Chapter 6
Sobolev Spaces

6.1 Weak Derivatives

Throughout this chapter, we denote by Ω an open subset of $\mathbb{R}^N$. We begin with an elementary computation.

Lemma 6.1.1 *Let* $1 \leq |\alpha| \leq m$ *and let* $f \in \mathcal{C}^m(\Omega)$. *Then for every* $u \in \mathcal{C}^m(\Omega) \cap \mathcal{K}(\Omega)$,

$$\int_\Omega f\, D^\alpha u\, dx = (-1)^{|\alpha|} \int_\Omega (D^\alpha f) u\, dx.$$

Proof We assume that $\alpha = (0, \ldots, 0, 1)$. Let $u \in \mathcal{C}^1(\Omega) \cap \mathcal{K}(\Omega)$, and define

$$\begin{aligned} g(x) &= f(x)u(x), \; x \in \Omega, \\ &= 0, \qquad\qquad\; x \in \mathbb{R}^N \setminus \Omega. \end{aligned}$$

The fundamental theorem of calculus implies that for every $x' \in \mathbb{R}^{N-1}$,

$$\int_{\mathbb{R}} D^\alpha g(x', x_N) dx_N = 0.$$

Fubini's theorem ensures that

$$\int_\Omega (f D^\alpha u + (D^\alpha f)u) dx = \int_{\mathbb{R}^N} D^\alpha g\, dx = \int_{\mathbb{R}^{N-1}} dx' \int_{\mathbb{R}} D^\alpha g\, dx_N = 0.$$

When $|\alpha| = 1$, the proof is similar. It is easy to conclude the proof by induction. $\square$

Weak derivatives were defined by S.L. Sobolev in 1938.

M. Willem, *Functional Analysis*, Cornerstones,
https://doi.org/10.1007/978-3-031-09149-0_6

Definition 6.1.2 Let $\alpha \in \mathbb{N}^N$ and $f \in L^1_{\text{loc}}(\Omega)$. By definition, the weak derivative of order α of f exists if there is $g \in L^1_{\text{loc}}(\Omega)$ such that for every $u \in \mathcal{D}(\Omega)$,

$$\int_\Omega f \, D^\alpha u \, dx = (-1)^{|\alpha|} \int_\Omega gu \, dx.$$

The function g, if it exists, will be denoted by $\partial^\alpha f$.

By the annulation theorem, the weak derivatives are well defined.

Proposition 6.1.3 *Assume that $\partial^\alpha f$ exists. On*

$$\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\},$$

we have that

$$D^\alpha(\rho_n * f) = \rho_n * \partial^\alpha f.$$

Proof We deduce from Proposition 4.3.6 and from the preceding definition that for every $x \in \Omega_n$,

$$\begin{aligned} D^\alpha(\rho_n * f)(x) &= \int_\Omega D_x^\alpha \rho_n(x-y) f(y) dy \\ &= (-1)^{|\alpha|} \int_\Omega D_y^\alpha \rho_n(x-y) f(y) dy \\ &= (-1)^{2|\alpha|} \int_\Omega \rho_n(x-y) \partial^\alpha f(y) dy \\ &= \rho_n * \partial^\alpha f(x). \end{aligned}$$

□

Theorem 6.1.4 (du Bois–Reymond Lemma) *Let $|\alpha| = 1$ and let $f \in C(\Omega)$ be such that $\partial^\alpha f \in C(\Omega)$. Then $D^\alpha f$ exists and $D^\alpha f = \partial^\alpha f$.*

Proof By the preceding proposition, we have

$$D^\alpha(\rho_n * f) = \rho_n * \partial^\alpha f.$$

The fundamental theorem of calculus implies then that

$$\rho_n * f(x + \varepsilon\alpha) = \rho_n * f(x) + \int_0^\varepsilon \rho_n * \partial^\alpha f(x + t\alpha) dt.$$

By the regularization theorem,

$$\rho_n * f \to f, \quad \rho_n * \partial^\alpha f \to \partial^\alpha f$$

uniformly on every compact subset of Ω. Hence we obtain

$$f(x+\varepsilon\alpha) = f(x) + \int_0^\varepsilon \partial^\alpha f(x+t\alpha)dt,$$

so that $\partial^\alpha f = D^\alpha f$ by the fundamental theorem of calculus. □

Notation From now on, the derivatives of a continuously differentiable function will also be denoted by ∂^α.

Let us prove the *closing lemma*. The *graph* of the weak derivative is closed in $L^1_{\text{loc}} \times L^1_{\text{loc}}$.

Lemma 6.1.5 *Let* $(f_n) \subset L^1_{\text{loc}}(\Omega)$ *and let* $\alpha \in \mathbb{N}^N$ *be such that in* $L^1_{\text{loc}}(\Omega)$*,*

$$f_n \to f, \quad \partial^\alpha f_n \to g.$$

Then $g = \partial^\alpha f$.

Proof For every $u \in \mathcal{D}(\Omega)$, we have by definition that

$$\int_\Omega f_n \partial^\alpha u\, dx = (-1)^{|\alpha|} \int_\Omega (\partial^\alpha f_n) u\, dx.$$

Since by assumption,

$$\left| \int_\Omega (f_n - f)\partial^\alpha u\, dx \right| \leq ||\partial^\alpha u||_\infty \int_{\text{spt } u} |f_n - f| dx \to 0$$

and

$$\left| \int_\Omega (\partial^\alpha f_n - g) u\, dx \right| \leq ||u||_\infty \int_{\text{spt } u} |\partial^\alpha f_n - g| dx \to 0,$$

we obtain

$$\int_\Omega f \partial^\alpha u\, dx = (-1)^{|\alpha|} \int_\Omega g u\, dx. \qquad \square$$

Example (Weak Derivative) If $-N < \lambda \leq 1$, the function $f(x) = |x|^\lambda$ is locally integrable on $\mathbb{R}^N$. We approximate f by

$$f_\varepsilon(x) = \left(|x|^2 + \varepsilon\right)^{\lambda/2}, \quad \varepsilon > 0.$$

Then $f_\varepsilon \in C^\infty(\mathbb{R}^N)$ and

$$\partial_k f_\varepsilon(x) = \lambda\, x_k \Big(|x|^2 + \varepsilon\Big)^{\frac{\lambda-2}{2}},$$

$$\left|\partial_k f_\varepsilon(x)\right| \le \lambda |x|^{\lambda-1}.$$

If $\lambda > 1 - N$, we obtain in $L^1_{\rm loc}(\mathbb{R}^N)$ that

$$f_\varepsilon(x) \to f(x) = |x|^\lambda,$$

$$\partial_k f_\varepsilon(x) \to g(x) = \lambda\, x_k |x|^{\lambda-2}.$$

Hence $\partial_k f(x) = \lambda\, |x|^{\lambda-2} x_k$.

Definition 6.1.6 The *gradient* of the (weakly) differentiable function u is defined by

$$\nabla u = (\partial_1 u, \ldots, \partial_N u).$$

The *divergence* of the (weakly) differentiable vector field $v = (v_1, \ldots, v_N)$ is defined by

$$\text{div } v = \partial_1 v_1 + \ldots + \partial_N v_N.$$

Let $1 \le p < \infty$ and $u \in L^1_{\rm loc}(\Omega)$ be such that $\partial_j u \in L^p(\Omega)$, $j = 1, \ldots, N$. We define

$$||\nabla u||_{L^p(\Omega)} = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p} = \left(\int_\Omega \left|\sum_{j=1}^N (\partial_j u)^2\right|^{p/2} dx\right)^{1/p}.$$

Theorem 6.1.7 *Let* $1 < p < \infty$ *and let* $(u_n) \subset L^1_{\rm loc}(\Omega)$ *be such that*

(a) $u_n \to u$ *in* $L^1_{\rm loc}(\Omega)$*;*
(b) for every n, $\nabla u_n \in L^p(\Omega; \mathbb{R}^N)$*;*
(c) $c = \sup_n ||\nabla u_n||_p < \infty$.

Then $\nabla u \in L^p(\Omega; \mathbb{R}^N)$ *and*

$$||\nabla u||_p \le \varliminf_{n\to\infty} ||\nabla u_n||_p.$$

Proof We define f on $\mathcal{D}(\Omega; \mathbb{R}^N)$ by

$$\langle f, v \rangle = \int_{\Omega} u \operatorname{div} v \, dx.$$

We have that

$$\begin{aligned}
|\langle f, v \rangle| &= \lim_{n\to\infty} |\int_{\Omega} u_n \operatorname{div} v \, dx| \\
&= \lim_{n\to\infty} |\int_{\Omega} \nabla u_n \cdot v \, dx| \\
&\leq \varliminf_{n\to\infty} ||\nabla u_n||_p \left(\int_{\Omega} |v|^{p'} dx \right)^{1/p'}.
\end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $L^{p'}(\Omega)$, Proposition 3.2.3 implies the existence of a continuous extension of f to $L^{p'}(\Omega; \mathbb{R}^N)$. By Riesz's representation theorem, there exists $g \in L^p(\Omega; \mathbb{R}^N)$ such that for every $v \in \mathcal{D}(\Omega; \mathbb{R}^N)$,

$$\int_{\Omega} g \cdot v \, dx = \langle f, v \rangle = \int_{\Omega} u \operatorname{div} v \, dx.$$

Hence $\nabla u = -g \in L^p(\Omega; \mathbb{R}^N)$. Choosing $v = |\nabla u|^{p-2} \nabla u$, we find that

$$\begin{aligned}
\int_{\Omega} |\nabla u|^p dx = \int_{\Omega} \nabla u \cdot v \, dx &\leq \varliminf_{n\to\infty} ||\nabla u_n||_p \left(\int_{\Omega} |v|^{p'} dx \right)^{1/p'} \\
&= \varliminf_{n\to\infty} ||\nabla u_n||_p \left(\int_{\Omega} |\nabla u|^p dx \right)^{1-1/p}.
\end{aligned}$$

□

Sobolev spaces are spaces of differentiable functions with integral norms. In order to define complete spaces, we use weak derivatives.

