID-19 pandemic with students in and out of isolation, which caused a few changes to the classes.

before starting the task. Thus, the topic of functions was rather new to the students, but the task was built on familiar geometrical concepts. We included high-achieving student pairs: (1) Because the errors they make are likely to be made by low achievers—not vice versa; (2) to compare these results with those from the first trial (Bach & Bikner-Ahsbahs, 2020). The teacher chose the student pairs by assessing their level of proficiency-based participation in class, assignments and national test results.

The data consists of videos from the classrooms, screencasts from the students' computers (filming their screens, recording sound and webcam), stand-alone videos of the pairs' collaboration and student worksheets. The worksheets were in paper form in two of the classes, while Word was used in the third one. The data was transcribed verbatim in Danish. Relevant episodes have been translated into English keeping the character of Danish wording as far as possible.

### *3.3 Presenting and Analysing Data*

In the following sections, we present three cases, each followed by analyses of mathematical communication competency, including communication genres and instrumentation profiles, respectively.

To identify the *communication genre* in the episodes, we consider the following three criteria based on the description of mathematical communication competency from a psycholinguistic perspective:

1. *Listener* and *speaker*: Do the students express themselves mathematically, and do they actively listen to one another (e.g., by turn-taking and listening)?
2. The media of communication and the relations between them (e.g., written, oral, visual or gestural).
3. The communication genres shown (mathematical conversation or empractical mathematical communication).

Finally, we conclude whether the students show mathematical communication competency or not.

To analyse the students' use of GeoGebra, we identify their *instrumentation profiles* based on three aspects:

1. The mathematical knowledge involved in solving the task (e.g., their understanding of the functions).
2. The information tools involved (e.g., mathematical theory, GeoGebra, peers, or paper and pencil).
3. Strategies for solving the task (e.g., trial and error) (Guin & Trouche, 1998).

## 4 Case A: Andrea and Bea

Two students, Andrea and Bea, work together on functions during the course. For the purpose of this case, the students first fill in their tables individually (Tables 2 and 3). When filling in the table, Andrea drags point $A$ until reaching the expected height of $AB$. Then she identifies the area. Bea fills in the table by looking at point $P(1, 3)$, and then she fills in the table by calculating: $3 \cdot 3 = 9; 4 \cdot 3 = 12; 7 \cdot 3 = 21; 10 \cdot 3 = 30$ without using GeoGebra. Finally, Bea opens GeoGebra again and drags point $A$ from 1 to 10, checking her results.

**Table 2** Andrea's results

| Height of AB (cm) | 1 | 3 | ⑤ | 7 | 10 |
|---|---|---|---|---|---|
| Area of the rectangle, ABCD | 3 | 9 | 12 | 21 | 30 |

**Table 3** Bea's results. Bea also adds extra heights with the corresponding areas

| Height of AB (cm) | 1 | 3 | 4 | 7 | 10 |
|---|---|---|---|---|---|
| Area of the rectangle, ABCD | 3 | 9 | 12 | 21 | 30 |

After filling in the tables, Andrea and Bea discuss the relationship between the rectangle and point $P$. They begin by discovering that Andrea made an error (Table 2). It is presented in Transcript 1.

| | | |
|---|---|---|
| 1 | Andrea | Mmh. I got 4 first. It is probably wrong. In the second one, I also got 9. |
| 2 | Bea | Okay. What about the fourth? [*Bea refers to the height of AB*]. |
| 3 | Andrea | 4? That one I haven't made. I think, [*for* $AB = 4$ in the table] But for the third, I also got 12 [*the field in the third column and second raw says 12, but it is wrong for* $AB = 5$]. |
| 4 | Bea | Okay. I got this. Well, because the width is 3 all the time, right? [*pointing with her mouse at line BC*]. So here now, it's on 2 [*dragging point A so AB becomes 2*], so it's just two times 3, four times 3, six times 3, 10 times 3 [*opening her Word document again*]. |
| 4 | Bea | Okay. Should we move forward? Construct the function for the relation between $AB$ and the area of the rectangle. What type of function is it? [*reading from the task description*]. |
| 5 | Andrea | Mmh. |
| 6 | Bea | Isn't it just something like every time the height of $AB$ increases, it's multiplied by 3? |
| 7 | Andrea | Mmh [*sounds like a yes/yes maybe*]. |
| 8 | Bea | Could it be something like $x$ times 3 is the answer? [*meanwhile, she writes: "x", then "xx3 ="*[3]]. |
| 9 | Andrea | Of $AB$? Yes. |

[3] In Denmark, students may write $x$ as a multiplication sign.

| | | |
|---|---|---|
| 10 | Bea | Yes, because then you take *x*. It can be any number. As an example, let us say 3, which you multiply by 3 and then you get the answer. And then, four times 3 and then you get the number. seven times 3 and so forth. |

Afterwards, Bea asks the teacher for help to define the function equation and draw the graph.

## *4.1 Analysing Mathematical Communication*

**The roles of listener and speaker**: Andrea and Bea respond to each other's expressions, by continuing and talking related to the other student's latest expression (e.g., lines 1–2, 2–3). When acting as listener, they participate by validating each other's utterances, showing open-endedness until line 10, as the dialogue stops. **Media of communication**: Andrea and Bea use many media forms: written, oral and visual. They both use tables (e.g., lines 1, 2, 3) and they use the GeoGebra template actively (e.g., line 4). The students explore the relationship between the numbers included in the table. However, Andrea's results in the table are wrong, but by communicating Andrea and Bea reach an agreement (see lines 6–10). **Communication genre**: The students communicate actively, practising turn-taking and negotiating the results (lines 3–4). They show open-endedness, as they move the conversation forward (e.g., lines 3–5), and verbal mathematical integrity, as they respect each other's mathematical expressions, and participate equally in the communication. They use mathematical notions (e.g., lines 4 and 8) and bring the use of GeoGebra actively into their communication (e.g., line 4). Thus, we may identify their communication genre as mathematical conversation putting mathematical communication competency into practice, as both participate actively by expressing themselves and interpreting each other's expressions.

## *4.2 Analysing Instrumentation Profiles*

**Mathematical knowledge**: Both students rely on mathematical knowledge and their understanding of the expected relationship at point *P*, between the height and the area of the rectangle (lines 6–10). Although Andrea made an error, she is aware of it (line 1). **Information tools**: The students, particularly Bea, use mathematical inference and compare the representations at play (lines 3, 4). Their mathematical knowledge and GeoGebra serve as information tools (e.g., line 4). **Strategies**: Checking the results from the calculations using previous reasoning shows that the students also have verification techniques, which they use to validate their results by means of the tool, too (line 4).

Andrea and Bea display a *theoretical work method* due to the use of mathematical knowledge and verification of results, and since they do not rely only on paper-and-pencil techniques or GeoGebra.

### *4.3 Summarising Case A*

Andrea and Bea show a combination of a theoretical work method and mathematical conversation. Their level of mathematical understanding and tool use denotes mathematical conversation, in which they draw mathematical inferences and compare results. GeoGebra as a tool is embedded in their conversation, yet their communication has the qualities of a conversation.

## 5 Case B: Caroline and Diane

Two students, Caroline and Diane, collaborate most of the time during the course about functions. However, Diane did not participate in the previous session concerning linear functions, and her knowledge about functions is therefore limited. Both students fill in the table (Table 1) correctly. To solve the subsequent task, in which the students have to define the function equation and draw the graph, they have the following dialogue.

| | | |
|---|---|---|
| 11 | Caroline | The function of *P*. So it is on 4 now, that is the side length [*Caroline just looks at the GeoGebra without changing anything.*]. *P*, it can… *P* is equal to… *AB* times… *P* is equal to *AB* times… *AB* times *x*. *AB* times *x*… *AB* times *x*. *AB*, side length. Height. *AB* is equal to the height. Height and then |
| 12 | Diane | Then we write that *AB* is equal to the height. |
| 13 | Caroline | I just need to figure out what we have… to do the function. |
| 14 | Caroline | *AB* is equal to the height and *x* is equal to … the height … *x* is equal to… |
| 15 | Diane | The width. |
| 16 | Caroline | By the *x*-axis, it is equal to |
| 17 | Diane | The width. |
| 18 | Caroline | The height. *x* is equal to the height on the axis [*x-axis, red.*]. |
| | | [… *Another student comes in, they talk about something irrelevant*]. |
| 19 | Caroline | *y*, it's 12. Ohh, it's so difficult. I don't know what to do. [*laughing and bringing her hands to her head*]. |
| 20 | Diane | Well, it's really difficult. |

| | | |
|---|---|---|
| 21 | Caroline | But it's easy to see. The side length is 4, like the point on the *x*-axis and the area of this figure is 12, and then that is y, so I think *y* is 12 in this example. Wait then. Now, I'm writing $x = 4$ and $y = 12$ and then the width is equal to, it's 3.<br>Then I just write four times 3 is 12, so *x* times 3 equals 12 [*The teacher comes in and chitchats and leaves again…*]. |
| 22 | Caroline | 3 times *x* is equal to y. *x* times 3 is equal to *y*. *y* is equal to *x* times 3. *x* times 3. *x* times 3. *x* times *A* times *B*. *x* times *A* times *B* is equal to, is equal to, *x* times *B* is equal to *y*, which is…<br>I don't know how to write it so it counts for all. |
| 23 | Caroline | [*after a short while*] What we know is that the side length, it's the position on the *x*-axis and the area is located on the *y*-axis [*she points with her pen at the coordinate system in GeoGebra*]. And 3, that's just a number we need to have since it is the width of this one [*points to the rectangle*]. |

After this talk, the students take a break.

## *5.1 Analysing Mathematical Communication*

**The roles of listener and speaker**: Caroline and Diane are not participating equally in the communication situation, and they do not have the same preconditions to do so, as Diane did not participate in the previous session. Diane expresses that it is difficult (line 20), and Caroline responds that "it's easy to see" (line 21). Caroline expresses herself mathematically, yet she lacks open-endedness as she does not listen (e.g., 12–13 and 15–18). In line 13, it sounds like Caroline is talking to herself and not to Diane as Caroline says, "I need to…". Diane listens and turn-takes (e.g., lines 15, 17 and 20). It seems that Caroline does not show verbal integrity, as she works on the task, and not responding to Diane's suggestions (e.g., lines 14–20). Differences in relation to mathematical understanding impair verbal integrity in this case. **Media of communication**: Caroline and Diane primarily communicate orally. Yet, they refer to and look at the GeoGebra template (e.g., line 21), but without taking advantage of the dynamicity embedded in GeoGebra (lines 11 and 23), and they write notes in the worksheets. **Communication genre**: The communication is embedded in the practical context, making it difficult to understand outside of this context (e.g., lines 11 and 22). Caroline and Diane's communication is hierarchical; turn-taking, open-endedness and verbal integrity are missing. Their communication is empractical in this situation, and they do not show mathematical communication competency. Lack of verbal integrity seems to be important, as communication competency entails mathematical communication with different people showing different levels of mathematical proficiency.

### *5.2 Analysing Instrumentation Profiles*

**Mathematical knowledge**: Caroline has more knowledge about functions than Diane, who did not participate in the prior lesson. Caroline and Diane (as a group) do semantic interpretations by translating the involved representations and comparing features of *P* with the properties of the rectangles (e.g., lines 11 and 21). Yet, they do not refer to their functional relationships. **Information tools**: Caroline and Diane both show insufficient information tools, meaning that they do not exploit the possibilities of the tool in the situation, as they do not interact with it. **Strategies**: Caroline and Diane's strategy is to solve the task without dragging in the tool (they do not drag in this situation). They compare the representations in GeoGebra and reason based on the template (static) and its representation (lines 14, 21 and 23).

Caroline and Diane display a *random work method* due to their avoidance/lack of knowledge of or engagement with the tool, and due to the limited mathematical knowledge they show concerning functions.

### *5.3 Summarising Case B*

Caroline and Diane show a combination of a *random work method* and *empractical communication.* The dynamic features of GeoGebra are not exploited—GeoGebra is idle in the situation. Their communication is empractical because it is hierarchical and lacks verbal integrity, turn-taking and open-endedness.

## 6 Case C: Emma and Frida

Two girls, Emma and Frida, communicate about the relationship between the rectangle and point *P.*

| | | |
|---|---|---|
| 24 | Emma | How do you know that *P* is an intersection? |
| 25 | Frida | Wait, mm, can I borrow your rubber, please? |
| 26 | Frida | Find the intersection *P* in… |
| 27 | Emma | How do you know it's an intersection, Frida? |
| 28 | Frida | It stands here [*Frida finds P in the algebra window where it is an intersection*]. Try to see if you go up here. *P* is an intersection. |
| 29 | Emma | Mmh. |
| 30 | Frida | [*Frida is humming her own melody*]. Mmh, I think, I don't know much about this. |
| 31 | Emma | Ugf [*Emma is dragging her rectangle up and down, reducing the height to zero.*]. |

| | | |
|---|---|---|
| 32 | Frida | I think that the intersection decides, how, decides, of… |
| 33 | Emma | It's like, if you want to draw. The rectangle's height and area. |
| 34 | Frida | [*Frida writes on paper*]. I think that the intersection is defined by… the rectangle [*reading out loud*]. |
| 35 | Frida | Yes, so if we change. |
| 36 | Emma | So *P* moves askew to the right [*grabs the mouse again*]. |
| 37 | Frida | And … |
| 38 | Frida | The rectangle's… area [*she drags point P so the figure and point P change*]. |
| 39 | Frida | Okay, then we change. |
| 40 | Emma | Well, if it moves askew to the right, more and more. |
| 41 | Frida | Okay, then wait two seconds, two seconds. If we only change the width, what happens then? |
| 42 | Emma | But then, no, no, if you change the width. You cannot drag point *D* up [*vertically, red.*]. *D* cannot physically move upwards. It can only move to the side like this. Because then it moves up and down and the others as well [*Emma is on point D in the rectangle in GeoGebra to illustrate*]. |

## *6.1 Analysing Mathematical Communication*

**The roles of listener and speaker**: Emma and Frida both participate in the activity, and they both express themselves and interpret each other. They build on and respond to each other's expressions (e.g., lines 24–25). This means that they both act as listener and speaker. Yet sometimes they lack open-endedness (e.g., lines 32–34). **Media of communication**: The pair communicates orally while discussing what to write on the worksheet. They use the GeoGebra template as they communicate and drag in GeoGebra (e.g., lines 28 and 42), but they seem to be having trouble when translating between the visual representations and verbal expression (lines 36 and 40). **Communication genre**: Emma and Frida use various mathematical notions, such as intersection[4] when they refer to point *P* (e.g., lines 32 and 34). The communication is context-dependent, focusing on the activities of dragging in GeoGebra rather than on communicating mathematically (e.g., line 38). Yet, the language also includes words that are normally not part of the mathematical vocabulary (e.g., *askew* in line 36), and their expressions are empractical. For instance, "it stands here" (line 28) and "So *P* moves askew to the right." (line 36). The expressions also include some kind of movement, e.g., they say that *P* is moving (lines 36 and 42). In this situation,

[4] 'Intersection' stems from their use of GeoGebra, and it is not part of the task, which complicates the situation.

their communication is empractically embedded in the use of GeoGebra with a lack of turn-taking and open-endedness, meaning that they do not explicitly interpret each other's mathematical expressions or react mathematically. Thus, the two do not express mathematical communication competency.

### *6.2 Analysing Instrumentation Profiles*

**Mathematical knowledge**: Emma and Frida do not express mathematical knowledge concerning the concept of function as covariation. They drag all points of the rectangle and explore their properties, not realising that the shape changes from a rectangle to a trapezoid (line 42). Emma and Frida are confused about $P$'s relation to and movement as regards the rectangle and the "intersection" as they try to define $P$ (lines 26, 28 and 34). Thus, the students are struggling with the mathematical content. **Information tools**: Emma and Frida use each other as well as GeoGebra as primary information resources trying to solve the task. For instance, by looking in the algebra window (line 28) and by dragging (lines 31, 38 and 42). **Strategies**: To find a solution, the students search in GeoGebra by dragging (lines 31, 38 and 42).

Emma and Frida follow a *mechanical work method* as they do not show enough mathematical knowledge, and they reason through using the tool practically. They are not yet capable of using the tool, adapting to its affordances as part of the instrumentation process.

### *6.3 Summarising Case C*

Emma and Frida follow a mechanical work method and communicate empractically. The use of the tool is embedded in their communication, and they try to find answers with respect to the tool.

## 7 Summarising the Results

Based on the results from Bach and Bikner-Ahsbahs (2020) (Fig. 1) and our analyses in this chapter, we found three kinds of relations between the use of the digital tool and the students' mathematical communication:

1. Tool-*embedded* mathematical conversation: The conversation shows verbal integrity, turn-taking, equal status of participants, etc. The students also have a theoretical and/or resourceful work method emphasised by the instrumentation of the DGE and their understanding of the mathematical content. E.g., Case A, Case 3 (Bach & Bikner-Ahsbahs, 2020).

2. Tool-*embedded* empractical mathematical communication: The communication is embedded in the context of the tool and focuses on an activity, where mathematics is addressed via the tool. Work methods are either mechanical or random, indicating that mathematical knowledge is not shown in the situation. For mechanical methods, students refer to the tool when reasoning. E.g., Case C, Case 1 (Bach & Bikner-Ahsbahs, 2020).
3. Tool-*idle* empractical mathematical communication: The dynamic features of the DGE are not exploited, but idle. It is empractical communication due to the existing hierarchy, reduced turn-taking and verbal mathematical integrity. The work method is random as the students show severe difficulties, both with paper and pencil and GeoGebra. E.g., Case B, Case 2 (Bach & Bikner-Ahsbahs, 2020).

The results from the second trial confirm the results from the first trial but in a differentiated way. Empractical communication is related to the random-mechanical instrumentation profiles, while mathematical conversation is related to the theoretical-resourceful instrumentation profiles. In the latter, communication competency is intensely shown, whereas communication competency is not observed in the former. When the students show empractical communication, the focus is on the practicality of engaging with DGE (RQ2) rather than communicating mathematically.

## 8 Reflections on the Theory Networking Conducted in This Study

As the prior section reveals, two different kinds of empractical mathematical communication appear—tool-embedded and tool-idle. In the former, the dynamicity of the DGE is exploited, whereas in the latter, the template is considered a static representation, its dynamic potential not being exploited but left idle. We will now reflect on these results by adopting a networking of theories approach. Thereby, we consider the two trials as cases of theory networking. We apply the strategy of *coordination* leading to *local integration* by "putting together a small number of theoretical approaches into a new framework" (Prediger & Bikner-Ahsbahs, 2014, p. 120). Our reflection is guided by the notion of theory introduced by Radford (2008). This notion allows us to operationalise *local integration* (see Sabena et al., 2014) as linking two theoretical approaches by elaborating a new concept. The aim is to provide an answer to a new paradigmatic research question (RQ1), founded by a linking principle and a linking methodology.

### 8.1 Four Types of Tool-Based Mathematical Communication

When using a tool, actions and cognition are related (Vérillon & Rabardel, 1995), and when communicating orally, speaking and listening are based on the cognitive-linguistic resources in use (O'Connell & Kowal, 2012). Hence, when communication is related to tool use, both cognitive resources and cognitive processes guide speaking and listening related to acting with the tool. We call such mathematical communication *tool-based*. The new concept of *tool-based mathematical communication* will now be conceptualised based on our empirical results.

Table 4 shows all the types of tool-based communication we found. Originally, two types of tool-based communication were evidenced in our data of the two trials—tool-embedded mathematical conversation and tool-embedded empractical mathematical communication. In the former, mathematical competency is put into practice, but not in the latter. However, when investigating empractical mathematical communication more in-depth, Case B does not fit well into the strong tool-embedded nature of empractical communication, because the students do not exploit the dynamic nature of GeoGebra. The dynamic potential of DGE rather *idles* as the students refer to the template as a static representation. This is similar to Case 2 elaborated by Bach and Bikner-Ahsbahs (2020).

**Table 4** Four types of tool-based mathematical communication

| | Embedded digital tool use<br>(related to the dynamic nature of representations) | Idle digital tool use<br>(related to static representations or the template given) |
|---|---|---|
| Mathematical conversation | *Tool-embedded* mathematical conversation (Cases: A & 1) | Conjecture: *Tool-idle* mathematical conversation |
| Empractical mathematical communication | *Tool-embedded* empractical mathematical communication (Cases: C & 3) | *Tool-idle* empractical mathematical communication (Cases: B & 2) |

As the instrumentation profiles are developed with CAS (Guin & Trouche, 1998), and dynamicity is missing in CAS, the distinction between tool-embedded and tool-idle talk is not relevant in CAS-related instrumentation profiles, but it is relevant for DGE. Therefore, we distinguish between two ways of utilising DGE: *tool-embedded* and *tool-idle*. When we relate the two types of utilising the tool to the two mathematical communication genres in a crossing table, four relations appear three of which are found in our data. *Tool-idle conversation* has not yet been observed in our data. However, we may predict that this type of communication exists empirically, for instance, when students show a *rational instrumentation profile* (Guin & Trouche, 1998), or if a task requires proving a conjecture, where mathematical knowledge is required rather than acting with the tool.

### 8.2 Tool-Based Mathematical Communication: A Case of Local Integration

We will now justify why the phenomenon of tool-based communication can be regarded as a case of local integration. The linking principle (*P*) assumes that tool-based mathematical communication is a particular type of communication linking cognitive-linguistic resources with cognitive action schemes developed during instrumentation, which emerge in connection with communicating and acting. For instance, in empractical communication, cognitive action schemes may activate cognitive-linguistic resources that are related to the practicality of the activity, whereas in conversation, the reverse may be the case. The four types of tool-based mathematical communication lead to a differentiated vision of four types of *tool-based mathematical communication* that answer our research question 1. This result raises a new kind of paradigmatic research question, which is: what types of tool-based mathematical communication can be found empirically with respect to a specific digital tool? The linking methodology (*M*) consists of three steps: (1) Analysing the cases from the two theoretical perspectives and linking them; (2) identifying dimensions for a crossing table, which allows structuring the empirical cases; (3) reflecting the results in the crossing table theoretically with respect to the affordances and possibilities of the tool in the communication.

However, in our networking case, empirical evidence is still scarce, as we only bring in three additional cases. Furthermore, tool-idle mathematical conversation could not be identified in the data, and tool-idle and tool-embedded empractical communication still require further elaboration based on empirical data. Thus, we have proceeded toward local integration of the two theoretical approaches into the phenomenon of tool-based mathematical communication, but we have only been partially successful.

## 9 Conclusion

Tool-based communication draws on two different kinds of cognitive resources—linguistic mathematical resources on the one hand and instrumental action schemes on the other. Our results indicate that when learning to use a digital tool, the students do not easily activate and practise both kinds of cognitive resources in parallel. They rather foreground one kind of resource, determining how to use the other. Therefore, expecting students to use a DGE template and to simultaneously show mathematical communication competency in class may set them in a complex situation, in which both mathematical knowledge and instrumentation are crucial. However, when using digital tools, the focus must at some point shift to the activity of handling the tool, which then would lead to practical context-embedded communication; that is, empractical mathematical communication not holding the qualities of a mathematical conversation. In empractical tool-based communication, mathematics comes into play via the tool, where activating and practising mathematical communication competency is hardly possible. Consequently, when expected to communicate in a tool-based manner, the students may be faced with a 'double-bind' situation in which they will barely be able to fulfil both requirements: learning to use a digital tool and simultaneously developing mathematical communication competency. This raises the critical question of how students can be enabled to develop mathematical communication competency in tool-based communication.

Additional research is needed to further explore the four types of tool-based mathematical communication, involving a variety of task situations and students, and investigate, in which kinds of communication genres mathematical communication competency can be fostered and under what conditions communication tilts to empractical talk. This would provide more theoretical as well as practical insight into how to improve mathematical communication competency when using digital tools.

## References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274. https://doi.org/10.1023/A:1022103903080

Bach, C. C., & Bikner-Ahsbahs, A. (2020). Students' experiences with dynamic geometry software and its mediation on mathematical communication competency. In A. Donevska-Todorova, E. Faggiona, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H.-G. Weigand (Eds.), *Proceedings of the tenth ERME topic conference on mathematics education in the digital age* (MEDA) (pp. 427–434). European Society for Research in Mathematics Education. https://hal.archives-ouvertes.fr/hal-02932218/document

Bühler, K. (1934/1990). *Theory of language: The representational function of language* (D. F. Goodwin, Trans.). John Benjamins Publishing Company.

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227. https://doi.org/10.1023/A:1009892720043

Johnson, H. L., & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educational Studies in Mathematics, 97*(3), 299–316. https://doi.org/10.1007/s10649-017-9799-7

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1–3), 55–85. https://doi.org/10.1023/A:1012789201736

Jungwirth, H. (2006). Everyday computer-based maths teaching: The predominance of practical activities. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 377–384). PME.

Klieme, E., Pauli, C., & Reusser, K. (2009). The Phythagoras study. Investigating effects of teaching and learning in Swiss and German mathematics classrooms. In T. Janik, & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom* (pp. 137–160). Waxmann.

Morgan, C., Craig, T., Schuette, M., & Wagner, D. (2014). Language and communication in mathematics education: An overview of research in the field. *ZDM – Mathematics Education, 46*(6), 843–853. https://doi.org/10.1007/s11858-014-0624-9

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

O'Connell, D. C., & Kowal, S. (2012). *Dialogical genres. Empractical and conversational listening and speaking*. Springer. https://doi.org/10.1007/978-1-4614-3529-7

Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 117–126). Springer. https://doi.org/10.1007/978-3-319-05389-9

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM - Mathematics Education, 40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Sabena, C., Arzarello, F., Bikner-Ahsbahs, A., & Schäfer, I. (2014). The epistemological gap: A case study on networking of APC and IDS. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 179–200). Springer. https://doi.org/10.1007/978-3-319-05389-9

Schacht, F. (2015). Student documentations in mathematics classrooms using digital tools: Theoretical considerations and empirical findings. *Electronic Journal of Mathematics & Technology, 9*(5), 320–339.

Schou, M. H., & Bikner-Ahsbahs, A. (2022). Unpacking hidden views: Seven ways to treat your formula. *Educational Studies in Mathematics*, *109*(3), 639–659 (2022). https://doi.org/10.1007/s10649-021-10092-7

Vérillon, P., & Rabardel, P. (1995). Cognition, and artifact: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology in Education, 10*(1), 77–101. https://doi.org/10.1007/BF03172796

# An Embodied Cognition View on the KOM-Framework's Aids and Tools Competency in Relation to Digital Technologies

**Morten Misfeldt, Uffe Thomas Jankvist, and Eirini Geraniou**

## 1 Introduction

The prevalence of and the role that digital technologies (DT) play in the mathematics programmes at all educational levels around the world is today significantly different than it was in 2002 when the KOM-framework (Niss & Højgaard, 2011, 2019) was first launched. Dynamic Geometry Software (DGS), such as *GeoGebra*, is increasingly used at both primary and secondary levels. Computer Algebra Systems (CAS), such as *Maple*, *TI-Nspire*, *WordMath*, etc., are an integral part of the upper secondary school mathematics programmes—even mandatory at the final national written assessments (Jankvist et al., 2021). In the light of the escalated situation concerning DT, there seems to be a need for providing a deepening of the DT aspects of mathematical competencies descriptions (e.g., Geraniou & Jankvist, 2019; Jankvist et al., 2018). And this is not only from a practice perspective but also from the perspective of doing research related to the use of technology in the mathematics programmes—or any other educational system relying on competencies descriptions of mathematics.

M. Misfeldt (✉)
University of Copenhagen, Copenhagen, Denmark
e-mail: misfeldt@ind.ku.dk

U. T. Jankvist
Aarhus University, Copenhagen, Denmark
e-mail: utj@edu.au.dk

E. Geraniou
UCL Institute of Education, London, UK
e-mail: e.geraniou@ucl.ac.uk



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_11

This chapter focuses on the KOM-framework's aids and tools competency and investigates its application in the digital era of mathematical learning. The aids and tools competency may be viewed as distinguishing between more classical material or physical tools (e.g., centicubes, rod systems, abacuses, rulers, compasses, protractors, specially lined paper, cardboard for folding or cutting) and digital tools (e.g., calculators, computers and mathematical software, such as CAS and DGE). Although this distinction, at first sight, may appear straightforward, new available software such as virtual manipulatives may somewhat blur this picture—not least since some of these also aim at illustrating and explaining actual physical manipulatives. In the literature, it is well argued that physical manipulatives may provide learners with an element of embodied cognition of a given mathematical process or object (e.g., Clements, 2000). Surely, the effects of adding physical and virtual manipulatives to mathematics teaching have been both discussed and researched. In the reform wave of the 1980s, manipulatives were considered important and promising and were thus widely implemented. This led to several reflections and precautions expressed as an awareness to "advise [teachers] to view the appropriateness and limitations of the materials for the purpose of leading to and authenticating a part of formal mathematics" (Hart, 1993, p. 27), as well as simply noticing that "Although kinaesthetic experience can enhance perception and thinking, understanding does not travel through the fingertips and up the arm" (Ball, 1992, p. 47). Despite these disappointments, manipulatives persist to be an important part of teaching mathematics, and their educational value—if used meaningfully—stands uncontested (Bartolini & Martignone, 2014).

The digital development in general, and the widespread use of digital technologies and tools in the mathematical classroom in particular, has led to developments of *virtual manipulatives*. Some research (e.g., Tran et al., 2017) discusses how to "mirror" the affordances of physical tools within virtual tools, while others simply compare the educational value of physical and virtual manipulatives (Hunt et al., 2011). Other studies concern newly invented bodily actions to be associated with certain mathematical processes, e.g., multiplication and conceptions (e.g., Drijvers, 2019; Mariotti & Montone, 2020; Sinclair et al., 2020). Largely, these studies draw on aspects of embodied cognition (e.g., with reference to the early works of Lakoff & Núñez, 2000), or even explicate a combination of embodied cognition and the instrumental approach—embodied instrumentation approach (Drijvers, 2019; Shvarts et al., 2021).

In a Mathematics Education in the Digital Age (MEDA) conference paper (Jankvist et al., 2018), we have previously argued for the potential combination of KOM's aids and tools competency with elements of the instrumental approach (e.g., Guin & Trouche, 2002), since these two complement each other in certain desirable ways, at least if we restrict ourselves to "usual" DT such as CAS and DGE. We include the previously presented example and analysis from the MEDA paper in this chapter, but expand our scope by taking into consideration also new virtual manipulatives (e.g., Mariotti & Montone, 2020; Soury-Lavergne, 2021). More precisely, we ask the question:

*What do the instrumental approach and the embodied instrumentation approach offer to the discussion of students' aids and tools competency in situations involving digital technologies?*

We provide two examples in this chapter. Firstly, one of constructing slope fields showing solutions to a differential equation, showcasing the differences between two different approaches relying on use of DT. Second, two different examples of virtual manipulatives: one that mirrors physical manipulatives, a ruler, in a virtual setting; and one that relies on newly invented finger motions to simulate a mathematical process, in this case, that of multiplication. These examples serve as input for discussing answers to the research question above—and their respective analyses to yet a discussion of the potential of connecting these theoretical constructs from a networking of theories perspective. And if not "truly" networked (Bikner-Ahsbahs & Prediger, 2014), then at least how they may complement each other in order to enrich our understanding of how the theoretical constructs of embodied cognition and instrumental genesis, respectively, can facilitate the understanding of the mathematical aids and tools competency in the digital era.

## 2 The Mathematical Competency of Aids and Tools

Niss and Højgaard (2019) define "*A mathematical competency* is someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations" (p. 14, italics in original). A mathematical competency "focuses on the activation of mathematics to deal with a specific sort of challenge that actually or potentially calls for "specific kinds of activation" of mathematics in order to answer questions, solve problems, understand phenomena, relationships or mechanisms, or to take a stance or make a decision" (p. 14).

The KOM-framework operates with eight distinct, yet mutually related, mathematical competencies: mathematical thinking; problem tackling; modelling; reasoning; representation; symbols and formalism; communication; aids and tools. Each of these competencies consists of a producing side and an analytical side. The *aids and tools competency*, "consists of, on the one hand, *having knowledge of* the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their *possibilities and limitations* in different sorts of contexts, and, on the other hand, being able to reflectively *use* such aids" (Niss & Højgaard, 2011, pp. 68–69, italics in original). It continues:

> Mathematics has always made use of diverse technical aids, both to represent and maintain mathematical entities and phenomena, and to deal with them, e.g., in relation to measurements and calculations. This is not just a reference to ICT, i.e., calculators and computers (including arithmetic programmes, graphic programmes, computer algebra and spreadsheets), but also to tables, slide rules, abacuses, rulers, compasses, protractors, logarithmic and normal distribution paper, etc. The competency is about being able to deal with and relate

> to such aids. […] Since each of these aids involves one or more types of mathematical representation, the aids and tools competency is closely linked to the representation competency. Furthermore, since using certain aids often involves submitting to rather definite "rules" and rests on particular mathematical assumptions, the aids and tools competency is also linked to the symbol and formalism competency. […] (p. 69)

In their recent account of the eight mathematical competencies, Niss and Højgaard (2019) further elaborate:

> For millennia mathematics has made use of material aids and tools for its undertakings, from carved bones, counting pebbles (calculi), blocks, cords, rulers, compasses, abaci, slide rulers, mechanical instruments and machines, calculators, computers, tablets, smartphones and so on and so forth. Such aids and tools offer particular kinds of material representations of mathematical objects and processes, which typically require separate and particular theoretical and practical introduction and training before being put to use. This, together with the fact that by being artefacts such aids and tools have many physical properties without any bearing on mathematics, gives rise to particular challenges for being able to deal with them in a thoughtful manner in mathematical contexts and situations. (p. 18)

Again, the specificities related to tools of a digital nature is somewhat subtle. However, we may obtain some insights into Niss's view on this matter from another text published after the KOM-framework's coming into being. Niss (2016, p. 248) states that DT on the one hand may "enhance a wide variety of mathematical capacities", but on the other hand, also may "replace some mathematical competencies", which is not desirable. Among enhancement of mathematical capacities, Niss (2016) mentions that DT can:

> […] help generate student experiences of mathematics-laden processes and phenomena that might be difficult to obtain by other means; create platforms and spaces for exploration in which mathematical entities can be investigated through manipulation and variation; produce static and dynamic images of objects, phenomena and processes that are otherwise difficult to capture and grasp; create connections between different representations of a given mathematical entity; help solve hard or otherwise inaccessible computational problems; perform rule-based symbolic transformations and manipulations; support the production of mathematical texts; and create platforms for individualised training and assessment. (p. 248)

Among the things that DT cannot do for the teaching and learning of mathematics, Niss (2016) mentions:

> […] replace students' creation of meaning and understanding of mathematical concepts and results; replace reasoning and sound and critical judgement; replace problem-solving competency; replace symbols and formalism competency, including the ability to perform basic computations; construct, interpret, or validate mathematical models; and replace the work needed to understand "what?," "how?," and "why?" in mathematics. (pp. 248–249)

## 3 The (Embodied) Instrumental Approach

One of the frameworks on DT in mathematics education that has previously been applied in connection to the Danish KOM-framework is the instrumental approach (e.g., Geraniou & Jankvist, 2019; Jankvist et al., 2018) (also sometimes referred to as the "theory of instrumental genesis"). Drijvers and colleagues (2013) present the instrumental approach in terms of three dualities.

Firstly, the *artefact-instrument duality* describes the lengthy process of an artefact becoming an instrument in the hands of a user, which is referred to as *instrumental genesis*.

Secondly, the *instrumentation-instrumentalisation duality* concerns the relationship between the artefact and the user, i.e., how the user's knowledge directs the use of an artefact (instrumentalisation), and how a tool can shape and affect the user's thinking and actions (instrumentation). The process of instrumentation is closely connected to the digital tool serving an *epistemic purpose*, which means that it is used to create understanding or support learning within the user's cognitive system. By contrast, when DT are used to create a difference in the world external to the user, it is said to serve a *pragmatic purpose* (Artigue, 2002; Lagrange, 2005; Trouche, 2005). DT serves of course both pragmatic and epistemic purposes, but any use which is only, or mainly, pragmatic is according to Artigue (2010) of little—or even negative—educational value.

Thirdly, the *scheme-technique duality* concerns "the relationships between thinking and gesture" (Drijvers et al., 2013, p. 26). From a practical perspective, techniques can be seen as "the observable part of the students' work on solving a given type of tasks (i.e., a set of organised gestures) and schemes as the cognitive foundations of these techniques that are not directly observable, but can be inferred from the regularities and patterns in students' activities" (ibid., p. 27). For Vergnaud (2009), concepts are psychological entities fundamentally related to actions. Vergnaud refers to this relation as a *scheme*, and it can be defined as implicit or explicit ways of organising behaviour, involving also the necessary knowledge to act meaningfully in certain situations. Hence, a scheme combines intentions and actions with conceptual knowledge. Furthermore, schemes enable us to understand the conceptualisation process by linking gestures and thoughts through the encountering of various situations. Conceptualisation here refers to the process in which learners develop concepts and make connections in their knowledge. Drijvers et al. (2013) define a scheme as "a more or less stable way to deal with specific situations or tasks, guided by developing knowledge" (p. 27). These three dualities can be used as analytical constructs in exploring how the use of artefacts, such as DT, can shape the learning (and teaching) of mathematics (e.g., Geraniou & Jankvist, 2019).

In the past, there have been some initiatives to use DT that offer embodied experiences. For example, the Calculator Based Ranger (CBR), a small motion detector to be connected to the TI Graphing Calculator, and therefore used in Distance Sensor Activities in mathematics lessons.[1] These provided students with an embodied experience to explore and interpret distance-time graphs, supported by the CBR tool. Students were given the opportunity to connect their physical activity of walking with its graphical representation and therefore support their learning about the rate of change of a linear graph and the motion of an object in terms of distance versus

[1] See: https://www.mydigitalchalkboard.org/portal/default/Resources/Viewer/ResourceViewer?action=2&resid=60193.

time. Taking such activities into account, we may certainly agree on how powerful embodied experiences are, and how technology may support such experiences and students' emerging cognition. Drijvers (2019) argues for the importance of an integrative approach between embodied cognition and instrumental genesis for enriching mathematical practices and learning, pointing out that we should consider the benefits and the limitations, the opportunities and the challenges when using physical tools as well as digital tools. Of course, there may be differences in how physical tools, in the light of the theory of embodied cognition, can influence and in fact transform mathematical thinking, compared to how mathematical thinking is transformed in the case of DT use in accordance to the instrumental approach. Embodied experiences lead to sensorimotor schemes (e.g., Arzarello et al., 2009; Maschietto & Bartolini-Bussi, 2009), whereas instrumental approaches are known to lead to instrumentation schemes as paved by the lengthy process of instrumental genesis (e.g., Roorda et al., 2016). In an effort to explore the co-existence of such schemes, Drijvers (2019) introduced the idea of *embodied instrumentation approach*, which is defined as learning that occurs during techno-physical interaction with any DT. This approach "explores the co-emergence of sensorimotor schemes, tool techniques and mathematical cognition, and offers a design heuristic for ICT activities which align the bodily foundations of cognition and the need for instrumental genesis" (p. 22). Some further research on embodied instrumentation has been carried out. For example, Alberto et al. (2019) explored how a student used both her own body and a digital tool to solve trigonometric equations, and thus showed how embodied instrumentation promoted trigonometric problem-solving.

### *3.1 Embodied Cognition*

In more general terms, *embodied cognition* can be described as the idea that body and mind are more intimately connected than we usually think. Embodied cognition contests the dualistic idea that the body and the mind are separate and replaces it with a conception of cognition as an entanglement of bodily experience and thought—seen as one joint experience instead of two separate processes (McNerney, 2011). An important consequence for mathematics—and mathematics education—thus consists in avoiding seeing mathematics and mathematical insights as the product of a "pure mind". Rather it should be viewed as a consequence of our human embodied cognition and social interaction. This means that human experience, language and representations become crucial elements when trying to understand the nature of mathematics as well as the experience of doing and comprehending mathematics (de Freitas & Sinclair, 2014; Menary, 2015).

In their book, "Where mathematics comes from", Lakoff and Núñez (2000) take the insights that cognitive science offers about the human mind as an outset for understanding mathematics as *a human endeavour*. Their key method is to point to generative conceptual metaphors that ground mathematical concepts in human experience. This leads to an analysis that roughly considers, for example, mathematical sets as grounded in the experience of collecting objects, and arithmetic and numbers as grounded in comparing such collections. Negative numbers are viewed as grounded in experience of direction, and infinity and continuity in experiences of iterative processes that go on and on. Furthermore, these concepts are linked metaphorically, e.g., the number line connects numbers with continuity and infinity.

Another important consequence of the embodied approach to cognition is that it highlights the importance of "the surroundings/environment" for cognitive work. By using language, representations or other artefacts with which humans are able to augment their cognitive capabilities, and—in relation to mathematics—shape their potential for mathematical insights. Such cognitive tools allow anchoring, generation of overview, conceptual blending and further investigation of mathematical phenomena (Johansen & Misfeldt, 2020; Menary, 2015). Menary (2015) describes this as a process of enculturation, where there is a delicate interplay, not only between the mind and the body but also between the cognitive (body-mind) processes and the surrounding environment. One shapes one's cognitive niche by active use of tools, representations and metaphors (Menary, 2015). The shaping does not happen in a vacuum but is sanctioned by both natural phenomena (such as those described by Núñez, 2009) and by logic, culture and tradition.

In their book, "Mathematics and the body", de Freitas and Sinclair (2014) criticised the approach to embodiment suggested by Lakoff and Núñez (2000), because the latter "remain committed to an immaterial mathematical concept" (de Freitas & Sinclair, 2014, p. 200). Instead, de Freitas and Sinclair take their departure in Barad's (2007) agential realism in order to develop an embodied theory of the role played by materiality (tools, manipulatives, language and representations) in school mathematics. Barad's key idea of agential realism (developed in an analysis of modern physics) can be seen as a strong rejection of the individualist "brain in a vat" approach to understand how humans make sense of the world. Rather she views phenomena as a constant interaction between various material and nonmaterial agents. In that sense, human understanding of mathematical concepts and phenomena are inseparable from the bodily experienced representations, and tools that mediate interaction between them. Furthermore, de Freitas and Sinclair's (2014) work builds on Châtelet's (2000) notion of *virtual* mathematical objects: "a state of being that is *both* physical and mathematical" (cited from de Freitas & Sinclair, 2014, p. 201), meaning that they are governed by both logical rules and by material experiences. The physical and mathematical aspects are seen as two different dimensions of a virtual concept, and these two dimensions are always present in mathematics.

## *3.2 Body-Artefact System*

In his paper on embodied instrumentation, Drijvers (2019) described a gap between the embodied and the instrumented strategies to understand mathematical cognition. He claimed that these approaches "share some similar theoretical bases, and can be coordinated and aligned in a meaningful way" and that an embodied instrumentation approach may "reconcile the embodied nature of instrumentation schemes and the instrumental nature of sensorimotor schemes" (pp. 21–22). In line with Drijvers (2019), we will look at embodiment and instrumentation as closely related, but rather than focusing on the interaction between these two approaches, we will study how the approaches both support and extend, our understanding of the use of aids and tools in mathematics.

Building on this theoretical approach, Shvarts et al. (2021) further developed the idea of a "body-artefact functional system" that takes departure in the interplay between goal-directed activities and the affordances that artefacts and the environment provide to meet one's bodily potential. This view combines instrumental genesis with considerations about what types of thinking that are possible from our bodily outset. In this sense, the body-artefact functional system views the body potential in the light of both the intentions in play and the environment and artefacts available.

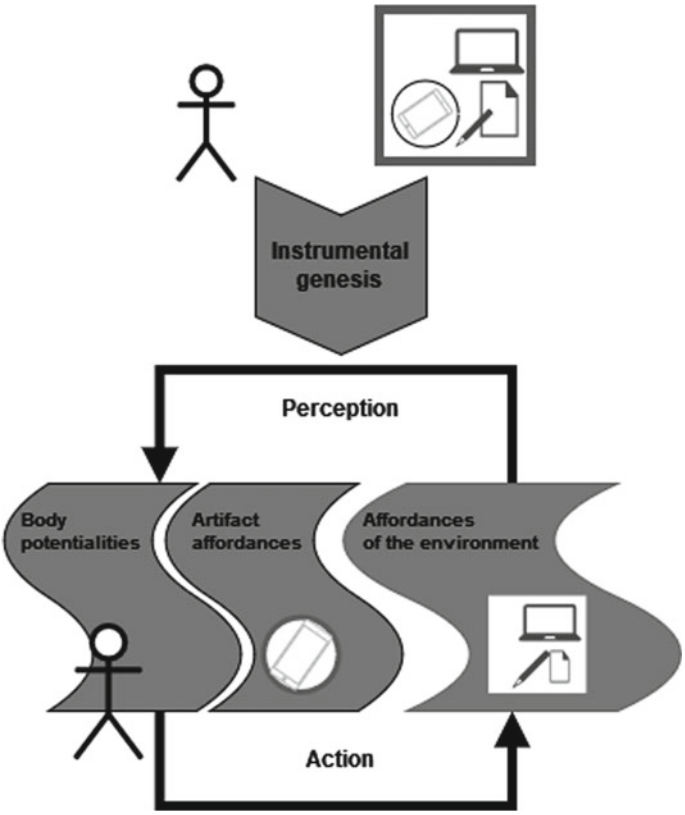



**Fig. 1** Instrumental genesis from a radical embodied point of view. This diagram shows the body-artefact functional system as defined by Shvarts et al. (2021), and is redrawn from that paper (their Fig. 3, p. 455)

Where the instrumental genesis approach studies the development of a person-artefact constellation in relation to goal-directed activity, the body-artefact functional system carves out the bodily potential for developing strategies and approaches with the artefact in the environment (Fig. 1).

