Chapter 8
Elliptic Problems

8.1 The Laplacian

The *Laplacian*, defined by

$$\Delta u = \text{div } \nabla u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_N^2},$$

is related to the mean of functions.

Definition 8.1.1 Let Ω be an open subset of $\mathbb{R}^N$ and $u \in L^1_{\text{loc}}(\Omega)$. The mean of u is defined on

$$D = \{(x, r) : x \in \Omega, 0 < r < d(x, \partial\Omega)\}$$

by

$$M(x, r) = V_N^{-1} \int_{B_N} u(x + ry)dy.$$

Lemma 8.1.2 *Let $u \in \mathcal{C}^2(\Omega)$. The mean of u satisfies on D the relation*

$$\lim_{r \downarrow 0} 2\frac{N+2}{r^2}[M(x, r) - u(x)] = \Delta u(x).$$

Proof Since we have uniformly for $|y| < 1$,

$$u(x + ry) = u(x) + r\nabla u(x) \cdot y + \frac{r^2}{2} D^2 u(x)(y, y) + o(r^2),$$

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we obtain by symmetry

$$\int_{B_N} x_j dx = 0, \int_{B_N} x_j x_k dx = 0, \, j \neq k, \int_{B_N} x_j^2 dx = \frac{V_N}{N+2},$$

and

$$M(x,r) = u(x) + \frac{r^2}{2}\frac{1}{N+2}\Delta u(x) + o(r^2). \qquad \square$$

Lemma 8.1.3 *Let $u \in \mathcal{C}^2(\Omega)$. The following properties are equivalent:*

(a) $\Delta u \leq 0$*;*
(b) for all $(x,r) \in D$*,* $M(x,r) \leq u(x)$*.*

Proof By the preceding lemma, (a) follows from (b).

Assume that (a) is satisfied. Differentiating under the integral sign and using the divergence theorem, we obtain

$$\frac{\partial M}{\partial r}(x,r) = V_N^{-1}\int_{B_N} \nabla u(x+ry)\cdot y dy = r V_N^{-1}\int_{B_N} \Delta u(x+ry)\frac{1-|y|^2}{2}dy \leq 0.$$

We conclude that

$$M(x,r) \leq \lim_{r \downarrow 0} M(x,r) = u(x). \qquad \square$$

Definition 8.1.4 Let $u \in L^1_{\text{loc}}(\Omega)$. The function u is superharmonic if for every $v \in \mathcal{D}(\Omega)$ such that $v \geq 0$, $\int_\Omega u \Delta v dx \leq 0$.

The function u is subharmonic if $-u$ is superharmonic.

The function u is harmonic if for every $v \in \mathcal{D}(\Omega)$, $\int_\Omega u\Delta v dx = 0$.

We extend Lemma 8.1.3 to locally integrable functions.

Theorem 8.1.5 (Mean-Value Inequality) *Let $u \in L^1_{\text{loc}}(\Omega)$. The following properties are equivalent:*

(a) u is superharmonic;
(b) for almost all $x \in \Omega$ and for all $0 < r < d(x, \partial\Omega)$*,* $M(x,r) \leq u(x)$*.*

Proof Let $u_n = \rho_n * u$. Property (a) is equivalent to

(c) for every n, $\Delta u_n \leq 0$ on Ω_n.

Property (b) is equivalent to

(d) for all $x \in \Omega_n$ and for all $0 < r < d(x, \partial\Omega_n)$, $V_N^{-1}\int_{B_N} u_n(x+ry)dy \leq u_n(x)$.

We conclude the proof using Lemma 8.1.3.

(a) ⇒ (c). By Proposition 4.3.6, we have on Ω_n that

$$\Delta u_n(x) = \Delta\rho_n * u(x) = \int_\Omega \big(\Delta\rho_n(x-y)\big)u(y)dy \le 0.$$

(c) ⇒ (a). It follows from the regularization theorem that for every $v \in \mathcal{D}(\Omega)$, $v \ge 0$,

$$\int_\Omega u\Delta v dx = \lim_{n\to\infty}\int_\Omega u_n\Delta v dx = \lim_{n\to\infty}\int_\Omega (\Delta u_n)v dx \le 0.$$

(b) ⇒ (d). We have on Ω_n that

$$V_N^{-1}\int_{B_N} u_n(x+ry)dy = V_N^{-1}\int_{B(0,1/n)} dz \int_{B_N} \rho_n(z)u(x+ry-z)dy$$

$$\le \int_{B(0,1/n)} \rho_n(z)u(x-z)dz = u_n(x).$$

(d) ⇒ (b). For $j \ge 1$, we define

$$\omega_j = \{x \in \Omega : d(x,\partial\Omega) > 1/j \text{ and } |x| < j\}.$$

Proposition 4.2.10 and the regularization theorem imply the existence of a subsequence (u_{n_k}) converging to u in $L^1(\omega_j)$ and almost everywhere on ω_j. Hence for almost all $x \in \omega_j$ and for all $0 < r < d(x,\partial\omega_j)$, $M(x,r) \le u(x)$. Since $\Omega = \bigcup_{j=1}^{\infty}\omega_j$, property (b) is satisfied. □

Theorem 8.1.6 (Maximum Principle) *Let Ω be an open connected subset of $\mathbb{R}^N$ and $u \in L^1_{\text{loc}}(\Omega)$ a superharmonic function such that $u \ge 0$ almost everywhere on Ω and $u = 0$ on a subset of Ω with positive measure. Then $u = 0$ almost everywhere on Ω.*

Proof Define

$$U_1 = \{x \in \Omega : \text{there exists } 0 < r < d(x,\partial\Omega) \text{ such that } M(x,r) = 0\}.$$

$$U_2 = \{x \in \Omega : \text{there exists } 0 < r < d(x,\partial\Omega) \text{ such that } M(x,r) > 0\}.$$

It is clear that U_1 and U_2 are open subsets of Ω such that $\Omega = U_1 \cup U_2$. By the preceding theorem, we obtain

$$U_2 = \{x \in \Omega : \text{for all } 0 < r < d(x, \partial\Omega), M(x, r) > 0\},$$

so that U_1 and U_2 are disjoint. If $\Omega = U_2$, then $u > 0$ almost everywhere on Ω by the preceding theorem. We conclude that $\Omega = U_1$ and $u = 0$ almost everywhere on Ω. □

8.2 Eigenfunctions

En nous servant de quelques conceptions de l'analyse fonctionnelle nous représentons notre problème dans une forme nouvelle et démontrons que dans cette forme le problème admet toujours une solution unique.
Si la solution cherchée existe dans le sens classique, alors notre solution se confond avec celle-ci.

