Chapter 4
Lebesgue Spaces

4.1 Convexity

The notion of convexity plays a basic role in functional analysis and in the theory of inequalities.

Definition 4.1.1 A subset C of a vector space X is convex if for every $u, v \in C$ and every $0 < \lambda < 1$, we have $(1-\lambda)x + \lambda y \in C$.

A point x of the convex set C is internal if for every $y \in X$, there exists $\varepsilon > 0$ such that $x + \varepsilon y \in C$. The set of internal points of C is denoted by int C.

A subset C of X is a cone if for every $x \in C$ and every $\lambda > 0$, we have $\lambda x \in C$.

Let C be a convex set. A function $F : C \to]-\infty, +\infty]$ is convex if for every $x, y \in C$ and every $0 < \lambda < 1$, we have $F((1-\lambda)x + \lambda y) \leq (1-\lambda)F(x) + \lambda F(y)$.

A function $F : C \to [-\infty, +\infty[$ is concave if $-F$ is convex.

Let C be a cone. A function $F : C \to]-\infty, +\infty]$ is positively homogeneous if for every $x \in C$ and every $\lambda > 0$, we have $F(\lambda x) = \lambda F(x)$.

Examples Every linear function is convex, concave, and positively homogeneous. Every norm is convex and positively homogeneous. Open balls and closed balls in a normed space are convex.

Proposition 4.1.2 *The upper envelope of a family of convex (respectively positively homogeneous) functions is convex (respectively positively homogeneous).*

Lemma 4.1.3 *Let Y be a hyperplane of a real vector space X, $f : Y \to \mathbb{R}$ linear and $F : X \to]-\infty, +\infty]$ convex and positively homogeneous such that $f \leq F$ on Y and*

$$Y \cap \operatorname{int}\{x \in X : F(x) < \infty\} \neq \phi.$$

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Then there exists $g : X \to \mathbb{R}$ *linear such that* $g \le F$ *on* X *and* $g\big|_Y = f$.

Proof There exists $z \in X$ such that $X = Y \oplus \mathbb{R}z$. We must prove the existence of $c \in \mathbb{R}$ such that for every $y \in Y$ and every $t \in \mathbb{R}$,

$$\langle f, y\rangle + ct \le F(y + tz).$$

Since F is positively homogeneous, it suffices to verify that for every $u, v \in Y$,

$$\langle f, u\rangle - F(u - z) \le c \le F(v + z) - \langle f, v\rangle.$$

For every $u, v \in Y$, we have by assumption that

$$\langle f, u\rangle + \langle f, v\rangle \le F(u + v) \le F(u - z) + F(v + z).$$

We define

$$a = \sup_{u\in Y}\langle f, u\rangle - F(u - z) \le b = \inf_{v\in Y} F(v + z) - \langle f, v\rangle.$$

Let $u \in Y \cap \operatorname{int}\{x \in X : F(x) < \infty\}$. For t large enough, $F(tu-z) = tF(u-z/t) < +\infty$. Hence $-\infty < a$. Similarly, $b < +\infty$. We can choose any $c \in [a, b]$. □

Let us state a cornerstone of functional analysis, the *Hahn–Banach theorem.*

Theorem 4.1.4 *Let* Y *be a subspace of a separable normed space* X*, and let* $f \in \mathcal{L}(Y, \mathbb{R})$*. Then there exists* $g \in \mathcal{L}(X, \mathbb{R})$ *such that* $||g|| = ||f||$ *and* $g\big|_Y = f$.

Proof Let (z_n) be a sequence dense in X. We define $f_0 = f$, $Y_0 = Y$, and $Y_n = Y_{n-1}+\mathbb{R}z_n$, $n \ge 1$. Let there be $f_n \in \mathcal{L}(Y_n, \mathbb{R})$ such that $||f_n|| = ||f||$ and $f_n\big|_{Y_{n-1}} = f_{n-1}$. If $Y_{n+1} = Y_n$, we define $f_{n+1} = f_n$. If this is not the case, the preceding lemma implies the existence of $f_{n+1} : Y_{n+1} \to \mathbb{R}$ linear such that $f_{n+1}\big|_{Y_n} = f_n$ and for every $x \in Y_{n+1}$,

$$\langle f_{n+1}, x\rangle \le ||f||\ ||x||.$$

On $Z = \bigcup_{n=0}^{\infty} Y_n$ we define h by $h\big|_{Y_n} = f_n$, $n \ge 0$. The space Z is dense in X, $h \in \mathcal{L}(Z, \mathbb{R})$, $||h|| = ||f||$, and $h\big|_Y = f$. Finally, by Proposition 3.2.3, there exists $g \in \mathcal{L}(X, \mathbb{R})$ such that $||g|| = ||h||$ and $g\big|_Z = h$. □

Notation The *dual* of a normed space X is defined by $X^* = \mathcal{L}(X, \mathbb{R})$. Let us recall that the norm on X^* is defined by

$$||g|| = \sup_{\substack{u \in X \\ \|u\| \le 1}} |\langle g, u \rangle| = \sup_{\substack{u \in X \\ \|u\| \le 1}} \langle g, u \rangle.$$

Theorem 4.1.5 *Let Z be a subspace of a separable normed space X, and let $u \in X \backslash \overline{Z}$. Then*

$$0 < d(u, Z) = \max\{\langle g, u \rangle : g \in X^*, ||g|| \le 1, g\Big|_Z = 0\}.$$

In particular if $u \in X \backslash \{0\}$, then

$$||u|| = \max_{\substack{g \in X^* \\ \|g\| \le 1}} \langle g, u \rangle = \max_{\substack{g \in X^* \\ \|g\| \le 1}} |\langle g, u \rangle|.$$

Proof Let us first prove that

$$c = \sup \left\{ \langle g, u \rangle : g \in X^* : ||g|| \le 1, g\Big|_Z = 0 \right\} \le \delta = d(u, Z).$$

Assume that $||g|| \le 1$ and $g\Big|_Z = 0$. Then, for every $z \in Z$,

$$\langle g, u \rangle = \langle g, u - z \rangle \le ||g|| \, ||u - z|| \le ||u - z||,$$

so that $\langle g, u \rangle \le \delta$ and $c \le \delta$.

It suffices then to prove the existence of $g \in X^*$ such that $||g|| \le 1$, $g\Big|_Z = 0$ and $\langle g, u \rangle = \delta$. Let us define the functional f on $Y = \mathbb{R}u \oplus Z$ by

$$\langle f, tu + z \rangle = t\delta.$$

Since, for $t \ne 0$,

$$\langle f, tu + z \rangle \le |t|\delta \le |t| \, ||u + z/t|| = ||tu + z||,$$

the functional f is such that $||f|| \le 1$. The preceding theorem implies the existence of $g \in X^*$ such that $||g|| = ||f|| \le 1$ and $g\Big|_Y = f$. In particular $\langle g, u \rangle = \delta$ and $g\Big|_Z = 0$. □

The next theorem is due to P. Roselli and the author. Let us define

$$C_+ = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \ge 0, x_2 \ge 0 \right\}.$$

Theorem 4.1.6 (Convexity Inequality) *Let $F : C_+ \to \mathbb{R}$ be a positively homogeneous function, and let $u_j \in L^1(\Omega, \mu)$ be such that $u_j \geq 0$, $\int_\Omega u_j d\mu > 0$, $j = 1, 2$. If F is convex, then*

$$F\left(\int_\Omega u_1 d\mu, \int_\Omega u_2 d\mu\right) \leq \int_\Omega F(u_1, u_2) d\mu.$$

If F is concave, the reverse inequality holds.