## 4 First Example: Slope Fields

The first example stems from our own practice. The first and second authors have both taught slope fields to the mathematics education students at the Danish School of Education, Aarhus University. Although the approach to teaching this topic has changed over the years, the aids and tools competency has always been in focus.[2] In 2009 and 2011, the approach was on learning how to programme a computer to create slope fields of simple differential equations. This was done both with the free CAS system, *Wiris*, and with the DGS, *GeoGebra*. The activities were based on a constructionist approach in the sense that there was a focus on students developing their own mathematical tools (Papert, 1980). In 2013, 2015 and 2017, the approach was changed to use *Wolfram Alpha* instead, i.e., to simply call commands that plot the slope fields. The reason for this change was multi-faceted, but one of the main problems experienced with the first approach was that it simply was too much work and effort to create the string of code required to plot a slope field. The amount of knowledge about loops/sequences, and about how to plot vectors in a lattice that are needed in order to develop one's own slope field plot with tools like *Wiris* or *GeoGebra*, did not seem to be worthwhile. Rather it—in this specific case—moved the students' focus away from the numerical solutions of differential equations.

One example showcasing the differences between the two approaches is how to create a slope field showing solutions to the equation: $\mathrm{d}y/\mathrm{d}x = \sin(x)\sin(y)$ (Fig. 2).

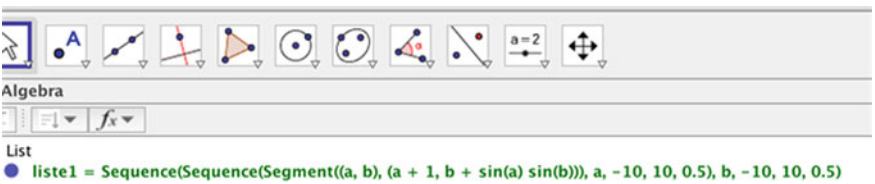



**Fig. 2** The command line in GeoGebra for a 'home made' slope field

[2] This example and the subsequent analysis in terms of both the aids and tools competency and the instrumental approach was initially presented at the 2018 MEDA conference (Jankvist et al., 2018).

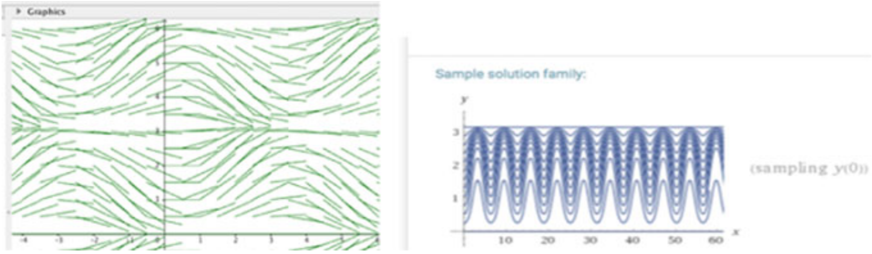



**Fig. 3** Coarse ‘home made’ GeoGebra slope field versus smooth Wolfram Alpha plot

The resulting slope field looked rather coarse. On the contrary, if the differential equation is typed into *Wolfram Alpha*, you immediately get the stepwise solution as well as an illustration of the solution curves. In this case, the image from *Wolfram Alpha* is more illustrative and detailed (see Fig. 3).

## *4.1 Analysis of the First Example*

From a competencies’ perspective, the example above calls for students to apply their aids and tools competency to, firstly, know that there exists DT for constructing slope fields and that this can be beneficial in cases where differential equations cannot be solved analytically. Secondly, the aids and tools competency may come into play, if students are to choose between the two different approaches laid out above, i.e., is it more beneficial, also from a learning point of view, to program one’s own plotter, or is it perfectly fine to use the already made app, e.g., that of *Wolfram Alpha*, knowing that it will blackbox several of the underlying processes?

In terms of the artefact-instrument duality, we see that the choice of technology for visualising slope fields has consequences reaching further than just this specific task and topic. The resulting instrumental genesis leads to students’ familiarity and control over the tool they use. Even though both tools are relevant from a mathematics education point of view, they are very different, and familiarity with each of these tools might influence the learning of mathematics further. The instrumentation-instrumentalisation duality, as well as the scheme-technique duality, can be used to look at the details in these differences. The case where *GeoGebra* is used to create slope fields clearly brings the tool to a use that might not be directly intended by the creators of *GeoGebra*, and it clearly pushes the software a little out of the usual scenario (for instance by creating a lattice as a sequence of sequences in order to place a line segment at each lattice point). This means, on the one hand, that the students will be required to instrumentalise *GeoGebra* and take control over it (and this can obviously benefit their future ability to use *GeoGebra*). On the other hand, the focus of the work with *GeoGebra* is on creating a lattice, and perhaps on controlling the length of the line elements (they can get rather large or small—making the image incomprehensible). Hence, this use of *GeoGebra* is instrumentalising the students to focus more on the procedure of creating the slope field (deciding on a lattice, and "programming" a procedure for setting line segments from each point) than on the actual layout of the slope field. The instrumented techniques obtained might focus on a number of technical concerns that are of little relevance to understanding the involved mathematics, which might (this is a hypothetical analysis) pollute the students' scheme of differential equations and (numerical) solutions to such. Hence, in this case, the instructor needs to pay special attention to bringing into play the students' schemes of numerical and analytical solutions to differential equations in relation to the slope field plot. Seen as an embodied activity the "sideward sliding" of the activity becomes even more obvious. The cognitive challenge dealing with placing a relevant lattice on the plane, and controlling the placement and size of the line segment really do require cognitive activity that can meaningfully build on embodied cognition—but the focus is far from being centred on differential equations, and more related to the practicalities of constructing slope fields.

The work with *Wolfram Alpha*, however, is focussed directly on the visuals of the slope fields, blackboxing everything leading to this image. Furthermore, the case of working with a differential equation and visualising the family of solutions does seem to be considered by the developers of *Wolfram Alpha.* Writing the differential equation into the system automatically gives access to the solution (including—in the premium version—a stepwise solution replicating a paper-and-pencil solution) as well as relevant visualisations of families of solutions. The students' instrumentation of *Wolfram Alpha* is thus almost salient. The instrumentalisation might go in different directions depending on the focus of the teaching and the abilities and preferences of the students. *Wolfram Alpha* allows for the development of a completely blackboxed trial-and-error technique, where the student simply tries various commands in the command field and sees if the input is somehow interpretable with regard to the task at hand. Such a technique might not lead to the development of a strong and relevant scheme for differential equations (for related discussions, see Jankvist & Misfeldt, 2015; Jankvist et al., 2019). However, the DT allows students to investigate and explore mathematics without the technical barriers that were experienced when programming in *GeoGebra.* This may lead students more directly to consider families of solutions to differential equations, which should force them to activate their schemes related to, for example, what it means to be a solution to a differential equation as well as, say, the difference between numerical and analytic solutions to differential equations.

## 5 Second Example: Virtual Manipulatives

As a second example, we concentrate on virtual manipulatives, and how such digital resources have been created to represent physical manipulatives and physical tools taking into account the affordances and limitations of DT. In the subsequent analysis,

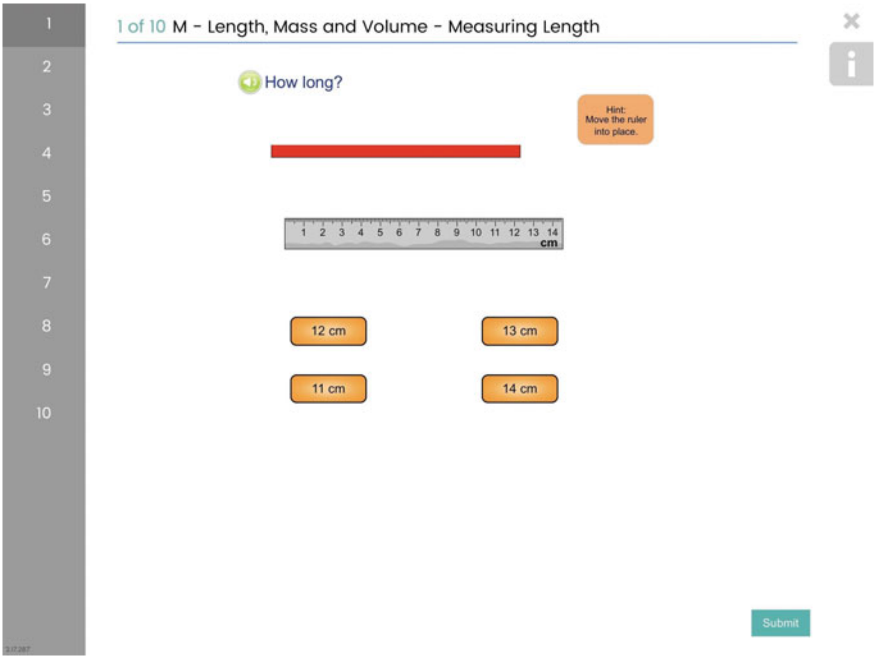



**Fig. 4** Students are expected to drag the ruler and place it beneath the red rod so as to measure its length and identify which is the correct answer out of the 4 provided

we discuss how these digital resources may create opportunities for learners to have embodied experiences with the support of DT (Drijvers, 2019).

Calculators, tablets, smartphones and smartwatches can be used as the "current" digital media for accessing mathematical features (e.g., Drijvers, 2019). Virtual manipulatives—or in other words "digital entities whose manipulation on the screen makes it possible to represent a mathematical concept, a relationship or a procedure" (Moyer-Packenham et al., 2002, as cited in Soury-Lavergne, 2021, p. 4)—have made more frequent appearances in the mathematics classroom in the last decade or so. A crucial affordance of these digital resources is how they enable the learner to "dynamically" interact with them (e.g., Moyer-Packenham, 2016), and therefore be supported in a mathematical task. For example, the "virtual" ruler in the *Mathletics* application,[3] in which students are prompted to drag and place the ruler beneath the red rod, so as to measure its length (see Figs. 4 and 5) is a virtual manipulative designed to support young students' measuring competency.

[3] See: https://www.mathletics.com/uk/.

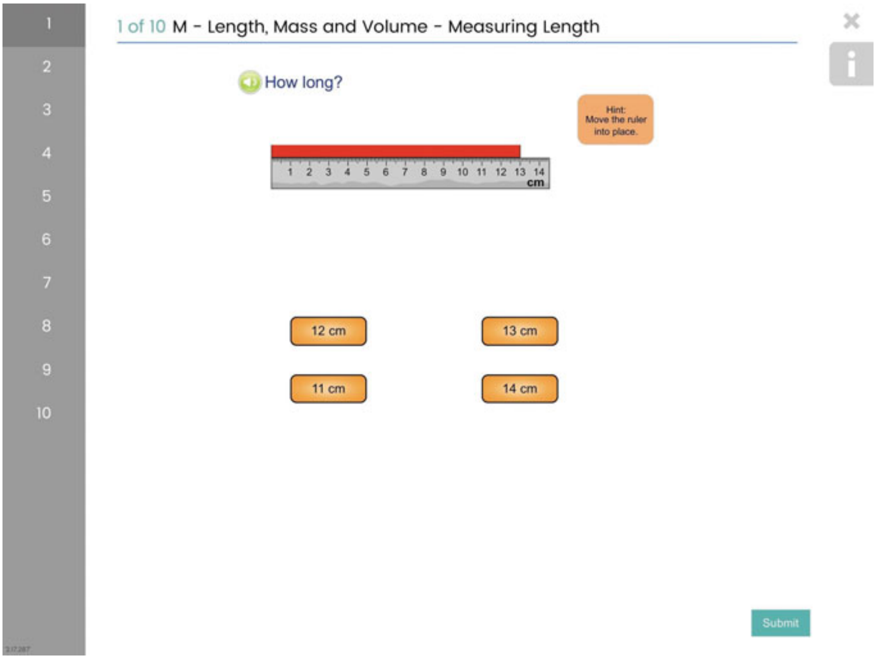



**Fig. 5** The student places the ruler under the red rod and can see that the length is 13 cm

Other examples of virtual manipulatives can be found in other online applications, such as the *Mathigon's polypad* from the *Mathigon* online interactive mathematics textbook.[4] This offers students the opportunity to manipulate visual representations of mathematical shapes and concepts involved in the topics of: geometry (such as 2D shapes for exploring their properties); number (such as number lines for understanding order of numbers); fractions (such as fraction bars or wall for interpreting equivalence of different fractions); algebra (such as balance scale for solving equations); and probability (such as coins, dice and spinners to use for exploring experimental probability).

There are also tools involving multi-touch screen technology that offer the opportunity to look into an embodied approach when a learner uses them. Such tools are *TouchCounts* (TC) and *TouchTimes* (TT), created by Nathalie Sinclair and Nicholas Jackiw.[5] TC is designed to support children in developing a strong number sense by enabling them to use their fingers, eyes and ears to create and manipulate their own numbers and therefore learn to count, add and subtract (e.g., Sinclair & Heyd-Metzuyanim, 2014). TT is designed to enable children to use their fingers to carry out

[4] See: https://mathigon.org/ and https://youtu.be/vwyIZsi0b98 by following the link: https://mathigon.org/polypad.

[5] See: http://touchcounts.ca/ and http://touchcounts.ca/touchtimes/index.html.

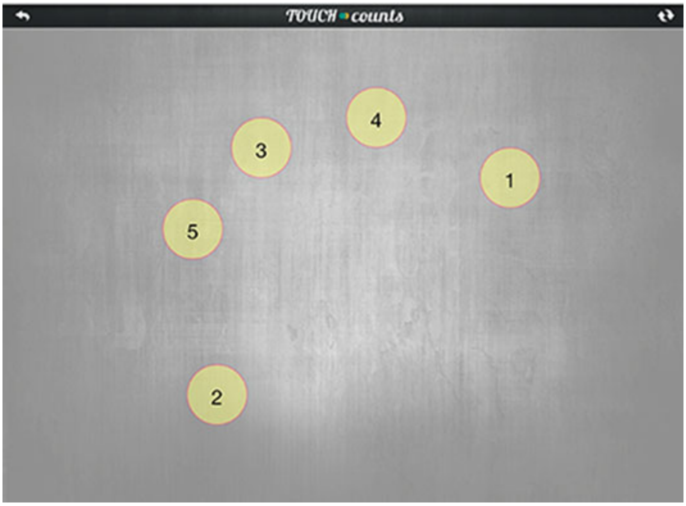



**Fig. 6** Numbers 1, 2, 3, 4 and 5 represented in TouchCounts

multiplications (e.g., Bakos & Sinclair, 2019; Sinclair et al., 2020). They can use two-handed gestures to create their own factors and products and therefore learn about multiplicative relationships. In both these applications, numbers are represented with a circle and the mathematical notation of the number in question presented inside that circle. So, for example, the numbers "1", "2", "3", "4" and "5" are represented in TC as shown in Fig. 6.

## *5.1 Analysis of the Second Example*

From a competencies' perspective, the examples presented above encourage students to apply their aids and tools competency to recognise a mathematical tool, as this is represented virtually in a digital tool. In the *Mathletics* example, such a tool is the ruler, and in the *Mathigon*, example, there are a number of virtual manipulatives designed as online interactive dynamic representations of the static real-life mathematical shape (2D shape on paper), object (such as a dice) or tool (such as a ruler). Even though all these virtual manipulatives are carefully designed to represent real-life manipulatives, students are still expected to activate their representation competency to be able to identify these in the digital tool. Or in other words, students are expected to have the knowledge of the properties of the tool even if it is presented in a "diverse" form of a digital resource. They are also expected to recognise how they are meant to use these virtual manipulatives in the digital learning environment in light of the affordances of that learning environment. The fact that these tools are designed to mirror the tangible object, their manipulation in the digital medium is different to the manipulation of the tangible object in real life. The example with the "virtual ruler" in the Mathletics application could be perceived as a virtual manipulative that just resembles a "real-life" tool and its proposed or expected use in this environment is an imitation of what the user/learner would do when using a physical ruler to measure one dimension of a real-life object or its two-dimensional representation on paper. Having the mathematical competency of using a ruler to measure a given length, provided that one is familiar with the functionality of "drag & drop" (digital competency), which exists in many digital tools used in education, then one should be able to carry out the given task with minimal support by the teacher. The digital ruler is actually designed for embodied cognition; it supports direct comparison between sizes and hence builds on some of the central embodied metaphors. It combines this with the aids and tools competency and a well-organised representation in a synergetic fashion. The ruler is a tool—and the *Mathletics* version of this tool supports the externalisation of cognitive processes together with the basic embodied experience of comparing sizes. Such a virtual ruler for training measurement can of course also be designed with more complex epistemic agendas. For example, if the ruler did not begin with zero (but for example with a random whole number) the manipulative would support more abstract thinking about measuring and measurement.

In the TT and TC applications, students are encouraged to recognise a visual representation of a number, similar to a 2D counter, that is a circle—but with the mathematical notation of that number added inside the circle. Children use their fingers to tap on the screen and create the next number in the numerical sequence, counting up. These actions are certainly directly linked to embodied instrumentation. The tools involve an action of embodied cognition that is empowered by the affordances of digital technologies. The connection to bodily experience is facilitated both by the way this tool enhances metaphors of sets and small numbers and because of the very direct way that the tool supports two-handed gestures and bodily memory. When placed in such a didactical situation, the learner's mathematical knowledge and competencies will influence the way the tool is used, involving certain "gestures-on-screen" and "dragging interactions" (instrumentalisation). At the same time, the way the tool is used will affect the mathematical learning process and the learning outcomes (instrumentation). The learner's mathematical knowledge and the aids and tools competency form a certain scheme, which then prescribes the learner's actions (as presented in the scheme-technique duality of the instrumental genesis). The dragging of a virtual manipulative can be perceived as the technique in the scheme-technique duality—that we take the liberty of referring to as a "virtual gesture". When carrying out a mathematical task in a digital learning environment, possessing aids and tools competency can result in acting meaningfully and dragging virtual manipulatives in a meaningful way, indicating an enriched mathematical learning experience. As Drijvers (2019) argued, "instrumental genesis is not just an individual process, but is part of social learning processes and institutionalisation within the specific educational context" (p. 16). In the second example, the specific educational context is the learning of counting and multiplying within the TC and

TT applications, and the use of the virtual manipulatives in *Mathletics* and the *Mathigon's Polypad*. Students' interactions with such virtual manipulatives, may well be influenced by past social experiences involving tangible manipulatives, which can lead to virtual manipulatives becoming institutionalised.

## 6 Discussion of Connecting Theoretical Perspectives

We began this chapter by asking what the instrumental approach and the embodied instrumentation approach have to offer to the discussion of students' aids and tools competency in situations involving digital technologies. We have replied to this question by providing examples of tool use in two different mathematics teaching situations. In relation to the first example, the KOM-framework offers a rather limited analysis in terms of the aids and tools competency. It does, however, articulate students' needs to know about the digital tools' strengths and weaknesses, not least in relation to specific mathematical representations, which the instrumental approach does not do as explicitly. The instrumental approach focuses more directly on the students' interactions with the DT, rather than merely addressing students' knowledge about these. The embedded notion of scheme enables us to address students' conceptual understanding in relation to differential equations and solutions of such. This insight is not new—as mentioned we have previously provided a similar analysis (Jankvist et al., 2018). Yet, as argued above, recent theoretical developments and foci within mathematics education point to the fact that the instrumental approach might provide a better understanding of students' work with DT, if it is related to discussions of embodiment—as for example, Drijvers (2019) suggests. Hence, in the second example, we see that the aids and tools competency builds also on the ability to use tools as a way to activate embodied cognitive metaphors, and as a way of supporting and designing the use of gestures in reasoning and meaning-making.

Furthermore, the interplay between the embodied instrumentation approach, as introduced by Drijvers (2019) and Shvarts et al. (2021), and KOM's aids and tools competency makes it interesting to ask the question of how aids and tools support embodied and distributed cognitive processes—simultaneously. For one, aids and tools can activate embodied cognition and make students use their bodily experience in mathematical meaning-making processes. Of course, aids and tools can automate and simplify mathematical cognitive processes. Yet, the entanglement of these two types of processes is, in our view, as delicate as it is important.

Rather than viewing distributing cognitive processes to aids and tools, as a process of toning down or even giving up embodied meaning-making, we believe that it should be viewed as a process of domesticating and balancing the degree to which embodied processes are foregrounded. Embodied cognition plays a vital role also in abstract mathematics. Yet, in order to work with abstract mathematics, competencies in working with aids, tools, symbols and representations in a goal-directed manner shaping these tools in light of the task at hand, are critically important. Some of the local processes might be much better described with the instrumental approach than with embodied cognition, but this should not lead to disregarding the importance of the body. The competency to use aids and tools to activate embodied cognitive processes is important, and it is entangled with the ability to steer the instrumental genesis process in a productive and meaningful direction obtaining a balance between distributing/disembodying and (re)embodying these tools.

Referring back to Niss' (2016) comments of caution when integrating DT into mathematics teaching and learning, it seems clear that an embodied cognition and instrumentation approach to using DT is not about "replacing mathematical competencies". It is about creating bodily means for generating meaning and understanding of mathematical concepts, results and processes. It is at the core of mathematical concept formation, mathematical thinking and reasoning. It is about becoming acquainted with the abstract. It is an embodiment of the "what?", "how?" and "why?" in mathematics.

Hence, from our perspective, the connection—or networking—of these perspectives appears both feasible and quite promising in relation to looking deeper into students' possession and development of mathematical competencies in the digital era. Surely, there are fundamental issues to be addressed, e.g., if the discussion of to what extent the underlying philosophies of learning of the theoretical approaches of mathematical competencies and embodied instrumentation are at all compatible, and if so, then on which terms and to what extent. Such discussion and analysis, however, goes beyond the scope of this chapter. Here we have merely pointed to the potential of connecting KOM's aids and tools competency with the constructs of embodied cognition and instrumentation and illustrated this through the analyses of two examples. The reason for this, as discussed previously, is the entrance of new virtual manipulatives, which challenge the existing frameworks "on the market" in relation to their explanatory power.

**Acknowledgements** This chapter was partly written in the frame of project 8018-00062B under *Independent Research Fund Denmark* and project NNF19OC0058651 under *Novo Nordisk Foundation.*

## References

Alberto, R., Bakker, A., Walker-van Aalst, O., Boon, P., & Drijvers, P. (2019). Networking theories in design research: An embodied instrumentation case study in trigonometry. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education* (pp. 3088–3095). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Artigue, M. (2010). The future of teaching and learning mathematics with digital technologies. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain. The 17th ICMI study* (pp. 463–475). Springer. https://doi.org/10.1007/978-1-4419-0146-0_23

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109. https://doi.org/10.1007/s10649-008-9163-z

Bakos, S., & Sinclair, S. (2019). Exploring the semiotic potential of TouchTimes with primary teachers. In J. Novotná, & H. Moraová (Eds.), *Proceedings of international symposium of elementary mathematics teaching: Opportunities in learning and teaching elementary mathematics* (pp. 53–62). Charles University, Faculty of Education.

Barad, K. (2007). *Meeting the universe halfway: Quantum physics and the entanglement of matter and meaning*. Duke University Press. https://doi.org/10.2307/j.ctv12101zq

Ball, D. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator, 16*(2), 14–18, 46–47.

Bartolini, M. G., & Martignone, F. (2014). Manipulatives in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education.* Springer. https://doi.org/10.1007/978-94-007-4978-8_93

Bikner-Ahsbahs, A., & Prediger, S. (2014). *Networking of theories as a research practice in mathematics education.* Springer. https://doi.org/10.1007/978-3-319-05389-9

Châtelet, G. (2000). *Figuring space philosophy, mathematics, and physics.* Kluwer Academic Publishers.

Clements, D. H. (2000). 'Concrete' manipulatives, concrete ideas. *Contemporary Issues in Early Childhood, 1*(1), 45–60. https://doi.org/10.2304/ciec.2000.1.1.7

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom.* Cambridge University Press. https://doi.org/10.1017/CBO9781139600378

Drijvers, P. (2019). Embodied instrumentation: Combining different views on using digital technology in mathematics education. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education* (pp. 8–28). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics, 82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: Necessity of instrumental orchestration. *Zentralblatt Für Didaktik Der Mathematik, 34*(5), 204–211. https://doi.org/10.1007/BF02655823

Hart, K. (1993). Confidence in success. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F.-L. Lin (Eds.), *Proceedings of psychology of mathematics education*, PMEXVII (Vol. 1, pp. 17–31). University of Tsukuba.

Hunt, A. W., Nipper, K. L., & Nash, L. E. (2011). Virtual vs. concrete manipulatives in mathematics teacher education: Is one type more effective than the other? *Current Issues in Middle Level Education*, *16*(2), 1–6.

Jankvist, U. T., Dreyøe, J., Geraniou, E., Weigand, H.-G., & Misfeldt, M. (2021). CAS from an assessment point of view: Challenges and potentials. In A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, J. Trgalova, & H.-G. Weigand (Eds.), *Mathematics education in the digital age: Learning, practice and theory* (pp. 99–120). Routledge.

Jankvist, U. T., Geraniou, E., & Misfeldt, M. (2018). The KOM framework's aids and tools competency in relation to digital technologies—A networking of theories perspective. In H.-G. Weigand, A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, N. Grønbæk, & J. Trgalova (Eds.), *Research proceedings of the fifth ERME topic conference (ETC 5) on mathematics education in the digital age (MEDA)* (pp. 123–130). University of Copenhagen and ERME.

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics? *For the Learning of Mathematics, 35*(1), 15–20.

Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). Tschirnhaus' transformation: Mathematical proof, history and CAS. In E. Barbin, U. T. Jankvist, T. H. Kjelsen, B. Smestad, & C. Tzanakis (Eds.), *Proceedings of the Eighth European Summer University on History and Epistemology in Mathematics Education ESU 8* (pp. 319–330). Oslo Metropolitan University.

Johansen, M. W., & Misfeldt, M. (2020). Material representations in mathematical research practice. *Synthese, 197*, 3721–3741. https://doi.org/10.1007/s11229-018-02033-4

Lagrange, J.-B. (2005). Using symbolic calculators to study mathematics. The case of tasks and techniques. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators. Turning a computational device into a mathematical instrument* (pp. 113–135). Springer.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Mariotti, M. A., & Montone, A. (2020). The potential synergy of digital and manipulative artefacts. *Digital Experiences in Mathematics Education, 6*(2), 109–122. https://doi.org/10.1007/s40751-020-00064-6

Maschietto, M., & Bartolini-Bussi, M. (2009). Working with artefacts: Gestures, drawings and speech in the construction of the mathematical meaning of the visual pyramid. *Educational Studies in Mathematics, 70*(2), 143–157. https://doi.org/10.1007/s10649-008-9162-0

McNerney, S. (2011). A brief guide to embodied cognition: Why you are not your brain. *Scientific American, 4* [Blog post].

Menary, R. (2015). Mathematical cognition—A case of enculturation. In T. Metzinger, & J. M. Windt (Eds.), *Open MIND*: *25*(T) (pp. 1–20). MIND Group. https://doi.org/10.15502/9783958570818

Moyer-Packenham, P. (2016). Revisiting the definition of a virtual manipulative. In P. Moyer-Packenham (Ed.), *International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 3–23). Springer. https://doi.org/10.1007/978-3-319-32718-1_1

Moyer-Packenham, P., Bolyard, J., & Spikell, M. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*(6), 372–377.

Niss, M. (2016). Mathematics standards and curricula under the influence of digital affordances: Different notions, meanings, and roles in different parts of the world. In M. Bates & Z. Usiskin (Eds.), *Digital curricula in school mathematics* (pp. 239–250). Information Age Publishing.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Núñez, R. (2009). Numbers and arithmetic: Neither hardwired nor out there. *Biological Theory, 4*(1), 68–83. https://doi.org/10.1162/biot.2009.4.1.68

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. Basic Books.

Roorda, G., Ros, P., Drijvers, P., & Goedhart, M. (2016). Solving rate of change tasks with a graphing calculator: A case study on instrumental genesis. *Digital Experiences in Mathematics Education, 2*(3), 228–252. https://doi.org/10.1007/s40751-016-0022-8

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics, 107*(3), 447–469. https://doi.org/10.1007/s10649-021-10053-0

Sinclair, N., Chorney, S., Güneş, C., & Bakos, S. (2020). Disruptions in meanings: Teachers' experiences of multiplication in TouchTimes. *ZDM—Mathematics Education, 52*(7), 1471–1482. https://doi.org/10.1007/s11858-020-01163-9

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with TouchCounts: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning, 19*(1–2), 81–99. https://doi.org/10.1007/s10758-014-9212-x

Soury-Lavergne, S. (2021). Duos of digital and tangible artefacts in didactical situations. *Digital Experiences in Mathematics Education, 7*(1), 1–21. https://doi.org/10.1007/s40751-021-00086-8

Tran, C., Smith, B., & Buschkuehl, M. (2017). Support of mathematical thinking through embodied cognition: Non-digital and digital approaches. *Cognitive Research: Principles and Implications, 2*, 16. https://doi.org/10.1186/s41235-017-0053-8

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators. Turning a computational device into a mathematical instrument* (pp. 197–230). Springer.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

# The Three Types of Overview and Judgement

# Mathematics in Action: On the *Who*, *Where* and *How* of the Constructions and Use of Mathematical Models in Society

**Raimundo Elicer and Morten Blomhøj**

## 1 Introduction

Mathematics, as a discipline, has several meanings and thus plays different roles in society. As a social activity, it may be conceived of as a pure or applied science, a school subject, or even an aesthetical position (Niss, 1994). Amongst this plurality of meanings, mathematics is also a "system of instruments, products as well as processes that can assist decisions and actions related to the mastering of extra-mathematical practice areas" (p. 367). Davis and Hersh (1986) portrayed this role as the prescriptive function of mathematical models in society, whereby, aside from its descriptive and predictive functions, it "leads to human action or automatically to some sort of technological action" (p. 120). Therefore, mathematics is embedded, more or less explicitly, in activities that transcend disciplinary boundaries, informing and shaping human action with real-world consequences. This recognition, known as *mathematics in action* (e.g., Skovsmose, 1994), has become one of Critical Mathematics Education's (CME) core preoccupations as a research programme (Ernest et al., 2016; Valero et al., 2015). Some remaining non-trivial research questions are whether, to what extent, and how such a critique of mathematics in action belongs in classroom praxis.

It is no surprise that the reality of mathematics in action is a reason to teach and learn mathematics and its applications in school. Niss (1996), however, made a clear distinction between (real) reasons and justifications or arguments. The former may

R. Elicer · M. Blomhøj (✉)
Danish School of Education, Aarhus University, Tuborgvej 164, 2400 Copenhagen, Denmark
e-mail: blomhoej@edu.au.dk

R. Elicer
e-mail: raimundo@edu.au.dk

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_12

be historical, implicit and fussy, whereas the latter becomes activated and explicit, and their solidity is subject to scrutiny. In this regard, Blum and Niss (1991) identified different types of justifications for teaching mathematical models and applications, among which the so-called *critical competence argument* focuses on (p. 43):

> … preparing students to live and act with integrity as private and social citizens, possessing a critical competence in a society, the shape and functioning of which are being increasingly influenced by the utilisation of mathematics through applications and modelling.

The critical argument or justification has mathematics in action as a premise and presumes that there is a teaching practice that can, in fact, activate a critical competence. In the context of teaching and learning mathematical modelling and applications, Kaiser et al. (2006) bundled justifications and teaching practices as *perspectives*. The critical perspective entails that "critiquing modelling is part of the learning that takes place in the process of doing modelling, and one of the aims is to produce critical, politically engaged citizens" (Barbosa, 2006, p. 296).

One way of positioning this perspective is by utilising the highly influential KOM framework. The KOM project, first reported in Danish (Niss & Jensen, 2002), is an effort to understand and categorise a general mathematical competen*ce* into eight interrelated mathematical competen*cies* and three types of overviews and judgements about mathematics (Niss & Højgaard, 2011). We deem the critical perspective on modelling to be framed by the KOM framework not as a competency but as an overview and judgement. It becomes necessary to distinguish it from the mathematical modelling competency that belongs to the same framework and is addressed, for example, in Blomhøj (2020), and Blomhøj and Jensen (2007). The critical perspective is distinct from others, as it calls not only to be critical *with* and *in* mathematical modelling but also *about* mathematics in action. In the KOM framework, modelling competency engages mainly with the first two prepositions. However, in the overviews and judgements, mathematics becomes 'the object of explicit treatment, reflection and articulation' (Niss & Højgaard, 2011, p. 74f). The three overviews and judgements concern the application of mathematics (OJ1), its historical development (see, e.g., Thomsen & Clark, 2022), and its nature as a discipline (see, e.g., Østergaard & Sun, 2022). Therefore, reflecting *about* mathematics in action is framed as OJ1, the overview and judgement about the actual application of mathematics within other disciplines and fields of practice.

In their description of overviews and judgements, Niss and Højgaard (2011, 2019) did not engage in any particularities of the digital era. Recent research has shed light on links between the use of digital technologies and some of the mathematical competencies, in particular about aids and tools (Jankvist et al., 2018), problem tackling (Geraniou & Jankvist, 2019) and representation (Pedersen et al., 2021). This chapter aims to build this up further and explore how digital tools enter the picture regarding OJ1. First, we discuss how to theoretically frame OJ1 as an observable construct by drawing on the distinction between reflections that are internal and external to the modelling process (Blomhøj & Kjeldsen, 2011). Secondly, we borrow the instrumentation-instrumentalisation duality from the theory of instrumental genesis (Artigue, 2002) to account for students' use of digital technologies in mathematical modelling. Lastly, we argue for the potential of digital technologies to develop OJ1 by networking on these theories (Prediger et al., 2008). We illustrate these challenges by reflecting on two projects from our teaching experience at the Department of Science and Environment at Roskilde University.

## 2 OJ1, Internal and External Reflections

As discussed above, the critical perspective on modelling stands on the premise of recognising mathematics in action. In the KOM framework, Niss and Højgaard (2019, p. 24) phrase this assumption as follows:

> Mathematics is widely used for extra-mathematical purposes in a large variety of everyday, occupational, societal, scholarly and scientific undertakings. This use is brought about by the explicit or implicit construction or utilisation of mathematical models.

From that point of departure, they provide a characterisation of this overview and judgement. Beyond the reasons and arguments, we need to address and frame what it means to experience it. Niss and Højgaard (2011) did so by proposing a set of archetypical questions quoted in Table 1, first column.

**Table 1** OJ1 in question form

| Niss and Højgaard (2011, p. 75) | Niss and Højgaard (2019, p. 24) |
|---|---|
| Who, outside mathematics itself, actually uses it for anything? | Exactly which people are in fact using mathematics? |
| What for?<br>Why?<br>On what conditions?<br>With what consequences? | When, and in what contexts and situations do they use it and for what purposes? |
| How?<br>By what means?<br>What is required to be able to use it? | In what ways do they use it, and what are the competencies they possess and activate for so doing? |

The second column in Table 1 shows how Niss and Højgaard (2019) organised these questions into three in revisiting the KOM framework, which we interpret as overarching objects of reflection. Activating and observing OJ1 requires a theoretical framing that accounts for such reflective stance towards the *who*, *where* and *how* of mathematics in action.

*Reflective knowing* is a theoretical construct derived from John Dewey's notion of *reflective inquiry* (Dewey, 2015/1938). Skovsmose (1992, 1994) rescued Dewey's philosophy as a fundamental step to blur the line between knowing and doing. However, he was critical of Dewey's optimism about the scientific method, thus proposing a reflective knowing that, additionally, implies critiquing what is learned (Alrø & Skovsmose, 2002). Instead of a definition, Skovsmose (1994, Chap. 6) delineated a set of entry points or worry questions to adjust the lens towards students' reflections. This framework allows for distinguishing mathematical-oriented, model-oriented and context-oriented reflections (Skovsmose, 1998).

As part of developing a didactical theory of mathematical modelling (Kaiser et al., 2006; Niss & Blum, 2020), Blomhøj and Kjeldsen (2011) focused on characterising reflections in the process of mathematical modelling. They defined reflection as a "deliberate act of thinking about some actual or potential action aiming at understanding or improving the action" (p. 386). Moreover, they acknowledged that they could only be analysed through communicative acts. From here, they distinguished between *internal and external reflections*, depending on whether they refer to steps within the modelling cycle (see, e.g., Blomhøj, 2004) or to the use and consequences of the model.

## *2.1 A Competency and an Overview*

We have previously argued that OJ1 and modelling competency are distinct. However, they are not disjoint. In the KOM framework, mathematical modelling competency is defined as "a person's insightful readiness to autonomously carry through all aspects of a mathematical modelling process and to reflect on the modelling process and on the actual or potential use of the model in a particular context" (Blomhøj & Jensen, 2007, p. 48). By this definition, mathematical modelling competency consists of a productive side—being able to perform modelling—and an analytical side—being able to grasp and critically analyse extant models (Blomhøj & Niss, 2021).

Both aspects are closely related to the modelling process and involve reflections related to the sub-processes in the modelling cycle (Blomhøj, 2004). This type of relations can be characterised as *internal reflections.* They can be an integrated part of the modeller's work or a retrospective analysis of a modelling process performed by people not involved in the modelling process. Examples of questions initiating internal reflections are: "Why did we formulate the problem as we did? … Which essential elements did we include in our system and why? … What were our reasons for mathematising the system as we did? On what grounds did we estimate the parameters? …Why do we think that the model is valid concerning our problem, or why not?" (Blomhøj & Kjeldsen, 2011, p. 558).

Additionally, exercising modelling competency may provoke reflections on the roles and functions of mathematical models in application contexts. These are characterised as *external reflections* because they are not related to the modelling cycle (Blomhøj & Kjeldsen, 2011). They can take the form of reflections on the effects of the application of a model. These could be from an ethical, political, or economic perspective, as indicated in some of the projects mentioned above. They may also address general side effects caused by applying a mathematical model in a societal context, as Skovsmose (1990) pinpoints. These are (1) a reformulation of the problem in hand to be suitable for investigation through mathematical modelling, (2) changes in the discourse about the problem in the direction that is either pro or contra to the model and possible adjustments of the model, (3) a limitation of the possible actions taken into consideration to those that can easily be evaluated in the model, and (4) a delimitation of the group of people who can take part in the discussion and act as a basis of critique. Awareness of and experiences with such phenomena provide grounds for students' external reflections concerning the use of mathematical models in societal contexts.

Overall, the notions of reflective knowing and external reflections allow us to frame OJ1 as an observable construct. In principle, the *who*, *where* and *how* of mathematical modelling in society are external to the learners' modelling process. However, we further need to frame the contribution of digital tools to the mathematical modelling process of OJ1.

## 3 Networking of Theories and Digital Tools

The proliferation of numerous theoretical frameworks developed for specific purposes is an ongoing challenge to mathematics education as a field of research (see, e.g., Goos, 2018). As a response, Prediger et al. (2008) and Prediger and Bikner-Ahsbahs (2014) introduced the notion of the *networking of theories*, a continuum of strategies ranging from ignoring others to the eventual consolidation of a unifying global theory. The choice of a strategy depends on the purpose and the particular phenomenon that the researchers attempt to illuminate.

One prominent approach to theorising the use of tools—digital or otherwise—in mathematics education is the theory of instrumental genesis (TIG, e.g., Trouche, 2005). The theory stands on the basis that "an instrument is a mixed entity, part artefact, part cognitive schemes that make it an instrument" (Artigue, 2002, p. 250). *Instrumental genesis* refers to the process by which an artefact turns into an instrument. This process is bidirectional. The subject provides potentialities and eventually transforms an artefact into an instrument through its *instrumentalisation*. Conversely, the instrument provides opportunities for appropriating new schemes of instrumented action through the *instrumentation* of the subject.

There are recent examples of networking KOM with other frameworks dedicated to digital technologies, such as TIG and other aspects of the broader *instrumental approach* (Vergnaud, 2009). For example, Jankvist et al. (2018) compared and combined KOM's tools and aids competency with TIG and the notion of scheme. To define a mathematical digital competency, Geraniou and Jankvist (2019) networked KOM's mathematical competencies with frameworks of digital competencies through the lenses of the TIG and the theory of conceptual fields (Vergnaud, 2013). One common finding is that mathematical competencies tend to be too broad to analyse students' interactions with digital tools in the learning process. Further, the instrumental approach does not account for learners' awareness and choice of appropriate tools, their affordances and limitations.

In what follows, we focus on networking the construct of internal and external reflections in mathematical modelling and the instrumentation-instrumentalisation duality from TIG. Our main argument is twofold. For one, these theoretical constructs may address the same unit of analysis, but, consistent with the literature, they account for observable aspects at a different granularity. However, TIG has the potential to provoke critical reflections about the use of mathematical models in society through an implemented anticipation concerning digital instruments. To illustrate our claims, we offer examples from our own teaching experiences in a natural science bachelor programme in two different settings: a course in statistical modelling and a bachelor students' modelling project. According to Blum and Niss (1991, p. 43), a critical competence enables students to "recognise, understand, analyse and assess representative examples of actual uses of mathematics". Therefore, we begin by describing the real-life events that contextualise each of the teaching experiences.

## 4 Statistical Models and the Use of Hydroxychloroquine

During the outbreak of the COVID-19 pandemic, fast publication of scientific research on the spread, diagnosis, and potential treatments ensued. Early studies reported anecdotal recovery when using the antimalarial agent hydroxychloroquine (HCQ) to treat COVID-19 patients. Media outlets and some political leaders echoed the promise of HCQ as a cure for the disease. Nonetheless, evidence of its efficacy is far from settled (Das et al., 2020). In response, the World Health Organisation (WHO) called for conducting randomised trial studies that provide more conclusive evidence.

The first author and his co-lecturer used the dataset from one such study in the course Statistical Models at Roskilde University in the fall of 2020. The contents of this bachelor's course include probability distributions, parametrical and non-parametrical hypothesis tests and linear regression. Apart from homework assignments, the evaluation involves developing and discussing a group project based on a selected dataset from actual scientific research. Moreover, due to institutional decisions, students used Python as a programming language and Jupyter Notebooks as working environments for the first time, while their prior courses had used MatLab. They received a brief description of the study and the dataset displayed in Table 2—COVID-19.

**Table 2** COVID-19 survival and age

| Treatment | Age | Survival | | Total |
|---|---|---|---|---|
| | | Dead | Alive | |
| Hydroxychloroquine | <50 | 19 | 316 | 335 |
| | 50–69 | 55 | 355 | 410 |
| | 70+ | 30 | 172 | 202 |
| Standard treatment | <50 | 19 | 298 | 317 |
| | 50–69 | 31 | 365 | 396 |
| | 70+ | 34 | 159 | 193 |

As a general rule, the dataset should not be able to be solved straight-forward with the methods available in the course. Students must handle a non-routine problem that entails manipulations, decisions and eventual trade-offs. In particular, students were familiar with contingency tables for two categorical variables, while the dataset in Table 2—COVID-19 represents three: treatment (HCQ or standard), age group (<50, 50–69 or 70 or above) and survival (dead or alive).

In summary, the students reached two conclusions. First, age was significantly associated with decreased survival. Second, only in the age group of 50–69 years was the type of treatment associated with survival, in that HCQ curbed it. They did so by performing a total of five $\chi^2$ hypothesis tests (choosing $\alpha = 0.05$) for contingency tables, separating the dataset into each treatment for the first conclusion and the three age groups for the latter.

As instructed, the group submitted a seven-page report and three pages of appended code. We now focus on traces of internal and external reflections related to OJ1 questions and whether instrumental genesis plays a role.

Regarding the *who* question, students only knew about the author of the study (WHO) through the dataset they received. The provided digital instruments did not mediate this information. However, they phrased their conclusion, positioning patients as decision-makers:

> … it definitely is not recommendable for patients to *choose* the hydroxychloroquine treatment over standard treatment *if* they are in the age between 50 and 69. (emphasis added)

The choice of treatment on behalf of patients—whether factual or not—is external to the modelling cycle. The appropriate conditioning of such a decision according to a demographic group is internal. Both reflections are coordinated and represent a crucial aspect of probabilistic literacy in the context of risk (Borovcnik, 2016). A digital instrument played a role here, although it was not a Jupyter Notebook but an Excel spreadsheet, as shown on the report screenshot in Fig. 1. Students used Jupyter Notebooks to solve class examples, but a new situation with three categorical variables challenged their associated schemes. The instrumentalisation of available

| less_5( | Dead | Alive | Total |
|---|---|---|---|
| Standard | 19 | 298 | 317 |
| Hydroxyc | 19 | 316 | 335 |
| Total | 38 | 614 | 1304 |

(a) Under 50

| 50_69 | Dead | Alive | Total |
|---|---|---|---|
| Standard | 31 | 365 | 396 |
| Hydroxychl | 55 | 355 | 410 |
| Total | 86 | 720 | 806 |

(b) Between 50 and 69

| 70_more | Dead | Alive | Total |
|---|---|---|---|
| Standard | 34 | 159 | 193 |
| Hydroxychl | 30 | 172 | 202 |
| Total | 64 | 331 | 395 |

(c) Over 70

**Fig. 1** Separation in the three age groups in Excel scripts was only possible through another instrument to divide and visualise the resulting datasets.