Definition 6.1.8 Let $k \geq 1$ and $1 \leq p < \infty$. On the Sobolev space

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \text{ for every } |\alpha| \leq k, \partial^\alpha u \in L^p(\Omega)\},$$

we define the norm

$$||u||_{W^{k,p}(\Omega)} = ||u||_{k,p} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha u|^p dx \right)^{1/p}.$$

On the space $H^k(\Omega) = W^{k,2}(\Omega)$, we define the scalar product

$$(u \mid v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} (\partial^\alpha u \mid \partial^\alpha v)_{L^2(\Omega)}.$$

The Sobolev space $W^{k,p}_{\text{loc}}(\Omega)$ is defined by

$$W^{k,p}_{\text{loc}}(\Omega) = \{u \in L^p_{\text{loc}}(\Omega) : \text{ for all } \omega \subset\subset \Omega, u\Big|_\omega \in W^{k,p}(\omega)\}.$$

A sequence (u_n) converges to u in $W^{k,p}_{\text{loc}}(\Omega)$ if for every $\omega \subset\subset \Omega$,

$$||u_n - u||_{W^{k,p}(\omega)} \to 0, \quad n \to \infty.$$

The space $W^{k,p}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$. We denote by $H^k_0(\Omega)$ the space $W^{k,2}_0(\Omega)$.

Theorem 6.1.9 *Let $k \geq 1$ and $1 \leq p < \infty$. Then the spaces $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$ are complete and separable.*

Proof Let $M = \sum_{|\alpha| \leq k} 1$. The space $L^p(\Omega; \mathbb{R}^M)$ with the norm

$$||(v_\alpha)||_p = \left(\sum_{|\alpha| \leq k} \int_\Omega |v_\alpha|^p dx \right)^{1/p}$$

is complete and separable. The map

$$\Phi : W^{k,p}(\Omega) \to L^p(\Omega; \mathbb{R}^M) : u \mapsto (\partial^\alpha u)_{|\alpha| \leq k}$$

is a linear isometry: $||\Phi(u)||_p = ||u||_{k,p}$. By the closing lemma, $\Phi(W^{k,p}(\Omega))$ is a closed subspace of $L^p(\Omega; \mathbb{R}^M)$. It follows that $W^{k,p}(\Omega)$ is complete and separable. Since $W^{k,p}_0(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$, it is also complete and separable. □

Theorem 6.1.10 *Let $1 \leq p < \infty$. Then $W^{1,p}_0(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.*

Proof It suffices to prove that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. We use regularization and truncation.

Regularization Let $u \in W^{1,p}(\mathbb{R}^N)$ and define $u_n = \rho_n * u$. By Proposition 4.3.6, $u_n \in C^\infty(\mathbb{R}^N)$. Proposition 4.3.14 implies that in $L^p(\mathbb{R}^N)$,

$$u_n \to u, \partial_k u_n = \rho_n * \partial_k u \to \partial_k u.$$

We conclude that $W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$.

Truncation Let $\theta \in C^\infty(\mathbb{R})$ be such that $0 \le \theta \le 1$ and

$$\begin{aligned}\theta(t) &= 1, \quad t \le 1,\\ &= 0, \quad t \ge 2.\end{aligned}$$

We define the sequence

$$\theta_n(x) = \theta(|x|/n).$$

Let $u \in W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$. It is clear that $u_n = \theta_n u \in \mathcal{D}(\mathbb{R}^N)$. It follows easily from Lebesgue's dominated convergence theorem that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$. □

We extend some rules of differential calculus to weak derivatives.

Proposition 6.1.11 (Change of Variables) *Let Ω and ω be open subsets of $\mathbb{R}^N$, $G : \omega \to \Omega$ a diffeomorphism, and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. Then $u \circ G \in W^{1,1}_{\mathrm{loc}}(\omega)$ and*

$$\frac{\partial}{\partial y_k}(u \circ G) = \sum_j \frac{\partial u}{\partial x_j} \circ G \ \frac{\partial G_j}{\partial y_k}.$$

Proof Let $v \in \mathcal{D}(\omega)$ and $u_n = \rho_n * u$. By Lemma 6.1.1, for n large enough, we have

$$\int_\omega u_n \circ G(y) \ \frac{\partial v}{\partial y_k}(y)dy = -\int_\omega \sum_j \frac{\partial u_n}{\partial x_j} \circ G(y) \ \frac{\partial G_j}{\partial y_k}(y) \ v(y)dy. \qquad (*)$$

It follows from Theorem 2.4.5 with $H = G^{-1}$ that

$$\begin{aligned}&\int_\Omega u_n(x)\frac{\partial v}{\partial y_k} \circ H(x)|\det H'(x)|dx\\ &\quad = -\int_\Omega \sum_j \frac{\partial u_n}{\partial x_j}(x)\frac{\partial G_j}{\partial y_k} \circ H(x) v \circ H(x)|\det H'(x)|dx. \qquad (**)\end{aligned}$$

The regularization theorem implies that in $L^1_{\mathrm{loc}}(\Omega)$,

$$u_n \to u, \quad \frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}.$$

Taking the limit, it is permitted to replace u_n by u in (∗∗). But then it is also permitted to replace u_n by u in (∗), and the proof is complete. □

Proposition 6.1.12 (Derivative of a Product) *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $f \in C^1(\Omega)$. Then $fu \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and*

$$\partial_k(fu) = f\partial_k u + (\partial_k f)u.$$

Proof Let $u_n = \rho_n * u$, so that by the classical rule of derivative of a product,

$$\partial_k(fu_n) = (\partial_k f)u_n + f\partial_k u_n.$$

It follows from the regularization theorem that

$$fu_n \to fu, \partial_k(fu_n) \to (\partial_k f)u + f\partial_k u$$

in $L^1_{\text{loc}}(\Omega)$. We conclude by invoking the closing lemma. □

Proposition 6.1.13 (Derivative of the Composition of Functions) *Let $u \in W^{1,1}_{\text{loc}}(\Omega)$, and let $f \in C^1(\mathbb{R})$ be such that $c = \sup_{\mathbb{R}}|f'| < \infty$. Then $f \circ u \in W^{1,1}_{\text{loc}}(\Omega)$ and*

$$\partial_k(f \circ u) = f' \circ u \ \partial_k u.$$

Proof We define $u_n = \rho_n * u$, so that by the classical rule,

$$\partial_k(f \circ u_n) = f' \circ u_n \ \partial_k u_n.$$

We choose $\omega \subset\subset \Omega$. By the regularization theorem, we have in $L^1(\omega)$,

$$u_n \to u, \quad \partial_k u_n \to \partial_k u.$$

By Proposition 4.2.10, taking if necessary a subsequence, we can assume that $u_n \to u$ almost everywhere on ω. We obtain

$$\int_\omega |f \circ u_n - f \circ u|dx \le c \int_\omega |u_n - u|dx \to 0,$$

$$\int_\omega |f' \circ u_n \ \partial_k u_n - f' \circ u \ \partial_k u|dx \le c \int_\omega |\partial_k u_n - \partial_k u|dx + \int_\omega |f' \circ u_n - f' \circ u| \, |\partial_k u|dx \to 0.$$

Hence in $L^1(\omega)$,

$$f \circ u_n \to f \circ u, \quad f' \circ u_n \ \partial_k u_n \to f' \circ u \ \partial_k u.$$

Since $\omega \subset\subset \Omega$ is arbitrary, we conclude the proof by invoking the closing lemma. □

On $\mathbb{R}$, we define

$$\begin{aligned} \text{sgn}(t) &= t/|t|, \quad t \neq 0 \\ &= 0, \quad\quad t = 0. \end{aligned}$$

Corollary 6.1.14 *Let $g : \mathbb{R} \to \mathbb{R}$ be such that $c = \sup_{\mathbb{R}} |g| < \infty$ and, for some sequence $(g_n) \subset C(\mathbb{R})$, $g(t) = \lim_{n \to \infty} g_n(t)$ everywhere on $\mathbb{R}$. Define*

$$f(t) = \int_0^t g(s)ds.$$

Then, for every $u \in W^{1,1}_{loc}(\Omega)$, $f \circ u \in W^{1,1}_{loc}(\Omega)$ and

$$\nabla(f \circ u) = (g \circ u)\nabla u.$$

In particular $u^+, u^-, |u| \in W^{1,1}_{loc}(\Omega)$ and

$$\nabla u^+ = \chi_{\{u>0\}}\nabla u, \nabla u^- = -\chi_{\{u<0\}}\nabla u, \chi_{\{u=0\}}\nabla u = 0, \nabla|u| = (\text{sgn } u)\nabla u.$$

Proof We can assume that $\sup_n \sup_{\mathbb{R}} |g_n| \leq c$. We define $f_n(t) = \int_0^t g_n(s)ds$. The preceding proposition implies that

$$\nabla(f_n \circ u) = (g_n \circ u)\nabla u.$$

Since, in $L^1_{\text{loc}}(\Omega)$, by Lebesgue's dominated convergence theorem,

$$f_n \circ u \to f \circ u, (g_n \circ u)\nabla u \to (g \circ u)\nabla u,$$

the closing lemma implies that

$$\nabla(f \circ u) = (g \circ u)\nabla u.$$

□

Corollary 6.1.15 *Let $1 \leq p < \infty$ and let $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ be such that $u = 0$ on $\partial\Omega$. Then $u \in W^{1,p}_0(\Omega)$.*

Proof It is easy to prove by regularization that $W^{1,p}(\Omega) \cap \mathcal{K}(\Omega) \subset W^{1,p}_0(\Omega)$.