S.L. Sobolev

Let Ω be a smooth bounded open subset of $\mathbb{R}^N$ with frontier Γ. An *eigenfunction* corresponding to the *eigenvalue* λ is a nonzero solution of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{$\mathcal{P}$}$$

We will use the following *weak formulation* of problem $(\mathcal{P})$: find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega uv \, dx.$$

Theorem 8.2.1 *There exist an unbounded sequence of eigenvalues of* $(\mathcal{P})$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

and a sequence of corresponding eigenfunctions that is a Hilbert basis of $H_0^1(\Omega)$.

Proof On the space $H_0^1(\Omega)$, we define the inner product

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and the corresponding norm $||u||_a = \sqrt{a(u, u)}$.

For every $u \in H_0^1(\Omega)$, there exists one and only one $Au \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$a(Au, v) = \int_\Omega uv\, dx.$$

Hence problem $(\mathcal{P})$ is equivalent to

$$\lambda^{-1}u = Au.$$

Since $a(Au, u) = \int_\Omega u^2 dx$, the eigenvalues of A are strictly positive. The operator A is symmetric, since

$$a(Au, v) = \int_\Omega uv\, dx = a(u, Av).$$

It follows from the Cauchy–Schwarz and Poincaré inequalities that

$$||Au||_a^2 = \int_\Omega u\ Au\ dx \le ||u||_{L^2(\Omega)}||Au||_{L^2(\Omega)} \le c||u||_{L^2(\Omega)}||Au||_a.$$

Hence

$$||Au||_a \le c||u||_{L^2(\Omega)}.$$

By the Rellich–Kondrachov theorem, the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact, so that the operator A is compact. We conclude using Theorem 3.4.8. □

Proposition 8.2.2 (Poincaré's Principle) *For every $n \ge 1$,*

$$\lambda_n = \min\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega u^2 dx = 1, \int_\Omega ue_1 dx = \ldots = \int_\Omega ue_{n-1} dx = 0\right\}.$$

Proof We deduce from Theorem 3.4.7 that

$$\lambda_n^{-1} = \max\left\{\frac{a(Au, u)}{a(u, u)} : u \in H_0^1(\Omega), u \ne 0, a(u, e_1) = \ldots = a(u, e_{n-1}) = 0\right\}.$$

Since e_k is an eigenfunction,

$$a(u, e_k) = 0 \Longleftrightarrow \int_\Omega ue_k dx = 0.$$

Hence we obtain

$$\lambda_n^{-1} = \max\left\{\frac{\int_\Omega u^2 dx}{\int_\Omega |\nabla u|^2 dx} : u \in H_0^1(\Omega), u \neq 0, \int_\Omega ue_1 dx = \ldots = \int_\Omega ue_{n-1} dx = 0\right\},$$

or

$$\lambda_n = \min\left\{\frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} : u \in H_0^1(\Omega), u \neq 0, \int_\Omega ue_1 dx = \ldots = \int_\Omega ue_{n-1} dx = 0\right\}. \square$$

Proposition 8.2.3 *Let $u \in H_0^1(\Omega)$ be such that $||u||_2 = 1$ and $||\nabla u||_2^2 = \lambda_1$. Then u is an eigenfunction corresponding to the eigenvalue λ_1.*

Proof Let $v \in H_0^1(\Omega)$. The function

$$g(\varepsilon) = ||\nabla(u + \varepsilon v)||_2^2 - \lambda_1 ||u + \varepsilon v||_2^2$$

reaches its minimum at $\varepsilon = 0$. Hence $g'(0) = 0$ and

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \lambda_1 \int_\Omega uv \, dx = 0. \qquad \square$$

Proposition 8.2.4 *Let Ω be a smooth bounded open connected subset of $\mathbb{R}^N$. Then the eigenvalue λ_1 of $(\mathcal{P})$ is simple, and e_1 is almost everywhere strictly positive on Ω.*

Proof Let u be an eigenfunction corresponding to λ_1 and such that $||u||_2 = 1$. By Corollary 6.1.14, $v = |u| \in H_0^1(\Omega)$ and $||\nabla v||_2^2 = ||\nabla u||_2^2 = \lambda_1$. Since $||v||_2 = ||u||_2 = 1$, the preceding proposition implies that v is an eigenfunction corresponding to λ_1. Assume that $u^+ \neq 0$. Then u^+ is an eigenfunction corresponding to λ_1, and by the maximum principle, $u^+ > 0$ almost everywhere on Ω. Hence $u = u^+$. Similarly, if $u^- \neq 0$, then $-u = u^- > 0$ almost everywhere on Ω. We can assume that $e_1 > 0$ almost everywhere on Ω. If e_2 corresponds to λ_1, then e_2 is either positive or negative, and $\int_\Omega e_1 e_2 dx = 0$. This is a contradiction. $\square$

Example Let $\Omega = \,]0, \pi[$. Then $(\mathcal{P})$ becomes

$$\begin{cases} -u'' = \lambda u & \text{in }]0, \pi[, \\ u(0) = u(\pi) = 0. \end{cases}$$

Sobolev's embedding theorem and the du Bois–Reymond lemma imply that $u \in \mathcal{C}^2(]0, \pi[) \cap \mathcal{C}([0, \pi])$. Hence $\lambda_n = n^2$ and $e_n = \sqrt{\frac{2}{\pi}} \frac{\sin nx}{n}$. The sequence (e_n) is a

Hilbert basis on $H_0^1(]0,\pi[)$ with scalar product $\int_0^\pi u'v'\,dx$, and the sequence (ne_n) is a Hilbert basis of $L^2(]0,\pi[)$ with scalar product $\int_0^\pi uv\,dx$.

Definition 8.2.5 Let G be a subgroup of the orthogonal group $\mathbf{O}(N)$. The open subset Ω of $\mathbb{R}^N$ is G-invariant if for every $g \in G$ and every $x \in \Omega$, $g^{-1}x \in \Omega$. Let Ω be G-invariant. The action of G on $H_0^1(\Omega)$ is defined by $gu(x) = u(g^{-1}x)$. The space of fixed points of G is defined by

$$\mathrm{Fix}(G) = \{u \in H_0^1(\Omega) : \text{for every } g \in G, gu = u\}.$$

A function $J : H_0^1(\Omega) \to \mathbb{R}$ is G-invariant if for every $g \in G$, $J \circ g = J$.

Proposition 8.2.6 *Let Ω be a G-invariant open subset of $\mathbb{R}^N$ satisfying the assumptions of Proposition 8.2.4. Then $e_1 \in \mathrm{Fix}(G)$.*

Proof By a direct computation, we obtain, for all $g \in G$,

$$||ge_1||_2 = ||e_1||_2 = 1, ||\nabla ge_1||_2^2 = ||\nabla e_1||_2^2 = \lambda_1.$$

Propositions 8.2.3 and 8.2.4 imply the existence of a scalar $\lambda(g)$ such that

$$e_1(g^{-1}x) = \lambda(g)e_1(x).$$

Integrating on Ω, we obtain $\lambda(g) = 1$. But then $ge_1 = e_1$. Since $g \in G$ is arbitrary, $e_1 \in \mathrm{Fix}(G)$. □

Example (Symmetry of the First Eigenfunction) For a ball or an annulus

$$\Omega = \{x \in \mathbb{R}^N : r < |x| < R\},$$

we choose $G = \mathbf{O}(N)$. Hence e_1 is a radial function.