Proof We define $F(x) = +\infty$, $x \in \mathbb{R}^2 \setminus C_+$, and $y_j = \int_\Omega u_j d\mu$, $j = 1, 2$. Lemma 4.1.3 implies the existence of $\alpha, \beta \in \mathbb{R}$ such that

$$F(y_1, y_2) = \alpha y_1 + \beta y_2 \text{ and, for all } x_1, x_2 \in \mathbb{R}, \alpha x_1 + \beta x_2 \leq F(x_1, x_2). \qquad (*)$$

For every $0 \leq \lambda \leq 1$, we have

$$\alpha(1-\lambda) + \beta\lambda \leq F(1-\lambda, \lambda) \leq (1-\lambda)F(1, 0) + \lambda F(0, 1),$$

so that $c = \sup_{0 \leq \lambda \leq 1} |F(1-\lambda, \lambda)| < \infty$. Since

$$\left|F(u_1, u_2)\right| \leq c(u_1 + u_2),$$

the comparison theorem implies that $F(u_1, u_2) \in L^1(\Omega, \mu)$. We conclude from $(*)$ that

$$\begin{aligned} F\left(\int_\Omega u_1 d\mu, \int_\Omega u_2 d\mu\right) &= \alpha \int_\Omega u_1 d\mu + \beta \int_\Omega u_2 d\mu \\ &= \int_\Omega \alpha u_1 + \beta u_2 d\mu \\ &\leq \int_\Omega F(u_1, u_2) d\mu. \end{aligned}$$ □

Lemma 4.1.7 *Let $F : C_+ \to \mathbb{R}$ be a continuous and positively homogeneous function. If $F(., 1)$ is convex (respectively concave), then F is convex (respectively concave).*

Proof Assume that $F(., 1)$ is convex. It suffices to prove that for every $x, y \in \overset{\circ}{C}_+$, $F(x + y) \leq F(x) + F(y)$. The preceding inequality is equivalent to

$$F\left(\frac{x_1 + y_1}{x_2 + y_2}, 1\right) \leq \frac{x_2}{x_2 + y_2} F\left(\frac{x_1}{x_2}, 1\right) + \frac{y_2}{x_2 + y_2} F\left(\frac{y_1}{y_2}, 1\right).$$ □

Remark Define F on $\mathbb{R}^2$ by

$$F(y,z) = -\sqrt{yz}, \qquad (y,z) \in C_+,$$
$$\qquad = +\infty, \qquad (y,z) \in \mathbb{R}^2 \setminus C_+.$$

The function F is positively homogeneous and, by the preceding lemma, is convex on C_+, hence on $\mathbb{R}^2$. It is clear that $0 = F$ on $Y = \mathbb{R} \times \{0\}$. There is no linear function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g \leq F$ on $\mathbb{R}^2$ and $g = 0$ on Y.

The convexity inequality implies a version of the Cauchy–Schwarz inequality: if $v, w \in L^1(\Omega, \mu)$, then

$$\int_\Omega |vw|^{1/2} d\mu \leq \left(\int_\Omega |v| d\mu\right)^{1/2} \left(\int_\Omega |w| d\mu\right)^{1/2}.$$

Definition 4.1.8 Let $1 < p < \infty$. The exponent p' conjugate to p is defined by $1/p + 1/p' = 1$. On the Lebesgue space

$$\mathcal{L}^p(\Omega, \mu) = \left\{u \in \mathcal{M}(\Omega, \mu) : \int_\Omega |u|^p d\mu < \infty\right\},$$

we define the functional $||u||_p = \left(\int_\Omega |u|^p d\mu\right)^{1/p}$.

Theorem 4.1.9 *Let* $1 < p < \infty$.

(a) (Hölder's inequality.) Let $v \in \mathcal{L}^p(\Omega, \mu)$ *and* $w \in \mathcal{L}^{p'}(\Omega, \mu)$. *Then*

$$\int_\Omega |vw| d\mu \leq ||v||_p ||w||_{p'}.$$

(b) (Minkowski's inequality.) Let $v, w \in \mathcal{L}^p(\Omega, \mu)$. *Then*

$$||v + w||_p \leq ||v||_p + ||w||_p.$$

(c) (Hanner's inequalities.) Let $v, w \in \mathcal{L}^p(\Omega, \mu)$. *If* $2 \leq p < \infty$, *then*

$$||v + w||_p^p + ||v - w||_p^p \leq (||v||_p + ||w||_p)^p + \big|||v||_p - ||w||_p\big|^p.$$

If $1 < p \leq 2$, *the reverse inequality holds.*

Proof On C_+, we define the continuous positively homogeneous functions

$$F(x_1, x_2) = x_1^{1/p} x_2^{1/p'},$$

$$G(x_1, x_2) = (x_1^{1/p} + x_2^{1/p})^p,$$
$$H(x_1, x_2) = (x_1^{1/p} + x_2^{1/p})^p + |x_1^{1/p} - x_2^{1/p}|^p.$$

Inequality (a) follows from the convexity inequality applied to F and $u = (|v|^p, |w|^{p'})$. Inequality (b) follows from the convexity inequality applied to G and $u = (|v|^p, |w|^p)$. Finally, inequalities (c) follow from the convexity inequality applied to H and $u = (|v|^p, |w|^p)$. When $v = 0$ or $w = 0$, the inequalities are obvious.

On $[0, +\infty[$, we define $f = F(., 1)$, $g = G(., 1)$, $h = H(., 1)$. It is easy to verify that

$$f''(x) = \frac{1-p}{p^2} x^{\frac{1}{p}-2},$$
$$g''(x) = \frac{1-p}{p} x^{-\frac{1}{p}-1}(x^{-\frac{1}{p}} + 1)^{p-2},$$
$$h''(x) = \frac{1-p}{p} x^{-\frac{1}{p}-1}\left[(x^{-\frac{1}{p}} + 1)^{p-2} - |x^{-\frac{1}{p}} - 1|^{p-2}\right].$$

Hence f and g are concave. If $2 \leq p < \infty$, then h is concave, and if $1 < p \leq 2$, then h is convex. It suffices then to use the preceding lemma. □

4.2 Lebesgue Spaces

Let $\mu : \mathcal{L} \to \mathbb{R}$ be a positive measure on the set Ω.

