Chapter 1
Distance

1.1 Real Numbers

Analysis is based on the real numbers.

Definition 1.1.1 Let S be a nonempty subset of $\mathbb{R}$. A real number x is an upper bound of S if for all $s \in S$, $s \leq x$. A real number x is the supremum of S if x is an upper bound of S, and for every upper bound y of S, $x \leq y$. A real number x is the maximum of S if x is the supremum of S and $x \in S$. The definitions of lower bound, infimum, and minimum are similar. We shall write $\sup S$, $\max S$, $\inf S$, and $\min S$.

Let us recall the fundamental property of $\mathbb{R}$.

Axiom 1.1.2 Every nonempty subset of $\mathbb{R}$ that has an upper bound has a supremum.

In the extended real number system, every subset of $\mathbb{R}$ has a supremum and an infimum.

Definition 1.1.3 The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ has the following properties:

(a) if $x \in \mathbb{R}$, then $-\infty < x < +\infty$ and $x+(+\infty) = +\infty+x = +\infty$, $x+(-\infty) = -\infty + x = -\infty$;
(b) if $x > 0$, then $x \cdot (+\infty) = (+\infty) \cdot x = +\infty$, $x \cdot (-\infty) = (-\infty) \cdot x = -\infty$;
(c) if $x < 0$, then $x \cdot (+\infty) = (+\infty) \cdot x = -\infty$, $x \cdot (-\infty) = (-\infty) \cdot x = +\infty$.

If $S \subset \mathbb{R}$ has no upper bound, then $\sup S = +\infty$. If S has no lower bound, then $\inf S = -\infty$. Finally, $\sup \phi = -\infty$ and $\inf \phi = +\infty$.

Definition 1.1.4 Let X be a set and $F : X \to \overline{\mathbb{R}}$. We define

$$\sup_X F = \sup_{x \in X} F(x) = \sup\{F(x) : x \in X\}, \inf_X F = \inf_{x \in X} F(x) = \inf\{F(x) : x \in X\}.$$

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Proposition 1.1.5 *Let X and Y be sets and $f : X \times Y \to \overline{\mathbb{R}}$. Then*

$$\sup_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\sup_{x\in X} f(x,y), \ \sup_{x\in X}\inf_{y\in Y} f(x,y) \le \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Definition 1.1.6 A sequence $(x_n) \subset \overline{\mathbb{R}}$ is increasing if for every n, $x_n \le x_{n+1}$. The sequence (x_n) is decreasing if for every n, $x_{n+1} \le x_n$. The sequence (x_n) is monotonic if it is increasing or decreasing.

Definition 1.1.7 The lower limit of $(x_n) \subset \overline{\mathbb{R}}$ is defined by $\underline{\lim}_{n\to\infty} x_n = \sup_k \inf_{n\ge k} x_n$. The upper limit of (x_n) is defined by $\overline{\lim}_{n\to\infty} x_n = \inf_k \sup_{n\ge k} x_n$.

Remarks

(a) The sequence $a_k = \inf_{n\ge k} x_n$ is increasing, and the sequence $b_k = \sup_{n\ge k} x_n$ is decreasing.

(b) The lower limit and the upper limit always exist, and

$$\underline{\lim}_{n\to\infty} x_n \le \overline{\lim}_{n\to\infty} x_n.$$

Proposition 1.1.8 *Let $(x_n), (y_n) \subset \,]-\infty, +\infty]$ be such that $-\infty < \underline{\lim}_{n\to\infty} x_n$ and $-\infty < \underline{\lim}_{n\to\infty} y_n$. Then*

$$\underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \le \underline{\lim}_{n\to\infty} (x_n + y_n).$$

Let $(x_n), (y_n) \subset [-\infty, +\infty[$ be such that $\overline{\lim}_{n\to\infty} x_n < +\infty$ and $\overline{\lim}_{n\to\infty} y_n < +\infty$. Then

$$\overline{\lim}_{n\to\infty} (x_n + y_n) \le \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n.$$

Definition 1.1.9 A sequence $(x_n) \subset \mathbb{R}$ converges to $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for every $n \ge m$, $|x_n - x| \le \varepsilon$. We then write $\lim_{n\to\infty} x_n = x$.

The sequence (x_n) is a Cauchy sequence if for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that for every $j, k \ge m$, $|x_j - x_k| \le \varepsilon$.

Theorem 1.1.10 *The following properties are equivalent:*

(a) (x_n) *converges;*
(b) (x_n) *is a Cauchy sequence;*

(c) $-\infty < \overline{\lim_{n\to\infty}} x_n \le \underline{\lim_{n\to\infty}} x_n < +\infty$.

If any and hence all of these properties hold, then $\lim_{n\to\infty} x_n = \underline{\lim_{n\to\infty}} x_n = \overline{\lim_{n\to\infty}} x_n$.

Let us give a sufficient condition for convergence.

Theorem 1.1.11 *Every increasing and majorized, or decreasing and minorized, sequence of real numbers converges.*

Remark Every increasing sequence of real numbers that is not majorized converges in $\overline{\mathbb{R}}$ to $+\infty$. Every decreasing sequence of real numbers that is not minorized converges in $\overline{\mathbb{R}}$ to $-\infty$. Hence, if (x_n) is increasing, then

$$\lim_{n\to\infty} x_n = \sup_n x_n,$$

and if (x_n) is decreasing, then

$$\lim_{n\to\infty} x_n = \inf_n x_n.$$

In particular, for every sequence $(x_n) \subset \overline{\mathbb{R}}$,

$$\underline{\lim_{n\to\infty}} x_n = \lim_{k\to\infty} \inf_{n\ge k} x_n$$

and

$$\overline{\lim_{n\to\infty}} x_n = \lim_{k\to\infty} \sup_{n\ge k} x_n.$$

Definition 1.1.12 The series $\sum_{n=0}^{\infty} x_n$ converges, and its sum is $x \in \mathbb{R}$ if the sequence $\sum_{n=0}^{k} x_n$ converges to x. We then write $\sum_{n=0}^{\infty} x_n = x$.