The *where* question addresses the extra-mathematical context, purposes and consequences. An explicit statement illustrates this external reflection when presenting their dataset:

> The goal of our study is to analyse the data and conclude if hydroxychloroquine has any effect on COVID-19 survival in these three specific age groups.

To some degree, the consequences are explicit in the conclusion quoted earlier. However, their project did not address the reach of such studies in national and international healthcare guidelines or the continuation of research on particular drugs.

The mechanisms and competencies accounting for the *how* question are evident in students' work. Their project consisted, after all, of a step-by-step journey of mathematical concepts and procedures applied to decision-making in an extra-mathematical context. Two aspects are worth mentioning.

First, probabilistic modelling is implicit, and the instrumentalisation of programming has pragmatic value instead of an epistemic one (Artigue, 2002). A hypothesis test presumes that, under the null hypothesis, a statistic resulting from what was observed and what is expected in a random experiment follows a mathematical model, that is, a distribution. Students did not incur such reflections, and their commented code in Fig. 2 illustrates it. For them, the expected frequencies under the null hypothesis and the $\chi^2$ statistic were mere results of using formulae.

```
for i in np.arange(n):
    for j in np.arange(m):
        e[i][j] = o[i][m]*o[n][j]/o[n][m]; # calculating expected using
        #formula

chi2 = 0.0;
for i in np.arange(n):
    for j in np.arange(m):
        chi2 += (o[i][j] - e[i][j])**2/e[i][j]; # calculating chi-squre
        #using formula
```

**Fig. 2** Sample code computing expected frequencies and test statistic

Second, and on a more positive note, students reflected on the role of hypothesis tests as grounds for their statistical reasoning. After displaying all *P*-values and respective conclusions, they commented:

> Considering that the research is observational, we can conclude only that the association is not due to chance.

A randomised controlled study is experimental, but the error forces them to explain what they are testing with their calculations, aside from cause and effect, that is the extent to which the differences in survival probabilities can be explained by chance alone. The careful phrasing of their conclusions was consistent with this reasoning. For example:

> For patients outside this age group [50–69], our test results could not provide *evidence* that one treatment is better than the other. (emphasis added)

Overall, the students instrumentalised Jupyter Notebooks pragmatically. However, it allowed them to reach concrete and consistent results, which, in turn, are the substance of their external reflections. This is mainly the case for the *who* and *how* questions of OJ1. Moreover, these reflections are intertwined or mediated by internal reflections connected to the modelling and reasoning competencies (Niss & Højgaard, 2011).

## 5 Epidemiology as a Domain for Students' Modelling Projects

Models and modelling are essential in the epidemiology of infectious diseases (Anderson & May, 1992). Fundamental concepts and phenomena related to epidemics caused by infectious diseases are defined and understood by means of mathematical models. These are indispensable when epidemics are described, predicted, or regulated through restrictions, testing or vaccination programmes (Bailey, 1986). Thus, the epidemiology of infectious diseases is an exemplary case of the role and function of mathematical modelling in an interdisciplinary field of research in the biomedical domain.

In Denmark, there are several cases of healthcare politics issues in which mathematical models have played important roles. The COVID-19 pandemic is a clear example. Figure 3 shows an iconic picture of the Danish Minister of Health,

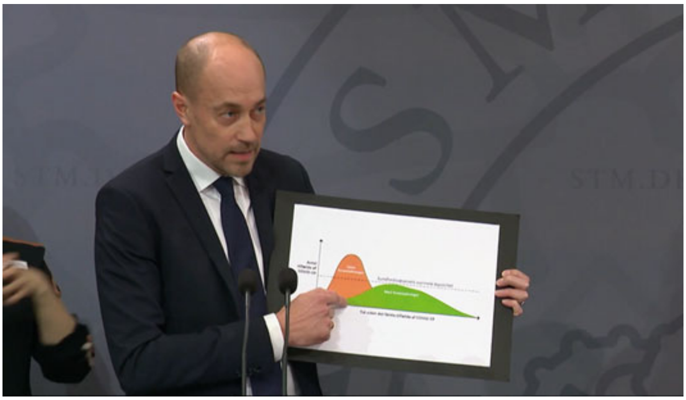


**Fig. 3** Minister of Health Magnus Heunicke at a press conference on March 10, 2020. Copyright © TV2

presenting what was hereafter referred to as *the red and green curves*. These curves represent the number of infected people expected in a scenario with and without restrictions on social and societal activities. The red curve exceeds the capacity for the intensive care of infected individuals, while the green curve stays below this capacity. The US health authorities have used the same diagram (Roberts, 2020). Although it is a conceptual model to support political decisions and communication about the necessity of installing restrictions, the diagram is presented as a result of a mathematical model.

The Danish vaccination programme against measles, mumps and rubella (MMR), launched in 1984, was decided based on a cost–benefit mathematical model. Later information campaigns for reaching a critical level of immunity in the population have also used model results. A mathematical model also showed that the prevalence of Chlamydia in young age groups is caused by a subpopulation with much higher contact rates than the average. Based on these results, in 2011, the municipality of Copenhagen launched a campaign for home testing against sexually transmitted Chlamydia infection targeting teenagers and young people and the tracing of contacts is done backwards to increase the chances of finding persons with high contact rates. In 2019, Denmark extended a vaccination programme against the human papillomavirus (HPV) to include boys and girls from 12 years of age. The programme is argued for based on a mathematical model that estimates it can cause a 90% reduction in the prevalence of cervical cancer (Statens Serum Institut, 2021).

Since 1992, the second author of this chapter has supervised students' mathematical modelling projects in the natural science bachelor's programme at Roskilde University. Blomhøj and Kjeldsen (2018) analysed in detail how this form of problem-oriented project work constitutes a solid and successful learning environment for developing students' mathematical modelling competency, including OJ1. This learning environment has been developed in parallel with two theoretical pillars. First, the theoretical understanding of mathematical modelling competency in the Danish KOM project (Niss & Højgaard, 2011, 2019) and, second, the theoretical understanding of a mathematical modelling process and related student reflections, including the Critical Mathematics Education's stance on mathematics in action (Blomhøj & Kjeldsen, 2011). Modelling projects in this programme can be seen as a rather extreme realisation of the holistic approach to mathematical modelling (Blomhøj & Jensen, 2007). Over the years, several projects in this programme have been modelling projects within the domain of epidemiology. A common point of departure for these projects has been the simple susceptible-infectious-recovered (SIR) model for the outbreak of an infectious disease caused by a virus in a closed population.

Typically, the students begin by reproducing the development of the SIR model from its basic assumptions, ending with three coupled ordinary differential equations that cannot be solved analytically. However, they are introduced earlier to MatLab, and some know how to use it to solve and analyse systems of differential equations numerically. As a case of implemented anticipation (Niss, 2010), the students need to develop a clear idea about how to analyse a system of differential equations numerically in MatLab before engaging in compartment modelling, which would result in such a mathematical object.

Through different analyses, it became clear to the students that the purpose of the simple SIR model is to understand the fundamental mechanisms and phenomena involved in infectious epidemics rather than to describe real epidemiological data or predict the courses of actual outbreaks (Blomhøj, 2020).

In this type of project, there is a close interplay between the technical level and the problem they want to illuminate and investigate in their project. The technical level lets students activate their modelling competency, particularly in compartment modelling and their tools and aids competency in using MatLab and other digital resources for implementing and analysing the model.

### *5.1 A Bachelor Project on the Modelling of Influenza Epidemics*

To illustrate the aforementioned interplay, we describe and analyse a project on influenza epidemics and the strategy in Denmark for vaccination against influenza. The Danish policy is to offer free influenza vaccines for people older than 65 and high-risk groups with specific chronic diseases. Students wanted to analyse this strategy and possibly suggest an alternative. The project was conducted by a group of four students and was documented in the project report (Jørgensen et al., 2002). Translations of excerpts from Danish are our own.

Their underlying hypothesis was that it might be more efficient, under a societal-economic rationality, to offer free vaccines to age groups with higher contact rates, try to reach the level of herd immunity, and protect the elderly part of the population indirectly.

Reflections internal to the modelling cycle led to model one generic Danish influenza season of 120 days representing the winter season—December, January, February and March—and to assume a constant population size. Moreover, a critical step in the modelling process was deciding how to represent different age groups and their interactions in the model. The group designed their model population according to five age groups: 1–5, 6–15, 16–25, 26–64 and 65–75 years. In the model, vaccination is given prior to an epidemic outbreak, and it is assumed to cause immunity in 75% of vaccinated individuals in all groups. They decided—supported by their supervisor—to develop the simplest possible model that utilised age-depending contact rates. Each of the five age groups consisted of a simple susceptible-infectious-recovered (SIR) model. In each time step in the numerical solution, the actual infection rate was calculated from the number of susceptible people in the group, the five different contact rates and the number of infectious people in the five groups.

An incursion into the *who* question came as a result of assigning parameter values. They assumed the cure rate, that is, 1/the average period of being infectious, to be

1/3 per day for all age groups. However, the group searched the literature for age-dependent contact rates and found that such rates have been estimated in a few cases based on previous epidemics in Asia (1958) and Hong Kong (1969):

> Therefore, we will make a point of departure in the contact rates developed by Longini et al. (1978) since they are *well-founded* and the *population structure is built-in.* (p. 51, emphasis added)

This internal reflection on their mathematisation step is grounded in the external reflection of the use of such models. By well-founded, they refer to Longini and colleagues' scientific work and healthcare authorities who collected relevant data. Moreover, as a case of students' instrumentation, they took advantage of working with a matrix representation of these rates in MatLab. The population structure is built in the estimates in the form of $5 \times 5$ matrices.

Students' reflections on the *where* of OJ1 stemmed directly from their problem formulation (p. 9):

> Is it possible to prevent influenza epidemics through a vaccination programme? And what is the best strategy for distributing vaccines to the population?

As a core result and answer to the first part, the group stated that, according to their model, it was possible to prevent an influenza epidemic through vaccination in a situation where the entire population was susceptible to the virus. If the vaccine were distributed equally across age groups, it required the immunisation of 1.5 million individuals. If the two younger age groups are vaccinated, herd immunity in the population could be reached with one million vaccinated individuals. Therefore, there are strong arguments for considering alternative strategies for vaccination against influenza.

However, the group discussed further social consequences in a section titled 'From Model to Action', calling to:

> …evaluate whether one should take action on the grounds of these results to change the current vaccination strategy. We can envision that a vaccination programme would find popular opposition. (p. 72)

They referred to the dilemma between using vaccination to protect individuals against possibly severe health conditions and using vaccination as a tool for optimising the societal handling of a healthcare problem. It is not evident that the work with the digital artefact prompted this reflection directly but indirectly through the results of its use, that is, as a consequence of its procedures.

Regarding the *how* of OJ1, the group reflected that their model could not directly answer the second part of their problem formulation. However, they used the model to compare and analyse different vaccination strategies. In particular, they showed that, according to the model, vaccination of the oldest age group reduced the attack rate in that group by 40% and the general attack rate by only 4%. If the same number of vaccines were given to the 6–15-year-old group, the general attack rate was reduced by 30%. Thus, for the oldest age group, it was only slightly better to have all vaccines than to give them to the 6–15 year age group, and for the population in general, this strategy would be nearly eight times more efficient.

The instrumentalisation of MatLab to solve and simulate made it possible for the students to investigate these scenarios relevant to their problems. In this process, the students were enabled to ask questions and suggest parameters that they otherwise would not have observed. However, the students developed their model in close interplay with ideas of how to implement it in MatLab. Aside from the case of the matrix representation of contact rates, the following disclaimer placed above the appended code is worth highlighting:

> Most of our programmes are built to simulate influenza over any given number of years. However, we have not made use of this feature, which may seem useless. However, *we value the opportunity to increase the simulation period*, which is why it is preserved in our programmes, while the models only run for one year. (p. 81, emphasis added)

All students in the group had some experience with MatLab, and two were particularly skilful programmers in general. This experience, illustrated by the openness of their code, suggests instrumentation of the students towards a broader understanding of the SIR model. The parameter values were simply choices of variables instances. During the project, and in interplay with their understanding of the SIR model, MatLab was developed into an instrument for the students, enabling them to set up and analyse a compartment model for influenza epidemics.

Overall, in modelling projects in epidemiology, students engaged in critical reflections about assumptions and choices in the modelling process (internal) and how mathematical models are used or can be used to support or evaluate healthcare policies (external). Students had opportunities to develop their modelling competency and their OJ1. The latter focused on the role and function of mathematical modelling in epidemiology, understood both as a research area and groundwork for political decisions.

## 6 Discussion

We have described two cases of students' projects from our teaching experience that used mathematical models to assist decisions in extra-mathematical contexts. In both cases, the instrumentalisation of digital artefacts—Jupyter Notebooks and MatLab—carry a pragmatic value (Artigue, 2002) in computing *P*-values for hypothesis tests, solving paired differential equations and simulating scenarios. However, these results and interpretations provoke and give substance to reflections associated with the actual application of mathematics. That is particularly the case of *how* mathematical models are constructed and used for making decisions, as students' task is to experience such a process. Students recognise *where* the application of mathematical models has intended and unintended consequences, but the mediating role of digital instruments is not evident. An opportunity may arise by using digital media to search for data and contextual information, as Radakovic (2015) highlights in his pedagogy of risk.

Nonetheless, students use digital instruments to visualise and mathematise the disaggregation of a sample or population into age groups. This feature is particular to the data involved in the problems; it concerns direct effects on people's wellbeing. In other words, the students' instrumentation can trigger reflections on agency issues, that is, on *who* is acting upon mathematical arguments.

Our approach focuses on students' reflections on *who*, *where* and *how* mathematics is applied, that is, on KOM's OJ1 archetypal questions (Niss & Højgaard, 2011, 2019). We distinguished between internal and external reflections (Blomhøj & Kjeldsen, 2011). Additionally, the theory of instrumental genesis provided the language that allowed us to pinpoint the role of digital instruments in such reflections, considering the duality of instrumentation and instrumentalisation (Artigue, 2002). This fussy endeavour illustrates the acknowledged issue of the proliferation of theories in mathematics education as a research field (Prediger & Bikner-Ahsbahs, 2014).

This diversity of theoretical constructs can be untangled through appropriate networking strategies (Prediger et al., 2008). The notion and distinction of internal and external reflections from the developing didactical theory of mathematical modelling *make* the OJ1 construct *understandable*. It is achieved by operationalising archetypal questions (*who*, *where* and *how*) as observable in communicative acts, whether oral or written. Furthermore, internal and external reflections, as well as TIG (Trouche, 2005), work as "a language of descriptions of an educational practice" (Silver & Herbst, 2007, p. 56). However, they described different aspects of such an educational practice. The former adjusts the lens towards the students' insights about their mathematical modelling process and eventual extra-mathematical uses; the latter focuses on the interaction between students and digital instruments (Python and MatLab programming). This juxtaposition of theoretical approaches is a *combination* in which elements need not fit as well as in a *coordination* strategy (Prediger et al., 2008).

As a final point, the COVID-19 pandemic has shown that mathematical models play an essential role in describing and predicting the pandemic's course and evaluating the consequences of possible interventions and treatments. However, the models and their basic assumptions have only been displayed and discussed in public to a minimal degree, calling for the need for external reflection and OJ1 in the general public. The connection with a coherent teaching practice is still a non-trivial enterprise (Blomhøj & Elicer, 2021). We hope this chapter encourages researchers to network the theoretical complexity of teaching for reflective knowing by integrating particular aspects of the digital era into the picture.

## References

Alrø, H., & Skovsmose, O. (2002). *Dialogue and learning in mathematics education: Intention, reflection, critique*. Kluwer.

Anderson, R. M., & May, R. M. (1992). *Infectious diseases of humans: Dynamics and control*. Oxford University Press.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274. https://doi.org/10.1023/A:1022103903080

Bailey, N. T. (1986). Macro-modelling and prediction of epidemic spread at community level. *Mathematical Modelling, 7*(5–8), 689–717.

Barbosa, J. C. (2006). Mathematical modelling in classroom: A socio-critical and discursive perspective. *ZDM Mathematics Education, 38*(3), 293–301. https://doi.org/10.1007/BF02652812

Blomhøj, M. (2004). Mathematical modelling—A theory for practice. In B. Clarke, et al. (Ed.), *International perspectives on learning and teaching mathematics* (pp. 145–160). National Centre for Mathematics Education.

Blomhøj, M. (2020). Characterising modelling competency in students' projects: Experiences from a natural science bachelor program. In G. Stillman, G. Kaiser, & C. E. Lampen (Eds.), *Mathematical modelling education and sense-making* (pp. 395–405). Springer. https://doi.org/10.1007/978-3-030-37673-4_34

Blomhøj, M., & Elicer, R. (2021). Epidemiological models and the problem of coherence from the critical justification to a practice of teaching mathematical modelling. *Quadrante*, *30*(2), 79–100. https://doi.org/10.48489/quadrante.23597

Blomhøj, M., & Jensen, T. H. (2007). What's all the fuss about competencies? Experiences with using a competence perspective on mathematics education to develop the teaching of mathematical modelling. In W. Blum, et al. (Eds.), *Modelling and applications in mathematics education. The ICMI-study 14* (pp. 45–56). Springer.

Blomhøj, M., & Kjeldsen, T. H. (2011): Students' reflections in mathematical modelling projects. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling. International perspectives on the teaching and learning* (Vol. 1, pp. 385–396). Springer. https://doi.org/10.1007/978-94-007-0910-2_38

Blomhøj, M., & Kjeldsen, T. H. (2018). Interdisciplinary problem oriented project work—A learning environment for mathematical modelling. In S. Schukajlow & W. Blum (Eds.), *Evaluierte Lernumgebungen zum Modellieren. Realitätsbezüge im Mathematikunterricht* (pp. 11–29). Springer. https://doi.org/10.1007/978-3-658-20325-2_2

Blomhøj, M., & Niss, M. (2021). Decoding, understanding, and evaluating extant mathematical models: What does that take? *Quadrante*, *30*(2), 9–36. https://doi.org/10.48489/quadrante.24129

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37–68. https://www.jstor.org/stable/3482238

Borovcnik, M. (2016). Probabilistic thinking and probability literacy in the context of risk. *Educação Matemática Pesquisa, 18*(3), 1491–1516.

Das, S., Bhowmick, S., Tiwari, S., & Sen, S. (2020). An updated systematic review of the therapeutic role of hydroxychloroquine in coronavirus disease-19 (COVID-19). *Clinical Drug Investigation, 40*, 591–601. https://doi.org/10.1007/s40261-020-00927-1

Davis, P. J., & Hersh, R. (1986). *Descartes' dream: The world according to mathematics*. Harvester Press.

Dewey, J. (2015). *Experience and education*. Free Press (original work published 1938).

Ernest, P., Skovsmose, O., van Bendegem, J. P., Bicudo, M., Miarka, R., Kvasz, L., & Moeller, R. (2016). The philosophy of mathematics education. In *The philosophy of mathematics education. ICME-13 topical surveys*. Springer. https://doi.org/10.1007/978-3-319-40569-8

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency." *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Goos, M. (2018). A review of research on understanding and promoting students' mathematical thinking published in educational studies in mathematics 2014–2018. In D. M. Gómez (Ed.), *Proceedings of the first PME regional conference: South America* (pp. 1–16). PME.

Jankvist, U. T., Geraniou, E., & Misfeldt, M. (2018). The KOM framework's aids and tools competency in relation to digital technologies: A networking of theories perspective. In H.-G. Weigand, A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, N. Grønbæk, & J. Trgalova (Eds.), *Proceedings of the fifth ERME topic Conference (ETC 5) on Mathematics Education in the Digital Age (MEDA, September 5–7, 2018)* (pp. 123–130). University of Copenhagen.

Jørgensen, C., Lohfert, C., Mikkelsen, M., & Nielsen, A. -L. H. (2002). Influenzaepidemier – et matematisk modelleringsprojekt [Influenza epidemics—A mathematical modelling project]. *IMFUFA-tekst*, nr. 410. Roskilde University. http://thiele.ruc.dk/imfufatekster/pdf/410.pdf

Kaiser, G., Blomhøj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modelling. *ZDM Mathematics Education, 38*(2), 82–85. https://doi.org/10.1007/BF02655882

Longini, I. M., Jr., Ackerman, E., & Elveback, L. R. (1978). An optimisation model for influenza A epidemics. *Mathematical Biosciences, 38*(1–2), 141–157.

Niss, M. (1994). Mathematics in society. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 367–378). Kluwer.

Niss, M. (1996). Goals of mathematics teaching. In A. Bishop, M. A. (Ken) Clements, C. Keitel-Kreidt, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 11–47). Kluwer. https://doi.org/10.1007/978-94-009-1465-0_2

Niss, M. (2010). Modeling a crucial aspect of students' mathematical modeling. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 43–59). Springer. https://doi.org/10.1007/978-1-4419-0561-1_4

Niss, M., & Blum, W. (2020). *The learning and teaching of mathematical modelling*. Routledge. https://doi.org/10.4324/9781315189314

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Østergaard, M. K., & Sun, D. (2022). The nature of mathematics as a discipline. In U. T. Jankvist, & E. Geraniou, (Eds.), *Mathematical competencies in the digital era* (pp. 257–270). Springer. https://doi.org/10.1007/978-3-031-10141-0_14

Pedersen, M. K., Bach, C. C., Gregersen, R. M., Højsted, I. H., & Jankvist, U. T. (2021). Mathematical representation competency in relation to use of digital technology and task design—A literature review. *Mathematics, 9*(4), 1–25. https://doi.org/10.3390/math9040444

Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 117–125). Springer. https://doi.org/10.1007/978-3-319-05389-9_8

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Radakovic, N. (2015). "People can go against the government": Risk-based decision making and high school students' concepts of society. *Canadian Journal of Science, Mathematics and Technology Education, 15*(3), 276–288. https://doi.org/10.1080/14926156.2015.1062938

Roberts, S. (2020, March 27). Flattening the coronavirus curve. *The New York Times*. https://www.nytimes.com/article/flatten-curve-coronavirus.html

Silver, E. A., & Herbst, P. G. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39–68). NCTM. https://doi.org/10.4065/81.11.1423

Skovsmose, O. (1990). Reflective knowledge: Its relation to the mathematical modelling process. *International Journal of Mathematical Education in Science and Technology, 21*(5), 765–779. https://doi.org/10.1080/0020739900210512

Skovsmose, O. (1992). Democratic competence and reflective knowing in mathematics. *For the Learning of Mathematics*, *12*(2), 2–11. https://www.jstor.org/stable/40248044

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education.* Kluwer.

Skovsmose, O. (1998). Linking mathematics education and democracy: Citizenship, mathematical archaeology, mathemacy and deliberative interaction. *Zentralblatt Für Didaktik Der Mathematik, 30*(6), 195–203. https://doi.org/10.1007/s11858-998-0010-6

Statens Serum Institut (2021, August 31). *Vaccination mod human papilloma virus (HPV)* [Vaccination against Human Papillomavirus (HPV)]. https://www.ssi.dk/vaccinationer/boernevaccination/vaccination-mod-livmoderhalskraeft

Thomsen, M., & Clark, K. M. (2022). Perspectives of embedding the historical development of mathematics in mathematical tasks. In U. T. Jankvist, & E. Geraniou (Eds.), *Mathematical competencies in the digital era* (pp. 239–255). Springer. https://doi.org/10.1007/978-3-031-10141-0_13

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). Springer. https://doi.org/10.1007/0-387-23435-7_9

Valero, P., Andrade-Molina, M., & Montecino, A. (2015). Lo político en la educación matemática: De la educación matemática crítica a la política cultural de la educación matemática [The political aspect in mathematics education: From critical mathematics education to the cultural politics of mathematics education]. *Revista Latinoamericana de Investigación En Matemática Educativa*, *18*(3), 7–20. https://doi.org/10.12802/relime.13.1830

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

Vergnaud, G. (2013). Conceptual development and learning. *Revista Qurriculum, 26*, 39–59.

# Perspectives on Embedding the Historical Development of Mathematics in Mathematical Tasks

**Marianne Thomsen and Kathleen M. Clark**

## 1 The Second Type of Overview and Judgement

The KOM framework includes three types of overview and judgements, which are seen as related to eight mathematical competencies. However, overview and judgement have a more comprehensive character; they regard mathematics as a discipline and its relationship with nature, society, and culture (Niss & Højgaard, 2019; Niss & Jensen, 2002). Within the KOM framework the mathematical competencies, overview and judgement, and the mathematical content areas are interconnected in different ways depending upon which competency, mathematical content, and type of overview and judgement are chosen to be in focus in a given teaching and learning unit. The second type of overview and judgement, *the historical development of mathematics, seen from internal as well as from socio-cultural perspectives* (Niss & Højgaard, 2019), is among other things, characterized as including these questions:

> What are the forces and mechanisms behind the historical development of mathematics in society and culture? In what respects and under what conditions and circumstances is the development of mathematics primarily influenced by internal forces, respectively by external forces? (p. 24)

This chapter's focus is on how these questions can be embedded in mathematical tasks concerning working with the same geometric content and various geometrical reasoning related to respectively an original source and GeoGebra. Our perspective of including original sources in mathematics classrooms is to support students to gain insights in "another *view* of the landscape of mathematical concepts, techniques, and

M. Thomsen (✉)
University College Copenhagen, Copenhagen, Denmark
e-mail: mart@kp.dk

K. M. Clark
Florida State University, Tallahassee, USA
e-mail: kclark@fsu.edu

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_13

theorems" (Fried, 2011, p. 22). From the second type of overview and judgement perspective, the word *another* takes on great significance. The various perspectives between working with an old original source and new digital technologies (e.g., GeoGebra) give opportunities to create situations where students and the teacher can discuss different ways of working with—as an example—types of geometrical proof (e.g., Thomsen & Olsen, 2019). This can be characterized as seeing the change in the possibilities for working with related mathematics in different ways—different concepts and techniques can be related to work with the same theorems. In this chapter we analyse and give examples from two different empirical cases. Both concern students' work with the interplay between GeoGebra and Euclid's Book IV, Proposition 6: *To inscribe a square in a given circle.* The first case stems from the first author's PhD project. The second one concerns in-service teachers in a graduate course module presented by the second author. From an internal view one might say that the mathematics does not change in these cases, but the way one can work with it and reason about it changes. From an external and a socio-cultural perspective, working with digital technologies in many countries is a part of current mathematics education and it affects students' learning possibilities (e.g., Artigue, 2002; Trouche et al., 2013). Among other things, it tends to require that students can benefit from reflecting upon what that means for their possibilities of working with mathematics and thereby support their development of the second type of overview and judgement. This can be exemplified by working with the similarities and differences between the text in the original source and the visual representations in GeoGebra. These visual representations can emerge by using buttons in GeoGebra, e.g., "regular polygon," which relate to the text in the original source as a rather complex figure. Such reflections can be seen as a part of the second type of overview and judgement.

## 2 The Anthropological Theory of the Didactic

In the perspective of the Anthropological Theory of the Didactic (ATD), the explicit focus is on the institutional condition for teaching. ATD emphasizes four types of didactic transpositions between four different types of knowledge: *Scholarly knowledge, Knowledge to be taught, Taught knowledge*, and *Learned/available knowledge.* The didactical transposition goes backward and forward between these four types of knowledge (Chevallard & Bosch, 2014). One might characterize the KOM-Project and Euclidean geometry as *Scholarly knowledge*; however, the way that this is interpreted in the framework of the Danish national curricula can be characterized as *Knowledge to be taught*, while teaching materials and the task designs can be categorized within the *Taught knowledge.* Finally, the way students are presented with opportunities and are able to work with these can be considered as a type of *Learned/available knowledge.* In this chapter we particularly focus on the type of learned/available knowledge related to the second type of overview and judgement. Each type of knowledge consists of a praxeology. A praxeology is divided into two blocks which are also connected: (1) A praxis block (i.e., a know-how block), including the type of task and the technique to solve the task and (2) A theoretical ("logos") block (i.e., a know-why block), including the technology and theory (Chevallard et al., 2015). Regarding the logos block Bosch and Gascón (2014) stated:

> The *logos* block contains two levels of description and justification of the *praxis*. The first level is called a "technology," using here the etymological sense of "discourse" (*logos*) of the technique (*technè*). The second level is simply called the "theory" and its main function is to provide a basis and support of the technological discourse. (p. 68)

It is possible to follow different didactical transpositions of the use of Euclid's *Elements* in school mathematics through time (Winsløw, 2012). In the case of using digital technologies such as GeoGebra, some of the Euclidean notions and the proofs are hidden from students and the propositions in Euclid's *Elements* can help to reveal these (e.g., Thomsen, 2021). In the view of a praxeology you can say that the logos block is hidden in many of the techniques GeoGebra offers, which affects the technology related to the praxis block. While one can characterize the propositions in Euclid's *Elements* as being the theory in the logos block, these are both related to the original source and GeoGebra. The theory can also rely on various possibilities of technologies, and this depends on which techniques the students use to work on the task. The dialectical relationship between the praxis and logos block does not automatically fit together (Chevallard et al., 2015).

With regard to what we present in this chapter, we subscribe to the idea that there might be a gap where it is possible for students while using digital technologies to know the "praxis" part of a praxeology, yet the "logos" part might not be as well developed. Working with the interplay between original sources and digital technologies might have the potential to support the dialectical relationship between the praxis and logos block.

## 3 Hypothesis and Research Goal

Our hypothesis is that being aware of the second type of overview and judgement and trying to embed that perspective in mathematical tasks—e.g., by working with the interplay between original mathematical sources and GeoGebra—might be one way to support the possibilities to observe and empower the dialectical relationship between the praxis and the logos block. To investigate this hypothesis, we use the networking strategy called *combining* and *coordinating*, which "are mostly used for a networked understanding of an empirical phenomenon or a piece of data" (Prediger & Bikner-Ahsbabs, 2014, p. 119). This means that the research goal of this chapter is both to create and describe a theoretical combination and to investigate our hypothesis. Both parts are related to the analysis of the two empirical cases we address in this chapter.

## 4 Combined and Coordinated Theories

Besides the didactical transposition and praxeology from ATD and the second type of overview and judgement from the KOM framework, we also combine and coordinate approaches, distinctions, and notions from the research fields of history of mathematics and digital technologies in mathematics education.

Within the research domain of history of mathematics in mathematics education, Jankvist (2009) distinguished between using history as a tool and as a goal. With regard to history as a tool, the focus is on students' learning of mathematics (in-issues) and when considering history as a goal, then meta-issues provide the turning point and the "focus is on the developmental and evolutionary aspects of mathematics as a discipline" (Jankvist, 2009, p. 239). Regarding Jankvist's (2009) categorization Fried et al. (2016) stated: "Thus, reading a source deepens the mathematical understanding on both levels, on that of *doing mathematics* and on that of *reflecting about mathematics*" (p. 218). An emphasis on the second type of overview and judgement is aligned with history as a goal and "it is obvious that if overview and judgement regarding this development is to have any weight or solidness, it must rest on concrete examples from the history of mathematics" (Jankvist, 2013, p. 639). In our empirical cases Euclid's Book IV, Proposition 6 is the concrete example, which is the setting for the teaching and learning sessions.

Within the research field of digital technologies, Artigue (2002) relies on Chevallard's term praxeologies and stresses the techniques to include the distinction and tension between having a *pragmatic value* (the productive potential) and an *epistemic value* (the supporting understanding potential). Artigue uses the term technology related to digital technologies, thereby the technology and theory are combined into a single "theoretical" component in the logos block within the former distinction (Artigue, 2002). Furthermore, Artigue (2002) uses the term instrumental genesis (see also Trouche, 2016; Verillon & Rabardel, 1995) for other interpretations of the instrumental genesis) when it comes to using digital technologies. Artigue (2002) emphasized:

> For a given individual, the artefact at the outset does not have an instrumental value. It becomes an instrument through a process, called *instrumental genesis*, involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes. (Artigue, 2002, p. 250)

Here Artigue described personal schemes as the appropriation of social pre-existing schemes. The construction of personal schemes and the instrumental genesis also relies on the discursive environment individuals are offered through their participation in different institutional structures.

Lagrange (2005) emphasized that digital technologies require new techniques and that "push button" techniques challenge the traditional paper and pencil techniques. The notion of instrumental distance (Haspekian, 2014) takes into account the distance between the didactical potential connected to respectively, paper–pencil mathematics and to the mathematics a specific digital tool enables one to work with, e.g., between the dynamic and static figures in geometry software and paper–pencil geometry, respectively:

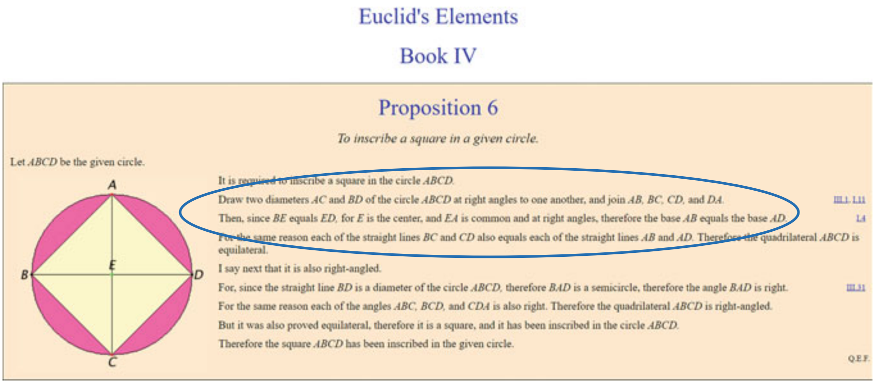
> It intends to take into account, beyond the "computer transposition" (Balacheff, 1994), the set of changes (cultural, epistemological or institutional) introduced by the use of a specific tool in mathematics "praxis." (Haspekian, 2014, p. 247)

Since this notion also relies on cultural, epistemological, or institutional changes, we think it fits well into our foundation of combining and coordinating within this presented theoretical line, which mainly relies on ATD. On the one hand, we consider that using original mathematical sources as a part of the task allows the cultural, epistemological, and institutional perspective of the source to be embedded in the task. On the other hand, using digital technologies—in this case GeoGebra—has the potential to work with the possible tension between the way one can work with the mathematics represented in the source and the way one can work with the same mathematics in GeoGebra. This affords potential to create situations where these—or perhaps more realistically, fragments of these—can be explicit and thus support more empowering institutional connections resulting from praxeological analysis. This allows us to analyse both a task and students' work with it from the perspective of an overall praxeology including two praxeologies. One is related to the part of the task, which focuses on the original source, and one is related to the use of GeoGebra. We investigate whether students' work with such kinds of tasks sharpens their reflections about different ways of working with geometrical reasoning through time and thereby support their development of the second type of overview and judgement. Furthermore, we wish to determine if this can be seen as a part of a bridge building between the praxis and logos blocks.

## 5 The Empirical Cases

In this section, we present two cases in light of ATD and from the perspective of both the second type of overview and judgement and from a Euclidean perspective. The first case we present stems from a teaching module in the beginning of a 7th grade (students aged 12–14 years old) mathematics class in a Danish elementary school. The module primarily focused on students' work with the interplay between Euclid's Book IV, Proposition 6 and GeoGebra. The teacher and students had not previously worked with an original source, but they seemed to be quite familiar in using GeoGebra. The duration of the entire teaching module consisted of six modules: one of 180 min in duration, two of 90 min each, and three of 45 min each. We focus on a pair of students who worked with a specific part of a task.

The second case concerns in-service teachers' work on a task that also focused on the interplay between Euclid's Book IV, Proposition 6 and GeoGebra. It involves three students who participated an asynchronous online course ("Using History in the Teaching of Mathematics") offered in Spring 2021. Fourteen students—all teachers at middle school, high school, or university—took the course, 12 of whom were pursuing a master's degree in Curriculum and Instruction, with a major in Mathematics Education. We focus on the experience of three students during a one-week module in the course focused on primary sources from Euclid's *Elements* (or, rather, versions of Euclid's *Elements*): one middle school teacher (StA) and one high school


Euclid's Elements

Book IV

Proposition 6

*To inscribe a square in a given circle.*

Let *ABCD* be the given circle.

It is required to inscribe a square in the circle *ABCD*.

Draw two diameters *AC* and *BD* of the circle *ABCD* at right angles to one another, and join *AB*, *BC*, *CD*, and *DA*. III.1, I.11

Then, since *BE* equals *ED*, for *E* is the center, and *EA* is common and at right angles, therefore the base *AB* equals the base *AD*. I.4

For the same reason each of the straight lines *BC* and *CD* also equals each of the straight lines *AB* and *AD*. Therefore the quadrilateral *ABCD* is equilateral.

I say next that it is also right-angled.

For, since the straight line *BD* is a diameter of the circle *ABCD*, therefore *BAD* is a semicircle, therefore the angle *BAD* is right. III.31

For the same reason each of the angles *ABC*, *BCD*, and *CDA* is also right. Therefore the quadrilateral *ABCD* is right-angled.

But it was also proved equilateral, therefore it is a square, and it has been inscribed in the circle *ABCD*.

Therefore the square *ABCD* has been inscribed in the given circle.

Q.E.F.



**Fig. 1** An example of an English translation of Euclid's elements, Book IV, Proposition 6 (Joyce, 1998)

teacher (StB), neither of whom had taught or were currently teaching a course on Geometry, and another high school teacher (StC), currently teaching Geometry (and had approximately eight years' experience teaching the subject). The three students chose to form a group to work on the content for the Euclid module during the course. This case also focuses on a task similar to the one in the first case.

The tasks we focus on in the empirical cases comprise an excerpt of a task connected to one aspect of Euclid's Book IV, Proposition 6 (see the circled excerpt of Proposition 6, Fig. 1). In the first case the students worked with Eibe's (1897) translation into Danish of Euclid's *Elements*. In the second case, students used Joyce's (1998) version of Book IV, Proposition 6 (given in Fig. 2a, b).

## *5.1 Analysis of the Empirical Cases*

The analysis is divided into two parts. The first part is related to the mathematical content of the tasks. The *learned/available knowledge* of the tasks concerns working with the interplay between this aspect of the proposition and GeoGebra, via a praxeological analysis. Because of the similarities of the tasks related to the first and second case (see Fig. 2a, b), we have chosen to merge both into a single analysis. The second part of the analysis is related to examples of students' responses to the last question in the task related to (1) Artigue's distinctions between epistemic and pragmatic value, (2) Lagrange's notion of "push button" techniques, and (3) Haspekian's concept of instrumental distance. This part of the analysis will be divided into two parts corresponding to each of the cases. Both the first and the second part of the analysis are connected to the second type of overview and judgement.

### 5.2 *Part 1—Tasks from the Perspective of Praxeologies*

The highlighted excerpt and the student tasks/questions are given in Fig. 2a, b.[1] We have chosen to focus on tasks associated with this particular part of the original source because it is the phase where Euclid begins the proof of why this is an inscribed square in a given circle.

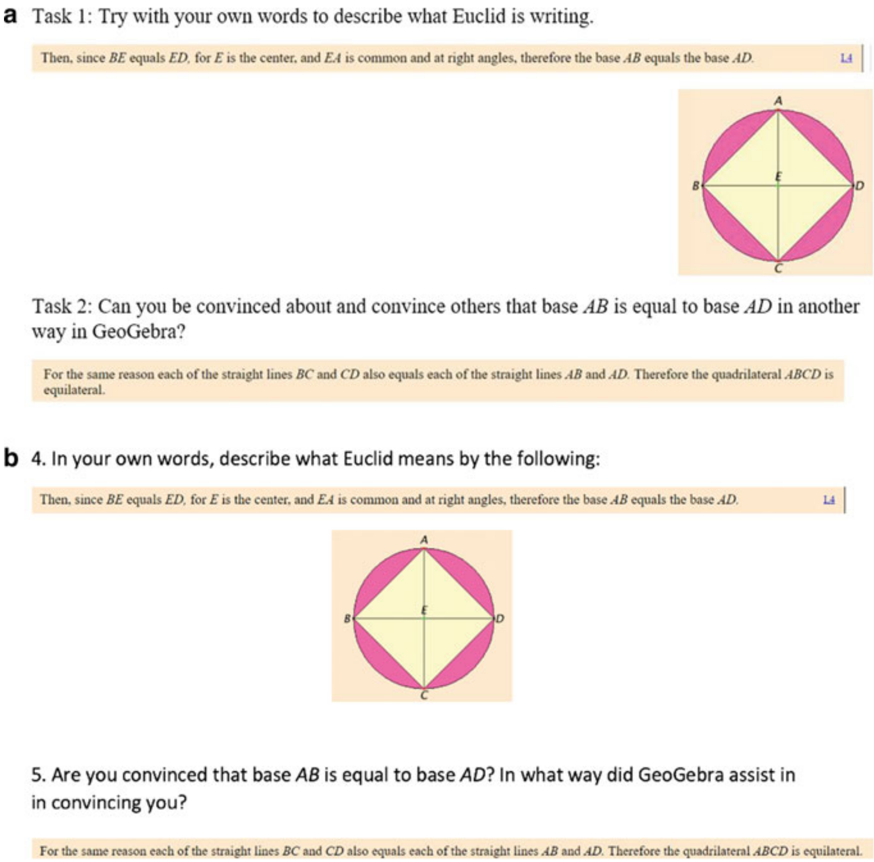



**Fig. 2** **a** Focal tasks for the first empirical case (a reconstruction from Danish into English). The text from Euclid's proposition related to task 1 states: "Then, since *BE* equals *ED*, for *E* is the center, and *EA* is common and at right angles, therefore the base *AB* equals the base *AD*." The text related to task 2 states: "For the same reason each of the straight lines *BC* and *CD* also equals each of the straight lines *AB* and *AD*. Therefore the quadrilateral *ABCD* is equilateral." (Joyce's (1998) English translation of Euclid's elements, Book IV, Proposition 6). **b** Focal tasks for the second empirical case. These excerpts use the same text as given in Fig. 2a from Joyce's (1998) English translation of Euclid's elements, Book IV, Proposition 6)

[1] This way of working with the interplay between an original source and GeoGebra was inspired to some extent by Olsen and Thomsen (2017).

Task 1 in case 1 and task 4 in case 2 are both related to the excerpt of the text in Euclid's proposition. However, tasks 2 (case 1) and 5 (case 2) are related to the use of GeoGebra in slightly different ways. In task 2 the students are asked to use GeoGebra to be convinced in another way than the source provides and in task 5 the students are asked to use GeoGebra to assist them in being convinced. We address this difference in part 2 of the analysis. First, we make a common praxeological analysis for the tasks presented in Fig. 2a, b.

If we consider the two tasks as parts of the same task, then we may also consider two praxeologies comprising a single praxeology: one which is closely connected to the Euclidean proposition and one which is related to investigating and becoming convinced by using GeoGebra. Both praxeologies relate to the same mathematical content and the same task, which is to be convinced and convince others (inspired by e.g., Harel & Sowder, 2007), that the bases *AB* and *AD* have the same length. The first part of the task is closely related to the original source text. Here, the language, notions, and way of building up mathematical arguments are connected to Euclid's proposition. The second part of the task is related to the use of GeoGebra and here students can answer the question in multiple ways. Consequently, it depends on the way they choose to use GeoGebra. However, it is possible for students to use GeoGebra to answer the first part of the task and to also use the original source to assist in answering the second part of the task.

If we focus on the praxis block in the praxeology related to task 1 and 4, the first part of the task, where students describe in their own words the mathematics in the excerpt of Euclid's text, the technique is to read, understand, and find words that describe the meaning of the text. When we examine the logos block, we might say that the technology is related to the mathematical content concerning: (1) radii in a circle have equal length from the center to the circumference and (2) two triangles with respectively two equal sides and the angles contained between them are equal—in this case two right angles—then the bases are equal. If we continue to look at the theory part, one might say that it is an expression of deductive reasoning, general arguments, and relying on axioms and concepts. This might be considered the static geometry of Euclid.

If we then focus on the praxeology related to task 2 and 5, which requests that students use GeoGebra to solve the task, our analysis takes a little different form, because here we use Artigue's (2002) linking of the technology and theory part of the logos block into a single combined theoretical block. Furthermore, we divide the technique into a part which has epistemic value and a part which has pragmatic value—in these cases the "push-button" technique can be seen as a part hereof. In these cases, the praxis block includes a type of task that can be formulated as convincing oneself and others by using GeoGebra. As well, the technique can be an expression of epistemic value if students use GeoGebra in different ways and formulate related convincing arguments. The technique can be an expression of pragmatic value if students use buttons in GeoGebra without further argumentation, thus only relying on the outcome of e.g., the "push-button" technique. The theoretical block, the combined logos block, can include various mathematical concepts and areas depending on which techniques students decide to use to solve the task. These could include given lengths with numbers, the grid, regular polygon, circle tools, etc.

