



Partial Vertex Cover on Graphs of Bounded Degeneracy

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Abstract. In the PARTIAL VERTEX COVER (PVC) problem, we are given an n -vertex graph G and a positive integer k , and the objective is to find a vertex subset S of size k maximizing the number of edges with at least one end-point in S . This problem is W[1]-hard on general graphs, but admits a parameterized subexponential time algorithm with running time $2^{O(\sqrt{k})}n^{O(1)}$ on planar and apex-minor free graphs [Fomin et al. (FSTTCS 2009, IPL 2011)], and a $k^{O(k)}n^{O(1)}$ time algorithm on bounded degeneracy graphs [Amini et al. (FSTTCS 2009, JCSS 2011)]. Graphs of bounded degeneracy contain many sparse graph classes like planar graphs, H -minor free graphs, and bounded tree-width graphs (see Fig. 1). In this work, we prove the following results:

- There are algorithms for PVC on graphs of degeneracy d with running time $2^{O(dk)}n^{O(1)}$ and $(e + ed)^k 2^{o(k)}n^{O(1)}$ which are improvements on the previous $k^{O(k)}n^{O(1)}$ time algorithm by Amini et al. [2]
- PVC admits a polynomial compression on graphs of bounded degeneracy, resolving an open problem posed by Amini et al. [2].

Keywords: Parameterized Algorithms · Partial Vertex Cover · Bounded Degeneracy · Planar Graphs

1 Introduction

In a covering problem, we are given a family \mathcal{F} of subsets of a universe U , and the objective is to find a minimum size subfamily of \mathcal{F} covering all the elements in U . Well known examples of covering problems are SET COVER, VERTEX COVER, DOMINATING SET, FACILITY LOCATION, k -MEDIAN, k -CENTER, etc. Covering problems are fundamental in combinatorial optimization and they are very well studied in all areas of algorithms and complexity.

Another variant of covering problems is partial covering problems. In a partial covering problem, the input is a family \mathcal{F} of subsets of a universe U and a positive integer k . The objective is to find a k size subset of \mathcal{F} that covers the maximum

number of elements in U . Two prominent examples of partial covering problems on graphs are PARTIAL VERTEX COVER (PVC) and PARTIAL DOMINATING SET (PDS), which has got considerable attention in the field of parameterized complexity¹.

PARTIAL VERTEX COVER (PVC) **Parameter:** k
Input: An undirected graph G and a positive integer k
Objective: Find a vertex subset S of size k such that the number of edges with at least one end-point in S is maximized

PARTIAL DOMINATING SET (PDS) **Parameter:** k
Input: An undirected graph G and a positive integer k
Objective: Find a vertex subset S of size k such that the size of the closed neighborhood of S is maximized

Even though there are many works on PVC and PDS in the realm of parameterized complexity, there are still some open questions about these problems. It is previously known that PVC is W[1]-hard [10] and PDS, as a more general problem of DOMINATING SET, is W[2]-hard. Amini et al. [2] proved that PVC can be solved in time $k^{O(k)}n^{O(1)}$ in bipartite graphs, triangle free graphs, planar graphs, H -minor free graphs (for a fixed H), and bounded degeneracy graphs. On planar graphs, they gave faster algorithms with running time $2^{O(k)}n^{O(1)}$ for PVC and PDS. Later, Fomin et al. [5] gave parameterized subexponential time algorithms with running time $2^{O(\sqrt{k})}n^{O(1)}$ for PVC and PDS on planar graphs and apex-minor free graphs. Also, unlike DOMINATING SET, which is known to be FPT [1] on bounded degeneracy graphs, PDS is W[1]-hard [9] in this class.

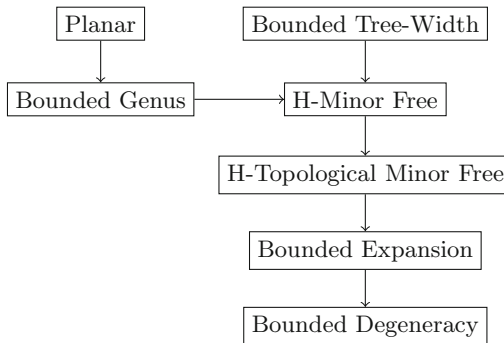


Fig. 1. Inclusion relation between various sparse graph classes.

¹ For basic definitions related to parameterized algorithms and complexity we refer to Sect. 2.1.