Assume that spt u is bounded. Let $f \in C^1(\mathbb{R})$ be such that $|f(t)| \leq |t|$ on $\mathbb{R}$,

$$\begin{aligned} f(t) &= 0, \quad |t| \leq 1, \\ &= t, \quad |t| \geq 2. \end{aligned}$$

Define $u_n = f(n\,u)/n$. Then $u_n \in \mathcal{K}(\Omega)$, and by the preceding proposition, $u_n \in W^{1,p}(\Omega)$. By Lebesgue's dominated convergence theorem, $u_n \to u$ in $W^{1,p}(\Omega)$, so that $u \in W_0^{1,p}(\Omega)$.

If spt u is unbounded, we define $u_n = \theta_n u$ where (θ_n) is defined in the proof of Theorem 6.1.10. Then spt u_n is bounded. By Lebesgue's dominated convergence theorem, $u_n \to u$ in $W^{1,p}(\Omega)$, so that $u \in W_0^{1,p}(\Omega)$. □

Proposition 6.1.16 *Let Ω be an open subset of $\mathbb{R}^N$. Then there exist a sequence (U_n) of open subsets of Ω and a sequence of functions $\psi_n \in \mathcal{D}(U_n)$ such that*

(a) for every n, $U_n \subset\subset \Omega$ and $\psi_n \geq 0$;
(b) $\sum_{n=1}^{\infty} \psi_n = 1$ on Ω;
(c) for every $\omega \subset\subset \Omega$ there exists m_ω such that for $n > m_\omega$ we have $U_n \cap \omega = \phi$.

Proof Let us define $\omega_{-1} = \omega_0 = \phi$, and for $n \geq 1$,

$$\begin{aligned} \omega_n &= \{x \in \Omega : d(x, \partial\Omega) > 1/n \text{ and } |x| < n\}, \\ K_n &= \overline{\omega_n} \setminus \omega_{n-1}, \\ U_n &= \omega_{n+1} \setminus \overline{\omega_{n-2}}. \end{aligned}$$

The theorem of partitions of unity implies the existence of $\varphi_n \in \mathcal{D}(U_n)$ such that $0 \leq \varphi_n \leq 1$ and $\varphi_n = 1$ on K_n. It suffices then to define

$$\psi_n = \varphi_n / \sum_{j=1}^{\infty} \varphi_j.$$

□

Theorem 6.1.17 (Hajłasz) *Let $1 \leq p < \infty$, $u \in W_{\text{loc}}^{1,p}(\Omega)$, and $\varepsilon > 0$. Then there exists $v \in C^\infty(\Omega)$ such that*

(a) $v - u \in W_0^{1,p}(\Omega)$;
(b) $||v - u||_{W^{1,p}(\Omega)} < \varepsilon$.

Proof Let (U_n) and (ψ_n) be given by the preceding proposition. For every $n \geq 1$, there exists k_n such that

$$v_n = \rho_{k_n} * (\psi_n u) \in \mathcal{D}(U_n)$$

and

$$||v_n - \psi_n u||_{1,p} < \varepsilon/2^n.$$

By Proposition 3.1.6, $\sum_{n=1}^{\infty}(v_n - \psi_n u)$ converges to w in $W_0^{1,p}(\Omega)$. On the other hand, we have on $\omega \subset\subset \Omega$ that

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{m_\omega} v_n \in C^\infty(\omega), \quad \sum_{n=1}^{\infty} \psi_n u = u.$$

Setting $v = \sum_{n=1}^{\infty} v_n$, we conclude that

$$||v - u||_{1,p} = ||w||_{1,p} \leq \sum_{n=1}^{\infty} ||v_n - \psi_n u||_{1,p} < \varepsilon. \qquad \square$$

Corollary 6.1.18 (Deny–Lions) *Let $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

6.2 Cylindrical Domains

Let U be an open subset of $\mathbb{R}^{N-1}$ and $0 < r \leq \infty$. Define

$$\Omega = U \times \,]-r, r[, \quad \Omega_+ = U \times \,]0, r[.$$

The extension by reflection of a function in $W^{1,p}(\Omega_+)$ is a function in $W^{1,p}(\Omega)$.

For every $u : \Omega_+ \to \mathbb{R}$, we define on Ω:

$$\rho u(x', x_N) = u\Big(x', |x_N|\Big), \quad \sigma u(x', x_N) = (\operatorname{sgn} x_N) u\Big(x', |x_N|\Big).$$

Lemma 6.2.1 (Extension by Reflection) *Let $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega_+)$. Then $\rho u \in W^{1,p}(\Omega)$, $\partial_k(\rho u) = \rho(\partial_k u)$, $1 \leq k \leq N-1$, and $\partial_N(\rho u) = \sigma(\partial_N u)$, so that*

$$||\rho u||_{L^p(\Omega)} = 2^{1/p}||u||_{L^p(\Omega_+)}, \quad ||\rho u||_{W^{1,p}(\Omega)} = 2^{1/p}||u||_{W^{1,p}(\Omega_+)}.$$

Proof Let $v \in \mathcal{D}(\Omega)$. Then by a change of variables,

$$\int_\Omega (\rho u)\partial_N v \, dx = \int_{\Omega_+} u \, \partial_N w \, dx, \qquad (*)$$

where

$$w(x', x_N) = v(x', x_N) - v(x', -x_N).$$

A truncation argument will be used. Let $\eta \in C^\infty(\mathbb{R})$ be such that

$$\begin{aligned} \eta(t) &= 0, \quad t < 1/2, \\ &= 1, \quad t > 1, \end{aligned}$$

and define η_n on Ω_+ by

$$\eta_n(x) = \eta(n\, x_N).$$

The definition of weak derivative ensures that

$$\int_{\Omega_+} u\, \partial_N(\eta_n w)dx = -\int_{\Omega_+} (\partial_N u)\eta_n w\, dx, \qquad (**)$$

where

$$\partial_N(\eta_n w) = \eta_n \partial_N w + n\eta'(n\, x_N)w.$$

Since $w(x', 0) = 0$, $w(x', x_N) = h(x', x_N)x_N$, where

$$h(x', x_N) = \int_0^1 \partial_N w(x', t\, x_N)dt.$$

Lebesgue's dominated convergence theorem implies that

$$\begin{aligned} \left| \int_{\Omega_+} n\, \eta'(n\, x_N)w\, u\, dx \right| &= \left| \int_{U\times\,]0,1/n[} n\, \eta'(n\, x_N)h\, x_N u\, dx \right| \\ &\le ||\eta'||_\infty \int_{U\times\,]0,1/n[} |hu| dx \to 0, \quad n \to \infty. \end{aligned}$$

Taking the limit in $(**)$, we obtain

$$\int_{\Omega_+} u\, \partial_N w\, dx = -\int_{\Omega_+} (\partial_N u)w\, dx = -\int_\Omega \sigma(\partial_N u)v\, dx.$$

It follows from $(*)$ that

$$\int_\Omega (\rho u)\partial_N v\, dx = -\int_\Omega \sigma(\partial_N u)v\, dx.$$

Since $v \in \mathcal{D}(\Omega)$ is arbitrary, $\partial_N(\rho u) = \sigma(\partial_N u)$. By a similar but simpler argument, $\partial_k(\rho u) = \rho(\partial_k u)$, $1 \le k \le N-1$. □

It makes no sense to define an L^p function on a set of measure zero. We will define the trace of a $W^{1,p}$ function on the boundary of a smooth domain. We first consider the case of $\mathbb{R}^N_+$.

Notation We define

$$\mathcal{D}(\overline{\Omega}) = \{u|_\Omega : u \in \mathcal{D}(\mathbb{R}^N)\},$$

$$\mathbb{R}^N_+ = \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}.$$

Lemma 6.2.2 (Trace Inequality) *Let* $1 \le p < \infty$. *Then for every* $u \in \mathcal{D}(\overline{\mathbb{R}^N_+})$,

$$\int_{\mathbb{R}^{N-1}} \left|u(x', 0)\right|^p dx' \le p||u||^{p-1}_{L^p(\mathbb{R}^N_+)} ||\partial_N u||_{L^p_{(\mathbb{R}^N_+)}}.$$

Proof The fundamental theorem of calculus implies that for all $x' \in \mathbb{R}^{N-1}$,

$$\left|u(x', 0)\right|^p \le p \int_0^\infty \left|u(x', x_N)\right|^{p-1} \left|\partial_N u(x', x_N)\right| dx_N.$$

When $1 < p < \infty$, using Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \left|u(x', 0)\right|^p dx' &\le p \int_{\mathbb{R}^N_+} |u|^{p-1} |\partial_N u| dx \\ &\le p \left(\int_{\mathbb{R}^N_+} |u|^{(p-1)p'} dx\right)^{1/p'} \left(\int_{\mathbb{R}^N_+} |\partial_N u|^p dx\right)^{1/p} \\ &= p \left(\int_{\mathbb{R}^N_+} |u|^p dx\right)^{1-1/p} \left(\int_{\mathbb{R}^N_+} |\partial_N u|^p dx\right)^{1/p}.\end{aligned}$$

The case $p = 1$ is similar. □

Proposition 6.2.3 *Let* $1 \le p < \infty$. *Then there exists one and only one continuous linear mapping* $\gamma_0 : W^{1,p}(\mathbb{R}^N_+) \to L^p(\mathbb{R}^{N-1})$ *such that for every* $u \in \mathcal{D}(\mathbb{R}^N_+)$, $\gamma_0 u = u(., 0)$.