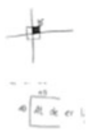
Here numbers or the grid show the length; the arguments can be based on examples or general argumentation and on both static and dynamic geometry. Since GeoGebra is based on Euclidean geometry, to some extent we can say that the theoretical part—the logos block—is based on Euclidean geometry, even though this might not explicitly and visibly rely on axiomatic deductive reasoning. These differences between the two praxeologies can be defined as the instrumental distance, which constitute the intersection space in the students' work with the interplay between the original source and GeoGebra.

The aim of asking students to work with the same mathematical statement (i.e., the mathematical content) in two different ways was to make it visible for students that this is in fact possible. Having a tool like GeoGebra can be seen as a historical and cultural development compared with the tools that are the foundation of Euclid's propositions. Emphasizing these differences can also be seen as a way of making it visible for students that the way of handling the mathematics and the mathematics itself has developed over time. We see this as an example of how the second type of overview and judgement can be embedded in tasks and by working through the interplay of original sources and GeoGebra.

### *5.3 Part 2—Case 1 Analysis*

This analysis focuses on a pair of grade 7 students' work. First, we present their written assignment (Table 1). Second, we show a screen capture from their work connected to their work with Euclid's Book IV, Proposition 6 (Fig. 3). And finally, we present a transcribed excerpt from their dialogue connected to their work with this proposition.

**Table 1** An example of a group's work (translation to English from their written answers in Danish)

| The questions in the task | The students' written answer | The students' supporting illustration |
|---|---|---|
| Try with your own words to describe what Euclid is writing | That $BE + ED$ is the same line |  |
| | They have the same length—On a square all sides are the same length |  |
| Can you be convinced about and convince others that base $AB$ is equal to base $AD$ in another way in GeoGebra? | "by eye"—"by thumb" |  |

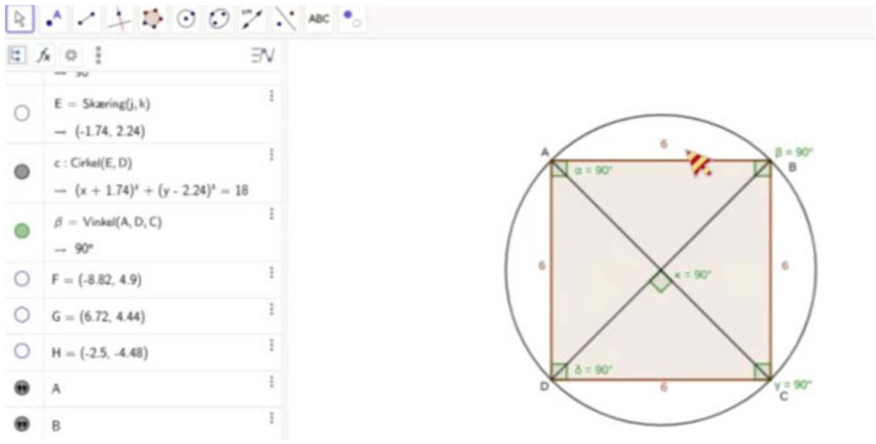



**Fig. 3** Group 1's GeoGebra screen capture (an excerpt)

The following transcribed excerpt is from the dialogue between the students at the beginning of their work with task 2 (Fig. 3b, the screen capture):

S1: (Reading the task aloud) For the same reason each of the straight lines *BC* and *CD* also equals each of the straight lines *AB* and *AD*. Therefore, the quadrilateral *ABCD* is equilateral.
S2: (Summarizes) Ok, they are all equal length. That is cheating a little because now I have (points at each of the sides of the square, which has a number for the length attached to each side).
Both: (They laugh and say together, while they point of each side) this is 6 cm, this is 6 cm, this is 6 cm, and this is 6 cm.
S1: And therefore, this is an equilateral square.
S2: (Refers to the written length in GeoGebra) What if it did not say so?
S1: Then it will still be equilateral, you can see that.

The students continued their investigation in GeoGebra, e.g., trying to draw an equilateral quadrilateral, without a grid. This work in GeoGebra seems to be based on a discussion of how they can use their hands to show the angle of 90°. Here they talk about the space between their thumb and the rest of the fingers while stretching the thumb as far out as possible from the rest of the fingers. They ended up writing as an answer to task 2: "by eye"—"by thumb." On the one hand this might indicate that they were not able to make a mathematical argumentation by working with the interplay between this statement of Book IV, Proposition 6, and GeoGebra. Instead, they lean on the visualization GeoGebra offers and their own judgement by assessing it from their immediate look at it. On the other hand, it indicates that they are aware of the fact that the angles in a square are 90 degrees and that they have to build their mathematical argument thereon.

We consider when it becomes clear to students that they actually just get the answer from using GeoGebra as an example of what Lagrange (2005) called "push button" techniques. In some way the embedded second type of overview and judgement in the tasks confront the students with the fact that there is an instrumental distance here between the way the proof is built in the original source and what GeoGebra offered. A sign of this is when S1 stated "that it is cheating a little" and referred to the value of the length attached to each side as given in GeoGebra. S2 seemed to completely agree with this statement. This, combined with their laughing and way of pointing at each side while stating the length of it, can be analysed as their acquired awareness, which is related to the second type of overview and judgement. If we look at it from the perspective of Artigue's notions of epistemic and pragmatic values, it would have actually been easy for students if they had just accepted the result because this provided a fulfilling answer. They could have stated that they were convinced by GeoGebra. In this way, GeoGebra could be characterized as having a pragmatic value on the students' "push button" technique to solve the task. However, the students do not seem to be completely satisfied with the answer, because it might seem like S2 prompted them to continue their investigation by asking, "What if it (the outcome in GeoGebra of the length of the side of the square) did not say so?" Therefore, we cannot deduce, though we can assume, that the students acquired an awareness of the instrumental distance between the way they worked with this task connected to the original source and then in GeoGebra. We also see this as a sign of having the second type of overview and judgement embedded in an explicit way in the tasks, while working with the interplay between original sources and GeoGebra, because it points out that the students can be convinced in another way by using GeoGebra rather than only through the original source.

### *5.4 Part 2—Case 2 Analysis*

In this case students were asked to explain in their own words what Euclid meant by his text, and they were also asked to explain how GeoGebra supported them in being convinced about that base *AB* is equal to base *AD*. In these tasks the second type of overview and judgement are embedded in a more hidden way, because the students were asked to use GeoGebra in a supportive manner connected to the original source.

The students' approach to task 4 (Fig. 2b) reflected their recent experience and familiarity with geometric ideas. In particular, StA[2] and StB, who had not previously taught geometry, explained the first part of Euclid's proof as:

StA: He is explaining how we know the side lengths of the square are all equal. By describing what we know about the construction, that is the evidence that they are equal and therefore a square.

[2] For the second empirical case we use StA, StB, and StC to designate Student A, B, and C, respectively.

StB: I think Euclid was proving why triangle *AEB* is congruent to *AED*. He outlined that since line segments *BE* and *DE* are congruent (since *E* is the center of the circle and bisects the diameter), *AE* is reflexive (shared between the two triangles), and the angle between those two line segments is a right angle, the two triangles are congruent (as we teach now: SAS congruency). Therefore, the remaining sides, *AB* and *AD*, are also congruent.

Whereas StB taught advanced courses in her high school and likely was more familiar with reasoning through mathematical arguments (as evidenced by her outline of the argument that Euclid made), StA provided more of a summary of what Euclid's argument entailed (the fact that all four sides of the quadrilateral were equal in length). However, as a high school geometry teacher, StC gave an explanation similar to StB:

StC: Since *E* is the center of the circle and also the midpoint of each diameter, segments *BE* and *ED* are congruent. In addition, we know that angles *BEA* and *DEA* are right angles due to the perpendicular diameters and we know *AE* is congruent to *AE* due to the reflexive property. Knowing these conditions, we can determine triangles *BAE* and *DAE* are congruent using either the side-angle-side postulate or the leg-leg theorem. From here, we can use the concept of corresponding parts of congruent triangles are congruent (CPCTC) and deduce that segments *AB* and *AD* are congruent.

StC's proof technique is similar to that which she would expect of her own students. However, unlike StB's response, which was situated to "describe what Euclid means" (as given in the task), StC set out to provide the details to prove base *AB* equals the base *AD*.

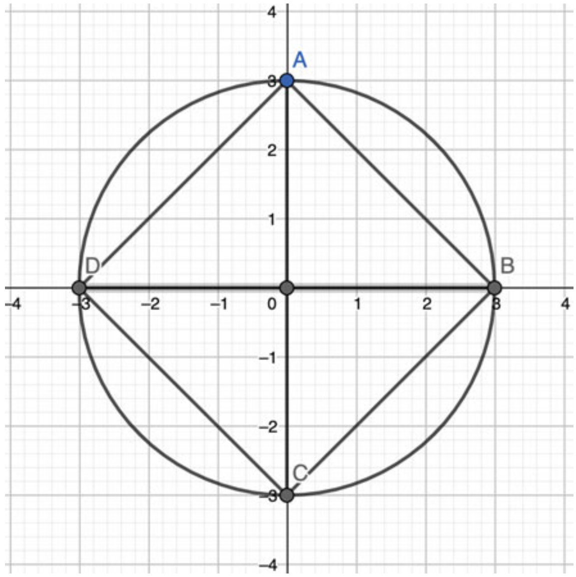
Interestingly, when the students were asked to discuss whether they were convinced that bases *AB* and *AD* are equal and in what way GeoGebra assisted in convincing them, only StB and StC recognized that GeoGebra provided a tool to confirm the argument they had just made in task 4. Although she stated she was convinced, StA was unable to provide a valid argument for the congruence of segments *AB* and *AD*.

StA: Yes, because the base *AB* and *AD* are both constructed from lines that are of equal length and extending from the same center point at the same angle.

StB: Based on Euclid's proposition and GeoGebra, I am convinced that *AB* and *AD* are congruent. I can visually see that the line segments are of equal lengths, but I also understand based on triangle congruency why that must work. GeoGebra was extremely helpful in confirming this because I was able to see the exact measurements of each line segment I constructed.

StC: I am convinced that base *AB* equals base *AD*, mainly due to my experience in geometry. In addition, GeoGebra shows me the segment measurements for as many constructions as I can make and visually proves the segments are congruent.

**Fig. 4** StB's GeoGebra figure to accompany task 5




Even though StA, StB, and StC completed the module tasks as a group (in both asynchronous and real-time sessions), GeoGebra held a different role for their sense-making while working the original source. For StA, GeoGebra served only as a measurement tool and not as a sense-making tool. Consequently, in this way we can consider the pragmatic value that GeoGebra possesses in terms of students' ability to employ "push button" techniques to perform the given task. StB found that GeoGebra provided her with confirmation of the results as well as relationships (e.g., equal lengths (results) and congruence) that the proof provided, thus we recognize that GeoGebra held more epistemic value for her. As well, StB was the only one of the three who provided her GeoGebra sketch to accompany her responses (Fig. 4).

Finally, StC had similar views to StB. However, in a follow up asynchronous discussion, she stated that she was "not a super fan [of GeoGebra], but it'll do." She also noted that "it can show the relationships I would typically show students." In this way, by using the verb "show," StC positions her view of GeoGebra as more of a demonstration tool, as opposed to a powerful sense-making tool and this serves as a similar example to what we detected in the first case; that is, StC's "showing" results or relationships as a type of "push button" technique. Furthermore, as the only Geometry teacher in the group, she was convinced "mainly due to [her] experience in geometry." This could be attributed to her intuition, her experience with similar numerical and conceptual arguments, or something else entirely. With regard to task 5 then, GeoGebra took a secondary role in her being convinced. However, it seemed to be common for the students that they—while working with the tasks—reflected upon the differences between the techniques associated with respectively (1) the original source, (2) their prior knowledge of geometry, and (3) their use of GeoGebra. Thus, we can assert that they also, to some degree, reflected in a manner related to the second type of overview and judgement.

## 6 Discussion and Conclusion

In both cases, students incorporated their prior knowledge of geometry in their work with the presented tasks. In the light of Artigue's (2002) description of how an artifact becomes an instrument and has an instrumental value to a given individual, we might say that both the original source and GeoGebra can be seen as artifacts placed in a context of a mathematical task, which the students draw upon their "appropriation of social pre-existing schemes" (Artigue, 2002, p. 250). At the same time, students' work with these tasks seem to influence their awareness of the different ways of working with a mathematical context related to different artifacts and therefore represent a "new part" of their "appropriation of social pre-existing schemes." In the two cases it seemed that students explicitly attended to tasks with an awareness of the historical development of mathematics seen from the inside (e.g., using the words and tools provided via the text of Euclid's proposition) and this awareness is different when considering other tools along with Euclid's text (e.g., the various tools and modes provided in GeoGebra). When examining the cases in light of ATD and the combined theoretical perspective we have presented and used to analyse them, we assert that this difference in the way one is able to work with the mathematics is a result of human activity, both in Euclid's proposition and in the construction of, for example, the "buttons" in GeoGebra and which can also be seen as a change from socio-cultural perspectives.

Our hypothesis was that working with the interplay between original sources and GeoGebra, by focusing on the second type of overview and judgement, might support an empowering of the dialectical relation between the praxis and logos block in a praxeology. Although we cannot definitively conclude this from the two cases presented, we do claim that subtle hints of this exist. In the first case, we witness students doubting if it is appropriate to just take the result of the "measurement button." We take students' laughing about how easy it seemed to solve the task in GeoGebra and their continued investigation as a sign of their awareness of the possible gap between the praxis block and logos block and as a sign that they wanted to use techniques in GeoGebra which have a more epistemic value. In other words: As a result of their awareness of their use of "push button" techniques and the "instrumental distance," they show signs of an empowered relation between the praxis block and the logos block, even though they did not find a really convincing mathematical argument, it continued to be based somewhat on their intuition related to the visualization GeoGebra offered.

In the second case, Students B and C reflected on how GeoGebra supported the mathematical proof in the original source and both refer to GeoGebra as a supporting tool because of the visualization and the measurement tool it offers. Student B saw GeoGebra as a tool which supported an alternative between an empirical example and the more general proof. In this sense it seemed like the interplay between an original source and GeoGebra supported an empowering of the relation between the praxis block and the logos block. Student C had a more critical stance regarding the use of GeoGebra, in which she appeared to rely on a previous critical attitude toward GeoGebra and it remained unchanged, for example, as a more critical and reflective approach to the work with GeoGebra throughout the module. Even though we assumed that embedding the second type of overview and judgement in the task, by focusing their work on the interplay between original sources, has the potential for empowering the relation between the praxis block and the logos block, it did not seem to just happen when students worked with these tasks. As a result, we conclude that the role of the teacher or the instructor seems to be important, and it is one of the reasons why we believe it is important to work with tasks like those presented in this chapter in both pre-service and in-service teacher education programs.

We have used the networking strategy called *combining* and *coordination* (Prediger & Bikner-Ahsbabs, 2014) of the included theories to analyse two particular cases. This combination of theories can also be used to analyse other cases, which concern working with the interplay between the history of mathematics—including original sources—and digital technologies. In one way, students are supported in being aware of the instrumental distance and when they use "push button" techniques while working with geometry in GeoGebra. This is in contrast to the way content is presented in the original source, which can be seen as a way of supporting students in developing the second type of overview and judgement. Furthermore, this can give them a view of whether the technique they use to solve a task has a pragmatic or an epistemic value. In our cases we did not *explicitly* use these distinctions and notions; this is one of the reasons we focused on when the second type of overview and judgement is *embedded* in the tasks.

We have chosen to consider some aspects of the KOM framework as a praxeology. Related to these cases, the praxis block consists of a task related to Euclidean geometry. In these cases, the technique is related to students' use of their reasoning competency, but we have not expounded on this part because our focus is on the second type of overview and judgement. In terms of a praxeology, we see that as an aspect of technology. Finally, we see these three levels as they are described in the KOM framework as included in supporting the development of students' mathematical mastery. In terms of a praxeology, this can be seen as the theory level and thus, connecting the different elements in the KOM framework can be seen as a didactical theory within the ATD framework. We have added Jankvist's (2009) distinction between using history as a goal and as a tool. To some extent this distinction makes visible, or rather, supports the assertion that we base this chapter on the second type of overview and judgement and have it in mind as a goal itself. Based upon the distinction between epistemic and pragmatic value (Artigue, 2002), the notions of "push-button" technique (Lagrange, 2005) and of instrumental distance (Haspekian, 2014) rely on and stress the notions of a praxeology. We also find that this theoretical combination has the potential to be developed into the networking strategies called synthesizing, in which one puts "together a small number of theoretical approaches into a new framework" (Prediger & Bikner-Ahsbabs, 2014, p. 120). Furthermore, this framework is based on theories, theoretical distinctions, and notions within the overarching field of research in mathematics education, the research field of digital technologies, and the research field of history of mathematics.

## References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274. https://doi.org/10.1023/A:1022103903080

Bosch, M., & Gascón, J. (2014). Introduction to the anthropological theory of the didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education. Advances in mathematics education* (pp. 67–84). Springer. https://doi.org/10.1007/978-3-319-05389-9_1

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 170–174). Springer. https://doi.org/10.1007/978-94-007-4978-8_48

Chevallard, Y., Bosch, M., & Kim, S. (2015). What is a theory according to the anthropological theory of the didactic? In K. Krainer & N. Vondrová (Eds.), *Proceedings of the ninth congress of the European Society for Research in Mathematics Education (CERME9)* (pp. 2614–2620). Charles University in Prague, Faculty of Education and ERME. https://hal.archives-ouvertes.fr/hal-01289424

Eibe, T. (1897). Euklids elementer III–IV [*Euclid's elements, Book 3 and 4*] (A. Danish, Trans.). Gyldendal.

Fried, M. N. (2011). History of mathematics in mathematics education: Problems and prospects. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and epistemology in mathematics education—Proceedings of the 6th European Summer University* (pp. 13–26). Verlag Holzhausen GmbH.

Fried, M. N., Guillemette, D., & Jahnke, H. N. (2016). Panel 1: Theoretical and/or conceptual frameworks for integrating history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 ICME satellite meeting of the international study group on the relations between the history and pedagogy of mathematics* (pp. 211–230). IREM de Montpellier. https://hal.archives-ouvertes.fr/hal-01349228

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Information Age Publishing.

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital area. Mathematics education in the digital era* (Vol. 2, pp. 241–275). Springer. https://doi.org/10.1007/978-94-007-4638-1_11

Jankvist, U. T. (2009). A categorization of the 'whys' and 'hows' of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261. https://doi.org/10.1007/s10649-008-9174-9

Jankvist, U. T. (2013). History, applications, and philosophy in mathematics education: HAPh—A use of primary sources. *Science and Education, 22*(3), 635–656. https://doi.org/10.1007/s11191-012-9470-8

Joyce, D. E. (1998). *Euclid's elements.* Book IV, Proposition 6. https://mathcs.clarku.edu/~djoyce/elements/bookIV/propIV6.html

Lagrange, J.-B. (2005). Using symbolic calculators to study mathematics. The case of tasks and techniques. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic*

*calculators. Turning a computational device into a mathematical instrument* (pp. 113–135). Springer. https://hal.archives-ouvertes.fr/hal-02379891
Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9
Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.
Olsen, I. M., & Thomsen, M. (2017). *Matematikhistorie og it i matematikundervisningen i grundskolen* [Unpublished master's thesis]. Danish School of Education, Aarhus University.
Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education. Advances in mathematics education* (pp. 117–126). Springer. https://doi.org/10.1007/978-3-319-05389-9_1
Thomsen, M. (2021). Working with Euclid's geometry in GeoGebra. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jesse, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, & A. Werneberg (Eds.), *Bringing nordic mathematics education into the future. Papers from NORMA 20. Proceedings of the ninth nordic conference on mathematics education.* SMDF.
Thomsen, M., & Olsen, I. M. (2019). Original sources, ICT and mathemacy. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education (CERME11)* (pp. 2060–2061). Freudenthal Group and Freudenthal Institute, Utrecht University and ERME.
Trouche, L. (2016). Didactics of mathematics: Concepts, roots, interactions and dynamics from France. In J. Monaghan, L. Trouche, & J. M. Borwein (Eds.), *Tools and mathematics* (pp. 219–256). Springer. https://doi.org/10.1007/978-3-319-02396-0_10
Trouche, L., Drijvers, P., Gueudet, G., & Sacristán, A. I. (2013). Technology driven developments and policy implications for mathematics education. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 753–789). Springer. https://doi.org/10.1007/978-1-4614-4684-2_24
Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101. https://doi.org/10.1007/BF03172796
Winsløw, C. (2012). Matematiklærerprofessionen i et institutionelt perspektiv. *Mona*, *4*, 7–22. https://tidsskrift.dk/mona/article/view/36002

# Facilitating Teachers' Reflections on the Nature of Mathematics Through an Online Community

**Maria Kirstine Østergaard and Dandan Sun**

## 1 Introduction

One of the key aspects used to capture mathematical mastery in the so-called KOM-project "Competencies and Mathematical Learning" (Niss & Højgaard, 2011, 2019[1]) is the overview and judgment connected to the nature of mathematics as a discipline. So part of a mathematics teacher's job is to develop students' understanding of the characteristic features of mathematics. This clearly requires that mathematics teachers themselves have a deep insight into specific traits of mathematical thought processes and activities (Niss & Højgaard, 2011). Then the question of how this insight should be acquired and developed during teacher training becomes important. This chapter addresses this issue. Specifically, digital technology is used as a key element to achieve this goal. The internet has changed the way we communicate with each other, and has furthermore considerably changed the way we relate to each other (Borba et al., 2016). Sharing knowledge and experience with others, which constitutes an essential part of effective professional learning, can take place not only face-to-face but also in online communities, allowing teachers to engage with each other in a convenient and flexible way. Recent studies of online teacher

[1] The report on the project was first published in Danish in 2002. Although the framework was recently revised in Niss and Højgaard (2019), here we refer to the 2011 edition because of the significantly more thorough description of mathematical overview and judgment.

M. K. Østergaard (✉)
Danish School of Education, Aarhus University, Copenhagen, Denmark
e-mail: mko@edu.au.dk

University College Copenhagen, Copenhagen, Denmark

D. Sun
School of Mathematics and Statistics, Shandong Normal University, Jinan, China



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_14

communities find that the supportive quality of online communities can effectively stimulate reflections on the professional practice by confronting teachers with new perspectives (e.g., Lantz-Andersson et al., 2018; Unwin, 2015).

In this chapter, we investigate how such an online community can facilitate teachers' reflections on the nature of mathematics as a discipline in a way that strengthens and connects their knowledge and beliefs. In a qualitative case study involving a Chinese teacher participating in a formally organized online community, we thus seek to address the following research question:

*When aiming to develop teachers' overview and judgment concerning the nature of mathematics, how can an online community provide opportunities for gaining experience with and reflecting on mathematics as a discipline?*

For this purpose, we combine the notion of overview and judgment with theoretical constructs connected to *beliefs*. We investigate the case of a teacher who is part of a larger in-service online teacher education program and engages in online discussions about the nature of mathematics (specifically about mathematical proof). Applying theories about mathematics-related beliefs to help us understand the teacher's overview and judgment thus provides us with a framework for analysis, both in regard to an assessment of the character of the teacher's views on the nature of mathematics and a possible development in his overview and judgment concerning this aspect.

The case explores in depth the development of the teacher's overview and judgment concerning the nature of mathematics in the digital environment. The first part of the chapter explains the notion of mathematical overview and judgment and its connection to both beliefs and knowledge. It is followed by a thorough description of the methodology, including the organization of the online teacher education program and the data collection. Subsequently, we present and analyze the case, leading to a discussion of the potentials of an online community for the professional development of teachers' overview and judgment.

## 2 Mathematical Overview and Judgment

Since 2002, when the first Danish edition of the KOM framework (Niss & Højgaard, 2002) was published, the competency framework it presents has had a significant impact on the Danish mathematics program on all educational levels, including teacher education. The report describes mathematical competence as relying on two pillars: eight action-oriented competencies (which are not to be confused with the overall concept of mathematical competence), and three forms of overview and judgment. While the competencies enable an individual to act appropriately in mathematical situations, overview and judgment provide the knowledge and views (or beliefs) about mathematics as a discipline that are necessary to make meaningful and appropriate choices by relating mathematics to nature, society, and culture. Thus, both aspects are required to master mathematics: Overview and judgment concerning mathematics as a discipline alone do not make a person mathematically competent; one must also be able to act in mathematical situations. Conversely, these actions draw on overall knowledge and ideas about mathematics as a discipline, although the purpose of mathematical overview and judgment exceeds a qualification of the ability to act:

> Having an overview and being able to exercise judgment are of significant importance for the creation of a balanced picture of mathematics, even though this is not behavioral in any simplistic way. The point is that the object of this judgment is mathematics as a whole and not specific mathematical situations or problems. (Niss & Højgaard, 2011, p. 74)

Niss and Højgaard (2019) identify mathematical overview and judgment as "insights into essential features of mathematics as a discipline". The three forms of overview and judgment (OJ) concern:

OJ1. The actual application of mathematics within other disciplines and fields of practice.
OJ2. The historical development of mathematics, seen from internal as well as from socio-cultural perspectives.
OJ3. The nature of mathematics as a discipline.

Our focus in this chapter is on OJ3; therefore, we will mainly elaborate on the character of this, and leave it to the reader to study the first two in depth.

Niss and Højgaard exemplify OJ3 with questions connected to the nature of mathematics (Niss & Højgaard, 2011, p. 77):

- "What is characteristic of mathematical problem formulation, thought, and methods?"
- "What types of results are produced and what are they used for?"
- "What science philosophical status does its concepts and results have?"
- "How is mathematics constructed?"
- "What is its connection to other disciplines?"
- "In what ways does it distinguish itself scientifically from other disciplines?"
- Etc.

Paying attention to teachers' overview and judgment concerning questions of this kind is an important part of teachers' professional development. In order to develop students' OJ3, teachers themselves should have insight into the characteristic features of mathematics. Moreover, teachers' OJ3 to some extent reflects what Ernest (1989) identifies as teachers' *philosophy of mathematics*, which has a huge influence on teachers' teaching in the classroom. To be more specific, teachers' OJ3 may embody their position with regard to three views: a problem-solving view, which characterizes mathematics as "a dynamic, continually expanding field of human creation and invention"; a Platonist view, which characterizes mathematics as "a static but unified body of knowledge"; and an instrumentalist view, according to which mathematics is perceived as "a set of unrelated but utilitarian rules and facts"

(Ernest, 1989, p. 249). Hence, when reflecting upon the questions connected to OJ3, the teacher in the case described in this chapter may to some extent modify his position with regard to these three views.

The case can be connected to many of these questions, as it describes a teacher's reactions to, thoughts about, and reflections on mathematical proof. Proof is regarded as central to the discipline of mathematics and the practice of mathematicians, an essential component of doing, communicating, and recording mathematics (Schoenfeld, 1994). It can be related to the nature of mathematics in many ways. Proof may be considered as a method and as a way of thinking and building mathematical arguments. In this sense, teachers' overview of proof is closely related to the characteristics of mathematical thought and methods as well as the construction of mathematics, i.e., the first and forth questions listed above. It may also be considered as a form of result, which is closely linked to the second question. Furthermore, the teacher in the case discusses more philosophically related issues about for example the validity and limitation of proofs—issues that relate to the third question in the list.

Niss and Højgaard (2011) emphasize the importance of making the characteristics of mathematics as a discipline a separate and independent subject of interest and consideration if a conscious and critical insight into these characteristics is to be developed. Hence, when seeking to develop in-service teachers' OJ3, it is necessary to consider how we can promote such interest in and consideration of the nature of mathematics through an online community. And since OJ is based on both knowledge and beliefs about mathematics, such considerations must include how this promotion might contribute to a change or a modification of both of these aspects and their interrelationship. We first turn our attention toward the concept of beliefs in order to understand how they might be changed.

## 3 Beliefs About Mathematics

As Furinghetti and Pehkonen (2002) pointed out, the concept of beliefs is defined in many—and sometimes even contradictory—ways in the field of mathematics education. Furthermore, related terms are often used as synonyms to beliefs, which can make it rather complicated to navigate in the affective field of educational research. McLeod's (1994) categorization of affect can be useful, as it divides affective factors into three dimensions: beliefs, attitudes, and emotions. This categorization is repeated by Philipp (2007), who defines beliefs as "lenses through which one looks when interpreting the world" (p. 258), and elaborates:

> (…) psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with various degree of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (Philipp, 2007, p. 259)

According to Green (1971), beliefs are always arranged in belief systems and related in a quasi-logical structure. This means that relations between beliefs do not necessarily follow an objective logic but follow the individual's subjective understanding of logical relations. However, due to the quasi-logical structure, these relations are not stable, and they may change. But not only the *relations* between beliefs are changeable. As stated in the citation above, beliefs themselves can be changed, although it might be difficult. The stability of beliefs is connected to their psychological importance to the person holding them, making central beliefs harder to change than more peripheral beliefs.

Another influential factor when it comes to changing beliefs is the way in which the beliefs are held (Green, 1971). *Evidentially* held beliefs are supported by evidence (e.g., experience) that can be verified, whereas *non-evidentially* held beliefs are not, even though some other beliefs may be given as reasons for them. When beliefs are evidentially held, they can be changed with reason or further evidence. In contrast, non-evidentially held beliefs are not as likely to be changed or modified by presenting new evidence or reasonable arguments. Green (1971, p. 48) exemplifies a non-evidentially held belief with the attitude: "Don't bother me with facts; I have made up my mind".

Regardless of the evidentiality of a person's beliefs, they can be both difficult and time consuming to change, depending on their centrality. Changing a belief is not easy because it will not only affect the belief in question, but also the belief(s) which it supports, and then the entire structure of a belief system might be in danger. However, because an evidentially held belief is based on evidence, so are its related and derived beliefs, and thereby the new evidence will be adaptable to these too. And if a change in beliefs is to last, it is essential that the experiences or reasons on which the beliefs are built become objects of reflection. It is in the reflection that relations between beliefs are established, considered, and assessed (Green, 1971). Thus, when aiming to change someone's beliefs, one must not only provide access to new information and opportunities for gaining new experience, but also make room for reflections upon these. Such considerations of course also apply when seeking a development in a person's OJ, since beliefs play an important part here. Niss and Højgaard also emphasize the importance of making the mathematical perspectives that are included in the three forms of OJ "the object of explicit treatment, reflection, and articulation" (Niss & Højgaard, 2011, pp. 74–75).

In the words of Dewey, reflection—or reflective thinking—can be defined as an "active, persistent and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it, and further conclusions to which it leads" (Dewey, 1933, p. 118). For a belief to be an object of reflection thus presupposes that it is in fact a *conscious* belief. However, many of our beliefs are held at an unconscious (or implicit) level and have never been articulated (Furinghetti, 1996). If we are not asked about our beliefs or presented with information that is related to them, we do not necessarily think about them or become aware of them (Lester, 2002). A way for unconscious beliefs to become conscious can be to be given the opportunity to articulate them, thus "awakening" the beliefs by external motivation (Furinghetti, 1996). In the case presented later in this chapter, one of the goals for the online community is to provide such opportunities for the teacher, thereby facilitating reflections that can lead to a development in his OJ3. In addition, reflections on the nature of mathematics may also stimulate the teacher to modify his position somewhat with regard to the abovementioned three views connected to the philosophy of mathematics (Ernest, 1989).

As seen above in the quote by Dewey, reflection is not only connected to beliefs, but also to (supposed) knowledge, which furthermore constitutes an essential part of mathematical OJ. In the next section, we focus on how these concepts are related.

## 4 The Relationship Between OJ, Beliefs, and Knowledge

As mentioned, Niss and Højgaard (2011) describe mathematical OJ as being based on knowledge as well as beliefs about mathematics as a discipline. Thus, when seeking to develop teachers' OJ, both aspects must be considered. The relationship between knowledge and beliefs is, however, a long-standing topic of discussion. The factor of truth plays an essential role: "knowledge is valid with a probability of 100%, whereas the corresponding probability for belief is usually less than 100%". (Furinghetti & Pehkonen, 2002, p. 43). In other words, a belief that is actually true but based on guessing or assumptions can never be classified as knowledge. Still, a person or a group of people can consider a belief to be true—and thus as knowledge—until it is proven wrong, which again complicates the distinction between beliefs and knowledge. As Green (1971) notes, a belief can only be(come) knowledge if it is based on evidence. However, when considering knowledge and beliefs, some researchers suggest a perspective that pays attention to how teachers view a statement instead of whether the statement is ontologically true (Philipp, 2007).

Although certainly not the same, beliefs and knowledge are closely linked (Thompson, 1992). Furinghetti and Pehkonen (2002) distinguish between "objective (official) knowledge that is accepted by a community and subjective (personal) knowledge that is not necessarily subject to an outsider's evaluation" (p. 43), and further argue that beliefs could be considered as subjective knowledge. Differentiating between knowledge and beliefs based on whether one can respect disagreeing positions, Philipp (2007) argues that one person's belief may be another person's knowledge. Jankvist (2015) states that "to decide whether a given answer or statement is a belief, or if it is in fact knowledge, is quite difficult, sometimes probably impossible" (p. 53). In terms of developing OJ, it may be productive to regard knowledge as evidence based on which beliefs may be held. Knowledge then not only constitutes an essential part of a person's OJ, but also functions both as evidence and as a basis for reflections on beliefs, making knowledge an important foundation in developing evidentially held beliefs. In the following sections, we present an example of how such a development can be facilitated through an online community for Chinese teachers.

# 5 Method

## *5.1 Context and Setting*

The research in this chapter was part of a larger in-service teacher education program in which teachers' beliefs about mathematics and mathematics teaching were challenged mainly by the history of mathematics, and in an online environment (Sun, 2021). The one-year program was divided into two parts: the first half focused on studying materials, and the second half focused on practicing in the classroom. All the activities revolved around nine topics, such as the sum of angles in a triangle, irrational numbers, and functions. Although the history of mathematics definitely played an important role in changing the teachers' beliefs, the research in this chapter focuses on the other noteworthy independent variable of the project, namely technology. And even though various factors may have influenced the professional development of the teachers during the program, especially their overview and judgment concerning the historical development of mathematics, i.e., OJ2, we here focus on their development of OJ3, thus illustrating the potential for developing OJ3 through an online community. Hence, in the case described below, the topic communicated through the online environment was chosen specifically to challenge the teachers' views on the nature of mathematics.

A mini-app (a sub-app embedded in the most popular Chinese multi-functional media app WeChat) was designed for sharing historical passages related to different mathematical topics with 63 voluntary lower secondary school teachers from 14 provinces in China. With this mini-app, the teachers could conveniently read materials chosen by a researcher in mathematics education (the second author of this chapter). The purpose of these materials was to offer the teachers new knowledge or make them aware of knowledge that they already possessed, perhaps adding new perspectives or placing their knowledge in new contexts.

Another important role of the mini-app was to provide the teachers with the possibility to express their views and thus articulate their beliefs, making these more conscious. In this way, the shared materials, as well as the following thought-provoking questions, were to function as the external motivation that "awakened" possible unconsciously held beliefs, and provided the knowledge (or evidence) on which they could be based. In the mini-app, teachers could leave a comment after the passages they read, and write down what came into their mind after reading, including ideas and doubts. More importantly, the mini-app enabled the teachers to have discussions with other teachers and made them aware of views that differed from their own, which helped them reflect on their beliefs. In a way, the technology—in this case the mini-app—made various opinions more easily available to the teachers than what would have been possible in real life.

In addition to the mini-app, another important element of the online environment was online seminars. These seminars took place in the first half year after the teachers had finished reading the materials about a specific topic, usually every other week. Although it might have been possible to find a similar group of teachers interested in the same topic to have a discussion without digital technology, it would definitely have been more challenging. It is both time and energy consuming for teachers to gather, even if they live in the same city. In a real-life setting, the biweekly seminars would thus have been almost impossible to organize. With the help of technology, the teachers could access the meetings just by clicking on a link, and then they could listen to and have discussions with others. Furthermore, the comments in the mini-app gave the teachers more time to reflect. They could pause anytime to think more about an interesting comment, they could come back later and read it again, and they could even discover what other teachers thought about the matter—all of which improved the conditions for reflection.

In the case described in this chapter, six short passages related to the topic "the sum of angles in a triangle" were sent to the participating teachers via the online mini-app, usually one passage every other day, in order to challenge their OJ3, especially concerning mathematical proof. Most of the passages were related to the history of "the sum of angles in a triangle", including thoughts about the way it was discovered, and different ways to argue or prove that the sum of angles in a triangle is 180°. They were chosen from historical books and papers, but reformulated by the second author in order to make them more accessible to the participating teachers. The time spent on the mini-app varied among the teachers. It is estimated that more than half of the teachers spent at least half an hour reading and commenting on this particular topic in the mini-app.

To instigate the teachers' reflection on the nature of mathematics and to establish a common ground for discussion, the teachers were presented with several thought-provoking questions at the end of some of the passages, for example: "Is this a proof? Why or why not?"; "What kind of argument is convincing in mathematics?"; "Why do we prove?"; "What is the role of proof?"; and "Is a proposition being proved always true?". We have previously argued that these questions and considerations about mathematical proof can be connected to OJ3. Furthermore, most of the questions were intended to both uncover and challenge the teachers' beliefs about proof. Also, when teachers argue why they hold a certain belief, knowledge is very likely to be involved.

Subsequently, the teachers were invited to a joint online meeting with the attendance of the second author of this chapter, who was managing the project. Here, teachers could have real-time communication to help deepen their reflections.

### *5.2 Data Sources and Analysis*

Various kinds of data were collected during the discussion, including comments and discussions in the mini-app and a video recording of the online meeting. The data thus indicate if and how the online community might facilitate the reflections that are needed to change beliefs as well as the possible changes in their OJ3. To reveal more possible reflections hidden in the visible conversation, we selected one teacher who participated in both the written and the oral dialogues as an example of the potential of this form of online community in relation to developing teachers' OJ3. The sampling here was information-oriented (Flyvbjerg, 2006), i.e., the case could provide a great deal of information to answer the research question.

A short interview with the teacher was conducted and recorded. Here, he was asked to elaborate on excerpts of the discussions and to give his overall opinion on the opportunity to see different opinions and communicate with other teachers and researchers in relation to fostering and supporting reflection.

We analyzed the data by applying the theoretical concepts described above. Signs of beliefs, knowledge, and reflection were identified and characterized in regard to their interrelations as well as centrality, evidentiality, and status of consciousness. In addition, we reviewed how digital technology might have facilitated and contributed to the teacher's awakening and articulation of his beliefs, his access to and use of knowledge, and his opportunity for reflection, i.e., his OJ.

## 6 The Case of Teacher Li

This case illustrates how a teacher (Li) reflected on an issue connected to the characteristics of mathematical problem formulation, thought, and methods. In the mini-app, the teachers were asked to give their view on the role of proof in mathematics. This led to a comprehensive exchange of opinions among the teachers. First, Li described his perception of the role of proof in mathematics and in teaching:

Li: It [proof] lays a foundation for mathematical research. It is also helpful in exercising logical thinking, and it affects language expression, way of thinking and work style.

Another teacher, Zhao, then asked him to elaborate on his comment, which encouraged Li to articulate his beliefs, making them more conscious or detailed:

Zhao: What do you mean when you say that proof can influence work style?
Li: When deductive proof becomes a kind of mathematical competency for someone, he or she will deal with problems in a strict and standardized way, with reasonable evidence, and step by step.... This means proof affects the style of doing things.

However, contributions from other teachers, e.g., Sung, made Li consider his understanding of the matter, which he stated later in another comment:

Li: Sung mentioned […] that new conclusions can be found by proof, which is different from my previous understanding of proof. I used to think that proof was only carried out within a fixed range, but now it seems that proof can also be used to find new conclusions outside the range.

The "fixed range" here means the range composed by the proposition we already know; that is to say, teacher Li used to think that we can only prove a known proposition with another known proposition. His statement indicates that his belief about proof only being "carried out within a fixed range" was held unconsciously. It seems that he had not thought about the matter before and did not become aware of this belief until Sung brought it up. Inspired by Sung, Li noticed his unconscious beliefs and reflected more on them, including trying to find evidence. Not only did this input cause Li to rethink his own perception, but he even chose to articulate and share his reflections with the others—a gesture which again might have encouraged others to express their thoughts and doubts. From Ernest's (1989) perspective of the philosophy of mathematics, teacher Li seemed to modify his view on the structure of mathematics from regarding mathematics as a more static but logic structure to regarding it as a continually expanding field. Thereby, it could be said that Li took a small step from a more Platonist view to a problem-solving view, or at least that the consistency with the problem-solving view increased in relation to this part of his OJ3.

The discussion continued at the real-time online meeting. Afterwards, Li made a new comment in the mini-app, this time also considering teaching perspectives:

Li: I used to think that deductive reasoning is not conducive to cultivating students' innovation and discovering new conclusions. After listening to other teachers' ideas, I found that proof is an important way to discover unknown conclusions.

The new belief that "proof can be used to discover" thus seemed to have become more central to Li in his belief system concerning teaching. He used to think that proof "cannot be used to discover" and it "is not conducive to cultivating students' innovation", which means that Li's belief about the role of proof itself was closely related to his belief about the role of proof in mathematics teaching and learning. Hence, Li reflected on the role of proof not only in a mathematical context, but also in relation to how his modified understanding of the role of proof might affect his teaching. It was not only in a mathematical sense that he changed his views on the role of proof, but he also considered what part it might play in developing his students' mathematical understanding, hence putting the role of proof in a didactical context.

Li indicated in the interview that he continued his reflection on the matter, connecting it to his mathematical knowledge and thus basing his changed beliefs on evidence:

Li: I'm thinking that it makes sense, because if you look through Euclid's *Elements*, many results are retrieved from just 5 axioms, right? And even though he wrote more than 400 theorems, it can go on and on. [...] There is no ceiling for proof. It is open. It can move forward and discover new things.

In summary, the process of Li's change in beliefs began with inputs from the researcher in charge of the app, in the form of texts and related questions. These questions started Li's own thinking, but it was the confrontation with other teachers' views, beliefs, and knowledge that contributed to this process by drawing his attention to unconsciously held beliefs and rethinking his understanding. He reflected on the reason for his beliefs and linked it to other related beliefs concerning teaching. Finally, he connected his reflections to his knowledge, making his beliefs evidentially held.

The question is if this process would have taken place without the availability of the app and the flexibility offered by this form of online community. Li described his general experience with the program in the interview:

Li: I am presented with many opinions on [mathematical issues]. Without this, […] I certainly would not have thought about it. I cannot consider such things every day. […] if one does not see other things or listen to other opinions, he will be the frog in the well.

The last remark refers to a famous Chinese fable about a frog who lives in a well and believes the sky is only as big as what he can see. In other words, Li clearly felt that the communication with other teachers broadened his perspective on mathematical issues. Furthermore, Li explained that the online environment made everything quite convenient, not limited by time and space. He pointed to an advantage of the combination of the mini-app and the online meeting:

Li: Before the real-time online discussion, we need to learn and think. The text communication in the mini-app is a prerequisite for ensuring an effective online discussion. First, we think, and then we can come up with insightful perspectives to communicate during the online meeting.

There is no doubt that Li experienced a development in his OJ3 through his participation in the project. Original beliefs were articulated, challenged, reevaluated, and reflected upon. The reflection process involved new knowledge and contexts, making his beliefs based on evidence. In this way, it appears that Li's knowledge and beliefs about the nature of mathematics have increased in quality, detail, and range.

## 7 Discussion

As beliefs act as filters that affect what one sees (Pajares, 1992), it is not easy for people to see things they do not believe, even with new experience. Therefore, different voices are of vital importance for critical reflection. As the presented case shows, the nature of the digital media provides excellent conditions for reflection: the teachers in the program are in a short-time span presented with many different questions, views, and perspectives on a subject, which is in line with existing research on online communities mentioned in the introduction. In this case, the teacher highlights how this form of communication increases his thinking and broadens his perspectives by providing access to inputs from many colleagues.

With the case of Li, we have illustrated in which ways digital technology in the form of an online community might contribute to teachers' reflection processes and furthermore to the development of their OJ3, making specific aspects of the nature of mathematics "the object of explicit treatment, reflection and articulation" (Niss & Højgaard, 2011, pp. 74–75). Firstly, the shared materials and the related thought-provoking questions from the researcher as well as the comments and questions from other teachers provide the external motivation needed to make the participants aware of unconscious beliefs, conflicting evidence, other possible contexts etc. This can foster an increased curiosity that might inspire further investigation and thought processes. Conflicting positions of the teachers thus incite them to check their own positions, making their beliefs more objective. Secondly, the discussions in the online community encourage teachers to articulate, explain, and argue for their beliefs, thus making unconscious beliefs conscious, and conscious beliefs more detailed and nuanced, as seen in the case of Li reflecting on the role of proof. Thirdly, Li connects his existing knowledge to his modified belief, thereby making it evidentially held. This is not caused directly by technology, but is triggered by the online environment. In addition, other teachers' knowledge about mathematics (and the way they connect it to their beliefs) is very likely to be shared in the online environment, which plays an important role in regard to developing evidentially held beliefs. When other perspectives and new knowledge are included in the teachers' reflections, they can be connected to the beliefs that might be in a process of change or modification and thereby become part of the evidential foundation. Moreover, these new perspectives can put existing beliefs into new contexts, changing or adding relations between beliefs.

