Markovian Models of Queuing Systems with Positive and Negative Replenishment Policies

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Abstract. The Markov models of queuing-inventory systems with infinite buffer were analyzed under different replenishment policies. Besides traditional positive replenishment, the negative replenishment is considered after which inventory level instantly decreases. Some customers are assumed to leave the system without acquiring an item after the service completion. The ergodicity conditions of the introduced systems, as well as, formulas for stationary distributions and performance measures were developed. Total cost minimization problems were solved for the different replenishment policies.

Keywords: Queuing-inventory systems · Markovian models · Positive and negative replenishment · Matrix-geomteric method

1 Introduction

Systems where the serving process consists of releasing (selling) resource units to incoming customers are called Queuing-Inventory Systems (QIS) [1]. The reason for that naming is that such systems have properties both of Queuing and Inventory systems. First papers on this subject are known to be [2,3]. QIS subject has been widely studying by different authors during last three decades. The current state of QIS theory and its applications were extensively discussed in review paper [4].

In the most papers on QIS the replenishment is assumed to be positive, that is upon its completion the inventory goes up by the given positive amount that is defined by the accepted policy. But in practice, due to different reasons (technical errors, human errors, etc.) the inventory level may immediately decrease. We call such QIS with negative replenishment (like in case with negative customers). To our best knowledge, this kind of models were not studied in the available literature.

It should be noted that these models look similar to QIS models with perishable inventory. But the main difference is that in latter models items perish

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after some time and inventory goes down, while in our models inventory level decreases immediately due to negative replenishment. So in our paper we introduce separate class of QIS models with positive and negative replenishment.

2 Model Description

We consider Markov models of QIS system with one server under one of the three replenishment policies: (s, S), (s, Q), $(S-1, S)$. Besides the traditional positive replenishment, we assume negative replenishment after which inventory level instantly decreases due to unexpected events. The negative replenishment events are described by Poisson point process with parameter κ. We assume that negative replenishment affect the items reserved for service as well. In each policy lead time is exponentially distributed with average ν^{-1}.

The customer income in all models are described by Poisson process with intensity λ. We assume that all customers require the identical item amount.

The customers are accepted for service if upon arrival the server is idle and inventory level is positive, otherwise customer joins the unlimited queue. Customers are assumed to join queue even if the inventory level is 0, i.e. according to Bernoulli scheme customer joins queue with probability ϕ_1 or leaves the system with complementary probability ϕ_2, where $\phi_1 + \phi_2 = 1$.

Customers in queue are considered impatient, when inventory level drops down to zero, customers leave the system independently after randomly distributed time that has exponential distribution with parameter τ^{-1}.

After the service completion customer according to Bernoulli scheme either acquires the item with probability σ_1 or leaves the system empty handed with probability σ_2, where $\sigma_1 + \sigma_2 = 1$. Average service times for both cases have exponential distribution with averages μ_1 and μ_2 accordingly.

3 Calculation of Stationary Distributions Under the Different Replenishment Policies

First let's consider the system under the (s, S) replenishment policy. The system is described with Two Dimensional Markov Chain, (2-D MC) with state vectors (m, n), where n represents the number of customers in the queue, $n = 0, 1, 2, ...$, while m represents the inventory level, $m = 0, 1, ..., S$. The state space is defined as follows:

$$E = \bigcup_{n=0}^{\infty} L(n)$$

where $L(n) = \{(n, 0), (n, 1), ..., (n, S)\}$ called the n^{th} level, $n = 0, 1, 2, \ldots$.