Proof Let $u \in \mathcal{D}(\overline{\mathbb{R}^N_+})$ and define $\gamma_0 u = u(., 0)$. The preceding lemma implies that

$$||\gamma_0 u||_{L^p(\mathbb{R}^{N-1})} \le p^{1/p} ||u||_{W^{1,p}(\mathbb{R}^N_+)}.$$

The space $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $W^{1,p}(\mathbb{R}^N_+)$ by Theorem 6.1.10 and Lemma 6.2.1. By Proposition 3.2.3, γ_0 has a unique continuous linear extension to $W^{1,p}(\mathbb{R}^N_+)$. □

Proposition 6.2.4 (Integration by Parts) *Let $1 \leq p < \infty$, $u \in W^{1,p}(\mathbb{R}^N_+)$, and $v \in \mathcal{D}(\overline{\mathbb{R}^N_+})$. Then*

$$\int_{\mathbb{R}^N_+} v\, \partial_N u\, dx = -\int_{\mathbb{R}^N_+} (\partial_N v) u\, dx - \int_{\mathbb{R}^{N-1}} \gamma_0 v\, \gamma_0 u\, dx',$$

and

$$\int_{\mathbb{R}^N_+} v \partial_k u\, dx = -\int_{\mathbb{R}^N_+} (\partial_k v) u\, dx, \quad 1 \leq k \leq N-1.$$

Proof Assume, moreover, that $u \in \mathcal{D}(\overline{\mathbb{R}^N_+})$. Integrating by parts, we obtain for all $x' \in \mathbb{R}^{N-1}$,

$$\int_0^\infty v(x', x_N) \partial_N u(x', x_N) dx_N = -\int_0^\infty \partial_N v(x', x_N) u(x', x_N) dx_N - v(x', 0) u(x', 0).$$

Fubini's theorem implies that

$$\int_{\mathbb{R}^N_+} v\, \partial_N u\, dx = -\int_{\mathbb{R}^N_+} \partial_N v u\, dx - \int_{\mathbb{R}^{N-1}} v(x', 0) u(x', 0) dx'.$$

Let $u \in W^{1,p}(\mathbb{R}^N_+)$. Since $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $W^{1,p}(\mathbb{R}^N_+)$, there exists a sequence $(u_n) \subset \mathcal{D}(\overline{\mathbb{R}^N_+})$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N_+)$. By the preceding lemma, $\gamma_0 u_n \to \gamma_0 u$ in $L^p(\mathbb{R}^{N-1})$. It is easy to finish the proof.

The proof of the last formulas is similar. □

Notation For every $u : \mathbb{R}^N_+ \to \mathbb{R}$, we define $\overline{u}$ on $\mathbb{R}^N$ by

$$\begin{aligned} \overline{u}(x', x_N) &= u(x', x_N), & x_N > 0, \\ &= 0, & x_N \leq 0. \end{aligned}$$

Proposition 6.2.5 *Let $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{R}^N_+)$. The following properties are equivalent:*

(a) $u \in W_0^{1,p}(\mathbb{R}^N_+)$;
(b) $\gamma_0 u = 0$;
(c) $\overline{u} \in W^{1,p}(\mathbb{R}^N)$ *and* $\partial_k \overline{u} = \overline{\partial_k u}$, $1 \leq k \leq N$.

Proof If $u \in W_0^{1,p}(\mathbb{R}_+^N)$, there exists $(u_n) \subset \mathcal{D}(\mathbb{R}_+^N)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}_+^N)$. Hence $\gamma_0 u_n = 0$ and $\gamma_0 u_n \to \gamma_0 u$ in $L^p(\mathbb{R}^{N-1})$, so that $\gamma_0 u = 0$.

If $\gamma_0 u = 0$, it follows from the preceding proposition that for every $v \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} v\, \overline{\partial_k u}\, dx = -\int_{\mathbb{R}^N} \partial_k v\, \overline{u}\, dx, \quad 1 \le k \le N.$$

We conclude that (c) is satisfied.

Assume that (c) is satisfied. We define $u_n = \theta_n \overline{u}$, where (θ_n) is defined in the proof of Theorem 6.1.10. It is clear that $u_n \to \overline{u}$ in $W^{1,p}(\mathbb{R}^N)$ and spt $u_n \subset B[0, 2n] \cap \overline{\mathbb{R}_+^N}$.

We can assume that spt u_n is a compact subset of $\overline{\mathbb{R}_+^N}$. We define $y_n = (0, \ldots, 0, 1/n)$ and $v_n = \tau_{y_n} \overline{u}$. Since $\partial_k v_n = \tau_{y_n} \partial_k \overline{u}$, the lemma of continuity of translations implies that $u_n \to u$ in $W^{1,p}(\mathbb{R}_+^N)$.

We can assume that spt u is a compact subset of $\mathbb{R}_+^N$. For n large enough, $\rho_n * u \in \mathcal{D}(\mathbb{R}_+^N)$. Since $\rho_n * u \to u$ is in $W^{1,p}(\mathbb{R}^N)$, we conclude that $u \in W_0^{1,p}(\mathbb{R}^N)$. □

6.3 Smooth Domains

In this section we consider an open subset $\Omega = \{\varphi < 0\}$ of $\mathbb{R}^N$ of class C^1 with a bounded boundary Γ. We use the notations of Definition 9.4.1.

Let $\gamma \in \Gamma$. Since $\nabla\varphi(\gamma) \neq 0$, we can assume that, after a permutation of variables, $\partial_N \varphi(\gamma) \neq 0$. By Theorem 9.1.1 there exist $r > 0$, $R > 0$, and

$$\beta \in C^1\big(B(\gamma', R) \times]-r, r[\big)$$

such that, for $|x' - \gamma'| < R$ and $|t| < r$, we have

$$\varphi(x', x_N) = t \quad \Leftrightarrow \quad x_N = \beta(x', t)$$

and the set

$$U_\gamma = \Big\{\big(x', \beta(x', t)\big) : |x' - \gamma'| < R, |t| < r\Big\}$$

is an open neighborhood of γ. Moreover

$$\Omega \cap U_\gamma = \Big\{\big(x', \beta(x', t)\big) : |x' - \gamma'| < R, -r < t < 0\Big\}$$

and

$$\Gamma \cap U_\gamma = \Big\{\big(x', \beta(x', 0)\big) : |x' - \gamma'| < R\Big\}.$$

The Borel–Lebesgue theorem implies the existence of a finite covering $U_1, \dots, U_k$ of Γ by open subsets satisfying the above properties. There exists a partition of unity $\psi_1, \dots, \psi_k$ subordinate to this covering.

Theorem 6.3.1 (Extension Theorem) *Let $1 \le p < \infty$ and let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary or the product of N open intervals. Then there exists a continuous linear mapping*

$$P : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

such that $Pu\big|_\Omega = u$.

Proof Let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary, and let $u \in W^{1,p}(\Omega)$. Proposition 6.1.11 and Lemma 6.2.1 imply that

$$P_U u(x) = u(x', \beta(x', -|\varphi(x', x_N)|)) \in W^{1,p}(U).$$

Moreover,

$$||P_U u||_{W^{1,p}(U)} \le a_U ||u||_{W^{1,p}(\Omega)}. \qquad (*)$$

We define $\psi_0 = 1 - \sum_{j=1}^{k} \psi_j$,

$$\begin{aligned} u_0 &= \psi_0 u, && x \in \Omega, \\ &= 0, && x \in \mathbb{R}^N \setminus \Omega, \end{aligned}$$

and for $1 \le j \le k$,

$$\begin{aligned} u_j &= P_{U_j}(\psi_j u), && x \in U_j, \\ &= 0, && x \in \mathbb{R}^N \setminus U_j. \end{aligned}$$

Formula $(*)$ and Proposition 6.1.12 ensure that for $0 \le j \le k$,

$$||u_j||_{W^{1,p}(\mathbb{R}^N)} \le b_j ||u||_{W^{1,p}(\Omega)}.$$

(The support of $\nabla \psi_0$ is compact!) Hence

$$Pu = \sum_{j=0}^{k} u_j \in W^{1,p}(\mathbb{R}^N), \quad ||Pu||_{W^{1,p}(\mathbb{R}^N)} \le c||u||_{W^{1,p}(\Omega)},$$

and for all $x \in \Omega$,

$$Pu(x) = \sum_{j=0}^{k} \psi_j(x)u(x) = u(x).$$

If Ω is the product of N open intervals, it suffices to use a finite number of extensions by reflections and a truncation. □

Theorem 6.3.2 (Density Theorem in Sobolev Spaces) *Let $1 \leq p < \infty$ and let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary or the product of N open intervals. Then the space $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

Proof Let $u \in W^{1,p}(\Omega)$. Theorem 6.1.10 implies the existence of a sequence $(v_n) \subset \mathcal{D}(\mathbb{R}^N)$ converging to Pu in $W^{1,p}(\mathbb{R}^N)$. Hence $u_n = v_n\big|_{\Omega}$ converges to u in $W^{1,p}(\Omega)$. □

Theorem 6.3.3 (Trace Inequality) *Let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary Γ. Then there exist $a > 0$ and $b > 0$ such that, for $1 \leq p < \infty$ and for every $u \in \mathcal{D}(\bar{\Omega})$,*

$$\int_{\Gamma} |u|^p d\gamma \leq a\|u\|^p_{L^p(\Omega)} + bp\|u\|^{p-1}_{L^p(\Omega)}\|\nabla u\|_{L^p(\Omega)}.$$

