Mathematical Model of the Tandem Retrial Queue M | GI | 1 | M | 1 with a Common Orbit

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Abstract. This paper considers a retrial tandem queue with single orbit, Poisson arrivals of incoming calls and without intermediate buffer. The first server provides services for incoming calls for an arbitrary random time, while the second server does for an exponentially distributed random time. Blocked customers at either the first server or the second server join the orbit and stay there for an exponentially distributed time before retrying to enter the first server again. Under an asymptotic condition when the mean of retrial intervals is extremely large, we derive a diffusion limit, which is further utilized to obtain an approximation to the number of customers in the orbit in stationary regime.

Keywords: tandem queue · retrial queue · diffusion limit

1 Introduction

The new feature of retrial queues in comparison with the conventional ones is that blocked customers that cannot find an idle server upon arrival join the orbit and retry for service after some random time. These models have been extensively studied in the literature; see the books [1,2] and survey papers [3,4]. The paper [4] summarizes major analytical results on retrial queues up to 1990 for both single server and multiserver models. Reference [3] presents a careful survey on single server retrial models with and without impatient customers. Furthermore, a survey of recent results for retrial queues is presented in [5].

The analysis of retrial queues is more difficult in comparison with that of counterparts with infinite buffer because each orbiting customer independently retries leading to a total retrial rate that is proportional to the number of customers in the orbit.

Tandem queues are simple networks of queues connected in a line topology are widely used in many applications such as computer communication, manufacturing and service systems. For example, in call centers, customers first connect

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to IVR (Interactive Voice Response) unit and then to operators [6]. Some other applications can be found in transmitting multimedia information [7], and in [8] for modelling a multi-agent robotic system, etc.

To our knowledge, only a little attention was paid the study of on tandem queues with retrials due to the complex of these models. In [9], the authors consider a tandem system of two sequentially connected servers without an intermediate buffer. In this system the blocking phenomenon occurs at the first server when a customer finishes the service a the first server but sees the second server busy. Customers that cannot enter the first server because the server is busy or blocked join the orbit and retry to enter the first server according to a constant retrial rate policy. Furthermore, [10] presents an approximate analysis for a tandem queue with a common orbit and constant retrial rate.

As a closely related paper, Phung-Duc [11] obtained an explicit solution for a simple model where only blocked customers the first server joins the orbit while blocked customers at the second server are lost. In this line, [12] presented a matrix-analytic solution for a model with Batch Markovian Arrival Process (BMAP) and general service time distribution at the first server and customers from the first server are lost if the second server is busy.

Furthermore, in our recent papers, we obtained the approximation of the stationary probability distribution of the number of calls in the orbit by methods of asymptotic analysis [13] and asymptotic diffusion analysis [14] for a special case with exponential distributions for service times in both servers. Further related papers can be found in [15,16]. In [16] a fixed point approximation is proposed for a tandem retrial queue. Pourbabai [15] investigates the tandem behavior in telecommunication systems with finite buffer and with repeated calls of constant retrial time. In [15], an approximation method is proposed.

In this paper, we study the two-phase tandem retrial queue system with one orbit and arbitrary service time distribution at the first server by the method of asymptotic diffusion analysis under the condition when the delay of calls in the orbit is extremely large. To the best of our knowledge, this is the first work dealing with a tandem retrial queue with classical (linear) retrial rate and arbitrary service time distribution at the first server, where blocked customers at the first or the second server enters orbit.

The remaining parts of the paper are organized as follows. In Sect. 2, we present the description of the model in detail. In Sect. 3, we write down the set of Kolmogorov differential equations while Sects. 4 and 5 show to the first order analysis (fluid limit) and the second order analysis (diffusion limit). Section 6 shows the use of the diffusion limit to approximate queue-length distribution in the orbit in the steady-state. Section 7 demonstrates some numerical examples.

2 Analytical Model

We consider a tandem retrial queue with two sequentially connected servers where customers arrive at the server according to a Poisson process with rate λ (see the Fig. 1). In this paper, customers and calls are interchangeably used.

If the server is idle upon the arrival of a call, the call occupies it immediately for a random time with the distribution function $B(x)$ and then moves to the second server. In the case that the second server is free, the call occupies it for a random time exponentially distributed with mean $1/\mu$. On the other hand, if the first server is busy upon arrival of a customer, this customer immediately goes to the orbit staying there for a period of time which is exponentially distributed with parameter σ and then tries to enter the first server again. Upon the service completion at the first server, if the second server is busy, the call immediately goes to the same orbit, staying there for a random period of time which is exponentially distributed mean $1/\sigma$ and trying to enter the first server for service again. This process is repeated until the call successfully receives services from both servers and leave the system.

