Involutions on Different Goguen L-fuzzy Sets

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Abstract. J.A. Goguen introduced the L-fuzzy sets (that are extension of Zadeh's fuzzy sets), in which the image set of the membership function is an ordered set $(L, \leq_L)$ (base set of the L-fuzzy set). Since then, a lot of papers studying different classes of those sets have been published. One of the main topics in this field has been the involutions in each set L, but these involutions are not totally independent. So the present paper is devoted to relate the involutions on the base sets L of some L-fuzzy sets, being L a certain bounded lattice. In particular, we consider L being the bounded lattices defined by $[0, 1]$ and $[0, 1]^2$, as well as by the base sets of the Atanassov's and interval-valued fuzzy sets.

Keywords: L-fuzzy sets · Bounded lattices · Order isomorphisms · Involutions

1 Introduction

The Goguen L-fuzzy sets on a universe X were introduced in [9], being those determined by a membership function $f : X \to L$, where $(L, \leq)$ is a partially ordered set (a poset). The set L is called set of membership degrees of an L-fuzzy set. Goguen L-fuzzy sets generalize Zadeh's ordinary fuzzy sets [11], in which $(L, \leq)$ is the set $[0, 1]$ with the usual order relation $\leq$ on the real numbers.

Since that initial definition, different L-fuzzy sets have been considered. Some examples are Atanassov's Intuitionistic fuzzy sets [1], interval-valued fuzzy sets [2], etc. In the first case the set L is defined as $L = \{(a, b) \in [0, 1]^2 \; ; \; a + b \leq 1\}$ which from now on we will denote as A^*. In the second case $L = \{[a, b] \subseteq [0, 1] \; ; \; a \leq b\}$ which from now on we will refer to as I^*.

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Many papers have studied in depth different Goguen L-fuzzy sets, analyzing the different connectives (t-norms, t-conorms,...) and other elements of their algebraic structures. In this sense, an essential topic is that of obtaining the complement of a given set. Involutions have usually been used for this task. Note that the definitions that we are going to introduce are applied to posets or bounded posets, although in our case we are going to use them for bounded lattices (that is, posets that have a minimum 0_L, a maximum 1_L, and every pair of elements has a supremum and an infimum). In fact, in this paper, we will focus on the posets $([0,1],\leq)$, $(A^*,\leq_{A^*})$, $(I^*,\leq_{I^*})$, and $([0,1]^2,\leq_{[0,1]^2})$, which also have a bounded lattice structure.

Definition 1. *Given a bounded poset* $(L,\leq,0_L,1_L)$*, a function* $\alpha : L \to L$ *is an involution if:*

1. $\alpha(0_L) = 1_L$, $\alpha(1_L) = 0_L$.
2. *It is strictly decreasing respect to the relation* $\leq$.
3. *It is involutive, that is, for all* $x \in L$, $\alpha(\alpha(x)) = x$.

Thus, if μ is an L-fuzzy set on X, its complement μ' can be obtained through an involution α on L, as follows:

$$\mu'(x) = \alpha(\mu(x)), \text{ for all } x \in X.$$

Hence the importance of analyzing the involutions in each bounded poset $(L,\leq,0_L,1_L)$. Many papers have been devoted to this study, some of them obtaining a characterization of involutions. However, a pending issue is the possible relationship between these involutions (in the corresponding bounded poset) and their characterizations. The present paper opens this study, obtaining some initial results. To do so, each Section will be devoted to compare and relate the involutions on the set of membership degrees of two Goguen L-fuzzy sets, all of them having a bounded lattice structure. Section 2 will consider Zadeh's and Attanasov's sets. In Sect. 3 the considered sets will be Attanasov's and Interval valued sets. Then, Sect. 4 will compare involutions in Interval valued sets with those defined in $[0,1]^2$. And finally, Sect. 5 will connect Zadeh's sets with those defined in $[0,1]^2$. To close the paper some conclusions are presented.

The analysis requires some background definitions that are presented here.

Definition 2. *A function* $n : ([0,1],\leq) \to ([0,1],\leq)$ *is a negation if*

1. n *is decreasing, and*
2. $n(0) = 1$ *and* $n(1) = 0$.