Let's rearrange state space E in lexicographical order as follows

$$(0, 0), (0, 1), ..., (0, S), (1, 0), (1, 1), ..., (1, S), ...$$

In that case we obtain Level Independent Quasi-Birth-Death Process (LIQBD) with the following generator:

$$G = \begin{pmatrix} B & A_0 & . & . & . \\ A_2 & A_1 & A_0 & . & . \\ . & A_2 & A_1 & A_0 & . \\ . & . & . & . & . \end{pmatrix} \tag{1}$$

All block matrices in (1) are square matrices of dimension $S+1$ and their elements $B = ||b_{ij}||$ and $A_k = ||a_{ij}^{(k)}||, i, j = 0, 1, ..., S$ are calculated as follows:

$$b_{ij} = \begin{cases} \nu, & \text{if } i \le s, j = S \\ \kappa & \text{if } i > s, j = i-1 \\ -(\nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -(\nu + \kappa + \lambda), & \text{if } 0 < i \le s, j = i \\ -(\kappa + \lambda), & \text{if } s < i \le S, j = i \\ 0, & \text{in other cases} \end{cases} \tag{2}$$

$$a_{ij}^{(0)} = \begin{cases} \lambda\phi_1, & \text{if } i = j = 0 \\ \lambda, & \text{if } i \ne 0, i = j \\ 0, & \text{in other cases} \end{cases} \tag{3}$$

$$a_{ij}^{(1)} = \begin{cases} \nu, & \text{if } 0 \le i \le s, j = S \\ \kappa, & \text{if } i > 0, j = i-1 \\ -(\tau + \nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -(\nu + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } 0 < i, j = i \\ 0, & \text{in other cases} \end{cases} \tag{4}$$

$$a_{ij}^{(2)} = \begin{cases} \tau & \text{if } i = j = 0 \\ \mu_1\sigma_1, & \text{if } i \ne 0, i = j \\ \mu_2\sigma_2, & \text{if } i > 0, j = i-1 \\ 0, & \text{in other cases} \end{cases} \tag{5}$$

Theorem 1. *Under the (s, S) replenishment policy system is ergodic if and only if the following inequality holds true:*

$$\lambda(1 - (1 - \phi_1)\pi(0)) < \tau\pi(0) + (\mu_1\sigma_1 + \mu_2\sigma_2)(1 - \pi(0)) \tag{6}$$

where

$$\pi(0) = \left(1 + (1 + a^{-1})((1 + a)^{s+1} - 1) + (S - s - 1)(1 + a)\right)^{-1},$$

$$a = \frac{\nu}{\mu_2\sigma_2 + \kappa}.$$

Proof. Let's designate stationary distribution corresponding to the generator $A = A_0 + A_1 + A_2$ by $\pi = (\pi(0), \pi(1), ..., \pi(S))$. These variables satisfies the following system of equations:

$$\pi A = 0, \pi e = 1 \tag{7}$$

where 0 is null row vector of dimension $S+1$ and e is column vector of dimension $S+1$ that contains only 1's. $\pi(m), m = 0, 1, ..., S$ is the probability of the state with inventory level equal to $m, m = 0, 1, ..., S$.

We conclude from (3)–(5) that elements of generator $A = ||a_{ij}||i, j = 0, 1, ..., S$ are calculated as follows:

$$a_{ij} = \begin{cases} -\nu, & \text{if } i = j = 0 \\ \nu, & \text{if } 0 \le i \le s, j = S \\ \mu_2\sigma_2 + \kappa & \text{if } i > 0, j = i - 1 \\ -(\mu_2\sigma_2 + \kappa + \nu), & \text{if } 0 < i \le s, j = i \\ -(\mu_2\sigma_2 + \kappa), & \text{if } i > s, j = i \\ 0, & \text{in other cases} \end{cases} \tag{8}$$

We conclude from (8) that system of linear equations (7) gets the following form:

$$(\nu + (\kappa + \mu_2\sigma_2)(1 - \delta_{m,0}))\pi(m) = (\kappa + \mu_2\sigma_2)\pi(m + 1), 0 \le m \le s; \tag{9}$$

$$(\kappa+\mu_2\sigma_2)\pi(m) = (\kappa+\mu_2\sigma_2)\pi(m+1)(1-\delta_{m,S})+\nu\sum_{i=0}^{S}\pi(i)\delta_{m,S}\,, s+1 \le m \le S. \tag{10}$$

