Analysis of Open Queueing Networks with Batch Services

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Abstract. In this work we analyze an open queueing network with batch services. In more detail, the arrival process is Poissonian and each node consists of a single server and an infinite waiting queue. Arrivals are served in fixed-size batches: if the number of customers in a node is less than the predefined batch size, the server remains idle, otherwise he will select the required number of customers, which then will be served as a unique batch with exponentially distributed service time. In this paper we show that, under suitable conditions on the routing matrix, such queueing network is equivalent, in terms of stationary distribution, to a Jackson network with single-server nodes and state-dependent service rates. Finally, the goodness of the proposed approach is confirmed by comparing analytical and simulation results.

Keywords: Open queueing networks · Analysis · Batch service

1 Introduction

Queueing systems and networks with batch services attract the interest of many researchers, since they permit to model and analyze various multi-user systems [1,2], large scale semiconductor manufacturing systems [3], cloud computing systems [4] and wireless sensor networks [5].

The analysis of any queueing network is aimed at obtaining expressions for its stationary characteristics, the most important of which is the stationary probability distribution of the states of the system. Since the equilibrium equations for queuing networks with batch services have a high dimensionality, the calculation of the stationary distribution as a numerical solution of these equations is computationally difficult. Therefore, special attention has been devoted to the search for product-form solutions.

It is worth noticing that the fundamental works on queueing networks with batch services are relatively recent, as they were published in 1990 [6,7]. In more

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detail, in [6] a continuous-time Markov chain is introduced to model queueing networks with simultaneous changes due to batch services, or discrete-time structure and clustering processes such as those arising in polymer chemistry. It is shown that if multiple instantaneous state transitions of the process are allowed and the Markov chain is reversible, then its stationary distribution has a product-form. In [7] a discrete-time closed queueing network with batch services is considered and the state of the network is defined by a vector with dimension equal to the number of customers. Each element of the status vector is associated with a specific customer and indicates the node occupied by that customer. So, customers transitions are reduced to changes of the corresponding labels, and it is assumed that the change of a label does not depend on the status of the labels of the other customers. It is shown that on an irreducible set of states and for arbitrary given functions of service and routing, there is a product-form for the stationary probability distribution of the queueing network states. Chao [8] and Economou [9] considered networks, for which the quasi-reversibility conditions are met and the groups of customers at the end of the service in one node always pass to another node together.

To analyze queuing networks with batch services and an arbitrary distribution of the service time that do not admit a product-form of the stationary distribution, in [3,10] it was proposed to use the decomposition method. Finally, in [11,12] the stationary distribution was calculated as the normalized solution of the system of equilibrium equations.

In this paper, we consider open queueing networks with service of fixed-size batches of customers and independent routing. It is assumed that the batch size is significantly smaller than the number of nodes to which the customers can be routed at the end of the service. Thus, the network nodes work independently and this consideration permits to simplify the analysis of the queueing network, which is reduced to the investigation of the individual queues in isolation. In more detail it is proposed to calculate the stationary state probability distribution of the open network in a product-form, similar to the case of birth-death processes after recalculating the transition rates. To the best of our knowledge, this approach is new. Until now, indeed, the probability generating function [4,13–15], the Laplace-Stieltjes transform [16], and the direct calculation of the stationary distribution as a solution of the Kolmogorov equations [17] have been mainly used to calculate the stationary characteristics of the queueing network.

The rest of the paper is organized as follows. Section 2 introduces the model of the queueing network, while in Sect. 3 an equivalent (in terms of stationary distribution) Jackson network with single-server nodes is proposed. In more detail, for such equivalent system state-dependent service rates as well as expressions for the stationary probability distribution are derived. Then, Sect. 4 compares the values of the analytical expression with the simulation results, and analyses the dependence of the characteristics of open queueing networks on different system parameters (batch size, arrival rate, service rate).