Definition 4.2.1 Let $1 \leq p < \infty$. The space $L^p(\Omega, \mu)$ is the quotient of $\mathcal{L}^p(\Omega, \mu)$ by the equivalence relation "equality almost everywhere." By definition,

$$||u||_{L^p(\Omega,\mu)} = ||u||_p = \left(\int_\Omega |u|^p d\mu\right)^{1/p}.$$

When Λ_N is the Lebesgue measure on the open subset Ω of $\mathbb{R}^N$, the space $L^p(\Omega, \Lambda_N)$ is denoted by $L^p(\Omega)$.

In practice, we identify the elements of $L^p(\Omega, \mu)$ and the functions of $\mathcal{L}^p(\Omega, \mu)$.

Proposition 4.2.2 *Let $1 \leq p < \infty$. Then the space $L^p(\Omega, \mu)$ with the norm $||.||_p$ is a normed space.*

Proof Minkowski's inequality implies that if $u, v \in L^p(\Omega, \mu)$, then $u + v \in L^p(\Omega, \mu)$ and

$$||u + v||_p \leq ||u||_p + ||v||_p.$$

It is clear that if $u \in L^p(\Omega, \mu)$ and $\lambda \in \mathbb{R}$, then $\lambda u \in L^p(\Omega, \mu)$ and $||\lambda u||_p = |\lambda| \, ||u||_p$. Finally, if $||u||_p = 0$, then $u = 0$ almost everywhere and $u = 0$ in $L^p(\Omega, \mu)$. □

The next inequalities follow from Hölder's inequality.

Proposition 4.2.3 (Generalized Hölder's Inequality) *Let* $1 < p_j < \infty$, $u_j \in L^{p_j}(\Omega, \mu)$, $1 \leq j \leq k$, *and* $1/p_1 + \ldots + 1/p_k = 1$. *Then* $\prod_{j=1}^{k} u_j \in L^1(\Omega, \mu)$ *and*

$$\int_\Omega \prod_{j=1}^{k} |u_j| d\mu \leq \prod_{j=1}^{k} ||u_j||_{p_j}.$$

Proposition 4.2.4 (Interpolation Inequality) *Let* $1 \leq p < q < r < \infty$,

$$\frac{1}{q} = \frac{1-\lambda}{p} + \frac{\lambda}{r},$$

and $u \in L^p(\Omega, \mu) \cap L^r(\Omega, \mu)$. *Then* $u \in L^q(\Omega, \mu)$ *and*

$$||u||_q \leq ||u||_p^{1-\lambda} ||u||_r^{\lambda}.$$

Proposition 4.2.5 *Let* $1 \leq p < q < \infty$, $\mu(\Omega) < \infty$, *and* $u \in L^q(\Omega, \mu)$. *Then* $u \in L^p(\Omega, \mu)$ *and*

$$||u||_p \leq \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} ||u||_q.$$

Proposition 4.2.6 *Let* $1 \leq p < \infty$ *and* $(u_n) \subset L^p(\Omega, \mu)$ *be such that*

(a) $||u_n||_p \to ||u||_p$, $n \to \infty$;
(b) u_n *converges to* u *almost everywhere.*

Then $||u_n - u||_p \to 0$, $n \to \infty$.

Proof Since almost everywhere

$$0 \leq 2^p(|u_n|^p + |u|^p) - |u_n - u|^p,$$

Fatou's lemma ensures that

$$2^{p+1} \int_{\Omega} |u|^p d\mu \leq \underline{\lim} \int_{\Omega} \left[2^p (|u_n|^p + |u|^p) - |u_n - u|^p \right] d\mu$$
$$= 2^{p+1} \int |u|^p d\mu - \overline{\lim} \int_{\Omega} |u_n - u|^p d\mu.$$

Hence $\overline{\lim} \, ||u_n - u||_p^p \leq 0$. □

The next result is more precise.

Theorem 4.2.7 (Brezis–Lieb Lemma) *Let $1 \leq p < \infty$ and let $(u_n) \subset L^p(\Omega, \mu)$ be such that*

(a) $c = \sup_n ||u_n||_p < \infty$*;*
(b) u_n *converges to* u *almost everywhere.*

Then $u \in L^p(\Omega, \mu)$ *and*

$$\lim_{n \to \infty} \left(||u_n||_p^p - ||u_n - u||_p^p \right) = ||u||_p^p.$$

Proof By Fatou's lemma, $||u||_p \leq c$. Let $\varepsilon > 0$. There exists, by homogeneity, $c(\varepsilon) > 0$ such that for every $a, b \in \mathbb{R}$,

$$\left| |a + b|^p - |a|^p - |b|^p \right| \leq \varepsilon |a|^p + c(\varepsilon) |b|^p.$$

We deduce from Fatou's lemma that

$$\int_{\Omega} c(\varepsilon) |u|^p d\mu \leq \underline{\lim}_{n \to \infty} \int_{\Omega} \varepsilon |u_n - u|^p + c(\varepsilon) |u|^p - \left| |u_n|^p - |u_n - u|^p - |u|^p \right| d\mu$$
$$\leq (2c)^p \varepsilon + \int_{\Omega} c(\varepsilon) |u|^p d\mu - \overline{\lim}_{n \to \infty} \int_{\Omega} \left| |u_n|^p - |u_n - u|^p - |u|^p \right| d\mu,$$

or

$$\overline{\lim}_{n \to \infty} \int_{\Omega} \left| |u_n|^p - |u_n - u|^p - |u|^p \right| d\mu \leq (2c)^p \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

We define

$$\begin{aligned} R_h(s) &= s + h, & s &\leq -h, \\ &= 0, & |s| &< h, \\ &= s - h, & s &\geq h. \end{aligned}$$

Theorem 4.2.8 (Degiovanni–Magrone) *Let $\mu(\Omega) < \infty$, $1 \leq p < \infty$, and $(u_n) \subset L^p(\Omega, \mu)$ be such that* *(a)* $c = \sup_n ||u_n||_p < \infty$;

(b) u_n *converges to* u *almost everywhere.*

Then

$$\lim_{n\to\infty}\Big(||u_n||_p^p - ||R_h u_n||_p^p\Big) = ||u||_p^p - ||R_h u||_p^p.$$

Proof Let us define

$$f(s) = |s|^p - |R_h(s)|^p.$$

For every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that

$$|f(s) - f(t)| \le \varepsilon\Big||s|^p + |t|^p\Big| + c(\varepsilon).$$

It follows from Fatou's lemma that

$$\begin{aligned}2\varepsilon\int_\Omega |u|^p d\mu + c(\varepsilon)m(\Omega) &\le \underline{\lim}_{n\to\infty}\int_\Omega \varepsilon\,\big(|u_n|^p+|u|^p\big)+c(\varepsilon)-\big|f(u_n)-f(u)\big|d\mu\\ &\le \varepsilon\, c^p + \varepsilon\int_\Omega |u|^p d\mu + c(\varepsilon)\mu(\Omega) - \overline{\lim}_{n\to\infty}\int_\Omega \big|f(u_n) - f(u)\big|d\mu.\end{aligned}$$

Hence

$$\overline{\lim}_{n\to\infty}\int_\Omega \big|f(u_n) - f(u)\big|d\mu \le \varepsilon\, c^p.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Theorem 4.2.9 (F. Riesz, 1910) *Let* $1 \le p < \infty$. *Then the space* $L^p(\Omega, \mu)$ *is complete.*