In this work, we give a parameterized single exponential time algorithm for PVC on d -degenerate graphs. Our algorithm also works for the more general weighted version of the problem.

Theorem 1.1. *Given $G = (V, E)$, a d -degenerate graph with edge weights $w : E \rightarrow \mathbb{R}^+$, and an integer $k > 0$, there is an algorithm that runs in time $2^{kd+k}(kd)^{O(\log(kd))}n^{O(1)}$ and finds a subset $S \subseteq V$ of size k , with maximum possible $E_G(S)$, i.e., the total weight of edges with at least one end-point in S .*

It is also possible to apply a slight change in the algorithm of Theorem 1.1 to get a faster running time.

Theorem 1.2. *Given $G = (V, E)$, a d -degenerate graph with edge weights $w : E \rightarrow \mathbb{R}^+$, and an integer $k > 0$, there is an algorithm that runs in time $(e + ed)^k 2^{o(k)}n^{O(1)}$ and finds a subset $S \subseteq V$ of size k , with maximum possible $E_G(S)$, i.e., the total weight of edges with at least one end-point in S .*

In [2], Amini et al. asked whether PVC and PDS admit polynomial kernels on planar graphs. We prove that PVC admits a polynomial compression on d -degenerate graphs, a more general class of sparse graphs. To get a better size bound for planar graphs, we prove the following general theorem.

Theorem 1.3. *Given a d -degenerate graph $G = (V, E)$ that does not contain any $K_{p,p}$ as a subgraph, and an integer $k > 0$, there is a polynomial-time algorithm that outputs a subgraph $H = (V' \subseteq V, E' \subseteq E)$ of G with $O(pd^2(2dk)^p)$ vertices and a weight function $\rho : V' \rightarrow \{0, \dots, 2^{dk}\}$ on the vertex set V' with the following properties.*

- For any vertex subset $S' \subseteq V' \subseteq V$, $E_G(S')$ is equal to $E_H(S') + \sum_{v \in S'} \rho(v)$.
- Let S be a partial vertex cover of size k in G covering at least t edges. Then there is a vertex set $S' \subseteq V'$ of size k such that $E_H(S') + \sum_{v \in S'} \rho(v)$ is at least t .

Since planar graphs are 5-degenerate and do not contain $K_{3,3}$ as a subgraph, we get the following corollary from Theorem 1.3.

Corollary 1.3.1. *There is a polynomial compression for PVC on planar graphs. Here, the compressed instance is a graph H with $O(k^3)$ vertices and a weight function on the vertex set of H where the weight of each vertex can be encoded using $O(k)$ bits.*

Because no d -degenerate graph contains $K_{d+1,d+1}$ as a subgraph, we also have the following corollary.

Corollary 1.3.2. *There is a polynomial compression for PVC on d -degenerate graphs. Here, the compressed instance is a graph H with $O(k^{d+1})$ vertices and a weight function on the vertex set of H , where the weight of each vertex can be encoded using at most kd bits.*

Independent of our work, Koana et al. [12] recently showed that PVC on d -degenerate graphs admits a kernel of size $k^{O(d)}$ and unless $\text{coNP} \subseteq \text{NP} \setminus \text{poly}$, it does not admit any kernel of size $O(k^{d-2-\epsilon})$.

Our Methods. First, we explain the overview of our FPT algorithm mentioned in Theorem 1.1, which is based on the following randomized process. Notice that for a d -degenerate graph, there is a sequence of vertices such that for any vertex v , the number of v 's neighbors at the right of it in the sequence is at most d . Let S be a solution for PVC and let S' be the set of vertices that are not in S , but they are a “right neighbor” of a vertex in S . Clearly, $|S \cup S'| \leq k + kd$. If we color each vertex red or blue uniformly at random, with probability at least $\frac{1}{2^{k+kd}}$, all the vertices in S would get red, and all the vertices in S' would get blue. Now we assign a value $val(v)$ to any vertex v , which is $|N_G(v)|$ minus the number of red “right neighbors” of v . This assignment of values ensures that each edge incident on a red vertex contributes to the value of exactly one red vertex. Observing that for every vertex in S all of its red “right neighbors” are also in S , the solution will be the k most valuable red vertices, and the number of edges covered by them will be the sum of their values. This algorithm can be derandomized using universal sets. In Sect. 3, we present the deterministic version of the algorithm.