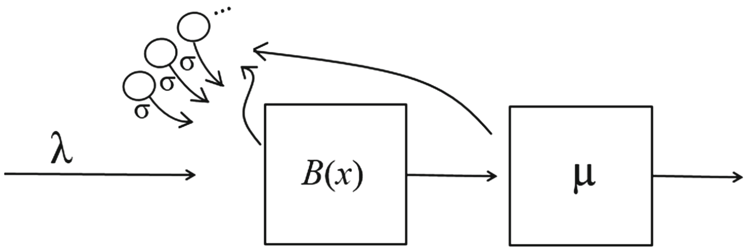

Fig. 1. The model

We define the following notations for further analysis.

The process $k(t)$ - the state of servers at time t: 0, if both servers are free; 1, if the first server is busy and the second one is free; 2, if the first server is free and the second one is busy; 3, if both servers are busy;

The process $z(t)$ - the remainder of service at the first server at time t;

The process $i(t)$ - number of retrial customers in the orbit at time t.

The purpose of the study is twofold: 1) to obtain the fluid and diffusion limit of $i(t)$ and 2) based on the diffusion limit, to build an approximation to the steady-state distribution of $i(t)$.

3 Kolmogorov Backward Equations

We define probabilities

$$P_k(i,t) = P\{k(t) = k, i(t) = i\}, k = 0, 2,$$

$$P_k(i,z,t) = P\{k(t) = k, i(t) = i, z(t) < z\}, k = 1, 3. \tag{1}$$

The process $\{k(t), i(t)\}$, $k = 0, 2$, $\{k(t), i(t), z(t)\}$, $k = 1, 3$ is a Markov chain. Kolmogorov backward equations for (1) are given as follows.

$$\frac{\partial P_0(i,t)}{\partial t} = -(\lambda + i\sigma)P_0(i,t) + \mu P_2(i,t),$$

$$\frac{\partial P_1(i,z,t)}{\partial t} = \frac{\partial P_1(i,z,t)}{\partial z} - \frac{\partial P_1(i,0,t)}{\partial z} - \lambda P_1(i,z,t)$$
$$+ (i+1)\sigma B(z) P_0(i+1,t) + \lambda P_1(i-1,z,t)$$
$$+ \lambda B(z) P_0(i,t) + P_3(i,z,t)\mu,$$
$$\frac{\partial P_2(i,t)}{\partial t} = \frac{\partial P_1(i,0,t)}{\partial z} + \frac{\partial P_3(i-1,0,t)}{\partial z} - (\lambda+\mu+i\sigma)P_2(i,t),$$
$$\frac{\partial P_3(i,z,t)}{\partial t} = \frac{\partial P_3(i,z,t)}{\partial z} + \frac{\partial P_3(i,0,t)}{\partial z} - (\lambda+\mu)P_3(i,z,t)$$
$$+ \lambda P_3(i-1,z,t) + \lambda B(z) P_2(i,t) + (i+1)\sigma B(z) P_2(i+1,t). \tag{2}$$

We define partial characteristic functions, using $j = \sqrt{-1}$

$$H_k(u,t) = \sum_{i=0}^{\infty} e^{jui} P_k(i,t), k = 0,2.$$

$$H_k(u,z,t) = \sum_{i=0}^{\infty} e^{jui} P_k(i,z,t), k = 1,3. \tag{3}$$

We rewrite (2) using $H_k(u,t), k = 0,2, H_k(u,z,t), k = 1,3$ and add all the resulted equations with $z \to \infty$. We obtain following equations for further research in next sections.

