Chapter 3
Norms

3.1 Banach Spaces

Since their creation by Banach in 1922, normed spaces have played a central role in functional analysis. Banach spaces are complete normed spaces. Completeness allows one to prove the convergence of a sequence or of a series without using the limit.

Definition 3.1.1 A norm on a real vector space X is a function

$$X \to \mathbb{R} : u \mapsto ||u||$$

such that

($\mathcal{N}_1$) for every $u \in X \setminus \{0\}$, $||u|| > 0$;
($\mathcal{N}_2$) for every $u \in X$ and for $\alpha \in \mathbb{R}$, $||\alpha u|| = |\alpha| \, ||u||$;
($\mathcal{N}_3$) (Minkowski's inequality) for every $u, v \in X$,

$$||u + v|| \leq ||u|| + ||v||.$$

A (real) normed space is a (real) vector space together with a norm on that space.

Examples 1. Let $(X, ||.||)$ be a normed space and let Y be a subspace of X. The space Y together with $||.||$ (restricted to Y) is a normed space.
2. Let $(X_1, ||.||_1)$, $(X_2, ||.||_2)$ be normed spaces. The space $X_1 \times X_2$ together with

$$||(u_1, u_2)|| = \max(||u_1||_1, ||u_2||_2)$$

is a normed space.

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3. We define the norm on the space $\mathbb{R}^N$ to be

$$|x|_\infty = \max\Big\{|x_1|, \dots, |x_N|\Big\}.$$

Every normed space is a metric space.

Proposition 3.1.2 *Let X be a normed space. The function*

$$X \times X \to \mathbb{R} : (u, v) \mapsto ||u - v||$$

is a distance on X. The following mappings are continuous:

$$\begin{aligned} &X \to \mathbb{R} : u \mapsto ||u||, \\ &X \times X \to X : (u, v) \mapsto u + v, \\ &\mathbb{R} \times X \to X : (\alpha, u) \mapsto \alpha u. \end{aligned}$$

Proof By $\mathcal{N}_1$ and $\mathcal{N}_2$,

$$d(u, v) = 0 \Longleftrightarrow u = v, \quad d(u, v) = ||-(u - v)|| = ||v - u|| = d(v, u).$$

Finally, by Minkowski's inequality,

$$d(u, w) \le d(u, v) + d(v, w).$$

Since by Minkowski's inequality,

$$\Big| ||u|| - ||v|| \Big| \le ||u - v||,$$

the norm is continuous on X. It is easy to verify the continuity of the sum and of the product by a scalar. □

Definition 3.1.3 Let X be a normed space and $(u_n) \subset X$. The series $\sum_{n=0}^{\infty} u_n$ converges, and its sum is $u \in X$ if the sequence $\sum_{n=0}^{k} u_n$ converges to u. We then write $\sum_{n=0}^{\infty} u_n = u$.

The series $\sum_{n=0}^{\infty} u_n$ converges normally if $\sum_{n=0}^{\infty} ||u_n|| < \infty$.

Definition 3.1.4 A Banach space is a complete normed space.

Proposition 3.1.5 *In a Banach space X, the following statements are equivalent:*

(a) $\sum_{n=0}^{\infty} u_n$ *converges;*

(b) $\lim_{\substack{j \to \infty \\ j < k}} \sum_{n=j+1}^{k} u_n = 0.$

Proof Define $S_k = \sum_{n=0}^{k} u_n$. Since X is complete, we have

$$(a) \Longleftrightarrow \lim_{\substack{j \to \infty \\ j < k}} ||S_k - S_j|| = 0 \Longleftrightarrow \lim_{\substack{j \to \infty \\ j < k}} \left\| \sum_{n=j+1}^{k} u_n \right\| = 0 \Longleftrightarrow b). \qquad \square$$

Proposition 3.1.6 *In a Banach space, every normally convergent series converges.*

Proof Let $\sum_{n=0}^{\infty} u_n$ be a normally convergent series in the Banach space X. Minkowski's inequality implies that for $j < k$,

$$\left\| \sum_{n=j+1}^{k} u_n \right\| \leq \sum_{n=j+1}^{k} ||u_n||.$$

Since the series is normally convergent,

$$\lim_{\substack{j \to \infty \\ j < k}} \sum_{n=j+1}^{k} ||u_n|| = 0.$$

It suffices then to use the preceding proposition. $\square$

Examples 1. The space of bounded continuous functions on the metric space X,

$$\mathcal{BC}(X) = \left\{ u \in \mathcal{C}(X) : \sup_{x \in X} |u(x)| < \infty \right\},$$

together with the norm

$$||u||_\infty = \sup_{x \in X} |u(x)|,$$

is a Banach space. Convergence with respect to $||.||_\infty$ is uniform convergence.

2. Let μ be a positive measure on Ω. We denote by $L^1(\Omega, \mu)$ the quotient of $\mathcal{L}^1(\Omega, \mu)$ by the equivalence relation "equality almost everywhere". We define the norm

$$||u||_1 = \int_\Omega |u| \, d\mu.$$

Convergence with respect to $||.||_1$ is convergence in mean. We will prove in Sect. 4.2, on Lebesgue spaces, that $L^1(\Omega, \mu)$ is a Banach space.

3. Let Λ_N be the Lebesgue measure on the open subset Ω of $\mathbb{R}^N$. We denote by $L^1(\Omega)$ the space $L^1(\Omega, \Lambda_N)$. Convergence in mean is not implied by simple convergence, and almost everywhere convergence is not implied by convergence in mean.

If $m(\Omega) < \infty$, the comparison theorem implies that for every $u \in \mathcal{BC}(\Omega)$,

$$||u||_1 = \int_\Omega |u| dx \leq m(\Omega) ||u||_\infty.$$

Hence $\mathcal{BC}(\Omega) \subset L^1(\Omega)$, and the canonical injection is continuous, since

$$||u - v||_1 \leq m(\Omega) ||u - v||_\infty.$$

In order to characterize the convergence in $L^1(\Omega, \mu)$ we shall define the notions of *convergence in measure* and of *equi-integrability*.

We consider a positive measure μ on Ω. We identify two μ-measurable functions on Ω when they are μ-almost everywhere equal.

Definition 3.1.7 A sequence of measurable functions (u_n) converges in measure to a measurable function u if for every $t > 0$,

$$\lim_{n \to \infty} \mu\{|u_n - u| > t\} = 0.$$

Proposition 3.1.8 *Assume that the sequence (u_n) converges in measure to u.*
Then there exists a subsequence (u_{n_k}) converging almost everywhere to u on Ω.

Proof There exists a subsequence (u_{n_k}) such that, for every k,

$$\mu\{|u_{n_k} - u| > 1/2^k\} \leq 1/2^k.$$

Let us define

$$A_k = \{|u_{n_k} - u| > 1/2^k\}, \quad B_k = \Omega \backslash A_k$$

and

$$A = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k, \qquad B = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} B_k$$

so that $A = \Omega \backslash B$. For every $x \in B$, there exists $j \geq 1$ such that

$$k \geq j \Rightarrow |u_{n_k}(x) - u(x)| \leq 1/2^k.$$

Hence, for every $x \in B$, $\lim\limits_{k \to \infty} u_{n_k}(x) = u(x)$.

Since, for every j,

$$\mu(A) \leq \mu\left(\bigcup_{k=j}^{\infty} A_k\right) \leq 2/2^j,$$

we conclude that $\mu(A) = 0$. □

Proposition 3.1.9 *Let (u_n) be a sequence of measurable functions such that*

(a) (u_n) converges to u almost everywhere on Ω,
(b) for every $\varepsilon > 0$, there exists a measurable subset B of Ω such that $\mu(B) < \infty$ and $\sup\limits_n \int_{\Omega \backslash B} |u_n| d\mu \leq \varepsilon$.