We define $v(|x|) = u(x)$. By a simple computation, we have

$$\frac{\partial^2}{\partial x_k^2}u(x) = v''(|x|)\frac{x_k^2}{|x|^2} + v'(|x|)\left(\frac{1}{|x|} - \frac{x_k^2}{|x|^3}\right).$$

Hence we obtain

$$\Delta u = v'' + (N-1)v'/|x|.$$

Let $\Omega = B(0,1) \subset \mathbb{R}^3$. The first eigenfunction, $u(x) = v(|x|)$, is a solution of

$$-v'' - 2v'/r = \lambda v.$$

The function $w = rv$ satisfies

$$-w'' = \lambda w,$$

so that

$$w(r) = a \sin(\sqrt{\lambda} r - b)$$

and

$$v(r) = a \frac{\sin(\sqrt{\lambda} r - b)}{r}.$$

Since $u \in H_0^1(\Omega) \subset L^6(\Omega)$, $b = 0$ and $\lambda = \pi^2$. Finally, we obtain

$$u(x) = a \frac{\sin(\pi |x|)}{|x|}.$$

It follows from Poincaré's principle that

$$\pi^2 = \min \left\{ ||\nabla u||^2_{L^2(\Omega)} / ||u||^2_{L^2(\Omega)} : u \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

Let us characterize the eigenvalues without using the eigenfunctions.

Theorem 8.2.7 (Max-inf Principle) *For every $n \geq 1$,*

$$\lambda_n = \max_{V \in \mathcal{V}_{n-1}} \inf_{\substack{u \in V^\perp \\ ||u||_{L^2} = 1}} \int_\Omega |\nabla u|^2 dx,$$

where $\mathcal{V}_{n-1}$ denotes the family of all $(n-1)$-dimensional subspaces of $H_0^1(\Omega)$.

Proof Let us denote by Λ_n the second member of the preceding equality. It follows from Poincaré's principle that $\lambda_n \leq \Lambda_n$.

Let $V \in \mathcal{V}_{n-1}$. Since the codimension of $V^\perp$ is equal to $n-1$, there exists $x \in \mathbb{R}^N \setminus \{0\}$ such that $u = \sum_{j=1}^{n} x_j e_j \in V^\perp$. Since

$$\int_\Omega |\nabla u|^2 dx = \sum_{j=1}^{n} \lambda_j x_j^2 \int_\Omega e_j^2 dx \leq \lambda_n \int_\Omega u^2 dx,$$

we obtain

$$\inf_{\substack{u \in V^\perp \\ ||u||_{L^2}=1}} \int_\Omega |\nabla u|^2 dx \le \lambda_n.$$

Since $V \in \mathcal{V}_{n-1}$ is arbitrary, we conclude that $\Lambda_n \le \lambda_n$. □

8.3 Symmetrization

> *La considération systématique des ensembles $E[a \le f < b]$ m'a été pratiquement utile parce qu'elle m'a toujours forcé à grouper les conditions donnant des effets voisins.*
>
> *Henri Lebesgue*

According to the *isodiametric inequality* in $\mathbb{R}^2$, among all domains with a fixed diameter, the disk has the largest area. A simple proof was given by J.E. Littlewood in 1953 in *A Mathematician's Miscellany*. We can assume that the domain Ω is convex and that the horizontal axis is tangent to Ω at the origin. We obtain

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} \rho^2(\theta) + \rho^2\left(\theta + \frac{\pi}{2}\right) d\theta \le \pi(d/2)^2.$$

We will prove the *isoperimetric inequality* in $\mathbb{R}^N$ using *Schwarz's symmetrization*.

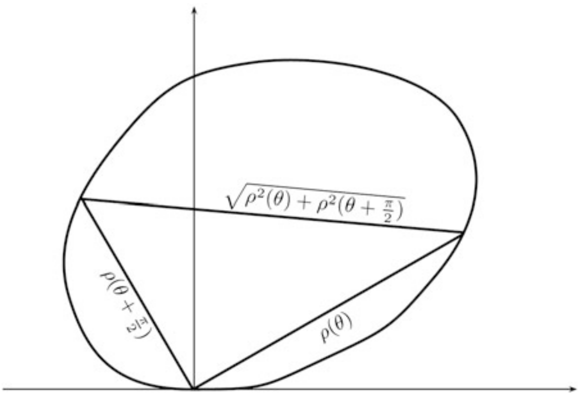

Fig. 8.1 Isodiametric inequality

In this section, we consider Lebesgue's measure on $\mathbb{R}^N$. We define

$$\begin{aligned}\mathcal{K}_+(\mathbb{R}^N) &= \{u \in \mathcal{K}(\mathbb{R}^N) : \text{for all } x \in \mathbb{R}^N, u(x) \geq 0\},\\ L^p_+(\mathbb{R}^N) &= \{u \in L^p(\mathbb{R}^N) : \text{for almost all } u(x) \geq 0\},\\ W^{1,p}_+(\mathbb{R}^N) &= W^{1,p}(\mathbb{R}^N) \cap L^p_+(\mathbb{R}^N),\\ BV_+(\mathbb{R}^N) &= BV(\mathbb{R}^N) \cap L^1_+(\mathbb{R}^N).\end{aligned}$$

Definition 8.3.1 Schwarz's symmetrization of a measurable subset A of $\mathbb{R}^N$ is defined by $A^* = \{x \in \mathbb{R}^N : |x|^N V_N < m(A)\}$. An admissible function $u : \mathbb{R}^N \to [0, +\infty]$ is a measurable function such that for all $t > 0, m_u(t) = m(\{u > t\}) < \infty$. Schwarz's symmetrization of an admissible function u is defined on $\mathbb{R}^N$ by

$$u^*(x) = \sup\{t \in \mathbb{R} : x \in \{u > t\}^*\}.$$

The following properties are clear:

(a) $\chi_{A^*} = \chi^*_A$;
(b) $m(A^* \setminus B^*) \leq m(A \setminus B)$;
(c) u^* is radially decreasing, $|x| \leq |y| \Rightarrow u^*(x) \geq u^*(y)$;
(d) $u \leq v \Rightarrow u^* \leq v^*$.