Next, we give a high-level idea of our polynomial compression algorithm. We prove that a “large” d -degenerate graph without any $K_{p,p}$ as a subgraph, has an independent set I of size $k + 1$ and a vertex subset \mathcal{C} such that for any distinct $x, y \in I$, $N_G(x) \cap N_G(y) = \mathcal{C}$. Then, we prove that there is a solution that does not contain the least degree vertex of I . This leads to a simple reduction rule as long as the number of vertices is not polynomially bounded in k . This algorithm is explained in Sect. 4.

Other Related Works. In [16] some generalization of vertex cover (e.g. PVC) parameterizing by tree-width is studied. Also, PVC parameterized by the number of covered edges is studied in [11]. There are also extensive works on the approximability of PVC on general graphs [13–15]. For example, Manurangsi in [14] presents a simple FPT approximation scheme that runs in $(1/\epsilon)^{O(k)} n^{O(1)}$ as well as an approximation kernelization scheme of $O(k/\epsilon)$ vertices for weighted PVC.

2 Preliminaries

For a graph $G = (V, E)$, we denote the number of vertices and edges by n and m , respectively. For a vertex $v \in V$ we denote the set of neighbors of v by $N_G(v)$ and the degree of v by $|N_G(v)|$. For $A \subseteq V$, we use $E_G(A)$ to denote the total number (weight) of edges with at least one end-point in A . We denote a complete bipartite graph with partitions of size p and q by $K_{p,q}$. We use $[n]$ to denote the set $\{1, 2, \dots, n\}$.

Definition 2.1 (*d -degenerate graph*). *An undirected graph G is said to be d -degenerate if every subgraph of G contains a vertex of degree at most d . The degeneracy of a graph is the smallest value of d for which it is d -degenerate.*

We use the following proposition to derive Corollary 1.3.1 from Theorem 1.3.

Proposition 1. *Planar graphs are 5-degenerate.*

Proof. By Euler’s formula, we know $m \leq 3n - 6$ for all $n \geq 3$. Therefore, $\sum_{v \in V} |N_G(v)| \leq 6n - 12$, and there is a vertex of a degree at most 5 in any planar graph. Since every subgraph of a planar graph is also planar, planar graphs are 5-degenerate. \square

For a graph $G = (V, E)$, let λ be an ordering of vertices of G ; i.e. $\lambda : [n] \rightarrow V$ is a bijective function. We say λ is *d-posterior*, if $\lambda(i)$ has at most d neighbors among $\lambda(i + 1), \lambda(i + 2), \dots, \lambda(n)$. Also, for $v = \lambda(i)$, we call $N_G(v) \cap \{\lambda(i + 1), \lambda(i + 2), \dots, \lambda(n)\}$ *posterior neighbors* of v and we denote them by $PN_\lambda(v)$. Note that since λ is a d -posterior ordering, we have $PN_\lambda(v) \leq d$ for all $v \in V$. Next, we will state some useful propositions about d -degenerate graphs.

Proposition 2. *There exists a d-posterior ordering for vertices of any d-degenerate graph G.*

Proof. Let $G_1 = G$, and for $2 \leq i \leq n$ construct G_i by removing the minimum degree vertex from G_{i-1} . Set $\lambda(i)$ to be a minimum degree vertex in G_i . \square

Proposition 3. *For a d-degenerate graph $G = (V, E)$, we have $m \leq nd$.*

Proof. Consider a d -posterior ordering λ and note that $m = \sum_{v \in V} PN_\lambda(v) \leq nd$ because $PN_\lambda(v) \leq d$ for any $v \in V$. \square

Proposition 4. *Let $G = (V, E)$ be a d-degenerate graph. Then, there is a $(d+1)$ -coloring for V such that for any $(u, v) \in E$, u and v get different colors; i.e., $f : V \rightarrow [d + 1]$ such that $f(u) \neq f(v)$ for all $(u, v) \in E$. Furthermore, one can construct this coloring in time $n^{O(1)}$.*

Proof. Let λ be a d -posterior ordering of V and for each i from n to 1, choose a color for $\lambda(i)$ which does not occur in $PN_\lambda(\lambda(i))$. \square

2.1 Parameterized Complexity

We state the following definitions slightly modified from the Kernelization book [8].