Here and in later formulas $\delta_{x,y}$ designates Kronecker symbols.

$$a_{ij} = \begin{cases} (1 + a)^m\pi(0), & \text{if } 1 \le m \le s + 1 \\ (1 + a)^{s+1}\pi(0), & \text{if } s + 1 < m \le S \end{cases} \tag{11}$$

where $\pi(0)$ is derived from the normalizing condition, $\pi(0)+\pi(1)+...+\pi(S) = 1$.

According [5] (Chapter 3, p. 81–83) LIQBD we are studying is ergodic iff:

$$\pi A_0 e < \pi A_2 e \tag{12}$$

Then from (3), (5) and (11) after applying some mathematical transformations we get (6) from (12).

Note 1. Ergodicity condition (6) has probabilistic meaning. Total summed intensity of incoming requests should be smaller than total summed intensity of outgoing requests. Condition (6) could be replaced with rough but easily checked condition: $\lambda < min(\tau, \mu_1\sigma_1 + \mu_2\sigma_2)$.

Let's replace stationary distribution corresponding to generator G with $p = (p_0, p_1, ...)$ where $p_n = (p(n, 0), p(n, 1), ..., p(n, S)), n = 0, 1, ...$ Assuming that ergodicity condition (6) holds true, stationary distributions may be calculated as follows:

$$p_n = p_0 R^n, n = 0, 1, ... \tag{13}$$

where R is minimal nonnegative solution of the following quadratic equation:

$$R^2 A_2 + R A_1 + A_0 = 0 \tag{14}$$

Probability of border states p_0 is calculated from the following system of equations:

$$p_0(B + RA_2) = 0 \tag{15}$$

$$p_0(I - R)^{-1} e = 1 \tag{16}$$

where I is unit matrix of size $S + 1$.

Now let's consider model with (s, Q) policy. State space of this model is also given by E, but corresponding generator matrix $\widetilde{G}$ is determined as follows:

$$\widetilde{G} = \begin{pmatrix} \widetilde{B} & A_0 & . & . & . \\ A_2 & \widetilde{A_1} & A_0 & . & . \\ . & A_2 & \widetilde{A_1} & A_0 & . \\ . & . & . & . & . \end{pmatrix}$$

where elements of matrices $\widetilde{B}$ and $\widetilde{A_1}$ are calculated as follows:

$$\widetilde{b}_{ij} = \begin{cases} \nu, & \text{if } j = i + S - s \\ \kappa, & \text{if } i > 0, j = i - 1 \\ -(\nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -(\nu + \kappa + \lambda), & \text{if } 0 < i \le s, j = i \\ -(\kappa + \lambda), & \text{if } s < i \le S, j = i \\ 0, & \text{in other cases} \end{cases} \tag{17}$$

$$\widetilde{a}_{ij}^{(1)} = \begin{cases} \nu, & \text{if } 0 \le i \le s, j = i + S - s \\ \kappa, & \text{if } i > 0, j = i - 1 \\ -(\tau + \nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -(\nu + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } 0 < i, j = i \\ 0, & \text{in other cases} \end{cases} \tag{18}$$

Theorem 2. *Under the (s, Q) replenishment policy system is ergodic if and only if the inequality (6) holds true, where*

$$\pi(0) = (1 + a)^{-(s+1)} \left(\frac{(1 + a)^{s+1} - 1}{a(1 + a)} + S - s - a^{-1}(1 - (1 + a)^{-s}) \right)^{-1}.$$