2 Statement of the Problem

Consider a continuous-time open queueing network N consisting of L nodes S_i, $i \in I$, $I = \{1, \dots, L\}$. Customers arrive to the queueing network N from an outside source (denoted in the following as S_0) according to a Poisson stream of rate λ_0. Customer transitions between nodes and the source are defined by the routing matrix $\Theta = (\theta_{ij})$, $i, j = 0, \dots, L$, where θ_{ij} is the transition probability from node S_i to node S_j. The state of the network is defined by a vector $s = (s_1, \dots, s_L)$, where s_i is the number of customers at node S_i. Denote by $X = \{s : s_i \geq 0\}$ the state space of the queueing network N.

Each node S_i, $i = 1, \dots, L$, operates as an infinite capacity single-server queue. Arriving customers are placed in the waiting queue if the server is busy. Customers are served in batches, and let b_i be the customer batch size for node S_i. The server remains idle until the required number b_i of customers arrives at the node and then the service of the batch starts immediately; otherwise, b_i customers are selected in any order for service, while the others remain in the queue. The service times of batches at node S_i are exponentially distributed with parameter μ_i, $i = 1, \dots, L$. After a batch finishes its service at node S_i, each customer will go, independently of the others, to node S_j with probability θ_{ij}, $i, j = 0, 1, \dots, L$.

Our aim is to find the stationary distribution $\pi(s) = (\pi_1(s_1), \dots, \pi_L(s_L))$, $s \in X$, for the queueing network N, where $\pi_i(s_i)$ represents the stationary distribution for node S_i, $s_i = 0, 1, \dots$, $i = 1, \dots, L$, starting from the analysis of a single node.

3 Analysis of the Model

In this paper we analyze large scale networks with individual routing of the customers, assuming that the number of possible destinations is significantly larger than the batch size. Hence the probability of the simultaneous arrival of two or more customers in a node can be neglected. Therefore, we will assume that each node in N is fed by a Poisson stream of customers.

First we will study the isolated node S_i, $i = 1, \dots, L$. It is known that the equilibrium equations for this node have the form

$$\begin{cases} \lambda_i \pi_i(n) = \mu_i \pi_i(b_i), \ n = 0, \\ \lambda_i \pi_i(n) = \lambda_i \pi_i(n-1) + \mu_i \pi_i(b_i + n), \ 1 \leq n \leq b_i - 1, \\ (\lambda_i + \mu_i) \pi_i(n) = \lambda_i \pi_i(n-1) + \mu_i \pi_i(b_i + n), \ n \geq b_i. \end{cases} \tag{1}$$

where λ_i denotes the arrival rate to node S_i, $i = 1, \dots, L$.

We define a birth-death process ξ_i, which will be equivalent in steady-state probabilities to the Markov process describing the node S_i. Let the process ξ_i be defined on a set of states $\{0, 1, \dots\}$, let $\lambda_i = \lambda_i(n)$ be the transition rate of the process ξ_i from state n to state $n+1$, which does not depend on the state n, $n \in \{0, 1, \dots\}$, and let $\widetilde{\mu}_i(n)$ be the transition rate of the process ξ_i from state

n to state $n-1$, where $n \in \{1, 2, \dots\}$. The states $\{0, 1, \dots\}$ and the parameter λ_i of the process ξ_i correspond to the states $\{0, 1, \dots\}$ and the parameter λ_i of node S_i. Let us find the rates $\widetilde{\mu}_i(n)$, $n = 1, 2, \dots$. To this aim, note that the steady-state probabilities of the birth–death process ξ_i are given by [18]

$$\pi_i(k) = \pi_i(0) \prod_{n=1}^{k} \frac{\lambda_i}{\widetilde{\mu}_i(n)}, \; k = 1, 2, \dots, \tag{2}$$

where

$$\pi_i(0) = \left(1 + \sum_{k=1}^{\infty} \prod_{n=1}^{k} \frac{\lambda_i}{\widetilde{\mu}_i(n)}\right)^{-1}, \; i = 1, \dots, L.$$