Proof Let (u_n) be a Cauchy sequence in $L^p(\Omega, \mu)$. There exists a subsequence $v_j = u_{n_j}$ such that for every j,

$$||v_{j+1} - v_j||_p \le 1/2^j.$$

We define the sequence

$$f_k = \sum_{j=1}^{k} |v_{j+1} - v_j|.$$

Minkowski's inequality ensures that

$$\int_\Omega f_k^p d\mu \leq \left(\sum_{j=1}^{k} 1/2^j\right)^p < 1.$$

Levi's theorem implies the almost everywhere convergence of f_k to $f \in L^p(\Omega, \mu)$. Hence v_k converges almost everywhere to a function u. For $m \geq k+1$, it follows from Minkowski's inequality that

$$\int_\Omega |v_m - v_k|^p d\mu \leq \left(\sum_{j=k}^{m-1} 1/2^j\right)^p \leq (2/2^k)^p.$$

By Fatou's lemma, we obtain

$$\int_\Omega |u - v_k|^p d\mu \leq (2/2^k)^p.$$

In particular, $u = u - v_1 + v_1 \in L^p(\Omega, \mu)$. We conclude by invoking the Cauchy condition:

$$\begin{aligned} ||u - u_k||_p \leq ||u - v_k||_p + ||v_k - u_k||_p &\leq 2/2^k \\ +||u_{n_k} - u_k||_p &\to 0, \quad k \to \infty. \end{aligned}$$ □

Proposition 4.2.10 *Let $1 \leq p < \infty$ and let $u_n \to u$ in $L^p(\Omega, \mu)$. Then there exist subsequences $v_j = u_{n_j}$ and $g \in L^p(\Omega, \mu)$ such that almost everywhere,*

$$|v_j| \leq g \text{ and } v_j \to u, \quad j \to \infty.$$

Proof If the sequence (u_n) converges in $L^p(\Omega, \mu)$, it satisfies the Cauchy condition by Proposition 1.2.3. The subsequence (v_j) in the proof of the preceding theorem converges almost everywhere to u, and for every j,

$$|v_j| \leq |v_1| + \sum_{j=1}^{\infty} |v_{j+1} - v_j| = |v_1| + f \in L^p(\Omega, \mu).$$ □

Theorem 4.2.11 (Density Theorem) *Let $1 \leq p < \infty$ and $\mathcal{L} \subset L^p(\Omega, \mu)$. Then $\mathcal{L}$ is dense in $L^p(\Omega, \mu)$.*

Proof Let $u \in L^p(\Omega, \mu)$. Since u is measurable with respect to μ on Ω, there exists a sequence $(u_n) \subset \mathcal{L}$ such that $u_n \to u$ almost everywhere. We define

$$v_n = \max(\min(|u_n|, u), -|u_n|).$$

By definition, $|v_n| \leq |u_n|$, and almost everywhere,

$$|v_n - u|^p \leq |u|^p \in L^1, |v_n - u|^p \to 0, \quad n \to \infty.$$

It follows from Lebesgue's dominated convergence theorem that $||v_n - u||_p \to 0$, $n \to \infty$. Hence

$$Y = \{u \in L^p(\Omega, \mu) : \text{there exists } f \in \mathcal{L} \text{ such that } |u| \leq \text{f almost everywhere}\}$$

is dense in $L^p(\Omega, \mu)$. It suffices to prove that $\mathcal{L}$ is dense in Y.

Let $u \in Y$, $f \in \mathcal{L}$ be such that $|u| \leq f$ almost everywhere and $(u_n) \subset \mathcal{L}$ such that $u_n \to u$ almost everywhere. We define

$$w_n = \max(\min(f, u_n), -f).$$

By definition, $w_n \in \mathcal{L}$ and, almost everywhere,

$$|w_n - u|^p \leq 2^p f^p \in L^1, |w_n - u|^p \to 0, \quad n \to \infty.$$

It follows from Lebesgue's dominated convergence theorem that $||w_n - u||_p \to 0$, $n \to \infty$. Hence $\mathcal{L}$ is dense in Y. □

Theorem 4.2.12 *Let Ω be open in $\mathbb{R}^N$ and $1 \leq p < \infty$. Then the space $L^p(\Omega)$ is separable.*

Proof By the preceding theorem, $\mathcal{K}(\Omega)$ is dense in $L^p(\Omega)$. Proposition 2.3.2 implies that for every $u \in \mathcal{K}(\Omega)$,

$$u_j = \sum_{k \in \mathbb{Z}^N} u(k/2^j) f_{j,k}$$

converges to u in $L^p(\Omega)$. We conclude the proof using Proposition 3.3.11. □

4.3 Regularization

La logique parfois engendre des monstres. Depuis un demi-siècle on a vu surgir une foule de fonctions bizarres qui semblent s'efforcer de ressembler aussi peu que possible aux honnêtes fonctions qui servent à quelque chose.

Henri Poincaré

Regularization by convolution allows one to approximate locally integrable functions by infinitely differentiable functions.

Definition 4.3.1 Let Ω be an open subset of $\mathbb{R}^N$. The space of test functions on Ω is defined by

$$\mathcal{D}(\Omega) = \{u \in C^\infty(\mathbb{R}^N) : \text{spt}\, u \text{ is a compact subset of } \Omega\}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ be a multi-index. By definition,

$$|\alpha| = \alpha_1 + \ldots + \alpha_N, \quad D^\alpha = \partial_1^{\alpha_1} \ldots \partial_N^{\alpha_N}, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Using a function defined by Cauchy in 1821, we shall verify that 0 is not the only element in $\mathcal{D}(\Omega)$.

Proposition 4.3.2 *The function defined on* $\mathbb{R}$ *by*

$$\begin{aligned} f(x) &= \exp(1/x), & x < 0, \\ &= 0, & x \geq 0, \end{aligned}$$

is infinitely differentiable.

Proof Let us prove by induction that for every n and every $x < 0$,

$$f^{(n)}(0) = 0, \quad f^{(n)}(x) = P_n(1/x)\exp(1/x),$$

where P_n is a polynomial. The statement is true for $n = 0$. Assume that it is true for n. We obtain

$$\lim_{x \to 0^-} \frac{f^{(n)}(x) - f^n(0)}{x} = \lim_{x \to 0^-} \frac{P_n(1/x)\exp(1/x)}{x} = 0.$$

Hence $f^{(n+1)}(0) = 0$. Finally, we have for $x < 0$,

$$f^{(n+1)}(x) = (-1/x^2)(P_n(1/x) + P_n'(1/x))\exp(1/x) = P_{n+1}(1/x)\exp(1/x). \quad \square$$

Definition 4.3.3 We define on $\mathbb{R}^N$ the function

$$\begin{aligned} \rho(x) &= c^{-1}\exp(1/(|x|^2 - 1)), & |x| < 1, \\ &= 0, & |x| \geq 1, \end{aligned}$$

where

$$c = \int_{B(0,1)} \exp(1/(|x|^2 - 1))dx.$$

The regularizing sequence $\rho_n(x) = n^N \rho(nx)$ is such that

$$\rho_n \in \mathcal{D}(\mathbb{R}^N), \quad \text{spt } \rho_n = B[0, 1/n], \quad \int_{\mathbb{R}^N} \rho_n \, dx = 1, \quad \rho_n \geq 0.$$