Theorem 1.1.13 *The following statements are equivalent:*

(a) $\sum_{n=0}^{\infty} x_n$ *converges;*

(b) $\lim_{\substack{j\to\infty \\ j<k}} \sum_{n=j+1}^{k} x_n = 0$.

Theorem 1.1.14 *Let* (x_n) *be such that* $\sum_{n=0}^{\infty}|x_n|$ *converges. Then* $\sum_{n=0}^{\infty}x_n$ *converges and*

$$\left|\sum_{n=0}^{\infty} x_n\right| \leq \sum_{n=0}^{\infty} |x_n|.$$

1.2 Metric Spaces

Metric spaces were created by Maurice Fréchet in 1906.

Definition 1.2.1 A distance on a set X is a function

$$X \times X \to \mathbb{R} : (u, v) \to d(u, v)$$

such that

($\mathcal{D}_1$) for every $u, v \in X$, $d(u, v) = 0 \iff u = v$;
($\mathcal{D}_2$) for every $u, v \in X$, $d(u, v) = d(v, u)$;
($\mathcal{D}_3$) (triangle inequality) for every $u, v, w \in X$, $d(u, w) \leq d(u, v) + d(v, w)$.

A metric space is a set together with a distance on that set.

Examples

1. Let (X, d) be a metric space and let $S \subset X$. The set S together with d (restricted to $S \times S$) is a metric space.
2. Let (X_1, d_1) and (X_2, d_2) be metric spaces. The set $X_1 \times X_2$ together with

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is a metric space.

3. We define the distance on the space $\mathbb{R}^N$ to be

$$d(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

4. We define the distance on the space $C([0, 1]) = \{u : [0, 1] \to \mathbb{R} : u$ is continuous$\}$ to be

$$d(u, v) = \max_{x \in [0,1]} |u(x) - v(x)|.$$

Definition 1.2.2 Let X be a metric space. A sequence $(u_n) \subset X$ converges to $u \in X$ if

$$\lim_{n \to \infty} d(u_n, u) = 0.$$

We then write $\lim\limits_{n \to \infty} u_n = u$ or $u_n \to u$, $n \to \infty$. The sequence (u_n) is a Cauchy sequence if

$$\lim_{j,k \to \infty} d(u_j, u_k) = 0.$$

The sequence (u_n) is bounded if

$$\sup_n d(u_0, u_n) < \infty.$$

Proposition 1.2.3 *Every convergent sequence is a Cauchy sequence. Every Cauchy sequence is a bounded sequence.*

Proof If (u_n) converges to u, then by the triangle inequality, it follows that

$$0 \leq d(u_j, u_k) \leq d(u_j, u) + d(u, u_k)$$

and $\lim\limits_{j,k \to \infty} d(u_j, u_k) = 0$.

If (u_n) is a Cauchy sequence, then there exists m such that for $j, k \geq m$, $d(u_j, u_k) \leq 1$. We obtain for every n that

$$d(u_0, u_n) \leq \max\{d(u_0, u_1), \ldots, d(u_0, u_{m-1}), d(u_0, u_m) + 1\}.$$ □

Definition 1.2.4 A sequence (u_{n_j}) is a subsequence of a sequence (u_n) if for every j, $n_j < n_{j+1}$.

Definition 1.2.5 Let X be a metric space. The space X is complete if every Cauchy sequence in X converges. The space X is precompact if every sequence in X contains a Cauchy subsequence. The space X is compact if every sequence in X contains a convergent subsequence.

Remark

(a) Completeness allows us to prove the convergence of a sequence without using the limit.
(b) Compactness will be used to prove existence theorems and to find hidden uniformities.

The proofs of the next propositions are left to the reader.

Proposition 1.2.6 *Every Cauchy sequence containing a convergent subsequence converges. Every subsequence of a convergent, Cauchy, or bounded sequence satisfies the same property.*

Proposition 1.2.7 *A metric space is compact if and only if it is precompact and complete.*

Theorem 1.2.8 *The real line* $\mathbb{R}$*, with the usual distance, is complete.*

Example (A Noncomplete Metric Space) We define the distance on $X = C([0, 1])$ to be

$$d(u, v) = \int_0^1 |u(x) - v(x)|\, dx.$$

Every sequence $(u_n) \subset X$ such that

(a) for every x and for every n, $u_n(x) \leq u_{n+1}(x)$;
(b) $\sup_n \int_0^1 u_n(x)dx = \lim_{n\to\infty} \int_0^1 u_n(x)dx < +\infty$;

is a Cauchy sequence. Indeed, we have that

$$\lim_{j,k\to\infty} \int_0^1 |u_j(x) - u_k(x)|dx = \lim_{j,k\to\infty} |\int_0^1 (u_j(x) - u_k(x))dx| = 0.$$

But X with d is not complete, since the sequence defined by

$$u_n(x) = \min\{n, 1/\sqrt{x}\}$$

satisfies (a) and (b) but is not convergent. Indeed, assuming that (u_n) converges to u in X, we obtain, for $0 < \varepsilon < 1$, that

$$\int_\varepsilon^1 |u(x) - 1/\sqrt{x}|dx = \lim_{n\to\infty} \int_\varepsilon^1 |u(x) - u_n(x)|dx \leq \lim_{n\to\infty} \int_0^1 |u(x) - u_n(x)|dx = 0.$$

But this is impossible, since $u(x) = 1/\sqrt{x}$ has no continuous extension at 0.