Definition 2.2 (FPT optimization problem). *A parameterized optimization problem Π is fixed parameter tractable (FPT) if there is an algorithm (called FPT algorithm) that solves Π , such that the running time of the algorithm on instances of size n with parameter k is upper bounded by $f(k) \cdot n^{O(1)}$ for a computable function f .*

Definition 2.3 (Polynomial-time preprocessing algorithm). *A polynomial-time preprocessing algorithm \mathcal{A} for a parameterized optimization problem Π is a pair of polynomial-time algorithms. The first one is called the **reduction algorithm**, and given an instance (I, k) of Π , the reduction algorithm outputs an instance $(I', k') = \mathcal{R}_\mathcal{A}(I, k)$ of a problem Π' . The second algorithm*

is called the **solution lifting** algorithm. This algorithm takes an instance (I, k) of Π , the output instance (I', k') of the reduction algorithm, and a solution s' to the instance (I', k') . The solution lifting algorithm works in time polynomial in $|I|, k, |I'|, k'$ and $|s'|$, and outputs a solution s to (I, k) such that if s' is an optimal solution to (I', k') then s is an optimal solution to (I, k) .

Definition 2.4 (Compression, Kernelization). A polynomial time preprocessing algorithm \mathcal{A} is called a compression, if $size_{\mathcal{A}}$ is upper bounded by a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ where $size_{\mathcal{A}}$ is defined as follows:

$$size_{\mathcal{A}}(k) = \sup\{|I'| + k' : (I', k') = \mathcal{R}_{\mathcal{A}}(I, k) \text{ for any instance } (I, k) \text{ of the problem}\}$$

If the upper bound $g(\cdot)$ is a polynomial function of k , we say \mathcal{A} is a polynomial compression. A compression (polynomial compression) is called a kernelization (polynomial kernelization) if the input and output of the reduction algorithm are instances of the same problem, i.e., $\Pi = \Pi'$.

3 FPT Algorithm for Weighted Partial Vertex Cover

In this section, we show that PVC can be solved in parameterized single exponential time on d -degenerate weighted graphs. That is, we prove Theorem 1.1.

We will use a *universal set* in our algorithm defined as follows (see also section 5.6.1 of [4]).

Definition 3.1 ((n, l) -universal set). An (n, l) -universal set is a family \mathcal{U} of subsets of $[n]$ such that for any $A \subseteq [n]$ of size l , the family $\{U \cap A : U \in \mathcal{U}\}$ contains all 2^l subsets of A .

Lemma 3.1 (Naor et al. [17]). For any $n, l \geq 1$, one can construct an (n, l) -universal set of size $2^l l^{O(\log l)} \log n$ in time $2^l l^{O(\log l)} n \log n$.

We now describe our FPT algorithm for solving PVC in the given d -degenerate weighted graph $G = (V, E)$. To give a better intuition, we first state the algorithm informally. Consider a d -posterior ordering for the vertices. Suppose we have an oracle that paints the vertices with blue and red, such that all vertices in the solution get red, all vertices that are not in the solution but are a posterior neighbor of a vertex in the solution get blue, and other vertices get either red or blue. Observe that the solution is a subset of red vertices such that for any vertex in the solution, its red posterior neighbors are also in the solution. Then we will assign a value to each vertex, such that the solution will be the set of k most valuable red vertices. In the algorithm, we use a universal set instead of the oracle. The following is the exact description of the algorithm.

Let λ be a d -posterior ordering of V and $l = \min(n, k + kd)$. First, we construct an (n, l) -universal set \mathcal{U} of subsets of V , and for each $U \in \mathcal{U}$ with size $\geq k$ and $v \in V$, we define the *value* of v with respect to U as:

$$val_U(v) = \sum_{u \in N_G(v) \setminus (PN_{\lambda}(v) \cap U)} w(u, v)$$

And we define $sol(U) \subseteq U$ as the set of k most valuable vertices in U , and we set the value of U to be $val(U) = \sum_{v \in sol(U)} val_U(v)$. Finally, we return $sol(U)$ for the most valuable U .

To prove Theorem 1.1, first we show the following lemmas.

Lemma 3.2. *For any $U \in \mathcal{U}$ and $A \subseteq U$, we have $\sum_{v \in A} val_U(v) \leq E_G(A)$.*

Proof. Recall that $E_G(A)$ is the total weight of edges with at least one end-point in A .