Proof Let $1 < p < \infty$, $u \in \mathcal{D}(\bar{\Omega})$, and $\nu \in C^{\infty}(\mathbb{R}^N; \mathbb{R}^N)$.
Since

$$\text{div}|u|^p\nu = |u|^p \,\text{div}\nu + pu|u|^{p-2}\nabla u \cdot \nu,$$

the divergence theorem implies that

$$\int_{\Gamma} |u|^p \nu \cdot n d\gamma = \int_{\Omega} \left[|u|^p \,\text{div}\nu + pu|u|^{p-2}\nabla u \cdot \nu\right] dx.$$

Assume that $1 \leq \nu \cdot n$ on Γ. Using Hölder's inequality, we obtain that, for $1 < p < \infty$,

$$\begin{aligned}\int_{\Gamma} |u|^p d\gamma \leq \int_{\Gamma} |u|^p \nu \cdot n d\gamma &\leq a \int_{\Omega} |u|^p dx + bp \int_{\Omega} |u|^{p-1}|\nabla u| dx \\ &\leq a \int_{\Omega} |u|^p dx + bp \left(\int_{\Omega} |u|^{(p-1)p'} dx\right)^{1/p'} \left(\int_{\Omega} |\nabla u|^p dx\right)^{1/p} \\ &= a \int_{\Omega} |u|^p dx + bp \left(\int_{\Omega} |u|^p dx\right)^{1-1/p} \left(\int_{\Omega} |\nabla u|^p dx\right)^{1/p},\end{aligned}$$

where $a = \|\text{div}\nu\|_{\infty}$ and $b = \|\nu\|_{\infty}$.

When $p \downarrow 1$, it follows from Lebesgue's dominated convergence theorem that

$$\int_\Gamma |u| d\gamma \le a \int_\Omega |u| dx + b \int_\Omega |\nabla u| dx.$$

Let us construct an admissible vector field ν. Let $U = \{x \in \mathbb{R}^N : \nabla\varphi(x) \neq 0\}$. The theorem of partition of unity implies the existence of $\psi \in \mathcal{D}(U)$ such that $\psi = 1$ on Γ. We define the vector field w by

$$\begin{aligned} w(x) &= \psi(x) \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}, \quad && x \in U \\ &= 0, && x \in \mathbb{R}^N \backslash U. \end{aligned}$$

For n large enough, the C^∞ vector field $\nu = 2\rho_n * w$ is such that $1 \le \nu \cdot n$ on Γ. □

Theorem 6.3.4 *Under the assumptions of Theorem 6.3.3, there exists one and only one continuous linear mapping*

$$\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$$

such that for all $u \in \mathcal{D}(\bar{\Omega})$, $\gamma_0 u = u\Big|_\Gamma$.

Proof It suffices to use the trace inequality, Proposition 3.2.3, and the density theorem in Sobolev spaces. □

Theorem 6.3.5 (Divergence Theorem) *Let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary Γ and $\nu \in W^{1,1}(\Omega; \mathbb{R}^N)$. Then*

$$\int_\Omega \operatorname{div} \nu dx = \int_\Gamma \gamma_0 \nu \cdot n d\gamma.$$

Proof When $\nu \in \mathcal{D}(\bar{\Omega}; \mathbb{R}^N)$, the proof is given in Section 9.4. In the general case, it suffices to use the density theorem in Sobolev spaces and the trace theorem. □

6.4 Embeddings

Let $1 \le p, q < \infty$. If there exists $c > 0$ such that for every $u \in \mathcal{D}(\mathbb{R}^N)$,

$$||u||_{L^q(\mathbb{R}^N)} \le c||\nabla u||_{L^p(\mathbb{R}^N)},$$

then necessarily

$$q = p^* = Np/(N-p).$$

Indeed, replacing $u(x)$ by $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$, we find that

$$||u||_{L^q(\mathbb{R}^N)} \le c\lambda^{\left(1+\frac{N}{q}-\frac{N}{p}\right)} ||\nabla u||_{L^p(\mathbb{R}^N)},$$

so that $q = p^*$.

We define for $1 \le j \le N$ and $x \in \mathbb{R}^N$,

$$\widehat{x_j} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N).$$

Lemma 6.4.1 (Gagliardo's Inequality) *Let* $N \ge 2$ *and* $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. *Then* $f(x) = \prod_{j=1}^{N} f_j(\widehat{x_j}) \in L^1(\mathbb{R}^N)$ *and*

$$||f||_{L^1(\mathbb{R}^N)} \le \prod_{j=1}^{N} ||f_j||_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Proof We use induction. When $N = 2$, the inequality is clear. Assume that the inequality holds for $N \ge 2$. Let $f_1, \ldots, f_{N+1} \in L^N(\mathbb{R}^N)$ and

$$f(x, x_{N+1}) = \prod_{j=1}^{N} f_j(\widehat{x_j}, x_{N+1}) f_{N+1}(x).$$

It follows from Hölder's inequality that for almost every $x_{N+1} \in \mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}^N} \left|f(x, x_{N+1})\right| dx &\le \left[\int_{\mathbb{R}^N} \prod_{j=1}^{N} \left|f_j(\widehat{x_j}, x_{N+1})\right|^{N'} dx\right]^{1/N'} ||f_{N+1}||_{L^N(\mathbb{R}^N)} \\ &\le \prod_{j=1}^{N} \left[\int_{\mathbb{R}^{N-1}} \left|f_j(\widehat{x_j}, x_{N+1})\right|^N d\widehat{x_j}\right]^{1/N} ||f_{N+1}||_{L^N(\mathbb{R}^N)}.\end{aligned}$$

The generalized Hölder inequality implies that

$$\begin{aligned}||f||_{L^1(\mathbb{R}^{N+1})} &\le \prod_{j=1}^{N}\left[\int_{\mathbb{R}^N}\left|f_j(\widehat{x_j},x_{N+1})\right|^N d\widehat{x_j}dx_{N+1}\right]^{1/N}||f_{N+1}||_{L^N(\mathbb{R}^N)}\\ &= \prod_{j=1}^{N+1}||f_j||_{L^N(\mathbb{R}^N)}.\end{aligned}$$ □

Lemma 6.4.2 (Sobolev's Inequalities) *Let $1 \le p < N$. Then there exists a constant $c = c(p,N)$ such that for every $u \in \mathcal{D}(\mathbb{R}^N)$,*

$$||u||_{L^{p^*}(\mathbb{R}^N)} \le c||\nabla u||_{L^p(\mathbb{R}^N)}.$$

Proof Let $u \in C^1(\mathbb{R}^N)$ be such that spt u is compact. It follows from the fundamental theorem of calculus that for $1 \le j \le N$ and $x \in \mathbb{R}^N$,

$$\left|u(x)\right| \le \frac{1}{2}\int_{\mathbb{R}}\left|\partial_j u(x)\right|dx_j.$$

By the preceding lemma,

$$\int_{\mathbb{R}^N}\left|u(x)\right|^{N/(N-1)}dx \le \prod_{j=1}^{N}\left[\frac{1}{2}\int_{\mathbb{R}^N}\left|\partial_j u(x)\right|dx\right]^{1/(N-1)}.$$

Hence we obtain

$$||u||_{N/(N-1)} \le \frac{1}{2}\prod_{j=1}^{N}||\partial_j u||_1^{1/N} \le c_N||\nabla u||_1.$$

For $p > 1$, we define $q = (N-1)p^*/N > 1$. Let $u \in \mathcal{D}(\mathbb{R}^N)$. The preceding inequality applied to $|u|^q$ and Hölder's inequality imply that

$$\begin{aligned}\left(\int |u|^{p^*}dx\right)^{\frac{N-1}{N}} &\le q\, c_N\int_{\mathbb{R}^N}|u|^{q-1}|\nabla u|dx\\ &\le q\, c_N\left(\int_{\mathbb{R}^N}|u|^{(q-1)p'}dx\right)^{1/p'}\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{1/p}.\end{aligned}$$

It is easy to conclude the proof. □

Lemma 6.4.3 (Morrey's Inequalities) *Let $N < p < \infty$ and $\lambda = 1 - N/p$. Then there exists a constant $c = c(p,N)$ such that for every $u \in \mathcal{D}(\mathbb{R}^N)$ and every $x, y \in \mathbb{R}^N$,*

$$\begin{aligned}&\big|u(x)-u(y)\big|\leq c|x-y|^{\lambda}\|\nabla u\|_{L^p(\mathbb{R}^N)},\\&\|u\|_\infty\leq c\|u\|_{W^{1,p}(\mathbb{R}^N)}.\end{aligned}$$

Proof Let $u\in\mathcal{D}(\mathbb{R}^N)$, and let us define $B=B(a,r)$, $a\in\mathbb{R}^N$, $r>0$, and

$$⨍ u=\frac{1}{m(B)}\int_B u\,dx.$$

We assume that $0\in\bar{B}$. It follows from the fundamental theorem of calculus and Fubini's theorem that

$$\begin{aligned}\left|⨍ u-u(0)\right|&\leq\frac{1}{m(B)}\int_B\big|u(x)-u(0)\big|dx\\&\leq\frac{1}{m(B)}\int_B dx\int_0^1\big|\nabla u(tx)\big|\,|x|dt\\&\leq\frac{2r}{m(B)}\int_0^1 dt\int_B\big|\nabla u(tx)\big|dx\\&=\frac{2r}{m(B)}\int_0^1\frac{dt}{t^N}\int_{B(ta,tr)}\big|\nabla u(y)\big|dy.\end{aligned}$$

Hölder's inequality implies that

$$\left|⨍ u-u(0)\right|\leq\frac{2r}{m(B)}\int_0^1 m\big(B(ta,tr)\big)^{1/p'}\frac{dt}{t^N}\|\nabla u\|_{L^p(B)}=\frac{2}{\lambda V_N^{1/p}}r^{\lambda}\|\nabla u\|_{L^p(B)}.$$