Then (u_n) converges in measure to u.

Proof Let $t > 0$ and let $\varepsilon > 0$. By assumption (b) there exists a measurable subset B of Ω such that $\mu(B) < \infty$ and $\sup\limits_n \int_{\Omega \backslash B} |u_n| d\mu \leq \varepsilon t/3$. It follows from Fatou's lemma that $\int_{\Omega \backslash B} |u| d\mu \leq \varepsilon t/3$. Lebesgue's dominated convergence theorem implies the existence of m such that

$$n \geq m \Rightarrow \int_B \chi_{|u_n - u| > t} \, d\mu \leq \varepsilon/3.$$

We conclude using Markov's inequality that, for $n \geq m$,

$$\begin{aligned} \mu\{|u_n - u| > t\} &\leq \int_B \chi_{|u_n - u| > t} \, d\mu + \frac{1}{t} \int_{\Omega \backslash B} |u_n - u| d\mu \\ &\leq \frac{\varepsilon}{3} + \frac{1}{t} \int_{\Omega \backslash B} |u_n| d\mu + \frac{1}{t} \int_{\Omega \backslash B} |u| d\mu \leq \varepsilon. \end{aligned}$$

□

Proposition 3.1.10 *Let $u \in L^1(\Omega, \mu)$ and let $\varepsilon > 0$. Then*

(a) there exists $\delta > 0$ such that, for every measurable subset A of Ω

$$\mu(A) \leq \delta \Rightarrow \int_A |u| d\mu \leq \varepsilon \ ;$$

(b) there exists a measurable subset B of Ω such that $\mu(B) < \infty$ and $\int_{\Omega \setminus B} |u| d\mu \leq \varepsilon$.

Proof (a) By Lebesgue's dominated convergence theorem, there exists m such that

$$\int_{|u|>m} |u| d\mu \leq \varepsilon/2.$$

Let $\delta = \varepsilon/(2m)$. For every measurable subset A of Ω such that $\mu(A) \leq \delta$, we have that

$$\int_A |u| d\mu \leq m\mu(A) + \int_{|u|>m} |u| d\mu \leq \varepsilon.$$

(b) By Lebesgue's dominated convergence theorem, there exists n such that

$$\int_{|u| \leq 1/n} |u| d\mu \leq \varepsilon.$$

The set $B = \{|u| > 1/n\}$ is such that $\mu(B) < \infty$ and $\int_{\Omega \setminus B} |u| d\mu \leq \varepsilon$. □

Definition 3.1.11 A subset S of $L^1(\Omega, \mu)$ is equi-integrable if

(a) for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every measurable subset A of Ω satisfying $\mu(A) \leq \delta$, $\sup_{u \in S} \int_A |u| d\mu \leq \varepsilon$,
(b) for every $\varepsilon > 0$, there exists a measurable subset B of Ω such that $\mu(B) < \infty$ and $\sup_{u \in S} \int_{\Omega \setminus B} |u| d\mu \leq \varepsilon$.

Theorem 3.1.12 (Vitali) *Let $(u_n) \subset L^1(\Omega, \mu)$ and let u be a measurable function. Then the following properties are equivalent:*

(a) $\|u_n - u\|_1 \to 0, n \to \infty$,
(b) (u_n) converges in measure to u and $\{u_n : n \in \mathbb{N}\}$ is equi-integrable.

Proof Assume that (a) is satisfied. Markov's inequality implies that, for every $t > 0$,

$$\mu\{|u_n - u| > t\} \le \frac{1}{t}\|u_n - u\|_1 \to 0, n \to \infty.$$

Let $\varepsilon > 0$. There exists m such that

$$n \ge m \Rightarrow \|u_n - u\|_1 \le \varepsilon/2.$$

In particular, for every measurable subset A of Ω and for every $n \ge m$,

$$\int_A |u_n| d\mu \le \int_A |u| d\mu + \int_A |u_n - u| d\mu \le \int_A |u| d\mu + \varepsilon/2.$$

Proposition 3.1.10 implies the existence of $\delta > 0$ such that, for every measurable subset A of Ω,

$$\mu(A) \le \delta \Rightarrow \int_A \sup\Big(2|u|, |u_1|, ..., |u_{m-1}|\Big) d\mu \le \varepsilon.$$

We conclude that, for every measurable subset A of Ω,

$$\mu(A) \le \delta \Rightarrow \sup_n \int_A |u_n| d\mu \le \varepsilon.$$

Similarly, Proposition 3.1.10 implies the existence of a measurable subset B of Ω such that $\mu(B) < \infty$ and

$$\int_{\Omega \setminus B} \sup\Big(2|u|, |u_1|, ..., |u_{m-1}|\Big) d\mu \le \varepsilon.$$

We conclude that $\sup_n \int_{\Omega \setminus B} |u_n| d\mu \le \varepsilon$.

Assume now that (b) is satisfied. Let $\varepsilon > 0$. By assumption, there exists $\delta > 0$ such that, for every measurable subset A of Ω,

$$\mu(A) \le \delta \Rightarrow \sup_n \int_A |u_n| d\mu \le \varepsilon,$$

and there exists a measurable subset B of Ω such that $\mu(B) < \infty$ and

$$\sup_n \int_{\Omega \setminus B} |u_n| d\mu \le \varepsilon.$$

We assume that $\mu(B) > 0$. The case $\mu(B) = 0$ is simpler. Since (u_n) converges in measure to u, Proposition 3.1.8 implies the existence of a subsequence (u_{n_k}) such that $u_{n_k} \to u$ almost everywhere on Ω. It follows from Fatou's lemma that, for every measurable subset A of Ω,

$$\mu(A) \leq \delta \Rightarrow \int_A |u| d\mu \leq \varepsilon,$$

and that

$$\int_{\Omega \setminus B} |u| d\mu \leq \varepsilon.$$

There exists also m such that

$$n \geq m \Rightarrow \mu\{|u_n - u| > \varepsilon/\mu(B)\} \leq \delta.$$

Let us define $A_n = \{|u_n - u| > \varepsilon/\mu(B)\}$, so that, for $n \geq m$, $\mu(A_n) \leq \delta$. For every $n \geq m$, we obtain

$$\int_\Omega |u_n - u| d\mu \leq \int_{\Omega \setminus B} |u_n| + |u| d\mu + \int_{A_n} |u_n| + |u| d\mu + \int_{B \setminus A_n} |u_n - u| d\mu$$
$$\leq 4\varepsilon + \int_{B \setminus A_n} \varepsilon/\mu(B) d\mu \leq 5\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

The following characterization is due to de la Vallée Poussin.