Definition 4.3.4 Let Ω be an open set of $\mathbb{R}^N$. By definition, $\omega \subset\subset \Omega$ if ω is open and $\overline{\omega}$ is a compact subset of Ω. We define, for $1 \leq p < \infty$,

$$L^p_{\text{loc}}(\Omega) = \{u : \Omega \to \mathbb{R} : \text{for all } \omega \subset\subset \Omega, u\Big|_\omega \in L^p(\omega)\}.$$

A sequence (u_n) converges to u in $L^p_{\text{loc}}(\Omega)$ if for every $\omega \subset\subset \Omega$,

$$\int_\omega |u_n - u|^p dx \to 0, \quad n \to \infty.$$

Definition 4.3.5 Let $u \in L^1_{\text{loc}}(\Omega)$ and $v \in \mathcal{K}(\mathbb{R}^N)$ be such that spt $v \subset B[0, 1/n]$. For $n \geq 1$, the convolution $v * u$ is defined on

$$\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$$

by

$$v * u(x) = \int_\Omega v(x - y)u(y)dy = \int_{B(0,1/n)} v(y)u(x - y)dy.$$

If $|y| < 1/n$, the translation of u by y is defined on Ω_n by $\tau_y u(x) = u(x - y)$.

Proposition 4.3.6 *Let $u \in L^1_{\text{loc}}(\Omega)$ and $v \in \mathcal{D}(\mathbb{R}^N)$ be such that spt $v \subset B[0, 1/n]$. Then $v * u \in C^\infty(\Omega_n)$, and for every $\alpha \in \mathbb{N}^N$, $D^\alpha(v * u) = (D^\alpha v) * u$.*

Proof Let $|\alpha| = 1$ and $x \in \Omega_n$. There exists $r > 0$ such that $B[x, r] \subset \Omega_n$. Hence

$$\omega = B(x, r + 1/n) \subset\subset \Omega,$$

and for $0 < |\varepsilon| < r$,

$$\frac{v * u(x + \varepsilon\alpha) - v * u(x)}{\varepsilon} = \int_\omega \frac{v(x + \varepsilon\alpha - y) - v(x - y)}{\varepsilon} u(y)dy.$$

But

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \neq 0}} \frac{v(x + \varepsilon\alpha - y) - v(x - y)}{\varepsilon} = D^\alpha v(x - y)$$

and

$$\sup_{\substack{y \in \omega \\ 0 < |\varepsilon| < r}} \left| \frac{v(x + \varepsilon\alpha - y) - v(x - y)}{\varepsilon} \right| < \infty.$$

Lebesgue's dominated convergence theorem implies that

$$D^\alpha (v * u)(x) = \int_\omega D^\alpha v(x - y) u(y) dy = (D^\alpha v) * u(x).$$

It is easy to conclude the proof by induction. □

Lemma 4.3.7 *Let $\omega \subset\subset \Omega$.*

(a) Let $u \in C(\Omega)$. Then for every n large enough,

$$\sup_{x \in \omega} |\rho_n * u(x) - u(x)| \leq \sup_{|y|<1/n} \sup_{x \in \omega} |\tau_y u(x) - u(x)|.$$

(b) Let $u \in L^p_{\text{loc}}(\Omega)$, $1 \leq p < \infty$. Then for every n large enough,

$$||\rho_n * u - u||_{L^p(\omega)} \leq \sup_{|y|<1/n} ||\tau_y u - u||_{L^p(\omega)}.$$

Proof For every n large enough, $\omega \subset\subset \Omega_n$. Let $u \in C(\Omega)$. Since

$$\int_{B(0,1/n)} \rho_n(y) dy = 1,$$

we obtain for every $x \in \omega$,

$$\begin{aligned} \big| \rho_n * u(x) - u(x) \big| &= \left| \int_{B(0,1/n)} \rho_n(y) \Big(u(x - y) - u(x) \Big) dy \right| \\ &\leq \sup_{|y|<1/n} \sup_{x \in \omega} \big| u(x - y) - u(x) \big|. \end{aligned}$$

Let $u \in L^p_{\text{loc}}(\Omega)$, $1 \leq p < \infty$. By Hölder's inequality, for every $x \in \omega$, we have

$$\left| \rho_n * u(x) - u(x) \right| = \left| \int_{B(0,1/n)} \rho_n(y) \Big(u(x-y) - u(x) \Big) dy \right|$$
$$\leq \left(\int_{B(0,1/n)} \rho_n(y) \left| u(x-y) - u(x) \right|^p dy \right)^{1/p}.$$

Fubini's theorem implies that

$$\int_\omega \left| \rho_n * u(x) - u(x) \right|^p dx \leq \int_\omega dx \int_{B(0,1/n)} \rho_n(y) \left| u(x-y) - u(x) \right|^p dy$$

$$= \int_{B(0,1/n)} dy \int_\omega \rho_n(y) \left| u(x-y) - u(x) \right|^p dx$$

$$\leq \sup_{|y|<1/n} \int_\omega \left| u(x-y) - u(x) \right|^p dx. \qquad \square$$

Lemma 4.3.8 (Continuity of Translations) *Let* $\omega \subset\subset \Omega$.

(a) Let $u \in \mathcal{C}(\Omega)$. *Then* $\lim_{y \to 0} \sup_{x \in \omega} |\tau_y u(x) - u(x)| = 0$.

(b) Let $u \in L^p_{\text{loc}}(\Omega)$, $1 \leq p < \infty$. *Then* $\lim_{y \to 0} ||\tau_y u - u||_{L^p(\omega)} = 0$.

Proof We choose an open subset U such that $\omega \subset\subset U \subset\subset \Omega$. If $u \in \mathcal{C}(\Omega)$, then property (a) follows from the uniform continuity of u on U.

Let $u \in L^p_{\text{loc}}(\Omega)$, $1 \leq p < \infty$, and $\varepsilon > 0$. The density theorem implies the existence of $v \in \mathcal{K}(U)$ such that $||u - v||_{L^p(U)} \leq \varepsilon$. By (a), there exists $0 < \delta < d(\omega, \partial U)$ such that for every $|y| < \delta$, $\sup_{x \in \omega} |\tau_y v(x) - v(x)| \leq \varepsilon$. We obtain for every $|y| < \delta$,

$$||\tau_y u - u||_{L^p(\omega)} \leq ||\tau_y u - \tau_y v||_{L^p(\omega)} + ||\tau_y v - v||_{L^p(\omega)} + ||v - u||_{L^p(\omega)}$$

$$\leq 2||u - v||_{L^p(U)} + m(\omega)^{1/p} \sup_{x \in \omega} |\tau_y v(x) - v(x)|$$

$$\leq (2 + m(\omega)^{1/p})\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. $\square$

We deduce from the preceding lemmas the following *regularization theorem*.