After a translation, we obtain that, for every $x\in B[a,r]$,

$$\left|⨍ u-u(x)\right|\leq c_\lambda r^{\lambda}\|\nabla u\|_{L^p(B)}.$$

Let $x\in\mathbb{R}^N$. Choosing $a=x$ and $r=1$, we find

$$|u(x)|\leq\left|⨍ u\right|+c_\lambda\|\nabla u\|_{L^p(B)}\leq c\big(\|u\|_{L^p(B)}+\|\nabla u\|_{L^p(B)}\big).$$

Let $x,y\in\mathbb{R}^N$. Choosing $a=(x+y)/2$ and $r=|x-y|/2$, we obtain

$$|u(x)-u(y)|\leq 2^{1-\lambda}c_\lambda|x-y|^{\lambda}\|\nabla u\|_{L^p(B)}.$$ □

Notation We define

$$C_0(\overline{\Omega}) = \{u\big|_{\Omega} : u \in C_0(\mathbb{R}^N)\}.$$

Theorem 6.4.4 (Sobolev's Embedding Theorem, 1936–1938) *Let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary or the product of N open intervals.*

(a) If $1 \le p < N$ and if $p \le q \le p^*$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$, and the canonical injection is continuous.
(b) If $N < p < \infty$ and $\lambda = 1 - N/p$, then $W^{1,p}(\Omega) \subset C_0(\overline{\Omega})$, the canonical injection is continuous, and there exists $c = c(p, \Omega)$ such that for every $u \in W^{1,p}(\Omega)$ and all $x, y \in \Omega$,

$$\big|u(x) - u(y)\big| \le c||u||_{W^{1,p}(\Omega)}|x - y|^{\lambda}.$$

Proof Let $1 \le p < N$ and $u \in W^{1,p}(\mathbb{R}^N)$. By Theorem 6.1.10, there exists a sequence $(u_n) \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$.

We can assume that $u_n \to u$ almost everywhere on $\mathbb{R}^N$. It follows from Fatou's lemma and Sobolev's inequality that

$$||u||_{L^{p^*}(\mathbb{R}^N)} \le \varliminf_{n\to\infty} ||u_n||_{L^{p^*}(\mathbb{R}^N)} \le c \lim_{n\to\infty} ||\nabla u_n||_{L^p(\mathbb{R}^N)} = c||\nabla u||_{L^p(\mathbb{R}^N)}.$$

Let P be the extension operator corresponding to Ω and $v \in W^{1,p}(\Omega)$. We have

$$||v||_{L^{p^*}(\Omega)} \le ||Pv||_{L^{p^*}(\mathbb{R}^N)} \le c||\nabla Pv||_{L^p(\mathbb{R}^N)} \le c_1||v||_{W^{1,p}(\Omega)}.$$

If $p \le q \le p^*$, we define $0 \le \lambda \le 1$ by

$$\frac{1}{q} = \frac{1-\lambda}{p} + \frac{\lambda}{p^*},$$

and we infer from the interpolation inequality that

$$||v||_{L^q(\Omega)} \le ||v||_{L^p(\Omega)}^{1-\lambda}||v||_{L^{p^*}(\Omega)}^{\lambda} \le c_1^{\lambda}||v||_{W^{1,p}(\Omega)}.$$

The case $p > N$ follows from Morrey's inequalities. □

Lemma 6.4.5 *Let Ω be an open subset of $\mathbb{R}^N$ such that $m(\Omega) < +\infty$, and let $1 \le p \le r < +\infty$. Assume that X is a closed subspace of $W^{1,p}(\Omega)$ such that $X \subset L^r(\Omega)$. Then, for every $1 \le q < r$, $X \subset L^q(\Omega)$ and the canonical injection is compact.*

Proof The closed graph theorem implies the existence of $c > 0$ such that, for every $u \in X$,

$$\|u\|_{L^r(\Omega)} \leq c\|u\|_{W^{1,p}(\Omega)}.$$

Our goal is to prove that

$$S = \{u \in X : \|u\|_{W^{1,p}(\Omega)} \leq 1\}$$

is precompact in $L^q(\Omega)$ for $1 \leq q < r$. Let $1/q = 1-\lambda+\lambda/r$. By the interpolation inequality, for every $u \in S$,

$$\|u\|_{L^q(\Omega)} \leq \|u\|^{\lambda}_{L^r(\Omega)}\|u\|^{1-\lambda}_{L^1(\Omega)} \leq c^{\lambda}\|u\|^{1-\lambda}_{L^1(\Omega)}.$$

Hence it suffices to prove that S is precompact in $L^1(\Omega)$.

Let us verify that S satisfies the assumptions of M. Riesz's theorem in $L^1(\Omega)$:

(a) It follows from Hölder's inequality that, for every $u \in S$,

$$\|u\|_{L^1(\Omega)} \leq \|u\|_{L^r(\Omega)} m(\Omega)^{1-1/r} \leq cm(\Omega)^{1-1/r}.$$

(b) Similarly, we have that, for every $u \in S$,

$$\int_{\Omega\backslash\omega_k} |u|dx \leq \|u\|_{L^r(\Omega)} m(\Omega\backslash\omega_k)^{1-1/r} \leq cm(\Omega\backslash\omega_k)^{1-1/r}$$

where

$$\omega_k = \{x \in \Omega : d(x, \partial\Omega) > 1/k\}.$$

Lebesgue's dominated convergence theorem implies that

$$\lim_{k\to\infty} m(\Omega\backslash\omega_k) = 0.$$

(c) Let $\omega \subset\subset \Omega$. Assume that $|y| < d(\omega, \partial\Omega)$ and $u \in C^{\infty}(\Omega) \cap W^{1,p}(\Omega)$.

Since, by the fundamental theorem of calculus,

$$\left|\tau_y u(x) - u(x)\right| = \left|\int_0^1 y \cdot \nabla u(x - ty)dt\right| \leq |y| \int_0^1 \left|\nabla u(x - ty)\right|dt,$$

we obtain

$$\begin{aligned}\|\tau_y u - u\|_{L^1(\omega)} &\le |y| \int_\omega dx \int_0^1 \Big|\nabla u(x - ty)\Big| dt \\ &= |y| \int_0^1 dt \int_\omega \Big|\nabla u(x - ty)\Big| dx \\ &= |y| \int_0^1 dt \int_{\omega - ty} \Big|\nabla u(z)\Big| dz \le |y|\ \|\nabla u\|_{L^1(\Omega)}.\end{aligned}$$

Using Corollary 6.1.18, we conclude by density that, for every $u \in S$,

$$\|\tau_y u - u\|_{L^1(\omega)} \le \|\nabla u\|_{L^1(\Omega)} |y| \le \|\nabla u\|_{L^p(\Omega)} m(\Omega)^{1-1/p} |y| \le c_1 |y|. \qquad \square$$

Theorem 6.4.6 (Rellich–Kondrachov Embedding Theorem) *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class C^1 or the product of N bounded open intervals:*

(a) If $1 \le p < N$ and $1 \le q < p^*$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$, and the canonical injection is compact.
(b) If $N < p < \infty$, then $W^{1,p} \subset C_0(\bar{\Omega})$, and the canonical injection is compact.

Proof Let $1 \le p < N, 1 \le q < p^*$. It suffices to use Sobolev's embedding theorem and the preceding lemma.

The case $p > N$ follows from Ascoli's theorem and Sobolev's embedding theorem. $\square$

We prove three fundamental inequalities.

Theorem 6.4.7 (Poincaré's Inequality in $W_0^{1,p}$) *Let $1 \le p < \infty$, and let Ω be an open subset of $\mathbb{R}^N$ such that $\Omega \subset \mathbb{R}^{N-1} \times]0, a[$. Then for every $u \in W_0^{1,p}(\Omega)$,*

$$||u||_{L^p(\Omega)} \le \frac{a}{2} ||\nabla u||_{L^p(\Omega)}.$$

Proof Let $1 < p < \infty$ and $v \in \mathcal{D}(]0, a[)$. The fundamental theorem of calculus and Hölder's inequality imply that for $0 < x < a$,

$$\big|v(x)\big| \le \frac{1}{2} \int_0^a \big|v'(t)\big| dt \le \frac{a^{1/p'}}{2} \Big| \int_0^a \big|v'(t)\big|^p dt \Big|^{1/p}.$$

Hence we obtain

$$\int_0^a \big|v(x)\big|^p dx \le \frac{a^{p/p'}}{2^p} a \int_0^a \big|v'(x)\big|^p dx = \frac{a^p}{2^p} \int_0^a \big|v'(x)\big|^p dx.$$

If $u \in \mathcal{D}(\Omega)$, we infer from the preceding inequality and from Fubini's theorem that

$$\begin{aligned}\int_\Omega |u|^p dx &= \int_{\mathbb{R}^{N-1}} dx' \int_0^a \left|u(x', x_N)\right|^p dx_N \\ &\le \frac{a^p}{2^p} \int_{\mathbb{R}^{N-1}} dx' \int_0^a \left|\partial_N u(x', x_N)\right|^p dx_N \\ &= \frac{a^p}{2^p} \int_\Omega |\partial_N u|^p dx.\end{aligned}$$

It is easy to conclude by density. The case $p = 1$ is similar. □

Definition 6.4.8 A metric space is connected if the only open and closed subsets of X are ϕ and X.