More, if $n(n(x)) = x$ *for all* $x \in [0,1]$, n *is an involution.*

Definition 3. *Given two partially ordered sets* $(X,\leq_X)$ *and* $(Y,\leq_Y)$*, a function* $\varphi : X \to Y$ *is an order homomorphism if* φ *is an order-preserving map.*

Definition 4. *Given two partially ordered sets* $(X,\leq_X)$ *and* $(Y,\leq_Y)$*, a bijective function* $\varphi : X \to Y$ *is an order isomorphism if* φ *and* φ^{-1} *are order-preserving maps.*

Remark 1. If $\varphi : X \to Y$ is an order isomorphism, from the above definition it directly follows:

- If X and Y are bounded partially ordered sets with X minimum element 0_X and maximum element 1_X, and with Y minimum element 0_Y and maximum element 1_Y, it is $\varphi(0_X) = 0_Y$ and $\varphi(1_X) = 1_Y$.
- If $(X, \leq_X)$ and $(Y, \leq_Y)$ are totally ordered sets, in order φ to be an order isomorphism, it suffices to be a bijective and increasing function.

Definition 5. *Let two partially ordered sets $(X, \leq_X)$ and $(Y, \leq_Y)$, and two functions $\varphi_X : X \to X$, $\varphi_Y : Y \to Y$. If $i : X \to Y$ is an inclusion (injective order homomorphism) of X in Y, we will say that φ_Y is an extension of φ_X, if for all $x \in X$, $\varphi_Y(i(x)) = i(\varphi_X(x))$.*

2 Involutions on $([0,1], \leq)$ and on $(A^*, \leq_{A^*})$

As previously mentioned, a particular case of Goguen L-fuzzy sets are the Zadeh's fuzzy sets, in which $(L, \leq)$ is the set $[0,1]$ with the usual relation $\leq$ in the real numbers. The involutions on this ordered set, also called strong negations, were characterized in [10] as follows: $n : [0,1] \to [0,1]$ is an involution if and only if there exists an order isomorphism $\varphi : ([0,1], \leq) \to ([0,1], \leq)$, such that for all $x \in [0,1]$ it is $n(x) = \varphi^{-1}(1 - \varphi(x))$. It should be noted that in the characterizations of involutions that will appear in this paper, involutions in $([0,1], \leq)$ will have an essential role.

Besides, Atanassov introduced in [1], the so called Intuitionistic fuzzy sets, which are L-fuzzy sets being $L = A^* = \{(a,b) \in [0,1]^2 \ ; \ a + b \leq 1\}$, and the order is defined as:

$$(a,b) \leq_{A^*} (c,d) \Leftrightarrow a \leq c \text{ and } b \geq d.$$

$(A^*, \leq_{A^*})$ is a bounded lattice, where the minimum and maximum elements are, respectively, $0_{A^*} = (0,1)$ and $1_{A^*} = (1,0)$.

Figure 1 represents A^*, showing two elements with $(a,b) \leq_{A*} (c,d)$.

Could we find any relationship between the involutions in $([0,1], \leq)$ and the involutions in $(A^*, \leq_{A^*})$?

To address this problem, we will first consider the set $[0,1]$ as a subset of A^*, for which we will define an inclusion of $[0,1]$ in A^*, which will naturally be:

$$i : [0,1] \to A^*, \text{ where } i(x) = (x, 0).$$

Let us note that i is an order isomorphism between $([0,1], \leq)$ and $(S, \leq_{A^*})$, where $S = \{(x,0) \ ; \ x \in [0,1]\} \subset A^*$.

If n is an involution in $([0,1], \leq)$, is there an extension $\mathcal{N}$ of n to A^*? That is, is there an involution $\mathcal{N}$ on A^*, such that $\mathcal{N} \circ i = i \circ n$? The answer is negative, since it must be $\mathcal{N}(1,0) = \mathcal{N}(i(1)) = i(n(1)) = i(0) = (0,0)$, while the image of $(1,0)$ by any involution in $(A^*, \leq_{A^*})$, should be $(0,1)$.

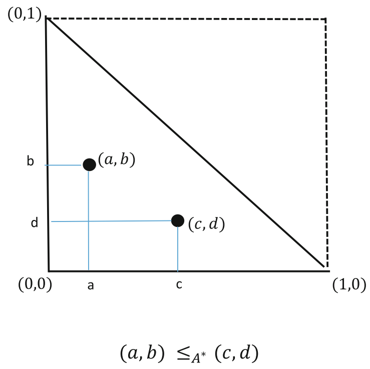

Fig. 1. The set A^* (Atanassov's Intuitionistic fuzzy sets) and two ordered elements.

However, the involutions in both sets are not totally independent, as it is demonstrated in [7], where a characterization of the involutions in $(A^*, \leq_{A^*})$ through the involutions in $([0,1], \leq)$ was obtained.