Any edge $e = (u, v)$ with exactly one end-point, say v , in A is counted at most once in $val_U(v)$ and since $u \notin A$, it is also counted at most once in $\sum_{v \in A} val_U(v)$.

For an edge $e' = (u', v')$ with both end-points in A , without loss of generality, suppose u' is later than v' in the ordering λ , i.e., $\lambda^{-1}(u') > \lambda^{-1}(v')$. Therefore, $u' \in PN_\lambda(v')$ and since $A \subseteq U$, $u' \in PN_\lambda(v') \cap U$ and e is not counted in $val_U(v')$. On the other hand, $v' \notin PN_\lambda(u')$, and e is counted in $val_U(u')$. Therefore, e is counted exactly once in $\sum_{v \in A} val_U(v)$.

Since the weights of edges are positive and all edges counted exactly once in $E_G(A)$ are counted at most once in $\sum_{v \in A} val_U(v)$, we have $\sum_{v \in A} val_U(v) \leq E_G(A)$. \square

Now, let S be a hypothetical solution, and define $\tilde{S} = S \cup (\bigcup_{v \in S} PN_\lambda(v))$. Note that:

$$|\tilde{S}| \leq |S| + \left| \bigcup_{v \in S} PN_\lambda(v) \right| \leq k + k \cdot d \quad (\text{since } |S| = k \text{ and } PN_\lambda(v) \leq d)$$

Therefore we have $|\tilde{S}| \leq l$. Consider a subset $T \subseteq V$ with size l such that $\tilde{S} \subseteq T$. According to Definition 3.1, there is a set $\tilde{U} \in \mathcal{U}$ such that $S = \tilde{U} \cap T$. Note that since $|S| = k$, size of \tilde{U} is $\geq k$, and $val_{\tilde{U}}$ and $sol(\tilde{U})$ are defined.

Lemma 3.3. $E_G(S) = \sum_{v \in S} val_{\tilde{U}}(v)$.

Proof. It is enough to show that each edge with at least one end-point in S is counted exactly once in $\sum_{v \in S} val_{\tilde{U}}(v)$.

Consider any $e = (u, v)$ with exactly one end-point, say v , in S . Note that $u \notin S$ and

$$(PN_\lambda(v) \cap \tilde{U}) \subseteq (\tilde{S} \cap \tilde{U}) \subseteq (T \cap \tilde{U}) = S$$

Therefore, $u \notin (PN_\lambda(v) \cap \tilde{U})$ and e is counted in $val_{\tilde{U}}(v)$. Since $u \notin S$, e is counted in $\sum_{v \in S} val_{\tilde{U}}(v)$ exactly once. For edges with two end-points in S , the proof is the same as the proof of Lemma 3.2. \square

We finally prove Theorem 1.1. For convenience, we restate the theorem here.

Theorem 1.1. *Given $G = (V, E)$, a d -degenerate graph with edge weights $w : E \rightarrow \mathbb{R}^+$, and an integer $k > 0$, there is an algorithm that runs in time $2^{kd+k}(kd)^{O(\log(kd))}n^{O(1)}$ and finds a subset $S \subseteq V$ of size k , with maximum possible $E_G(S)$, i.e., the total weight of edges with at least one end-point in S .*

Proof. By Lemma 3.2 and optimality of S , $val(U) = \sum_{v \in sol(U)} val_U(v) \leq E_G(sol(U)) \leq E_G(S)$ for all $U \in \mathcal{U}$ with size $\geq k$. Also, note:

$$\begin{aligned} val(\tilde{U}) &= \sum_{v \in sol(\tilde{U})} val_{\tilde{U}}(v) && \text{(definition of } val(\tilde{U})\text{)} \\ &\geq \sum_{v \in S} val_{\tilde{U}}(v) && \text{(definition of } sol(\tilde{U}) \text{ and since } S \subseteq \tilde{U}\text{)} \\ &= E_G(S) && \text{(Lemma 3.3)} \end{aligned}$$

Therefore, for the most valuable U , $val(U) = E_G(S)$. Since $val(U) \leq E_G(sol(U)) \leq E_G(S)$, $sol(U)$ is also a solution and $E_G(sol(U)) = val(U)$. This implies the algorithm’s correctness and shows that the weight of the edges covered by the solution is equal to $val(U)$.