$$\frac{\partial H_0(u,t)}{\partial t} = -\lambda H_0(u,t) + j\sigma \frac{\partial H_0(u,t)}{\partial u} + \mu H_2(u,t),$$
$$\frac{\partial H_1(u,z,t)}{\partial t} = \frac{\partial H_1(u,z,t)}{\partial z} - \frac{\partial H_1(u,0,t)}{\partial z} - j\sigma e^{-ju} \frac{\partial H_0(u,t)}{\partial u} B(z)$$
$$+ \lambda(e^{ju}-1)H_1(u,z,t) + \lambda B(z) H_0(u,t) + \mu H_3(u,z,t),$$
$$\frac{\partial H_2(u,t)}{\partial t} = \frac{\partial H_1(u,0,t)}{\partial z} + e^{ju}\frac{\partial H_3(u,0,t)}{\partial z} + j\sigma \frac{\partial H_2(u,t)}{\partial u} - (\lambda+\mu)H_2(u,t),$$
$$\frac{\partial H_3(u,z,t)}{\partial t} = \frac{\partial H_3(u,z,t)}{\partial z} - \frac{\partial H_3(u,0,t)}{\partial z} - j\sigma e^{-ju} B(z) \frac{\partial H_2(u,t)}{\partial u}$$
$$+ (\lambda(e^{ju}-1)-\mu)H_3(u,z,t) + \lambda B(z) H_2(u,t),$$
$$\frac{\partial H(u,t)}{\partial t} = (e^{ju}-1)\left\{ j\sigma e^{-ju}\left(\frac{\partial H_0(u,t)}{\partial u} + \frac{\partial H_2(u,t)}{\partial u}\right)\right.$$
$$\left. + \lambda(H_1(u,t) + H_3(u,t)) + \frac{\partial H_3(u,0,t)}{\partial z}\right\}. \tag{4}$$

We are going to solve (4) under $\sigma \to 0$.

4 Fluid Limit

By denoting $\sigma = \varepsilon$ and performing substitution in (4)

$$\tau = t\varepsilon, u = \varepsilon w, H_k(u,t) = F_k(w,\tau,\varepsilon,),$$
$$H_k(u,z,t) = F_k(w,z,\tau,\varepsilon), \tag{5}$$

we obtain

$$\varepsilon\frac{\partial F_0(w,\tau,\varepsilon)}{\partial\tau} = -\lambda F_0(w,\tau,\varepsilon) + j\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w} + \mu F_2(w,\tau,\varepsilon),$$
$$\varepsilon\frac{\partial F_1(w,z,\tau,\varepsilon)}{\partial\tau} = \frac{\partial F_1(w,z,\tau,\varepsilon)}{\partial z} - \frac{\partial F_1(w,0,\tau,\varepsilon)}{\partial z} - je^{-jw\varepsilon}\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w}B(z)$$
$$+ \lambda(e^{jw\varepsilon}-1)F_1(w,z,\tau,\varepsilon) + \lambda B(z)F_0(w,\tau,\varepsilon) + \mu F_3(w,z,\tau,\varepsilon),$$
$$\varepsilon\frac{\partial F_2(w,\tau,\varepsilon)}{\partial\tau} = \frac{\partial F_1(w,0,\tau,\varepsilon)}{\partial z} + e^{jw\varepsilon}\frac{\partial F_3(w,0,\tau,\varepsilon)}{\partial z}$$
$$+ j\frac{\partial F_2(w,\tau,\varepsilon)}{\partial w} - (\lambda+\mu)F_2(w,\tau,\varepsilon),$$
$$\varepsilon\frac{\partial F_3(w,z,\tau,\varepsilon)}{\partial\tau} = \frac{\partial F_3(w,z,\tau,\varepsilon)}{\partial z} - \frac{\partial F_3(w,0,\tau,\varepsilon)}{\partial z} - je^{-jw\varepsilon}B(z)\frac{\partial F_2(w,\tau,\varepsilon)}{\partial w}$$
$$+ (\lambda(e^{jw\varepsilon}-1)-\mu)F_3(w,z,\tau,\varepsilon) + \lambda B(z)F_2(w,\tau,\varepsilon),$$
$$\varepsilon\frac{\partial F(w,\tau,\varepsilon)}{\partial\tau} = (e^{jw\varepsilon}-1)\left\{je^{-jw\varepsilon}\left(\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w} + \frac{\partial F_2(w,\tau,\varepsilon)}{\partial w}\right)\right.$$
$$\left. + \lambda(F_1(w,\tau,\varepsilon) + F_3(w,\tau,\varepsilon)) + \frac{\partial F_3(w,0,\tau,\varepsilon)}{\partial z}\right\}, \tag{6}$$

which we will solve under the assumption that functions $F_k(w,\tau,\varepsilon)$, $F_k(w,z,\tau,\varepsilon)$ and their derivatives have limits as $\varepsilon \to 0$.

