Optimization of the Transmission of Messages Divided into Different Shares, Transmitted on Two Different Channels

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Abstract. The paper considers the transmission of messages with demultiplexing over two communication channels with different throughput capacities. The channel with the highest throughput receives the largest chunks of messages resulting from the demultiplexing, and the channel with the smallest throughput receives the smallest chunks. The problem of calculating the optimal channels throughputs is solved by taking into account the characteristics of the transmitted traffic.

Keywords: data flow distribution · communication channel throughput · demultiplexing · multiplexing · secret sharing scheme

1 Introduction

Due to the self-isolation and quarantine regimes implemented during the current COVID-19 pandemic, there is an increased demand for Internet connection services, data transfer speeds augmentation, throughput expansion, and additional communication channels purchase [1–3]. The most popular transmitted content is video data, for example, online broadcasts of cinemas, educational webinars. Broadcasting is carried out using client-server applications, in which the content can be pre-transformed using any algorithms, and only then transmitted to the user. The preliminary content transformation can be carried out in order to compress it, in other words, to reduce the transmitted traffic, as well as to ensure confidentiality, i.e., to perform cryptographic transformations. In such situations, even the choice of optimal cryptographic algorithms can lead to significant delays in the playback of the video data stream due to the fact that the reverse cryptographic conversion must be performed on the client side. The use of an additional communication channel makes it possible to organize distributed data transmission, which allows to solve the problem of ensuring confidentiality, but there arise some questions related to the efficiency of the use of computing resources, optimization of channel throughput, synchronization of transmitted streams, etc.

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2 Problem Statement

Let the sender (a person or an automatic device) transmit to the receiver a high-quality uncompressed media stream, which is a sequence of images (frames). The transmission is carried out over the Internet, and it is required that no one except the recipient can access the contents of the transmitted data. To meet this requirement, it is possible to organize secure media data streaming using cryptographic methods. However, when using such methods, there may arise problems related to ensuring the stability of the selected algorithms and, accordingly, with the availability of sufficiently powerful computing resources on the receiving side to guarantee timely data decryption. In such a situation, it is advisable to consider the possibility of using other methods of information protection that are not related to classical cryptography, e.g., secret sharing schemes (SSS) [4], demultiplexing.

Algorithms for dividing video data into unequal shares are proposed in [5–7] which will allow the sender and receiver to carry out the separation of the transmitted TCP / IP traffic over these channels, using two communication channels with different throughput, as, for example, it is described in [8–11]. Further, we will assume that SSS for unequal shares can be used not only for transmitting video frames, but also for transmitting streams of any messages, and all the transformations described in [5–7] are performed directly on the bit representation of these messages. When messages are divided into unequal shares, a smaller share of each message is transmitted over a lower throughput channel, while a larger share is transmitted over a higher throughput channel. Such message transmission from the sender to the recipient is carried out at the transport level of the seven-level OSI network model [12], where the TCP protocol provides guaranteed data delivery. When using two communication channels at the same time, there arise questions related to the efficiency of computing resources, optimization of channel throughput, synchronization of transmitted streams, buffering. These issues can be solved by implementing appropriate client-server applications and optimizing the throughput of communication channels. It is advisable to optimize the throughput capacities according to cost minimization criteria, one part of which is associated with message delays in the network (the growth of which leads to a delay in the recipient's response to messages and corresponding losses), the other part is related to the payment for channel throughput, which increases with throughput growth.

As a mathematical model for optimizing a two-channel SSS, a network with splitting requests (S-network) with two single-channel queuing systems (QS) is proposed (see Fig. 1). In terms of queuing theory (QT), we will call messages and their parts *requests*, demultiplexing messages - *splitting requests*, multiplexing messages *assembling requests*. Two requests corresponding to two parts of the same divided message will be referred to as *conjugate requests*. We define the discipline of servicing queues in front of the channels as FIFO (first in - first out) discipline. Requests are transmitted over two channels with different throughputs C_1, C_2 measured, for example, in Kbit/s. Unlike traditional QS networks, at