Definition 1.2.9 Let X be a metric space, $u \in X$, and $r > 0$. The open and closed balls of center u and radius r are defined by

$$B(u, r) = \{v \in X : d(v, u) < r\}, \quad B[u, r] = \{v \in X : d(v, u) \leq r\}.$$

The subset S of X is open if for all $u \in S$, there exists $r > 0$ such that $B(u, r) \subset S$. The subset S of X is closed if $X \setminus S$ is open.

Example Open balls are open; closed balls are closed.

Proposition 1.2.10 *The union of every family of open sets is open. The intersection of a finite number of open sets is open. The intersection of every family of closed sets is closed. The union of a finite number of closed sets is closed.*

Proof The properties of open sets follow from the definition. The properties of closed sets follow by considering complements. □

Definition 1.2.11 Let S be a subset of a metric space X. The interior of S, denoted by $\overset{\circ}{S}$, is the largest open set of X contained in S. The closure of S, denoted by $\overline{S}$, is the smallest closed set of X containing S. The boundary of S is defined by $\partial S = \overline{S} \setminus \overset{\circ}{S}$. The set S is dense if $\overline{S} = X$.

Proposition 1.2.12 *Let X be a metric space, $S \subset X$, and $u \in X$. Then the following properties are equivalent:*

(a) $u \in \overline{S}$;
(b) *for all* $r > 0$, $B(u, r) \cap S \neq \phi$;
(c) *there exists* $(u_n) \subset S$ *such that* $u_n \to u$.

Proof It is clear that (b) ⇔ (c). Assume that $u \notin \overline{S}$. Then there exists a closed subset F of X such that $u \notin F$ and $S \subset F$. By definition, then exists $r > 0$ such that $B(u, r) \cap S = \phi$. Hence (b) implies (a). If there exists $r > 0$ such that $B(u, r) \cap S = \phi$, then $F = X \setminus B(u, r)$ is a closed subset containing S. We conclude that $u \notin \overline{S}$. Hence (a) implies (b). □

Theorem 1.2.13 (Baire's Theorem) *In a complete metric space, every intersection of a sequence of open dense subsets is dense.*

Proof Let (U_n) be a sequence of dense open subsets of a complete metric space X. We must prove that for every open ball B of X, $B \cap \left(\cap_{n=0}^{\infty} U_n\right) \neq \phi$. Since $B \cap U_0$ is open (Proposition 1.2.10) and nonempty (density of U_0), there is a closed ball $B[u_0, r_0] \subset B \cap U_0$. By induction, for every n, there is a closed ball

$$B[u_n, r_n] \subset B(u_{n-1}, r_{n-1}) \cap U_n$$

such that $r_n \leq 1/n$. Then (u_n) is a Cauchy sequence. Indeed, for $j, k \geq n$, $d(u_j, u_k) \leq 2/n$. Since X is complete, (u_n) converges to $u \in X$. For $j \geq n$, $u_j \in B[u_n, r_n]$, so that for every n, $u \in B[u_n, r_n]$. It follows that $u \in B \cap \left(\cap_{n=0}^{\infty} U_n\right)$. □

Example Let us prove that $\mathbb{R}$ is uncountable. Assume that (r_n) is an enumeration of $\mathbb{R}$. Then for every n, the set $U_n = \mathbb{R} \setminus \{r_n\}$ is open and dense. But then $\bigcap_{n=1}^{\infty} U_n$ is dense and empty. This is a contradiction.

Definition 1.2.14 Let X be a metric space with distance d and let $S \subset X$. The subset S is complete, precompact, or compact if S with distance d is complete, precompact, or compact. A covering of S is a family $\mathcal{F}$ of subsets of X such that the union of $\mathcal{F}$ contains S.

Proposition 1.2.15 *Let X be a complete metric space and let $S \subset X$. Then S is closed if and only if S is complete.*

Proof It suffices to use Proposition 1.2.12 and the preceding definition. □

Theorem 1.2.16 (Fréchet's Criterion, 1910) *Let X be a metric space and let $S \subset X$. The following properties are equivalent:*

(a) S is precompact;
(b) for every $\varepsilon > 0$, there is a finite covering of S by balls of radius ε.

Proof Assume that S satisfies (b). We must prove that every sequence $(u_n) \subset S$ contains a Cauchy subsequence. Cantor's diagonal argument will be used. There is a ball B_1 of radius 1 containing a subsequence $(u_{1,n})$ from (u_n). By induction, for every k, there is a ball B_k of radius $1/k$ containing a subsequence $(u_{k,n})$ from $(u_{k-1,n})$. The sequence $v_n = u_{n,n}$ is a Cauchy sequence. Indeed, for $m, n \geq k$, $v_m, v_n \in B_k$ and $d(v_m, v_n) \leq 2/k$.

Assume that (b) is not satisfied. There then exists $\varepsilon > 0$ such that S has no finite covering by balls of radius ε. Let $u_0 \in S$. There is $u_1 \in S \setminus B[u_0, \varepsilon]$. By induction, for every k, there is

$$u_k \in S \setminus \bigcup_{j=0}^{k-1} B[u_j, \varepsilon].$$

Hence for $j < k$, $d(u_j, u_k) \geq \varepsilon$, and the sequence (u_n) contains no Cauchy subsequence. □

Every precompact space is *separable*.

Definition 1.2.17 A metric space is separable if it contains a countable dense subset.

Proposition 1.2.18 *Let X and Y be separable metric spaces, and let S be a subset of X.*

(a) The space $X \times Y$ is separable.
(b) The space S is separable.