Theorem 3.1.13 *Let* $S \subset L^1(\Omega, \mu)$ *be such that* $c = \sup_{u \in S} \|u\|_1 < +\infty$. *The following properties are equivalent:*

(a) for every $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for every measurable subset A of* Ω

$$\mu(A) \leq \delta \Rightarrow \sup_{u \in S} \int_A |u| d\mu \leq \varepsilon,$$

(b) there exists a strictly increasing convex function $F : [0, +\infty[\to [0, +\infty[$ *such that*

$$\lim_{t \to \infty} F(t)/t = +\infty, \quad M = \sup_{u \in S} \int_\Omega F(|u|) d\mu < +\infty.$$

Proof Since, by Markov's inequality

$$\sup_{u \in S} \mu\{|u| > t\} \leq c/t,$$

assumption (a) implies the existence of a sequence (n_k) of integers such that, for every k,

$$n_k < n_{k+1} \quad \text{and} \quad \sup_{u \in S} \int_{|u|>n_k} |u| d\mu \leq 1/2^k.$$

Let us define $F(t) = t + \sum_{k=1}^{\infty} (t - n_k)^+$. It is clear that F is strictly increasing and convex. Moreover, for every j,

$$t > 2n_{2j} \Rightarrow j \leq F(t)/t$$

and, for every $u \in S$, by Levi's theorem,

$$\int_\Omega F(|u|) d\mu = \int_\Omega |u| d\mu + \sum_{k=1}^{\infty} \int_\Omega (|u| - n_k)^+ d\mu \leq \int_\Omega |u| d\mu + \sum_{k=1}^{\infty} \int_{|u|>n_k} |u| d\mu \leq c+1,$$

so that S satisfies (b).

Assume now that S satisfies (b). Let $\varepsilon > 0$. There exists $s > 0$ such that for every $t \geq s$, $F(t)/t \geq 2M/\varepsilon$. Hence for every $u \in S$ we have that

$$\int_{|u|>s} |u| d\mu \leq \frac{\varepsilon}{2M} \int_{|u|>s} F(|u|) d\mu \leq \varepsilon/2.$$

We choose $\delta = \varepsilon/(2s)$. For every measurable subset A of Ω such that $\mu(A) \leq \delta$ and for every $u \in S$, we obtain

$$\int_A |u| d\mu \leq s\mu(A) + \int_{|u|>s} |u| d\mu \leq \varepsilon. \qquad \square$$

3.2 Continuous Linear Mappings

On a le droit de faire la théorie générale des opérations sans définir l'opération que l'on considère, de même qu'on fait la théorie de l'addition sans définir la nature des termes à additionner.

Henri Poincaré

In general, linear mappings between normed spaces are not continuous.

Proposition 3.2.1 *Let X and Y be normed spaces and $A : X \to Y$ a linear mapping. The following properties are equivalent:*

(a) *A is continuous;*
(b) $c = \sup\limits_{\substack{u \in X \\ u \neq 0}} \dfrac{||Au||}{||u||} < \infty.$

Proof If $c < \infty$, we obtain

$$||Au - Av|| = ||A(u - v)|| \leq c||u - v||.$$

Hence A is continuous.

If A is continuous, there exists $\delta > 0$ such that for every $u \in X$,

$$||u|| = ||u - 0|| \leq \delta \Rightarrow ||Au|| = ||Au - A0|| \leq 1.$$

Hence for every $u \in X \setminus \{0\}$,

$$||Au|| = \frac{||u||}{\delta} ||A\left(\frac{\delta}{||u||}u\right)|| \leq \frac{||u||}{\delta}. \qquad \square$$

Proposition 3.2.2 *The function*

$$||A|| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{||Au||}{||u||} = \sup_{\substack{u \in X \\ ||u||=1}} ||Au||$$

defines a norm on the space $\mathcal{L}(X, Y) = \{A : X \to Y : A$ is linear and continuous$\}$.

Proof By the preceding proposition, if $A \in \mathcal{L}(X, Y)$, then $0 \leq ||A|| < \infty$. If $A \neq 0$, it is clear that $||A|| > 0$. It follows from axiom $\mathcal{N}_2$ that

$$||\alpha A|| = \sup_{\substack{u \in X \\ ||u|| = 1}} ||\alpha Au|| = \sup_{\substack{u \in X \\ ||u|| = 1}} |\alpha| \, ||Au|| = |\alpha| \, ||A||.$$

It follows from Minkowski's inequality that

$$||A + B|| = \sup_{\substack{u \in X \\ ||u|| = 1}} ||Au + Bu|| \leq \sup_{\substack{u \in X \\ ||u|| = 1}} (||Au|| + ||Bu||) \leq ||A|| + ||B||. \qquad \square$$

Proposition 3.2.3 (Extension by density) *Let Z be a dense subspace of a normed space X, Y a Banach space, and $A \in \mathcal{L}(Z, Y)$. Then there exists a unique mapping $B \in \mathcal{L}(X, Y)$ such that $B\big|_Z = A$. Moreover, $||B|| = ||A||$.*

Proof Let $u \in X$. There exists a sequence $(u_n) \subset Z$ such that $u_n \to u$. The sequence (Au_n) is a Cauchy sequence, since

$$||Au_j - Au_k|| \le ||A||\,||u_j - u_k|| \to 0, \quad j, k \to \infty$$

by Proposition 1.2.3. We denote by f its limit. Let $(v_n) \subset Z$ be such that $v_n \to u$. We have

$$||Av_n - Au_n|| \le ||A||\,||v_n - u_n|| \le ||A||\,(||v_n - u|| + ||u - u_n||) \to 0, \quad n \to \infty.$$

Hence $Av_n \to f$, and we define $Bu = f$. By Proposition 3.1.2, B is linear. Since for every n,

$$||Au_n|| \le ||A||\,||u_n||,$$

we obtain by Proposition 3.1.2 that

$$||Bu|| \le ||A||\,||u||.$$

Hence B is continuous and $||B|| \le ||A||$. It is clear that $||A|| \le ||B||$. Hence $||A|| = ||B||$.

If $C \in \mathcal{L}(X, Y)$ is such that $C\big|_Z = A$, we obtain

$$Cu = \lim_{n\to\infty} Cu_n = \lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = Bu. \qquad \square$$

Proposition 3.2.4 *Let X and Y be normed spaces, and let $(A_n) \subset \mathcal{L}(X, Y)$ and $A \in \mathcal{L}(X, Y)$ be such that $||A_n - A|| \to 0$. Then (A_n) converges simply to A.*

Proof For every $u \in X$, we have

$$||A_n u - Au|| = ||(A_n - A)u|| \le ||A_n - A||\,||u||. \qquad \square$$

Proposition 3.2.5 *Let Z be a dense subset of a normed space X, let Y be a Banach space, and let $(A_n) \subset \mathcal{L}(X, Y)$ be such that*

(a) $c = \sup_n ||A_n|| < \infty$;

(b) for every $v \in Z$, $(A_n v)$ converges.

Then A_n converges simply to $A \in \mathcal{L}(X, Y)$, and

$$||A|| \le \underline{\lim}_{n\to\infty} ||A_n||.$$

Proof Let $u \in X$ and $\varepsilon > 0$. By density, there exists $v \in B(u, \varepsilon) \cap Z$. Since $(A_n v)$ converges, Proposition 1.2.3 implies the existence of n such that

$$j, k \geq n \Rightarrow ||A_j v - A_k v|| \leq \varepsilon.$$

Hence for $j, k \geq n$, we have

$$\begin{aligned} ||A_j u - A_k u|| &\leq ||A_j u - A_j v|| + ||A_j v - A_k v|| + ||A_k v - A_k u|| \\ &\leq 2c\,||u - v|| + \varepsilon \\ &= (2c + 1)\varepsilon. \end{aligned}$$

The sequence $(A_n u)$ is a Cauchy sequence, since $\varepsilon > 0$ is arbitrary. Hence $(A_n u)$ converges to a limit Au in the complete space Y. It follows from Proposition 3.1.2 that A is linear and that

$$||Au|| = \lim_{n\to\infty} ||A_n u|| \leq \varliminf_{n\to\infty} ||A_n||\,||u||.$$