On top of this, the combination of the app and the online meeting seems to work very well in facilitating teachers' reflection, as noticed by Li. The app provides an essential factor in relation to deep reflection, namely *time*. It allows teachers to read inputs as many times as they need, consider them, and read them again. When they meet for the online discussion session, they have had an excellent opportunity to think about the issues and the different perspectives of the issues. This is a very good basis for a peer discussion and might be the factor that enables them to discuss quite complex matters about the nature of mathematics and relate them to their teaching. It provides an opportunity for a very thorough and profound reflection on their own knowledge and beliefs and the relations between these.

One might argue that all of these aspects apply to any of the three forms of OJ, maybe even to any subject. This may be true. However, the nature of mathematics is often an implicit concept that is seldom a basis for discussion, and teachers can normally perform teaching without thinking about it consciously. In this case, the formal organization of the online community provides opportunities for the teachers to focus on the underlying characteristics of their subject, for example the construction of mathematics, mathematical thought processes, or the character of mathematical results. It allows them to discuss the nature of mathematics at an explicit, deep, and philosophical level, and to discover implicit knowledge and unconscious beliefs. And most importantly, they are given room to reflect, which is essential for developing overview and judgment concerning the nature of mathematics as a discipline. To some extent, the reflections on OJ3 furthermore influence teacher Li's philosophy of mathematics, which is a more implicit concept and very difficult to change, which reflects the importance of making the OJ3 the object of reflection.

With that said, the teachers' OJ2 on the historical development of mathematics might also be influenced because of the teachers' engagement with and reflections on the historical development of mathematics. Furthermore, their OJ2 has actually played a role in the reflections and beliefs that teachers had about the nature of mathematics as a discipline. Moreover, the teachers in the case not only discussed mathematical topics and issues in a mathematical context, but they also related these issues to the other frame of reference that they have in common, namely the teaching of mathematics, making this an object of discussion and reflection as well. Hence, the online community and the communication between the teachers may contribute to the development of another part of the KOM-framework: the didactic and pedagogical competencies related to teaching mathematics, perhaps particularly the professional development competency.

## 8 Final Remarks on Networking of Theories

In this research, teachers' development of OJ3 is the main variable in focus. However, the KOM-framework alone cannot explain why this development might occur. In other words, questions or dimensions of OJ3, like the characteristics of mathematical thought and methods, and the construction of mathematics, are helpful in making researchers gain an insight into what changes take place in teachers' reflections on the nature of mathematics, but they do not give any information about the mechanism behind this change. As OJ3 is a combination of knowledge and belief, we have applied constructs of beliefs theory from psychology, like unconscious beliefs, beliefs with evidence, and the relationship between knowledge and belief. Combining theories from psychology and mathematics education can contribute to understanding beliefs more deeply (Rolka & Roesken-Winter, 2015). Hence, by using elements of networking, this combination of theoretical constructs thus contributes to understanding how an online community provides opportunities for gaining experience with and reflecting on mathematics as a discipline, thereby influencing the development of overview and judgment.

## References

Borba, M. C., Askar, P., Engelbrecht, J., Gadanidis, G., Llinares, S., & Aguilar, M. S. (2016). Blended learning, e-learning and mobile learning in mathematics education. *ZDM – Mathematics Education, 48*(5), 589–610. https://doi.org/10.1007/s11858-016-0798-4

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Heath.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching, 15*(1), 13–33. https://doi.org/10.1080/0260747890150102

Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry, 12*(2), 219–245. https://doi.org/10.1177/1077800405284363

Furinghetti, F. (1996). A theoretical framework for teachers' conceptions. In E. Pehkonen (Ed.), *Current state of research on mathematical beliefs III. Proceedings of the MAVI-3 workshop* (pp. 19–25).

Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 39–58). Kluwer Academic. https://doi.org/10.1007/0-306-47958-3_3

Green, T. F. (1971). *The activities of teaching*. McGraw-Hill.

Jankvist, U. T. (2015). History, application, and philosophy of mathematics in mathematics education: Accessing and assessing students' overview and judgment. In S. J. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 383–404). Springer. https://doi.org/10.1007/978-3-319-17187-6_22

Lantz-Andersson, A., Lundin, M., & Selwyn, N. (2018). Twenty years of online teacher communities: A systematic review of formally-organized and informally-developed professional learning groups. *Teaching and Teacher Education, 75*, 302–315. https://doi.org/10.1016/j.tate.2018.07.008

Lester, F. K. (2002). Implications of research on students' beliefs for classroom practice. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 545–354). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47958-3_20

McLeod, D. B. (1994). Research on affect and mathematics learning in the JRME: 1970 to the present. *Journal for Research in Mathematics Education, 25*(6), 637–647. https://doi.org/10.5951/jresematheduc.25.6.0637

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*, 307–332. https://doi.org/10.3102/00346543062003307

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Information Age Publishing.

Rolka, K., & Roesken-Winter, B. (2015). Networking theories to understand beliefs and their crucial role in mathematics education. In B. Pepin & B. Roesken-Winter (Eds.), *From beliefs to dynamic affect systems in mathematics education* (pp. 73–93). Springer. https://doi.org/10.1007/978-3-319-06808-4_4

Schoenfeld, A. (1994). What do we know about mathematics curricula? *The Journal of Mathematical Behavior, 13*, 55–80. https://doi.org/10.1016/0732-3123(94)90035-3

Sun, D. D. (2021). 基于数学史网络研修的在职初中数学教师观念发展研究 [*Research on the development of junior middle school mathematics teachers' conception in an online teacher education program based on the history of mathematics*] [Doctoral dissertation]. East China Normal University, Shanghai, China.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 127–146). Macmillan Publishing Co.

Unwin, A. (2015). Developing new teacher inquiry and criticality: The role of online discussions. *British Journal of Educational Technology, 46*(6), 1214–1222. https://doi.org/10.1111/bjet.12194

# Broadening the Scene

# Teachers' Facilitation of Students' Mathematical Reasoning in a Dynamic Geometry Environment: An Analysis Through Three Lenses

**Ingi Heinesen Højsted, Eirini Geraniou, and Uffe Thomas Jankvist**

## 1 Introduction

In mathematics education research, numerous theoretical constructs and frameworks have been developed to describe the complex processes involved in teaching and learning mathematics. These constructs and frameworks may be classified in terms of the scope or the level of detail, which they provide concerning certain phenomena (Bikner-Ahsbahs & Prediger, 2010). An example of a broad analytical framework is the Danish KOM[1] framework (Niss & Højgaard, 2011, 2019), which is best known for its description of what mathematical mastery entails through its specification of eight mathematical competencies. The notion of mathematical competencies has gained substantial traction in mathematics education research in recent years, and has also played a role in relation to the framework of the international PISA assessments (Niss et al., 2016). At a national level—in Denmark—the KOM framework has permeated the national mathematics programmes and curricula at primary, lower secondary and upper secondary school levels as well as in mathematics teacher education since it saw the light of day in 2002 (in Danish).

[1] The KOM acronym translates to "Competencies and Mathematical Learning". The framework was originally published in Danish in 2002 and in English in 2011.

I. H. Højsted (✉)
Faculty of Education, University of the Faroe Islands, Tórshavn, Faroe Islands
e-mail: ingih@setur.fo

E. Geraniou
Institute of Education, University College London, London, UK
e-mail: e.geraniou@ucl.ac.uk

U. T. Jankvist
Danish School of Education, Aarhus University, Copenhagen, Denmark
e-mail: utj@edu.au.dk



U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_15

Something else, which now permeates the mathematics programmes in Denmark, is the use of digital technologies, typically Computer Algebra Systems (CAS) in upper secondary school and Dynamic Geometry Environments (DGE) in lower secondary and primary school (Jankvist et al., 2019). Except for mentioning mathematical software in relation to mathematical tools and aids, the KOM framework does not allocate much attention to the role of digital technologies as part of the teaching and learning of mathematics (Jankvist et al., 2018). Due to the number of challenges surrounding teaching with digital technologies in the Danish classrooms (e.g., Iversen et al., 2018; Jankvist et al., 2016), it makes sense—also at a theoretical level—to address the potential of combining or even networking the KOM's description of teacher competencies with a framework that deals explicitly with the introduction of digital technology into the mathematics classroom, and the role of the teacher in orchestrating the teaching/learning process. The Theory of Instrumental Orchestration (TIO) (Drijvers et al., 2010; Trouche, 2004), is an example of a fine-grained theoretical framework, which concerns the specific role of the teacher in supporting students' development of instruments (from an instrumental genesis perspective) in a technology-rich mathematics classroom.

By coordinating elements and constructs from the KOM framework and TIO, we examine the role of a teacher through a series of episodes from a teaching sequence, where the educational aim is to support students' development of mathematical reasoning competency in a context involving a DGE. We use the notion of justificational mediations (JM) (Jankvist & Misfeldt, 2019, 2021; Misfeldt & Jankvist, 2018) to articulate the coordination of KOM and TIO. The question that we seek to answer through our analysis is:

How can we understand the role of the teacher in a DGE teaching sequence that aims to support students' development of mathematical reasoning competency through three different lenses, namely KOM, TIO and JM?

Next, we present the two theoretical frameworks of KOM and TIO, including also the notion of justificational mediations, since this shall serve as a "bridge" between the two theoretical perspectives for our specific case.

## 2 KOM's Mathematics Teacher Competencies

We begin with the KOM framework and its mathematics teacher competencies, also sometimes referred to as didactico-pedagogical competencies. The Danish KOM framework (Niss & Højgaard, 2011, 2019) introduces a competency-based approach to describe what mathematical mastery entails, across mathematical topics and educational levels. The framework comprises eight mathematical competencies, one of which is the reasoning competency, which at its core, concerns the ability "to analyse or produce arguments (i.e., chains of statements linked by inferences) put forward in oral or written form to justify mathematical claims" (Niss & Højgaard, 2019, p. 16). We will take a closer look at the reasoning competency later—for now, however, we address the lesser-known part of the framework, namely its six competencies of mathematics teachers. These outline "a range of specific mathematical didactic and pedagogical competencies" (Niss & Højgaard, 2011, p. 85), which a mathematics teacher should possess:

- *Curriculum competency* entails being able to acquire knowledge about the curriculum and to analyse and evaluate its significance for teaching, as well as being able to develop and implement course plans adhering to curriculum and overarching frameworks.
- *Teaching competency* involves being able to develop and carry out teaching sequences. This includes the orchestration of different teaching and learning situations, with consideration to the target students, and being able to choose appropriate tasks and teaching materials, being able to motivate and inspire students and incorporate their ideas in the lessons.
- *Competency of revealing learning* consists of being able to understand and interpret student learning and to which degree they possess the mathematical competencies. Furthermore, it covers being able to understand the cognitive and affective cause for the students' conceptions and beliefs regarding mathematics.
- *Assessment competency* comprises being able to choose or develop formative and summative evaluation methods to reveal students' possession of mathematical competencies, and knowing the limitations of said methods. In addition, being able to communicate the results and help students to improve accordingly.
- *Cooperation competency* consists of being able to cooperate with other mathematics teachers and colleagues in other subjects, as well as with agents outside the institutions such as parents or authorities, on issues relevant for mathematics education.
- *Professional development competency* is a meta-competency in relation to the five above-mentioned competencies, as it concerns being able to develop one's competency as a teacher. This includes being able to reflect on one's developmental needs and being able to choose activities that will promote this development.

## 3 Instrumental Orchestration

In the instrumental approach to tool use (Artigue, 2002) a distinction is made between artefacts and instruments. Technological tools such as DGE are viewed as artefacts that may become instruments for an individual, as the artefact is appropriated for usage. An instrument is the psychological construct involving the utilization of artefacts in (mathematical) situations. The artefact is not an instrument from the outset, but in the process of instrumental genesis, cognitive schemes are developed (Vergnaud, 2009), which intertwine technical and mathematical knowledge, so that the individual can use the artefact purposefully, hence transforming the artefact into an instrument.

Trouche (2004) coined the term instrumental orchestration to highlight the necessary external steering of students' instrumental genesis, which is, in the context of mathematics education, usually performed by a teacher. Trouche (ibid.) described two central elements of instrumental orchestration, didactic configurations and exploitation modes. *Didactic configurations* comprise the teacher's organization of the artefacts in the teaching/learning environment in relation to the treatment of the mathematical situation at hand. It includes organization of students' or teachers' workspace and time spent on activities. *Exploitation modes* concerns the manner in which the teacher utilizes the didactical configuration in order to reach the goals set out for the activity. This comprises deliberations on the choice of mathematical tasks, the role of the artefact in these tasks and how this may support students' development of the intended instrument.

The didactic configurations and exploitation modes concern aspects of teaching sequences, which may be prepared before a teaching sequence is undertaken. Drijvers et al. (2010) added a third element to instrumental orchestration, by introducing the term *didactical performance*, which addresses the instrumental orchestration created by teachers "on the spot" during a teaching sequence. It entails the ad hoc decisions made, in order to deal with emerging situations in the learning/teaching environment, such as students' questions, dealing with unexpected situations regarding the artefact or the mathematical task.

## 4 Justificational Mediations

The use of digital technologies—in particular CAS but also DGE—in mathematics education is often analyzed using the instrumental approach to mathematics learning (e.g., Trouche, 2005). One critical concept in the instrumental approach is that of *mediation* that designates how tools support goal directed activities, and hence mediate between a student and his/her goal. The literature highlights a critical distinction between epistemic and pragmatic mediations referring to whether the students aim at understanding certain phenomena or at solving specific tasks (Artigue, 2002; Rabardel & Bourmaud, 2003).

When working with the use of CAS in the context of proofs and proving activities, Misfeldt and Jankvist (2018) describe four core questions/aspects about how CAS mediate proving. Aligning with the instrumental approach, this can be conceptualized as using CAS for justificational mediations (Jankvist & Misfeldt, 2019, 2021; Misfeldt & Jankvist, 2018). The notion of justificational mediations is equally relevant when using DGE, such as GeoGebra, for reasoning and proving activities (e.g., Gregersen & Baccaglini-Frank, 2020). In the context of DGE, Misfeldt and Jankvist's (2018) four questions/aspects can be reformulated into:

- (JM1). Do the teachers or students use DGE's features to establish truth? To what extent does the DGE's output act as a warrant in an argument?
- (JM2). Does the teachers' or the students' use of DGE allow interaction and experimentation? This highlights the degree to which the teacher or the students can change parameters, explore phenomena etc., and therefore to what extent the students still have agency, when working with DGE in relation to proofs.

- (JM3). Is the argumentation inductive, deductive or authoritarian? What type of conviction (or proof scheme, see Harel & Sowder, 2007) is in play and what type of warrant does DGE provide?
- (JM4). Does the argument highlight important aspects of the proof or the mathematical relationships?

Both the second and the fourth aspects are to some extent related to epistemic mediations. Misfeldt and Jankvist (2018) argue that these four aspects of justificational mediation more or less capture the important aspects of using digital tools in proving activities.

## 5 Methods

The study took place in a Danish lower secondary school in a grade 8 mathematics classroom (13–14 year old students). The preparation for the study consisted of a researcher and the teacher meeting to discuss the material in order to develop a shared understanding of the aim and learning goals of the teaching sequence. These preparatory discussions also served in unveiling any preconditions of the students and the teacher, in particular concerning their GeoGebra experience. The teacher described himself and his students as basic/intermediate users of GeoGebra with knowledge about the structure of the geometry part of the programme, e.g., how to use the tools for construction and measuring.

The researcher assumed a teacher-supporting role during the teaching sequence, especially during the start-up phase. In the first lesson, which we report on in this chapter, the researcher supported the class teacher in orchestrating the class discussions that took place.

The data was collected in the form of screencast recordings from every pair of students and external video of three chosen groups as they were working on the tasks. These recordings also captured audio from student–teacher interaction during task work, and from the class discussions. In addition, the students' written answers were collected.

## 6 The Teaching Sequence

The design entailed a three-week DGE teaching sequence, in which the mathematical aim was to support students' development of the reasoning competency (Niss & Højgaard, 2019). The teaching sequence included 15 tasks that were handed to the students in a printed version. The students worked in pairs sharing one computer, using GeoGebra to solve the tasks, and writing their answers in the handout. The students were paired and asked to work together and argue about their solving steps for two reasons; on the one hand, being able to communicate and justify one's reasoning to others is a characteristic of reasoning competency and hence a learning goal of the teaching sequence, on the other hand, it helps us as researchers to have a "window" into the students' thinking processes when it comes to interacting with the DGE. At the end of every lesson, class discussions followed to further elaborate crucial mathematical ideas, but also share students' solving strategies and understanding.

In the design of the teaching sequence, certain potentials of DGE were utilized. A seminal feature of DGE is the possibility of representing geometrical properties visually in the form of invariants that can be observed during the use of the drag mode. The properties of the figure are locked in a hierarchy of dependencies, which are determined by the construction procedure; the environment that is mimicking Euclidean geometry; and design choices by the software developers. In the literature, dragging is highlighted as a means to support the development of mathematical reasoning and conjecturing in geometry (e.g., Arzarello et al., 2002; Baccaglini-Frank & Mariotti, 2010; Edwards et al., 2014; Laborde, 2001; Leung, 2015). The design intention was to exploit the affordance of dragging so that students could investigate constructions in order to produce and investigate conjectures, which they made beforehand.

Reasoning in DGE in relation to the drag mode, involves awareness of the fact that it is dependencies between objects, which decide the outcome of dragging. Therefore, in order for the students to be able to interpret what they see on screen during dragging, the objective of the initial tasks of the teaching sequence was to divert students' attention to the fact that invariants, which are observable during dragging, reveal logical relationships between geometrical objects in GeoGebra.

Task 1 was a "dependency task" (Fig. 1). In the construction part, the students were encouraged to use the GeoGebra tool "midpoint", which causes the midpoint $C$ to be derived from points $A$ and $B$.

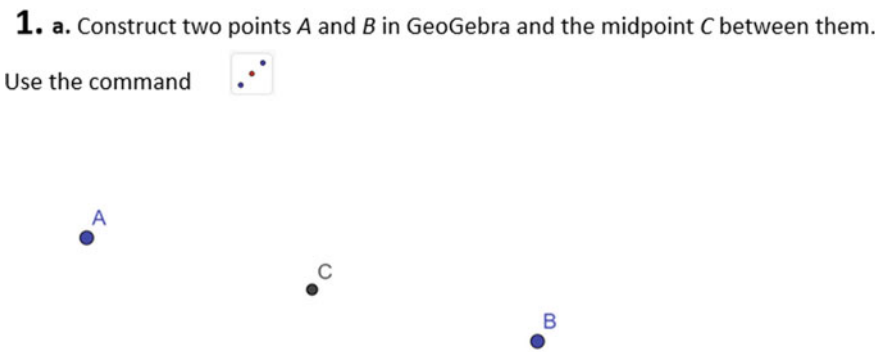



**Fig. 1** Task 1 of the teaching sequence, where question 1(b) is "What do you think happens to the other points when you drag point $A$? Guess first, and justify your guess to your partner", followed by the same question in relation to dragging points $B$ and then $C$

Afterwards the students are asked to predict-observe-explain (White & Gunstone, 2014) what happens during dragging in three sub questions concerning each element of the construction.

Most students predicted correctly in relation to dragging points *A* and *B*, whereas no students in the data collected predicted correctly in relation to dragging point *C*. In GeoGebra, derived objects can be locked so that they may not influence objects from which they are derived—in the literature sometimes referred to as parent/child relationship (e.g., Højsted & Mariotti, 2021; Talmon & Yerushalmy, 2004). Predominantly, a "child" cannot influence its parent in GeoGebra, hence, *A* and *B* are free objects in the student's construction, and may be dragged, while *C* may not be dragged. The teacher seemed not to be aware of this or at least he does not explain or help the students explain why it is not possible to drag point *C*.

Task 4 required students to construct a robust right-angled triangle, as shown in Fig. 2. By "robust", we refer to a triangle that will maintain certain properties, in this case it will remain a right-angled triangle, even when you drag its vertices and enlarge it or change its position on GeoGebra's workspace (Healy, 2000; Laborde, 2005). The task design is in line with what Mariotti (2012) coined "construction tasks", which encourage students to construct robust figures with specified invariants. The students are then required to describe their construction method and explain why the figure retains its properties. This is related to justifying a mathematical claim, which is a characteristic of reasoning competency (Niss & Højgaard, 2019, p. 16).

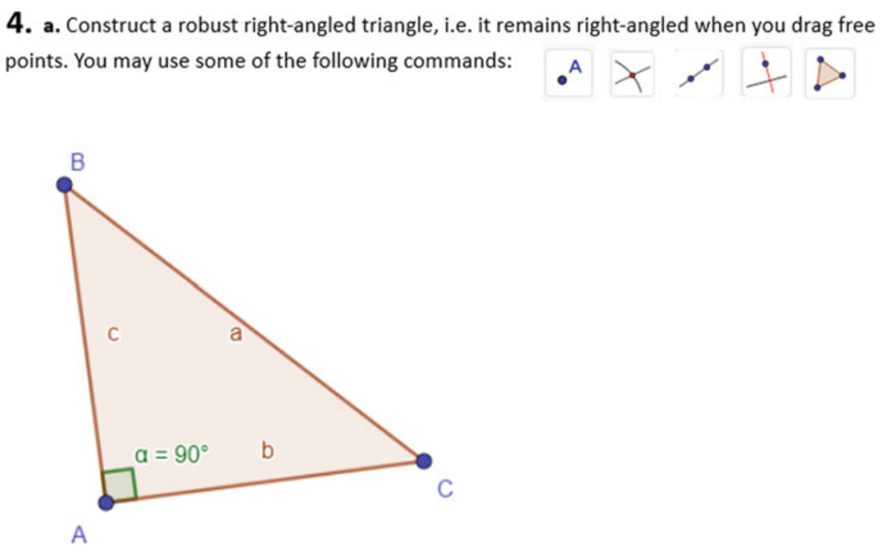



**Fig. 2** Task 4 of the teaching sequence, where question 4(b) is "Describe how you made it" and 4(c) is "Why does it remain a right-angled triangle when you drag the points? Explain to your partner"

In the next section, we present a series of episodes taken from two maths lessons in the form of student–teacher interaction that occurs while two students, Susan and Oliver, are working on task 1, task 4 and on the subsequent class discussions.

## 7 A Series of Episodes in Light of Justificational Mediations

In these episodes, the class teacher and the researcher (the first author of this chapter) worked with a class of 20 students. We focus on a group of two students and their interactions with the teacher (T) and researcher (R). Reflecting upon the justification mediation perspective discussed earlier, we will be looking at how the teacher and the researcher addressed students' mathematical reasoning focusing on the four questions/aspects related to justificational mediations.

In the following, we see the dialogue that emerges between Susan, Oliver and the teacher as they are working on task 1d. Susan and Oliver are puzzled that they could not move point *C*, so the teacher intervenes.[2]

(124) **T**: What do you think is possible to move?
(125) **Oliver**: So, the two other points.
(126) **Susan**: *A* and *B*.
(127) **T**: Yes, you can move *A* and *B* because that was what he (the researcher) said, you know, it's a dynamic program. That is, *C* will always be the midpoint, that is, *C* is automatically moved if *A* and *B* are moved.
(128) **Oliver**: Yes, exactly.

In terms of justificational mediation, the teacher showcases that GeoGebra is indeed a dynamic environment that allows interaction and experimentation (JM2). However, his comments here may not be considered as a scaffold for utilizing GeoGebra (considering its restrictive functionality, i.e., point *C* not being draggable) and therefore reach a mathematically valid justification for the phenomenon in question, neither does his additional comment:

(124) **T**: If you do not move A and B, then C stays the same. (This explanation does not make it clearer, why *C* should not move.)
(125) **Susan**: Okay.

In the above comments, the teacher also puts emphasis on the mathematical relationship between the three points, *A*, *B* and *C* (element of JM4). However, we cannot claim with much confidence that his argumentation here explains "why" this phenomenon occurs. On the contrary, his comments indicate how GeoGebra becomes the authority that convinces both the teacher and the students of the correctness of their claim (line 127). The GeoGebra tool therefore provides an element of an authoritarian argumentation (JM3).

[2] The numbering of lines corresponds to the full transcript in Højsted (2021).

In the next lesson, students were working on task 4. The teacher approached two students to take a closer look at their screens.

(562) **Susan**: We made them like this.
(563) **T**: Ah, so you have made the line segments invisible. (He means lines.)
(564) **Susan**: Yes, we have made it in two different ways.
(565) **T**: Yes?
(566) **Susan**: Oliver has made such an angle of a given size.
(567) **Oliver**: I just made…
(568) **T**: If you do that, right.

(The teacher takes over the computer.)

(569) **T**: Is it just me or does one of them not get locked then?

(Tries to drag point $E'$.)

Oliver used the tool "angle with given size" clicked twice on the screen and chose an angle of 90° (points $E$, $F$ and $E'$ emerge). Oliver then connected the points using the tool "polygon". From this method, $E$ and $F$ are free points, while $E'$ is locked as the 90° anticlockwise rotation of $E$ about $F$ (Fig. 3).

(570) **Susan**: Yes, this one gets locked (referring to point $E'$).
(571) **Oliver**: Yes, but here the $D$ point is locked.
(572) **T**: Yes, and that's the right-angled triangle. So, one can say that this one has both. So it's another method to

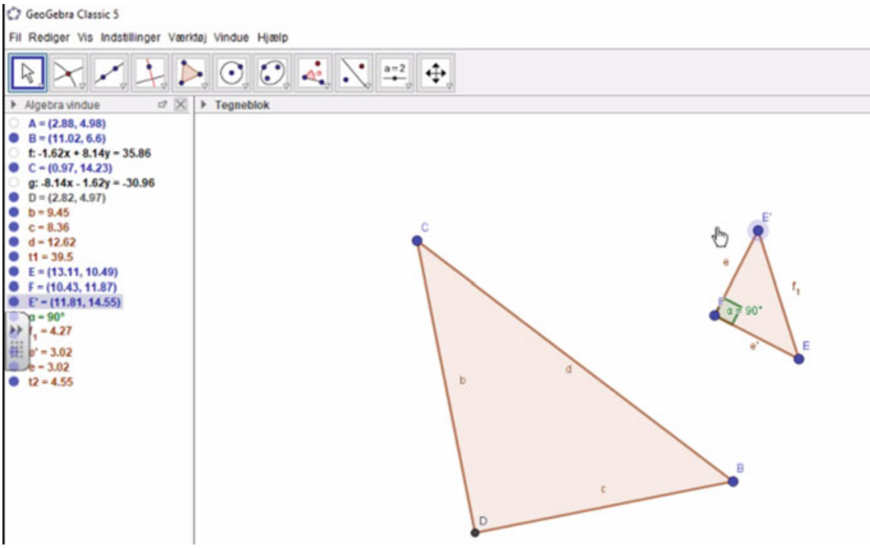



**Fig. 3** From the screencast recording, we see Oliver's constructed right-angled triangle in GeoGebra (on the right), using the "angle with given size" feature. Susan is illustrating that point $E'$ is locked. Point $F$ is barely visible in the corner of the right angle in the smaller triangle

make it. But here it is also locked (mouse marker on point *E*), because it must be 90° (mouse marker on point at point *F*), right? (His explanation does not clarify why it is different in the two constructions.)

The teacher's explanations continue to rely heavily on the construction process, focusing on the DGE construction process output as a way to justify (JM1). There seems to be a missed opportunity here for some justificational mediation. We would expect the teacher to ask probing questions such as: Why is this angle always 90°? What happens to point $E'$ when we drag point *E*? Which relationships are induced when we use the angle with a given size tool?

The same pair of students is then visited by the researcher (R), who asks about the two different constructions they made.

(614) **Susan**: We have made it in two ways.
(615) **R**: Cool. How did you do it?
(616) **Susan**: Well, one of them, so we made this one first (points at line *f* on the screen).
(617) **R**: A line?
(618) **Susan**: The line *f*. And then we made a parallel. Excuse me, not a parallel, one that intersects perpendicularly.
(619) **R**: Aha….

In more detail, Susan used the tool "line" by clicking twice on the screen (hence points *A* and *B* emerge and the line *f* through them) and then the tool "perpendicular line" clicking on the plane (point *C* emerges) then on line *f* (line *g* emerges) (see Fig. 4). She then uses the GeoGebra feature "intersect" between lines *f* and *g* (point *D* emerges). Finally, she connects the points *CBD* using the features "polygon" and "hides" lines *g* and *f* as well as point *A*. From this method, point *D* is derived and hence locked in GeoGebra, while *C* and *B* are free.

Susan is not finished explaining how her construction was completed, but Oliver wants to explain his, so he interrupts. Both students seem to be excited about their constructions and keen to share their methods for creating them.

(620) **Oliver**: And then another one we used [is an] angle of a given size, made a right angle, and we just put a polygon in it.
(621) **R**: Smart.
(622) **Susan**: Yes, in addition, we put an intersection between the two (lines *f* and *g*) and then we formed a polygon.

In the above dialogue, Oliver described how he used GeoGebra's features to create the required construction, but also implicitly claim that it is a correct construction as he used the GeoGebra tool to create the mathematical objects, i.e., perpendicular lines, 90° angle and a polygon. The GeoGebra's authority is yet again used for the student's argumentation (JM3). We could of course debate about the extent to

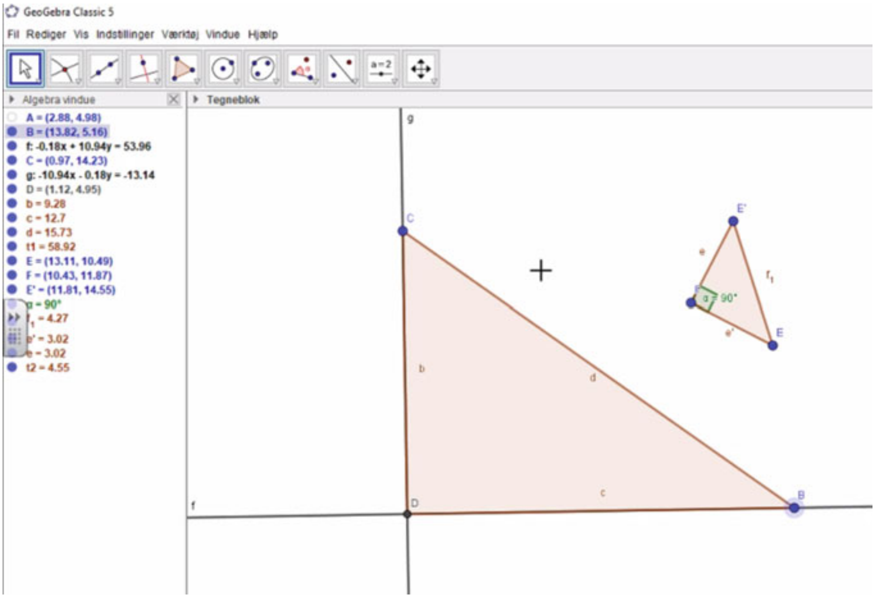



**Fig. 4** Susan's construction of the right-angled triangle in GeoGebra (on the left), using the perpendicular lines tool to create a robust right angle

which the GeoGebra output could act as a warrant in arguing the "correctness" of the students' construction. It is also worth mentioning how students' knowledge of creating a right angle by constructing perpendicular lines helped them in their solution, which indicates that they may be able to justify the correctness of their construction by highlighting mathematical relationships (indications of JM4). We cannot, however, make any strong claims about their argumentation, since they did not explicitly elaborate their thinking, offering any reasoning, but instead assumed that the researcher would immediately "see" what they were doing.

In the next dialogue, the second aspect of justification mediation becomes more evident (JM2). The researcher and the students display how their use of GeoGebra allows experimentation and discuss the impact of dragging certain points of their construction. Such experimentation allows them to investigate special cases and also reflect upon special mathematical relations of objects within their constructions and may lead to mathematically valid arguments and the formation of a mathematical proof (JM3 and JM4).

(623) **R**: And how, ehh… Can you also drag point *C*?

(624) **Susan**: Yes, these two I can drag both of them (*C* and *B* from her construction).

(625) **Susan**: Here you can't drag both (marks point *E*' from Oliver's construction).

(626) **Oliver**: You can't drag in *D* (from Susan's construction).

(They are trying to advocate their own constructions, while downgrading the other construction.)

(627) **R**: Okay.

(628) **Oliver**: And in C (the third point *F* in his construction, where the right angle is).

(629) **Susan**: But here you can drag this (*F*). You can't do that here (point *D* in her triangle, where the right angle is).

(630) **R**: No, because there… This is the intersection of *g* and *f*.

(631) **Susan**: Yes, exactly.

(632) **R**: Okay, super.

(633) **Susan**: Yes.

In the above episode, the students were preoccupied with who made the best construction. They discussed which construction had the most free points and sort of agreed that the constructions were equally good because there were two free points (they did not notice the constraint of Oliver's triangle always being isosceles). From the point of view of the aim of the task, both constructions were acceptable—the important part was that the students seemed to have developed an awareness of the fact that dependencies between elements of figures decide the outcome of dragging in GeoGebra. Therefore, they were able to describe why their figure remained a right-angled triangle, referring back to the construction method and using a geometrical interpretation. Once again, this could be linked to the first justification mediation (JM1), since the students use GeoGebra's features to establish truth, but also to the fourth one (JM4), since they refer to (or argue about) important aspects of the mathematical relationships between their constructed objects in GeoGebra. One could also say that their interactions so far imply using GeoGebra as the authority for an authoritarian argumentation justificational mediation (JM3).

There was a plenary discussion after each lesson, during which the teacher encouraged students to reflect upon all the tasks that they worked on during the GeoGebra activity. At the end of the first lesson, they started off by discussing task 1 on constructing two points, *A* and *B*, and their midpoint, *C*. The teacher repeated the task situation, and asked the class to describe what happens when point *A* is dragged.

(386) **Susan** (In plenary): *B* stays the same place, but *C* will move after point *A* to stay in the middle.

(387) **R** (In plenary)*:* Yes.

(388) **R** (In plenary): *A*, *B* and *C* (writing on the whiteboard). So, when I drag *A*, then you say that *B* stays and point *C* moves, so if I move it here, down here, then *C* moves down here too. Why does it do that?

(389) **Susan** (In plenary): Because *C* must stay in the middle.

(390) **R** (In plenary): Because *C* must remain in the middle. That is exactly right. It is the relation between these objects, as I call them. The relation between these points, that is, we said to GeoGebra that we want the midpoint between *A* and *B*. Therefore, when you drag in *A*, the point *C* remains the midpoint all of the time. Because you have given the command to the program that we must have the relation that *C* must be the midpoint between *A* and *B*.

It seems that the student relies on the fact that the point $C$ is the midpoint and by construction must remain in the middle. The teacher elaborates on Susan's point by emphasizing how they used GeoGebra's features to construct the midpoint, which induces geometrical properties that are maintained during dragging, and therefore it is "true by construction". During the plenary discussions, the teacher on several occasions adopted the strategy of elaborating students' constructions by reiterating and explaining their methods of constructions in GeoGebra.

At the end of the second lesson, they discussed the task concerning the robust right-angled triangle:

(753) **T** (In plenary): Can anyone help with the word robust. What was that? Susan?

(754) **Susan** (In plenary): So, even if you, for example, have a right-angled triangle, if you move some of the points, then it remains right-angled.

(755) **T** (In plenary): Yes. We have at least one or two free points that we can drag, and it still remains a right-angled triangle no matter what we do, right? That means that one cannot just put three random points and make a polygon, even though one might say, okay, this one is actually right-angled, because then when I start dragging it, I also drag at the right-angled angle, right? Ehhh, how did you make it? There are several different solutions. Can you try to put words on how you did?

Three pairs of students put forward their construction method, while the teacher mirrored the students' description on the whole class screen. The teacher discussed the different ways in which students constructed the right-angled triangle and showcased the relevant GeoGebra features that are exploited in an implicit attempt to justify the correctness of their constructions. The teacher still seemed to rely heavily on "how" constructions were made in GeoGebra and subsequently justified why the triangle remains a right-angled one based on the way it was constructed. The GeoGebra features therefore, once again, became the warrant for the correctness of the construction and the justification for the triangle remaining a right-angled one (JM1).

Reflecting upon all the interactions discussed above, the teacher seemed to mainly ask for what the students have done, and consequently what happens during dragging. There were a few occasions when the teacher and especially the researcher would ask the students to explain why they have done it and why that, which they observe, happens. For example, the researcher asked Susan to explain why the point $C$ moves (line 388). From the view of understanding, asking more "why" questions may encourage a deeper understanding of the onscreen phenomena, and provide an anchoring for a possible mathematical interpretation, i.e., explaining in terms of the induced geometrical properties and logical dependencies (JM4).

## 8 Analysis from a TIO and JM Perspective

From the point of view of TIO, we will relate the data presented previously to the notions of didactic configurations, exploitation modes and didactical performance, while referring to the previous JM analysis.

We described earlier, the organization of the artefact (DGE) in the teaching sequence, which in TIO terminology refers to the *didactic configuration* of the sequence. This preparation was mainly performed by the researcher prior to meeting with the teacher to present the teaching sequence and answer any questions that arose (prior but also during the study). As mentioned, the didactic configuration included: the organization into student pairs that were to work together using one artefact in order to foster explicit reasoning; the organization of the lessons into iterative cycles consisting of the students working on tasks using the artefact, followed by classroom discussions, in which the artefact was projected onto the whiteboard in order for solving strategies and key learning aims to be elaborated upon; and a lesson plan (time schedule for each task) for the three week teaching sequence.

The *exploitation modes* of the didactic configuration are also described in the teaching sequence section. The tasks were designed with the intention to exploit the artefact, specifically the drag mode in DGE, in order to foster students' development of mathematical reasoning competency. Dragging in DGE was the instrument intended to be developed by the students in order to conjecture, investigate, justify and reason about geometrical properties of figures using the task design heuristic of predict-observe-explain. Hence, the task design is consistent with the aim of using DGE according to JM2. The exploitation modes were predominantly prepared by the researcher prior to the sequence and discussed with the teacher, which also received the exploitation modes in the form of written guidelines concerning the educational goal of the teaching sequence.

While the above-mentioned exploitation modes were planned before the teaching sequence was undertaken, our data revealed that additional exploitation modes were used. For example, the teacher seemed to use probing and clarifying questions in order to reveal the students' solving steps related to the artefact so that the teacher could guide them further (e.g., lines 562–572), often referring to the DGE construction process output (JM1). Another exploitation mode, which was evident throughout the episodes, is how both the teacher and the researcher were expected to use well-articulated justifications related to the DGE in an effort to promote the use of similar justifications by the students, e.g., mirroring the students' utterances in classroom discussions and elaborating on their DGE mediated justifications (lines 386–390).

These strategies adopted by the teacher without prior instruction, may also be viewed as the teacher's *didactical performance*, since they are "on the spot decisions" made by the teacher in order to deal with emerging situations concerning the artefact and the tasks. Several examples of the teacher's didactical performance were evident in the episodes, for example asking the students to explain their construction method (lines 562–572, 614–633) (JM1) and posing probing questions, apparently in order to help the students understand, e.g., why the midpoint $C$ could not be dragged (lines 124–129). However, the teacher did in fact not help them understand the phenomena of derived points not being draggable in GeoGebra, but rather gave an authoritarian argument "you know, it's a dynamic program" (line 127), consistent with JM3. In terms of TIO, the incident may be considered a missed opportunity in relation to the students' development of the intended instrument, and therefore an ineffective didactical performance. From the data analyses we also notice that the didactical performance often involved asking the students "how" questions related to the construction process rather than "why" questions. Of course, the construction process is of essential importance, since it is where the theoretical properties of the construction are induced, which decide the outcome of dragging. However, the students are meant to use a tool as an instrument that allows them to interpret and make sense of what they perceive in the DGE as they try to drag, we propose that a strategy (*didactical performance*), which may prove useful, is to ask more why questions and therefore require justifications and understanding of what happens on screen.

Reflecting on our analysis from the point of view of TIO and justificational mediations, we notice that our attention becomes centred on the development of the instrument and to what extent the preparatory steps of the sequence, such as task design, classroom and lesson structure, are aligned with the development of the intended instrument, and the types of justificational mediations that emerge. In addition, we consider to what degree the on the spot actions performed by the teacher, e.g., posing probing questions, are supportive of the instrumental genesis and justificational mediations that are consistent with the educational aim of the activity.

## 9 Analysis from a KOM Teacher Competencies and JM Perspective

We will now relate the data from the teaching sequence to the six teacher competencies from the KOM framework: curriculum competency; teaching competency; competency of revealing learning; assessment competency; cooperation competency; and professional development competency. The analysis is linked with the previous JM analysis.

We explained the rationale behind the a priori design of the teaching sequence in relation to its educational aim. The overarching goal of developing mathematical reasoning competency is consistent with the Danish lower secondary school curriculum, *mathematics common goals* (Børne- og Undervisningsministeriet, 2019). Therefore, the design of the teaching sequence involves aspects related to *curriculum competency*. However, in this case it is the researcher's curriculum competency that is in effect. The outlining of the sequence and task design is clearly related to *teaching competency*, which involves being able to develop teaching sequences to meet learning aims, including the orchestration of teaching/learning situations and choosing adequate tasks. Again, it is the researcher's teaching competency that was activated.

When the teacher asks the students to explain their construction method (lines 562–572, 614–633), using mainly the DGE output as a warrant (JM1), or when the teacher mirrors and refines the students' DGE mediated justifications in classroom discussions (lines 386–390), these actions may be regarded as a realization of the teachers' teaching competency, in particular the part of the competency that concerns the incorporation of students' ideas in the lesson. The teaching competency is defined in rather broad terms, in fact, the phrase "being able to [...] carry out concrete teaching sequences with different purposes and aims" (Niss & Højgaard, 2011, p. 86) in the characterization of the competency, may be interpreted so broadly that it can encompass every action performed by the teacher in the presented episodes as well as each of the types of justificational mediations. From this point of view, we suggest that most teacher actions showcased a reasonably developed teaching competency, however, the missed learning opportunity concerning hierarchical dependencies in GeoGebra (lines 124–129) indicates a weakness in the teacher's teaching competency, in which the teacher employed an authoritarian argument (JM3). The teacher seemed unaware of the hierarchical dependencies, which serves to illustrate that teaching competency is intricately connected to knowledge of the artefact being used and its potentialities in relation to the overarching learning goal of the teaching sequence—mathematical reasoning competency. Since we do not have data concerning the teacher's ability to justify and discuss his choices regarding the teaching approach, we cannot analyze in relation to this part of teaching competency.

The teacher showcased some level of *competency of revealing learning* when he asked probing and clarifying questions, seemingly in order to reveal the students' solution process related to the artefact (e.g. lines 614–633). However, for the most part the teacher asked "what" questions related to the construction process (indicating JM1), which by itself does not necessarily give insight into the level of understanding possessed by the students, whereas more "why" questions might further have revealed the level of student learning. From the data, we cannot infer to what extent the teacher understands the affective causes for the students' conceptions and beliefs concerning the mathematics at play in the lessons, which is a characteristic of the competency of revealing learning.

Since *assessment competency* involves the ability to choose evaluation methods that reveal the students' possession of mathematical competencies, we can argue that the teacher's ability is partially lacking in this regard. Inquiring about the construction method in "how" questions only scratches the surface (e.g., lines 562–572), and may not allow assessment of whether or not the students have developed awareness of the fact that there are dependencies between objects in dynamical figures that decide the outcome of dragging.

We are unable to significantly utilize the notions of *cooperation competency* and *professional development competency* in our case. In order to evaluate the teacher's ability in relation to cooperation competency, we would need more data concerning the teacher's cooperation with other colleagues. What we can report is that the teacher showed the ability to cooperate with the researcher in order to pursue the learning goal of the sequence. Since professional development competency concerns the teacher's ability to identify and reflect on his own developmental needs in relation to the other five teaching competencies, we would need more information from the teacher to make a meaningful analysis, for example by carrying out a longitudinal research study where we observe the teacher's professional development over time.

## 10 Potentials for Networking KOM and TIO Bridged by JM

The previous analyses in terms of KOM, TIO and justificational mediations may be seen as a first step in potentially reaching a greater degree of integration and synthesizing of these differently focused theoretical constructs, as per a networking of theories approach (Bikner-Ahsbahs & Prediger, 2014). While KOM's competencies for teaching are of a more general nature, and in no way explicitly deal with the circumstances surrounding the use of digital technologies in teaching, instrumental orchestration does exactly this. When different theoretical frameworks are combined—or networked—attention should of course be paid to the frameworks' reciprocal coherence, and in case this exists, then at which level the different frameworks are integrated. Prediger et al. (2008) introduce a "scale" of networking strategies stretching from "ignoring other theories" to "unifying globally". In terms of potential networking, our study would be located in between, consisting of the strategies for coordinating and combining, i.e., strategies mostly used for a networked understanding of an empirical phenomenon or piece of data, or strategies for synthesizing and integrating (locally) which is when "theoretical approaches are coordinated carefully and in a reflected way [that] goes beyond understanding a special empirical phenomenon" (Bikner-Ahsbahs & Prediger, 2010, p. 496). The instrumental approach has proven well-suited for networking with other theoretical constructs (Drijvers et al., 2013); also in relation to the KOM framework (e.g., Geraniou & Jankvist, 2019; Højsted, 2019, 2020). For that reason also, the instrumental approach's construct for teaching, i.e., the instrumental orchestration, appears an obvious choice for networking with KOM's six teacher competencies. Still, as the analysis of the empirical case illustrates, when it comes to networking aspects of these two aspects locally, a mediator may be needed. For us, the notion of justificational mediations served as such "glue" or better yet, as a way of bridging the frameworks of KOM and TIO.