Proof. The elements of generator $\widetilde{A} = A_0 + \widetilde{A}_1 + A_2$ are calculated as follows:

$$\widetilde{a}_{ij} = \begin{cases} -\nu, & \text{if } i = j = 0 \\ \nu, & \text{if } 0 \leq i \leq s, j = i + S - s \\ \mu_2\sigma_2 + \kappa, & \text{if } i > 0, j = i - 1 \\ -(\mu_2\sigma_2 + \kappa + \nu), & \text{if } 0 < i \leq s, j = i \\ -(\mu_2\sigma_2 + \kappa), & \text{if } i > s, j = i \\ 0, & \text{in other cases} \end{cases} \tag{19}$$

We conclude from (19) that system of linear equations (7) corresponding to generator $\widetilde{A}$ has the following form:

$$(\nu + (\kappa + \mu_2\sigma_2)(1 - \delta_{m,0}))\pi(m) = (\kappa + \mu_2\sigma_2)(\pi(m+1), 0 \leq m \leq s; \tag{20}$$

$$(\kappa + \mu_2\sigma_2)\pi(m) = (\kappa + \mu_2\sigma_2)(\pi(m+1)(1 - \delta_{m,0}) + \nu\pi(m - S + s)\delta_{m,S}, s + 1 \leq m \leq S; \tag{21}$$

Then from (24) and (21) we get:

$$\pi_m = \begin{cases} (1+a)^{m-(s+1)}\pi(s+1) & \text{if } 0 \leq m \leq s \\ \pi(s+1), & \text{if } s + 1 \leq m \leq S - s \\ (1 - (1+a)^{m-(S-1)})\pi(s+1), & \text{if } S - s + 1 \leq m \leq S \end{cases} \tag{22}$$

where $\pi(s+1)$ is calculated from normalizing condition:

$$\pi(s+1) = \left(\frac{(1+a)^{s+1} - 1}{a(1+a)} + S - s - a^{-1}(1 - (1+a)^{-s})\right)^{-1}.$$

Taking into consideration (3), (5) and (22) and after applying some transformations from (12) we conclude that Theorem 2 is true.

Finally, let's consider model with $(S-1, S)$ policy. Elements for corresponding generator matrix $\widetilde{\widetilde{G}}$ is calculated as follows:

$$\widetilde{\widetilde{G}} = \begin{pmatrix} \widetilde{\widetilde{B}} & A_0 & . & . & . \\ A_2 & \widetilde{\widetilde{A_1}} & A_0 & . & . \\ . & A_2 & \widetilde{\widetilde{A_1}} & A_0 & . \\ . & . & . & . & . \end{pmatrix}$$

where elements of matrices $\widetilde{\widetilde{B}}$ and $\widetilde{\widetilde{A_1}}$ are calculated as follows:

$$\widetilde{\widetilde{b}}_{ij} = \begin{cases} (S-i)\nu, & \text{if } 0 \leq i \leq S-1, j = i+1 \\ \kappa, & \text{if } i > 0, j = i - 1 \\ -(S\nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -((S-i)\nu + \kappa + \lambda), & \text{if } 0 < i \leq s, j = i \\ 0, & \text{in other cases} \end{cases}$$

$$\tilde{\tilde{a}}_{ij}^{(1)} = \begin{cases} (S-i)\nu, & \text{if } 0 \le i \le S-1 j = i+1 \\ \kappa, & \text{if } i > 0, j = i-1 \\ -(\tau + S\nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -((S-i)\nu + \kappa + \lambda + \mu_1\sigma_1 + \mu_2\sigma_2), & \text{if } 0 < i, j = i \\ 0, & \text{in other cases} \end{cases}$$