By substituting (2) in (1), we get the expressions that define $\widetilde{\mu}_i(n)$, $n = 1, 2, \dots$,

$$\begin{cases} \widetilde{\mu}_i(n) = \lambda_i - \mu_i \dfrac{\lambda_i^{b_i}}{\widetilde{\mu}_i(n+1) \cdot \ldots \cdot \widetilde{\mu}_i(b_i+n)}, \; 1 \le n \le b_i - 1, \\ \widetilde{\mu}_i(n) = \lambda_i + \mu_i - \mu_i \dfrac{\lambda_i^{b_i}}{\widetilde{\mu}_i(n+1) \cdot \ldots \cdot \widetilde{\mu}_i(b_i+n)}, \; n \ge b_i. \end{cases} \tag{3}$$

Let $M_i = \lim_{n \to \infty} \widetilde{\mu}_i(n)$; if the limit exists, then:

$$\mu_i \lambda_i^{b_i} = (\lambda_i + \mu_i - M_i) M_i^{b_i}$$

or

$$M_i^{b_i+1} - (\lambda_i + \mu_i) M_i^{b_i} + \lambda_i^{b_i} \mu_i = 0. \tag{4}$$

The existence of the equivalent birth-death process ξ_i requires that the previous equation has a positive solution, fulfilling the stability condition for each node S_i.

The answer is provided by the following theorem (without loss of generality we denote the generic M_i, for $i \in I$ by x).

Theorem 1. *The equation*

$$x^{b+1} - (\lambda + \mu) x^b + \lambda^b \mu = 0 \tag{5}$$

has two positive roots, the largest of which belongs to the interval

$$\left(\frac{b(\lambda + \mu)}{b+1}, \frac{(\lambda + \mu)^{b+1} - \lambda^b \mu}{(\lambda + \mu)^b} \right).$$

Proof. Consider the function

$$f(x) = x^{b+1} - (\lambda + \mu) x^b + \lambda^b \mu$$

for $\lambda < b\mu$ and $b \ge 1$.

It is easy to verify that $f(x)$ is continuous for any $x \in R$ and $x_1 = \lambda$ is a root of $f(x)$. To determine the existence of other roots let us consider the first derivative of $f(x)$:

$$f'(x) = (b+1)x^{b-1}\left(x - \frac{b}{b+1}(\lambda+\mu)\right). \tag{6}$$

The equation $f'(x) = 0$ has only one positive root

$$x^* = \frac{b(\lambda+\mu)}{b+1},$$

with $x^* > x_1$. Indeed,

$$x^* - x_1 = \frac{b}{b+1}(\lambda+\mu) - \lambda = \frac{b\mu - \lambda}{b+1} > 0,$$

since $\lambda < b\mu$ and $b \geq 1$. Since $f'(x) > 0$ for

$$x \in \left(\frac{b(\lambda+\mu)}{b+1}, \lambda+\mu\right),$$

then the function $f(x)$ is increasing in such interval. Moreover,

$$f\left(\frac{b(\lambda+\mu)}{b+1}\right) < 0$$

and $f(\lambda+\mu) > 0$, hence in the interval $\left(\frac{b(\lambda+\mu)}{b+1}, \lambda+\mu\right)$ there is a value of x such that $f(x) = 0$.

To further refine the estimation of the root, let us note that in the above-mentioned interval the function $f(x)$ is convex, since

$$f''(x) = bx^{b-2}((b+1)x - (b-1)(\lambda+\mu)) > 0$$

for

$$x > \frac{b(\lambda+\mu)}{b+1} > \frac{(b-1)(\lambda+\mu)}{b+1}.$$

The tangent line to $f(x)$ at the point $x = \lambda + \mu$ is

$$y(x) = \lambda^b\mu + (\lambda+\mu)^b(x - (\lambda+\mu))$$

and its intersection with the horizontal axis is

$$x_0 = (\lambda+\mu) - \frac{\lambda^b\mu}{(\lambda+\mu)^b}.$$

Since the function $f(x)$ is convex, x_0 is an upper bound for the roots of $f(x)$, and this implies that the largest root of Eq. (5) belongs to the interval

$$\left(\frac{b(\lambda+\mu)}{b+1}, \frac{(\lambda+\mu)^{b+1} - \lambda^b\mu}{(\lambda+\mu)^b}\right).$$