But then A is continuous and $||A|| \leq \varliminf_{n\to\infty} ||A_n||$. □

Theorem 3.2.6 (Banach–Steinhaus theorem) *Let X be a Banach space, let Y be a normed space, and let $(A_n) \subset \mathcal{L}(X, Y)$ be such that for every $u \in X$,*

$$\sup_n ||A_n u|| < \infty.$$

Then

$$\sup_n ||A_n|| < \infty.$$

First Proof Theorem 1.3.13 applied to the sequence $F_n : u \mapsto ||A_n u||$ implies the existence of a ball $B(v, r)$ such that

$$c = \sup_n \sup_{u \in B(v,r)} ||A_n u|| < \infty.$$

It is clear that for every $y, z \in Y$,

$$||y|| \leq \max\{||z + y||, ||z - y||\}. \qquad (*)$$

Hence for every n and for every $w \in B(0, r)$, $||A_n w|| \leq c$, so that

$$\sup_n ||A_n|| \leq c/r.$$

Second Proof Assume to obtain a contradiction that $\sup_n ||A_n|| = +\infty$. By considering a subsequence, we assume that $n\, 3^n \le ||A_n||$. Let us define inductively a sequence (u_n). We choose $u_0 = 0$. There exists v_n such that $||v_n|| = 3^{-n}$ and $\frac{3}{4}3^{-n}||A_n|| \le ||A_n v_n||$. By $(*)$, replacing if necessary v_n by $-v_n$, we obtain

$$\frac{3}{4}3^{-n}||A_n|| \le ||A_n v_n|| \le ||A_n(u_{n-1} + v_n)||.$$

We define $u_n = u_{n-1} + v_n$, so that $||u_n - u_{n-1}|| = 3^{-n}$. It follows that for every $k \ge n$,

$$||u_k - u_n|| \le 3^{-n}/2.$$

Hence (u_n) is a Cauchy sequence that converges to u in the complete space X. Moreover,

$$||u - u_n|| \le 3^{-n}/2.$$

We conclude that

$$\begin{aligned} ||A_n u|| &\ge ||A_n u_n|| - ||A_n(u_n - u)|| \\ &\ge ||A_n|| \left[\frac{3}{4}3^{-n} - ||u_n - u||\right] \\ &\ge n\, 3^n \left[\frac{3}{4}3^{-n} - \frac{1}{2}3^{-n}\right] = n/4. \end{aligned}$$

□

Corollary 3.2.7 *Let X be a Banach space, Y a normed space, and $(A_n) \subset \mathcal{L}(X, Y)$ a sequence converging simply to A. Then (A_n) is bounded, $A \in \mathcal{L}(X, Y)$, and*

$$||A|| \le \varliminf_{n\to\infty} ||A_n||.$$

Proof For every $u \in X$, the sequence $(A_n u)$ is convergent, hence bounded, by Proposition 1.2.3. The Banach–Steinhaus theorem implies that $\sup_n ||A_n|| < \infty$. It follows from Proposition 3.1.2 that A is linear and

$$||Au|| = \lim_{n\to\infty} ||A_n u|| \le \varliminf_{n\to\infty} ||A_n||\, ||u||,$$

so that A is continuous and $||A|| \le \varliminf_{n\to\infty} ||A_n||$. □

The preceding corollary explains why every natural linear mapping defined on a Banach space is continuous.

Examples (Convergence of functionals) We define the linear continuous functionals f_n on $L^1(]0, 1[)$ to be

$$\langle f_n, u\rangle = \int_0^1 u(x)x^n\, dx.$$

Since for every $u \in L^1(]0, 1[)$ such that $||u||_1 = 1$, we have

$$|\langle f_n, u\rangle| < \int_0^1 |u(x)|dx = 1,$$

it is clear that

$$||f_n|| = \sup_{\substack{u \in L^1 \\ ||u||_1 = 1}} |\langle f_n, u\rangle| \leq 1.$$

Choosing $v_k(x) = (k+1)x^k$, we obtain

$$\lim_{k\to\infty} \langle f_n, v_k\rangle = \lim_{k\to\infty} \frac{k+1}{k+n+1} = 1.$$

It follows that $||f_n|| = 1$, and for every $u \in L^1(]0, 1[)$ such that $||u||_1 = 1$,

$$|\langle f_n, u\rangle| < ||f_n||.$$

Lebesgue's dominated convergence theorem implies that (f_n) converges simply to $f = 0$. Observe that

$$||f|| < \varliminf_{n\to\infty} ||f_n||.$$

Definition 3.2.8 A seminorm on a real vector space X is a function $F\colon X \to [0, +\infty[$ such that

(a) for every $u \in X$ and for every $\alpha \in \mathbb{R}$, $F(\alpha u) = |\alpha| F(u)$, (positive homogeneity);
(b) for every $u, v \in X$, $F(u + v) \leq F(u) + F(v)$, (subadditivity).

Examples (a) Any norm is a seminorm.
(b) Let X be a real vector space, Y a normed space, and $A\colon X \to Y$ a linear mapping. The function F defined on X by $F(u) = \|Au\|$ is a seminorm.
(c) Let X be a normed space, Y a real vector space, and $A\colon X \to Y$ a surjective linear mapping. The function F defined on Y by

$$F(v) = \inf\Big\{\|u\| : Au = v\Big\}$$

is a seminorm.

Proposition 3.2.9 *Let F be a seminorm defined on a normed space X. The following properties are equivalent*

(a) F is continuous;
(b) $c = \sup\limits_{\substack{u \in X \\ \|u\|=1}} F(u) < \infty$.

Proof If F satisfies (b), then

$$\big|F(u) - F(v)\big| \leq F(u - v) \leq c\|u - v\|,$$

so that F is continuous.

It is easy to prove that the continuity of F at 0 implies (b). □

Let F be a seminorm on the normed space X and consider a convergent series $\sum\limits_{k=1}^{\infty} u_k$. For every n,

$$F\left(\sum_{k=1}^{n} u_k\right) \quad \leq \sum_{k=1}^{n} F(u_k).$$

If, moreover, F is continuous, it follows that

$$F\left(\sum_{k=1}^{\infty} u_k\right) \quad \leq \sum_{k=1}^{\infty} F(u_k) \leq +\infty.$$

Zabreiko's theorem asserts that the converse is valid when X is a Banach space.

Theorem 3.2.10 *Let X be a Banach space and let $F : X \to [0, +\infty[$ be a seminorm such that, for any convergent series $\sum\limits_{k=1}^{\infty} u_k$,*

$$F\left(\sum_{k=1}^{\infty} u_k\right) \quad \leq \sum_{k=1}^{\infty} F(u_k) \leq +\infty.$$

Then F is continuous.