Finally, the running time of constructing the family \mathcal{U} is

$$2^{kd+k} (kd + k)^{O(\log(kd+k))} n^{O(1)}.$$

Moreover, we only have a polynomial process for each $U \in \mathcal{U}$. Since, the size of \mathcal{U} is $2^{kd+k} (kd + k)^{O(\log(kd+k))} \log n$, the total running time is

$$2^{kd+k} (kd + k)^{O(\log(kd+k))} n^{O(1)}.$$

□

3.1 Improved Running Time Using Lopsided Universal Sets

One can use lopsided universal sets instead of universal sets in the above algorithm to get the running time $(e + ed)^k 2^{o(k)} n^{O(1)}$. In the following we briefly introduce lopsided universal sets.

Definition 3.2 ((n, p, q)-lopsided universal set). *An (n, p, q)-lopsided universal set is a family \mathcal{U} of subsets of $[n]$ such that for any $A \subseteq [n]$ of size p and $B \subseteq [n] \setminus A$ of size q , there is a $U \in \mathcal{U}$ that $A \subseteq U$ and $B \cap U = \emptyset$.*

Lemma 3.4 (Fomin et al. [6]). *There is an algorithm that given n, p and q constructs an (n, p, q)-lopsided universal set \mathcal{U} of size $\binom{p+q}{p} \cdot 2^{o(p+q)} \log n$ in time $O\left(\binom{p+q}{p} \cdot 2^{o(p+q)} n \log n\right)$.*

Now, let $p = k$ and $q = \min(n - k, kd)$ and follow the same steps as the aforementioned algorithm using an (n, p, q)-lopsided universal set \mathcal{U} instead of an (n, l)-universal set. The proof of the correctness is also the same, except that for Lemma 3.3, $\tilde{U} \in \mathcal{U}$ should be such that $S \subseteq \tilde{U}$ and $(\tilde{S} \setminus S) \cap \tilde{U} = \emptyset$. Finally, for the running time, we have the following:

Proposition 5. *For any r and n such that $1 \leq r \leq n$ we have $\binom{n}{r} \leq \left(\frac{en}{r}\right)^r$.*

Using the same analysis as the first algorithm, the running time of this algorithm would be $\binom{k+kd}{k} \cdot 2^{o(k)} n^{O(1)}$ which by the Proposition 5 is $(e + ed)^k 2^{o(k)} n^{O(1)}$. This proves Theorem 1.2.

4 Polynomial Compression for Partial Vertex Cover

In this section, we present a polynomial compression for PVC in families of graphs with bounded degeneracy. That is, we prove Theorem 1.3.

For convenience we will allow self-loops for this part, but not parallel edges. For a vertex v with self-loops, we will not count v in $N_G(v)$ so $v \notin N_G(v)$, and we will use $L_G(v)$ to denote the number of self-loops of v . Therefore, the given graph $G = (V, E)$ is undirected, unweighted and any $v \in V$ might have several self-loops. Also, G does not contain $K_{p,p}$ and without considering self-loops, it is d -degenerate.

We say a subset $U \subseteq V$ of size $k + 1$ is *nice* if U is an independent set and there is a subset $\mathcal{C} \subseteq V$ such that for any $u, u' \in U$, $N_G(u) \cap N_G(u') = \mathcal{C}$. For each $u \in U$, we call $N_G(u) \setminus \mathcal{C}$ *private neighbors* of u with respect to U , and we denote it by $PV_U(u)$. Figure 2 shows a *nice* subset.