Theorem 4.3.9

(a) Let $u \in \mathcal{C}(\Omega)$. Then $\rho_n * u$ converges uniformly to u on every compact subset of Ω.
(b) Let $u \in L^p_{\text{loc}}(\Omega)$, $1 \leq p < \infty$. Then $\rho_n * u$ converges to u in $L^p_{\text{loc}}(\Omega)$.

The following consequences are fundamental.

Theorem 4.3.10 (Annulation Theorem) *Let $u \in L^1_{\text{loc}}(\Omega)$ be such that for every $v \in \mathcal{D}(\Omega)$,*

$$\int_\Omega v(x)u(x)dx = 0.$$

Then $u = 0$ almost everywhere on Ω.

Proof By assumption, for every n, $\rho_n * u = 0$ on Ω_n. □

Theorem 4.3.11 *Let $1 \leq p < \infty$. Then $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.*

Proof By the density theorem, $\mathcal{K}(\Omega)$ is dense in $L^p(\Omega)$. Let $u \in \mathcal{K}(\Omega)$. There exists an open set ω such that spt $u \subset \omega \subset\subset \Omega$. For j large enough, the support of $u_j = \rho_j * u$ is contained in ω. Since $u_j \in \mathcal{C}^\infty(\mathbb{R}^N)$ by Proposition 4.3.6, $u_j \in \mathcal{D}(\Omega)$. The regularization theorem ensures that $u_j \to u$ in $L^p(\Omega)$. □

Definition 4.3.12 A partition of unity subordinate to the covering of the compact subset Γ of $\mathbb{R}^N$ by the open sets $U_1, \ldots, U_k$ is a sequence $\psi_1, \ldots, \psi_k$ such that

(a) $\psi_j \in \mathcal{D}(U_j)$, $\psi_j \geq 0$, $j = 1, \ldots, k$;
(b) $\sum_{j=1}^k \psi_j = 1$ on Γ, $\sum_{j=1}^k \psi_j \leq 1$ on $\mathbb{R}^N$.

Let us prove the *theorem of partition of unity*.

Theorem 4.3.13 *Let $U_1, \ldots, U_k$ be a covering by open sets of the compact subset Γ of $\mathbb{R}^N$. Then there exists a partition of unity subordinates to $U_1, \ldots, U_k$.*

Proof Let K be a compact subset of the open subset U of $\mathbb{R}^N$. We choose an open set ω such that $K \subset \omega \subset\subset U$. For n large enough, $\varphi = \rho_n * \chi_\omega$ is such that $\varphi \in \mathcal{D}(U)$, $\varphi = 1$ on K and $0 \leq \varphi \leq 1$ on $\mathbb{R}^N$.

For n large enough, the finite sequence

$$F_j = \{x : d(x, \mathbb{R}^N \setminus U_j) \geq 1/n\}, \quad j = 1, \ldots, k$$

is a covering of Γ by closed sets. Indeed if this is not the case, there exists, by the compactness of Γ, $x \subset \Gamma \setminus \bigcup_{j=1}^k U_j$. This is a contradiction.

By the first part of the proof, there exists, for $j = 1, \ldots, k$, $\varphi_j \in \mathcal{D}(U_j)$ such that $\varphi_j = 1$ on $\Gamma \bigcap F_j$ and $0 \leq \varphi_j \leq 1$ on $\mathbb{R}^N$. Let us define the functions

$$\begin{aligned}
\psi_1 &= \varphi_1,\\
\psi_2 &= \varphi_2(1-\varphi_1),\\
&\ldots\\
\psi_k &= \varphi_k(1-\varphi_1)\ldots(1-\varphi_{k-1}).
\end{aligned}$$

It is easy to prove, by a finite induction, that

$$\psi_1 + \ldots + \psi_k = 1 - (1-\varphi_1)\ldots(1-\varphi_k).$$

Assume that $x \in \Gamma$. There exists j such that $x \in F_j$. By definition, we conclude that $\varphi_j(x) = 1$ and $\psi_1(x) + \ldots + \psi_k(x) = 1$. □

Now we consider Euclidean space.

Proposition 4.3.14 *Let* $1 \leq p < \infty$ *and* $u \in L^p(\mathbb{R}^N)$. *Then* $||\rho_n * u||_p \leq ||u||_p$ *and* $\rho_n * u \to u$ *in* $L^p(\mathbb{R}^N)$.

Proof It follows from Hölder's inequality that

$$\left|\rho_n * u(x)\right| = \left|\int_{\mathbb{R}^N} u(y)\rho_n(x-y)dy\right| \leq \left|\int_{\mathbb{R}^N} \left|u(y)\right|^p \rho_n(x-y)dy\right|^{1/p}.$$

Fubini's theorem implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\rho_n * u(x)\right|^p dx &\leq \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \left|u(y)\right|^p \rho_n(x-y)dy\\
&= \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} \left|u(y)\right|^p \rho_n(x-y)dx\\
&= \int_{\mathbb{R}^N} \left|u(y)\right|^p dy.
\end{aligned}$$

Hence $||\rho_n * u||_p \leq ||u||_p$.

Let $u \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$. The density theorem implies the existence of $v \in \mathcal{K}(\mathbb{R}^N)$ such that $||u - v||_p \leq \varepsilon$. By the regularization theorem, $\rho_n * v \to v$ in $L^p(\mathbb{R}^N)$. Hence there exists m such that for every $n \geq m$, $||\rho_n * v - v||_p \leq \varepsilon$. We obtain for every $n \geq m$ that

$$||\rho_n * u - u||_p \leq ||\rho_n * (u - v)||_p + ||\rho_n * v - v||_p + ||v - u||_p \leq 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Proposition 4.3.15 *Let* $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^N)$, *and* $g \in \mathcal{K}(\mathbb{R}^N)$. *Then*

$$\int_{\mathbb{R}^N} (\rho_n * f) g \, dx = \int_{\mathbb{R}^N} f(\rho_n * g) dx.$$

Proof Fubini's theorem and the parity of ρ imply that

$$\begin{aligned}\int_{\mathbb{R}^N} (\rho_n * f)(x) g(x) dx &= \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \rho_n(x-y) f(y) g(x) dy \\ &= \int_{\mathbb{R}^N} dy \int_{\mathbb{R}^N} \rho_n(x-y) f(y) g(x) dx \\ &= \int_{\mathbb{R}^N} (\rho_n * g)(y) f(y) dy.\end{aligned}$$

□

4.4 Compactness

We prove a variant of *Ascoli's theorem.*

Theorem 4.4.1 *Let X be a precompact metric space, and let S be a set of uniformly continuous functions on X such that*

(a) $c = \sup_{u \in S} \sup_{x \in X} |u(x)| < \infty$;
(b) *for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\sup_{u \in S} \omega_u(\delta) \leq \varepsilon$.

Then S is precompact in $\mathcal{BC}(X)$.