Proof Let (e_n) and (f_n) be sequences dense in X and Y. The family $\{(e_n, f_k) : (n, k) \in \mathbb{N}^2\}$ is countable and dense in $X \times Y$. Let

$$\mathcal{F} = \{(n, k) \in \mathbb{N}^2 : k \geq 1, B(e_n, 1/k) \cap S \neq \phi\}.$$

For every $(n, k) \in \mathcal{F}$, we choose $f_{n,k} \in B(e_n, 1/k) \cap S$. The family $\{f_{n,k} : (n, k) \in \mathcal{F}\}$ is countable and dense in S. □

1.3 Continuity

Let us define continuity using distances.

Definition 1.3.1 Let X and Y be metric spaces. A mapping $u : X \to Y$ is continuous at $y \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{d_Y(u(x), u(y)) : x \in X, d_X(x, y) \leq \delta\} \leq \varepsilon. \tag{$*$}$$

The mapping u is continuous if it is continuous at every point of X. The mapping u is uniformly continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\omega_u(\delta) = \sup\{d_Y(u(x), u(y)) : x, y \in X, d_X(x, y) \leq \delta\} \leq \varepsilon.$$

The function ω_u is the modulus of continuity of u.

Remark It is clear that uniform continuity implies continuity. In general, the converse is false. We shall prove the converse when the domain of the mapping is a compact space.

Example The distance $d : X \times X \to \mathbb{R}$ is uniformly continuous, since

$$|d(x_1, x_2) - d(y_1, y_2)| \leq 2 \max\{d(x_1, y_1), d(x_2, y_2)\}.$$

Lemma 1.3.2 *Let X and Y be metric spaces, $u : X \to Y$, and $y \in X$. The following properties are equivalent:*

(a) u is continuous at y;
(b) if (y_n) converges to y in X, then $(u(y_n))$ converges to $u(y)$ in Y.

Proof Assume that u is not continuous at y. Then there is $\varepsilon > 0$ such that for every n, there exists $y_n \in X$ such that

$$d_X(y_n, y) \leq 1/n \quad \text{and} \quad d_Y(u(y_n), u(y)) > \varepsilon.$$

But then (y_n) converges to y in X and $(u(y_n))$ is not convergent to $u(y)$.

Let u be continuous at y and (y_n) converging to y. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $(*)$ is satisfied, and there exists m such that for every $n \geq m$, $d_X(y_n, y) \leq \delta$. Hence for $n \geq m$, $d_Y(u(y_n), u(y)) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(u(y_n))$ converges to $u(y)$. □

Proposition 1.3.3 *Let X and Y be metric spaces, K a compact subset of X, and $u : X \to Y$ a continuous mapping, constant on $X \setminus K$. Then u is uniformly continuous.*

Proof Assume that u is not uniformly continuous. Then there is $\varepsilon > 0$ such that for every n, there exist $x_n \in X$ and $y_n \in K$ such that

$$d_X(x_n, y_n) \leq 1/n \text{ and } d_Y(u(x_n), u(y_n)) > \varepsilon.$$

By compactness, there is a subsequence (y_{n_k}) converging to y. Hence (x_{n_k}) converges also to y. It follows from the continuity of u at y and from the preceding lemma that

$$\begin{aligned}\varepsilon &\leq \overline{\lim_{k\to\infty}}\, d_Y(u(x_{n_k}), u(y_{n_k}))\\ &\leq \lim_{k\to\infty} d_Y(u(x_{n_k}), u(y)) + \lim_{k\to\infty} d_Y(u(y), u(y_{n_k})) = 0.\end{aligned}$$

This is a contradiction. □

Lemma 1.3.4 *Let X be a set and $F : X \to]-\infty, +\infty]$ a function. Then there exists a sequence $(y_n) \subset X$ such that $\lim_{n\to\infty} F(y_n) = \inf_X F$. The sequence (y_n) is called a minimizing sequence.*

Proof If $c = \inf_X F \in \mathbb{R}$, then for every $n \geq 1$, there exists $y_n \in X$ such that

$$c \leq F(y_n) \leq c + 1/n.$$

If $c = -\infty$, then for every $n \geq 1$, there exists $y_n \in X$ such that

$$F(y_n) \leq -n.$$

In both cases, the sequence (y_n) is a minimizing sequence. If $c = +\infty$, the result is obvious. □

Proposition 1.3.5 *Let X be a compact metric space, and let $F : X \to \mathbb{R}$ be a continuous function. Then F is bounded, and there exists $y, z \in X$ such that*

$$F(y) = \min_X F, \quad F(z) = \max_X F.$$

Proof Let $(y_n) \subset X$ be a minimizing sequence: $\lim_{n\to\infty} F(y_n) = \inf_X F$. There is a subsequence (y_{n_k}) converging to y. We obtain

$$F(y) = \lim_{k\to\infty} F(y_{n_k}) = \inf_X F.$$

Hence y minimizes F on X. To prove the existence of z, consider $-F$. □

The preceding proof suggests a generalization of continuity.

Definition 1.3.6 Let X be a metric space. A function $F : X \to]-\infty, +\infty]$ is lower semicontinuous (l.s.c.) at $y \in X$ if for every sequence (y_n) converging to y in X,

$$F(y) \le \underline{\lim}_{n\to\infty} F(y_n).$$

The function F is lower semicontinuous if it is lower semicontinuous at every point of X. A function $F : X \to [-\infty, +\infty[$ is upper semicontinuous (u.s.c.) at $y \in X$ if for every sequence (y_n) converging to y in X,

$$\overline{\lim}_{n\to\infty} F(y_n) \le F(y).$$

The function F is upper semicontinuous if it is upper semicontinuous at every point of X.