Taking into account the previous theorem and the stability condition of the equivalent birth-death process, Eq. (4) has a unique root, located in the interval $(\lambda_i, \lambda_i + \mu_i)$, which can be determined numerically (explicit closed-for solutions can be easily derived only for $b = 1$ and $b = 2$). From the system of Eqs. (3) it follows

$$\widetilde{\mu}_i(b_i) = \widetilde{\mu}_i(b_i + 1) = \widetilde{\mu}_i(b_i + 2) = \cdots = M_i,$$

and then the service rates $\widetilde{\mu}_i(b_i - 1)$, $\widetilde{\mu}_i(b_i - 2)$,..., $\widetilde{\mu}_i(1)$ can be easily calculated. Thus, the rates $\widetilde{\mu}_i(n)$ are determined for each state n of process ξ_i.

The results obtained for the process ξ_i can be applied to any node, and so we can create an open queueing network $\widetilde{N}$ with nodes $\widetilde{S}_i$ and service rates $\widetilde{\mu}_i(n)$, where n is the number of customers in the node $\widetilde{S}_i$, $n = 1, 2, \ldots,$ $i = 1, \ldots, L$. The other parameters of $\widetilde{N}$ coincide with the corresponding parameters of the original queueing network N.

$\widetilde{N}$ is equivalent in stationary distribution to the queueing network N with batch services and is a Jackson network.

The arrival rates in nodes S_i are determined by the following equations

$$\lambda_i = \frac{\omega_i}{\omega_0}\lambda_0,\ i = 1, \ldots, L,$$

where the vector of visitation rates $\omega = (\omega_1, \ldots, \omega_L)$ is the solution of the equation $\omega\Theta = \omega$ with the normalization condition $\sum_{i=0}^{L} \omega_i = 1$.

The queueing network N and its equivalent network $\widetilde{N}$ are stable if the utilization coefficient in the node S_i, $i = 1, \ldots, L$,

$$\rho_i = \frac{\lambda_i}{b_i\mu_i} < 1,$$

and, under such conditions, we can compute the stationary distribution for $\widetilde{N}$. We obtain

$$\pi(s) = \prod_{i=1}^{L} \pi_i(s_i),\ s \in X,$$

where

$$\pi_i(s_i) = \pi_i(0) \prod_{n=1}^{s_i} \frac{\lambda_i}{\widetilde{\mu}_i(n)}.$$

Then, the average number of customers in the node S_i, $i = 1, \ldots, L$, is given by

$$\bar{s}_i = \sum_{n=1}^{\infty} n\pi_i(n),$$

the average sojourn time in the node S_i, $i = 1, \ldots, L$, is

$$\bar{u}_i = \frac{\bar{s}_i}{\lambda_i},$$

and the average response time of the queueing network is

$$\bar{\tau} = \frac{1}{\lambda_0} \sum_{i=1}^{L} \lambda_i \bar{u}_i.$$

4 Numerical Examples

Numerical examples are reported in this section to verify the goodness of the product-form approximation for complex networks and investigate the dependence of their characteristics on different system parameters (batch size, arrival rate, service rate). Although different topologies have been investigated, for sake of brevity just one network topology is considered, focusing on overall system performance parameters as well as on characteristics of single queues.

Consider the queueing network N with the following parameters (unless otherwise stated): $L = 14$, $b = (3,\ 2,\ 2,\ 3,\ 2,\ 2,\ 3,\ 3,\ 2,\ 2,\ 3,\ 2,\ 2,\ 3)$, $\mu = (0.8,\ 0.6,\ 0.9,\ 0.6,\ 0.8,\ 0.8,\ 0.9,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 0.7,\ 0.7)$, and

$$\Theta = \begin{pmatrix}
0.0 & 0.3 & 0.4 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.2 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.2 & 0.1 & 0.2 & 0.1 & 0.1 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.2 & 0.2 & 0.1 \\
0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.2 \\
0.2 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.2 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 \\
0.2 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.1 \\
0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 \\
0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.0 \\
0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 \\
0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.0 \\
0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0 & 0.0 & 0.1 & 0.1 & 0.0 & 0.1 \\
0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.0
\end{pmatrix}.$$