Theorem 3. *Under the $(S, S-1)$ replenishment policy system is ergodic if and only if the inequality (6) holds true, where*

$$\pi(0) = \left(\sum_{m=0}^{S} \frac{S! a^m}{(S-m)!} \right)^{-1}.$$

Proof. The elements of generator $\tilde{\tilde{A}} = A_0 + \tilde{\tilde{A}}_1 + A_2$ are calculated as follows:

$$\tilde{\tilde{a}}_{ij} = \begin{cases} -(S\nu + \lambda\phi_1), & \text{if } i = j = 0 \\ -(\mu_2\sigma_2 + \kappa + (S-i)\nu), & \text{if } 0 < i \le S, j = i \\ (S-i)\nu, & \text{if } 0 \le i \le S-1, j = i+1 \\ \mu_2\sigma_2 + \kappa, & \text{if } 0 < i \le S, j = i-1 \\ 0, & \text{in other cases} \end{cases} \tag{23}$$

We conclude from (23) that system of linear equations (7) corresponding to generator $\tilde{\tilde{A}}$ is the same as balance equations for one-dimensional birth-death process, where death intensity is equal to $\mu_2\sigma_2 + \kappa$ and birth intensity of state m is equal to $(S-m)v, m = 0, 1, ...S$. Therefore, we get the following:

$$\pi(m) = \frac{S!}{(S-m)!} a^m \pi(0), m = 0, 1, ..., S \tag{24}$$

where $\pi(0)$ is calculated from normalizing condition.

Then taking into consideration (3), (5) and (23) after applying some transformation to (12) we conclude that Theorem 3 is true.

4 Calculation of Performance Measures

In each replenishment policy the performance measures are calculated through corresponding state probabilities. So average inventory level S_{av} is calculated as follows:

$$S_{av} = \sum_{m=1}^{S} m \sum_{n=0}^{\infty} p(n, m) \tag{25}$$

Average reorder quantity V_{av} under (s, S) policy:

$$V_{av} = \sum_{m=S-s}^{S} m \sum_{n=0}^{\infty} p(n, S-m) \tag{26}$$

Note 2. Average reorder quantities under (s,Q) and $(S,S-1)$ policies are constants and equal to $Q=S-s$ and 1 correspondingly. Average queue length L_{av} under all policies is calculated as follows:

$$L_{av}=\sum_{n=1}^{\infty} n \sum_{m=0}^{S} p(n,m) \tag{27}$$

Average reorder rate RR under (s,Q) and (s,S) policies is determined as follows:

$$RR=\kappa p(0,s+1)+(\mu_2\sigma_2+\kappa)\sum_{n=1}^{\infty} p(n,s+1) \tag{28}$$

RR under $(S,S-1)$ policy is calculated as follows:

$$RR=\kappa \sum_{m=1}^{S} p(0,m)+(\mu_2\sigma_2+\kappa)\sum_{m=1}^{S}\sum_{n=1}^{\infty} p(n,m) \tag{29}$$

Total loss probability PL is calculated as follows:

Under (s,S) and (s,Q) policies:

$$PL=\phi_2\sum_{n=0}^{\infty} p(n,0)+\frac{\tau}{\tau+\phi_2\lambda+\nu}\sum_{n=1}^{\infty} p(n,0) \tag{30}$$

Under $(S-1,S)$ policy:

$$PL=\phi_2\sum_{n=0}^{\infty} p(n,0)+\frac{\tau}{\tau+\phi_2\lambda+S\nu}\sum_{n=1}^{\infty} p(n,0) \tag{31}$$

First operand in formulas (30) and (31) refers to the loss due to the empty inventory, while the second operand refers to the loss due to customer impatience.

5 Numerical Results

In this section results of numerical experiments will be presented and discussed. The behavior of performance measures vs s under (s,S) and (s,Q) policies are depicted in Fig. 1 and Fig. 2.