Remark A function $F : X \to \mathbb{R}$ is continuous at $y \in X$ if and only if F is both l.s.c. and u.s.c. at y.

Let us generalize the preceding proposition.

Proposition 1.3.7 *Let X be a compact metric space and let $F : X \to]-\infty, \infty]$ be an l.s.c. function. Then F is bounded from below, and there exists $y \in X$ such that*

$$F(y) = \min_X F.$$

Proof Let $(y_n) \subset X$ be a minimizing sequence. There is a subsequence (y_{n_k}) converging to y. We obtain

$$F(y) \le \underline{\lim}_{k\to\infty} F(y_{n_k}) = \inf_X F.$$

Hence y minimizes F on X. □

When X is not compact, the situation is more delicate.

Theorem 1.3.8 (Ekeland's Variational Principle) *Let X be a complete metric space, and let $F : X \to]-\infty, +\infty]$ be an l.s.c. function such that $c = \inf_X F \in \mathbb{R}$. Assume that $\varepsilon > 0$ and $z \in X$ are such that*

$$F(z) \le \inf_X F + \varepsilon.$$

Then there exists $y \in X$ such that

(a) $F(y) \le F(z)$;
(b) $d(y, z) \le 1$;

(c) for every $x \in X \setminus \{y\}$, $F(y) - \varepsilon\, d(x, y) < F(x)$.

Proof Let us define inductively a sequence (y_n). We choose $y_0 = z$ and

$$y_{n+1} \in S_n = \{x \in X : F(x) \leq F(y_n) - \varepsilon\, d(y_n, x)\}$$

such that

$$F(y_{n+1}) - \inf_{S_n} F \leq \frac{1}{2}\left[F(y_n) - \inf_{S_n} F\right]. \qquad (*)$$

Since for every n,

$$\varepsilon\, d(y_n, y_{n+1}) \leq F(y_n) - F(y_{n+1}),$$

we obtain

$$c \leq F(y_{n+1}) \leq F(y_n) \leq F(y_0) = F(z),$$

and for every $k \geq n$,

$$\varepsilon\, d(y_n, y_k) \leq F(y_n) - F(y_k). \qquad (**)$$

Hence

$$\lim_{\substack{n\to\infty \\ k\geq n}} d(y_n, y_k) = 0.$$

Since X is complete, the sequence (y_n) converges to $y \in X$. Since F is l.s.c., we have

$$F(y) \leq \lim_{n\to\infty} F(y_n) \leq F(z).$$

It follows from $(**)$ that for every n,

$$\varepsilon\, d(y_n, y) \leq F(y_n) - F(y).$$

In particular, for every n, $y \in S_n$, and for $n = 0$,

$$\varepsilon\, d(z, y) \leq F(z) - F(y) \leq c + \varepsilon - c = \varepsilon.$$

Finally, assume that

$$F(x) \leq F(y) - \varepsilon\, d(x, y).$$

The fact that $y \in S_n$ implies that $x \in S_n$. By $(*)$, we have

$$2F(y_{n+1}) - F(y_n) \leq \inf_{S_n} F \leq F(x),$$

so that

$$F(y) \leq \lim_{n\to\infty} F(y_n) \leq F(x).$$

We conclude that $x = y$, because

$$\varepsilon\, d(x, y) \leq F(y) - F(x) \leq 0. \qquad \square$$

Definition 1.3.9 Let X be a set. The upper envelope of a family of functions $F_j : X \to]-\infty, \infty]$, $j \in J$, is defined by

$$\left(\sup_{j\in J} F_j\right)(x) = \sup_{j\in J} F_j(x).$$

Proposition 1.3.10 *The upper envelope of a family of l.s.c. functions at a point of a metric space is l.s.c. at that point.*

Proof Let $F_j : X \to]-\infty, +\infty]$ be a family of l.s.c. functions at y. By Proposition 1.1.5, we have, for every sequence (y_n) converging to y,

$$\sup_j F_j(y) \leq \sup_j \underline{\lim}_{n\to\infty} F_j(y_n) = \sup_j \sup_k \inf_m F_j(y_{m+k})$$

$$\leq \sup_k \inf_m \sup_j F_j(y_{m+k}) = \underline{\lim}_{n\to\infty} \sup_j F_j(y_n).$$

Hence $\sup_j F_j$ is l.s.c. at y. $\square$

Proposition 1.3.11 *The sum of two l.s.c. functions at a point of a metric space is l.s.c. at this point.*

Proof Let $F, G : X \to]-\infty, \infty]$ be l.s.c. at y. By Proposition 1.1.8, we have for every sequence (y_n) converging to y that

$$F(y) + G(y) \leq \underline{\lim}_{n\to\infty} F(y_n) + \underline{\lim}_{n\to\infty} G(y_n) \leq \underline{\lim}_{n\to\infty} (F(y_n) + G(y_n)).$$

Hence $F + G$ is l.s.c. at y. $\square$

Proposition 1.3.12 *Let $F : X \to]-\infty, \infty]$. The following properties are equivalent:*

(a) F is l.s.c;
(b) for every $t \in \mathbb{R}$, $\{F > t\} = \{x \in X : F(x) > t\}$ is open.

Proof Assume that F is not l.s.c. Then there exists a sequence (x_n) converging to x in X, and there exists $t \in \mathbb{R}$ such that

$$\underline{\lim}_{n\to\infty} F(x_n) < t < F(x).$$

Hence for every $r > 0$, $B(x, r) \not\subset \{F > t\}$, and $\{F > t\}$ is not open.