Lemma 8.3.2 *Let (A_n) be an increasing sequence of measurable sets. Then*

$$\bigcup_{n=1}^{\infty} A^*_n = \left(\bigcup_{n=1}^{\infty} A_n\right)^*.$$

Proof By definition, $A^*_n = B(0, r_n)$, $\left(\bigcup_{n=1}^{\infty} A_n\right)^* = B(0, r)$, where $r^N_n V_N = m(A_n)$, $r^N V_N = m\left(\bigcup_{n=1}^{\infty} A_n\right)$. It suffices to observe that by Proposition 2.2.26,

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$ □

Theorem 8.3.3 *Let u be an admissible function. Then u^* is lower semicontinuous, and for all $t > 0$, $\{u > t\}^* = \{u^* > t\}$ and $m_u(t) = m_{u^*}(t)$.*

Proof Let $t > 0$. Using the preceding lemma, we obtain

$$\{u > t\}^* = \left(\bigcup_{s>t}\{u > s\}\right)^* = \bigcup_{s>t}\{u > s\}^* \subset \{u^* > t\} \subset \{u > t\}^*.$$

In particular, $\{u^* > t\}$ is open and $m\{u > t\} = m\{u^* > t\}$. □

Proposition 8.3.4 *Let* $1 \leq p < \infty$ *and* $u, v \in L^p_+(\mathbb{R}^N)$. *Then* $u^*, v^* \in L^p_+(\mathbb{R}^N)$ *and*

$$||u^*||_p = ||u||_p, \, ||u^* - v^*||_p \leq ||u - v||_p.$$

Proof Using Cavalieri's principle and the preceding theorem, we obtain

$$||u^*||_p^p = \int_0^\infty m_{(u^*)^p}(t)dt = \int_0^\infty m_{u^p}(t)dt = ||u||_p^p.$$

Assume that $p \geq 2$, and define $g(t) = |t|^p$, so that g is convex, even, of class $\mathcal{C}^2$, and $g(0) = g'(0) = 0$. For $a < b$, the fundamental theorem of calculus implies that

$$g(b - a) = \int_a^b ds \int_s^b g''(t - s)dt.$$

Hence we have that

$$g(u - v) = \int_0^\infty ds \int_s^\infty g''(t - s) \left[\chi_{\{u>t\}}(1 - \chi_{\{v>s\}}) + \chi_{\{v>t\}}(1 - \chi_{\{u>s\}})\right] dt.$$

Integrating on $\mathbb{R}^N$ and using Fubini's theorem, we find that

$$\int_{\mathbb{R}^N} g(u-v)dx = \int_0^\infty ds \int_s^\infty g''(t-s)[m(\{u > t\} \setminus \{v > s\}) + m(\{v > t\} \setminus \{u > s\})]dt.$$

Finally, we obtain

$$\int_{\mathbb{R}^N} g(u^* - v^*)dx \leq \int_{\mathbb{R}^N} g(u - v)dx.$$

If $1 \leq p < 2$, it suffices to approximate $|t|^p$ by $g_\varepsilon(t) = (t^2 + \varepsilon^2)^{p/2} - \varepsilon^p$, $\varepsilon > 0$. □

Approximating *Schwarz's symmetrizations* by *polarizations*, we will prove that if $u \in W^{1,p}_+(\mathbb{R}^N)$, then $u^* \in W^{1,p}_+(\mathbb{R}^N)$ and $||\nabla u^*||_p \leq ||\nabla u||_p$.

Definition 8.3.5 Let σ_H be the reflection with respect to the frontier of a closed affine half-space H of $\mathbb{R}^N$. The polarization (with respect to H) of a function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\begin{aligned} u^H(x) &= \max\{u(x), u(\sigma_H(x))\}, \quad & x \in H, \\ &= \min\{u(x), u(\sigma_H(x))\}, \quad & x \in \mathbb{R}^N \setminus H. \end{aligned}$$

The polarization A^H of $A \subset \mathbb{R}^N$ is defined by $\chi_{A^H} = \chi_A^H$. We denote by $\mathcal{H}$ the family of all closed affine half-spaces of $\mathbb{R}^N$ containing 0.

Let us recall that a closed affine half-space of $\mathbb{R}^N$ is defined by

$$H = \{x \in \mathbb{R}^N : a \cdot x \leq b\},$$

where $a \in \mathbb{S}^{N-1}$ and $b \in \mathbb{R}$. It is clear that

$$\sigma_H(x) = x + 2(b - a \cdot x)a.$$

The following properties are easy to prove:

(a) if A is a measurable subset of $\mathbb{R}^N$, then $m(A^H) = m(A)$;
(b) $\{u^H > t\} = \{u > t\}^H$;
(c) if u is admissible, $(u^H)^* = u^*$;
(d) if moreover, $H \in \mathcal{H}$, $(u^*)^H = u^*$.

Lemma 8.3.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and $a \leq b$, $c \leq d$. Then*

$$f(b - d) + f(a - c) \leq f(a - d) + f(b - c).$$

Proof Define $x = b - d$, $y = b - a$, and $z = d - c$. By convexity, we have

$$f(x) - f(x - y) \leq f(x + z) - f(x + z - y). \qquad \square$$

Proposition 8.3.7 *Let $1 \leq p < \infty$ and $u, v \in L^p(\mathbb{R}^N)$. Then $u^H, v^H \in L^p(\mathbb{R}^N)$, and*

$$||u^H||_p = ||u||_p, \quad ||u^H - v^H||_p \leq ||u - v||_p.$$

Proof Observe that

$$\int_{\mathbb{R}^N} |u(x)|^p dx = \int_H |u(x)|^p + |u(\sigma_H(x))|^p dx$$

$$= \int_H |u^H(x)|^p + |u^H(\sigma_H(x))|^p dx = \int_{\mathbb{R}^N} |u^H(x)|^p dx.$$

Using the preceding lemma, it is easy to verify that for all $x \in H$,

$$|u^H(x) - v^H(x)|^p + |u^H(\sigma_H(x)) - v^H(\sigma_H(x))|^p$$
$$\leq |u(x) - v(x)|^p + |u(\sigma_H(x)) - v(\sigma_H(x))|^p.$$

It suffices then to integrate over H. $\square$

Lemma 8.3.8 *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a uniformly continuous function. Then the function $u^H : \mathbb{R}^N \to \mathbb{R}$ is uniformly continuous, and for all $\delta > 0$, $\omega_{u^H}(\delta) \leq \omega_u(\delta)$.*

Proof Let $\delta > 0$ and $x, y \in \mathbb{R}^N$ be such that $|x - y| \leq \delta$. If $x, y \in H$ or if $x, y \in \mathbb{R}^N \setminus H$, we have

$$|\sigma_H(x) - \sigma_H(y)| = |x - y| \leq \delta$$

and

$$|u^H(x) - u^H(y)| \leq \max(|u(x) - u(y)|, |u(\sigma_H(x)) - u(\sigma_H(y))|) \leq \omega_u(\delta).$$