Proof Let us define, for any $t > 0$, $G_t = \{u \in X : F(u) \leq t\}$. Since $X = \bigcup_{n=1}^{\infty} \overline{G}_n$, Baire's theorem implies the existence of m such that $\overline{G}_m$ contains a closed ball $B[a, r]$. Using the propreties of F, we obtain

$$B[0, r] \subset \frac{1}{2}B[a, r] + \frac{1}{2}B[-a, r] \subset \overline{G}_{m/2} + \overline{G}_{m/2} \subset \overline{G}_m.$$

Let us define $t = m/r$, so that $B[0, 1]$ is contained in $\overline{G}_t$, and, for every k, $B[0, 1/2^k]$ is contained in $\overline{G}_{t/2^k}$. Let $u \in B[0, 1]$. There exists $u_1 \in G_t$ such that $\|u - u_1\| \leq 1/2$. We construct by induction a sequence (u_k) such that

$$u_k \in G_{t/2^{k-1}}\ ,\ \|u - u_1 - \ldots - u_k\| \leq 1/2^k.$$

By assumption

$$F(u) = F\left(\sum_{k=1}^{\infty} u_k\right) \leq \sum_{k=1}^{\infty} F(u_k) \leq \sum_{k=1}^{\infty} t/2^{k-1} = 2t.$$

Since $u \in B[0, 1]$ is arbitrary, we obtain

$$\sup_{\substack{u \in X \\ \|u\|=1}} F(u) \leq 2t.$$

It suffices then to use Proposition 3.2.9. □

Let A be a linear mapping between two normed spaces X and Y. If A is continuous, then the graph of A is closed in $X \times Y$:

$$u_n \xrightarrow{X} u,\ Au_n \xrightarrow{Y} v \quad \Rightarrow \quad v = Au.$$

The *closed graph theorem*, proven by S. Banach in 1932, asserts that the converse is valid when X and Y are Banach spaces.

Theorem 3.2.11 *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear mapping with a closed graph. Then A is continuous.*

Proof Let us define on X the seminorm $F(u) = \|Au\|$. Assume that the series $\sum_{k=1}^{\infty} u_k$ converges to u in X and that $\sum_{k=1}^{\infty} F(u_k) < +\infty$. Since Y is a Banach space, $\sum_{k=1}^{\infty} Au_k$ converges to v in Y. But the graph of the linear mapping A is closed, so that $v = Au$ and

$$F(u) = \|Au\| = \|v\| = \|\sum_{k=1}^{\infty} Au_k\| \leq \sum_{k=1}^{\infty} \|Au_k\| = \sum_{k=1}^{\infty} F(u_k).$$

We conclude using Zabreiko's theorem:

$$\sup_{\substack{u \in X \\ \|u\|=1}} \|Au\| = \sup_{\substack{u \in X \\ \|u\|=1}} F(u) < +\infty.$$

□

The *open mapping theorem* was proved by J. Schauder in 1930.

Theorem 3.2.12 *Let X and Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$ be surjective. Then $\{Au : u \in X, \|u\| < 1\}$ is open in Y.*

Proof Let us define on Y the seminorm $F(v) = \inf\{\|u\| : Au = v\}$. Assume that the series $\sum_{k=1}^{\infty} v_k$ converges to v in Y and that $\sum_{k=1}^{\infty} F(v_k) < +\infty$. Let $\varepsilon > 0$. For every k, there exists $u_k \in X$ such that

$$\|u_k\| \leq F(v_k) + \varepsilon/2^k \quad \text{and} \quad Au_k = v_k.$$

Since X is a Banach space, the series $\sum_{k=1}^{\infty} u_k$ converges to u in X. Hence we obtain

$$\|u\| \leq \sum_{k=1}^{\infty} \|u_k\| \leq \sum_{k=1}^{\infty} F(v_k) + \varepsilon$$

and

$$Au = \sum_{k=1}^{\infty} Au_k = \sum_{k=1}^{\infty} v_k = v,$$

so that $F(v) \leq \sum_{k=1}^{\infty} F(v_k) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $F(v) \leq \sum_{k=1}^{\infty} F(v_k)$. Zabreiko's theorem implies that

$$\{Au : u \in X, \|u\| < 1\} = \{v \in Y : F(v) < 1\}$$

is open in Y. □

3.3 Hilbert Spaces

Hilbert spaces are Banach spaces with a norm derived from a scalar product.

Definition 3.3.1 A scalar product on the (real) vector space X is a function

$$X \times X \to \mathbb{R} : (u, v) \mapsto (u|v)$$

such that

($\mathcal{S}_1$) for every $u \in X \setminus \{0\}$, $(u|u) > 0$;
($\mathcal{S}_2$) for every $u, v, w \in X$ and for every $\alpha, \beta \in \mathbb{R}$, $(\alpha u + \beta v|w) = \alpha(u|w) + \beta(v|w)$;
($\mathcal{S}_3$) for every $u, v \in X$, $(u|v) = (v|u)$.

We define $||u|| = \sqrt{(u|u)}$. A (real) pre-Hilbert space is a (real) vector space together with a scalar product on that space.

Proposition 3.3.2 *Let $u, v, w \in X$ and let $\alpha, \beta \in \mathbb{R}$. Then*

(a) $(u|\alpha v + \beta w) = \alpha(u|v) + \beta(u|w)$;
(b) $||\alpha u|| = |\alpha| \, ||u||$.

Proposition 3.3.3 *Let X be a pre-Hilbert space and let $u, v \in X$. Then*

(a) (parallelogram identity) $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$;
(b) (polarization identity) $(u|v) = \frac{1}{4}||u + v||^2 - \frac{1}{4}||u - v||^2$;
(c) (Pythagorean identity) $(u|v) = 0 \iff ||u + v||^2 = ||u||^2 + ||v||^2$.

Proof Observe that

$$||u + v||^2 = ||u||^2 + 2(u|v) + ||v||^2, \tag{$*$}$$

$$||u - v||^2 = ||u||^2 - 2(u|v) + ||v||^2. \tag{$**$}$$

By adding and subtracting, we obtain parallelogram and polarization identities. The Pythagorean identity is clear. □

Proposition 3.3.4 *Let X be a pre-Hilbert space and let $u, v \in X$. Then*

(a) (Cauchy–Schwarz inequality) $|(u|v)| \le ||u|| \, ||v||$;
(b) (Minkowski's inequality) $||u + v|| \le ||u|| + ||v||$.

Proof It follows from $(*)$ and $(**)$ that for $||u|| = ||v|| = 1$,

$$|(u|v)| \le \frac{1}{2}\Big(||u||^2 + ||v||^2\Big) = 1.$$

Hence for $u \neq 0 \neq v$, we obtain

$$\frac{|(u|v)|}{||u||\ ||v||} = \left|\left(\frac{u}{||u||}\Big|\frac{v}{||v||}\right)\right| \leq 1.$$

By $(*)$ and the Cauchy–Schwarz inequality, we have

$$||u+v||^2 \leq ||u||^2 + 2||u||\ ||v|| + ||v||^2 = \Big(||u|| + ||v||\Big)^2.$$ □

Corollary 3.3.5 *(a) The function $||u|| = \sqrt{(u|u)}$ defines a norm on the pre-Hilbert space X.*
(b) The function

$$X \times X \to \mathbb{R} : (u, v) \mapsto (u|v)$$

is continuous.