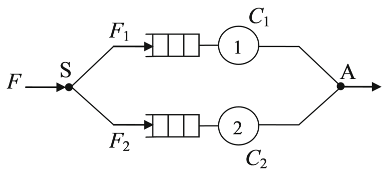

Fig. 1. Network with splitting requests. S - split point, A - assemble point

Point S, the request does not go to one of the branches, but is split into two requests, one of which arrives at QS1 and the other goes to QS2. At Point A, two conjugate requests "merge" and turn into one request. Accordingly, the incoming Traffic F (Kbit/s) is divided into two parts F_1 and F_2, where $F_1 + F_2 = F$. The moment of entry of the request into the network is simultaneously the moment of its splitting and the moment of entry of the resulting conjugate requests into each of the two branches of the network (into each of the two QSs). Consequently, the Intensities λ, λ_1 and λ_2 of the request flows entering the network are the same in QS1 and in the QS2, respectively, and all three request flows are described by the same probabilistic law. Another feature of the considered S-network, not shown in Fig. 1, is that before Point A, two more queues are formed (one on each branch) - synchronization queues. At the moment of exit from the first (second) channel, the request enters the first (second) synchronization queue before Point A, where it remains until its conjugate "half" is found in another synchronization queue. In other words, one of the two conjugate requests that arrived first at Point A waits for the second conjugate request to arrive. At the moment of its arrival, both conjugate requests are merged into one request leaving the network, and the transfer of the request is completed.

Note that all requests arrive in each synchronization queue in the same sequence that they enter the network. Therefore, if at least one request is pending in one synchronization queue, the other synchronization queue is empty. At any finite time interval, either the first synchronization queue is empty, or the second, or both queues are empty. Both of these queues can be non-empty at the same time only at one point in time: when the condition "at the selection point there is a pair of requests conjugated with each other" is fulfilled. It follows from this that of the two conjugate requests, the one that arrives later is not delayed in the synchronization queue. Consequently, the Time u of the message transmission (in terms of QT, the time the request is in the network, i.e., the time elapsed from the moment the request arrives in Point S until the moment it leaves Point A) is determined by the formula:

$$u = \max(u_1, u_2), \tag{1}$$

where u_1 is the time the request was in QS1:

$$u_1 = w_1 + x_1, \tag{2}$$

u_2 is the sojourn time of the conjugate request in QS2:

$$u_2 = w_2 + x_2, \tag{3}$$

w_1 is the request waiting time in Queue 1, x_1 is the request service time in Channel 1, w_2 is the waiting time of the conjugate request in queue 2, x_2 is the service time of the conjugate request in Channel 2.

The average Time U of staying in the S-network, according to (1), is expressed by the formula:

$$U = M\left[\max(u_1, u_2)\right] = M\left[\max(w_1 + x_1, w_2 + x_2)\right]. \tag{4}$$

Time U depends on the Throughputs C_1, C_2:

$$U = U(C_1, C_2).$$

Let the price of the throughput of any channel, calculated for the network operation time, be equal to m c.u./(Kbit/s). Then the problem of optimizing Throughputs C_1, C_2 of the S-network channels (or, in other words, the problem of optimizing the S-network) can be formulated as follows:

$$f = lU\left(C_1, C_2\right) + mC_1 + mC_2 \to \min_{C_1, C_2}, \tag{5}$$

$$\begin{cases} C_1 \geq F_1, \\ C_2 \geq F_2, \end{cases} \tag{6}$$

where $U(C_1, C_2) = M\left[max(u_1, u_2)\right]$, l (c.u./s) is the cost of the average network delay per second. Cost Coefficient l is equal to losses (arising from waiting for applications) calculated for the period of network operation.

Thus, the problem (5), (6) is posed as the problem of minimizing the average costs over the network operation time. A network with optimal channel capacity will be called optimal.

The non-triviality of the problem posed is due to the absence in the QT of explicit formulas that allow, directly or by means of appropriate transformations, to accurately calculate the average Time U of requests in the S-network under some general and natural assumptions about the incoming flow of requests and methods for their splitting. To solve this problem, it is necessary to develop appropriate exact or approximate methods. Further development and research of such methods is ongoing.

3 Exponential Network with Independent Branches

3.1 Problem Statement

Consider a network with independent branches (Fig. 2), which makes sense to study as a simplified first approximation of the S-network shown in Fig. 1.

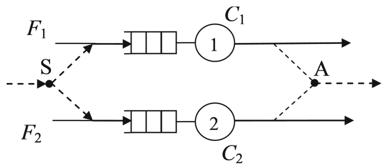

Fig. 2. S-network with independent branches

In this network with independent branches, QS1 and QS2 operate independently, each serving its own stream of requests. Each of these QSs is individually equivalent to the corresponding QS in an S-network, but a pair of QSs in a network with independent branches (Fig. 2) is not equivalent to a pair of QSs in an S-network (Fig. 1). In a network with independent branches, the QSs function independently; in an S-network the processes in one and the other QS are statistically dependent. Statistical independence of the network branches in Fig. 2 simplifies its analysis.

The dashed lines in Fig. 2 show a single passage through the network of a single request, divided into parts, in a stationary mode of network operation at a random time. At the moment of its arrival, the conjugate parts of the split request arrive at the corresponding QSs, then pass through the queues and service, and are assembled into one request, as described above. The travel time of the split request through the network is expressed by formula (1), the optimization problem for such a network is posed in the form (5), (6). In order to solve this problem by analytical methods, let us express $M\left[max(u_1, u_2)\right]$, in terms of Channel Throughputs C_1, C_2.