If $x \in H$ and $y \in \mathbb{R}^N \setminus H$, we have

$$|x - \sigma_H(y)| = |\sigma_H(x) - y| \leq |\sigma_H(x) - \sigma_H(y)| = |x - y| \leq \delta$$

and

$$\begin{aligned}|u^H(x) - u^H(y)| \leq \max(&|u(x) - u(\sigma_H(y))|, |u(\sigma_H(x)) - u(y)|, \\ &|u(\sigma_H(x)) - u(\sigma_H(y))|, |u(x) - u(y)|) \leq \omega_u(\delta).\end{aligned}$$

We conclude that

$$\omega_{u^H}(\delta) = \sup_{|x-y|\leq\delta} |u^H(x) - u^H(y)| \leq \omega_u(\delta). \qquad \square$$

Lemma 8.3.9 *Let $1 \leq p < \infty$, $u \in L^p(\mathbb{R}^N)$, and $H \in \mathcal{H}$. Define $g(x) = e^{-|x|^2}$. Then*

$$\int_{\mathbb{R}^N} ug\, dx \leq \int_{\mathbb{R}^N} u^H g\, dx. \qquad (*)$$

If, moreover, $0 \in \overset{o}{H}$ and

$$\int_{\mathbb{R}^N} ug\, dx = \int_{\mathbb{R}^N} u^H g\, dx, \qquad (**)$$

then $u^H = u$.

Proof For all $x \in H$, we have

$$u(x)g(x) + u(\sigma_H(x))g(\sigma_H(x)) \leq u^H(x)g(x) + u^H(\sigma_H(x))g(\sigma_H(x)).$$

It suffices then to integrate over H to prove $(*)$.

If $(**)$ holds, we obtain, almost everywhere on H,

$$u(x)g(x) + u(\sigma_H(x))g(\sigma_H(x)) = u^H(x)g(x) + u^H(\sigma_H(x))g(\sigma_H(x)).$$

If $0 \in \overset{o}{H}$, then $g(\sigma_H(x)) < g(x)$ for all $x \in \overset{o}{H}$, so that

$$u(x) = u^H(x), u(\sigma_H(x)) = u^H(\sigma_H(x)).$$ □

Lemma 8.3.10 *Let $u \in L^p(\mathbb{R}^N) \bigcap C(\mathbb{R}^N) (1 \leq p < \infty)$ be such that, for all $H \in \mathcal{H}$, $u^H = u$. Then $u \geq 0$ and $u = u^*$.*

Proof Let $x, y \in \mathbb{R}^N$ be such that $x \neq y$ and $|x| \leq |y|$. There exists $H \in \mathcal{H}$ such that $x \in H$ and $y = \sigma_H(x)$. By assumption, we have

$$u(y) = u^H(y) \leq u^H(x) = u(x).$$

Hence

$$|x| \leq |y| \Rightarrow u(y) \leq u(x).$$

We conclude that there exists a (continuous) decreasing function $v : [0, +\infty[\to \mathbb{R}$ such that $u(x) = v(|x|)$. Since $u \in L^p(\mathbb{R}^N)$, it is clear that

$$\lim_{r \to +\infty} v(r) = 0.$$

Hence $u \geq 0$ and for all $t > 0$, $\{u > t\} = \{u^* > t\}$, so that $u = u^*$. □

Consider a sequence of closed affine half-spaces

$$H_n = \{x \in \mathbb{R}^N : a_n \cdot x \leq b_n\}$$

such that $((a_n, b_n))$ is dense in $\mathbb{S}^{N-1} \times\ ]0, +\infty[$.

The following result is due to J. Van Schaftingen.

Theorem 8.3.11 *Let $1 \leq p < \infty$ and $u \in L^p_+(\mathbb{R}^N)$. Define*

$$u_0 = u,$$
$$u_{n+1} = u_n^{H_1 \ldots H_{n+1}}.$$

Then the sequence (u_n) converges to u^* in $L^p(\mathbb{R}^N)$.

Proof Assume that $u \in \mathcal{K}_+(\mathbb{R}^N)$. There exists $r > 0$ such that spt $u \subset B[0, r]$. Hence for all n,

$$\text{spt}\, u_n \subset B[0, r].$$

The sequence (u_n) is precompact in $\mathcal{C}(B[0,r])$ by Ascoli's theorem, since

(a) for every n, $||u_n||_\infty = ||u||_\infty$;
(b) for every $\varepsilon > 0$, there exists $\delta > 0$, such that for every n, $\omega_{u_n}(\delta) \le \omega_u(\delta) \le \varepsilon$.

Assume that (u_{n_k}) converges uniformly to v. Observe that

$$\text{spt } v \subset B[0,r].$$

We shall prove that $v = u^*$. Since by Proposition 8.3.4,

$$||u^* - v^*||_1 = ||u^*_{n_k} - v^*||_1 \le ||u_{n_k} - v||_1 \to 0, \quad k \to \infty,$$

it suffices to prove that $v = v^*$.

Let $m \ge 1$. For every $n_k \ge m$, we have

$$u_{n_{k+1}} = u_{n_k}^{H_1 \dots H_m \dots H_{n_{k+1}}}.$$

Lemma 8.3.9 implies that

$$\int_{\mathbb{R}^N} u_{n_k}^{H_1 \dots H_m} g \, dx \le \int_{\mathbb{R}^N} u_{n_{k+1}} g \, dx.$$

It follows from Proposition 8.3.7 that

$$\int_{\mathbb{R}^N} v^{H_1 \dots H_m} g \, dx \le \int_{\mathbb{R}^N} vg \, dx.$$

By Lemma 8.3.9, $v^{H_1} = v$, and by induction, $v^{H_m} = v$.

Let $a \in \mathbb{S}^{N-1}$, $b \ge 0$, and $H = \{x \in \mathbb{R}^N : a \cdot x \le b\}$. There exists a sequence (n_k) such that $(a_{n_k}, b_{n_k}) \to (a,b)$. We deduce from Lebesgue's dominated convergence theorem that

$$||v^H - v||_1 = ||v^H - v^{H_{n_k}}||_1 \to 0, \quad k \to \infty.$$

Hence for all $H \in \mathcal{H}$, $v = v^H$. Lemma 8.3.10 ensures that $v = v^*$.

Let $u \in L^p_+(\mathbb{R}^N)$ and $\varepsilon > 0$. The density theorem implies the existence of $w \in \mathcal{K}_+(\mathbb{R}^N)$ such that $||u - w||_p \le \varepsilon$. By the preceding step, the sequence

$$\begin{aligned} w_0 &= w, \\ w_{n+1} &= w_n^{H_1 \dots H_{n+1}}, \end{aligned}$$

converges to w^* in $L^p(\mathbb{R}^N)$. Hence there exists m such that for $n \ge m$, $||w_n - w^*||_p \le \varepsilon$. It follows from Propositions 8.3.4 and 8.3.7 that for $n \ge m$,

$$||u_n - u^*||_p \le ||u_n - w_n||_p + ||w_n - w^*||_p + ||w^* - u^*||_p \le 2||u - w||_p + \varepsilon \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Proposition 8.3.12 *Let* $1 \le p < \infty$ *and* $u \in W^{1,p}(\mathbb{R}^N)$. *Then* $u^H \in W^{1,p}(\mathbb{R}^N)$ *and* $||\nabla u^H||_p = ||\nabla u||_p$.