We used the following parameters for numerical experiments:

$$\lambda=30, \phi_1=0.5, \phi_2=0.5, \sigma_1=0.4, \sigma_2=0.6, \mu_1=45, \mu_2=35,$$

$$\nu=8, \kappa=6, \tau=20, S=20$$

S_{av} under (s,S) policy is increasing with the increase of s and is a little bit higher than (s,Q). This behavior is expected as with higher s the inventory is replenished more frequently up to S which results in higher average inventory level. But under (s,Q) the replenishment amount is fixed $(S-s)$ and becomes

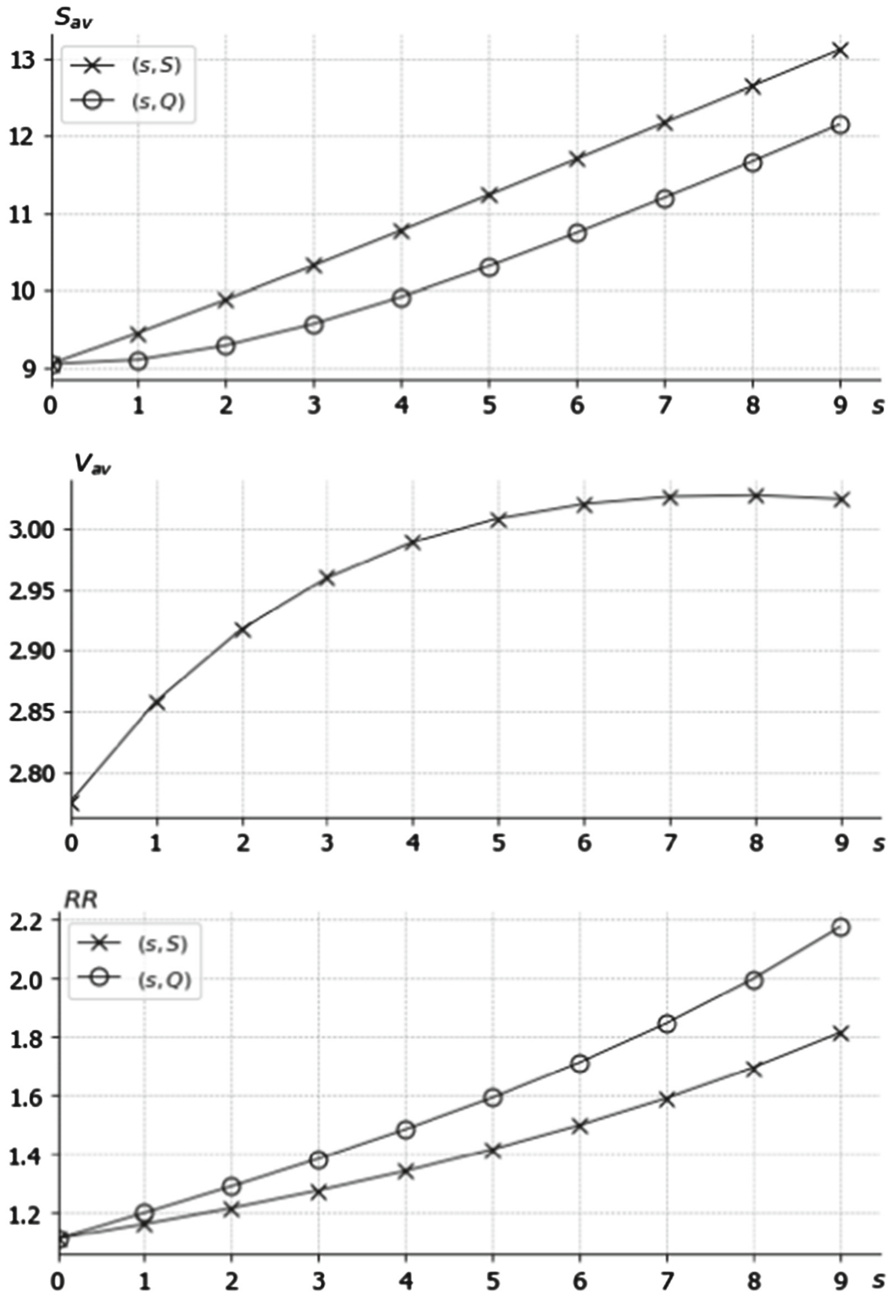

Fig. 1. Dependence of inventory related performance measures on the reorder level s under (s, S), (s, Q) policies