Assume that $\{F > t\}$ is not open. Then there exists a sequence (x_n) converging to x in X such that for every n,

$$F(x_n) \le t < F(x).$$

Hence $\underline{\lim}_{n\to\infty} F(x_n) < F(x)$ and F is not l.s.c. at x. □

Theorem 1.3.13 *Let X be a complete metric space, and let $(F_j : X \to \mathbb{R})_{j\in J}$ be a family of l.s.c. functions such that for every $x \in X$,*

$$\sup_{j\in J} F_j(x) < +\infty. \qquad (*)$$

Then there exists a nonempty open subset V of X such that

$$\sup_{j\in J} \sup_{x\in V} F_j(x) < +\infty.$$

Proof By Proposition 1.3.10, the function $F = \sup_{j\in J} F_j$ is l.s.c. The preceding proposition implies that for every n, $U_n = \{F > n\}$ is open. By $(*)$, $\bigcap_{n=1}^{\infty} U_n = \phi$. Baire's theorem implies the existence of n such that U_n is not dense. But then $\{F \le n\}$ contains a nonempty open subset V. □

Definition 1.3.14 The characteristic function of $A \subset X$ is defined by

$$\begin{aligned} \chi_A(x) &= 1, \quad x \in A, \\ &= 0, \quad x \in X \setminus A. \end{aligned}$$

Proposition 1.3.15 *Let X be a metric space and $A \subset X$. Then*

$$A \text{ is open} \iff \chi_A \text{ is l.s.c.}; \quad A \text{ is closed} \iff \chi_A \text{ is u.s.c.}$$

Definition 1.3.16 Let S be a nonempty subset of a metric space X. The distance of x to S is defined on X by $d(x, S) = \inf_{s \in S} d(x, s)$.

Proposition 1.3.17 *The function "distance to S" is uniformly continuous on X.*

Proof Let $x, y \in X$ and $s \in S$. Since $d(x, s) \leq d(x, y) + d(y, s)$, we obtain

$$d(x, S) \leq \inf_{s \in S} (d(x, y) + d(y, s)) = d(x, y) + d(y, S).$$

We conclude by symmetry that $|d(x, S) - d(y, S)| \leq d(x, y)$. □

Definition 1.3.18 Let Y and Z be subsets of a metric space. The distance from Y to Z is defined by $d(Y, Z) = \inf\{d(y, z) : y \in Y, z \in Z\}$.

Proposition 1.3.19 *Let Y be a compact subset, and let Z be a closed subset of a metric space X such that $Y \cap Z = \phi$. Then $d(Y, Z) > 0$.*

Proof Assume that $d(Y, Z) = 0$. Then there exist sequences $(y_n) \subset Y$ and $(z_n) \subset Z$ such that $d(y_n, z_n) \to 0$. By passing, if necessary, to a subsequence, we can assume that $y_n \to y$. But then $d(y, z_n) \to 0$ and $y \in Y \cap Z$. □

1.4 Convergence

Definition 1.4.1 Let X be a set and let Y be a metric space. A sequence of mappings $u_n : X \to Y$ converges simply to $u : X \to Y$ if for every $x \in X$,

$$\lim_{n \to \infty} d(u_n(x), u(x)) = 0.$$

The sequence (u_n) converges uniformly to u if

$$\lim_{n \to \infty} \sup_{x \in X} d(u_n(x), u(x)) = 0.$$

Remarks

(a) Clearly, uniform convergence implies simple convergence.
(b) The converse is false in general. Let $X =]0, 1[$, $Y = \mathbb{R}$, and $u_n(x) = x^n$. The sequence (u_n) converges simply but not uniformly to 0.
(c) We shall prove a partial converse due to Dini.

Notation Let $u_n : X \to \overline{\mathbb{R}}$ be a sequence of functions. We write $u_n \uparrow u$ when for every x and for every n, $u_n(x) \leq u_{n+1}(x)$ and

$$u(x) = \sup_n u_n(x) = \lim_{n \to \infty} u_n(x).$$

We write $u_n \downarrow u$ when for every x and every n, $u_{n+1}(x) \leq u_n(x)$ and

$$u(x) = \inf_n u_n(x) = \lim_{n \to \infty} u_n(x).$$

Theorem 1.4.2 (Dini) *Let X be a compact metric space, and let $u_n : X \to \mathbb{R}$ be a sequence of continuous functions such that*

(a) $u_n \uparrow u$ or $u_n \downarrow u$;
(b) $u : X \to \mathbb{R}$ is continuous.

Then (u_n) converges uniformly to u.

Proof Assume that

$$0 < \lim_{n \to \infty} \sup_{x \in X} |u_n(x) - u(x)| = \inf_{n \geq 0} \sup_{x \in X} |u_n(x) - u(x)|.$$

There exist $\varepsilon > 0$ and a sequence $(x_n) \subset X$ such that for every n,

$$\varepsilon \leq |u_n(x_n) - u(x_n)|.$$

By monotonicity, we have for $0 \leq m \leq n$ that

$$\varepsilon \leq |u_m(x_n) - u(x_n)|.$$

By compactness, there exists a sequence (x_{n_k}) converging to x. By continuity, we obtain for every $m \geq 0$,

$$\varepsilon \leq |u_m(x) - u(x)|.$$

But then (u_n) is not simply convergent to u. □

Example (Dirichlet Function) Let us show by an example that two simple limits suffice to destroy *every* point of continuity. Dirichlet's function

$$u(x) = \lim_{m \to \infty} \lim_{n \to \infty} (\cos \pi m! x)^{2n}$$

is equal to 1 when x is rational and to 0 when x is irrational. This function is everywhere discontinuous. Let us prove that uniform convergence preserves continuity.