The considered network satisfies the assumptions introduced above. Indeed, the network consists of a relatively large number of nodes, the size of the batches that are served together is significantly less than the number of possible output nodes and the routing probabilities are of the same order of magnitude (there is no privileged path through the network). Hence, the Poissonian assumption can be reasonably assumed for any node of the network.

The first two sets of tests investigated the accuracy of the developed method by comparing the analytical values with the results of discrete-event simulation. In more detail, in the first experiment we analysed the (overall) average response time as a function of the input rate λ_0.

Table 1 shows that the largest difference in the values of $\bar{\tau}$ is observed for $\lambda_0 = 0.1$ and does not exceed 10.2%, while for the other values of λ_0, the

Table 1. Average response time of the queueing network.

λ_0	0.1	0.5	1.0	1.5	2.0	2.5	2.7
Approximation	107.32	27.57	18.29	16.20	16.76	20.62	24.06
Simulation	118.25	28.78	18.61	16.35	16.86	20.82	25.24

deviation is no more than 5%. Note that the intensity of the flow $\lambda_0 = 2.7$ is almost the maximum for the network under consideration, since for such value the stability condition for node S_9 is still met.

In the second example we focused on a specific node (the queue S_7), considering the average number of customers (Tables 2) as well as the average sojourn time in the node (Tables 3) for different values of the service rate μ_7 with fixed arrival rate $\lambda_0 = 1.5$.

Table 2. Average number of customers in the node S_7.

μ_7	0.2	0.4	0.6	0.8	0.9	1.0	1.2
Approximation	8.72	2.53	1.89	1.63	1.56	1.5	1.41
Simulation	8.66	2.53	1.91	1.66	1.58	1.52	1.43

Table 3. Average sojourn time in the node S_7.

μ_7	0.2	0.4	0.6	0.8	0.9	1.0	1.2
Approximation	18.26	5.31	3.95	3.42	3.26	3.13	2.95
Simulation	18.13	5.30	3.99	3.47	3.31	3.18	3.00

The characteristics of the node S_7, derived by discrete-event simulation, were calculated in stationary conditions with a confidence interval of 0.001 and a confidence level higher than 0.95.

In the third experiment we investigated the dependence of the stationary characteristics of the nodes S_2, S_6 and S_{11} on the intensity of the incoming flow λ_0 (see Fig. 1 and 2). The characteristics of the other nodes are not shown in the graphs for sake of clarity, since their behavior does not differ qualitatively form the reported ones.

Figure 1 shows that the average number of customers in all systems monotonically increases with λ_0. Instead, the average (node) sojourn time reaches a minimum for some value of λ_0 as highlighted by Fig. 2. This can be explained as follows. When λ_0 is close to zero, the device is idle for a long time, and the customers forming an "incomplete" batch have to wait in the buffer until

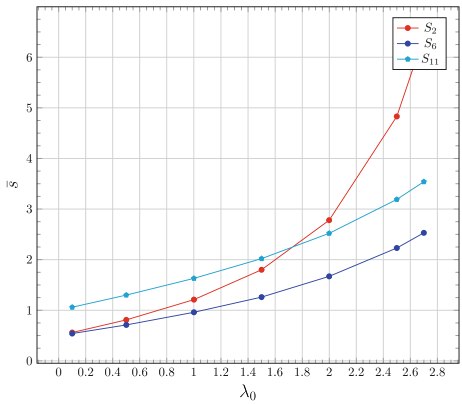

Fig. 1. Average number of customers in the nodes S_2, S_6 and S_{11}.