More precisely, the justificational mediations we observed the teacher using provided us with a window on which didactico-pedagogical competencies the teacher relied upon as well as choices of strategies to use, when orchestrating the teaching practices and students' learning involving GeoGebra. We observed the teacher using GeoGebra's features as a warrant to establish truth, but also observed students' and teacher's interactions involving experimentation by changing parameters in a given task and exploring the "instant feedback" received by GeoGebra. Furthermore, we noticed the teacher using GeoGebra as a tool for convincing students of certain mathematical facts, but also to highlight mathematical relationships. Such justificational mediations supported the teacher in promoting students' mathematical reasoning by utilizing important features of the GeoGebra dynamic geometry environment.

In that way, since justificational mediations pertain to the role of technology (here DGE) in relation to justifications, it became an analytical hinge between the instrument and the mathematical reasoning competency in play. It thus qualified our analysis of the teacher actions, allowing us to analyze which warrants and justifications were appreciated in the DGE teaching/learning situation, and if these were coherent with the learning aim of the teaching sequence—to utilize potentialities of the instrument to support development of mathematical reasoning competency. For example, when the teacher resorts to an authoritarian argument (JM3) in order to justify that midpoint *C* cannot move (line 572) it allows us to analyze that as an ineffective didactical performance (TIO) and a weakness in relation to teaching competency (KOM), because the action does not support the students to understand that derived points are locked in GeoGebra, i.e., the action does not support the intended instrumental genesis, and the justification is not coherent with the development of reasoning competency. The example also served to illustrate that teaching competency and effective didactical performances are intricately connected with the teacher's own knowledge of the potentialities of the instrument in relation to the mathematical aim, in this case, reasoning competency.

Surely, the construct of justificational mediations need not be the only possible bridge, when combining KOM and TIO. Yet, in a teaching and learning situation where focus is on students' work with DGE in relation to mathematical reasoning this made sense. If, however, focus had been more closely related to any other of the eight mathematical competencies of KOM (cf. earlier), then a different theoretical construct would potentially serve better as the "glue". For example, if concerned with mathematical representations, Duval's (2006) classification of semiotic registers and register shifts might make for a more obvious choice (e.g., Jankvist & Geraniou, 2019).

## 11 Conclusion

Our aim was to consider the development of a somewhat networked frame to encapsulate the mathematics teacher's practice in a technology rich environment. We wanted to reflect on how different theoretical frameworks and constructs (KOM, TIO and justificational mediations) help us capture what is at stake, but also how they complement each other in reaching successful learning outcomes for both students and their teachers. In order to argue that JM can act as a mediator between KOM and TIO, we showed, on the one hand, how JM can be used naturally in KOM and TIO separately, while on the other hand, how JM can be used as a "hinge" between KOM and TIO. As Prediger et al. (2008) claim:

> The *empirical scope* of theories can be widened by coordinating and integrating new aspects into its empirical component, seldom changing its core. One instance of this effect is the coordination of theories of different grain sizes as presented by Halverscheid (2008), Gellert (2008) and Arzarello et al. (2008) who could *capitalize on the research* of other traditions. (Prediger et al., 2008, p.165)

Using KOM allowed us to maintain a focus on the didactico-pedagogical competencies for teaching. Using TIO supported us in maintaining a focus on the artefact/instrument, i.e., GeoGebra. Finally, using justificational mediations allowed us to reflect upon the teacher's mediations, when relying on their own mathematical competencies and their orchestrations in light of GeoGebra's functionalities and the learning goals for students. Applying those three lenses, KOM, TIO and justificational mediations, enabled us to capture different levels of analysis, which could be a synthesizing result, as follows: KOM's teacher competencies provide concepts to describe the practice of a mathematics teacher in broad terms, highlighting the competencies any mathematics teacher should possess. However, using the view of Instrumental Orchestration, it is possible to make a more fine-grained analysis of the context specific practice of the teacher in a technology rich environment. The empirical scope of the KOM framework can thereby be widened by integrating aspects from TIO. Justificational mediations then support teachers in identifying the best strategies (or mediations) in utilizing both their mathematical competencies and their instrumental orchestrations in supporting students' mathematical learning, which in the given empirical case, comprises reasoning competency.

**Acknowledgements** This book chapter was partly written in the frame of project 8018-00062B under *Independent Research Fund Denmark*.

## References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Arzarello, F., Bosch, M., Gascón, J., & Sabena, C. (2008). The ostensive dimension through the lenses of two didactic approaches. *ZDM—Mathematics Education, 40*(2), 179–188. https://doi.org/10.1007/s11858-008-0084-1

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik, 34*(3), 66–72. https://doi.org/10.1007/BF02655708

Baccaglini-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The maintaining-dragging model. *International Journal of Computers for Mathematical Learning, 15*(3), 225–253.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education* (pp. 483–506). Springer.

Bikner-Ahsbahs, A., & Prediger, S. (Eds.). (2014). *Networking of theories as a research practice in mathematics education.* Springer. https://doi.org/10.1007/978-3-319-05389-9

Børne- og Undervisningsministeriet. (2019). Matematik Fælles Mål [*Mathematics common aims*]. Author.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1), 103–131. Springer. https://doi.org/10.1007/s10649-006-0400-z

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234. https://doi.org/10.1007/s10649-010-9254-5

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses. A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics 82*(1), 23–49. https://doi.org/10.1007/s10649-012-9416-8

Edwards, M. T., Harper, S. R., Cox, D. C., Quinlan, J., & Phelps, S. (2014). Cultivating deductive thinking with angle chasing. *The Mathematics Teacher, 107*(6), 426–431. https://doi.org/10.5951/mathteacher.107.6.0426

Gellert, U. (2008). Validity and relevance: Comparing and combining two sociological perspectives on mathematics classroom practice. *ZDM—Mathematics Education 40*(2), 215–224. https://doi.org/10.1007/s11858-008-0074-3

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of "mathematical digital competency". *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Gregersen, R. M., & Baccaglini-Frank. A. (2020). Developing an analytical tool of the processes of justificational mediation. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H. G. Weigand (Eds.), *Proceedings of the 10th ERME topic conference (ETC10) mathematics education in the digital age (MEDA) 16–18 September 2020 in Linz, Austria* (pp. 451–458). Johannes Kepler University. https://hal.archives-ouvertes.fr/hal-02932218

Halverscheid, S. (2008). Building a local conceptual framework for epistemic actions in a modelling environment with experiments. *ZDM—Mathematics Education 40*(2), 225–234. https://doi.org/10.1007/s11858-008-0088-x

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Information Age Publishing.

Healy, L. (2000). Identifying and explaining geometrical relationship: Interactions with robust and soft Cabri constructions. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (pp. 103–117). Hiroshima University.

Højsted, I. H. (2019). Guidelines for design of didactic sequences utilizing dynamic geometry affordances related to mathematical reasoning competency. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education (CERME11, February 6–10, 2019)* (pp. 2850–2851). Freudenthal Group and Freudenthal Institute, Utrecht University and ERME.

Højsted, I. H. (2020). Guidelines for utilizing affordances of dynamic geometry environments to support development of reasoning competency. *Nordic Studies in Mathematics Education, 25*(2), 71–98.

Højsted, I. H. (2021) *Toward marvels in dynamic geometry teaching and learning: Developing guidelines for the design of didactic sequences that exploit potentials of dynamic geometry to foster students' development of mathematical reasoning competency* [Doctoral dissertation, Aarhus University]. AU Library Scholarly Publishing Services. https://doi.org/10.7146/aul.413

Højsted, I. H., & Mariotti, M. A. (2021). Signs emerging from students' work on a designed dependency task in dynamic geometry. In Y. Liljekvist, L. Björklund Boistrup, J. Häggström, L. Mattsson, O. Olande, & H. Palmér (Eds.), *Sustainable mathematics education in a digitalized world. Proceedings of MADIF12. The twelfth research seminar of the Swedish Society for*

*Research in Mathematics Education, January 14–15 in Växjö* (pp. 111–120). Swedish Society for Research in Mathematics Education.
Iversen, S. M., Misfeldt, M., & Jankvist, U. T. (2018). Instrumental mediations and students' identities. *Recherches en Didactique des Mathématiques, 38*(2), 133–155.
Jankvist, U. T., & Geraniou, E. (2019). Digital technologies as a way of making original sources more accessible to students. In E. Barbin, U. T. Jankvist, T. H. Kjelsen, B. Smestad, & C. Tzanakis (Eds.), *Proceedings of the eighth European Summer University on history and epistemology in mathematics education ESU 8* (pp. 107–130). Oslo Metropolitan University.
Jankvist, U. T., Geraniou, E., & Misfeldt, M. (2018). The KOM framework's aids and tools competency in relation to digital technologies: A networking of theories perspective. In H.-G. Weigand, A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, N. Grønbæk, & J. Trgalova (Eds.), *Proceedings of the fifth ERME topic conference (ETC 5) on mathematics education in the digital age (MEDA)* (pp. 123–130). University of Copenhagen.
Jankvist, U. T., & Misfeldt, M. (2019). CAS assisted proofs in upper secondary school mathematics textbooks. *REDIMAT—Journal of Research in Mathematics Education*, *8*(3), 232–266. https://doi.org/10.17583/redimat.2019.3315
Jankvist, U. T., & Misfeldt, M. (2021). Old frameworks—New technologies. *Canadian Journal of Science, Mathematics and Technology Education, 21*(1), 441–455. https://doi.org/10.1007/s42330-021-00164-4
Jankvist, U. T., Misfeldt, M., & Aguilar, M. S. (2019). What happens when CAS-procedures are objectified? The case of "solve" and "desolve." *Educational Studies in Mathematics, 101*(1), 67–81. https://doi.org/10.1007/s10649-019-09888-5
Jankvist, U. T., Misfeldt, M., & Marcussen, A. (2016). The didactical contract surrounding CAS when changing teachers in the classroom. *REDIMAT—Journal of Research in Mathematics Education, 5*(3), 263–286. https://doi.org/10.17583/redimat.2016.2013
Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-geometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.
Laborde, C. (2005). Robust and soft constructions: Two sides of the use of dynamic geometry environments. In *Proceedings of the 10th Asian technology conference in mathematics* (pp. 22–35). Korea National University of Education.
Leung, A. (2015). Discernment and reasoning in dynamic geometry environments. In S. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 451–469). Springer.
Mariotti, M. A. (2012). Proof and proving in the classroom: Dynamic geometry systems as tools of semiotic mediation. *Research in Mathematics Education, 14*(2), 163–185. https://doi.org/10.1080/14794802.2012.694282
Misfeldt, M., & Jankvist, U. T. (2018). Instrumental genesis and proof: Understanding the use of computer algebra systems in proofs in textbook. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach & C. Vale (Eds.), *Uses of technology in K-12 mathematics education: Tools, topics and trends* (pp. 375–385). Springer Verlag.
Niss, M., Bruder, R., Planas, N., Turner, R., & Villa-Ochoa, J. A. (2016). Survey team on: Conceptualisation of the role of competencies, knowing and knowledge in mathematics education research. *ZDM—Mathematics Education, 48*(5), 611–632. https://doi.org/10.1007/s11858-016-0799-3
Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).
Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9
Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM—Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*(5), 665–691. https://doi.org/10.1016/S0953-5438(03)00058-4

Talmon, V., & Yerushalmy, M. (2004). Understanding dynamic behavior: Parent-child relations in DGE. *Educational Studies in Mathematics, 57*(1), 91–119. https://doi.org/10.1023/B:EDUC.0000047052.57084.d8

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*, 281. https://doi.org/10.1007/s10758-004-3468-5

Trouche, L. (2005). An instrumental approach to mathematics learning in symbolic calculator environments. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators* (pp. 137–162). Springer.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.2307/26764894

White, R. T., & Gunstone, R. F. (2014). *Probing understanding*. Routledge.

# Mathematical Competencies and Programming: The Swedish Case

**Kajsa Bråting, Cecilia Kilhamn, and Lennart Rolandsson**

## 1 Introduction

The ongoing implementation of programming in school curricula around the world has been staged in various ways. For instance, in England, programming was made part of a new subject, "Computing", whilst Finland and Sweden have adopted a blend of cross-curriculum and single subject integration with the strongest link to mathematics (Bocconi et al., 2018; Mannila et al., 2014). In this chapter, we focus on the latter, that is, when programming is integrated in school mathematics. Specifically, we zoom in on the Swedish case where programming has been included in the mathematics syllabus in close connection to the core content of algebra, which makes the Swedish case unique (Kilhamn & Bråting, 2019). The aim of this chapter is to gain knowledge about how the Swedish way of implementing programming affects students' opportunities to develop mathematical competencies.

We will report on three different substudies from an ongoing Swedish research project concerning the implementation of programming in school mathematics (Bråting et al., 2021), discussing the new programming content in relation to some of the mathematical competencies developed by Niss and Højgaard (2019). The project as a whole is theoretically embedded in Chevallard's (2006) framework of

K. Bråting (✉) · L. Rolandsson
Uppsala University, Uppsala, Sweden
e-mail: kajsa.brating@edu.uu.se

L. Rolandsson
e-mail: lennart.rolandsson@edu.uu.se

C. Kilhamn
University of Gothenburg, Gothenburg, Sweden
e-mail: cecilia.kilhamn@ped.gu.se

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_16

how knowledge is transposed between different instances of the educational system. Given that the implementation of programming is at an early stage, the project focuses on how the national curriculum is interpreted and operationalized in textbooks and teaching materials, and further by the teachers.

In the first substudy, a programming task in a recently published Swedish textbook for secondary school is analysed in detail, focusing on semantic and syntactic aspects of concepts appearing in both mathematics and programming. In the second substudy, we report on Swedish mathematics teachers' views of how programming can enhance students' development of mathematical competencies. The substudy is based on interviews with twenty Swedish mathematics teachers who, as early adopters, teach programming within the frames of their ordinary mathematics lessons. Finally, in the third substudy, we describe how one of these teachers develops a lesson using programming for mathematical modelling.

To explore in what ways programming could contribute to learning in mathematics we have used the KOM framework (Niss & Højgaard, 2019) as an analytical lens, looking for aspects of the described competencies in our data. Five of the eight KOM competencies were clearly visible in the data:

- *Mathematical thinking competency* is described as "involving the competence to relate to and pose the kinds of generic questions that are characteristic of mathematics" (Niss & Højgaard, 2019, p. 15) and to use and relate to mathematical statements in various roles and contexts.
- *Mathematical problem handling competency* includes posing and solving mathematical problems, using different strategies and evaluating solutions.
- *Mathematical modelling competency* concerns situations where mathematics is used in contexts outside mathematics through construction and evaluation of mathematical models.
- *Mathematical representation competency* consists of the ability to interpret, make use of and move between a variety of representations, such as verbal, material, symbolic or graphic.
- *Mathematical aids and tools competency* concerns the ability to handle aids and tools for mathematical activity, including both traditional tools and a variety of digital technologies.

We commence with a brief historical survey of how programming and digitalization have made their way into the Swedish school curriculum. In order to clarify the Swedish case, we thereafter provide a comparison between the competencies from the KOM project based on Niss and Højgaard (2019) and the so-called abilities included in the Swedish national curriculum (Boesen et al., 2014; Swedish National Agency of Education, 2011). Finally, the three substudies are described and discussed. Each substudy ends with short takeaways summarizing the most important conclusions.

## 2 Programming in the Swedish School Curricula

The earliest signs of programming in Swedish school curricula appeared in the 1980s. The two topics Computer literacy (Swedish: Datalära) and Computing (Swedish: Datakunskap) were implemented in lower secondary school (grades 7–9) and upper secondary school (grades 10–12), respectively (Swedish National Board of Education, 1983; 1984). In both the topics, societal aspects were emphasized whilst technicalities in coding were more or less suppressed (Rolandsson & Skogh, 2014).

*Computer literacy* was taught at lower secondary and first year of upper secondary school within the two subjects Civics and Mathematics. At the time, it was unusual to find more than five computers per school (Söderlund, 2000), therefore the subject mainly focused on theory and coding with pencil and paper. In a detailed syllabus from 1984 (Swedish National Board of Education, 1984), it was made clear that the coding part should not receive too much attention. Instead, mathematics was to be in focus, in connection with problem-solving, and the time allocated to syntax and logics in programming languages was minimized. However, the syllabus embraced other topics as well: measuring and communicating with computers, computers in society and working with digital tools, that is, in modern terminology databases, word processing and spreadsheets. Considering that schools had to invest in computer technology, and that teachers were in desperate need of in-service training, Computer literacy received a slow start. Teachers also criticized the subject, arguing that it mainly concerned the translation of mathematical algorithms to code without any further reflections (Riis, 1987).

*Computing* was taught at upper secondary school as a subject on its own. It comprised a diverse palette of topics, including fundamental knowledge about computers and computer programming principles, software development, numerical methods and statistics, and their connection with society. The technical aspect of the subject was discussed amongst curriculum developers, who initiated the development of a structural programming language (Comal) and a school computer (Compis) as a way to strengthen this aspect. At the time, software development was not part of any teacher training, which gave rise to a revised syllabus in 1984 where an extensive number of commentaries were added to the subject Computing.

In the 1994 Swedish curriculum reform, Computer literacy was removed and the subject Computing was replaced with Programming, but only for a minority of students at upper secondary level (Swedish National Ministry of Education, 1994a, 1994b). Computers were mentioned in various subjects, with a focus on the computer's potentials for calculations, data processing, word processing and information retrieval. It was not until 2018 in connection to a revision of the Swedish 2011 curriculum that programming reappeared as a mandatory topic, now through all school levels as an aspect of digital competence (Swedish National Agency for Education, 2018). Digital tools are emphasized across the revised curriculum, whereas programming is pointed out specifically as a new content in the subjects Mathematics and Technology. In particular, digital tools and programming are included in connection with mathematical concepts and methods:

> […] pupils should be given opportunities to develop knowledge in using digital tools and programming to explore problems and mathematical concepts, make calculations and to present and interpret data. (Swedish National Agency of Education, 2018, p. 58)

In the mathematics syllabus, programming is integrated in the core content of problem-solving in both lower and upper secondary school and in the core content of algebra in grades 1–9. Within the algebra content, stepwise instructions as a base for programming are introduced in grades 1–3, whilst in grades 4–9 the focus is on how algorithms can be created and used in programming. Moreover, in grades 4–6, programming takes place in visual environments and in grades 7–9 in textual environments.

## 3 Relating the KOM Competencies to the Swedish National Curriculum

The Swedish national curriculum documents include a syllabus for mathematics that all schools are obliged to follow. Since 2011, the syllabus starts with a description of the overarching aims and a specific definition of five *abilities* (Swedish National Agency of Education, 2011; 2018). The formulation of these five abilities was much influenced by the development of theories about mathematical competencies that were prominent during the first decade of the twenty-first century (Boesen et al., 2014; Kilpatrick et al., 2001; Niss & Jensen, 2002). The word ability (Swedish: förmåga) is in this context to be understood as a skill or a competence that develops over time through teaching and practice, not as an innate quality.

To highlight similarities, we have compared the abilities defined in the Swedish syllabus with the competencies in the Danish competency framework (KOM) described by Niss and Højgaard (2019), and display an overview of the comparison in Table 1.

The two categories of competencies described in the KOM framework (left column in Table 1) are visible, although implicit, in the Swedish syllabus. However, we note that in category 1, dealing with asking and answering questions in, with and about mathematics, the mathematical thinking competency does not have an equivalent in the Swedish syllabus. The ability closest related to a mathematical thinking competency is the "ability to use and analyse mathematical concepts and their interrelationships", which, of course, does not cover all that could be included in the KOM competency.

In category 2, dealing with mathematical language and tools, procedural skills and the use of mathematical tools are described in quite different ways in the two documents. The KOM framework does not include procedural skills, claiming that "procedural skills are necessary but not sufficient for the exercise of a competency" (Niss & Højgaard, 2019, p. 20), whereas in the Swedish syllabus procedural skills are explicitly included in the abilities. In the ability to choose and use appropriate

**Table 1** A comparison between the Swedish syllabus for compulsory school, grades 1–9 (Swedish National Agency of Education, 2018) and the Danish competency framework (Niss & Højgaard, 2019)

| Competency in the KOM project | | Mathematical ability in the Swedish syllabus |
|---|---|---|
| Category 1: Posing and answering questions in and by means of mathematics | Problem handling competency | Ability to formulate and solve problems using mathematics and assess selected strategies and methods (include the choice and use of tools) |
| | Modelling competency | |
| | Reasoning competency | Ability to apply and follow mathematical reasoning |
| | Mathematical thinking competency | Ability to use and analyse mathematical concepts and their interrelationships |
| Category 2: Handling the language, constructs and tools of mathematics language and tools | Representation competency | Ability to use mathematical forms of expression to discuss, reason and give an account of questions, calculations and conclusions |
| | Symbols and formalism competency | |
| | Communication competency | |
| | Aids and tools competency | Ability to choose and use appropriate mathematical methods* to perform calculations and solve routine tasks |

* Methods include the choice and use of tools, for example, digital tools and programming

mathematical methods, mathematical representations are intertwined with the artefacts that are used to produce and work with them. What is meant by a mathematical method in the syllabus is unclear, but there are some examples given in the list of knowledge requirements, for example: "using symbols and concrete materials or diagrams" or "measurements, comparisons and estimates of length, mass, volume and times" (Swedish National Agency of Education, 2018, pp. 60–61). Such activities require the use of tools of different kinds, and students are assessed on their ability to choose, apply, account for and discuss their methods.

In summary, the Swedish syllabus covers many of the competencies in the KOM framework, but describes them in terms of abilities, with an emphasis on the *use* of these abilities. Mathematical thinking is an implicit aim in the descriptions of abilities, but not explicitly mentioned, and the Aids and tools competency is implicitly included in the notion of "mathematical methods".

Considering the rapid increase of digital tools, including programming, in school mathematics, it is important to bear in mind how Niss and Højgaard describe digital tools as "particular kinds of material representations of mathematical objects and processes" (2019, p. 18). Some such tools have been developed primarily for mathematics education, such as dynamic geometry software like GeoGebra and Desmos. Others have been developed for programming, such as Scratch and Python. The fact that programming has been included in mathematics in the Swedish curriculum, gives rise to questions about in what way the tools used have bearing on mathematics and how their incorporation into the classroom practice will transform mathematics teaching and learning.

# 4 Research Results

In this section, three substudies taken from our research project about the integration of programming in Swedish school mathematics are presented and discussed in view of mathematical content and mathematical competencies. In the first substudy, we analyse a programming task in a recently published Swedish textbook of programming in mathematics, intended for both lower and upper secondary school. In the second substudy, we report on Swedish teachers' views on which mathematical competencies they think students can develop through programming. Finally, in the third substudy, we look closer at one of these teachers who demonstrates an activity where students learn patterns by using programming as a tool.

## *4.1 Substudy 1: Programming in Teaching Materials for Mathematics*

This substudy is included in our project's investigation of government produced teaching materials and commercially produced textbooks in mathematics. The selection of the material is made according to their popularity and diversity. Qualitative content analyses are conducted with respect to the programming content. Here, we will focus on a task in a newly published Swedish textbook about programming to be used within the subject mathematics (Sanoma Utbildning, 2018). The textbook offers students opportunities to learn the basics of the programming language Python whilst solving mathematical problems. The textbook consists of twelve lessons with examples and tasks, for which the students do not need any prior programming knowledge. The task considered here belongs to the lesson on nested instructions, that is, instructions that include other instructions. We selected the task as it demonstrates semantic and syntactic differences between programming and mathematics. These differences will be described and discussed with respect to the mathematical thinking competency and the representation competency.

The task consists of a program that finds all twin primes in the closed interval from 2 to 100. A twin prime is defined as a pair of primes that differ by 2, for instance (3, 5) and (41, 43). As Fig. 1 (lines 1–5) shows, the program begins to define a function

```
Twin primes.py
1  def isPrime(n):
2      for x in range(2, n):
3          if n % x == 0:
4              return False
5      return True
6
7  for a in range(2, 101):
8      b = a + 2
9      if isPrime(a) and isPrime(b):
10         print(a, " and ", b, " are twin primes")
```

**Fig. 1** A program in Python that finds twin primes in the closed interval 2–100

`isPrime(n)` which tells whether a given integer $n$ is a prime or not ($n$ % $x$ means $n$ modulo $x$). Thereafter, in lines 7–10, the program loops through the interval 2–100 and uses the function `isPrime` to check for pairs of primes that differ by 2. Whenever twin primes are found by the code, the output will be displayed on the screen.

### 4.1.1 Mathematical Thinking Competency

We begin to take a closer look at the function `isPrime` (line 1–5). This function determines whether a given integer $n$ is a prime by checking all possible dividers up to $n$ itself. This is realized in a `for` loop where the function checks whether the result of the modulo operation $n$ % $x$ is equal to 0. That is, the function checks whether the remainder is equal to 0 after dividing the input value $n$ with the variable $x$ in the loop. The function returns the Boolean values `True` or `False` depending on whether the input value is a prime or not. Further down in the code, in line 9, the program calls the function `isPrime` to find out if the values $a$ and $b$, respectively, are primes.

Let us discuss the function `isPrime` in view of the mathematical thinking competency, especially the part that involves the ability to relate to mathematical concepts in different contexts (see Niss & Højgaard, 2019). As Table 1 shows, this part aligns with the Swedish ability to “use and analyse mathematical concepts”. The function in this program shows some important semantic differences of a function in the contexts of programming and mathematics. In school mathematics, a function is always a relation between two sets of numbers, whereas in programming a function can return values that are non-numbers. In our task the function `isPrime` returns a Boolean, that is, either `True` or `False` (lines 4 and 5). Furthermore, in programming a function does not need to be a relation. In this textbook, a function is defined as a number of lines of code that you can call later, which is exactly what happens in line 9 in our program. The tricky thing is that in programming a function can mean different things; sometimes it can refer to a mathematical function and sometimes to something that more resembles a process (Bråting & Kilhamn, 2021).

In the Swedish case with programming integrated in the mathematics syllabus, the semantic difference between functions in mathematics and programming gets crucial. In Niss and Højgaard's (2019) terminology, the scope of the function concept gets expanded when the domain where the function concept was first introduced gets enlarged. What they have in mind here is an expansion from functions defined by explicit algebraic expressions to functions defined in set-theoretic terms. That is, the meaning of the concept is the same, even though we can define functions in different ways. This is not the case in our example where the function concept actually has different meanings in the contexts of mathematics and programming.

### 4.1.2 Representation Competency

Another important semantic, as well as syntactic, issue is the usage of the equal sign in the program in Fig. 1. In line 8, the equal sign (=) in $b = a + 2$ represents an assignment. In programming, an assignment is realized by first evaluating the expression to the right-hand side of the equal sign. The resulting value is then assigned to the variable on the left-hand side. In our task, $b$ is *assigned* the value you get by evaluating the expression $a + 2$ (line 8). This usage of the equal sign differs from how it is normally used in mathematics, where the equal sign represents an equivalence *relation*. That is, the symbol "=" is included in both programming and mathematical notation, although representing different meanings (see also Altadmri & Brown, 2015; Bråting & Kilhamn, 2021).

To clarify this difference, the assignment $b = a + 2$ in line 8 in Fig. 1 will be compared with the relation $b = a + 2$ in the following mathematical problem:

*Alice (a) is 2 years younger than her sister Bea (b), together they are 18 years old. How old are Alice and Bea, respectively?*

This problem can be solved by letting $b = a + 2$ and $a + b = 18$, which leads to the solution $a = 8$ and $b = 10$. Here, the expression $b = a + 2$ represents the relation between Alice's and Bea's ages. It might be tempting to interpret this as an assignment as in line 8 in Fig. 1, but it is not. The difference is that, in contrast to an assignment, a relation holds all the time. Alice is always two years younger than Bea. The assignment $b = a + 2$ in line 8 is instead temporal. The first time we enter the loop, in line 7, $a$ is equal to 2 and $b$ is nothing. When we move on to line 8, $a$ is still equal to 2 but $b$ is now assigned the value 4. The second time we run the loop, in line 7, $a$ is equal to 3 but $b$ is still 4. In line 8, $a$ is still equal to 3 but $b$ is now assigned the value 5. That is, $a$ and $b$ do not always differ by 2. An essential difference between programming and mathematics is that in programming everything is realized in stepwise instructions executed in order, one after the other. In mathematics, there exists no such time aspect (see Bråting & Kilhamn, 2021).

Let us now focus on the two consecutive equal signs (==) in line 3 in Fig. 1. In Python, the symbol == is used as a relational operator that evaluates if two entities are equal. In our task the program tests if the result of the modulo operation `n % x` is equal to 0, that is, if the remainder is 0 after dividing the input value $n$ with the loop variable $x$ (line 3). Here, the meaning of the double equal sign (==) is similar to the meaning of the relational equal sign (=) in mathematical notation. This can be summarized as follows: In programming, the double equal sign (==) often represents a relational equality whilst the single equal sign (=) represents an assignment. In contrast, in mathematics the single equal sign (=) always represents a relational equality (see Bråting & Kilhamn, 2021).

### 4.1.3 Takeaway from Substudy 1

The program in Fig. 1 demonstrates both semantic and syntactic differences between mathematics and programming. First, the meaning of the function `isPrime` clearly differs from the meaning of a mathematical function. Second, the equal signs show that the same symbol can represent different meanings in programming and mathematics. Third, the meaning of the double equal sign == corresponds to the meaning of the equal sign (=) in mathematical notation, that is, different symbols can represent the same meaning. An additional example of the third issue is the modulus operator in our program. It has the same meaning in mathematics and programming but is represented with different symbols; % in programming and *mod* in mathematics. Figure 2 summarizes this as it illustrates similarities and differences regarding the meaning of some operations (the semantics) and the symbolic representations of these operations (the syntax) in mathematics and programming.

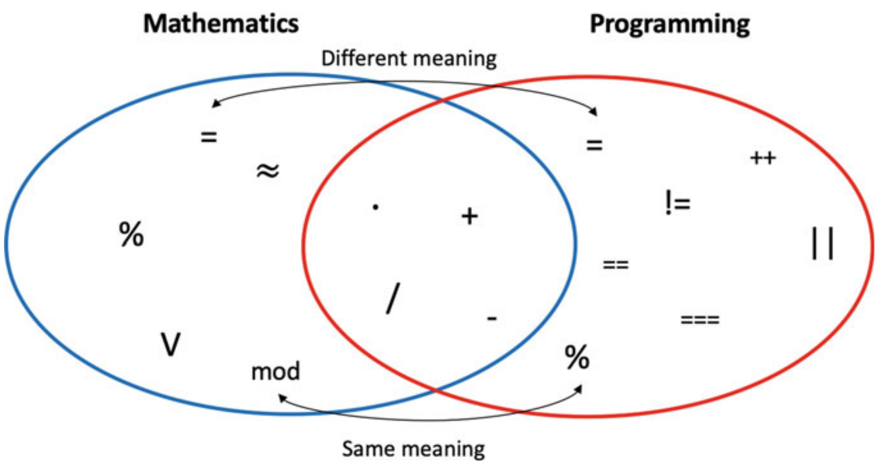



**Fig. 2** Syntactic and semantic similarities and differences of operators in mathematics and programming

Our analysis shows that Niss and Højgaard's description of representation competency in terms of "the ability to move between a wide range of representations of objects and symbols" (2019, p. 17) is essential when programming is integrated in mathematics.

## 4.2 Substudy 2: Teachers' Views on Programming in Mathematics

When programming was new in the Swedish mathematics syllabus, we interviewed twenty teachers who were identified as "early adopters" because they were enthusiastic about the implementation of programming in school and had started teaching it on their own accord. The teachers represented grades 1 through 9 (age 7–15) from schools in 14 different municipalities in various parts of Sweden, teaching mathematics and sometimes also technology. Many of them had extended responsibility for implementing digital technology in their schools. Each interview took around 30 min, was audio recorded and transcribed. The interviews were semi-structured focussing the following general topics: teacher background and experience; the role of programming in mathematics; important concepts; sources of inspiration; an example of a good lesson (Kilhamn et al., 2021). Each teacher was asked to describe at least one programming activity s/he identified as a good activity in mathematics, and to talk about what students learned through the activity.

Programming, as it is described by the teachers, mostly refers to creating, debugging or tinkering with a code, either unplugged (for example writing instructions for a friend or using a mini-robot with simple buttons), block coding in Scratch or text coding in Python. A few of the teachers do not make a distinction between "programming" and "digital tools" such as Excel or GeoGebra, whilst others make that distinction clearly, emphasizing the fact that programming is new to them although digital tools in general are not, as one teacher says: "It is easier to make use of digital tools than actual programming".

In an attempt to understand which mathematical competencies these early adopters think students can develop through programming, a summative content analysis was conducted (Hsiu & Shannon, 2005), beginning with a word search on the transcripts. Every time one of the chosen words was mentioned by an interviewee, an excerpt was saved with the word embedded in its context. The words included in the search were words connected to competencies in the KOM framework or abilities described in the Swedish syllabus, that is, various inflections of the words displayed in Table 1 (i.e., thinking/to think). All excerpts including each specific word were then compiled, and a detailed analysis was made of the excerpts for the two most frequently used words. For each teacher, the excerpts were summarized and encapsulated in a condensed statement in English that captured the essence of his/her point of view. The statements were then compared and similar views were merged to create an overview of the different views that were represented in the data (Table 2).

**Table 2** Total number of instances of words found in the transcripts of 20 interviews

| | Number of teachers ($n = 20$) | Total number of instances |
|---|---|---|
| Problem-solving | 17 (85%) | 35 |
| Thinking | 16 (75%) | 29 |
| Tool, digital tool | 8 | 14 |
| Competence, digital competence* | 7 | 13 |
| Concept, conception | 7 | 13 |
| Ability (Swedish: förmåga) | 5 | 8 |
| Communication | 3 | 3 |
| Calculation | 2 | 2 |
| method | 1 | 1 |
| Reasoning, representation, forms of expression, model/modelling, procedure | 0 | 0 |

* Including the Swedish term "digitalisering" when referring to students

### 4.2.1 Mathematical Problem Handling Competency and Mathematical Aids and Tools Competency

Problem-solving is mentioned in relation to programming 35 times in the data. Some teachers highlight that programming develops problem-solving ability, others talk about programming as a tool for problem-solving.

1. Programming is mainly about problem-solving

Teachers emphasize that students develop problem-solving strategies and ways of thinking that are valuable when solving problems in mathematics. This view includes descriptions of the problem-solving process as well as connections to logical thinking, exemplified here by a teacher who works in grades 1–3:

> Programming is logic as an aspect of problem-solving. Having a complicated problem that you do not quite understand, and then breaking it up into smaller sequences and solving one at a time. This is often too abstract in mathematics problems, but when it comes to programming it is the natural approach. When it doesn't work they [students] find out why. They see the problem-solving process in a new way. I think this way of thinking can be useful even when working traditionally with pen and paper.

2. Programming can be a tool for problem-solving

Teachers who present this view emphasize that students first need to learn how to use the tool, which means that it is not until upper secondary school that students will actually use programming as a problem-solving tool. A teacher for grades 7–9 says:

> We solve problems using programming, but on this level you actually do not need programming to solve the problems. Often pen and paper is actually quicker, even for the students. But later on, when they get to more complex problems, they might need programming. We do it on simple tasks to learn how, for later use.

In addition to the two views described above, some teachers also talk about the different roles programming plays in different school subjects. In the Swedish curriculum programming is also included in technology, where teachers refer to the use of programming to solve real-world problems of a different, more practical character:

> Programming in technology is used more to solve problems. In mathematics it is more a tool for mathematical understanding, making it more difficult to find good examples in maths.

Another teacher trivializes the role of programming in mathematics to that of coding, that is learning the tool, and then later applying the tool in technology:

> They learn coding in mathematics, so that they can use it later to program larger systems in technology.

### 4.2.2 Mathematical Thinking Competency

Although thinking is not mentioned at all in the Swedish syllabus, 75% of the early adopters talk about thinking in various ways connected to programming. It is clear that they see the development of students' thinking as an important goal. They use different words to describe the ways of thinking that were supported by programming, where *logical thinking* and *computational thinking* were the most frequent. Here are some examples:

> Working with programming in a textbook usually misses the goal. The goal is logical thinking and problem-solving. Textbooks are often too simple and too superficial. (teacher, grades 1–3)
>
> Logical thinking is an ability in mathematics. (teacher, grades 4–6)
>
> In the early grades it is mostly about computational thinking. Unplugged programming activities. They look for mistakes, they adjust, they try again, over and over. Once they get into computational thinking they can start programming and learn the tool. (teacher, grades 4–6)

Other aspects of mathematical thinking that are mentioned by the teachers in relation to programming are: sequential thinking, analytical thinking and algorithmic thinking. Furthermore, the teachers who talk about thinking often refer to several types of thinking at the same time, and frequently also in relation to problem-solving. A typical example is the following quote:

> Programming is all about logical thinking. I work with the six steps of computational thinking. Computational thinking, and working in small steps, that is precisely how we work with problem-solving.

### 4.2.3 Takeaway from Substudy 2

Teaching goals related to logical thinking and problem-solving seem to stand out when teachers embrace programming in mathematics. According to the teachers in this study, programming can enhance students' problem-solving in different ways.

Whether or not this is actually useful in school mathematics and enhances what Niss and Højgaard (2019) describes as Mathematical problem handling competency is still to be seen, since some teachers express concerns that programming is more useful for problem-solving in higher grades and/or in technology. It is clear that the inclusion of programming adds a new tool that students need to learn, thus providing opportunities for developing the Mathematical aids and tools competency. Furthermore, teachers acknowledge that programming provides many opportunities for students to develop Mathematical thinking competency.

A difference between the teachers' views and the curriculum is that the teachers do not emphasize using programming as a procedural method or as a tool for calculations, nor as a form of expression or a new system of representation. The results imply that the teachers and the curriculum are not very well aligned.

### *4.3 Substudy 3: A Mathematical Pattern in Code, a Teacher's Example*

In this substudy, we look closer at John, and his work on developing students' modelling competency (Niss & Højgaard, 2019). John is one of the early adopters from the previous substudy. He has a long experience working with computers and shows confidence in using programming as a tool for teaching mathematics. In the interviews (substudy 2), we noticed that John's teaching about patterns and general expressions differed from how it is normally done. We found that interesting and suitable for the scope of this chapter.

Over the years, John and his students have used different programming environments such as Scratch, Micro:Bit and Arduino. Before the interview, John attended an in-service training course in programming. Further below, we describe a programming activity adapted to students in grade 7, designed by John himself. It draws from the technical features included in Colaboratory, an online programming environment by Google. In previous lessons the students have worked with other patterns by hand in a traditional way. Together they also constructed a code in Python to describe a pattern where a configuration of markers grew each time with an additional three markers, starting with 5. Figure 3 shows the code they came up with: a sequence of instructions that computes and prints the figure number and its corresponding number of markers.

```
figure_number = int(input("What figure do you want the number of markers for? "))
number_markers = figure_number*3 + 2
print("Figure number ", figure_number , "has ", number_markers, "markers")
```

**Fig. 3** Code for a pattern constructed in collaboration

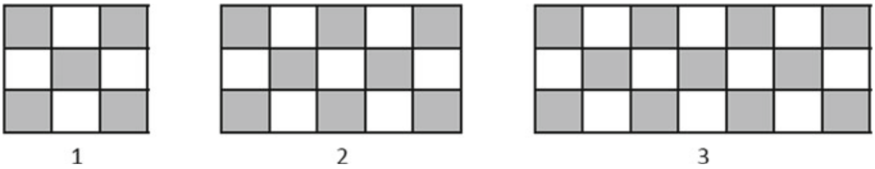



**Fig. 4** Figures 1–3 depicting the pattern

In the following activity, the students are asked to create a program that counts the number of white squares in the pattern illustrated in Fig. 4, for any given figure number.

According to John, the students usually accept the activity and deploy the programming tools to compare the outcome with the number of white squares in each figure. John expects the students to explore and redesign the code through many iterations, and also wants them to add comments that explain each line in the code. Most of the students start with the previous code (see Fig. 3) and tinker with it, eventually working out the correct formula for the number of white squares (`white_squares = 3* figure_number + 1`).

John then asks the students to change the code slightly, making it print the figure number for a large number of white squares, for example the figure number with 508 white squares. According to John, this is a pivotal point as the activity transforms into a need for the mathematical abstraction of $n$. The students usually accept the challenge, even if the number of the figure is high and distant. The magic in the activity emerges as John tells them how to remix the code such that it traverses all the figures from one to a distant number (lines 1–3, Fig. 5). The abundance of information on the screen draws the students' attention, as the answer can be extracted. One solution, commonly suggested by John's students, is shown in Fig. 5.

Finally, at the end of the lesson, John has prepared a revised version of the code (Fig. 6). It is a version where the user submits two numbers of white squares in two consecutive figures (stored in the two variables `n1` and `n2`), and the value of the nth figure into another variable (`theNthFigure`). The revised version prints the number of white squares in the nth figure. Although John considers that version too advanced for most of his students, he believes it will encourage them to see what is possible with code.

```
1 figure_number = 0
2 while figure_number < 100:
3   figure_number = figure_number + 1
4   white_squares = 3 * figure_number + 1
5   print("The figure ", figure_number, "has ", white_squares, "white squares")
```

**Fig. 5** The code traverses all the figures from one to one hundred

```
n1 = int(input ("The number of white squares in fig 1"))
n2 = int(input ("The number of white squares in fig 2"))
theNthFigure = int(input ("Tell me the figure number"))
white_squares = theNthFigure * (n2 - n1) + (n1 - (n2 - n1))
print("The figure ", theNthFigure,"has ", white_squares, "white squares")
```

**Fig. 6** The code handed to the students at the end of the lesson

### 4.3.1 Modelling Competency

According to John, textbooks in school commonly unfold the abstraction of the general number *n* in a pattern task without any deeper implications for students' learning. It is common, he says, to start with some questions about the number of white squares in figures 1, 3, 10 and so on, then to ask, for example, for the figure number that holds 50 white squares. Finally, the textbooks usually ask for the general expression that gives the number of white squares in figure $n$. John claims that such activities make no sense to the students, and often they give up before they reach the general expression.

In contrast, he encourages the students in the activity to discover and implement the mathematical abstraction implicitly, as they accept a challenge to tinker with the code to make it become the representation of the pattern. In that constructionist spirit, John expects the students to embrace the expression early on in the process, and develop a profound understanding of the general number $n$. In that manner, the students discover on their own that (1) there are actually different ways of writing the formula, (2) once you have the formula the program will give you all the other answers in no time. In this way, a general expression becomes a meaningful starting point rather than a meaningless end point of an activity. Furthermore, according to Duval (2006) the mere work of translating between the language of the code, natural language and conventional mathematical notation will increase the students' understanding of the general expression.

In the interview, John also emphasizes the importance of other competencies from computer science, as the students have to handle code errors (syntax, runtime and logic). According to John, skills in code debugging are important for learning mathematics, as such the students develop competencies related to perseverance and being meticulous. John challenges his students to initiate their own enquiries, and appreciates when some of the students take it to a higher level adding more code to suppress runtime and logic errors such as division by zero or input of non-numbers. Syntax errors do not necessarily have to do with mathematics, but can reveal insights as they draw attention to the details in the algorithm. Overall, an important feature of the activity is that it embraces error handling that challenges students to expand their knowledge as they develop mathematical modelling competency.

### 4.3.2 Takeaway from Substudy 3

In this substudy, the teacher has exchanged the traditional way of introducing the idea of the *n*th number with an activity based on code remixing. Instead of letting the general expression become the end point of the task the teacher asks them to write a program where the general expression is the starting point. In the process of modifying the code and critically analysing and evaluating what the code actually does, students have to deal with runtime and logical errors before the code could be used beneficially. The activity is a good example on how students apply a programming environment and a programming language to construct a mathematical model, and how they analyse, evaluate and make changes to that model.

## 5 Concluding Remarks

Considering the rapid implementation of programming in Swedish school mathematics and the fact that the teachers who now teach programming are educated mathematics teachers with very limited programming skills, we believe that the Swedish case is challenging. In this chapter, we have highlighted that Swedish teachers must have a thorough understanding of how mathematical concepts can be used and denoted in programming environments. Especially since semantics and syntax in mathematics and programming sometimes differ and sometimes overlap. When programming is integrated in the mathematics syllabus, the mathematical content obviously gets enlarged, but as we have demonstrated, the formation of mathematical competencies and abilities is also affected.

All three substudies have shown how programming can be fruitfully employed to develop some of the mathematical competencies. In particular, we have found that teachers who are early adopters explicitly talk about problem-solving and mathematical thinking competencies. The emphasis on mathematical thinking and logic is strikingly different from the learning goals described in the curriculum. The current Swedish national curriculum has been strongly influenced by standard-based curriculum reforms, where measurable outcomes are emphasized. Such a curriculum framework "gives precise accounts of the knowledge and skills that students are to achieve; a focus on assessment criteria that are aligned to this framework; and the introduction of high-stakes tests, such as exams based on specified performance requirements" (Sundberg & Wahlström, 2012, p. 348). Since mathematical thinking competency is not easy to define in terms of knowledge requirements and assessment criteria, it follows that this competency is left out, in favour of the outcomes of thinking practices, expressed in terms of for example applying, choosing, using and evaluating. With programming now entering the scene, we see two quite different possible paths outlined by the curriculum and the early adopters: the curriculum emphasizes programming as a set of tools that students are to learn, whereas the teachers see it as an opportunity to develop mathematical thinking and problem-solving skills. We hope the teachers' voice will be the strongest in this choice of path.