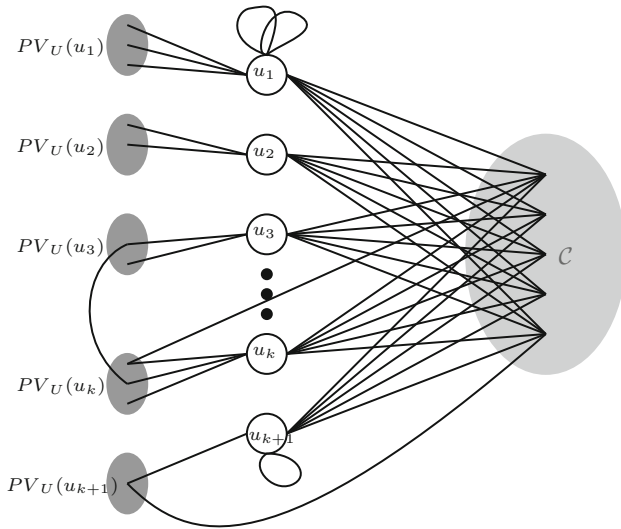


Fig. 2. A nice subset $U = \{u_1, u_2, \dots, u_{k+1}\}$

Lemma 4.1. *Let $G = (V, E)$ be an undirected graph with possible self-loops. For integers $h, p \geq 1$, suppose $I \subseteq V$ is an independent set of size $t > p \cdot (hk)^p$, such that $|N_G(v)| \leq h$ for all $v \in I$. Then either there is a nice $U \subseteq I$ or G contains a $K_{p,p}$. Furthermore, having G and I , we can find a nice subset or a $K_{p,p}$ in polynomial time.*

Proof. First, we show by induction that for each $0 \leq i \leq p$, either (a) there is a nice subset $U \subseteq I$, or (b) there is a $U_i \subseteq I$ of size $t_i > p \cdot (hk)^{p-i}$ such that $Q_i = \bigcap_{u \in U_i} N_G(u)$ has size $\geq i$.

For $i = 0$, clearly $U_0 = I$. If $i \geq 1$, by induction we know one of (a) or (b) is true for $i - 1$. If (a) is true, then we are done. So there is a $U_{i-1} \subseteq I$ with conditions as mentioned earlier. If there was a vertex $v \in V \setminus Q_{i-1}$ with $> p \cdot (hk)^{p-i}$ neighbors in U_{i-1} , let U_i be $U_{i-1} \cap N_G(v)$ and (b) will be true for i . Otherwise, all vertices in $V \setminus Q_{i-1}$ have $\leq p \cdot (hk)^{p-i}$ neighbors in U_{i-1} , and we do the following:

As long as there is an unmarked vertex in U_{i-1} , we pick an unmarked vertex $u \in U_{i-1}$ and mark all vertices in U_{i-1} that have a neighbor in $N_G(u) \setminus Q_{i-1}$.

Since $N_G(u) \leq h$ and each vertex in $V \setminus Q_{i-1}$ has $\leq p \cdot (hk)^{p-i}$ neighbors in U_{i-1} , at most $ph^{p-i+1}k^{p-i}$ vertices would get marked after picking u . Therefore, we would pick at least $\frac{|U_{i-1}|}{ph^{p-i+1}k^{p-i}} > \frac{p \cdot (hk)^{p-i+1}}{ph^{p-i+1}k^{p-i}} = k$ vertices. Since these vertices are independent, their number is at least $k + 1$, they are neighbors of Q_{i-1} , and they do not have common neighbors out of Q_{i-1} , each of their subsets of size $k + 1$ forms a *nice* subset and (a) will be true.

If $i = p$, the above proposition implies that either there is a *nice* subset $U \subseteq I$ or a $K_{p,p}$. In the same way as the induction, we also can construct U_i and Q_i using U_{i-1} and Q_{i-1} . This is easily doable by checking all vertices in $V \setminus Q_{i-1}$ to see whether they have $> p \cdot (hk)^{p-i}$ neighbors in U_{i-1} . If we could not find such a vertex, then we can find a *nice* subset like the induction by marking vertices. If we could construct all U_i s, then we can easily find a $K_{p,p}$ in the induced subgraph of $(U_p \cup Q_p)$. \square

Lemma 4.2. *Let $G = (V, E)$ be a d -degenerate graph with possible self-loops. Then there are $\geq \frac{n}{2d+1}$ vertices v with $|N_G(v)| \leq 2d$.*

Proof. Note $\sum_{v \in V} |N_G(v)| = 2(m - \sum_{v \in V} L_G(v)) \leq 2nd$ that the inequality is by Proposition 3. Suppose number of vertices like v with $|N_G(v)| > 2d$ is t . Then we have:

$$t(2d + 1) \leq \sum_{v \in V} |N_G(v)| \leq 2nd$$

This implies that

$$t \leq \frac{2nd}{2d + 1}, \text{ and}$$

$$n - t \geq \frac{n}{2d + 1}$$

This completes the proof of the lemma. \square

Lemma 4.3. *Any d -degenerate graph $G = (V, E)$ with possible self-loops has an independent set I with size $\geq \frac{n}{(d+1)(2d+1)}$ such that $|N_G(v)| \leq 2d$ for all $v \in I$ and one can find such an independent set in time $(n + m)^{O(1)}$.*

Proof. First, construct a $(d + 1)$ -coloring for V in $n^{O(1)}$ using Proposition 4. By Lemma 4.2 there are $\geq \frac{n}{(2d+1)}$ vertices with $|N_G| \leq 2d$ and therefore, there are $\geq \frac{n}{(2d+1)(d+1)}$ vertices with $|N_G| \leq 2d$ and the same color, which means they form an independent set. \square

Now, we are ready to describe the kernel. As long as, $n > p(d + 1)(2d + 1)(2dk)^p$, we apply the following reduction rule.