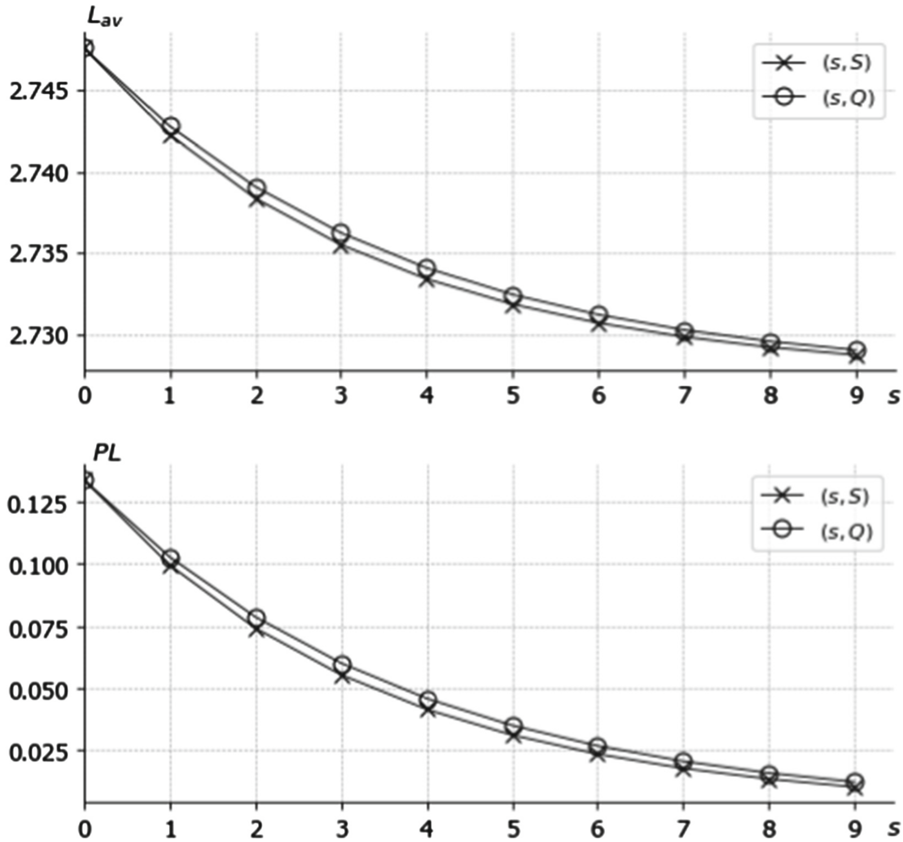

Fig. 2. Dependence of customer related performance measures on the reorder level s under (s, S), (s, Q) policies

lower with higher s which in turn results in lower average inventory level. Average order size V_{av} is also proportional to s which is reflected in graph. We excluded (s, Q) series from V_{av} as it is fixed for given s. RR is also lower under (s, S) policy due to higher average inventory level.

The average number of customers L_{av} in queue is almost the same for both policies and increase with s. Customer loss probabilities decrease for higher values of s due to higher S_{av} under both policies.

Behavior of the performance measures against maximum inventory size S under $(S-1, S)$ policy is depicted in Fig. 3. The inventory related performance measures S_{av} and RR intuitively increases, while L_{av} and PL decreases because with larger inventory system could serve more customers.