## References


Altadmri, A., & Brown, N. C. C. (2015). 37 Million compilations: Investigating novice programming mistakes in large-scale student data. In *Proceedings of the 46th ACM technical symposium on computer science education (SIGCSE'15)* (pp. 522–527). ACM. https://doi.org/10.1145/2676723.2677258

Bocconi, S., Chioccariello, A., & Earp, J. (2018). *The Nordic approach to introducing computational thinking and programming in compulsory education.* Report prepared for the Nordic@BETT2018 Steering Group. https://doi.org/10.17471/54007

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., & Palmgren, B. (2014). Developing mathematical competence: From the intended to the enacted curriculum. *The Journal of Mathematical Behavior, 33*, 72–87. https://doi.org/10.1016/j.jmathb.2013.10.001

Bråting, K., & Kilhamn, C. (2021). Exploring the intersection of algebraic and computational thinking. *Mathematical Thinking and Learning, 23*(2), 170–185. https://doi.org/10.1080/10986065.2020.1779012

Bråting, K., Kilhamn, C. & Rolandsson, L. (2021). Integrating programming in Swedish school mathematics: Description of a research project. In Y. Liljekvist, L. Björklund Boistrup, J. Häggström, L. Mattsson, O. Olande, & H. Palmér (Eds.), *Sustainable mathematics education in a digitalized world. Proceedings of MADIF11* (pp. 101–110). NCM & SMDF.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the fourth congress of the European Society for Research in Mathematics Education, CERME 4* (pp. 21–30). FUNDEMI IQS-Universitat Ramon Llull.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131. https://doi.org/10.1007/s10649-006-0400-z

Hsieh, H.-F., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15*(9), 1277–1288. https://doi.org/10.1177/1049732305276687

Kilhamn, C., & Bråting, K. (2019). Algebraic thinking in the shadow of programming. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European Society for Research in Mathematics Education, CERME11* (pp. 566–573). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Kilhamn, C., Bråting, K. & Rolandsson, L. (2021). Teachers' arguments for including programming in mathematics education. In G. A. Nortvedt, N. F. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, H. K. Nilsen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, A. Werneberg (Eds.), *Bringing Nordic mathematics education into the future. Preceedings of NORMA 20, the ninth nordic conference on mathematics education* (pp. 169–176). SMDF, Nr 14.

Kilpatrick, J., Swaffard, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* National Academy Press.

Mannila, L., Dagiene, V., Demo, B., Grgurina, N., Mirolo, C., Rolandsson, L., & Settle, A. (2014). Computational thinking in K-9 education. In A. Clear & R. Lister (Eds.), *Proceedings of working group reports of the 2014 on Innovation & Technology in computer science education conference* (pp. 1–29). ACM.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics Education, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Riis, U. (1987). *Datalära på grundskolans högstadium – Utvärdering av en treårssatsning.* [*Information literacy in primary school—Evaluation of a three-year initiative*]. Linköping University.

Rolandsson, L., & Skogh, I.-B. (2014). Programming in school: Look back to move forward. *ACM Transactions of Computing Education, 14*(2), 1–25. https://doi.org/10.1145/2602487

Sanoma Utbildning (2018). *Programmering i matematik. Räkna med kod* [*Programming in mathematics. Count with code*] Stockholm.

Söderlund, A. (2000). *Det långa mötet - IT och skolan: om spridning och anammande av IT i den svenska skolan* [*The long meeting—IT and the school: On the spread and adoption of IT in the Swedish school*] [Doctoral thesis, Luleå University of Technology].

Sundberg, D., & Wahlström, N. (2012). Standards-based curricula in a denationalised conception of education: The case of Sweden. *European Educational Research Journal*, *11*(3), 342–356. https://journals.sagepub.com/doi/pdf/10.2304/eerj.2012.11.3.342

Swedish Ministry of Education. (1994a). Läroplan för det obligatoriska skolväsendet, förskoleklassen och fritidshemmet, Lpo 94 [National curriculum of Sweden]. Stockholm: Fritzes.

Swedish Ministry of Education. (1994b). Läroplan för de frivilliga skolformerna, Lpf 94. Gymnasieskolan, gymnasiesärskolan, den kommunala vuxenutbildningen, statens skolor för vuxna och vuxenutbildningen för utvecklingsstörda. 94 [National curriculum of Sweden]. Fritzes.

Swedish Ministry of Education. (2000). Läroplan för de frivilliga skolformerna, Lpf 94 Gymnasieskolan, gymnasiesärskolan, den kommunala vuxenutbildningen, statens skolor för vuxna och vuxenutbildningen för utvecklingsstörda. 94 [National curriculum of Sweden]. Fritzes.

Swedish National Agency for Education. (2011). Läroplan för grundskolan, förskoleklassen och fritidshemmet, Lgr11 [National curriculum of Sweden]. Fritzes.

Swedish National Agency for Education. (2018). Läroplan för grundskolan, förskoleklassen och fritidshemmet, Lgr11 [National curriculum of Sweden]. Revised 2018. Elanders Sverige AB.

Swedish National Board of Education [Skolöverstyrelsen]. (1983). *Läroplan för gymnasieskolan 2 Supplement, 95 Datakunskap* [*Curriculum for upper secondary school 2 Supplement, 95 Computing*]. Liber Utbildningsförlaget.

Swedish National Board of Education [Skolöverstyrelsen]. (1984). *Läroplan för grundskolan Datalära i grundskolan: studieplan* [*Curriculum for compulsory school Computer literacy: syllabus*]. Liber Läromedel/Utbildningsförlaget.

# Coordinating Mathematical Competencies and Computational Thinking Practices from a Networking of Theories Point of View

**Andreas Lindenskov Tamborg and Kim André Stavenæs Refvik**

## 1 Introduction

Recently, we have seen renewed interest in computational thinking (CT) as a subject of relevance in mathematics education. For example, Norway, France, and Sweden now include CT and/or programming in their mathematics curricula. In research, one can also see studies of CT elements in mathematics education from both theoretical (e.g., Brennan & Resnick, 2012; Weintrop et al., 2016) and empirical perspectives (e.g., Benton et al., 2017, 2018; Borg et al., 2020; Buteau et al., 2020). These efforts to connect CT and mathematics are important if we are to constructively embrace the potential of CT with the continuation of existing practices in mathematics. Especially since CT remains an ambiguous term with several definitions—of which the majority give primary emphasis to computer science (e.g., Wing, 2006)—this chapter seeks to identify the most relevant mathematical competencies (MCs) for coordination with CT practices from a networking of theories point of view. Our starting point for this effort is Weintrop et al.'s (2016) taxonomy for computational thinking practices and Niss and Højgaard's (2002, 2019) mathematical MCs. These frameworks are particularly well-suited to this purpose, since Weintrop et al.'s (2016) definitions are developed in science and mathematics contexts, and the KOM framework has become a widely acknowledged common point of reference for articulating MCs (Niss & Højgaard, 2019). By coordinating these theories, we identify and articulate

A. L. Tamborg (✉)
Department of Science Education, University of Copenhagen, Copenhagen, Denmark
e-mail: andreas_tamborg@ind.ku.dk

K. A. S. Refvik
Department of Science and Mathematics, Volda University College, Volda, Norway
e-mail: refvikki@hivolda.no

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_17

the specific MCs that are most relevant when students engage in the CT practices described by Weintrop et al. (2016). We consider this an initial step in building an overview of the possible relevant connections to pursue when introducing CT to the mathematics classroom. We thus seek to answer the following research question: What MC and CT practices are relevant to coordinate from a networking of theories approach?

We begin the paper by outlining existing studies on the relation between CT and mathematics education and situating the contribution of this paper in this context. Next, we describe the frameworks we intend to network and our approach to coordinating them. Finally, we analyze the most relevant MCs in the CT practices described by Weintrop et al. (2016) as well as discuss the affordances and constraints of the individual frameworks as well as the benefits achieved from coordinating them.

## 2 Related Work

CT was first coined in Papert's (1980) book, *Mindstorms*, where he introduced a simple programming language called LOGO. LOGO is a mathematical micro world where students can navigate an interactive landscape by means of mathematical thinking. For instance, students can navigate a turtle who would leave a trail in the micro world, which could be used to draw geometrical figures. Influenced by Piaget's concept of constructivism, Papert's main idea was that LOGO would enable students to learn mathematics by expressing their ideas in the micro world. Papert's ideas, however, never had a mainstream break-through, and it was not until Wing (2006, 2008, 2011) re-introduced CT as a relevant component of education that CT again gained traction. Wing (2006) modified Papert's thoughts and defined CT as "the thought processes involved in formulating a problem and expressing its solution in a way that a computer—human or machine—can effectively carry out" (p. 33). Wing's (2006) definition thus emphasizes computer science concepts, such as abstraction and algorithms; however, the impact of CT on core topic teaching and learning has increased the relevance of studying CT's relation to mathematics. As argued by Geraniou and Jankvist (2019), digital competencies and MCs are not typically seen as a connected whole but as separate entities. They bridge this gap by networking theories [instrumental genesis (Drijvers et al., 2013) and conceptual fields (Vergnaud, 2009)] for studying mathematics learning in technology-rich contexts. While this work considers digital MCs broadly, other studies have focused explicitly on CT in mathematics education. Pérez (2018) has e.g., argued that mathematics and CT historically shared a focus on logical structures and modelling (Gadanidis et al., 2017) but that mathematics and CT education exist within two distinct epistemological frames. The frame of mathematics is associated with being a mathematician and engaging in mathematical practices, while the frame of computational thinking emphasizes productive actions and their role in task optimization. However, several studies have indicated that integrating CT and mathematics can support students' understanding of mathematical concepts (Calder et al., 2018; Caspersen et al., 2018; Cui & Ng, 2021; Ejsing-Duun & Misfeldt, 2015; Husain et al., 2017), mathematical argumentation (Kaufmann & Stenseth, 2021), and problem-solving (Ng & Cui, 2021). Most existing research however studies CT in relation to specific mathematical subject matter areas. The work of Weintrop et al. (2016), on which we build this chapter, is one exception. Their taxonomy identifies relevant CT practices in a mathematics and science context identified from tasks and interviews. This work describes CT practices that could be considered relevant for inclusion in mathematics teaching, with a secondary focus on how these practices relate to existing mathematics curricula. This chapter contributes to related literature by identifying the most relevant MCs for coordinating with these practices. Below, we introduce the frameworks we draw on to achieve this goal.

## 3 CT and KOM

Weintrop et al. (2016) define a taxonomy for CT practices in mathematics and science education. Their practices focus on computer science applications and aim to develop a foundation for discussing how CT relates to mathematics education. Their definition provides a set of actionable guidelines for teachers to follow when introducing CT to mathematics education as well as a shared language for teachers to better understand and systematically develop their teaching practices (Weintrop et al., 2016). The definition features four main practices—data practices, modelling and simulation practices, computational problem-handling practices, and systems thinking practices. Each practice includes five to seven taxonomic levels built from existing literature, educational activities and standards documents from the United States, and interviews with mathematicians and scientists (Weintrop et al., 2016). The practices are developed for a high school teacher audience and address their need to 'prepare students for potential careers in these fields' (Weintrop et al., 2016, p. 128). By focusing on CT practices, the authors reach a definition that emphasizes what is relevant for highschool students to learn. Building on this work, this chapter seeks to identify the relevant MCs when students engage in these practices and thereby develop a stronger foundation for connecting CT to mathematics education. To do so, we coordinate these practices with the KOM framework.

The KOM framework was introduced in 2002 and developed "the concepts of mathematical competence and mathematical competencies with particular regard to their possible roles in the teaching and learning of mathematics" (Niss & Højgaard, 2019, p. 10). KOM includes eight MCs: mathematical thinking, problem handling, modelling, reasoning, representation, communication, tools and aids, and symbols and formalism. Niss and Højgaard (2002) describe their focus on competencies as an alternative to a traditional focus on curriculum content and as a framework that can be applied both descriptively and normatively. KOM is not intended to stand alone when designing lessons but to be combined with other subject matter areas, where the MCs can be weighted differently depending on the context (Niss & Højgaard, 2019). The KOM framework does not pay explicit attention to the role of digital technology in mathematics (although technology plays a role in, e.g., the tools and aids competency); rather, it is concerned with identifying the components of being mathematically competent, and has led to the wide theoretical and empirical usage of the framework in research (see, e.g., Niss & Højgaard, 2011). In this chapter, we consider the KOM framework as a natural starting point from which to articulate possible relations between mathematics and CT due to KOM's wide use and since it is recognizable to both educators and researchers in mathematics education.

## 4 Method

In our investigation of the most relevant MCs in CT practices, we build on the networking of theories tradition (Prediger & Bikner-Ahsbahs, 2014; Prediger et al., 2008), which describes a number of ways (and degrees to which) of how different theories can be brought into the dialogue. We mainly draw on the coordinating theories approach, which is described as "well-fitting elements from different theories" (Prediger et al., 2008, p. 11), and it is recommended to include an analysis of the relationships between the different elements of these theories. Weintrop et al. (2016) describe new CT practices enabled by computational tools and their potential for students' learning. The KOM framework, on the other hand, defines a number of general MCs that can be brought into play in a variety of combinations depending on the subject to be taught. We coordinate these theories using Weintrop et al.'s (2016) practices as a starting point, from which we analyze the most relevant MCs for coordinating. At each taxonomic level for the four practices, we carefully study each of the eight competencies. Through this process, we identify relevant competencies based on both an immediate resemblance in wording (e.g., 'problem-solving', a term that appears in both KOM and CT) and on similar work processes or activities described in the two frameworks. An example of the latter is data visualization (level 5 in the data practices taxonomy), in which we identified aspects of the mathematical representation competency to be highly relevant (e.g., when choosing an adequate representation of specific data).

The KOM framework focuses on competencies, while the CT taxonomy focuses on practices. Consequently, it is important to point to the differences between competence and practice as concepts to coordinate them adequately. The KOM framework describes a competence as "someone's insightful readiness to act appropriately in response to the challenges of given situations" (Niss & Højgaard, 2019, p. 12). In this respect, competencies exceed a mere skill or knowledge in that the competence concept emphasizes one's ability to appropriately use abilities in a particular situation. In comparison, Weintrop et al. (2016) describe their understanding of practices as similar to that of the Next Generation Science Standards (NGSS):

> Following the example set by the NGSS, we have chosen to call these 'practices' as opposed to 'skills' or 'concepts' in order 'to emphasize that engaging in scientific investigation requires not only skill but also knowledge that is specific to each practice. (NGSS Lead States, 2013, p. 30)

This definition of practices regards the ability to activate a combination of skills and knowledge. In this respect, CT and KOM differ in the emphasis given to appropriately adapting abilities to a specific situation. What can be gained from coordinating CT and KOM is identifying how MCs can be relevant in ensuring students' engagement in CT practices is adapted to the specific situation in question. In addition to our analysis of the frameworks described above, we also include an example of how such coordination takes place (or could take place) in concrete mathematical tasks. We have found these tasks from relevant international sources to exemplify our claims.

## 5 Analysis—Coordinating CT Practices and KOM

### *5.1 Data Practices*

Weintrop et al.'s (2016) definition of data practices include students' ability to propose systematic data collection, run simulations to create data, manipulate data with computational tools, analyze data, draw conclusions based on findings, and use computational tools to produce data visualizations. KOM does not treat data as an independent mathematical competency, but it is a subject area of statistics (Niss & Højgaard, 2002) and has been mentioned as relevant for the modelling competency (Niss & Højgaard, 2019) (Table 1).

As seen above, we argue that the mathematical thinking and tools and aids competencies are relevant at all the taxonomic levels. Computational tools have changed how, at what speed, and what type data can be collected, and data collection uses computational tools for defining collection protocols as well as recording and storing data (Weintrop et al., 2016). Mastering the data practice will enable students to propose systematic data collection protocols and identify "how these can be automated with computational tools when appropriate" (Weintrop et al., 2016, p. 136). We find the tools and aids competency relevant to coordinate, since this MC focuses on choosing between different available computational tools and the ability to make

**Table 1** An overview of the relation between data practices and MCs

| | Thinking | Problem handling | Modelling | Reasoning | Representation | Symbols and formalism | Communication | Tools and aids |
|---|---|---|---|---|---|---|---|---|
| Data collection | ■ | | | | | | | ■ |
| Creating data | ■ | | ■ | | | | | ■ |
| Manipulating data | ■ | | ■ | | ■ | | ■ | ■ |
| Analyzing data | ■ | | ■ | ■ | | | ■ | ■ |
| Visualizing data | ■ | | ■ | ■ | ■ | | ■ | ■ |

constructive use of those chosen in a specific situation (Niss & Højgaard, 2019). Different tools are relevant in different phases of data collection (e.g., designing the collection protocol and data recording), and the tools and aids competency is relevant in determining the appropriate phase for individual tools.

Collecting data also involves identifying adequate data sources and collection strategies. Here, we find the mathematical thinking competency relevant, since this MC includes the ability to relate to general questions specific to mathematics as well as to the types of possible answers to such questions. Data collection is typically associated with searching for an answer to a given question, and, to define appropriate data sources and collection strategies, the ability to relate to mathematical questions and answers and the relation between them is relevant.

We find the modelling competency relevant in, for instance, creating data, which refers to using computational tools to generate data about phenomena that are theoretical in nature and cannot be easily observed/measured from existing data (Weintrop et al., 2016). This practice will typically involve aspects of modelling, since the phenomena data is created to explore needs to be described using mathematical concepts and language. This can be seen as constructing mathematical models of extra-mathematical situations by "taking various aspects data, facts, features and properties of the extra-mathematical domain being modelled into account" (Niss & Højgaard, 2019, p. 16).

Manipulating data includes "sorting, filtering, cleaning, normalizing, and joining disparate datasets" (Weintrop et al., 2016, p. 136) that may serve analysis or communication purposes, which enables students to reshape datasets into more useful formats. When manipulating data for further analysis, we find it relevant to coordinate with the representation competency. This MC includes the ability to interpret and translate representations of "mathematical objects, phenomena, relationships and processes, as well as of the ability to reflectively choose and make use of one or several such representations" (Niss & Højgaard, 2019, p. 17). It is relevant since, for instance, joining disparate datasets often includes identifying an appropriate representation for the multiple data sources in question. If the purpose is for students to communicate data more efficiently with others, we also find the mathematical communication competency relevant, which includes the ability to more clearly communicate data manipulations in various formats.

The mathematical reasoning competency is, for example, relevant in data analysis. Data analysis can be achieved using different strategies, including "looking for patterns or anomalies, defining rules to categorize data, and identifying trends and correlations" (Weintrop et al., 2016, p. 136). Computational tools make it possible to conduct more reliable and efficient data analysis, which has become increasingly relevant due to the amount of large datasets. We also find the mathematical reasoning competency relevant to coordinate, since it includes the ability to analyze and develop arguments to justify mathematical data analysis claims—either orally or via writing (Niss & Højgaard, 2019)—and since identifying trends and correlations typically is accompanied by justifications.

**Example**

An example of how modelling and simulation practices and MCs are combined can be found in a course developed in Denmark called 'Update Dices'.[1] This course instructs students on how to model dice with different numbers of sides, run simulations, and perform statistical analyses of how often more than half of the differently numbered dice will have the same pips under different conditions (e.g., where the dice has 3 or 6 sides and that there are 6, 100 or 150 dice). The task instructs students in modelling and simulating these operations using Excel spreadsheets and GeoGebra, in which there are also different representation tools to visualize the results of the simulations (e.g., histograms). Students are also introduced to the theorem in probability theory called the Law of Large Numbers, which states that the results of trials will move closer to the expected value of the trials with the more trials performed. By increasing the number of dice rolls in their simulations, students are able to explore this theorem in their own work. Students are thus both engaged in practices of collecting, creating, and analyzing data regarding dice rolls. In particular, the tools and aids and modelling competency are relevant here, since students use Excel to model the dice rolls, changing the model of the dice to represent a different number of pips with each trial.

## *5.2 Modelling and Simulation Practices*

Modelling and simulation practices concern students' abilities to 'create, refine, and use models of a phenomenon' (Weintrop et al., 2016, p. 136). Using computational models, students can design, build and assess mathematics and science models to understand a mathematical or scientific concept, find and test solutions, and assess the computational model's capabilities and limitations. In this practice, we particularly found the tools and aids, modelling, and problem-handling competencies to be relevant (Table 2).

The modelling competency is relevant to coordinate when using computational tools to understand a concept, which includes that new tools allow students to systematically interact and inquire phenomena in realistic, virtual environments with far more control than in the natural world (Weintrop et al., 2016). The modelling competency is relevant here, since using a computational tool to understand a concept includes analyzing the foundation and properties of the model as well as assessing the range of information that can be extracted from the model (Niss & Højgaard, 2019).

[1] https://xn--tekforsget-6cb.dk/wp-content/uploads/2020/09/Opdater-terninger-8.kl_.-matematik-22.09.20.pdf.

**Table 2** An overview of the relation between modelling and simulation practices and MCs

| | Thinking | Problem handling | Modelling | Reasoning | Representation | Symbols and formalism | Communication | Tools and aids |
|---|---|---|---|---|---|---|---|---|
| Using computational models to understand a concept | | | ■ | | ■ | | | ■ |
| Using computational models to find and test solutions | | ■ | ■ | | | | | ■ |
| Designing computational models | | | ■ | ■ | | | | ■ |
| Constructing computational models | | ■ | ■ | | | | | ■ |
| Troubleshooting and debuggin | | ■ | ■ | | | | | ■ |

We also find the representation competency relevant, since using a model to decode a concept typically includes interpreting the mathematical representation of that concept in the model (Niss & Højgaard, 2019). The tools and aids competency is also relevant, since using a computational model often involves considering the affordances and constraints of the tools used to model the concept as well as reflectively operating and navigating within the model (Niss & Højgaard, 2019).

Moreover, we find the reasoning competency relevant to coordinate with modelling and simulation practices. This MC is relevant at the third taxonomic level, assessing computational models, which relates to students' understanding of the relation between a model and what the model represents. This taxonomic level also concerns whether elements of a phenomenon have been omitted, simplified, augmented, or decreased in a particular model. This practice allows students to understand and articulate a model's validity by 'identifying assumptions built into the model' (Weintrop et al., 2016, p 137). Also, the reasoning competency is relevant here since it includes analyzing or producing arguments (orally or in writing) to justify mathematical claims and considers both the ability to provide justifications as well as critically assess and analyze these proposed justifications (Niss & Højgaard, 2019). Specifically, the ability to critically analyze and assess a proposed justification is relevant when students need to understand and articulate a model's validity from the assumptions built into it.

Lastly, we find the problem-solving competency relevant to coordinate with constructing computational models. This CT practice concerns encoding a model to aid computer interpretation—either in conventional programming or by "manipulating graphical interfaces or defining sets of rules to be followed" (Weintrop et al., 2016, p. 138)—which allows students to build models of their own ideas instead of re-using the work of others. The problem-handling competency concerns the ability to pose and solve mathematical problems, critically analyze and assess suggested and attempted solutions, and devise and implement strategies to solve problems. We find this competency relevant since constructing a computational model includes posing and solving mathematical ideas as well as developing and iteratively devising strategies to solve the specific problem at hand (Niss & Højgaard, 2019).

**Example**

An example of the relevance of coordinating of MCs and modelling and simulation practices can be found in a Danish task where grade 8 students were asked to construct a digital artifact that can detect and signal when the level of noise in a given room is too loud.[2] Students were encouraged to measure the level of noise produced by different sounds and explore the relative noise of these sounds in different settings using a sound meter downloaded onto their mobile devices. They were also encouraged to develop a digital design based on the sound meter that prompts people to lower their voices when a high level of noise is detected. The task took the students through a number of iterations in which they assessed the digital artifact's functionality. Through these iterations, the students tested, assessed the (re-) design of the artifact, and constructed computational models while simultaneously modelling the sounds into a measurable unit under the specific properties of the environment, which is at the core of the modelling competency. This task also exemplifies the relevance of the problem-handling competency, since, through the design iterations, the students critically analyzed and assessed their own attempted solutions to the problem of noise at their school and subsequently devised and implemented an improved strategy.

## *5.3 Computational Problem-Solving Practices*

Computational problem-solving practice has a strong relation with computer science, and it is central to mathematical inquiry (Weintrop et al., 2016). Weintrop et al.'s (2016) version of problem-solving is derived from computer science and focuses on practices where computational tools play a central role. Engaging in these practices includes reframing the problems to be solved using computational tools as well as understanding, modifying, and creating relevant computer programs; assessing different approaches and solutions; developing reusable solutions; identifying, creating, and using computational abstractions; and troubleshooting and debugging (Table 3).

In our view, the problem-handling and tools and aids competencies are particularly relevant to coordinate with the computational problem-solving practice. The problem-handling competency includes being able to pose or solve mathematical problems, the critical analysis and evaluation of one's own and others attempts to solve a problem, and devising and implementing problem-solving strategies (Niss &

[2] https://xn--tekforsget-6cb.dk/wp-content/uploads/2020/09/R%C3%B8de-%C3%B8rer-5.kl-matematik-22.09.20.pdf.

**Table 3** An overview of the relation between computational problem-solving practices and MCs

| | Thinking | Problem handling | Modelling | Reasoning | Representation | Symbols and formalism | Communication | Tools and aids |
|---|---|---|---|---|---|---|---|---|
| Preparing problems for computational solutions | | ■ | | | | | | ■ |
| Programming | | ■ | | | | ■ | | ■ |
| Choosing effective computational tools | | ■ | | | | | | ■ |
| Assessing different approaches/solutions to a problem | | ■ | | | ■ | | | ■ |
| Developing modular computational solutions | | ■ | ■ | | ■ | | | |
| Creating computational abstractions | ■ | | | | ■ | | ■ | |
| Troubleshooting and debuggin | ■ | ■ | ■ | | ■ | ■ | ■ | ■ |

Højgaard, 2019). Weintrop et al.'s (2016) computational problem-solving practice does explicitly include elements in this regard (e.g., modifying and creating computer programs capable of solving problems, assessing approaches and solutions, and reframing problems to be solved through computational tools and debugging); however, these practices always take place in digital environments or involve digital tools. Therefore, the mathematical tools and aids competency is equally relevant in this practice. Specifically, this competency involves constructively using tools and aids in mathematical work and choosing between different types based on their affordances and limitations. This MC is pivotal in computational solving practices and at the very core of several of this practice's taxonomic levels—especially the third taxonomic level (choosing computational effective tools) in it's considering 'requirements and constraints of the problem and the available resources and tools' (Weintrop et al., 2016, p. 139). This level thus clearly relates to students' ability to pay attention to the affordances and limitations of tools and aids in relation to a particular problem at hand. At the fourth taxonomic level (assessing different approaches/solutions to a problem) the problem handling and tools and aids competencies are likewise relevant, since assessing computational problem-solving strategies will facilitate students' ability to consider how the strategies' use of tools relate to the problem at hand.

As shown in Table 3, we argue that, at the second taxonomic level (programming), the symbols and formalism competency is relevant to coordinate in addition to the problem handling and tools and aids competencies. In Weintrop et al.'s (2016) work, programming includes the ability to understand and create programs—both from scratch and by modifying programs developed by others and understanding general programming concepts (i.e., conditional logic, iterative logic, and recursion). In this practice, we consider the symbols and formalism competency relevant to coordinate, since it concerns "the ability to relate to and deal with mathematical symbols, symbolic expressions and transformations, as well as with the rules and theoretical frameworks (formalisms) that govern them" (Niss & Højgaard, 2019, p. 17). Programming languages are an integral part of programming, and, although the symbols in programming languages are not identical to mathematical symbols, students require the ability to decode and employ symbolic expressions and transformations in a mathematical context, which relates to the symbols and formalism competency (Niss & Højgaard, 2019).

The competencies relevant to coordinate with this CT practice is in developing modular computational solutions, which Weintrop et al. (2016) describe as making it easier to 'incrementally construct solutions, test components independently, and increase the likelihood that components will be useful in future problems' (p. 139). This level concerns students' practice of focusing on the individual parts of a solution to a problem as well as understanding how they affect the overall solution and can be integrated as components of solutions in future problems. Mastering this practice will enable students to "develop solutions that consist of modular, reusable components and take advantage of the modularity of their solutions in both working on the current problem and reusing pieces of previous solution when confronting new challenges" (Weintrop et al., 2016, p. 139). Here, we find the modelling competency relevant. This MC focuses on implementing mathematics when dealing with 'extra-mathematical questions, contexts and situation' (Niss & Højgaard, 2019, p. 16) and includes both the ability to construct as well as critically analyze and evaluate models by considering the data, facts, features, and properties of the extra-mathematical domain. In the practice of developing modular computational solutions, this competency is relevant, since students need to consider how to modularly address the data, properties, and features of the problem to modularly solve it.

Creating computational abstractions is defined as "the ability to conceptualize and represent an idea or a process in more general terms by foregrounding the important aspects of the idea while backgrounding less important features" (Weintrop et al., 2016, p. 139), and it enables students to 'identify, create and use computational abstraction' (ibid., p. 140). Here, we find the mathematical thinking, representation, and communication competencies relevant. According to Niss and Højgaard (2019), the mathematical thinking competency involves "relating to and proposing abstractions of concepts and theories (…) of claims" (p. 15). In creating computational abstractions, the ability to interpret, translate, and move between representations, which are key aspects of the representation competency (Niss & Højgaard, 2019), is relevant. Moreover, when students are required to engage in written, oral, or visual mathematical communication in different genres to construct abstractions of an idea, the communication competency is also relevant.

**Example**

To exemplify our findings, we draw on a mathematical problem from a Swedish textbook in mathematics[3] that challenges students to use Python as a computational tool to solve it. The problem we used is called 'pattern with coins'. The students are

[3] Translated from 'Mönster med Mynt' from the book, *Räkna med kod*, published by Sanoma Utbildning (2018).

first introduced to three figures with the coins (The first figure has one 10-coins and four 1-coins, the second figure has two 10-coins and six 1-coins, the third figure has three 10-coins and eight 1-coins).

The problem poses six tasks for the students. In tasks 1 and 2, students are asked to figure out how many coins the fourth and the fifth figure would have. The third task presents students with a number sequence (5–8–11–14), which matches the number of coins in the figures, and a program written in Python, which prints the ten first numbers in the number sequence. The students are asked to modify the program to print the 100 first numbers in the sequence. Furthermore, after that, students are to modify the program to write only the numbers in the sequence that are less than 100. In the fourth task, the students are asked to find how many coins are used in figure number 100. For the fifth task, students are asked to develop a program where one puts in the figure number, and the program prints the number of coins in the figure and the value of the coins in the figure. The sixth and final task is to identify how many coins one needs to build all the figures from number 1 to 100 and find how much the coins are worth.

A central part of the task is that students solve it using Python; it therefore relates to preparing the problem for computational solving and programming. The task also exemplifies the relevance of the problem handling and tools and aids competencies; the problem-handling competency is relevant, since it features the ability to solve both open and closed problems formulated individually and in groups. Since the problem is solved using Python, the tools and aids competency is also relevant in terms of students making constructive use of it to handle the mathematical problem in question.

### *5.4 Systems Thinking Practices*

Systems thinking practices concern understanding, assessing, and designing systems and students' ability to systematically understand a complex system and the relationships among its parts. It includes communicating information about, defining, and managing the complexity of systems. While the other CT practices in Weintrop et al.'s (2016) framework seems to have several clear overlaps with mathematics education, this practice perhaps appears slightly more foreign to our understanding of conventional mathematics education. Likewise, and contrary to the other taxonomic levels of CT, systems thinking is not addressed explicitly in the KOM framework. However, we find several MCs to coordinate with this practice, specifically the modelling, tools and aids, reasoning, and communication competencies. The modelling competency features the ability to construct and critically analyze/evaluate existing models (Niss & Højgaard, 2019). As mathematical models often have several parts, this MC is relevant, since constructing or analyzing models will include the ability to manage these parts' relations. Table 4 illustrates our interpretation of the most relevant MCs to coordinate with Weintrop et al.'s (2016) systems thinking practices.

**Table 4** An overview of the relation between systems thinking practices and MCs

| | Thinking | Problem handling | Modelling | Reasoning | Representation | Symbols and formalism | Communication | Tools and aids |
|---|---|---|---|---|---|---|---|---|
| Investi gating complex systems as a whole | | | ■ | | | | | ■ |
| Understanding the relationship within a system | | | ■ | ■ | | | | ■ |
| Thinking in levels | | | ■ | ■ | | | | ■ |
| Communicating information about a system | | | ■ | ■ | | | ■ | ■ |
| Defining systems and managing complexity | | | ■ | ■ | | | ■ | ■ |

As evident in Table 4, the great relevance of coordinating the reasoning competency with systems thinking practices can be observed. Specifically, the reasoning competency involves the ability "to analyse or produce arguments (i.e., chains of statements linked by inference) put forward in oral or written form to justify mathematical claims" (Niss & Højgaard, 2019, p. 16). We consider this ability to be relevant in justifications of how the different elements of a system interact (the second taxonomic level), which Weintrop et al. (2016) defines as "the ability to identify the different elements of a system and articulate the nature of their interactions" (p. 141).

The communication competency is relevant to coordinate with the taxonomic level, communicating information about a system. Weintrop et al. (2016) describe this practice as "the ability to communicate information of what has been learned about a system in a way that makes the information accessible to viewers who do not know the exact details of the system from which the information was drawn" (p. 141), and it also includes developing effective visualization and infographics. Niss and Højgaard (2019) describe this competency as engaging in communication about different mathematical concepts with precision. When taught in a mathematics education context, the communication competency is thus relevant to maintain this precision. As this taxonomic level also includes efficiently using visual communication to justify a chain of statements about the system, the tools and aids competency is also relevant to the use of various tools for effective visual communication as well as the selection of an appropriate tool in this regard based on in its affordances and constraints.

## Example

In this section, we elaborate on the connections between the KOM framework and systems thinking practices at each taxonomic level, exemplifying some of these connections using a specific problem (Fig. 3) from PISA's 2003[4] problem-solving

[4] https://www.oecd.org/education/school/programmeforinternationalstudentassessmentpisa/testquestions-pisa2003.htm.

section. In the presented problem, students are instructed to develop a library system using a flowchart, which we consider a systems thinking practice (as described by Weintrop et al., 2016). The main practice involved in this problem is understanding a system as a whole and communicating information about that system. In order to develop a solution to this problem, students need to understand how the different parts of the system are included in the flowchart and therefore understand the system as a whole while also creating a clear visual presentation of their solution. In this problem, we view the mathematical communication and reasoning competencies as relevant, as the communication competency could help the students with precision in visualizing the solution and use the visualization to efficiently make a justification for that solution, which could then invoke the mathematical reasoning competency.

## 6 Discussion

Weintrop et al. (2016) explicitly aim to augment existing pedagogy and curricula rather than radically change existing practices. From a mathematics education point of view, their descriptions of CT practices are relevant in mathematics teaching and there are expected benefits of teaching them to students. Their definition does not, however, explicitly address what mathematical capabilities are required of students to engage in these practices and achieve their goals. While the taxonomy helps in de-mystifying CT in mathematics and science contexts by identifying the concrete practices it encompasses, it leaves the relations among these practices and existing mathematics education relatively un-articulated. Thus, for mathematics teachers, it is still not obvious how and to what extent their current teaching is relevant for CT. KOM, on the other hand, focuses on MCs from a relatively mathematics-centred and a historic point of view. It is in part this aspect of the framework that makes it such a useful place for stakeholders in mathematics education to meet conceptually. Since 2002, when KOM was first published, it has become clear that mathematics education is likely to be subjugated to changes outside of mathematics, exemplified by the role of CT in the mathematics curriculum of several countries. CT is, however, still an ambiguous term with several competing definitions, which makes it problematic to compare it with traditional, well-known subject matter areas, such as algebra and statistics. By building on Weintrop et al.'s (2016) definition of CT practices specific to mathematics and science teaching and coordinating this with KOM, we have taken the initial steps in articulating the MCs that would enable stronger cohesion and greater synergy when integrating CT in the mathematics classroom. Overall, our analyses found several MCs to be relevant in this regard, with the tools and aids, modelling, and the problem-handling competencies identified as specifically important. In the tools and aids competency, the ability to assess usage and reflectively choose among several available tools has become increasingly important as the question is often rather *which* tool to use rather than *whether* to use a tool. Modelling also appears as a relevant competency for most of Weintrop et al.'s (2016) practices—and not only the modelling and systems thinking practice, as might be assumed. There are thus several overlaps with both constructing and analyzing computational models "whilst taking purposes, data, facts, features and properties of the extra-mathematical domain being modelled into account" (Niss & Højgaard, 2019, p. 16).

We have used example activities and problems to show how the connection between MCs and CT practices could be relevant. However, the connections we propose are not explicit within the examples, meaning that teachers themselves have to identify the possible MCs invoked while engaging in different CT practices. We believe that our work contributes to teachers' ability to identify the possible links between CT and KOM within a mathematical context. However, we see this as a challenge for them due to their reported lack of confidence in teaching CT in mathematics (Vinnervik, 2022). Therefore, we argue that this chapter can help change the focus on specific tools to instead focus on what is in it for mathematics and how CT could play a role in developing students' MC.

Previous studies connecting CT and mathematics have found that there is no guarantee that combining these subjects will increase students' mathematical capabilities; however, systematic efforts to train students in CT using high-quality teaching materials can lead to progress in students' CT capabilities (Boylan et al., 2018). As shown in this chapter, mathematical competencies have the potential to support students in their appropriate use of CT practices, tools, and work processes in a situation-specific manner. Likewise, the current trends in international education policy are likely to call for more studies on how mathematics education can benefit from CT. As indicated by previous studies in contexts where CT is integrated in the mathematics curriculum, teachers report not feeling confident teaching this subject (Vinnervik, 2022). This chapter has investigated and articulated the relations between CT practices and MCs, providing an initial theoretical basis from which to conduct a more in-depth analysis into domain-specific areas of mathematics. It is thus our hope that this chapter can represent a first step toward investigating what existing mathematics education practices that are relevant in a CT context, thus making CT more accessible to teachers than what seems to be the current case.

## 7 Conclusion

This chapter sought to address the following research question: What MCs and CT practices are relevant to coordinate from a networking of theories approach?

The most relevant MCs activated when engaging with the CT practices described by Weintrop et al. (2016) were found to be mathematical aids and tools, mathematical modelling, and problem handling. The CT taxonomy focuses on how computational tools can support and enrich students' mathematics and science development. This leads to the mathematical aids and tools competency playing a vital role when engaging in CT practices, especially digital technology. From investigating relevant MCs to coordinate with CT practices, we see that CT practices could reach their full potential through the activation of different MCs. We also found that the mathematical representation, thinking, reasoning, representation, and communication competencies could also potentially complement students' abilities in the practices described in the CT taxonomy.

This chapter provides insight into articulating CT in the context of mathematics education and MCs. In our analysis, we found that there is potentially useful connections between CT practices and MCs from a mathematical education perspective. However, there is a need for more empirical studies investigating how engaging in CT practices plays a role in developing students' MCs and the relation between these within the mathematical classroom.

## References


Benton, L., Hoyles, C., Kalas, I., & Noss, R. (2017). Bridging primary programming and mathematics: Some findings of design research in England. *Digital Experiences in Mathematics Education, 3*(2), 115–138. https://doi.org/10.1007/s40751-017-0028-x

Benton, L., Kalas, I., Saunders, P., Hoyles, C., & Noss, R. (2018). Beyond jam sandwiches and cups of tea: An exploration of primary pupils' algorithm-evaluation strategies. *Journal of Computer Assisted Learning, 34*(5), 590–601. https://doi.org/10.1111/jcal.12266

Borg, A., Fahlgren, M. & Ruthven, K. (2020). Programming as a mathematical instrument: The implementation of an analytic framework. In *Proceedings of Mathematics Education in the Digital Era, Linz, Austria* (pp. 435–443). Retrieved from https://hal.archives-ouvertes.fr/hal-02932218

Boylan, M., Demack, S., Wolstenholme, C., Reidy, J., & Reaney, S. (2018). *ScratchMaths: Evaluation report and executive summary*. Sheffield Hallam University. Retrieved from https://shura.shu.ac.uk/23758/1/ScratchMaths%20evaluation%20report.pdf

Brennan, K., & Resnick, M. (2012). New frameworks for studying and assessing the development of computational thinking. In *Proceedings of the 2012 annual meeting of the American Educational Research Association, Vancouver, Canada* (pp. 1–25). http://scratched.gse.harvard.edu/ct/files/AERA2012.pdf

Buteau, C., Muller, E., Rodriguez, M. S., Gueudet, G., Mgombelo, J., & Sacristán, A. I. (2020). Instrumental orchestration of using programming for mathematics investigations. In *Proceedings of mathematics education in the digital era, Linz, Austria* (pp. 443–441). Retrieved from https://hal.archives-ouvertes.fr/hal-02932218

Calder, M., Craig, C., Culley, D., De Cani, R., Donnelly, C. A., Douglas, R., ... & Wilson, A. (2018). Computational modelling for decision-making: Where, why, what, who and how. *Royal Society Open Science*, *5*(6), 172096

Caspersen, M. E., Gal-Ezer, J., McGettrick, A., & Nardelli, E. (2018). *Informatics for all the strategy*. ACM.

Cui, Z., & Ng, O. (2021). The interplay between mathematical and computational thinking in primary schools students' mathematical problem-solving within a programming environment. *Journal of Educational Computing Research, 59*(5), 988–1012. https://doi.org/10.1177/0735633120979930

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, *82*(1), 23–49. http://www.jstor.org/stable/23434838

Ejsing-Duun, S., & Misfeldt, M. (2015). Tema 1: Programmering af robotenheder i grundskolen. *Tidsskrift Læring Og Medier (LOM), 8*(14), 1–19. https://doi.org/10.7146/lom.v8i14.21615

Gadanidis, G., Cendros, R., Floyd, L., & Namukasa, I. (2017). Computational thinking in mathematics teacher education. *Contemporary Issues in Technology and Teacher Education*, *17*(4), 458-477.

Geraniou, E., & Jankvist, U. T. (2019). Towards a definition of 'mathematical digital competency.' *Educational Studies in Mathematics, 102*(1), 29–45. https://doi.org/10.1007/s10649-019-09893-8

Husain, H., Kamal, N., Ibrahim, M. F., Huddin, A. B., & Alim A. A. (2017). Engendering problem solving skills and mathematical knowledge via programming. *Journal of Engineering Science and Technology* 1–11 (Special issue on PEKA 2017)

Kaufmann, O. T., & Stenseth, B. (2021). Programming in mathematics education. *International Journal of Mathematical Education in Science and Technology, 52*(7), 1029–1047. https://doi.org/10.1080/0020739X.2020.1736349

Ng, O. L., & Cui, Z. (2021). Examining primary students' mathematical problem-solving in a programming context: Towards computationally enhanced mathematics education. *ZDM–Mathematics Education, 53*(4), 847–860. https://doi.org/10.1007/s11858-020-01200-7

NGSS Lead States. (2013). *Next generation science standards: For states, by states*. The National Academies Press, Washington DC.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (No. 485). IMFUFA, Roskilde University. English translation of part I-VI of Niss and Jensen (2002).

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

Niss, M., & Jensen, T. H. (2002). *Kompetencer og matematiklæring—Ideer og inspiration til udvikling af matematikundervisning i Danmark* [Competencies and mathematical learning—Ideas and inspiration for the development of mathematics teaching and learning in Denmark]. The Ministry of Education.

Papert, S. (1980). *Mindstorm*. Harvester Press.

Pérez, A. (2018). A framework for computational thinking dispositions in mathematics education. *Journal for Research in Mathematics Education, 49*(4), 424–461. https://doi.org/10.5951/jresematheduc.49.4.0424

Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In *Networking of theories as a research practice in mathematics education* (pp. 117–125). Springer, Cham.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://doi.org/10.1159/000202727

Vinnervik, P. (2022). Implementing programming in school mathematics and technology: Teachers' intrinsic and extrinsic challenges. *International Journal of Technology and Design Education, 32*, 213–242. https://doi.org/10.1007/s10798-020-09602-0

Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal of Science Education and Technology, 25*(1), 127–147. https://doi.org/10.1007/s10956-015-9581-5

Wing, J. (2006). Computational thinking. *Communications of the ACM, 49*(3), 33–35. https://doi.org/10.1145/1118178.1118215

Wing, J. (2008). Computational thinking and thinking about computing. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 366*, 3717–3725. https://doi.org/10.1098/rsta.2008.0118

Wing, J. (2011). Research notebook: Computational thinking—What and why? *The Link* [The magazine of Carnegie Mellon University's School of Computer Science]. Retrieved from https://www.cs.cmu.edu/link/research-notebook-computational-thinking-what-and-why

# A Rich View of Mathematics Education and Assessment: Mathematical Competencies

**Ross Turner, Dave Tout, and Jim Spithill**

## 1 Introduction

As this chapter discusses and uses terminology about test development, and closely references PISA, this section provides some background information. Every three years the PISA survey provides comparative data on the performance of 15-year-olds in reading, mathematics and science literacy. In each PISA survey administration, one of the domains is the focus. Its framework and assessment are reviewed and new assessment content developed. In PISA 2012, mathematics was the major domain, and new framework and test development occurred, with the test development shared across seven different test development teams spread across the globe.