Theorem 1. *We have*

$$\lim_{\sigma\to 0} Me^{jw\sigma i\left(\frac{\tau}{\sigma}\right)} = e^{jwx(\tau)}, \tag{7}$$

where $x = x(\tau)$ satisfies

$$x'(\tau) = (1 + b_1(\lambda+x))^{-1}\left(\lambda b_1(\lambda+x) - x + B^*(\mu)\frac{(\lambda+x)^2}{\mu+\lambda+x}\right), \tag{8}$$

and where $b_1 = \int_0^\infty x dB(x)$ and $B^*(\mu) = \int_0^\infty e^{-\mu x} dB(x)$.

Proof. We take the limit $\varepsilon \to 0$ in (6)

$$-\lambda F_0(w,\tau) + j\frac{\partial F_0(w,\tau)}{\partial w} + \mu F_2(w,\tau) = 0,$$

$$\frac{\partial F_1(w,z,\tau)}{\partial z} - \frac{\partial F_1(w,0,\tau)}{\partial z} - j\frac{\partial F_0(w,\tau)}{\partial w}B(z)$$
$$+ \lambda B(z)F_0(w,\tau) + \mu F_3(w,z,\tau) = 0,$$

$$\frac{\partial F_1(w,0,\tau)}{\partial z} + \frac{\partial F_3(w,0,\tau)}{\partial z} + j\frac{\partial F_2(w,\tau)}{\partial w} - (\lambda+\mu)F_2(w,\tau) = 0,$$

$$\frac{\partial F_3(w,z,\tau)}{\partial z} - \frac{\partial F_3(w,0,\tau)}{\partial z} - jB(z)\frac{\partial F_2(w,\tau)}{\partial w}$$
$$+ (\lambda-\mu)F_3(w,z,\tau) + \lambda B(z)F_2(w,\tau) = 0,$$

$$\frac{\partial F(w,\tau)}{\partial \tau} = jw\left\{ j\left(\frac{\partial F_0(w,\tau)}{\partial w} + \frac{\partial F_2(w,\tau)}{\partial w}\right)\right.$$
$$\left. + \lambda(F_1(w,\tau) + F_3(w,\tau)) + \frac{\partial F_3(w,0,\tau)}{\partial z}\right\}. \quad (9)$$

We assume that (9) has a solution in the form

$$F_k(w,\tau) = r(x)e^{jwx(\tau)}, k = 0,2,\ F_k(w,z,\tau) = r(z,x)e^{jwx(\tau)}, k = 1,3, \quad (10)$$

where $x = x(\tau)$ expresses $\lim_{\sigma\to 0}\sigma i(\tau/\sigma)$. Substituting (10) into (9), we obtain

$$-(\lambda + x)r_0(x) + \mu r_2(x) = 0,$$

$$\frac{\partial r_1(z,x)}{\partial z} - \frac{\partial r_1(0,x)}{\partial z} + (\lambda+x)B(z)r_0(x) + +\mu r_3(z,x) = 0,$$

$$\frac{\partial r_1(0,x)}{\partial z} + \frac{\partial r_3(0,x)}{\partial z} - (\lambda+\mu+x)r_2(x) = 0,$$

$$\frac{\partial r_3(z,x)}{\partial z} - \frac{\partial r_3(0,x)}{\partial z} - \mu r_3(z,x) + (\lambda+\mu)B(z)r_2(x) = 0, \quad (11)$$

$$x'(\tau) = \lambda(r_1(x) + r_3(x)) - x(r_0(x) + r_2(x)) + \frac{\partial r_3(0,x)}{\partial z}. \quad (12)$$

Summing up the first equation with the third, the second equation with the fourth of (11), we have

$$\frac{\partial r_1(z,x)}{\partial z} + \frac{\partial r_3(0,x)}{\partial z} = (\lambda+x)(r_0(x) + r_2(x))$$

$$\frac{\partial r_3(z,x)}{\partial z} - \frac{\partial r_3(0,x)}{\partial z} + \frac{\partial r_1(z,x)}{\partial z} - \frac{\partial r_1(0,x)}{\partial z}$$
$$+ (\lambda+x)B(z)(r_0(x) + r_2(x)) = 0. \quad (13)$$