Proposition 1.4.3 *Let X and Y be metric spaces, $y \in X$, and $u_n : X \to Y$ a sequence such that*

(a) (u_n) *converges uniformly to* u *on* X*;*
(b) *for every* n*,* u_n *is continuous at* y*.*

Then u *is continuous at* y*.*

Proof Let $\varepsilon > 0$. By assumption, there exist n and $\delta > 0$ such that

$$\sup_{x\in X} d(u_n(x), u(x)) \leq \varepsilon \text{ and } \sup_{x\in B[y,\delta]} d(u_n(x), u_n(y)) \leq \varepsilon.$$

Hence for every $x \in B[y, \delta]$,

$$d(u(x), u(y)) \leq d(u(x), u_n(x)) + d(u_n(x), u_n(y)) + d(u_n(y), u(y)) \leq 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, u is continuous at y. □

Definition 1.4.4 Let X be a set and let Y be a metric space. On the space of bounded mappings from X to Y,

$$\mathcal{B}(X, Y) = \{u : X \to Y : \sup_{x,y\in X} d(u(x), u(y)) < \infty\},$$

we define the distance of uniform convergence

$$d(u, v) = \sup_{x\in X} d(u(x), v(x)).$$

Proposition 1.4.5 *Let* X *be a set and let* Y *be a complete metric space. Then the space* $\mathcal{B}(X, Y)$ *is complete.*

Proof Assume that (u_n) is such that

$$\lim_{j,k\to\infty} \sup_{x\in X} d(u_j(x), u_k(x)) = 0.$$

Then for every $x \in X$,

$$\lim_{j,k\to\infty} d(u_j(x), u_k(x)) = 0,$$

and the sequence $(u_n(x))$ converges to a limit $u(x)$. Let $\varepsilon > 0$. There exists m such that for $j, k \geq m$ and $x \in X$,

$$d(u_j(x), u_k(x)) \leq \varepsilon.$$

By continuity of the distance, we obtain, for $k \geq m$ and $x \in X$,

$$d(u(x), u_k(x)) \leq \varepsilon.$$

Hence for $k \geq m$,

$$\sup_{x \in X} d(u(x), u_k(x)) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (u_n) converges uniformly to u. It is clear that u is bounded. □

Corollary 1.4.6 (Weierstrass Test) *Let X be a set, and let $u_n : X \to \mathbb{R}$ be a sequence of functions such that*

$$c = \sum_{n=1}^{\infty} \sup_{x \in X} |u_n(x)| < +\infty.$$

Then the series $\sum_{n=1}^{\infty} u_n$ converges absolutely and uniformly on X.

Proof It is clear that for every $x \in X$, $\sum_{n=1}^{\infty} |u_n(x)| \leq c < \infty$. Let us write $v_j = \sum_{n=1}^{j} u_n$. By assumption, we have for $j < k$ that

$$\sup_{x \in X} |v_j(x) - v_k(x)| = \sup_{x \in X} |\sum_{n=j+1}^{k} u_n(x)| \leq \sum_{n=j+1}^{k} \sup_{x \in X} |u_n(x)| \to 0, \quad j \to \infty.$$

Hence $\lim_{j,k \to \infty} d(v_j, v_k) = 0$, and (v_j) converges uniformly on X. □

Example (Lebesgue Function) Let us show by an example that a uniform limit suffices to destroy *every* point of differentiability. Let us define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin 2^{n^2} x = \sum_{n=1}^{\infty} u_n(x).$$

Since for every n, $\sup_{x \in \mathbb{R}} |u_n(x)| = 2^{-n}$, the convergence is uniform, and the function f is continuous on $\mathbb{R}$. Let $x \in \mathbb{R}$ and $h_{\pm} = \pm\pi/2^{m^2+1}$. A simple computation shows that for $n \geq m+1$, $u_n(x + h_{\pm}) - u_n(x) = 0$ and

$$\frac{u_m(x + h_{\pm}) - u_m(x)}{h_{\pm}} = \frac{2^{m^2-m+1}}{\pi} [\cos 2^{m^2} x \mp \sin 2^{m^2} x].$$

Let us choose $h = h_+$ or $h = h_-$ such that the absolute value of the expression in brackets is greater than or equal to 1. By the mean value theorem,

$$\left|\sum_{n=1}^{m-1} \frac{u_n(x+h) - u_n(x)}{h}\right| \le \sum_{n=1}^{m-1} 2^{n^2-n} < 2^{(m-1)^2-(m-1)+1} = 2^{m^2-3m+3}.$$

Hence

$$\frac{2^{m^2-m+1}}{\pi} - 2^{m^2-3m+3} \le \left|\sum_{n=1}^{m} \frac{u_n(x+h) - u_n(x)}{h}\right| = \left|\frac{f(x+h) - f(x)}{h}\right|,$$

and for every $\varepsilon > 0$,

$$\sup_{0<|h|<\varepsilon} \left|\frac{f(x+h) - f(x)}{h}\right| = +\infty.$$

The Lebesgue function is everywhere continuous and nowhere differentiable. Uniform convergence of the *derivatives* preserves differentiability.

1.5 Comments

Our main references on functional analysis are the three classical works

- S. Banach, *Théorie des opérations linéaires* [6],
- F. Riesz and B.S. Nagy, *Leç ons d'analyse fonctionnelle* [62],
- H. Brezis, *Analyse fonctionnelle, théorie et applications* [8].

The proof of Ekeland's variational principle [20] in Sect. 1.3 is due to Crandall [21].

The proof of Baire's theorem, Theorem 1.2.13, depends implicitly on the axiom of choice. We need only the following weak form.

Axiom of Dependent Choices Let S be a nonempty set, and let $R \subset S \times S$ be such that for each $a \in S$, there exists $b \in S$ satisfying $(a, b) \in S$. Then there is a sequence $(a_n) \subset S$ such that $(a_{n-1}, a_n) \in R$, $n = 1, 2, \ldots$.