Proof Define $v = u \circ \sigma_H$. Observe that

$$\begin{aligned} u^H &= \frac{1}{2}(u+v) + \frac{1}{2}|u-v|, \quad \text{on } H, \\ &= \frac{1}{2}(u+v) - \frac{1}{2}|u-v|, \quad \text{on } \mathbb{R}^N \setminus H. \end{aligned}$$

Since the trace of $|u - v|$ is equal to 0 on ∂H, $u^H \in W^{1,p}(\mathbb{R}^N)$. Let $x \in H$. Corollary 6.1.14 implies that for $u(x) \ge v(x)$,

$$\nabla u^H(x) = \nabla u(x), \nabla u^H(\sigma_H(x)) = \nabla u(\sigma_H(x)),$$

and for $u(x) < v(x)$,

$$\nabla u^H(x) = \nabla v(x), \nabla u^H(\sigma_H(x)) = \nabla v(\sigma_H(x)).$$

We conclude that on H,

$$|\nabla u^H(x)|^p + |\nabla u^H(\sigma_H(x))|^p = |\nabla u(x)|^p + |\nabla u(\sigma_H(x))|^p.$$ □

Proposition 8.3.13 *Let* $u \in BV(\mathbb{R}^N)$. *Then* $u^H \in BV(\mathbb{R}^N)$ *and* $||Du^H|| \le ||Du||$.

Proof Let $u_n = \rho_n * u$. Propositions 4.3.14 and 8.3.7 imply that $u_n \to u$ and $u_n^H \to u^H$ in $L^1(\mathbb{R}^N)$. Theorem 7.3.3 and Proposition 8.3.12 ensure that

$$||Du_n^H|| = ||\nabla u_n^H||_1 = ||\nabla u_n||_1.$$

We conclude by Theorem 7.3.2 and Lemma 7.3.6 that

$$||Du^H|| \le \lim ||Du_n^H|| = \lim ||\nabla u_n||_1 = ||Du||.$$ □

Theorem 8.3.14 (Pólya–Szegő Inequality) *Let* $1 < p < \infty$ *and* $u \in W_+^{1,p}(\mathbb{R}^N)$. *Then* $u^* \in W_+^{1,p}(\mathbb{R}^N)$ *and* $||\nabla u^*||_p \le ||\nabla u||_p$.

Proof The sequence (u_n) given by Theorem 8.3.11 converges to u^* in $L^p(\mathbb{R}^N)$. By Proposition 8.3.12, for every n, $||\nabla u_n||_p = ||\nabla u||_p$. It follows from Theorem 6.1.7

that

$$||\nabla u^*||_p \leq \lim ||\nabla u_n||_p = ||\nabla u||_p. \qquad \square$$

Theorem 8.3.15 (Hilden's Inequality, 1976) *Let $u \in BV_+(\mathbb{R}^N)$. Then $u^* \in BV_+(\mathbb{R}^N)$ and $||Du^*|| \leq ||Du||$.*

Proof The sequence (u_n) given by Theorem 8.3.11 converges to u^* in $L^{1^*}(\mathbb{R}^N)$. By Proposition 8.3.13, for every n,

$$||Du_{n+1}|| \leq ||Du_n|| \leq ||Du||.$$

It follows from Theorem 7.3.2 that

$$||Du^*|| \leq \lim ||Du_n|| \leq ||Du||. \qquad \square$$

Theorem 8.3.16 (De Giorgi's Isoperimetric Inequality) *Let $N \geq 2$, and let A be a measurable subset of $\mathbb{R}^N$ with finite measure. Then*

$$N V_N^{1/N} (m(A))^{1-1/N} \leq p(A).$$

Proof If $p(A) = +\infty$, the inequality is clear. If this is not the case, then $\chi_A \in BV_+(\mathbb{R}^N)$. By definition of Schwarz's symmetrization,

$$A^* = B(0, r),\ V_N r^N = m(A).$$

Theorems 7.4.1 and 8.3.15 imply that

$$N V_N r^{N-1} = p(A^*) = ||D\chi_{A^*}||_{\mathbb{R}^N} = ||D\chi_A^*||_{\mathbb{R}^N} \leq ||D\chi_A||_{\mathbb{R}^N} = p(A).$$

It is easy to conclude the proof. $\square$

Using scaling invariance, we obtain the following version of the isoperimetric inequality.

Corollary 8.3.17 *Let A be a measurable subset of $\mathbb{R}^N$ with finite measure, and let B be an open ball of $\mathbb{R}^N$. Then*

$$p(B)/m(B)^{1-1/N} \leq p(A)/m(A)^{1-1/N}.$$

The constant $N V_N^{1/N}$, corresponding to the characteristic function of a ball, is the optimal constant for the Gagliardo–Nirenberg inequality.

Theorem 8.3.18 *Let $N \geq 2$ and $u \in L^{N/(N-1)}$ such that $||Du|| < +\infty$. Then*

$$NV_N^{1/N}||u||_{N/(N-1)} \leq ||Du||.$$

Proof

(a) Let $p = N/(N-1)$, $v \in L^p(\mathbb{R}^N)$, $v \geq 0$, and $g \in L^{p'}(\mathbb{R}^N)$. If $||g||_{p'} = 1$, we deduce from Fubini's theorem and Hölder's inequality that

$$\int_{\mathbb{R}^N} gv dx = \int_{\mathbb{R}^N} dx \int_0^\infty g\chi_{v>t} dt \quad = \int_0^\infty dt \int_{\mathbb{R}^N} g\chi_{v>t} dx \leq \int_0^\infty m(\{v > t\})^{1/p} dt.$$

Hence we obtain

$$||v||_p = \max_{||g||_{p'}=1} \int_{\mathbb{R}^N} gv dx \leq \int_0^\infty m(\{v > t\})^{1/p} dt. \qquad (*)$$

(b) Let $u \in \mathcal{D}(\Omega)$. Using inequality $(*)$, the Morse–Sard theorem (Theorem 9.3.1), the coarea formula (Theorem 9.3.3), and the isoperimetric inequality, we obtain

$$\begin{aligned} NV_N^{1/N}||u||_p &\leq NV_N^{1/N}[||u^+||_p + ||u^-||_p] \\ &\leq NV_N^{1/N}\left[\int_0^\infty m(\{u > t\})^{1/p} dt + \int_{-\infty}^0 m(\{u < t\})^{1/p} dt\right] \\ &\leq \int_0^\infty dt \int_{u=t} d\gamma + \int_{-\infty}^0 dt \int_{u=t} d\gamma = \int_{\mathbb{R}^N} |\nabla u| dx. \end{aligned}$$