Theorem 6.4.9 (Poincaré's Inequality in $W^{1,p}$) *Let $1 \le p < \infty$, and let Ω be a bounded open connected subset of $\mathbb{R}^N$. Assume that Ω is of class C^1. Then there exists $c = c(p, \Omega)$, such that, for every $u \in W^{1,p}(\Omega)$,*

$$\left\| u - \fint u \right\|_{L^p(\Omega)} \le c \|\nabla u\|_{L^p(\Omega)},$$

where

$$\fint u = \frac{1}{m(\Omega)} \int_\Omega u \, dx.$$

Assume that Ω is convex. Then, for every $u \in W^{1,p}(\Omega)$,

$$\left\| u - \fint u \right\|_{L^p(\Omega)} \le 2^{N/p} \, d \, \|\nabla u\|_{L^p(\Omega)},$$

where $d = \sup_{x,y \in \Omega} |x - y|$.

Proof Assume that Ω is of class C^1. It suffices to prove that

$$\lambda = \inf\left\{ \|\nabla u\|_p : u \in W^{1,p}(\Omega), \fint u = 0, \|u\|_p = 1 \right\} > 0.$$

Let $(u_n) \subset W^{1,p}(\Omega)$ be a minimizing sequence :

$$\|u_n\|_p = 1, \quad \fint u_n = 0, \quad \|\nabla u_n\|_p \to \lambda.$$

By the Rellich–Kondrachov theorem, we can assume that $u_n \to u$ in $L^p(\Omega)$. Hence $\|u\|_p = 1$ and $⨍ u = 0$. If $\lambda = 0$, then, by the closing lemma, $\nabla u = 0$. Since Ω is connected, $u = ⨍ u = 0$. This is a contradiction.

Assume now that Ω is convex and that $u \in C^\infty(\Omega) \bigcap W^{1,p}(\Omega)$. Hölder's inequality implies that

$$\int_\Omega \left|u(y) - ⨍ u\right|^p dy \le \int_\Omega dy \left[\int_\Omega \frac{|u(x) - u(y)|}{m(\Omega)} dx\right]^p$$
$$\le \frac{1}{m(\Omega)} \int_\Omega dy \int_\Omega \left|u(x) - u(y)\right|^p dx.$$

It follows from the fundamental theorem of calculus and Hölder's inequality that

$$\begin{aligned}\int_\Omega dy \int_\Omega \left|u(x) - u(y)\right|^p dx &\le d^p \int_\Omega dy \int_\Omega dx \left[\int_0^1 \left|\nabla u((1-t)x + ty)\right| dt\right]^p \\ &\le d^p \int_\Omega dy \int_\Omega dx \int_0^1 \left|\nabla u((1-t)x + ty)\right|^p dt \\ &= 2d^p \int_\Omega dy \int_\Omega dx \int_0^{1/2} \left|\nabla u((1-t)x + ty)\right|^p dt \\ &= 2d^p \int_\Omega dy \int_0^{1/2} dt \int_\Omega \left|\nabla u((1-t)x + ty)\right|^p dx \\ &\le 2^N d^p \int_\Omega dy \int_\Omega \left|\nabla u(z)\right|^p dz.\end{aligned}$$

We obtain that

$$\int_\Omega \left|u(y) - ⨍ u\right|^p dy \le 2^N d^p \int_\Omega \left|\nabla u(y)\right|^p dy.$$

We conclude by density, using Corollary 6.1.18. □

Theorem 6.4.10 (Hardy's Inequality) *Let $1 < p < N$. Then for every $u \in W^{1,p}(\mathbb{R}^N)$, $u/|x| \in L^p(\mathbb{R}^N)$ and*

$$||u/|x|||_{L^p(\mathbb{R}^N)} \le \frac{p}{N-p} ||\nabla u||_{L^p(\mathbb{R}^N)}.$$

Proof Let $u \in \mathcal{D}(\mathbb{R}^N)$ and $v \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N)$. We infer from Lemma 6.1.1 that

$$\int_{\mathbb{R}^N} |u|^p \text{div}\, v\, dx = -p \int_{\mathbb{R}^N} |u|^{p-2} u \nabla u \cdot v\, dx.$$

Approximating $v(x) = x/|x|^p$ by $v_\varepsilon(x) = x/(|x|^2 + \varepsilon)^{p/2}$, we obtain

$$(N-p)\int_{\mathbb{R}^N}|u|^p/|x|^p dx = -p\int_{\mathbb{R}^N}|u|^{p-2}u\nabla u\cdot x/|x|^p dx.$$

Hölder's inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p/|x|^p dx &\leq \frac{p}{N-p}\left(\int_{\mathbb{R}^N}|u|^{(p-1)p'}/|x|^p dx\right)^{1/p'}\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{1/p}\\ &= \frac{p}{N-p}\left(\int_{\mathbb{R}^N}|u|^p/|x|^p dx\right)^{1-1/p}\left(\int_{\mathbb{R}^N}|\nabla u|^p dx\right)^{1/p}.\end{aligned}$$

We have thus proved Hardy's inequality in $\mathcal{D}(\mathbb{R}^N)$. Let $u \in W^{1,p}(\mathbb{R}^N)$. Theorem 6.1.10 ensures the existence of a sequence $(u_n) \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$. We can assume that $u_n \to u$ almost everywhere on $\mathbb{R}^N$. We conclude using Fatou's lemma that

$$||u/|x|||_p \leq \varliminf_{n\to\infty}||u_n/|x|||_p \leq \frac{p}{N-p}\lim_{n\to\infty}||\nabla u_n||_p = \frac{p}{N-p}||\nabla u||_p. \quad \square$$

Fractional Sobolev spaces are interpolation spaces between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Definition 6.4.11 Let $1 \leq p < \infty$, $0 < s < 1$, and $u \in L^p(\Omega)$. We define

$$|u|_{W^{s,p}(\Omega)} = |u|_{s,p} = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p} \leq +\infty.$$

On the fractional Sobolev space

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : |u|_{W^{s,p}(\Omega)} < +\infty\},$$

we define the norm

$$||u||_{W^{s,p}(\Omega)} = ||u||_{s,p} = ||u||_{L^p(\Omega)} + |u|_{W^{s,p}(\Omega)}.$$

We give, without proof, the characterization of traces due to Gagliardo [26].

Theorem 6.4.12 *Let* $1 < p < \infty$.

(a) For every $u \in W^{1,p}(\mathbb{R}^N)$, $\gamma_0 u \in W^{1-1/p,p}(\mathbb{R}^{N-1})$.
(b) The mapping $\gamma_0 : W^{1,p}(\mathbb{R}^N) \to W^{1-1/p,p}(\mathbb{R}^{N-1})$ *is continuous and surjective.*
(c) The mapping $\gamma_0 : W^{1,1}(\mathbb{R}^N) \to L^1(\mathbb{R}^{N-1})$ *is continuous and surjective.*

6.5 Comments

The main references on Sobolev spaces are the books:

– R. Adams and J. Fournier, *Sobolev spaces* [1]
– H. Brezis, *Analyse fonctionnelle, théorie et applications* [8]
– V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations* [51]

Our proof of the *trace inequality* follows closely:

– A.C. Ponce, *Elliptic PDEs, measures, and capacities*, European Mathematical Society, 2016

The theory of partial differential equations was at the origin of Sobolev spaces. We recommend [9] on the history of partial differential equations and [55] on the prehistory of Sobolev spaces.

Because of Poincaré's inequalities, for every smooth, bounded open connected set Ω, we have that

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), \int_\Omega u^2 dx = 1\right\} > 0,$$

$$\mu_2(\Omega) = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H^1(\Omega), \int_\Omega u^2 dx = 1, \int_\Omega u dx = 0\right\} > 0.$$

Hence the first eigenvalue $\lambda_1(\Omega)$ of Dirichlet's problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the second eigenvalue $\mu_2(\Omega)$ of the Neumann problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ n \cdot \nabla u = 0 & \text{on } \partial\Omega, \end{cases}$$

are strictly positive. Let us denote by B an open ball such that $m(B) = m(\Omega)$. Then

$$\begin{aligned} &\lambda_1(B) \leq \lambda_1(\Omega) && \text{(Faber–Krahn inequality)}, \\ &\mu_2(\Omega) \leq \mu_2(B) && \text{(Weinberger, 1956)}. \end{aligned}$$

Moreover, if Ω is convex with diameter d, then

$$\pi^2/d^2 \leq \mu_2(\Omega) \quad \text{(Payne–Weinberger, 1960)}.$$

We prove in Theorem 6.4.9 the weaker estimate

$$1/(2^N d^2) \le \mu_2(\Omega).$$

There exists a bounded, connected open set $\Omega \subset \mathbb{R}^2$ such that $\mu_2(\Omega) = 0$. Consider on two sides of a square Q, two infinite sequences of small squares connected to Q by very thin pipes.