3.2 Network with Independent Branches Optimization

The calculation of the system in Fig. 2 contains the following steps. First, we can find the distribution functions of the sojourn time of entire requests in QS1, QS2. Since these QSs are exponential, the required distribution functions are known. Since Quantities u_1 and u_2 are independent, we find the distribution of the maximum of these quantities, and through it the desired $M\left[max(u_1, u_2)\right]$.

The distribution function of the sojourn time in QS1 has the form [13]:

$$P(u_1 \leq t) = 1 - e^{-\mu_1(1-\rho_1)t}, \tag{7}$$

similarly in QS2 it is described as:

$$P(u_2 \leq t) = 1 - e^{-\mu_2(1-\rho_2)t}, \tag{8}$$

where μ_1, μ_2 are the service intensity in the first and second channels, ρ_1, ρ_2 are the load factors of the first and second channels.

We find the Distribution Function $max(u_1, u_2)$ as the probability of simultaneous occurrence of two independent events: as the probability that the first value does not exceed t and that the second value does not exceed t:

$$P\left[max(u_1, u_2) \leq t\right] = \left[1 - e^{-\mu_1(1-\rho_1)t}\right]\left[1 - e^{-\mu_2(1-\rho_2)t}\right]. \qquad (9)$$

Using the well-known formula for calculating the mathematical expectation of a positive random variable (without calculating the probability density), we find

$$M\left[max\left(u_1, u_2\right)\right] = \int_0^\infty \left[1 - \left(1 - e^{-\mu_1(1-\rho_1)t}\right)\left(1 - e^{-\mu_2(1-\rho_2)t}\right)\right] dt = \\ = \frac{1}{\mu_1(1-\rho_1)} + \frac{1}{\mu_2(1-\rho_2)} - \frac{1}{\mu_1(1-\rho_1) + \mu_2(1-\rho_2)}. \qquad (10)$$

The resulting expression can be substituted into the problem (5), (6) to solve it for a network with independent branches by analytical methods. Before doing this, let us move on to the expression (10) and the parameters used in the problem (5), (6): $\rho_1 = F_1/C_1$, $\rho_2 = F_2/C_2$, $\mu_1 = C_1/H_1$, $\mu_2 = C_2/H_2$. Moving on to these designations in the expression (10) and substituting it into the problem (5), (6), we obtain:

$$f = \frac{lH_1}{(C_1 - F_1)} + \frac{lH_2}{(C_2 - F_2)} - \frac{l}{H_1^{-1}(C_1 - F_1) + H_2^{-1}(C_2 - F_2)} \\ + mC_1 + mC_2 \to \min_{C_1, C_2}, \qquad (11)$$

$$\begin{cases} C_1 \geq F_1, \\ C_2 \geq F_2. \end{cases} \qquad (12)$$

The point (C_1, C_2) of the local minimum of this positive function can be found from the system of equations

$$\frac{\partial f}{\partial C_1} = 0, \\ \frac{\partial f}{\partial C_2} = 0,$$

that is, from the equations

$$-\frac{lH_1}{(C_1 - F_1)^2} + \frac{lH_1^{-1}}{\left(H_1^{-1}(C_1 - F_1) + H_2^{-1}(C_2 - F_2)\right)^2} + m = 0, \qquad (13)$$

$$-\frac{lH_2}{(C_2 - F_2)^2} + \frac{lH_2^{-1}}{\left(H_1^{-1}(C_1 - F_1) + H_2^{-1}(C_2 - F_2)\right)^2} + m = 0. \qquad (14)$$

Let us denote $C_1 - F_1$ by x and $C_2 - F_2$ by y. As a result, the system (13), (14) takes the following form, which is well solved by numerical methods:

$$-\frac{lH_1}{x^2} + \frac{lH_1^{-1}}{\left(H_1^{-1}x + H_2^{-1}y\right)^2} + m = 0, \qquad (15)$$

$$-\frac{lH_2}{y^2}+\frac{lH_2^{-1}}{\left(H_1^{-1}x+H_2^{-1}y\right)^2}+m=0. \quad (16)$$

One of the main methods for reducing the delay time of conjugate requests in synchronization queues is to introduce the maximum positive correlation between the processes of moving conjugate requests along the network branches. This method is explored in the following sections of the article.