Definition 3.3.6 A family $(e_j)_{j \in J}$ in a pre-Hilbert space X is orthonormal if

$$\begin{aligned}(e_j|e_k) &= 1, \qquad j = k,\\ &= 0, \qquad j \neq k.\end{aligned}$$

Proposition 3.3.7 (Bessel's inequality) *Let (e_n) be an orthonormal sequence in a pre-Hilbert space X and let $u \in X$. Then*

$$\sum_{n=0}^{\infty} \big|(u|e_n)\big|^2 \leq ||u||^2.$$

Proof It follows from the Pythagorean identity that

$$\begin{aligned}||u||^2 &= \left\|u - \sum_{n=0}^{k}(u|e_n)e_n + \sum_{n=0}^{k}(u|e_n)e_n\right\|^2\\ &= \left\|u - \sum_{n=0}^{k}(u|e_n)e_n\right\|^2 + \sum_{n=0}^{k}\big|(u|e_n)\big|^2\\ &\geq \sum_{n=0}^{k}\big|(u|e_n)\big|^2.\end{aligned}$$ □

Proposition 3.3.8 *Let* $(e_0, \ldots, e_k)$ *be a finite orthonormal sequence in a pre-Hilbert space* X*,* $u \in X$*, and* $x_0, \ldots, x_k \in \mathbb{R}$*. Then*

$$\left\| u - \sum_{n=0}^{k} (u \mid e_n) e_n \right\| \le \left\| u - \sum_{n=0}^{k} x_n e_n \right\|.$$

Proof It follows from the Pythagorean identity that

$$\begin{aligned} \left\| u - \sum_{n=0}^{k} x_n e_n \right\|^2 &= \left\| u - \sum_{n=0}^{k} (u \mid e_n) e_n + \sum_{n=0}^{k} ((u \mid e_n) - x_n) e_n \right\|^2 \\ &= \left\| u - \sum_{n=0}^{k} (u \mid e_n) e_n \right\|^2 + \sum_{n=0}^{k} \left| (u \mid e_n) - x_n \right|^2. \end{aligned}$$ □

Definition 3.3.9 A Hilbert basis of a pre-Hilbert space X is an orthonormal sequence generating a dense subspace of X.

Proposition 3.3.10 *Let* (e_n) *be a Hilbert basis of a pre-Hilbert space* X *and let* $u \in X$*. Then*

(a) $u = \sum_{n=0}^{\infty} (u \mid e_n) e_n$*;*

(b) (Parseval's identity) $||u||^2 = \sum_{n=0}^{\infty} |(u \mid e_n)|^2$.

Proof Let $\varepsilon > 0$. By definition, there exists a sequence $x_0, \ldots, x_j \in \mathbb{R}$ such that

$$||u - \sum_{n=0}^{j} x_n e_n|| < \varepsilon.$$

It follows from the preceding proposition that for $k \ge j$,

$$||u - \sum_{n=0}^{k} (u \mid e_n) e_n|| < \varepsilon.$$

Hence $u = \sum_{n=0}^{\infty} (u \mid e_n) e_n$, and by Proposition 3.1.2,

$$\left\| \lim_{k\to\infty} \sum_{n=0}^{k} (u \mid e_n) e_n \right\|^2 = \lim_{k\to\infty} \left\| \sum_{n=0}^{k} (u \mid e_n) e_n \right\|^2 = \lim_{k\to\infty} \sum_{n=0}^{k} \left|(u \mid e_n)\right|^2 = \sum_{n=0}^{\infty} \left|(u \mid e_n)\right|^2.$$

□

We characterize pre-Hilbert spaces having a Hilbert basis.

Proposition 3.3.11 *Assume the existence of a sequence (f_j) generating a dense subset of the normed space X. Then X is separable.*

Proof By assumption, the space of (finite) linear combinations of (f_j) is dense in X. Hence the space of (finite) linear combinations with rational coefficients of (f_j) is dense in X. Since this space is countable, X is separable. □

Proposition 3.3.12 *Let X be an infinite-dimensional pre-Hilbert space. The following properties are equivalent:*

(a) X is separable;
(b) X has a Hilbert basis.

Proof By the preceding proposition, (b) implies (a).

If X is separable, it contains a sequence (f_j) generating a dense subspace. We may assume that (f_j) is free. Since the dimension of X is infinite, the sequence (f_j) is infinite. We define by induction the sequences (g_n) and (e_n):

$$e_0 = f_0/||f_0||,$$

$$g_n = f_n - \sum_{j=0}^{n-1} (f_n | e_j) e_j, \, e_n = g_n/||g_n||, \quad n \geq 1.$$

The sequence (e_n) generated from (f_n) by the Gram–Schmidt orthonormalization process is a Hilbert basis of X. □

Definition 3.3.13 A Hilbert space is a complete pre-Hilbert space.

Theorem 3.3.14 (Riesz–Fischer) *Let (e_n) be an orthonormal sequence in the Hilbert space X. The series $\sum_{n=0}^{\infty} c_n e_n$ converges if and only if $\sum_{n=0}^{\infty} c_n^2 < \infty$. Then*

$$\left\| \sum_{n=0}^{\infty} c_n e_n \right\|^2 = \sum_{n=0}^{\infty} c_n^2.$$

Proof Define $S_k = \sum_{n=0}^{k} c_n e_n$. The Pythagorean identity implies that for $j < k$,

$$||S_k - S_j||^2 = \left\| \sum_{n=j+1}^{k} c_n e_n \right\|^2 = \sum_{n=j+1}^{k} c_n^2.$$

Hence

$$\lim_{\substack{j \to \infty \\ j < k}} ||S_k - S_j||^2 = 0 \Longleftrightarrow \lim_{\substack{j \to \infty \\ j < k}} \sum_{n=j+1}^{k} c_n^2 = 0 \Longleftrightarrow \sum_{n=0}^{\infty} c_n^2 < \infty.$$

Since X is complete, (S_k) converges if and only if $\sum_{n=0}^{\infty} c_n^2 < \infty$. Then $\sum_{n=0}^{\infty} c_n e_n = \lim_{k \to \infty} S_k$, and by Proposition 3.1.2,

$$|| \lim_{k \to \infty} S_k ||^2 = \lim_{k \to \infty} ||S_k||^2 = \lim_{k \to \infty} \sum_{n=0}^{k} c_n^2 = \sum_{n=0}^{\infty} c_n^2.$$ □

Examples 1. Let μ be a positive measure on Ω. We denote by $L^2(\Omega, \mu)$ the quotient of

$$\mathcal{L}^2(\Omega, \mu) = \left\{ u \in \mathcal{M}(\Omega, \mu) : \int_{\Omega} |u|^2 d\mu < \infty \right\}$$

by the equivalence relation "equality almost everywhere." If $u, v \in L^2(\Omega, \mu)$, then $u + v \in L^2(\Omega, \mu)$. Indeed, almost everywhere on Ω, we have

$$|u(x) + v(x)|^2 \leq 2(|u(x)|^2 + |v(x)|^2).$$

We define the scalar product

$$(u|v) = \int_{\Omega} uv \, d\mu$$

on the space $L^2(\Omega, \mu)$.

The scalar product is well defined, since almost everywhere on Ω,

$$|u(x) \, v(x)| \leq \frac{1}{2}(|u(x)|^2 + |v(x)|^2).$$

By definition,

$$||u||_2 = \left(\int_\Omega |u|^2 d\mu \right)^{1/2}.$$

Convergence with respect to $||.||_2$ is convergence in quadratic mean. We will prove in Sect. 4.2, on Lebesgue spaces, that $L^2(\Omega, \mu)$ is a Hilbert space. If $\mu(\Omega) < \infty$, it follows from the Cauchy–Schwarz inequality that for every $u \in L^2(\Omega, \mu)$,

$$||u||_1 = \int_\Omega |u|\, d\mu \le \mu(\Omega)^{1/2} ||u||_2.$$

Hence $L^2(\Omega, \mu) \subset L^1(\Omega, \mu)$, and the canonical injection is continuous.

2. Let Λ_N be the Lebesgue measure on the open subset Ω of $\mathbb{R}^N$. We denote by $L^2(\Omega)$ the space $L^2(\Omega, \Lambda_N)$. Observe that

$$\frac{1}{x} \in L^2(]1, \infty[) \setminus L^1(]1, \infty[) \text{ and } \frac{1}{\sqrt{x}} \in L^1(]0, 1[) \setminus L^2(]0, 1[).$$

If $m(\Omega) < \infty$, the comparison theorem implies that for every $u \in \mathcal{BC}(\Omega)$,

$$||u||_2^2 = \int_\Omega u^2 dx \le m(\Omega) ||u||_\infty^2.$$

Hence $\mathcal{BC}(\Omega) \subset L^2(\Omega)$, and the canonical injection is continuous.