We denote

$$r_{02}(x) = r_0(x) + r_2(x),$$

$$r_{31}(z,x) = r_1(z,x) + r_3(z,x),$$

$$r_{31}(0,x) = r_1(0,x) + r_3(0,x).$$

Then from (13) we obtain

$$r_{31}(z,x) = (\lambda + x)r_{02}(x)\int\limits_0^z (1 - B(s))ds.$$

Letting $z \to \infty$ and denoting $r_k(\infty, x) = r_k(x)$, $k = 1, 3$, we have

$$r_1(x) + r_3(x) = (\lambda + x)b_1(r_0(x) + r_2(x)),$$

where $b_1 = \int_0^\infty x dB(x)$. Because $r_0(x)+r_1(x)+r_2(x)+r_3(x) = 1$, from the last equality we obtain

$$r_1(x) + r_3(x) = \frac{(\lambda + x)b_1}{1 + (\lambda + x)b_1},$$
$$r_0(x) + r_2(x) = \frac{1}{1 + (\lambda + x)b_1}.$$

Taking into account the first equation of (11), we write

$$r_0(x) = \frac{\mu}{\lambda + x} r_2(x).$$

We write the solution of the fourth differential equation of system (11) in the form

$$r_3(z,x) = e^{\mu z}\int\limits_0^z e^{-\mu s}\left(\frac{\partial r_3(0,x)}{\partial z} - (\lambda + x)B(s)r_2(x)\right)ds. \tag{14}$$

Let us send $z \to \infty$ in this equation to have

$$\mu\int\limits_0^\infty e^{-\mu s}\left(\frac{\partial r_3(0,x)}{\partial z} - (\lambda + x)B(s)r_2(x)\right)ds = 0.$$

The integrand satisfies the condition

$$\frac{\partial r_3(0,x)}{\partial z} = (\lambda + x)r_2(x)B^*(\mu), \tag{15}$$

where $B^*(\mu) = \int_0^\infty e^{-\mu x}dB(x)$. Solution (14), taking into account (15), we rewrite under $z \to \infty$ in the form

$$r_3(x) = (\lambda + x)r_2(x)(1 - B^*(\mu)).$$

We obtain equations for the stationary probability distribution $r_k(x)$, $k = \overline{0,3}$ of the states of servers

$$r_0(x) = \frac{\mu}{\mu+\lambda+x}(1+b_1(\lambda+x))^{-1},$$

$$r_1(x) = (1+b_1(\lambda+x))^{-1}(\lambda+x)\left(b_1 - \frac{1}{\mu}\frac{\lambda+x}{\mu+\lambda+x}\left(1-B^*(\mu)\right)\right),$$

$$r_2(x) = \frac{\lambda+x}{\mu+\lambda+x}(1+b_1(\lambda+x))^{-1},$$

$$r_3(x) = \frac{1}{\mu}\frac{(\lambda+x)^2}{\mu+\lambda+x}\left(1-B^*(\mu)\right)(1+b_1(\lambda+x))^{-1}. \tag{16}$$

Let us substitute $r_k(x)$ into (12) in order to obtain

$$x'(\tau) = (1+b_1(\lambda+x))^{-1}\left(\lambda b_1(\lambda+x) - x + B^*(\mu)\frac{(\lambda+x)^2}{\mu+\lambda+x}\right), \tag{17}$$

which coincides with (8).

Since $x(\tau)$ represents the asymptotic value ($\varepsilon \to 0$) of $\sigma i(\tau/\sigma)$, (7) holds. So, Theorem 1 is proved.

Let us denote

$$a(x) = x'(\tau) = (1+b_1(\lambda+x))^{-1}\left(\lambda b_1(\lambda+x) - x + B^*(\mu)\frac{(\lambda+x)^2}{\mu+\lambda+x}\right). \tag{18}$$

$a(x)$ plays an important role for our analysis. First, as it is shown in Theorem 1, $a(x)$ represents the dynamic of $x(\tau)$, which is the limit under $\sigma \to 0$ for $\sigma i(\tau/\sigma)$. Second, as it will be shown, $a(x)$ expresses the drift coefficient for the diffusion process that represents a scaled version of $i(t)$.

5 Diffusion Limit

We carry out the following substitution in (4)

$$H_k(u,t) = e^{j\frac{u}{\sigma}x(\sigma t)}H_k^{(1)}(u,t), k = 0,2$$

$$H_k(u,z,t) = e^{j\frac{u}{\sigma}x(\sigma t)}H_k^{(1)}(u,z,t), k = 1,3. \tag{19}$$