(c) By density, we obtain, for every $u \in \mathcal{D}^{1,1}(\mathbb{R}^N)$,

$$NV_N^{1/N}||u||_p \leq ||\nabla u||_1.$$

We conclude using Proposition 4.3.14 and Lemma 7.3.6. □

Definition 8.3.19 Let Ω be an open subset of $\mathbb{R}^N$. We define

$$\lambda_1(\Omega) = \inf\left\{||\nabla u||_2^2/||u||_2^2 : u \in W_0^{1,2}(\Omega) \setminus \{0\}\right\}.$$

Theorem 8.3.20 (Faber–Krahn Inequality) *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Then $\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$.*

Proof Define $Q(u) = ||\nabla u||_2^2/||u||_2^2$. Let $u \in W_0^{1,2}(\Omega) \setminus \{0\}$ and $v = |u|$. By Corollary 6.1.14, $Q(v) = Q(u)$. Proposition 8.3.4 and the Pólya–Szegő inequality imply that $Q(v^*) \leq Q(v)$. It is easy to verify that $v^* \in W_0^{1,2}(\Omega^*) \setminus \{0\}$. Hence we obtain

$$\lambda_1(\Omega^*) \leq Q(v^*) \leq Q(v) = Q(u).$$

Since $u \in W_0^{1,2}(\Omega) \setminus \{0\}$ is arbitrary, it is easy to conclude the proof. □

Using scaling invariance, we obtain the following version of the Faber–Krahn inequality.

Corollary 8.3.21 *Let Ω be an open subset of $\mathbb{R}^N$, and let B be an open ball of $\mathbb{R}^N$. Then*

$$\lambda_1(B)m(B)^{2/N} \leq \lambda_1(\Omega)m(\Omega)^{2/N}.$$

Remark Equality in the isoperimetric inequality or in the Faber–Krahn inequality is achieved only when the corresponding domain is a ball.

8.4 Elementary Solutions

There exists no locally integrable function corresponding to the *Dirac measure*.

Definition 8.4.1 The Dirac measure is defined on $\mathcal{K}(\mathbb{R}^N)$ by

$$\langle \delta, u \rangle = u(0).$$

Definition 8.4.2 The elementary solutions of the Laplacian are defined on $\mathbb{R}^N \setminus \{0\}$ by

$$E_N(x) = \frac{1}{2\pi} \log \frac{1}{|x|}, \qquad N = 2,$$
$$E_N(x) = \frac{1}{(N-2)NV_N} \frac{1}{|x|^{N-2}}, \quad N \geq 3.$$

Theorem 8.4.3 *Let $N \geq 2$. In $\mathcal{D}^*(\mathbb{R}^N)$, we have*

$$-\Delta E_N = \delta.$$

Proof Define $v(x) = w(|x|)$. Since

$$\Delta v = w'' + (N-1)w'/|x|,$$

it is easy to verify that on $\mathbb{R}^N \backslash \{0\}$, $\Delta E_N = 0$. It is clear that $E_N \in L^1_{\text{loc}}(\mathbb{R}^N)$.

Let $u \in \mathcal{D}(\mathbb{R}^N)$ and $R > 0$ be such that spt $u \subset B(0, R)$. We have to verify that

$$-u(0) = \int_{\mathbb{R}^N} E_N \Delta u \, dx = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < R} E_N \Delta u \, dx.$$

We obtain using the divergence theorem that

$$f(\varepsilon) = \int_{\varepsilon<|x|<R} (E_N \Delta u - u \Delta E_N) \, dx = \int_{\partial B(0,\varepsilon)} \left(u \nabla E_N \cdot \frac{\gamma}{|\gamma|} - E_N \nabla u \cdot \frac{\gamma}{|\gamma}| \right) \, d\gamma.$$

By a simple computation,

$$\int_{\partial B(0,\varepsilon)} \nabla E_N \cdot \frac{\gamma}{|\gamma|} = -1, \quad \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} E_N d\gamma = 0,$$

so that $\lim\limits_{\varepsilon \to 0} f(\varepsilon) = -u(0)$. □

Definition 8.4.4 Let $f, g \in \mathcal{D}^*(\Omega)$. By definition, $f \leq g$ if for every $u \in \mathcal{D}(\Omega)$ such that $u \geq 0$, $\langle f, u \rangle \leq \langle g, u \rangle$.

Theorem 8.4.5 (Kato's Inequality) *Let $g \in L^1_{\text{loc}}(\Omega)$ be such that $\Delta g \in L^1_{\text{loc}}(\Omega)$. Then*

$$(\text{sgn } g) \; \Delta g \leq \Delta |g|.$$

Proof Let $u \in \mathcal{D}(\Omega)$ and $\omega \subset\subset \Omega$ be such that $u \geq 0$ and spt $u \subset \omega$. Define $g_n = \rho_n * g$, and for $\varepsilon > 0$, $f_\varepsilon(t) = (t^2 + \varepsilon^2)^{1/2}$. Since $g_n \to g$ in $L^1(\omega)$, we can assume, passing if necessary to a subsequence, that $g_n \to g$ almost everywhere on ω.

For all $\varepsilon > 0$ and for n large enough, we have

$$\int_\Omega f'_\varepsilon(g_n)(\Delta g_n) u \, dx \leq \int_\Omega (\Delta f_\varepsilon(g_n)) u \, dx = \int_\Omega f_\varepsilon(g_n) \Delta u \, dx.$$

When $n \to \infty$, we find that

$$\int_\Omega f'_\varepsilon(g)(\Delta g) u \, dx \leq \int_\Omega f_\varepsilon(g) \Delta u \, dx.$$

When $\varepsilon \downarrow 0$, we obtain

$$\int_{\Omega} (\text{sgn } g)(\Delta g)u \, dx \leq \int_{\Omega} |g| \, \Delta u \, dx. \qquad \square$$

8.5 Comments

The notion of polarization of sets appeared in 1952, in a paper by Wolontis [87]. Polarizations of functions were first used by Baernstein and Taylor to approximate symmetrization of functions on the sphere in the remarkable paper [3]. The explicit approximation of Schwarz's symmetrization by polarizations is due to Van Schaftingen [84]. See [73, 85] for other aspects of polarizations. The proof of Proposition 8.3.4 uses a device of Alberti [2]. The notion of symmetrization, and more generally, the use of reflections to prove symmetry, goes back to Jakob Steiner [79].

The elegant proof of Theorem 8.3.18 is due to O.S. Rothaus, *J. Funct. Anal. 64 (1985) 296–313*.