We use the notation of Theorem 1.2.13. On

$$S = \{(m, u, r) : m \in \mathbb{N}, u \in X, r > 0, B(u, r) \subset B\},$$

we define the relation R by

$$((m, u, r) \,,\ (n, v, s)) \in R$$

if and only if $n = m + 1$, $s \leq 1/n$, and

$$B[v, s] \subset B(u, r) \cap (\bigcap_{j=1}^{n} U_j).$$

Baire's theorem follows then directly from the axiom of dependent choices.

In 1977, C.E. Blair proved that Baire's theorem implies the axiom of dependent choices, *Bull. Acad. Polon. Sci. Série Sc. Math. Astr. Phys. 25 (1977) 933–934*.

The reader will verify that the axiom of dependent choices is the only principle of choice that we use in this book.

1.6 Exercises for Chap. 1

La mathématique est une science de problèmes.
Georges Bouligand

1. Every sequence of real numbers contains a monotonic subsequence. *Hint*: Let

$$E = \{n \in \mathbb{N} : \text{for every } k \geq n, x_k \leq x_n\}.$$

If E is infinite, (x_n) contains a decreasing subsequence. If E is finite, (x_n) contains an increasing subsequence.

2. Every bounded sequence of real numbers contains a convergent subsequence.
3. Let (K_n) be a decreasing sequence of compact sets and U an open set in a metric space such that $\bigcap_{n=1}^{\infty} K_n \subset U$. Then there exists n such that $K_n \subset U$.
4. Let (U_n) be an increasing sequence of open sets and K a compact set in a metric space such that $K \subset \bigcup_{n=1}^{\infty} U_n$. Then there exists n such that $K \subset U_n$.
5. Define a sequence (S_n) of dense subsets of $\mathbb{R}$ such that $\bigcap_{n=1}^{\infty} S_n = \phi$. Define a family $(U_j)_{j \in J}$ of open dense subsets of $\mathbb{R}$ such that $\bigcap_{j \in J} U_j = \phi$.
6. In a complete metric space, every countable union of closed sets with empty interior has an empty interior. *Hint*: Use Baire's theorem.
7. Dirichlet's function is l.s.c. on $\mathbb{R} \setminus \mathbb{Q}$ and u.s.c. on $\mathbb{Q}$.
8. Let (u_n) be a sequence of functions defined on $[a, b]$ and such that for every n,

$$a \leq x \leq y \leq b \Rightarrow u_n(x) \leq u_n(y).$$

Assume that (u_n) converges simply to $u \in C([a,b])$. Then (u_n) converges uniformly to u.

9. (Banach fixed-point theorem) Let X be a complete metric space, and let $f : X \to X$ be such that

$$\mathrm{Lip}(f) = \sup\{d(f(x), f(y))/d(x, y) : x, y \in X, x \neq y\} < 1.$$

Then there exists one and only one $x \in X$ such that $f(x) = x$. *Hint*: Consider a sequence defined by $x_0 \in X$, $x_{n+1} = f(x_n)$.

10. (McShane's extension theorem) Let Y be a subset of a metric space X, and let $f : Y \to \mathbb{R}$ be such that

$$\lambda = \mathrm{Lip}(f) = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in Y, x \neq y\} < +\infty.$$

Define on X

$$g(x) = \sup\{f(y) - \lambda d(x, y) : y \in Y\}.$$

Then $g\big|_y = f$ and

$$\mathrm{Lip}(g) = \sup\{|g(x) - g(y)/d(x, y) : x, y \in X, x \neq y\} = \mathrm{Lip}(f).$$

11. (Fréchet's extension theorem) Let Y be a dense subset of a metric space X, and let $f : Y \to [0, +\infty]$ be an l.s.c. function. Define on X

$$g(x) = \inf\left\{ \varliminf_{n\to\infty} f(x_n) : (x_n) \subset Y \text{ and } x_n \to x \right\}.$$

Then g is l.s.c., $g\big|_Y = f$, and for every l.s.c. function $h : X \to [0, +\infty]$ such that $h\big|_Y = f$, $h \leq g$.

12. Let X be a metric space and $u : X \to [0, +\infty]$ an l.s.c. function such that $u \not\equiv +\infty$. Define

$$u_n(x) = \inf\{u(y) + n\, d(x, y) : y \in X\}.$$

Then $u_n \uparrow u$, and for every $x, y \in X$, $|u_n(x) - u_n(y)| \leq n\, d(x, y)$.

13. Let X be a metric space and $v : X \to]-\infty, \infty]$. Then v is l.s.c. if and only if there exists a sequence $(v_n) \subset C(X)$ such that $v_n \uparrow v$. *Hint*: Consider the function $u = \frac{\pi}{2} + \tan^{-1} v$.

14. (Sierpiński, 1921.) Let X be a metric space and $u : X \to \mathbb{R}$. The following properties are equivalent:

(a) There exists $(u_n) \subset C(X)$ such that for every $x \in X$, $\sum_{n=1}^{\infty} |u_n(x)| < \infty$ and $u(x) = \sum_{n=1}^{\infty} u_n(x)$.
(b) There exists $f, g : X \to [0, +\infty[$ l.s.c. such that for every $x \in X$, $u(x) = f(x) - g(x)$.

15. We define

$$X = \{u :]0, 1[\to \mathbb{R} : u \text{ is bounded and continuous}\}.$$

We define the distance on X to be

$$d(u, v) = \sup_{x \in]0,1[} |u(x) - v(x)|.$$

What are the interior and the closure of

$$Y = \{u \in X : u \text{ is uniformly continuous}\}?$$