4 S-Networks with Synchronous Branches

4.1 Fundamentals

Definition. Consider an S-network in which each incoming request has a random size h (Kbit) and is split into two conjugate requests so that the same proportion is always maintained between their sizes h_1 and h_2 (where $h_1+h_2=h$):

$$h_2/h_1=\gamma=const. \quad (17)$$

If, in this case, the throughput of the channels is connected by the condition

$$C_2=\gamma C_1, \quad (18)$$

then the service time x_1 of the request in Channel 1 and the Service Time x_2 of the conjugate request in Channel 2 coincide:

$$\begin{gathered} x_1=h_1/C_1, \\ x_2=h_2/C_2=(\gamma h_1)/(\gamma C_1)=h_1/C_1=x_1. \end{gathered} \quad (19)$$

Since the equality (19) is satisfied for each pair of conjugate requests, then in each pair both conjugate requests enter the queues to the channels, into the channels, and to the assembly point simultaneously. We call such a network an S-network with synchronous branches or an Ss-network.

It is easy to see that in the Ss network, $C_2=\gamma C_1$ implies $C=(1+\gamma)C_1$, where $C=C_1+C_2$ is the total throughput of the channels. Similarly, from $h_2/h_1=\gamma$ it follows that Traffic F_2 entering the second branch and all Traffic F entering the network are expressed in terms of F_1 by the relations $F_2=\gamma F_1$, $F=(1+\gamma)F_1$.

Ss-Network Optimization Problem . The problem (5), (6) of optimizing an S-network with synchronous branches can be solved exactly for any flow of requests for which both QSs in the branches can be calculated using exact QT methods.

Indeed, since Sojourn Time u_1 of any request in QS1 and Sojourn Time u_2 of the corresponding conjugate request in QS2 in the Ss-network coincide, then in (1) we have $u=\max(u_1,u_2)=u_1$ and, therefore, in (4) $U=M\left[\max(u_1,u_2)\right]=M\left[u_1\right]=U_1$, where U_1 is the average sojourn time of the request in QS1. Performing the substitutions $U(C_1,C_2)=U_1(C_1)$, $C_2=\gamma C_1$ and $F_2=\gamma F_1$ in the problem (5), (6), we obtain its equivalent formulation using only one variable parameter C_1:

$$f = lU_1(C_1) + m(1+\gamma)C_1 \rightarrow \min_{C_1}, \tag{20}$$

$$C_1 \geq F_1. \tag{21}$$

Note that after these substitutions are performed, the second constraint in (6) becomes equivalent to the first and, therefore, is absent in the constraints (21).

The solution to the problem (20), (21) of optimization of the Ss-network determines the optimal throughput C_1 of the first channel and, at the same time, the corresponding throughput $C_2 = \gamma C_1$ and $C = (1+\gamma)C_1$.

4.2 Ss-Network with Regular Incoming Flow and Fixed Order Size

Optimization. With a regular incoming flow, Time τ between arrivals of requests to the network (and, therefore, to each of the two branches) is constant: $\tau = const$.The size of requests arriving in QS1 is also fixed ($h_1 = const$), so the service time $x_1 = h_1/C_1$ is also fixed in QS1. It follows from the restriction $C_1 \geq F_1$ that $x_1 \leq \tau_1$, i.e., each request arriving in QS1 is serviced before the next one arrives. Therefore, the queue in front of Channel 1 is not formed, and the average sojourn time of the request in $U_1 = x_1 = h_1/C_1$. Substituting this expression for U_1 in (20), (21) instead of $U_1(C_1)$, we obtain the problem

$$f = l\frac{h_1}{C_1} + m(1+\gamma)C_1 \rightarrow \min_{C_1}, \tag{22}$$

$$C_1 \geq F_1, \tag{23}$$

whose solution is reduced to solving the algebraic equation

$$\frac{\partial f(C_1)}{\partial C_1} = 0 \ or \ -l\frac{h_1}{C_1^2} + m(1+\gamma) = 0,$$

determining the point of the local minimum

$$C_1 = \sqrt{\frac{lh_1}{m(1+\gamma)}}. \tag{24}$$

If the obtained value C_1 satisfies the constraint $C_1 \geq F_1$, then (24) is the solution to problem (22), (23). Otherwise, the solution to this problem is the smallest value C_1 closest to the point (24) that satisfies the constraint $C_1 \geq F_1$, i.e., value $C_1 = F_1$.

Comparison with the Single-Channel Version. The two-channel implementation of the SSS, compared to the single-channel implementation, significantly increases the security of the transmitted data from unauthorized use. To estimate the losses due to which this is achieved, let us compare the costs obtained in the optimal Ss-network with the costs characterizing the corresponding basic single-channel optimal system.