Reduction PVC 1. Use Lemma 4.3 to find an independent set I of size $\geq \frac{n}{(d+1)(2d+1)} > p \cdot (2dk)^p$ such that $|N_G(v)| \leq 2d$ for all $v \in I$. Then, since the given graph G does not contain any $K_{p,p}$, by setting $h = 2d$ and using Lemma 4.1, find a nice subset $U \subseteq I$. Then remove $u \in U$ that minimizes $|N_G(u)| + L_G(u)$ and add a self-loop to each vertex of $N_G(u)$.

To show the soundness of the reduction rule, we prove the following lemma.

Lemma 4.4. Suppose $G = (V, E)$ is a graph with possible self-loops, and $U \subseteq V$ is nice. Then, for any $u \in U$ with the minimum $|N_G(u)| + L_G(u)$, there is a solution for PVC which does not contain u .

Proof. Consider any solution S containing u . Since $|S| = k$, there is a $u' \in U$ such that (i) $(\{u'\} \cup PV_U(u')) \cap S = \emptyset$. Therefore, we have:

$$\begin{aligned} E_G(S \setminus \{u\} \cup \{u'\}) &\geq E_G(S) - (|N_G(u)| + L_G(u)) + (|N_G(u')| + L_G(u')) \\ &\hspace{15em} \text{(by (i))} \\ &\geq E_G(S) \quad (\text{since } |N_G(u)| + L_G(u) \leq |N_G(u')| + L_G(u')) \end{aligned}$$

This implies that $S \setminus \{u\} \cup \{u'\}$ is a solution that does not contain u . □

We finally prove Theorem 1.3. For convenience, we restate the theorem here.

Theorem 1.3. Given a d -degenerate graph $G = (V, E)$ that does not contain any $K_{p,p}$ as a subgraph, and an integer $k > 0$, there is a polynomial-time algorithm that outputs a subgraph $H = (V' \subseteq V, E' \subseteq E)$ of G with $O(pd^2(2dk)^p)$ vertices and a weight function $\rho: V' \rightarrow \{0, \dots, 2^{dk}\}$ on the vertex set V' with the following properties.

- For any vertex subset $S' \subseteq V' \subseteq V$, $E_G(S')$ is equal to $E_H(S') + \sum_{v \in S'} \rho(v)$.
- Let S be a partial vertex cover of size k in G covering at least t edges. Then there is a vertex set $S' \subseteq V'$ of size k such that $E_H(S') + \sum_{v \in S'} \rho(v)$ is at least t .

Proof. The running time of the described algorithm is polynomial by Lemma 4.1 and 4.3, and the reduction rule is safe by Lemma 4.4. The number of vertices in the kernel is $\leq p(d + 1)(2d + 1)(2dk)^p$, which is $O(pd^2(2dk)^p)$. Although the number of self-loops may be large, notice that the number of self-loops on a vertex will be at most n . We may remove the self-loops and add it as a weight on the vertex. Thus, each weight can be represented using at most $\log n$ bits. Since we have an algorithm for the problem with running time $2^{O(kd)}n^{O(1)}$, i.e., Theorem 1.1, when $kd \leq \log n$ the algorithm runs in polynomial time and thereby, it would be a compression itself. Otherwise, we have that $\log n \leq kd$ that guarantees the weight function ρ mentioned in the theorem statement. □

5 Conclusion

In this work we gave a single exponential parameterized algorithm and a polynomial compression for PVC on graphs of bounded degeneracy that include many sparse graph classes like planar graphs and H -minor free graphs. Is it possible to get similar results on biclique free graphs, a superclass of bounded degeneracy graphs? Notice that there is a linear kernel for DOMINATING SET on planar graphs, H -minor free graphs, and apex-minor free graphs [3, 7]. Can we get a linear kernel or compression for PVC on planar graphs?

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