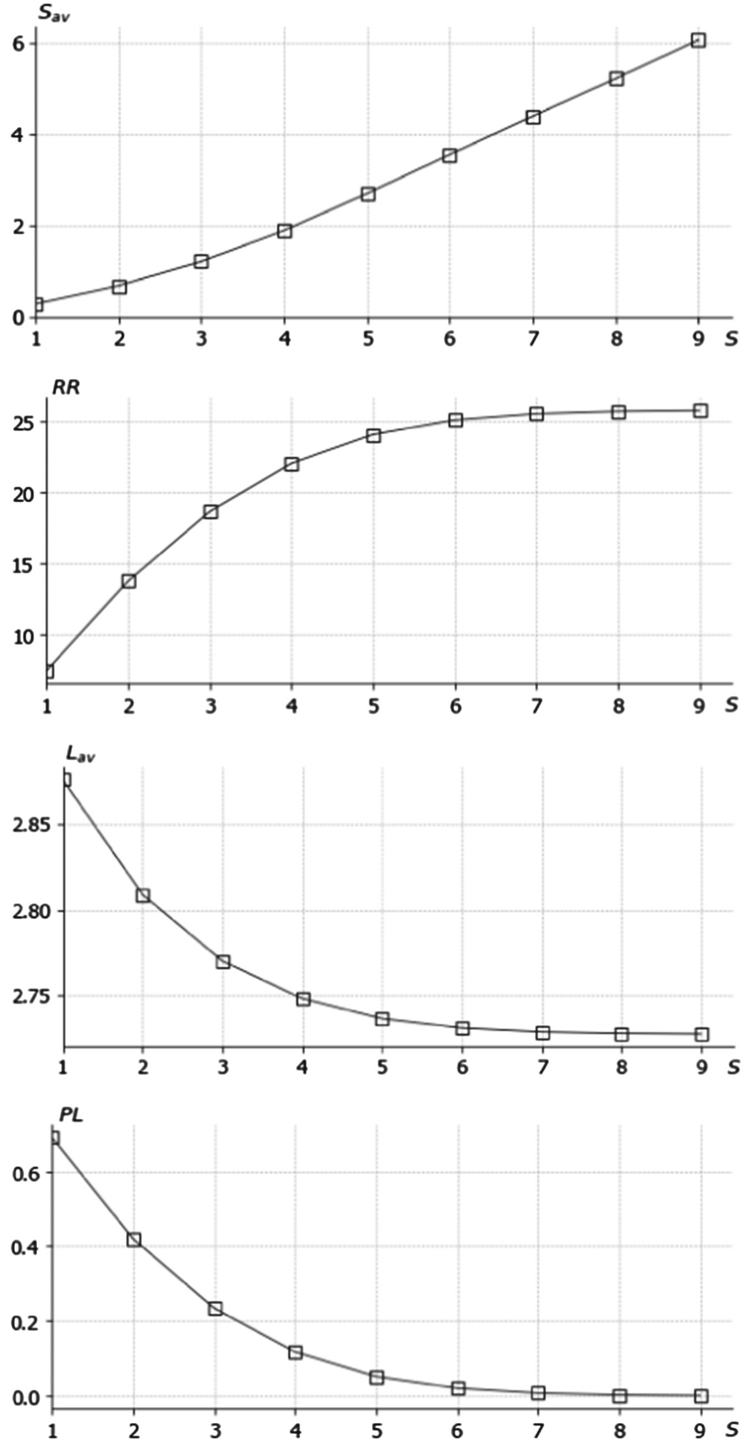

Fig. 3. Performance measures vs inventory size S under $(S-1, S)$ policy

6 Conclusion

The models of queuing-inventory systems with impatient customers and infinite buffer were studied under (s, S), (s, Q) and $(S, S-1)$ replenishment policies. The negative replenishment were considered that decreases the inventory level. Customer enters the system even when the inventory level is zero. We assume that customers after being served according to Bernoulli scheme either leaves the system empty handed or with an item from inventory. We used 2D Markov chains with tridiagonal generator matrices for mathematical modeling of the system. Ergodicity conditions were found and the algorithm for calculation of system performance measure was developed. Numerical experiments were performed and behavior of performance measures was analyzed under different policies.

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