In the considered case of a regular incoming flow and a fixed size of requests, the average Time U of message transmission over one channel for $C \geq F$ is equal to the average request service time (since there is no queue in front of the channel). I.e. $U = h/C$, where $h = h_1 + h_2 = const$ is the size of requests. Therefore, the optimization problem for a basic single-channel system takes the form

$$f = lh/C + mC \to \min_{C}, \tag{25}$$

$$C \geq F. \tag{26}$$

The local minimum of the objective function (25) is attained at the point

$$C = \sqrt{\frac{lh}{m}}. \tag{27}$$

Theorem 1. Transmission of a regular flow of fixed-size requests through the optimal Ss-network leads to the same costs as transmission through the optimal single-channel system. In this case, the total throughput of the optimal Ss-network is equal to the throughput of the optimal single-channel system.

Proof of the Theorem. To prove the theorem, it suffices to note that the statement of the problem (22), (23) for optimizing the Ss-network differs from the statement of the problem (25), (26) for optimizing a single-channel system only due to the formulation of the problem (22), (23) in terms of the optimal choice of throughputs abilities C_1. But since any of the parameters C_1, C_2, and C uniquely determine the other two parameters in the Ss-network, the problem of its optimization can be formulated in terms of the optimal choice of any of these three parameters. When choosing the variable C as a variable parameter - the total throughput of the channels - the formulation of the Ss-network optimization problem becomes equivalent to the formulation of the optimization problem for a single-channel system.

Indeed, in the problem (22), (23) $C_1 = C/(1+\gamma)$, $F_1 = F/(1+\gamma)$ and $h_1 = h/(1+\gamma)$. Carrying out the corresponding changes in the problem (22), (23), we obtain its formulation

$$f = l\frac{h/(1+\gamma)}{C/(1+\gamma)} + m(1+\gamma)C/(1+\gamma) \to \min_{C},$$
$$C/(1+\gamma) \geq F/(1+\gamma).$$

equivalent to the formulation of the problem (25), (26). From the equivalence of the formulations of the two fundamentally different problems under consideration, the numerical coincidence of their solutions follows. The theorem is proved.

The two problems under consideration are optimization problems for two different systems, the Ss-network and a single-channel system. The formal coincidence of their solutions means that the transmission of a regular incoming flow with a fixed request size through the optimal Ss-network leads to exactly the same costs as its transmission through the optimal single-channel system.

4.3 Ss-Network with an Arbitrary Incoming Flow of Requests

Theorem 1 is generalized by the following theorem.

Theorem 2. The transmission of any request flow through the optimal Ss-network leads to the same costs as its transmission through the optimal single-channel system. In this case, the total throughput of the optimal Ss-network is equal to the throughput of the optimal single-channel system.

The proof of the theorem is based on a comparison of the processes of passing through the Ss-network and through a single-channel system of the same implementation of the incoming request flow. Then, provided that the throughput of the single-channel system is equal to the throughput of the Ss-network, the advancement of requests in each branch of the Ss-network occurs synchronously with the advancement of requests in the single-channel system. Therefore:

- the equivalence of the network optimization problem and the optimization problem for a single-channel system (including when they are considered independently, i.e. when independent implementations of the same request flow are fed to the network input and to the single-channel system input);
- the coincidence of the total throughput of the optimal network channels with the throughput of the optimal single-channel system;
- the coincidence of the costs calculated for the period of operation of the optimal Ss-network and the costs of the optimal single-channel system for the same period.

A detailed presentation of the proof is beyond the scope of this article.

4.4 Example of Exponential Ss Network Optimization

Definition. An Ss network is said to be exponential if it includes a Poisson request flow and the request sizes are distributed exponentially. Accordingly, both QSs in such a network are M/M/1 systems. Their calculation is carried out according to the well-known formulas [13].

Ss-Network Optimization. The average Sojourn Time U_1 in QS1 of the exponential Ss-network is [13]:

$$U_1 = \frac{1/\mu_1}{1-\rho_1} = \frac{1}{\mu_1 - \lambda_1} = \frac{H_1}{H_1\mu_1 - H_1\lambda_1} = \frac{H_1}{C_1 - F_1}, \tag{28}$$

where μ_1 is the intensity of servicing requests in the QS1, $\rho_1 = \lambda_1/\mu_1 = F_1/C_1-$ is the load factor of QS1, $H_1 = M(h_1)$ is the average size of requests arriving in QS1.

Therefore, the problem (20), (21) as applied to the exponential Ss-network is specified as follows:

$$f = l\frac{H_1}{C_1 - F_1} + m(1+\gamma)C_1 \rightarrow \min_{C_1}, \tag{29}$$

$$C_1 \geq F_1. \tag{30}$$

The only minimum of objective function (29), determined from the equation

$$\frac{\partial f}{\partial C_1} = -\frac{lH_1}{(C_1 - F_1)^2} + m(1+\gamma) = 0 \tag{31}$$

is reached at the point

$$C_1 = F_1 + \sqrt{\frac{lH_1}{m(1+\gamma)}} \tag{32}$$

and is the solution to the problem (29), (30),since it satisfies the constraint (30).