6.6 Exercises for Chap. 6

1. Let $\Omega = B(0,1) \subset \mathbb{R}^N$. Then for $\lambda \neq 0$,

$$(\lambda - 1)p + N > 0 \iff |x|^\lambda \in W^{1,p}(\Omega),$$

$$\lambda p + N < 0 \iff |x|^\lambda \in W^{1,p}(\mathbb{R}^N \setminus \overline{\Omega}),$$

$$p < N \iff \frac{x}{|x|} \in W^{1,p}(\Omega; \mathbb{R}^N).$$

2. Let $1 < p < \infty$ and $u \in L^p(\Omega)$. The following properties are equivalent:
   (a) $u \in W^{1,p}(\Omega)$;
   (b) $\sup\left\{\int_\Omega u \operatorname{div} v \, dx : v \in \mathcal{D}(\Omega, \mathbb{R}^N), ||v||_{L^{p'}(\Omega)} = 1\right\} < \infty$;
   (c) there exists $c > 0$ such that for every $\omega \subset\subset \Omega$ and for every $y \in \mathbb{R}^N$ such that $|y| < d(\omega, \partial\Omega)$,

$$||\tau_y u - u||_{L^p(\omega)} \le c|y|.$$

3. Let $1 \le p < N$ and let Ω be an open subset of $\mathbb{R}^N$. Define

$$S(\Omega) = \inf_{\substack{u \in \mathcal{D}(\Omega) \\ ||u||_{L^{p^*}(\Omega)} = 1}} ||\nabla u||_{L^p(\Omega)}.$$

   Then $S(\Omega) = S(\mathbb{R}^N)$.
4. Let $1 \le p < N$. Then

$$\frac{1}{2^N} S(\mathbb{R}^N) = \inf\left\{||\nabla u||_{L^p(\mathbb{R}^N_+)}/||u||_{L^{p^*}(\mathbb{R}^N_+)} : u \in H^1(\mathbb{R}^N_+) \setminus \{0\}\right\}.$$

5. Poincaré–Sobolev inequality.
   (a) Let $1 < p < N$, and let Ω be an open bounded connected subset of $\mathbb{R}^N$ of class C^1. Then there exists $c > 0$ such that for every $u \in W^{1,p}(\Omega)$,

$$\left\| u - \fint u \right\|_{L^{p^*}(\Omega)} \leq c||\nabla u||_{L^p(\Omega)},$$

where $\fint u = \dfrac{1}{m(\Omega)} \int_\Omega u\,dx$. *Hint*: Apply Theorem 6.4.4 to $u - \fint u$.

(b) Let $A = \{u = 0\}$ and assume that $m(A) > 0$. Then

$$\|u\|_{L^{p^*}(\Omega)} \leq c\left(1 + \left[\frac{m(\Omega)}{m(A)}\right]^{1/p^*}\right)\|\nabla u\|_{L^p(\Omega)}.$$

Hint:

$$\left|\fint u\right| m(A)^{1/p^*} \leq \|u - \fint u\|_{L^{p^*}(\Omega)}.$$

6. Nash's inequality. Let $N \geq 3$. Then for every $u \in \mathcal{D}(\mathbb{R}^N)$,

$$||u||_2^{2+4/N} \leq c||u||_1^{4/N}||\nabla u||_2^2.$$

Hint: Use the interpolation inequality.

7. Let $1 \leq p < N$ and $q = p(N-1)/(N-p)$. Then for every $u \in \mathcal{D}(\overline{\mathbb{R}^N_+})$,

$$\int_{\mathbb{R}^{N-1}} \left|u(x', 0)\right|^q dx' \leq q||u||^{q-1}_{L^{p^*}(\mathbb{R}^N_+)}||\partial_N u||_{L^p(\mathbb{R}^N_+)}.$$

8. Verify that Hardy's inequality is optimal using the family

$$\begin{aligned} u_\varepsilon(x) &= 1, & |x| &\leq 1, \\ &= |x|^{\frac{p-N}{p}-\varepsilon}, & |x| &> 1. \end{aligned}$$

9. Let $1 \leq p < N$. Then $\mathcal{D}(\mathbb{R}^N \setminus \{0\})$ is dense in $W^{1,p}(\mathbb{R}^N)$.
10. Let $2 \leq N < p < \infty$. Then for every $u \in W_0^{1,p}(\mathbb{R}^N \setminus \{0\})$, $u/|x| \in L^p(\mathbb{R}^N)$ and

$$||u/|x|||_{L^p(\mathbb{R}^N)} \leq \frac{p}{p-N}||\nabla u||_{L^p(\mathbb{R}^N)}.$$

11. Let $1 \leq p < \infty$. Verify that the embedding $W^{1,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is not compact. Let $1 \leq p < N$. Verify that the embedding $W_0^{1,p}(B(0,1)) \subset L^{p^*}(B(0,1))$ is not compact.
12. Let us denote by $\mathcal{D}_r(\mathbb{R}^N)$ the space of radial functions in $\mathcal{D}(\mathbb{R}^N)$. Let $N \geq 2$ and $1 \leq p < \infty$. Then there exists $c(N, p) > 0$ such that for every $u \in \mathcal{D}_r(\mathbb{R}^N)$,

$$\big|u(x)\big| \le c(N,p)||u||_p^{1/p'}||\nabla u||_p^{1/p}|x|^{(1-N)/p}.$$

Let $1 \le p < N$. Then there exists $d(N,p) > 0$ such that for every $u \in \mathcal{D}_r(\mathbb{R}^N)$,

$$\big|u(x)\big| \le d(N,p)||\nabla u||_p|x|^{(p-N)/p}.$$

Hint: Let us write $u(x) = u(r)$, $r = |x|$, so that

$$r^{N-1}\big|u(r)\big|^p \le p\int_r^\infty \big|u(s)\big|^{p-1}\big|\frac{du}{dr}(s)\big|s^{N-1}ds,$$

$$\big|u(r)\big| \le \int_r^\infty \Big|\frac{du}{dr}(s)\Big|ds.$$

13. Let us denote by $W_r^{1,p}(\mathbb{R}^N)$ the space of radial functions in $W^{1,p}(\mathbb{R}^N)$. Verify that the space $\mathcal{D}_r(\mathbb{R}^N)$ is dense in $W_r^{1,p}(\mathbb{R}^N)$.
14. Let $1 \le p < N$ and $p < q < p^*$. Verify that the embedding $W_r^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ is compact. Verify also that the embedding $W_r^{1,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is not compact.
15. Let $1 \le p < \infty$ and let Ω be an open subset of $\mathbb{R}^N$. Prove that the map

$$W^{1,p}(\Omega) \to W^{1,p}(\Omega) : u \mapsto u^+$$

is continuous. *Hint*: $\nabla u^+ = H(u)\nabla u$, where

$$\begin{aligned} H(t) &= 1, \quad t > 0, \\ &= 0, \quad t \le 0. \end{aligned}$$

16. Sobolev implies Poincaré. Let Ω be an open subset of $\mathbb{R}^N$ $(N \ge 2)$ such that $m(\Omega) < +\infty$, and let $1 \le p < +\infty$. Then there exists $c = c(p,N)$ such that, for every $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_p \le c\, m(\Omega)^{1/N}\|\nabla u\|_p.$$

Hint. (a) If $1 \le p < N$, then

$$\|u\|_p \le m(\Omega)^{1/N}\|u\|_{p^*} \le c\, m(\Omega)^{1/N}\|\nabla u\|_p.$$

(b) If $p \ge N$, then

$$\|u\|_p = \|u\|_{q^*} \le c\|\nabla u\|_q \le c\, m(\Omega)^{1/N}\|\nabla u\|_p.$$

17. Let Ω be an open bounded convex subset of $\mathbb{R}^N, N \geq 2$, and $u \in C^1(\Omega) \bigcap W^{1,1}(\Omega)$. Then, for every $x \in \Omega$,

$$\left| u(x) - \fint u \right| \leq \frac{1}{N} \frac{d^N}{m(\Omega)} \int_\Omega \frac{|\nabla u(y)|}{|y-x|^{N-1}} dy,$$

where $\fint u = \dfrac{1}{m(\Omega)} \displaystyle\int_\Omega u(x)dx$ and $d = \sup_{x,y \in \Omega} |y-x|$.
Hint. Define

$$\begin{aligned} v(y) &= |\nabla u(y)| \quad &, y \in \Omega, \\ &= 0 &, y \in \mathbb{R}^N \backslash \Omega. \end{aligned}$$

(a) $u(x) - u(y) = \displaystyle\int_0^{|y-x|} \nabla u(x + r\sigma) \cdot \sigma dr, \ \sigma = \frac{y-x}{|y-x|}$.
(b)

$$\begin{aligned} m(\Omega)\left|u(x) - \fint u\right| &\leq \int_\Omega dy \int_0^{|y-x|} v(x+r\sigma)dr \\ &= \int_{\omega - x} dz \int_0^{|z|} v\left(x + r\frac{z}{|z|}\right) dr \\ &\leq \int_{\mathbb{S}^{N-1}} d\sigma \int_0^d \rho^{N-1} d\rho \int_0^\infty v(x+r\sigma)dr \\ &= \frac{d^N}{N} \int_{\mathbb{R}^N} \frac{v(x+z)}{|z|^{N-1}} dz. \end{aligned}$$

18. Let us define, for every bounded connected open subset Ω of $\mathbb{R}^N$, and for $1 \leq p < \infty$,

$$\lambda(p, \Omega) = \inf \left\{ \|\nabla u\|_p : u \in W^{1,p}(\Omega), \fint u = 0, \|u\|_p = 1 \right\}.$$

For every $1 \leq p < \infty$, there exists a bounded connected open subset Ω of $\mathbb{R}^2$ such that $\lambda(p, \Omega) = 0$.
Hint. Consider on two sides of a square Q two infinite sequences of small squares connected to Q by very thin pipes.

19. Prove that, for every $1 \leq p < \infty$,

$$\inf \left\{ \lambda(p, \Omega) : \Omega \text{ is a smooth bounded connected open subset of } \mathbb{R}^2, m(\Omega) = 1 \right\} = 0.$$

Hint. Consider a sequence of pairs of disks smoothly connected by very thin pipes.

20. Generalized Poincaré's inequality. Let $1 \leq p < \infty$, let Ω be a smooth bounded connected open subset of $\mathbb{R}^N$, and let $f \in [W^{1,p}(\Omega)]^*$ be such that

$$< f, 1 >= 1.$$

Then there exists $c > 0$ such that, for every $u \in W^{1,p}(\Omega)$,

$$\|u- < f, u > \|_p \leq c\|\nabla u\|_p.$$