For $H_k^{(1)}(u,t)$ and $H_k^{(1)}(u,z,t)$, $k = \overline{0,3}$, considering (18), we obtain

$$\frac{\partial H_0^{(1)}(u,t)}{\partial t} = -(\lambda + jua(x) + x)H_0^{(1)}(u,t)$$

$$+ j\sigma\frac{\partial H_0^{(1)}(u,t)}{\partial u} + \mu H_2^{(1)}(u,t),$$

$$\frac{\partial H_1^{(1)}(u,z,t)}{\partial t} = \frac{\partial H_1^{(1)}(u,z,t)}{\partial z} - \frac{\partial H_1^{(1)}(u,0,t)}{\partial z} - j\sigma e^{-ju}\frac{\partial H_0^{(1)}(u,t)}{\partial u}B(z)$$
$$+ (\lambda(e^{ju}-1) - jua(x))H_1^{(1)}(u,z,t) + (\lambda + xe^{-ju})B(z)H_0^{(1)}(u,t) + \mu H_3^{(1)}(u,z,t),$$
$$\frac{\partial H_2^{(1)}(u,t)}{\partial t} = \frac{\partial H_1^{(1)}(u,0,t)}{\partial z} + e^{ju}\frac{\partial H_3^{(1)}(u,0,t)}{\partial z} + j\sigma\frac{\partial H_2^{(1)}(u,t)}{\partial u}$$
$$-(\lambda + \mu + jua(x) + x)H_2^{(1)}(u,t),$$
$$\frac{\partial H_3^{(1)}(u,z,t)}{\partial t} = \frac{\partial H_3^{(1)}(u,z,t)}{\partial z} - \frac{\partial H_3^{(1)}(u,0,t)}{\partial z} - j\sigma e^{-ju}B(z)\frac{\partial H_2^{(1)}(u,t)}{\partial u}$$
$$+ (\lambda(e^{ju}-1) - \mu - jua(x))H_3^{(1)}(u,z,t) + (\lambda + xe^{-ju})B(z)H_2^{(1)}(u,t),$$
$$\frac{\partial H^{(1)}(u,t)}{\partial t} + jua(x)H^{(1)}(u,t)$$
$$= (e^{ju}-1)\left\{j\sigma e^{-ju}\left(\frac{\partial H_0^{(1)}(u,t)}{\partial u} + \frac{\partial H_2^{(1)}(u,t)}{\partial u}\right)\right.$$
$$- xe^{-ju}(H_0^{(1)}(u,t) + H_2^{(1)}(u,t))$$
$$\left. + \lambda(H_1^{(1)}(u,t) + H_3^{(1)}(u,t)) + \frac{\partial H_3^{(1)}(u,0,t)}{\partial z}\right\}. \tag{20}$$

Because $H^{(1)}(u,t)$ is the characteristic function of $i(t) - \frac{1}{\sigma}x(\sigma t)$, we make the substitutions as follows.

By defining $\sigma = \varepsilon^2$ in (20) and substituting

$$\tau = t\varepsilon^2,\ u = w\varepsilon, H_k^{(1)}(u,t) = F_k^{(1)}(w,\tau,\varepsilon),\ k = 0,2,$$
$$H_k^{(1)}(u,z,t) = F_k^{(1)}(w,z,\tau,\varepsilon),\ k = 1,3, \tag{21}$$

we obtain

$$\varepsilon^2\frac{\partial F_0^{(1)}(w,\tau,\varepsilon)}{\partial \tau} = -(\lambda + j\varepsilon wa(x) + x)F_0^{(1)}(w,z,\tau,\varepsilon)$$
$$+ j\varepsilon\frac{\partial F_0^{(1)}(w,\tau,\varepsilon)}{\partial w} + \mu F_2^{(1)}(w,\tau,\varepsilon),$$
$$\varepsilon^2\frac{\partial F_1^{(1)}(w,z,\tau,\varepsilon)}{\partial \tau} = \frac{\partial F_1^{(1)}(w,z,\tau,\varepsilon)}{\partial z} - \frac{\partial F_1^{(1)}(w,0,\tau,\varepsilon)}{\partial z}$$
$$- j\varepsilon e^{-jw\varepsilon}\frac{\partial F_0^{(1)}(w,\tau,\varepsilon)}{\partial w}B(z) + (\lambda(e^{jw\varepsilon}-1) - j\varepsilon wa(x))F_1^{(1)}(w,z,\tau,\varepsilon)$$
$$+ (\lambda + xe^{-jw\varepsilon})B(z)F_0^{(1)}(w,\tau,\varepsilon) + \mu F_3^{(1)}(w,z,\tau,\varepsilon),$$
$$\varepsilon^2\frac{\partial F_2^{(1)}(w,\tau,\varepsilon)}{\partial \tau} = \frac{\partial F_1^{(1)}(w,0,\tau,\varepsilon)}{\partial z} + e^{jw\varepsilon}\frac{\partial F_3^{(1)}(w,0,\tau,\varepsilon)}{\partial z}$$