Substituting the throughput (32) into the objective function expression (29), we find the costs of using the optimal exponential Ss-network:

$$f = l\frac{H_1}{\sqrt{\frac{lH_1}{m(1+\gamma)}}} + m(1+\gamma)\sqrt{\frac{lH_1}{m(1+\gamma)}} = 2\sqrt{(1+\gamma)mlH_1}. \tag{33}$$

Optimization of a Single-Channel Exponential System. The flow of requests included in the considered exponential Ss-network has intensity $\lambda = F/H = F_1/H_1$ and average request size $H = (1+\gamma)H_1$. When this stream is transmitted over a single-channel system, the average request transmission time is $U = \frac{1/\mu}{1-\rho} = \frac{H}{C-F}$. The optimization problem for such a single-channel QS has the form

$$f(C) = l\frac{H}{C-F} + mC \rightarrow \min_{C}, \tag{34}$$

$$C \geq F \tag{35}$$

and determines the throughput

$$C = F + \sqrt{\frac{lH}{m}}, \tag{36}$$

at which the average total costs (33) are minimal and amount to

$$f(C) = l\frac{H}{\sqrt{\frac{lH}{m}}} + m\left(F + \sqrt{\frac{lH}{m}}\right) = 2\sqrt{mlH} + mF. \tag{37}$$

Comparison of the Optimal Exponential Ss-Network and the Corresponding Single-Channel QS. To compare the solution (32) to the problem (29), (30) with the solution (36), we rewrite the solution (32) in terms of the total throughput of the Ss-network. Carrying out the substitutions $C_1 = C/(1+\gamma)$, $F_1 = F/(1+\gamma)$, $H_1 = H/(1+\gamma)$ in (32) equivalent for any Ss-network, we obtain the expression

$$\frac{C}{1+\gamma} = \frac{F}{1+\gamma} + \sqrt{\frac{lH}{m(1+\gamma)^2}}, \tag{38}$$

and, simplifying it, we find the total throughput of the optimal exponential Ss-network

$$C = F + \sqrt{\frac{lH}{m}},$$

coinciding, as we see, with the throughput (35) of the optimal single-channel exponential QS.

Similarly, performing the replacement $H_1 = H/(1+\gamma)$ in (33) equivalent for Ss-networks, we make sure that the costs associated with the use of the optimal exponential Ss-network coincide with the costs associated with the use of the optimal single-channel exponential system. Thus, the solutions to the exponential Ss-network optimization problem, and the optimization problem for the corresponding single-channel exponential QS, obtained in general form, confirm and illustrate Theorem 2 formulated above, proved for any request flow.

5 Networks with Splitting Requests in Constant Proportion

5.1 Fundamentals

Definition. S-networks in which the condition $h_2/h_1 = \gamma = const$ is satisfied when splitting requests, but the condition $C_2 = \gamma C_1$ is not imposed, we will call networks with split requests in equal proportions, or Se-networks. Thus, the Ss-networks considered above are a subset of Se-networks in which both conditions (17), (18) are satisfied.

Optimization of Se-Networks. The optimization problem for Se-networks has certain specific features. It is written, like the general problem (5), (6) of optimization of S-networks, in the form

$$f = lU(C_1, C_2) + mC_1 + mC_2 \to \min_{C_1, C_2}, \tag{39}$$

$$\begin{cases} C_1 \geq F_1, \\ C_2 \geq F_2, \end{cases} \tag{40}$$

(where $U(C_1, C_2) = M\left[\max(u_1, u_2)\right]$, and inherits the property (as opposed to the network with independent branches, see Fig. 2) that in the general case, in an elementary function, the inexpressible dependence of the mathematical expectation $M\left[\max(u_1, u_2)\right]$ on the variable parameters C_1, C_2 is determined here on stochastically interdependent random variables u_1, u_2 (sojourn time in QS1 and QS2).

In contrast to the optimization problem for Ss-networks, the variable parameters C_1, C_2 in (39), (40) are independent and not related by the condition $C_2 = \gamma C_1$, therefore, both inequalities are preserved in the constraints (40). And so, the progress of requests in the network branches is generally asynchronous here, which makes it difficult to find an explicit formula that accurately expresses time in terms of $U(C_1, C_2) = M\left[\max(u_1, u_2)\right]$ network parameters.