Proof Let $\varepsilon > 0$ and let δ corresponds to ε by (b). There exists a finite covering of the precompact space X by balls $B[x_1, \delta], \ldots, B[x_k, \delta]$. There exists also a finite covering of $[-c, c]$ by intervals $[y_1 - \varepsilon, y_1 + \varepsilon], \ldots, [y_n - \varepsilon, y_n + \varepsilon]$. Let us denote by J the (finite) set of mappings from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$. For every $j \in J$, we define

$$S_j = \{u \in S : |u(x_1) - y_{j(1)}| \leq \varepsilon, \ldots, |u(x_k) - y_{j(k)}| \leq \varepsilon\}.$$

By definition, $(S_j)_{j \in J}$ is a covering of S. Let $u, v \in S_j$ and $x \in X$. There exists m such that $d(x, x_m) \leq \delta$. We have

$$\left|u(x_m) - y_{j(m)}\right| \leq \varepsilon, \quad \left|v(x_m) - y_{j(m)}\right| \leq \varepsilon$$

and, by (b),

$$\left|u(x) - u(x_m)\right| \leq \varepsilon, \quad \left|v(x) - v(x_m)\right| \leq \varepsilon.$$

Hence $|u(x) - v(x)| \leq 4\varepsilon$, and since $x \in X$ is arbitrary, $||u - v||_\infty \leq 4\varepsilon$. If S_j is nonempty, then $S_j \subset B[u, 4\varepsilon]$. Since $\varepsilon > 0$ is arbitrary, S is precompact in $\mathcal{BC}(X)$ by Fréchet's criterion. □

We prove a variant of *M. Riesz's theorem* (1933).

Theorem 4.4.2 *Let Ω be an open subset of $\mathbb{R}^N$, $1 \leq p < \infty$, and let $S \subset L^p(\Omega)$ be such that*

(a) $c = \sup_{u \in S} ||u||_{L^p(\Omega)} < \infty$;

(b) for every $\varepsilon > 0$, there exists $\omega \subset\subset \Omega$ such that $\sup_{u \in S} \int_{\Omega \setminus \omega} |u|^p dx \leq \varepsilon^p$;

(c) for every $\omega \subset\subset \Omega$, $\lim_{y \to 0} \sup_{u \in S} ||\tau_y u - u||_{L^p(\omega)} = 0$.

Then S is precompact in $L^p(\Omega)$.

Proof Let $\varepsilon > 0$ and let ω corresponds to ε by (b). Assumption (c) implies the existence of $0 < \delta < d(\omega, \partial\Omega)$ such that for every $|y| \leq \delta$,

$$\sup_{u \in S} ||\tau_y u - u \,||_{L^p(\omega)} \leq \varepsilon.$$

We choose $n > 1/\delta$. We deduce from Lemma 4.3.7 that

$$\sup_{u \in S} ||\rho_n * u - u||_{L^p(\omega)} \leq \sup_{u \in S} \sup_{|y|<1/n} ||\tau_y u - u||_{L^p(\omega)} \leq \varepsilon. \quad (*)$$

We define

$$U = \{x \in \mathbb{R}^N : d(x, \omega) < 1/n\} \subset\subset \Omega.$$

Let us prove that the family $R = \{\rho_n * u\big|_\omega : u \in S\}$ satisfies the assumptions of Ascoli's theorem in $\mathcal{BC}(\omega)$.

1. By (a), for every $u \in S$ and for every $x \in \omega$, we have

$$\big|\rho_n * u(x)\big| \leq \int_U \rho_n(x - z)\big|u(z)\big|dz \leq \sup_{\mathbb{R}^N} |\rho_n| \; ||u||_{L^1(U)} \leq c_1.$$

2. By (a), for every $u \in S$ and for every $x, y \in \omega$, we have

$$\begin{aligned} \big|\rho_n * u(x) - \rho_n * u(y)\big| &\leq \int_U \big|\rho_n(x - z) - \rho_n(y - z)\big| \; \big|u(z)\big|dz \\ &\leq \sup_z \big|\rho_n(x - z) - \rho_n(y - z)\big| \; ||u||_{L^1(U)} \leq c_2|x - y|. \end{aligned}$$

Hence R is precompact in $\mathcal{BC}(\omega)$. Since

$$||v||_{L^p(\omega)} \leq m(\omega)^{1/p} \sup_{\omega} |v|,$$

R is precompact in $L^p(\omega)$. But then (∗) implies the existence of a finite covering of $S\big|_{\omega}$ in $L^p(\omega)$ by balls of radius 2ε. Assumption (b) ensures the existence of a finite covering of S in $L^p(\Omega)$ by balls of radius 3ε. Since $\varepsilon > 0$ is arbitrary, S is precompact in $L^p(\Omega)$ by Fréchet's criterion. □

4.5 Comments

Figure 4.1 gives a geometric interpretation of Lemma 4.1.3. It is contained in the *Lectures on Analysis* by G. Choquet (W.A. Benjamin, New York, 1969).

Proofs of the Hahn–Banach theorem without the axiom of choice (in separable spaces) are given in the treatise by Garnir et al. [28] and in the lectures by Favard [22].

The convexity inequality is due to Roselli and the author [64]. In contrast to Jensen's inequality [36], it is not restricted to probability measures. But we have to consider positively homogeneous functions. See [16] for the relations between convexity and lower semicontinuity.

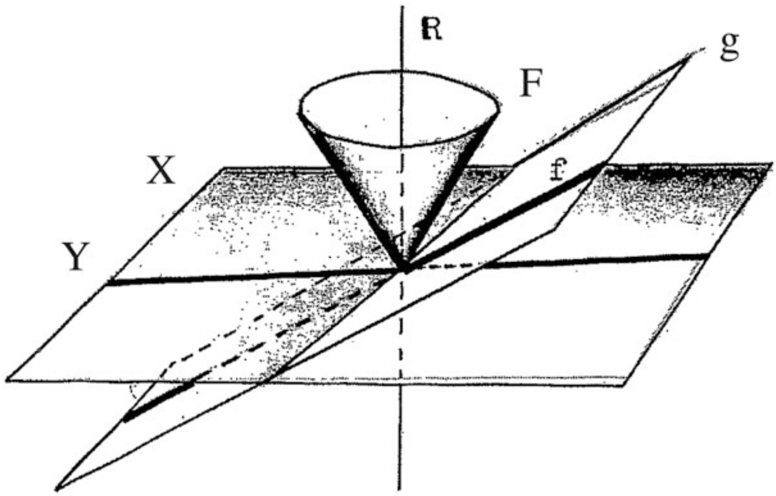

Fig. 4.1 Lemma of the Hahn-Banach theorem

4.6 Exercises for Chap. 4

1. (Young's inequality.) Let $1 < p < \infty$. Then for every $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

First proof: $A = \ell n\ a^p$, $B = \ell n\ b^{p'}$, $\exp\left(\frac{A}{p} + \frac{B}{p'}\right) \leq \frac{\exp A}{p} + \frac{\exp B}{p'}$.

Second proof: $\frac{b^{p'}}{p'} = \sup_{a \geq 0} \left(ab - \frac{a^p}{p}\right)$.