Theorem 3.3.15 (Vitali 1921, Dalzell 1945) *Let (e_n) be an orthonormal sequence in $L^2(]a, b[)$. The following properties are equivalent:*

(a) (e_n) is a Hilbert basis;

(b) for every $a \le t \le b$, $\displaystyle\sum_{n=1}^{\infty} \left(\int_a^t e_n(x) dx \right)^2 = t - a$;

(c) $\displaystyle\sum_{n=1}^{\infty} \int_a^b \left(\int_a^t e_n(x) dx \right)^2 dt = \frac{(b-a)^2}{2}$.

Proof Property (b) follows from (a) and Parseval's identity applied to $\chi_{[a,t]}$. Property (c) follows from (b) and Levi's theorem. The converse is left to the reader. □

Example The sequence $e_n(x) = \sqrt{\frac{2}{\pi}} \sin n\, x$ is orthonormal in $L^2(]0, \pi[)$. Since

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^\pi \left(\int_0^t \sin n\, x\, dx \right)^2 dt = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and since by a classical identity due to Euler,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

the sequence (e_n) is a Hilbert basis of $L^2(]0, \pi[)$.

3.4 Spectral Theory

Spectral theory allows one to diagonalize symmetric compact operators.

Definition 3.4.1 Let X be a real vector space and let $A : X \to X$ be a linear mapping. The eigenvectors corresponding to the eigenvalue $\lambda \in \mathbb{R}$ are the nonzero solutions of

$$Au = \lambda u.$$

The multiplicity of λ is the dimension of the space of solutions. The eigenvalue λ is simple if its multiplicity is equal to 1. The rank of A is the dimension of the range of A.

Definition 3.4.2 Let X be a pre-Hilbert space. A symmetric operator is a linear mapping $A : X \to X$ such that for every $u, v \in X$, $(Au|v) = (u|Av)$.

Proposition 3.4.3 *Let X be a pre-Hilbert space and $A : X \to X$ a symmetric continuous operator. Then*

$$||A|| = \sup_{\substack{u \in X \\ ||u|| = 1}} |(Au|u)|.$$

Proof It is clear that

$$a = \sup_{\substack{u \in X \\ ||u|| = 1}} |(Au|u)| \leq b = \sup_{\substack{u, v \in X \\ ||u|| = ||v|| = 1}} |(Au|v)| = ||A||.$$

If $||u|| = ||v|| = 1$, it follows from the parallelogram identity that

$$
\begin{aligned}
|(Au|v)| &= \frac{1}{4}|(A(u+v)|u+v) - (A(u-v)|u-v)| \\
&\le \frac{a}{4}[||u+v||^2 + ||u-v||^2] \\
&= \frac{a}{4}[2||u||^2 + 2||v||^2] = a.
\end{aligned}
$$

Hence $b = a$. □

Corollary 3.4.4 *Under the assumptions of the preceding proposition, there exists a sequence* $(u_n) \subset X$ *such that*

$$||u_n|| = 1, ||Au_n - \lambda u_n|| \to 0, |\lambda_1| = ||A||.$$

Proof Consider a maximizing sequence (u_n):

$$||u_n|| = 1, |(Au_n|u_n)| \to \sup_{\substack{u \in X \\ ||u||=1}} |(Au|u)| = ||A||.$$

By passing if necessary to a subsequence, we can assume that $(Au_n|u_n) \to \lambda_1$, $|\lambda_1| = ||A||$. Hence

$$
\begin{aligned}
0 \le ||Au_n - \lambda_1 u_n||^2 &= ||Au_n||^2 - 2\lambda_1(Au_n|u_n) + \lambda_1^2||u_n||^2 \\
&\le 2\lambda_1^2 - 2\lambda_1(Au_n|u_n) \to 0, \quad n \to \infty.
\end{aligned}
$$ □

Definition 3.4.5 Let X and Y be normed spaces. A mapping $A\colon X \to Y$ is compact if the set $\{Au\colon u \in X, ||u|| \le 1\}$ is precompact in Y.

By Proposition 3.2.1, every linear compact mapping is continuous.

Theorem 3.4.6 *Let X be a Hilbert space and let $A\colon X \to X$ be a symmetric compact operator. Then there exists an eigenvalue λ_1 of A such that $|\lambda_1| = ||A||$.*

Proof We can assume that $A \ne 0$. The preceding corollary implies the existence of a sequence $(u_n) \subset X$ such that

$$||u_n|| = 1, ||Au_n - \lambda_1 u_n|| \to 0, |\lambda_1| = ||A||.$$

Passing if necessary to a subsequence, we can assume that $Au_n \to v$. Hence $u_n \to u = \lambda_1^{-1} v$, $||u|| = 1$, and $Au = \lambda_1 u$. □

Theorem 3.4.7 (Poincaré's principle) *Let X be a Hilbert space and $A : X \to X$ a symmetric compact operator with infinite rank. Let there be given the eigenvectors $(e_1, \ldots, e_{n-1})$ and the corresponding eigenvalues $(\lambda_1, \ldots, \lambda_{n-1})$. Then there exists an eigenvalue λ_n of A such that*

$$|\lambda_n| = \max\left\{|(Au|u)| : u \in X, ||u|| = 1, (u|e_1) = \ldots = (u|e_{n-1}) = 0\right\}$$

and $\lambda_n \to 0$, $n \to \infty$.

Proof The closed subspace of X

$$X_n = \left\{u \in X : (u|e_1) = \ldots = (u|e_{n-1}) = 0\right\}$$

is invariant by A. Indeed, if $u \in X_n$ and $1 \le j \le n-1$, then

$$(Au|e_j) = (u|Ae_j) = \lambda_j(u|e_j) = 0.$$

Hence $A_n = A\Big|_{X_n}$ is a nonzero symmetric compact operator, and there exist an eigenvalue λ_n of A_n such that $|\lambda_n| = ||A_n||$ and a corresponding eigenvector $e_n \in X_n$ such that $||e_n|| = 1$. By construction, the sequence (e_n) is orthonormal, and the sequence $(|\lambda_n|)$ is decreasing. Hence $|\lambda_n| \to d$, $n \to \infty$, and for $j \neq k$,

$$||Ae_j - Ae_k||^2 = \lambda_j^2 + \lambda_k^2 \to 2d^2, \quad j, k \to \infty.$$

Since A is compact, $d = 0$. □

Theorem 3.4.8 *Under the assumptions of the preceding theorem, for every $u \in X$, the series $\sum_{n=1}^{\infty}(u|e_n)e_n$ converges and $u - \sum_{n=1}^{\infty}(u|e_n)e_n$ belongs to the kernel of A:*

$$Au = \sum_{n=1}^{\infty} \lambda_n (u|e_n)e_n. \qquad (*)$$

Proof For every $k \ge 1$, $u - \sum_{n=1}^{k}(u|e_n)e_n \in X_{k+1}$. It follows from Proposition 3.3.8. that

$$\left|\left|Au - \sum_{n=1}^{k} \lambda_n(u|e_n)e_n\right|\right| \le ||A_{k+1}|| \ \left|\left| u - \sum_{n=1}^{k}(u|e_n)e_n\right|\right| \le ||A_{k+1}|| \ ||u|| \to 0, \ k \to \infty.$$

Bessel's inequality implies that $\sum_{n=1}^{\infty}|(u|e_n)|^2 \leq ||u||^2$. We deduce from the Riesz–Fischer theorem that $\sum_{n=1}^{\infty}(u|e_n)e_n$ converges to $v \in X$. Since A is continuous,

$$Av = \sum_{n=1}^{\infty} \lambda_n (u|e_n)e_n = Au$$

and $A(u - v) = 0$. □

Formula $(*)$ is the diagonalization of symmetric compact operators.