At the same time, it is very important to find the exact solution to the problem (39), (40). This is due to the following considerations. The previously considered problem This is due to the following considerations. The previously considered problem (20), (21) of optimizing Ss-networks is the problem of finding the conditional minimum of the objective function (39), since it connects the arguments of function (39) with an additional condition $C_2 = \gamma C_1$, i.e. limits in the coordinate system $(C_1, 0, C_2)$ the search area for the minimum f to a one-dimensional set of points of the straight line $C_2 = \gamma C_1$. And when solving the problem (20), (21), we found the solution on this straight line that does not increase the costs of a two-channel SSS implementation in comparison with a single-channel implementation. And if in the two-dimensional region of feasible solutions to the problem (39), (40) the only minimum point of the objective function (39) is outside Straight Line $C_2 = \gamma C_1$, then the solution to the problem (39), (40) will be better than the solution to the problem (20), (21). The substantive meaning of such a solution will be to discover the possibility of switching to a two-channel SSS implementation not only without increasing costs (see Theorem 1), but also with their accompanying decrease.

In the next two sections, it is established that such a possibility is excluded in the class of Se-networks: the optimal solutions of the problem (39), (40) with independent throughputs always lie on Straight Line $C_2 = \gamma C_1$.

5.2 Se-Network with Regular Incoming Flow and Fixed Request Size

Theorem 3. The Se-network that is optimal for transmitting a regular flow of fixed-size requests is an Ss-network, i.e., when transmitting a regular flow of requests of a fixed size, the optimal solution to the problem (39), (40) always lies on Straight Line $C_2 = \gamma C_1$.

Proof of the Theorem. Taking into account the condition $h_2/h_1 = \gamma = const$ which defines the Se-network, we represent the domain of feasible solutions to the problem (39), (40) in the form of a union of two domains, R_1 and R_2 (Fig. 3).

Domain R_1 is determined by conditions $C_1 \geq F_1$, $C_2 \geq \gamma C_1$, Domain R_2 is determined by conditions $C_2 \geq F_2$, $C_2 \leq \gamma C_1$. Line $C_2 = \gamma C_1$ for $C_1 \geq F_1$ belongs to both domains. The point $(F_1,\ F_2)$ lies on this line, since $F_2/F_1 =$

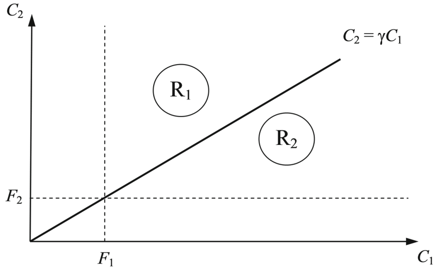

Fig. 3. Domain of feasible solutions to problem (39), (40)

$\lambda h_2/\lambda h_1 = \gamma$ (the coordinates of the point satisfy the equation of the line). Any point $(C_1,\ C_2)$ belonging to the domain of feasible solutions determines the service time of requests $x_1 = b_1 = h_1/C_1 \leq h_1/F_1 = \tau_1$ for QS1, i.e., the constant service time of requests in the first branch of the network does not exceed the constant period τ_1 of the arrival of requests in this branch. There is no queue to the channel in the first branch, Time u_1 of the sojourn of requests in QS1 is constant and equal to h_1/C_1. Similarly, we find that there is no queue to the channel in the second branch of the network, and Time u_2 of the sojourn of requests in QS2 is equal to constant h_2/C_2.

We use the method of proof by contradiction and assume that the least value of objective function (39) is attained at Point $\overline{C} = (C_1, C_2)$, which does not lie on Straight Line $C_2 = \gamma C_1$. Then the required point lies either in Domain R_1 and above this line, or in Domain R_2 and to the right of this line.

In the first case, the coordinates of Point $\overline{C}$ which lies in the domain of feasible solutions and delivers the smallest value of the objective function, have the form $(C_1, C_2) = (C_1, \gamma C_1 + \varepsilon)$, where $\varepsilon > 0$. In this case, the periodic process of servicing requests in QS1 is characterized by the Service Time $x_1 = h_1/C_1$, in QS2 - by the Service Time $x_2 = h_2/C_2 = \gamma h_1/(\gamma C_1 + \varepsilon) = h_1/(C_1 + \varepsilon/\gamma) < x_1$. Each request in the first branch and its conjugate request in the second branch starts to be served at the same time. The request in the second branch is served earlier and waits for the completion of the service of the conjugate request, which should come from QS1. As a result, Time u spent by the request to the network (until the moment of assembly) becomes equal to $x_1 = h_1/C_1$. But we get the same sojourn time at Point $\overline{C}^* = (C_1, \gamma C_1)$, at which $x_2 = x_1$. And, since at Point $\overline{C}^*$ with the same sojourn time and the same throughput C_1, the throughput of C_2 is lower than at Point $\overline{C}$, we get $f(\overline{C}^*) < f(\overline{C})$ (39). The resulting contradiction excludes the possibility of finding a solution to the problem (39), (40) in Domain R_1 outside Straight Line $C_2 = \gamma C_1$.