$$+ j\varepsilon \frac{\partial F_2^{(1)}(w,\tau,\varepsilon)}{\partial w} - (\lambda + \mu + j\varepsilon w a(x) + x) F_2^{(1)}(w,\tau,\varepsilon),$$

$$\varepsilon^2 \frac{\partial F_3^{(1)}(w,z,\tau,\varepsilon)}{\partial \tau} = \frac{\partial F_3^{(1)}(w,z,\tau,\varepsilon)}{\partial z} - \frac{\partial F_3^{(1)}(w,0,\tau,\varepsilon)}{\partial z}$$

$$- j\varepsilon e^{-jw\varepsilon} B(z) \frac{\partial F_2^{(1)}(w,\tau,\varepsilon)}{\partial w} + (\lambda(e^{jw\varepsilon} - 1) - \mu - j\varepsilon w a(x)) F_3^{(1)}(w,z,\tau,\varepsilon)$$

$$+ (\lambda + x e^{-jw\varepsilon}) B(z) F_2^{(1)}(w,\tau,\varepsilon),$$

$$\varepsilon^2 \frac{\partial F^{(1)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F^{(1)}(w,\tau,\varepsilon)$$

$$= (e^{jw\varepsilon} - 1) \Bigg\{ j\varepsilon e^{-jw\varepsilon} \left(\frac{\partial F_0^{(1)}(w,\tau,\varepsilon)}{\partial w} + \frac{\partial F_2^{(1)}(w,\tau,\varepsilon)}{\partial w} \right)$$

$$- x e^{-jw\varepsilon} (F_1^{(1)}(w,\tau,\varepsilon) + F_2^{(1)}(w,\tau,\varepsilon))$$

$$+ \lambda (F_1^{(1)}(w,\tau,\varepsilon) + F_3^{(1)}(w,\tau,\varepsilon)) + \frac{\partial F_3^{(1)}(w,0,\varepsilon)}{\partial z} \Bigg\}. \tag{22}$$

which we will solve under the assumption that $F_k^{(1)}(w,\tau,\varepsilon)$, $F_k^{(1)}(w,z,\tau,\varepsilon)$ and their derivatives have limits as $\varepsilon \to 0$.

Theorem 2. *$F_k^{(1)}(w,\tau)$ is given by*

$$F_k^{(1)}(w,\tau) = \Phi(w,\tau) r_k(x), k = \overline{0,3} \tag{23}$$

where $\Phi(w,\tau)$ satisfies

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = a'(x) w \frac{\partial \Phi(w,\tau)}{\partial w} + b(x) \frac{(jw)^2}{2} \Phi(w,\tau) \tag{24}$$

and $r_k(x)$ is defined in (16). $a(x)$ is defined by (18) and $b(x)$ is given by

$$b(x) = a(x) + 2(\lambda(g_1(x) + g_3(x)) + g_3'(0,x) - x(g_0(x) + g_2(x) - r_0(x) - r_2(x))), \tag{25}$$

where

$$g_3'(0,x) = (\lambda + x) B^*(\mu) g_2(x) + ((a(x) - \lambda)(\lambda + x) B^{*\prime}(\mu) - x B^*(\mu)) \tag{26}$$

and $g_k(x)$, $k = \overline{0,3}$ are defined by

$$-(\lambda + x) g_0(x) + \mu g_2(x) = a(x) r_0(x),$$

$$(\lambda + x) g_0(x) + ((\lambda + x)(B^*(\mu) - 1) + \mu) g_2(x) + \mu g_3(x)$$

$$= x r_0(x) + (a(x) - \lambda) r_1(x) - ((a(x) - \lambda)(\lambda + x) B^{*\prime}(\mu) - a(x) + \lambda B^*(\mu)) r_2(x),$$

$$g_1(x) + g_3(x) - (\lambda + x) b_1 (g_2(x) + g_0(x))$$

$$= \left((\lambda - a(x))(\lambda + x) \frac{b_2}{2} - x b_1 \right) (r_0(x) + r_2(x)),$$

$$g_0(x) + g_1(x) + g_2(x) + g_3(x) = 0, \tag{27}$$

and where $b_2 = \int_0^\infty x^2 dB(x)$.