3.5 Comments

The de la Vallée Poussin criterion was proved in the beautiful paper [17].

The first proof of the Banach–Steinhaus theorem in Sect. 3.2 is due to Favard [22], and the second proof to Royden [66].

Theorem 3.2.10 is due to P.P. Zabreiko, *Funct. Anal. and Appl. 3 (1969) 70-72*.

Let us recall the elegant notion of vector space over the reals used by S. Banach in [6] :

> Suppose that a non-empty set E is given, and that to each ordered pair (x, y) of elements of E there corresponds an element $x + y$ of E (called the *sum* of x and y) and that for each number t and $x \in E$ an element tx of E (called the *product* of the number t with the element x) is defined in such a way that these operations, namely *addition* and *scalar multiplication* satisfy the following conditions (where x, y and z denote arbitrary elements of E and a, b are numbers):
>
> 1) $x + y = y + x$,
> 2) $x + (y + z) = (x + y) + z$,
> 3) $x + y = x + z$ *implies* $y = z$,
> 4) $a(x + y) = ax + ay$,
> 5) $(a + b)x = ax + bx$,
> 6) $a(bx) = (ab)x$,
> 7) $1 \cdot x = x$.
>
> Under these hypotheses, we say that the set E constitutes a *vector* or *linear* space. It is easy to see that there then exists exactly one element, which we denote by Θ, such that $x + \Theta = x$ for all $x \in E$ and that the equality $ax = bx$ where $x \neq \Theta$ yields $a = b$; furthermore, that the equality $ax = ay$ where $a \neq 0$ implies $x = y$.
>
> Put, further, by definition :
>
> $$-x = (-1)x \quad \text{and} \quad x - y = x + (-y).$$

The space $\mathcal{L}^1(\mathbb{R}^N)$ with the *pointwise sum*

$$(u + v)(x) = u(x) + v(x),$$

and the *scalar multiplication*

$$(a \cdot u)(x) = a\, u(x),$$

is *not* a vector space. Indeed one has in general to allow $-\infty$ and $+\infty$ as values of the elements of $\mathcal{L}^1(\mathbb{R}^N)$. Hence the pointwise sum and the scalar multiplication by 0 are not, in general, well defined. On the other hand the space $L^1(\Omega, \mu)$, with the pointwise sum and the scalar multiplication, is a vector space since it consists of equivalence classes of μ-almost everywhere defined and finite function on Ω.

3.6 Exercises for Chap. 3

1. Prove that $\mathcal{BC}(\Omega) \cap L^1(\Omega) \subset L^2(\Omega)$.
2. Define a sequence $(u_n) \subset \mathcal{BC}(]0, 1[)$ such that $||u_n||_1 \to 0$, $||u_n||_2 = 1$, and $||u_n||_\infty \to \infty$.
3. Define a sequence $(u_n) \subset \mathcal{BC}(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $||u_n||_1 \to \infty$, $||u_n||_2 = 1$ and $||u_n||_\infty \to 0$.
4. Define a sequence $(u_n) \subset \mathcal{BC}(]0, 1[)$ converging simply to u such that $||u_n||_\infty = ||u||_\infty = ||u_n - u||_\infty = 1$.
5. Define a sequence $(u_n) \subset L^1(]0, 1[)$ such that $||u_n||_1 \to 0$ and for every $0 < x < 1$, $\overline{\lim_{n\to\infty}}\, u_n(x) = 1$. *Hint*: Use characteristic functions of intervals.
6. On the space $\mathcal{C}([0, 1])$ with the norm $||u||_1 = \int_0^1 |u(x)|dx$, is the linear functional

$$f : \mathcal{C}([0, 1]) \to \mathbb{R} : u \mapsto u(1/2)$$

   continuous?
7. Let X be a normed space such that every normally convergent series converges. Prove that X is a Banach space.
8. A linear functional defined on a normed space is continuous if and only if its kernel is closed. If this is not the case, the kernel is dense.
9. Is it possible to derive the norm on $L^1(]0, 1[)$ (respectively $\mathcal{BC}(]0, 1[)$) from a scalar product?
10. Prove *Lagrange's identity* in pre-Hilbert spaces:

$$\Big|\Big| ||v||u - ||u||v \Big|\Big|^2 = 2||u||^2||v||^2 - 2||u||\, ||v||(u|v).$$

11. Let X be a pre-Hilbert space and $u, v \in X \setminus \{0\}$. Then

$$\left|\left| \frac{u}{||u||^2} - \frac{v}{||v||^2} \right|\right| = \frac{||u - v||}{||u||\, ||v||}.$$

Let $f, g, h \in X$. Prove *Ptolemy's inequality*:

$$||f||\,||g-h|| \leq ||h||\,||f-g|| + ||g||\,||h-f||.$$

12. (The Jordan–von Neumann theorem.) Assume that the parallelogram identity is valid in the normed space X. Then it is possible to derive the norm from a scalar product. Define

$$(u|v) = \frac{1}{4}\left(||u+v||^2 - ||u-v||^2\right).$$

Verify that

$$(f+g|h) + (f-g|h) = 2(f|h),$$

$$(u|h) + (v|h) = 2\left(\frac{u+v}{2}|h\right) = (u+v|h).$$

13. Let f be a linear functional on $L^2(]0,1[)$ such that $u \geq 0 \Rightarrow \langle f, u\rangle \geq 0$. Prove, by contradiction, that f is continuous with respect to the norm $||.||_2$. Prove that f is not necessarily continuous with respect to the norm $||.||_1$.
14. Prove that every symmetric operator defined on a Hilbert space is continuous. *Hint*: If this were not the case, there would exist a sequence (u_n) such that $||u_n|| = 1$ and $||Au_n|| \to \infty$. Then use the Banach–Steinhaus theorem to obtain a contradiction.
15. In a Banach space an algebraic basis is either finite or uncountable. *Hint*: Use Baire's theorem.
16. Assume that $\mu(\Omega) < \infty$. Let $(u_n) \subset L^1(\Omega, \mu)$ be such that

    (a) $\sup_n \int_\Omega |u_n| \ell n(1+|u_n|)d\mu < +\infty$;
    (b) (u_n) converges almost everywhere to u.

    Then $u_n \to u$ in $L^1(\Omega, \mu)$.
17. Let us define, for $n \geq 1$, $u_n(x) = \dfrac{\cos 3^n x}{n}$.

    (a) The series $\sum_{n=1}^{\infty} u_n$ converges in $L^2(]0, 2\pi[)$.
    (b) For every $x \in A = \{2k\pi/3^j : j \in \mathbb{N}, k \in \mathbb{Z}\}$, $\sum_{n=1}^{\infty} u_n(x) = +\infty$.
    (c) For every $x \in B = \{(2k+1)\pi/3^j : j \in \mathbb{N}, k \in \mathbb{Z}\}$, $\sum_{n=1}^{\infty} u_n(x) = -\infty$.
    (d) The sets A and B are dense in $\mathbb{R}$.