A similar contradiction is caused by the assumption that it is possible to find the required minimum point in Domain R_2 outside Straight Line $C_2 = \gamma C_1$ leads to a similar contradiction.

Thus, the solution to the problem (39), (40) always lies on Straight Line $C_2 = \gamma C_1$. With the optimal choice of throughput, the considered Se-network (with a regular flow of fixed-size requests) becomes an Ss-network. The optimal throughput of such a network is found in Sect. 3.2 of the article. It was also shown there that the transmission of such a stream through the optimal Ss-network leads to exactly the same costs as its transmission through the optimal single-channel system. Now the corresponding conclusion also applies to optimal Se-networks.

5.3 Se-Network with Arbitrary Incoming Flow of Requests

Theorem 4. The Se-network that is optimal for transmitting any flow of requests is an Ss-network; Optimal Throughputs C_1, C_2 of the Se-network lie on Straight Line $C_2 = \gamma C_1$.

The Proof of the Theorem is based on the comparison of the Se-network request processes passing through its branches and establishing the fact that if C_1, C_2 are not connected by condition $C_2 = \gamma C_1$, i.e., if the service time of two conjugate requests does not coincide, then the random sojourn time $u^i = \max(u_1^i, u_2^i)$ of the i-th request in the Se-network is determined by the formula

$$u^i = \max(u_1^i, u_2^i) = \begin{cases} u_1^i, \; if \; C_2 \geq \gamma C_1, \\ u_2^i, \; if \; C_2 \leq \gamma C_1. \end{cases} \tag{41}$$

For $C_2 = \gamma C_1$ for all i we obtain $u_1^i = u_2^i$.

Averaging the sojourn time (41) over all requests, we obtain an expression that is valid for any Se-networks:

$$U = \begin{cases} U_1, \; if \; C_2 \geq \gamma C_1, \\ U_2, \; if \; C_2 \leq \gamma C_1. \end{cases} \tag{42}$$

And then we complete the establishment of the validity of Theorem 4 by proving it by contradiction (by analogy with the proof of Theorem 3).

We note two important corollaries of Theorem 4:

- for any incoming flow, the Ss-network is the optimal Se-network;
- the optimal two-channel SSS implementation in the form of a Se-network does not increase operating costs in comparison with the optimal (and, therefore, compared to any) single-channel SSS implementation.

Theorem 4 greatly simplifies the solution of the complex problem of nonlinear optimization of Se-networks, since it reduces the search for the optimal values of two variable parameters C_1, C_2, which provide the minimum of the function of two variables, to the search for the optimal value of one variable parameter, e.g., C_1 (the other is determined from the relation $C_2 = \gamma C_1$), that provides the minimum of the objective function. This is especially important when it is necessary to use simulation modeling to calculate the objective function, e.g., when QS in the network branches belongs to G/G/1 class systems for which there are no exact formulas in QT that express the average sojourn time in QS in terms of its parameters.

The average delay in both synchronization queues is zero. Hence it follows that for $C_2 = \gamma C_1$ in Se-networks $u_1 = u_2$ and is determined by the formula (4) $U = U_1$. Consequently, the optimal value of C_1 and the minimum cost (5) can be easily calculated analytically if QS1 is, for example, a system of class M/M /1, M/G/1, etc.

6 Discussion of Research Results (Conclusion)

The article introduces and investigates a mathematical model for the transmission of messages, divided into smaller and larger shares, transmitted over different channels with different throughput. In terms of queuing theory, a network with split requests (S-network) is defined as such a model. This network takes into account the transmission of a split request over two different channels, the formation of queues in front of the channels, and the assembly of split requests on the receiving side of the channel. The mathematical problem of optimizing the throughput of two channels of the S-network is posed in a general form. Methods for solving this problem (methods for optimizing S- network) are formulated and investigated. Analytical methods have solved the problem of optimizing an exponential S-network with independent branches. In practice, the solution to this problem can be used to optimize S-networks, in which the transmission of split requests constitutes a small part of the total traffic transmitted over two channels.

Special methods are used to study optimization problems for such S-networks, in which the transmission of split requests constitutes the main load transmitted over two channels. Four theorems are proved, which makes it possible to reduce two-dimensional optimization problems of such S-networks (i.e.,. problems with two variable parameters) to one-dimensional ones. Several S-networks with sequentially more complex properties are considered, as a result of which the possibility of reducing a two-dimensional optimization problem to a one-dimensional one for S-networks with the most general assumptions regarding the type of flow of requests arriving in the S-network and the distribution laws of the sizes of requests has been established.

All analytical solutions presented in the article have been verified and confirmed by simulation modeling.

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