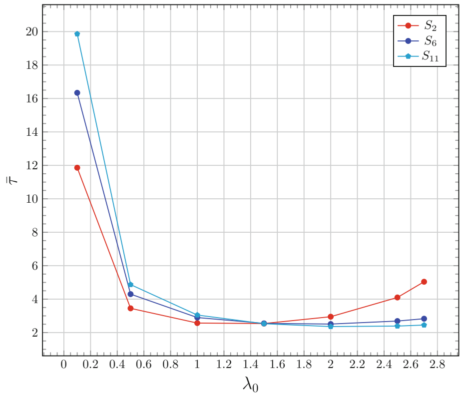

Fig. 2. Average sojourn time in the nodes S_2, S_6 and S_{11}.

the last element of the batch enters the system. Instead, when the arrival rate into the considered system approaches its service rate, the average waiting time increases significantly. Thus, there is an optimal value of the arrival rate, at which the average sojourn time in the node is minimal.

The fourth experiment is devoted to the study of stationary characteristics of the nodes S_7 and S_9 for different sizes b of the batch in these systems. The input rate in this experiment is again $\lambda_0 = 1.5$.

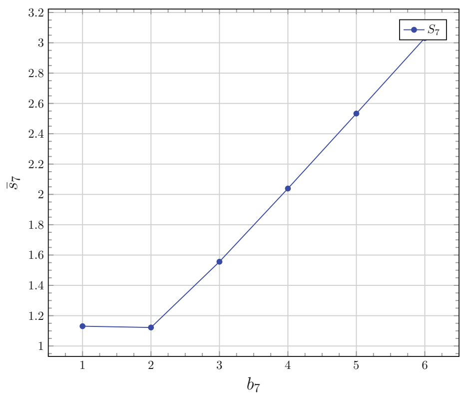

Fig. 3. Average number of customers in the node S_7.

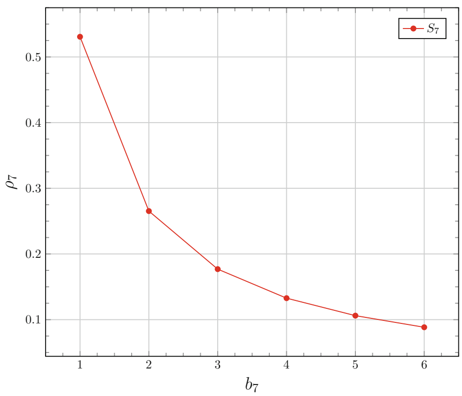

Fig. 4. Utilization coefficient of the node S_7.

The minimum value of the average number of customers in both nodes is achieved when the batch size is two (Fig. 3 and 5), while the utilization coefficient is a monotone decreasing function of b (Fig. 4 and 6), but its numerical value

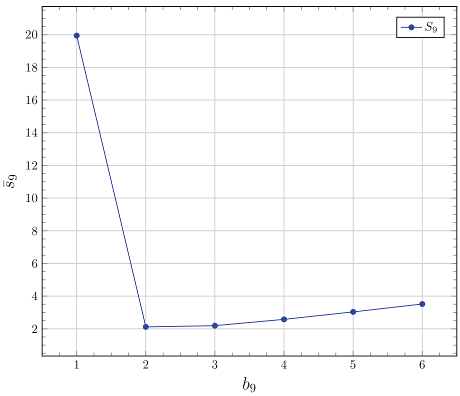

Fig. 5. Average number of customers in the node S_9.

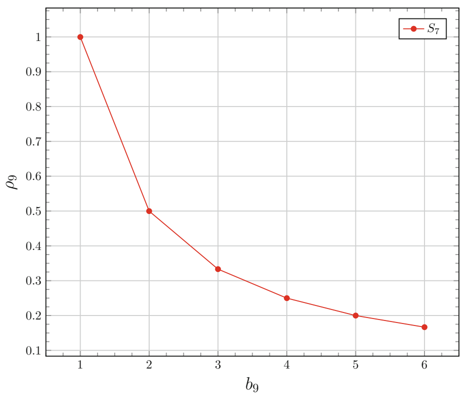

Fig. 6. Utilization coefficient of the node S_9.