PISA defined mathematical literacy as 'An individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens.' (OECD, 2013a).

The framework that sits behind PISA establishes the parameters that need to be met by the test developers in writing the assessment tasks and questions. In PISA, and in other assessments, the tasks begin with a stimulus (see, for example, the three sample assessment questions in Figs. 1, 2 and 3). One or more questions then follow based on the same stimulus material. The set of questions that derive from the same

R. Turner (✉) · D. Tout · J. Spithill
Australian Council for Educational Research, Melbourne, Australia
e-mail: ross.turner@acer.org

D. Tout
e-mail: dave@multifangled.com.au

J. Spithill
e-mail: jimspithill48@gmail.com

 
U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*,
Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0_18

## CLIMBING MOUNT FUJI

Mount Fuji is a famous dormant volcano in Japan.

**CLIMBING MOUNT FUJI: Question 2**

The Gotemba walking trail up Mount Fuji is about 9 kilometres (km) long.

Walkers need to return from the 18 km walk by 8 pm.

Toshi estimates that he can walk up the mountain at 1.5 kilometres per hour on average, and down at twice that speed. These speeds take into account meal breaks and rest times.

Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8 pm?

..........................................................................................................................



**Fig. 1** Climbing Mount Fuji: question 2 (OECD, 2013b, pp. 19–21)

stimulus makes up what is called a *unit* or *task*. Test developers often use the word *item* to refer to the stimulus and each individual associated question within a *unit*, and this term is used throughout this chapter. In PISA, for the main focus domain, hundreds of new items are written. These items are then used in what is called a *Field Trial* in all participating countries. This is critical in the quality assurance processes used in PISA, as it enables all the tasks and items to be checked for their measurement properties and checked regarding language and cultural factors. The quality assurance process also saw all participating countries review all proposed test material, providing feedback on such matters as accessibility to their 15-year-old learners, and hence having input into the items finally selected for inclusion in the survey. Only the best-performing items are used in the final, *Main Study*.

The other critical role of the Field Trial is that the statistical methods used allow a large number of questions (items) to be placed on the same scale of difficulty relative to each other, independent of the students taking the test. This is referred to as the *item difficulty*. This is critical because it enables a broad, representative spread of items to be selected for the Main Study so that all students, whatever their assumed abilities, can be catered for across the many, varied countries that participate in PISA. After the Main Study, new item difficulties are calculated, and this allows the construction of *described proficiency scales*, used for reporting of PISA outcomes, and the profile of aggregated abilities of the students in each participating country can be estimated more accurately against different *levels* or *bands* on the reporting scale.

## 2 PISA and the KOM Competencies

Significant initial collaboration occurred between researchers working across both PISA and the KOM framework. PISA was first administered in the year 2000, and this was also the year in which the 'KOM task group' was established, as introduced and described in Niss (2003).

> … a committee was appointed in Denmark in 2000, by the Ministry of Education and other official bodies, to conduct a project to explore the terrain of mathematics teaching and learning and to see what could be done to improve the state of affairs. The project was given the name 'the KOM project' (KOM – in Danish – stands for 'Competencies and the Learning of Mathematics'). The committee, which was chaired by the author of this paper, published its official report in October 2002. (p. 117)

An endnote in that paper refers to the close connection between the KOM project and the then-nascent PISA assessment of mathematics:

> It should be noted that the thinking behind and before the Danish KOM project has influenced the mathematics domain of OECD's PISA project, partly because the author is a member of the mathematics expert group for that project. That influence is reflected in PISA's notion of mathematical literacy and its constituents. (Niss, 2003, p. 124)

To clarify the terminology in use in this chapter about the competencies, we note following Niss and Højgaard (2019) that overall mathematical competence comprises a set of eight specific mathematical competencies, through which an individual enacts their mathematical knowledge and skill in response to some mathematical challenge. This volume provides a perspective on some of the links between those eight 'KOM competencies' and the development and implementation of the PISA mathematics framework and assessment, particularly over its early years of operation. We also note a somewhat different and shifting terminology used in the PISA context to capture more or less the same ideas.

For the current purpose, we note the close similarity between the KOM competencies and the original conception of competencies within PISA (OECD, 1999). However, we wish to focus largely on the KOM-inspired set of competencies that were referred to by the OECD assessment framework for PISA 2012 (OECD, 2013a) as 'fundamental mathematical capabilities'. These competencies underpinned the PISA 2012 assessment of mathematics, including both the component delivered in traditional paper-based form in each PISA survey administration to date, delivered to 65 participating countries/economies and the innovative computer-based assessment of mathematics (CBAM) that was undertaken in 2012 by 32 participating countries/economies. Our objective is to show how the KOM competencies were used in the PISA assessment context, and, for example, how this may help reduce the risk of a narrow focus on lower-order mathematical procedural skills and knowledge that is a hallmark of so much of elementary mathematics teaching, learning and assessment.

The influence on PISA mathematics of the thinking and work underlying the KOM project can be seen in the initial PISA mathematics assessment framework, and in the several subsequent frameworks that guided PISA implementation over its first two decades (see OECD, 1999, 2004a, 2004b, 2013a, 2019).

For further background to the design and operation of PISA in its early years, an overview of PISA is provided in Turner and Adams (2007).

## 3 The Appearance and Application of Competencies in the PISA Context

The opening stanza of the OECD's framework for the inaugural PISA mathematics assessment that took place in 2000 reads as follows:

> The mathematical literacy domain is concerned with the capacity of students to draw upon their mathematical competencies to meet the challenges of the future. (OECD, 1999, p. 41)

Eight mathematical competencies are then listed and described (OECD, 1999, p. 43), which is essentially an early version of what became the KOM competencies. The eight initial PISA competencies are labelled thus: mathematical thinking skill ; mathematical argumentation skill; modelling skill; problem posing and solving skill; representation skill; symbolic, formal and technical skill; communication skill; and aids and tools skill. With some modifications,[1] that set of mathematical competencies remained as one of the major organising principles underpinning the assessment frameworks for the triennial PISA assessments of mathematics from 2000 through to 2018.

As the implementation of PISA unfolded during the first decade of this millennium, the mathematics component of PISA underwent something of a change in its emphasis. In the first PISA implementations, particularly reflected in the 2003 assessment when mathematics first took its turn as the major assessment domain,

[1] For the PISA 2012 framework, seven 'fundamental mathematical capabilities' are listed (communication; mathematising; representation; reasoning and argument; devising strategies for solving problems; using symbolic, formal, and technical language and operations; using mathematical tools). These directly reflect seven of the eight KOM competencies, with the KOM 'mathematical thinking competency' not included.

part of the development of the mathematics component focused on three 'competency clusters' as the key way of thinking about cognitive mathematical demand. After the 2003 assessment, in the years leading up to the 2012 implementation when the mathematics domain again took centre stage as the main component of PISA, the individual competencies were placed much closer to the centre of activity. This affected at least two aspects of the PISA survey: the way student accomplishment on the PISA mathematics survey was reported; and the approach taken by the test developers to their assessment item development task.

Note that the set of competencies referred to in the PISA 2012 framework were labelled as 'fundamental mathematical capabilities'. In this chapter, 'competencies' is the term used to refer both to the formulation in documents from PISA 2012 and its subsequent surveys, and as a more generic reference reflecting the KOM usage.

### *3.1 Use of Competencies to Describe Mathematical Accomplishments in PISA*

As mentioned above, the OECD reports and describes the aggregated levels of mathematical proficiency demonstrated by surveyed learners in each participating country through a set of described proficiency scales (OECD, 2004b). Technical aspects of those proficiency scales are described in detail in the various PISA technical reports (e.g., Adams & Wu, 2003). The scales developed and used for PISA 2003 show something of the way the competencies were used to craft language describing different levels of mathematical proficiency. Reporting of PISA outcomes, in particular descriptions of the cognitive demands of PISA mathematics assessment tasks as seen in the various OECD documents providing commentary on the survey material, drew on analysis of the degree of activation of the competencies required in the solution of assessment questions (OECD, 2004a, 2004b). The shift in emphasis from 'competency clusters' to the individual component competencies can readily be seen in the way the item commentaries published by OECD following the 2003 and 2012 PISA surveys respectively were phrased (see OECD 2004a, 2004b).

### *3.2 Research, Exploring the Difficulty of PISA Assessment Questions Against the Different Mathematical Competencies*

Leading up to the PISA 2012 survey, the development of the new PISA mathematics assessment framework and tasks was informed by an expert evaluation of the extent to which tasks required activation of various mathematical competencies as students attempted to solve the assessment questions.

An extended research investigation was undertaken by a working group of members of the PISA mathematics expert group, led by one of the current authors, in order to test the importance of the competencies in relation to the cognitive demands of PISA mathematics tasks, and in particular to ascertain the usefulness of the competencies to inform predictions of the relative difficulty of the assessment items used in PISA.

> … experts involved in PISA implementation have looked closely at PISA mathematics survey questions and have judged the extent to which successfully answering those questions demands activation of various mathematical competencies that reflect the PISA framework. For the purpose of that investigation, six competencies have been given operational definitions and each of these competencies has been described at four levels. It is recognised that the six chosen competencies overlap to some extent and that they frequently operate in concert and interact with each other; nevertheless, the goal has been to treat each competency as distinctly as possible. (Turner, 2012, p. 3)

That material (the six selected competencies, their operational definitions and level descriptions) was used as the basis of a purpose-built rating scheme, by a number of mathematics education experts, to independently rate a large number of PISA mathematics assessment items. Items were rated against each of the six competencies, on a four-point scale. The ratings were then analysed together with empirical data on the PISA item difficulties. The objectives of the analysis were to answer these questions: What is the level of agreement among raters when they apply the competency rubric? Does each of the competencies capture different dimensions of cognitive complexity in the tasks? To what extent do ratings of the cognitive complexity account for (predict) the difficulty of the tasks for students? (Turner & Adams, 2012, p. 2).

Some early outcomes of the working group were presented as part of a symposium at the 2012 annual meeting of the American Educational Research Association (AERA), in Turner (2012) and in Turner and Adams (2012). The analysis presented in the latter paper indicated support for key elements of the research investigation. It was noted that while the expert ratings generated indicate considerable consistency, the data do not support reliance on the competency ratings made by any one individual alone. If the scheme is to support such an outcome then further work still needs to be done on the content—possibly on both the definition of the categories and the description of the rating levels (Turner & Adams, 2012, p. 5). It was further concluded that the variables [competencies] appear generally to be picking up somewhat distinct aspects of item demand (Turner & Adams, 2012, p. 12); and that the expert ratings from application of the scheme by a small number of experts can predict 70–80% of the difficulty of PISA survey items (Turner & Adams, 2012, p. 13).

The preliminary work provided support for the research team to continue as well as evidence that could be used as the basis of formulating adjustments to the operational definitions, and the descriptions of activation levels for each competency. This was subsequently done, with a view to increasing the consistency with which the material could be used by different independent raters. A further report of this work is in Turner et al. (2013), with a subsequent report in Turner et al. (2015) in which updated operational definitions and level descriptions are published. The version that was used to guide test development for PISA 2012 is provided in the Appendix to this chapter.

## 4 Use of the Competencies in PISA 2012 Test Development

The PISA 2012 test development process made extensive use of the thinking and tools developed during that previous research investigation into the relation between PISA item difficulty and the potential need for activation of a selected set of mathematical competencies in order to undertake particular mathematics tasks.

There are many challenges in writing quality test items in numeracy or mathematical literacy (e.g., see Tout & Spithill, 2015). One key challenge for large-scale assessments, such as in PISA, which is aimed at broad and diverse populations, is the need to create an assessment with items that are suitable for estimating and describing multiple proficiency levels within the target populations. This means that the assessment content needs to contain items that range across different aspects of mathematical literacy and vary in their cognitive demand from very easy and accessible through to difficult and challenging in order to be able to scale and describe student performance across the full range of the expected proficiency continuum of learners in the PISA assessment cohort.

Use of the PISA 2012 version of the KOM competencies supported and enhanced the ability of test developers to successfully create a comprehensive and broad set of mathematical literacy assessment tasks for PISA 2012. Turner et al. (2015) outlined the theoretical and practical issues associated with the development and use of these competencies as the basis of a rating scheme for the purpose of analysing mathematical problems.

### *4.1 Targeting the Difficulty Level of Assessment Tasks*

The PISA 2012 test developers were given training in the use of the rating scheme. They used the 2012 version of the scheme (see Appendix) which consisted of operational definitions of six competencies (communication; devising strategies; mathematisation; representation; using symbols, operations and formal language; and reasoning and argument), along with descriptions of four levels of activation of each competency from a minimum score of 0 through to a maximum score of 3. As discussed below, a seventh competency, using tools, which is one of the eight 'KOM competencies', was also used for reviewing the separate computer-based assessment items used in PISA 2012.

Application of the rating scheme to provide a total relative competency score for each item from 0 through to 18 assisted test developers to better estimate and anticipate each question's relative level of difficulty in advance, before any collection of empirical data or evidence about the actual performance of the items in the field. Predicting the level of difficulty of items at an early stage helps facilitate a spread of items across the breadth of the expected levels of skills of the target population. Knowledge about such factors that affect item performance and difficulty is very useful for test developers.

### *4.2 Enhancing the Spread of Cognitive Demands Across Assessment Tasks*

As well, knowledge about a range of different factors that can separately be used to target different cognitive aspects of a mathematical literacy task can be used to create a more diverse set of test items that help identify and describe a broad spectrum of student performance. The insights and knowledge provided by the competencies and their different levels of applicability to each mathematical context, situation and underpinning mathematics skills build test developers' awareness of different aspects of each task and helps guarantee that different cognitive aspects can be highlighted, or not, in particular questions. This is elaborated further below with some examples of how this can work with some sample questions.

Such insights and knowledge also help to reduce extraneous factors that can cause an item to be harder than it should (e.g., overload of text, or high-level formal representations) and this helps improve item reliability and validity. In parallel with this, awareness of the role of the competencies helped reduce the risk of a high emphasis on lower order, procedural mathematical skills and knowledge, by highlighting that there were several different competencies that needed to be addressed across the pool of items. For example, the Using Symbolic, Formal and Technical Language and Operations competency most explicitly address procedural actions and it is only the lowest level, level 0, that addresses such low-level procedures and processes and requires the activation of 'only elementary mathematical facts, rules, terms, symbolic expressions, or definitions (for example, arithmetic calculations are few and involve only easily tractable numbers)'. Having items that needed to cover the other levels of activation and across multiple competencies helps avoid an over-representation of standard mathematical actions.

As mentioned above, the mathematical competencies also helped in the interpretation and description of the resulting statistics about actual student performance and how to explain and describe the relative levels of performance of students across different PISA proficiency levels (see OECD, 2013; Turner et al., 2015).

### 4.3 Applying the Competencies in PISA 2012

The teams of test developers engaged in developing the PISA 2012 mathematics assessment items were initially informed about the set of competencies and trained in how to use and apply them when writing test items, typically focusing on one competency at a time and controlling the level of demand for activation of that competency. Templates for writing the test items included a table for each test developer to review, rate and score each item against the competencies. As part of the quality assurance processes used, the teams of writers panelled the test items (see Tout & Spithill, 2015) and at these panels, the ratings against the competencies were reviewed and moderated. This reviewing and revision process not only improved and enhanced the test questions themselves but also enabled the test developers to come to agreed understandings, positions and perspectives regarding the different competencies and their detailed descriptions.

The total score of the competency ratings was used as an estimate of each item's difficulty. The lead test developer monitored the item pool as it was built up and would advise test developers about any gaps identified—both in terms of the different PISA framework parameters as specified in the blueprint that needed to be met across the pool of test items, but also in relation to the need to target the different competencies. It should be noted that the latter competency targets were not specified within the PISA framework and were hence not formally measured or reported against.

## 5 Using the Competencies to Highlight Some Differences Between Digital and Paper-Based Assessment Tasks

As part of the PISA assessment of mathematical literacy in 2012, a parallel, optional computer-based assessment of mathematics (CBAM) instrument was constructed. Given the compulsory paper-based component existed, it was not the aim to just make a digital version of the paper-based assessment. Instead, the new set of CBAM items aimed to reflect the use and application of mathematics within authentic twenty-first century contexts and to also use the technology to ask different types of questions. CBAM attempted to emphasise different aspects of mathematical literacy which were not as readily assessed in a paper-based format. For instance, the assessment incorporated calculations that could be automated 'behind the scenes'; utilised spatial and visual simulations and manipulatives not possible in pencil-and-paper formats; and enabled problem-solving strategies based on the observation of patterns and trends and of the effect of manipulations and actions. As well, items were able to simulate computer-based applications such as spreadsheets, drawing tools and graphing tools (for further information about CBAM in PISA 2012 see Bardini, 2015; Hoogland & Tout, 2018; Stacey & Turner, 2015; Tout & Spithill, 2015).

Using the competencies as a lens, this section undertakes a new review of two sets of 2012 test items, in order to see if there were any noticeable differences in cognitive demands between the paper-based and computer-based items in PISA 2012. Using publicly available paper-based PISA items from 2012 (see OECD, 2013b) and a set of PISA 2012 CBAM units that are available to the authors, two sets of comparable items were used. There were 22 items in the CBAM item set and 23 items in the paper-based set and they were quite similar in their overall difficulty level and spread across the PISA construct parameters of context, content and cognitive processes. The authors had access to the agreed, moderated competency ratings for each item.

Two of the competencies rated very similarly across both the paper-based and CBAM sets of items (*Communication* and *Devising Strategies*); while the paper-based items were slightly higher in *Using Symbolic, Formal and Technical Language and Operations* and *Mathematising*. The CBAM item set was slightly higher in *reasoning and argument*. There appeared to be a significant difference in only one common competency and that was in *Representation* where the CBAM items were rated at much higher levels overall.

### *5.1 The* Representation *Competency*

In the samples chosen, most of the CBAM items (15/22) were rated as 2 or 3 in this competency, compared with the paper-based items where only seven items of 23 were rated at that level. As well, quite a few rated only as 0 in the *Representation* competency. The average value rating across the two sets was 1.0 for paper-based versus 1.9 for CBAM items, respectively. The Appendix shows the description for this *Representation* competency and also the different levels of activation of this competency.

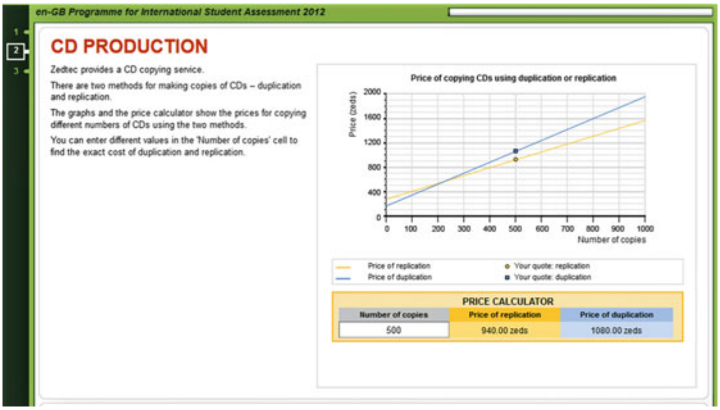
### *5.2 A Comparison of Two Items*

Using this competency as a lens, two 2012 PISA items—one a paper-based item and the other a CBAM item—are used as examples to see if it is possible to illustrate what the differences are and to consider why this difference might exist.

The paper-based item is question 2 in the Climbing Mount Fuji unit and the CBAM item is question 2 in the unit called CD Production (see Table 1; Fig. 2). Both units arose out of the test developer's personal experiences or interests. Climbing Mount Fuji was written by a test developer who had an active interest in walking that enabled

**Table 1** PISA 2012 score data for two items (OECD, 2014b, pp. 406–420)

| Question | Scale score | Proficiency level | OECD average % correct |
|---|---|---|---|
| Mount Fuji question 2 | 642 | 5 | 14.3 |
| CD production question 2 | 686 | 6 | 8.4 |




**CD PRODUCTION: Question 2**

Use the graphs and price calculator to find the rule for how the price of **replication** is determined.

Write the two missing values in the rule below to show how price, *P*, relates to number of copies made, *n*, for replication.

$$P = \ldots\ldots n + \ldots\ldots\ldots$$



**Fig. 2** CD production: question 2 (OECD, 2011a)

him to create an authentic scenario incorporating skills related to speed, distance and time. The CD Production unit arose out of the test developer's own experiences in having a CD produced for a resource he was writing and getting published at the time, so the charges and formulae sitting behind the scenario are based on actual costs and charges (adapted to the fictional world of Zedland). Both items covered the same PISA content area (Change & relationships) and PISA cognitive process (Formulate). Use of these categories is described in the PISA Mathematics Framework (OECD, 2013a).

In the paper-based *Climbing Mount Fuji* item, the main cognitive demands in terms of the *Representation* competency relate to understanding and working within the one representation—the text. There are no other representations used—and the illustration is decorative and is not necessary for solving the problem at hand. While it is possible that in order to solve the problem, a student may construct a simple representation of the given information to help understand and think through the embedded relationships, this was not required and therefore should not be counted as part of the item's representational demand. In this case, level 0 is appropriate for the *Representation* competency.

The CBAM *CD Production* item incorporates a number of interactive elements and different representations not available in a paper-based assessment task. For example, in this item, the menial, calculation-dependent work of creating and interpreting a table of values has been automated 'behind the scenes' (hence invisible to the test-taker) and the values of the variables and functions for the two linear graphs can be easily manipulated to generate values and be seen visually on the screen. This enables a stronger focus on understanding and interpreting the mathematics sitting behind the context.

As with the *Climbing Mt Fuji* example, *CD Production* requires reading and understanding the text, however, it also requires a broader understanding of the different representations used—both the graphical representations and the accompanying application, the price calculator. This interaction, along with the three different representational styles and sources of critical information, means this item requires much more substantial decoding and interpretation; and translating between and using, different representations of mathematical situations and information. The question also requires the student to provide an answer in a more formal mathematical representation that has not been used elsewhere. It could be argued that the item pushes up towards Level 3, although it does not really require the student to devise a representation that captures a complex mathematical situation. For this item, it is appropriate to rate it therefore at Level 2 for *Representation.*

As illustrated by these two examples and looking at the other CBAM items, the requirement to be able to interpret, decode, use, understand and move between different representations is much more common in the CBAM items compared with the paper-based items. This is enabled by the ability in the CBAM items to incorporate a range of different representations, including dynamic and interactive actions and processes, like the graph and the calculator app in the *CD Production* item. By their very nature, the paper-based items are static.

There are other aspects of the set of PISA 2012 CBAM items that also support the use and understanding of higher levels of representation, such as enabling different representations including simulations such as spreadsheets, drawing tools, graphing tools; the use of manipulatives; being able to observe changing patterns and trends; and handling information based in a web environment. This aspect is also illustrated in the example discussed below, *Star Points* (see Fig. 3). This also means that the CBAM items are more representative of the twenty-first century digital world we live and work in and their associated mathematical actions using different technologies, different tools and applications. This indicates that representation in the digital world is critical and that this aspect could be considered in the use and enhancement of the existing KOM competencies, where digital and technological aspects is mainly referenced under the mathematical aids and tools competency.

The application of the *Representation* competency in CBAM items highlighted benefits of using the mathematical competencies to underpin the test development in PISA 2012. It gave the test developers the ability to focus on the different aspects of the digital platform through making them think how they could more explicitly address different representations of mathematical information in a digital environment compared with a traditional paper-based environment. In addition, it helped explicitly incorporate the use and application of different 21st Century representations, which also helped make the items more authentic.

### 5.3 Using Mathematical Tools

A seventh mathematical competency, *Using mathematical tools*, was not used in the first iterations of the paper-based PISA mathematical literacy framework and its set of mathematical competencies. The *Using mathematical tools* competency was described as encompassing physical tools such as measuring instruments, as well as calculators and computer-based tools and applications that are becoming more widely available as we move further into the twenty-first century. This competency, also reflecting a corresponding KOM competency, involves knowing about and being able to make use of various tools that assist mathematical activity and also knowing about the limitations of such tools.

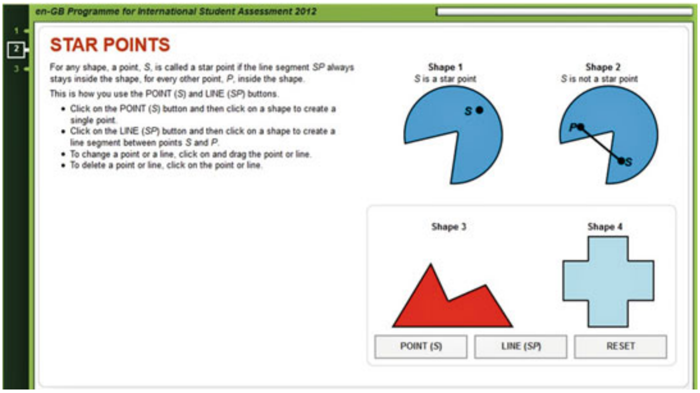
Based on the development of the CBAM components of PISA 2012, the *Using mathematical tools* competency was partially reintroduced. The 2012 PISA framework stated:

> Previously it has been possible to include the use of tools in paper-based PISA surveys in only a very minor way. The optional computer-based component of the PISA 2012 mathematics assessment will provide more opportunities for students to use mathematical tools and to include observations about the way tools are used as part of the assessment. (OECD, 2013a, p. 31)

As a consequence of the introduction and development of the CBAM component of PISA 2012, the test development team rated the CBAM items against three levels of this competency. The ratings varied from a score of 0 where there was no use of tools or their use was not relevant, through to a maximum score of 2, where the understanding and use of one or more tools or applications was a significant component of solving the problem. In the sample set of 22 CBAM items, 17 were rated at being at levels 1 or 2. Some of the mathematical tools and applications used in the CBAM items included measuring tools, spreadsheets, online calculators and related applications (e.g., currency converters), simulations and manipulatives were made available, alongside drawing and graphing tools. As well there was an inbuilt, online calculator incorporated into the PISA CBAM application that could be accessed when needed by the student.

The *CD Production* item discussed above required the understanding and use of mathematical tools through the manipulation and input of values into the Price Calculator application and the resulting illustrated points on the linear graphs. Without that ability to use and interpret the provided interactive tools, the student would have been unable to determine the solution to the question. *Using mathematical tools* was a critical cognitive component of solving this item.

As a further illustration of the types of tools used in the PISA 2012 CBAM items, Fig. 3 shows one of the items in the unit called *Star Points.*




**STAR POINTS: Question 1**

Shown above are four flat shapes. In Shape 1, the point *S* is a star point because, wherever you place *P*, the line *SP* always stays within the shape. But in Shape 2, the point *S* is **not** a star point because there are some lines *SP*, as in the example shown, that go **outside** the shape.

Create a star point for Shape 3 and a point that is **not** a star point for Shape 4.



**Fig. 3** STAR POINTS: question 1 (OECD, 2011b)

In this item, students needed to familiarise themselves with an interactive spatial tool provided in the unit to draw a point or a line, learn how to manipulate the tool and then to use it to understand what was defined as a Star Point. The latter questions in this unit focused on the use of this spatial concept to resolve issues related to the best locations for positioning security cameras in a building. The solution of the question was entirely predicated on the successful use, understanding and application of this manipulative, interactive tool.

The performance of this item in PISA 2012 is summarised in Table 2.

**Table 2** PISA 2012 score data for star points question 1 (OECD, 2014b, pp. 406–420)

| Question | Scale score | Proficiency level | OECD average % correct |
|---|---|---|---|
| Star points question 1 | 562 | 4 | 29.6 |

The fact that the *Using mathematical tools* competency was only used for writing and reviewing the 2012 CBAM items and not the paper-based items, in itself, tells a story about the breadth of skills that were able to be incorporated into the CBAM items that could not as easily be addressed in the paper-based items. This competency gave the test developers the ability to address and include a range of different tools and applications that could only be made available within a digital platform, in contrast to limitations they had in the paper-based items. It also helped explicitly incorporate the use and application of different 21st Century technologies, tools and applications, which once again helped make the items more authentic.

## 6 Rating a Task Using the Competencies

The Appendix describes the version of six KOM competencies as used in PISA 2012. It is essentially a rubric for assigning Levels 0 to 3 to each competency. Using the six competencies each with a maximum score of 3, alongside a maximum score of 2 for *Using mathematical tools*, the theoretical maximum sum of the competency ratings was 20. In the sample of paper-based and CBAM items referenced here, the highest score given was 15.

For the six easiest items as determined by the psychometric analysis of the PISA field trial (a step in the item development process) the mean competency rating sum score was 7.8, with a range of 4–13 out of 20. For the six hardest items the mean was 12.7, with a range of 11–15. It is instructive to note that none of the items came anywhere near the maximum possible score of 20. This makes sense when choosing items that had to be doable in the short time frame of the PISA test administration. Any possible items that might be scored at Level 3 in more than three of the competency rating criteria would likely be too complex for students to analyse and complete in just a few minutes. Such items would have a very low correct response rate and would therefore not provide useful information about the performance of the wider student cohort.

Table 3 show the breakdown in competency rating levels for the three sample items analysed above.

Test developers made a point of considering the rating scores in more detail, in terms of the Level descriptions presented in the Appendix:

- Communication ('COM'): Climbing Mount Fuji at Level 1 is a more familiar and relatable context that can likely be understood without 'repeated cycling' to 'decode and link multiple elements' which is required at Level 2 in the other two items.

**Table 3** Competency ratings for three items

| Unit name | PB/CB | Item ID | COM | STR | MAT | REP | SYF | RES | TLS | SUM |
|---|---|---|---|---|---|---|---|---|---|---|
| CD production | CB | CM015Q02 | 2 | 3 | 0 | 2 | 3 | 2 | 2 | 14 |
| Star points | CB | CM020Q01 | 2 | 3 | 0 | 3 | 0 | 3 | 2 | 13 |
| Climbing Mount Fuji | PB | PM942Q02 | 1 | 3 | 2 | 0 | 2 | 2 | 0 | 10 |

- Devising Strategies ('STR'): In each of the three items, a 'multi-stage strategy' is needed and with an aspect where students 'compare strategies' (Level 3).
- Mathematising ('MAT'): The two computer-based items are Level 0; in CD Production the mathematical model sits in the background as a given and in Star Points the 'the situation is purely intra-mathematical'. In Climbing Mount Fuji the 'variables, relationships and constraints are clear' (Level 2).
- Representation ('REP'): CD Production is Level 2, requiring students to 'translate between and use different standard representations of a mathematical situation'. Star Points is Level 3 as it requires students to 'compare' cases that distinguish what a star point is or is not. Climbing Mount Fuji could be solved by systematically tabulating times and stages which would involve a minimal representation (Level 0).
- Using Symbolic, Formal and Technical Language and Operations ('SYF'): CD Production is a formal mathematical context requiring 'mathematical technique and knowledge to produce results' (Level 3), even as the calculations are provided in the computer-based model. Star Points is essentially a geometrical rather than a symbolic task that does not require any calculations, so Level 0 applies. The classification of Climbing Mount Fuji would depend on the approach taken by a student; adopting a more formal algebraic approach whereby they 'construct a representation' would be Level 2. There is a trade-off here with the score for Representation, which can happen when trying to separate closely related competencies.
- Reasoning and Argument ('RES'): CD Production and Climbing Mount Fuji require students to 'reason from linked information sources', so Level 2. Star Points requires students to 'create chains of reasoning to check or justify inferences' and in a novel geometric context, which pushes it to Level 3.
- Using mathematical tools ('TLS'): This is straightforward, with CB items having tools that are integral to finding a solution (both rated at Level 2) and the paper-based item being Level 0.

## 7 Benefits of Using the Competencies as a Lens in PISA 2012

Overall, the use of the PISA version of the KOM competencies enhanced the implementation of the mathematical literacy assessment in PISA 2012. Not only did the research on the relation between the difficulty of PISA assessment items and the activation of different mathematical competencies support the development and description of the proficiency descriptions, but the associated rating scheme based on the competencies gave the PISA test development team a lens by which to assess and review their test items and supported their ability to develop a broad range of test items that incorporated different aspects of seven different competencies.

As illustrated by the above examples, using the competencies as a lens enables an analysis and comparison of sets of test items and the underpinning mathematical competencies needed to solve them. The ratings and descriptions of each competency enable this to be done coherently. The authors believe this is a proactive and constructive way that the competencies can be used and probably not just in relation to assessment development.

This goes back to the original conception of mathematical literacy from PISA 2000: the capacity of students to draw upon their mathematical competencies to meet the challenges of the future. It looks behind and beyond an immediate assessment to a contemporary view of the potential future of mathematics education, particularly in schools. In an extensive discussion of this issue, Gravemeijer et al. (2017) draw attention to the bigger picture as follows:

> We might add that, today, basically all mathematical operations that are taught in primary, secondary and tertiary education can be performed by computers and are performed by computers in the world outside school. This reveals a tension between what is going on in society and what is going on in schools… This does not mean that there is no need any more for learning mathematics, but what mathematics is important to learn changes… In this respect, we may re-emphasise our earlier remark that we have to shift away from teaching competencies that compete with what computers can do and start focusing on competencies that complement computer capabilities. (p. 107)

Having the competencies as a lens certainly helped PISA address the shortcomings of a 'procedural knowledge and skills' view of what mathematics education is about.

## 8 Conclusions

The mathematical competencies deployed as part of the OECD's PISA survey borrowed heavily from the KOM competencies framework and have had a major influence in several spheres of activity of mathematics education practitioners and researchers. The competencies have provided a key organising structure in the way PISA defined mathematical literacy and in the way, mathematical proficiency was analysed and understood within the PISA context over the first two decades of its global operation. The competencies have been central to the discourse about the cognitive demand of PISA mathematics tasks. They have been central in the way different levels of mathematical proficiency have been described in the reporting of PISA outcomes.

Using research that shows a strong relationship between the demand for activation of the mathematical competencies as learners undertake PISA survey items and the difficulty of those assessment items, test developers have been able to use a structured approach to item design for both paper-based and digital assessment instruments. That approach helped test developers to ensure the most important cognitive elements were present in the suite of tasks they developed for both online and off-line delivery and informed their expectations regarding task difficulty. The competencies provided researchers with a lens through which to see fundamental aspects of mathematical competence playing out in an international large-scale assessment context.

The new, brief, analysis undertaken above that used the competencies to compare different aspects of the digital platform compared with a traditional paper-based delivery, highlighted some significant differences between two of the competencies: *Representation* and *Using tools*. In both cases, CBAM digital items were of a much higher cognitive level than similar items in the paper-based format. The competencies helped explicitly address the use and application of 21st Century representations and interactive digital tools and applications, as discussed by Hoogland and Tout (2018).

The competencies helped the PISA survey to assess a wide range of mathematical understandings and applications, avoiding any potential over-emphasis on the use and reproduction of procedural knowledge and skills in an assessment context. The set of competencies was seen as an effective, developmental framework that was dynamic and evolving in parallel with the PISA mathematical literacy assessment and as such can play an important and critical role in moving forward in mathematics education in a digital world.

The lessons learned through these research and development efforts inspired by the KOM competency framework are providing tools and approaches with applicability in other mathematics-related contexts, especially in the way learner progress is understood and described.

The KOM competencies give researchers and assessment professionals a valuable and independent lens through which their efforts can become more systematic and purposeful.

## Appendix: Item Rating Scheme Using PISA Fundamental Mathematical Capabilities (27 Feb, 2012[2])

[2] Previously unpublished internal working draft.

**Table** Communication

| Variable | Level 0 | Level 1 | Level 2 | Level 3 |
|---|---|---|---|---|
| Reading, decoding and ***interpreting*** statements, questions, instructions, tasks and objects; ***imagining*** and understanding the situation presented and ***making sense*** of the information provided including the mathematical terms referred to; ***presenting and explaining*** one's mathematical work or reasoning | Understand a short sentence or phrase relating to concepts that give immediate access to the context, where all information is directly relevant to the task and where the order of information matches the steps of thought required to understand the task. Constructive communication involves only presentation of a single word or numeric result | Identify, select and directly combine relevant elements of the information provided, for example by cycling once within the text or between the text and other related representation/s. Any constructive communication required is simple and may involve writing a short statement or calculation, or expressing an interval or a range of values | Select and identify elements to be combined and use repeated cycling to understand instructions, or decode and link multiple elements of the context or task. Any constructive communication involves providing a brief description or explanation or presenting a sequence of calculation steps | Recognise and interpret logically complex relations (such as conditional or nested statements) involving the combining of multiple elements and connections. Any constructive communication would involve presenting an explanation or argumentation that links multiple elements of the problem |

**Table** Devising Strategies

| Variable | Level 0 | Level 1 | Level 2 | Level 3 |
|---|---|---|---|---|
| Selecting or devising, as well as controlling the implementation of, a mathematical strategy to solve problems arising from the task or context | Take direct actions, where the strategy needed is explicitly stated or obvious | Find a straightforward strategy (usually of a single-stage) that combines the relevant given information to reach a result or conclusions | Devise a straightforward multi-stage strategy, or use an identified strategy repeatedly, where using the strategy requires targeted and controlled processing, in order to transform given information to reach a conclusion | Devise a multi-stage strategy, where using the strategy involves substantial monitoring and control of the solution process in order to find a conclusion; or evaluate or compare strategies |

**Table** Mathematising

| Variable | Level 0 | Level 1 | Level 2 | Level 3 |
|---|---|---|---|---|
| ***Mathematising*** an extra-mathematical situation (which includes structuring, idealising, making assumptions, building a model), or ***making use*** of a given or constructed model by ***interpreting*** or validating it in relation to the context | Either the situation is purely intra-mathematical, or the relationship between the extra-mathematical situation and the model is not relevant to the problem | Make an inference about the situation directly from a given model; translate directly from a situation into mathematics where the structure, variables and relationships are given | Modify or use a given model to satisfy changed conditions or interpret inferred relationships; or identify and use a familiar model within limited and clearly articulated constraints; or create a model where the required variables, relationships and constraints are clear | Link, compare, evaluate or choose between different given models; or create a model in a situation where the assumptions, variables, relationships and constraints are to be identified or defined and check that the model satisfies the requirements of the task |

**Table** Representation

| Variable | Level 0 | Level 1 | Level 2 | Level 3 |
|---|---|---|---|---|
| ***Interpreting***, translating between and ***making use*** of given mathematical representations; ***selecting*** or ***devising*** representations to capture the situation or to present one's work. The representations referred to are depictions of mathematical objects or relationships, which include symbolic or verbal equations or formulae, graphs, tables, diagrams | Directly operate on a given representation where minimal interpretation is required in relation to the situation, for example going directly from text to numbers, reading a value directly from a graph or table | Explore or use a given standard representation in relation to a mathematical situation, for example, to compare data, to depict or interpret trends or relationships | Understand and use a representation that requires substantial decoding and interpretation; or translate between and use different standard representations of a mathematical situation, including modifying a representation; or construct a representation of a mathematical situation | Understand and use multiple representations that require substantial decoding and interpretation; or compare or evaluate representations; or link representations of different mathematical entities; or devise a representation that captures a complex mathematical situation |

**Table** Using Symbolic, Formal and Technical Language and Operations

| Variable | Level 0 | Level 1 | Level 2 | Level 3 |
|---|---|---|---|---|
| Understanding and ***implementing*** mathematical procedures and language (including symbolic expressions and arithmetic operations), governed by mathematical ***conventions and rules***; understanding and ***utilising constructs*** based on definitions, results, rules and ***formal systems*** | Activate only elementary mathematical facts, rules, terms, symbolic expressions or definitions (for example, arithmetic calculations are few and involve only easily tractable numbers) | Make direct use of a simple formally expressed mathematical relationship (for example, familiar linear relationships); use formal mathematical symbols (for example, by direct substitution or sustained arithmetic calculations involving fractions and decimals); use repeated or sustained calculations from level 0; or activate and directly use a formal mathematical definition, fact, convention or symbolic concept | Use and manipulate symbols (for example, by algebraically rearranging a formula); activate and use formally expressed mathematical relationships having multiple components; employ rules, definitions, results, conventions, procedures or formulae using a combination of multiple relationships or symbolic concepts; use repeated or sustained calculations from level 1 | Apply multi-step formal mathematical procedures; work flexibly with functional or involved algebraic relationships; use both mathematical technique and knowledge to produce results; use repeated or sustained calculations from level 2 |

**Table** Reasoning and Argument

| Variable | Level 0 | Level 1 | Level 2 | Level 3 |
|---|---|---|---|---|
| Logically rooted thought processes that explore and connect problem elements so as to ***make inferences*** from them, or to ***check a justification that is given*** or ***provide a justification*** of statements | Make direct inferences from the information and instructions given | Join information in order to make inferences, (for example to link separate components present in the problem, or to use direct reasoning within one aspect of the problem) | Analyse information (for example to connect several variables) to follow or create a multi-step argument; reason from linked information sources | Synthesise and evaluate, use or create chains of reasoning to check or justify inferences or to make generalisations, drawing on and combining multiple elements of information in a sustained and directed way |

## References

Adams, R., & Wu, M. (Eds.). (2003). *PISA programme for international student assessment (PISA), PISA 2000 technical report.* OECD Publishing.

Bardini, C. (2015). Computer-based assessment of mathematics in PISA 2012. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 173–188). Springer.

Gravemeijer, K., Stephan, M., Julie, C., Lin, F. L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education, 15*(suppl. 1), 105–123.

Hoogland, K., & Tout, D. (2018). Computer-based assessment of mathematics into the twenty-first century: Pressures and tensions. *ZDM—Mathematics Education*, *50*(4), 675–686. https://doi.org/10.1007/s11858-018-0944-2

Niss, M. A. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In A. Gagatsis, & S. Papastavridis (Eds.), *3rd Mediterranean conference on mathematical education, Athens, Hellas*, January 3–5, 2003 (pp. 116-124). Hellenic Mathematical Society.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28. https://doi.org/10.1007/s10649-019-09903-9

OECD. (1999). *Measuring student knowledge and skills: A new framework for assessment*. OECD Publications.

OECD. (2004a). *The PISA 2003 assessment framework: Mathematics.* PISA, OECD Publishing. https://doi.org/10.1787/9789264101739-en

OECD. (2004b). *Learning for tomorrow's world: First results from PISA 2003.* PISA, OECD Publishing. https://doi.org/10.1787/9789264006416-en

OECD (2011a). *CD production question 2: CM015Q02.* ACER. Available internally at http://cbasq.acer.edu.au/

OECD. (2011b). *Star points question 1: CM020Q01.* ACER. Available internally at http://cbasq.acer.edu.au/

OECD. (2013a). *PISA 2012 assessment and analytical framework: Mathematics.* PISA, OECD Publishing. https://doi.org/10.1787/9789264190511-en

OECD. (2013b). *PISA 2012 released mathematics items.* OECD Publishing. https://www.oecd.org/pisa/pisaproducts/pisa2012-2006-rel-items-maths-ENG.pdf

OECD. (2014a). *PISA 2012 results: What students know and can do (Vol. I, Revised edition, February 2014): Student performance in mathematics.* PISA, OECD Publishing. https://doi.org/10.1787/9789264208780-en

OECD. (2014b). *PISA 2012 technical report*. OECD Publishing.

OECD. (2019). *PISA 2018 assessment and analytical framework.* PISA, OECD Publishing. https://doi.org/10.1787/b25efab8-en

Stacey, K., & Turner, R. (2015). The evolution and key concepts of the PISA mathematics frameworks. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 5–33). Springer.

Tout, D., & Spithill, J. (2015). The challenges and complexities of writing items to test mathematical literacy. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 145–171). Springer.

Turner, R. (April, 2012). *Some drivers of test item difficulty in mathematics.* Paper presented at the annual meeting of the American Educational Research Association (AERA), Vancouver, April 13–17, 2012. http://research.acer.edu.au/pisa/4/

Turner, R., & Adams, R. J. (2007). The programme for international student assessment: An overview. *Journal of Applied Measurement, 8*(3), 237–248.

Turner, R., & Adams, R. J. (2012). *Some drivers of test item difficulty in mathematics: An analysis of the competency rubric.* Paper presented at the annual meeting of the American Educational Research Association (AERA), Vancouver. American Educational Research Association. http://research.acer.edu.au/pisa/7/

Turner, R., Blum, W., & Niss, M. (2015). Using competencies to explain mathematical item demand: A work in progress. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 85–115). Springer.

Turner, R., Dossey, J., Blum, W., & Niss, M. (2013). Using mathematics competencies to predict item difficulty in PISA. In M. Prenzel, M. Kobarg, K. Schöps, & S. Rönnebeck (Eds.), *Research on PISA: Research outcomes of the PISA research conference 2009* (pp. 23–37). Springer Science and Business Media.

# Index





U. T. Jankvist and E. Geraniou (eds.), *Mathematical Competencies in the Digital Era*, Mathematics Education in the Digital Era 20,
https://doi.org/10.1007/978-3-031-10141-0