8.6 Exercises for Chap. 8

1. Let $u \in \mathcal{C}(\Omega)$. The *spherical means* of u are defined on D by

$$S(x, r) = (NV_N)^{-1} \int_{\mathbb{S}^{N-1}} u(x + r\sigma) d\sigma.$$

   Verify that when $u \in \mathcal{C}^2(\Omega)$,

$$\lim_{r \downarrow 0} \frac{2N}{r^2} [S(x, r) - u(x)] = \Delta u(x).$$

2. Let $u \in \mathcal{C}(\Omega)$ be such that for every $(x, r) \in D$, $u(x) = M(x, r)$. Then for every $x \in \Omega_n$, $\rho_n * u = u$.

   The argument is due to A. Ponce:

$$\rho_n * u(x) = \int_{\mathbb{R}^N} \rho_n(x - y)u(y)dy = \int_0^\infty dt \int_{\rho(x-y)>t} u(y)dy$$
$$= u(x) \int_0^\infty dt \int_{\rho(x-y)>t} dy = u(x).$$

3. (Weyl's theorem.) Let $u \in L^1_{\text{loc}}(\Omega)$. The following properties are equivalent:

   (a) u is harmonic;

(b) for almost all $x \in \Omega$ and for all $0 < r < d(x, \partial\Omega)$, $u(x) = M(x, r)$;
(c) there exists $v \in \mathcal{C}^\infty(\Omega)$, almost everywhere equal to u, such that $\Delta v = 0$.

4. Let $u \in \mathcal{C}^2(\Omega)$ be a harmonic function. Assume that $u \geq 0$ on $B[0, R] \subset \Omega$. Then for every $0 < r < R$ and $|y| < R - r$, we have

$$|u(y) - u(0)| \leq \frac{1}{r^N V_N} \int_{r-|y|<|x|<r+|y|} u(x)dx$$

$$= \frac{(r + |y|)^N - (r - |y|)^N}{r^N} u(0).$$

Hint: Use the mean-value property.

5. (Liouville's theorem.) Let $u \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a harmonic function, bounded from below on $\mathbb{R}^N$. Then u is constant.
6. Let Ω be an open connected subset of $\mathbb{R}^N$, and let $u \in \mathcal{C}^\infty(\Omega)$ be a harmonic function such that for some $x \in \Omega$, $u(x) = \inf_\Omega u$. Then u is constant.
7. If $u \in \mathcal{D}(]0, \pi[)$, then

$$\int_0^\pi \left|\frac{du}{dx}\right|^2 - u^2 dx = \int_0^\pi \left|\frac{du}{dx} - \frac{\cos x}{\sin x} u\right|^2 dx.$$

Hence

$$\min_{\substack{u \in H_0^1(]0,\pi[) \\ ||u||_2 = 1}} \int_0^\pi \left|\frac{du}{dx}\right|^2 dx = 1.$$

8. (Min–max principle.) For every $n \geq 1$,

$$\lambda_n = \min_{V \in \mathcal{V}_n} \max_{\substack{u \in V \\ ||u||_2 = 1}} \int_\Omega |\nabla u|^2 dx,$$

where $\mathcal{V}_n$ denotes the family of all n-dimensional subspaces of $H_0^1(\Omega)$.

9. Let us recall that

$$\lambda_1(G) = \inf \left\{ ||\nabla u||_2^2 / ||u||_2^2 : u \in W_0^{1,2}(G) \setminus \{0\} \right\}.$$

Let Ω be an open subset of $\mathbb{R}^M$, and ω an open subset of $\mathbb{R}^N$. Then:

(a) $\lambda_1(\Omega \times \omega) = \lambda_1(\Omega) + \lambda_1(\omega)$;
(b) $\lambda_1(\mathbb{R}^N) = 0$;
(c) $\lambda_1(\Omega \times \mathbb{R}^N) = \lambda_1(\Omega)$.

10. Define $u \in \mathcal{D}_+(\mathbb{R}^N)$ such that for every $y \in \mathbb{R}^N$, $\tau_y u \neq u^*$, and for $1 \leq p < \infty$, $||\nabla u||_p = ||\nabla u^*||_p$. *Hint*: Consider two functions v and w such that $v = v^*$, $w = w^*$, $v \equiv 1$ on $B(0, 1)$, and spt $w \subset B[0, 1/2]$, and define $u = v + \tau_y w$.
11. (Hardy–Littlewood inequality.) Let $1 < p < \infty$, $u \in L^p_+(\mathbb{R}^N)$, and $v \in L^{p'}_+(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} u\, v\, dx \leq \int_{\mathbb{R}^N} u^* v^* dx.$$

12. Let $1 \leq p < \infty$ and $u, v \in L^p_+(\mathbb{R}^N)$. Then

$$||u + v||_p \leq ||u^* + v^*||_p.$$

Hint: Assume first that $p > 1$. Observe that

$$||u + v||_p = \sup_{\substack{w \in L^{p'} \\ ||w||_{p'}}} \int_{\mathbb{R}^N} (u + v) w\, dx.$$

13. Let Ω be a domain in $\mathbb{R}^N$ invariant under rotations. A function $u : \Omega \to \mathbb{R}$ is *foliated Schwarz's symmetric* with respect to $e \in \mathbb{S}^{N-1}$ if $u(x)$ depends only on $(r, \theta) = (|x|, \cos^{-1}(\frac{x}{|x|} \cdot e))$ and is decreasing in θ.

Let $e \in \mathbb{S}^{N-1}$. We denote by $\mathcal{H}_e$ the family of closed half-spaces H in $\mathbb{R}^N$ such that $0 \in \partial H$ and $e \in H$.

Prove that a function $u : \Omega \to \mathbb{R}$ is foliated Schwarz's symmetric with respect to e if and only if for every $H \in \mathcal{H}_e$, $u^H = u$.
14. Let $u \in L^p(\mathbb{R}^N)(1 \leq p < \infty)$, and let the closed affine half-space $H \subset \mathbb{R}^N$ be such that $u^H = u$. Then, for every $n \geq 1$, $(\rho_n * u)^H = \rho_n * u$.
Hint. For every $x, y \in H$, we have

$$|x - y| = \big|\sigma_H(x) - \sigma_H(y)\big| \leq \big|x - \sigma_H(y)\big| = \big|\sigma_H(x) - y\big|.$$

Hence we obtain, for every $x \in H$,

$$\begin{aligned} &\rho_n * u(x) - \rho_n * u\big(\sigma_H(x)\big) \\ &= \int_H \Big[u(y) - u\big(\sigma_H(y)\big)\Big] \big[\rho_n(x - y) - \rho_n\big(\sigma_H(x) - y\big)\big] dy \geq 0. \end{aligned}$$

15. Let $u \in L^p(\mathbb{R}^N)(1 \leq p < \infty)$ be such that, for all $H \in \mathcal{H}, u^H = u$. Then $u \geq 0$ and $u = u^*$.