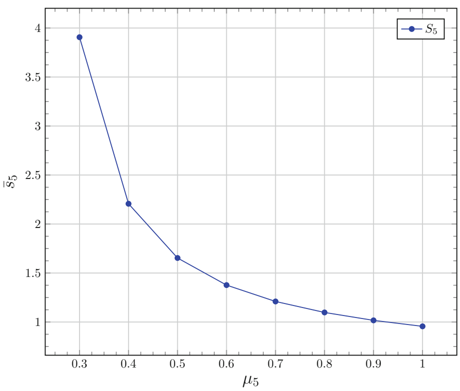

Fig. 7. Average number of customers in the node S_5.

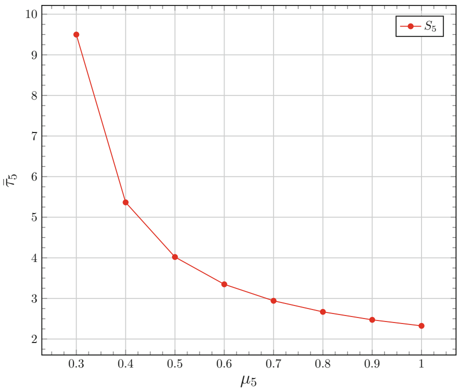

Fig. 8. Average sojourn time in the node S_5.

depends on the arrival rate at the considered node (in our example ρ_9 is almost twice ρ_7). It is worth noticing (see Fig. 6) that for $b_9 = 1$, the utilization ρ_9 of the node S_9 is close to 1 and this is confirmed by the high value of the number of customers in the system ($\bar{s}_9 \approx 20$ as shown in Fig. 5). When $b_9 = 2$, then $\bar{s}_9 \approx 2$, while the increment of b_9 leads to a slight increase in $\bar{s}_9$. Thus, the increase of the batch size can significantly improve the basic average characteristics of service systems. Actually, as shown by numerical experiments, the minimum value of both the average sojourn time and average number of customers in the system can be assumed at different values b, depending on the network topology and the routing matrix.

Finally, we calculated the stationary characteristics of the nodes S_5 and S_9 for different values of the service rate in these systems (assuming, as before, $\lambda_0 = 1.5$).

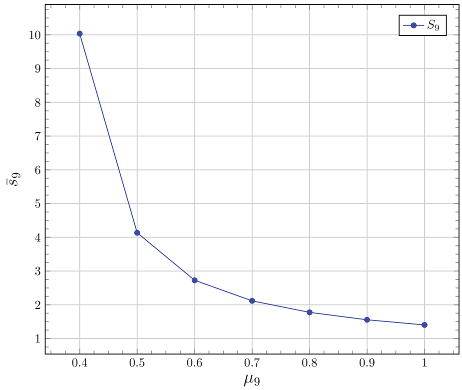

Fig. 9. Average number of customers in the node S_9.

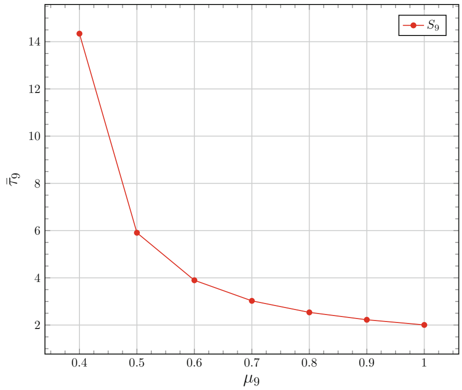

Fig. 10. Average sojourn time in the node S_9.

The graphs shown in Fig. 7, 8, 9 and 10 decrease monotonically with the growth of μ and asymptotically tend to their limit values.

5 Conclusions

In this paper large-size open queueing networks with batch services are considered. Under the assumption that the number of output nodes is significantly more than the batch size, it is shown that the stationary distribution of the queueing network can be expressed in product-form. Then, the parameters of the equivalent queueing network are derived and the goodness of the approximation is verified by means of discrete-event simulation.

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