2. (Hölder's inequality.) Let $1 < p < \infty$. If $||u||_p \neq 0 \neq ||v||_{p'}$, then by Young's inequality,

$$\int_\Omega \left|\frac{u}{||u||_p} \frac{v}{||v||_{p'}}\right| d\mu \leq 1.$$

3. (Minkowski's inequality.) Prove that

   (a) $||u||_p = \sup_{||w||_{p'}=1} \int_\Omega uw\ d\mu$

   (b) $||u + v||_p \leq ||u||_p + ||v||_p$

4. (Minkowski's inequality.) Let $1 < p < \infty$ and define, on $L^p(\Omega, \mu)$, the convex function $G(u) = \int_\Omega |u|^p d\mu$. Then with $\lambda = ||v||_p/(||u||_p + ||v||_p)$,

$$G\left(\frac{u+v}{||u||_p + ||v||_p}\right) = G\left((1-\lambda)\frac{u}{||u||_p} + \lambda \frac{v}{||v||_p}\right)$$

$$\leq (1-\lambda) G\left(\frac{u}{||u||_p}\right) + \lambda G\left(\frac{v}{||v||_p}\right) = 1.$$

Hence $||u + v||_p \leq ||u||_p + ||v||_p$.

5. (Jensen's inequality)

   (a) Let $f : [0, +\infty[\to \mathbb{R}$ be a convex function and $y > 0$. There exists $\alpha, \beta \in \mathbb{R}$ such that

$$f(y) = \alpha y + \beta \text{ and, for all } x \geq 0, \alpha x + \beta \leq f(x).$$

   (b) Let $f : [0, +\infty[\to \mathbb{R}$ be a convex function. Let μ be a positive measure on Ω such that $\mu(\Omega) = 1$, and let $u \in L^1(\Omega, \mu)$ be such that $u \geq 0$ and $\int_\Omega u\ d\mu > 0$. Then

$$f\left(\int_{\Omega} u\, d\mu\right) \leq \int_{\Omega} f(u) d\mu \leq +\infty.$$

If f is concave, the reverse inequality holds.

6. Assume that $\mu(\Omega) = 1$. Then for every $u \in L^1(\Omega, \mu)$, $u \geq 0$,

$$0 \leq \exp \int_{\Omega} \ell n\, u\, d\mu \leq \int_{\Omega} u\, d\mu \leq \ell n \int_{\Omega} \exp u\, d\mu \leq +\infty.$$

7. Let $\Omega = B(0, 1) \subset \mathbb{R}^N$. Then

$$\lambda p + N > 0 \Longleftrightarrow |x|^{\lambda} \in L^p(\Omega), \lambda p + N < 0 \Longleftrightarrow |x|^{\lambda} \in L^p(\mathbb{R}^N \setminus \overline{\Omega}).$$

8. A differentiable function $u : \mathbb{R} \to \mathbb{R}$ satisfies

$$x^2 u'(x) + u(x) = 0$$

if and only if $u(x) = cf(x)$, where $c \in \mathbb{R}$ and f is the function defined in Proposition 4.3.2.

9. Let $1 < p < \infty$, $(u_n) \subset L^1(\Omega, \mu)$ and let $u \colon \Omega \to \mathbb{R}$ be μ-measurable. Then the following properties are equivalent:

   (a) $\|u_n - u\|_p \to 0, n \to \infty$;
   (b) (u_n) converges in measure to u and $\{|u_n|^p : n \in \mathbb{N}\}$ is equi-integrable.

10. (Rising sun lemma, F. Riesz, 1932.) Let $g \in C([a, b])$. The set

$$E = \left\{ a < x < b : g(x) < \max_{[x,b]} g \right\}$$

consists of a finite or countable union of disjoint intervals $]a_k, b_k[$ such that $g(a_k) \leq g(b_k)$. *Hint*: If $a_k < x < b_k$, then $g(x) < g(b_k)$.

11. (Maximal inequality, Hardy–Littlewood, 1930.) Let $u \in L^1(]a, b[)$, $u \geq 0$. The *maximal function* defined on $]a, b[$ by

$$Mu(x) = \sup_{x<y<b} \frac{1}{y - x} \int_x^y u(s) ds$$

satisfies, for every $t > 0$,

$$|\{Mu > t\}| \leq t^{-1} \int_a^b u(s) ds.$$

Hint: Use the rising sun lemma with

$$g(x) = \int_a^x u(s)ds - tx.$$

12. (Lebesgue's differentiability theorem) Let $u \in L^1(]a, b[)$. Prove that for almost every $a < x < b$,

$$\lim_{\substack{y \to x \\ y > x}} \frac{1}{y - x} \int_x^y |u(s) - u(x)| ds = 0.$$

Hint: Use Theorem 4.3.11 and the maximal inequality.
13. (Godunova's inequality) Let $f: [0, +\infty[\to [0, +\infty[$ be convex, and let $u: \mathbb{R} \to [0, +\infty[$ be Lebesgue-measurable. Then

$$\int_0^\infty f\left(\int_0^x u(t)\frac{dt}{x}\right)\frac{dx}{x} \le \int_0^\infty f(u(x))\frac{dx}{x} \le +\infty.$$

Hint:

$$\begin{aligned}\int_0^\infty f\left(\int_0^x u(t)\frac{dt}{x}\right)\frac{dx}{x} &\le \int_0^\infty dx \int_0^x f(u(t))\frac{dt}{x^2} \\ &= \int_0^\infty dt \int_t^\infty f(u(t))\frac{dx}{x^2} \\ &= \int_0^\infty f(u(t))\frac{dt}{t}.\end{aligned}$$

14. (Hardy's inequality) Let $1 < p < \infty$ and $v: \mathbb{R} \to [0, +\infty[$ be Lebesgue-measurable. Then

$$\int_0^\infty \left[\int_0^x v(t)\frac{dt}{x}\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty v^p(x)dx \le +\infty. \qquad (*)$$

Verify that this inequality is optimal using the family

$$\begin{aligned} f_\varepsilon(x) &= 1, & 0 < x \le 1, \\ &= x^{-\varepsilon - 1/p}, & x > 1. \end{aligned}$$

Hint. Godunova's inequality

$$\int_0^\infty \left[\int_0^x u(t)\frac{dt}{x}\right]^p \frac{dx}{x} \le \int_0^\infty u^p(x)\frac{dx}{x}$$

is equivalent to $(*)$ where

$$v(x) = x^{-1/p} u(x^{1-1/p}).$$

15. (Knopp's inequality) Let $v: \mathbb{R} \to [0, +\infty[$ be Lebesgue-measurable. Then

$$\int_0^\infty \exp\left(\int_0^x v(t)\frac{dt}{x}\right) dx \le e \int_0^\infty \exp v(x) dx \le +\infty. \qquad (**)$$

Hint. Godunova's inequality

$$\int_0^\infty \exp\left(\int_0^x u(t)\frac{dt}{x}\right) \frac{dx}{x} \le \int_0^\infty \exp u(x) \frac{dx}{x}$$

is equivalent to $(**)$ where

$$v(x) = u(x) - \ln x.$$