Proof. The methodology of the proof is similar to that used in paper [14] before.

As it will be shown $a(x)$ in (18) and $b(x)$ in (25) are coefficients of a diffusion process. Later we will show their role in the approximation of the stationary distribution of $i(t)$.

Remark 1. The results in Theorem 2 show that in the heavy traffic regime ($\sigma \to 0$) $i(t)$ and the state of the servers are independent as their joint characteristic function is decomposed as a product of the orbit part and the server part.

6 Approximation of the Stationary Distribution Based on Diffusion Limit

In this section, we apply the diffusion limit to find the probability distribution of $i(t)$ under $\sigma \to 0$ in our system. This general method is also used other related work e.g. [14].

Lemma 1. *Under* $\sigma \to 0$

$$y(\tau) = \lim_{\sigma \to 0} \sqrt{\sigma} \left\{ i(\tau/\sigma) - \frac{1}{\sigma} x(\tau) \right\}, \tag{28}$$

is the solution of

$$dy(\tau) = a'(x) y d\tau + \sqrt{b(x)} dw(\tau). \tag{29}$$

We consider

$$l(\tau) = x(\tau) + \varepsilon y(\tau),$$

where $\varepsilon = \sqrt{\sigma}$ as before.

Lemma 2. *The process* $l(\tau)$ *is the solution of*

$$dl(\tau) = a(l) d\tau + \sqrt{\sigma b(l)} dw(\tau) \tag{30}$$

up to an infinitesimal of order ϵ^2.

Under the steady-state regime, we consider $l(\tau)$

$$s(l, \tau) = s(l) = \frac{\partial P\{l(\tau) < l\}}{\partial l}. \tag{31}$$

Theorem 3. *The density* $s(l)$ *of* $l(\tau)$ *is given by*

$$s(l) = \frac{C}{b(l)} \exp\left\{ \frac{2}{\sigma} \int_0^l \frac{a(x)}{b(x)} dx \right\}, \tag{32}$$

where C *is some constant that satisfies the normalization condition.*

7 Numerical Examples

Let us consider $G(i)$ in the form

$$G(i) = \frac{C}{b(\sigma i)} \exp\left\{ \frac{2}{\sigma} \int\limits_{0}^{\sigma i} \frac{a(x)}{b(x)} dx \right\}, \tag{33}$$

and define $P(i)$ as

$$P(i) = \frac{G(i)}{\sum\limits_{i=0}^{\infty} G(i)}. \tag{34}$$

We use $P(i)$ to approximate $P\{i(t) = i\}$.

We consider a particular case of $B(x)$ as a Gamma distribution with parameters of shape $\alpha = 2$ and of scale $\beta = 2$. We consider $\lambda = 0.5$ and $\mu = 1$.

Figure 2 presents the approximation of the probability distribution of the $i(t)$ with different values of calls' delay time in the orbit: P1 - the approximation with $\sigma = 0.5$, P2 - the approximation with $\sigma = 0.3$, P3 - the approximation with $\sigma = 0.1$.

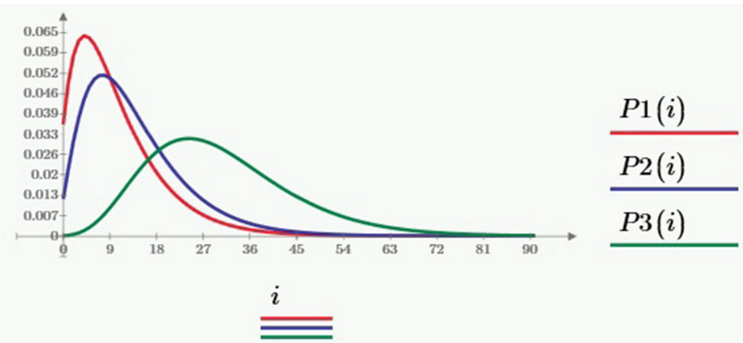

Fig. 2. The probability distribution $i(t)$

This figure shows the feasibility of our proposed approach.

8 Conclusion

In this paper, we have investigated the tandem retrial queue with two connected servers and without intermediate buffer. The first server provides services for calls for an arbitrary random time, while the second does for an exponentially distributed random time. Under the condition that $\sigma \to 0$, we have obtained diffusion limit of a scaled version of $i(t)$. The stationary probability density distribution of this diffusion process is used to approximate the stationary distribution of $i(t)$.

In further research, we plan to compare our approximate results with simulation

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