$\mathcal{N} : A^* \to A^*$ is an involution in $(A^*, \leq_{A^*})$, if and only if there exists an involution $n : [0,1] \to [0,1]$ satisfying $\mathcal{N}(a,b) = (n(1-b), 1-n(a))$ for all $(a,b) \in A^*$. More, in this case, it is $n(a) = pr_1\mathcal{N}(a, 1-a)$, where pr_1 is the projection on the first variable, that is, $pr_1(x,y) = x$.

This result suggests us to try to find another different inclusion, that provides us with the extension sought.

Let us consider $i^* : [0,1] \to A^*$, where $i^*(x) = (x, 1-x)$. i^* is an order isomorphism between $([0,1], \leq)$ and $(M, \leq_{A^*})$, where $M = \{(x, 1-x) \; ; \; x \in [0,1]\} \subset A^*$. And let n be an involution on $([0,1], \leq)$. Let us define $\mathcal{N}(a,b) = (n(1-b), 1-n(a))$.

In fact, according to the previous result, $\mathcal{N}$ is an involution on $(A^*, \leq_{A^*})$, and $\mathcal{N}(i^*(x)) = \mathcal{N}(x, 1-x) = (n(x), 1-n(x)) = i^*(n(x))$.

Figure 2 shows the inclusions i and i^* of $[0,1]$ in A^*. In the first one the image of x is $(x, 0)$ and the image of 0 is $(0,0)$, while in the second one the image of x is $(x, 1-x)$, and the image of 0 is $(0,1)$.

3 Involutions on $(A^*, \leq_{A^*})$ and on $(I^*, \leq_{I^*})$

The interval-valued Goguen L-fuzzy sets, are those in which a closed subinterval of $[0,1]$ is assigned to each element of the universe. So, in this case $L = I^* = \{[a,b] \subseteq [0,1]\}$, or equivalently, $I^* = \{(a,b) \in [0,1]^2 \; ; \; a \leq b\}$. Now, the order relation in this set is defined as

$$(a,b) \leq_{I^*} (c,d) \Leftrightarrow a \leq c \text{ and } b \leq d.$$

Then $(I^*, \leq_{I^*})$ is a bounded lattice, whose minimum and maximum elements are, respectively, $0_{I^*} = (0,0)$ and $1_{I^*} = (1,1)$.

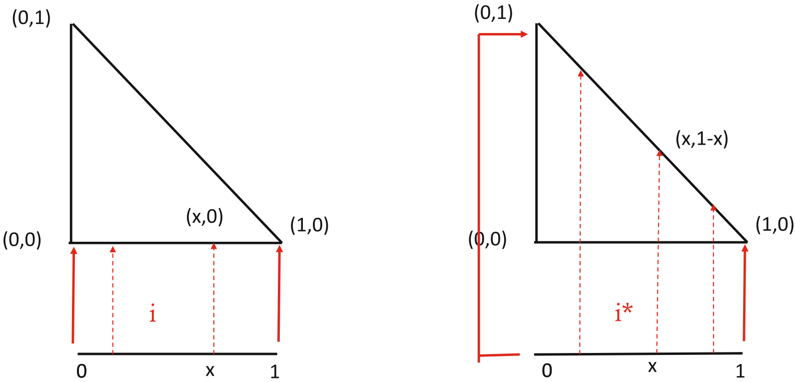

Fig. 2. Inclusions i and i^* of $([0,1], \leq)$ in $(A^*, \leq_{A^*})$

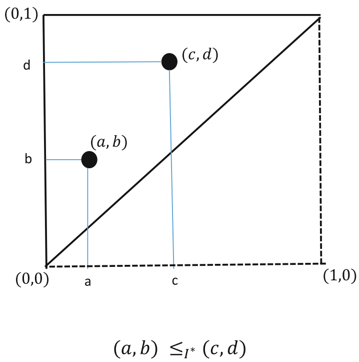

Fig. 3. Bounded partially ordered set $(I^*, \leq_{I^*})$ showing two ordered elements.

Figure 3 represents the set $(I^*, \leq_{I^*})$, and two ordered elements $(a, b) \leq_{I^*} (c, d)$.

Newly, we have that involutions in this ordered set are characterized [2,4]. Nevertheless, could we get a characterization of the involutions in $(I^*, \leq_{I^*})$, from the characterization of the involutions in $(A^*, \leq_{A^*})$?

For this, it is useful to note the existence of an order isomorphism between these two sets. In fact, the map $\varphi : (I^*, \leq_{I^*}) \to (A^*, \leq_{A^*})$, where $\varphi(a, b) = (a, 1 - b)$, is an order isomorphism [8]. Note that $\varphi^{-1} = \varphi$.

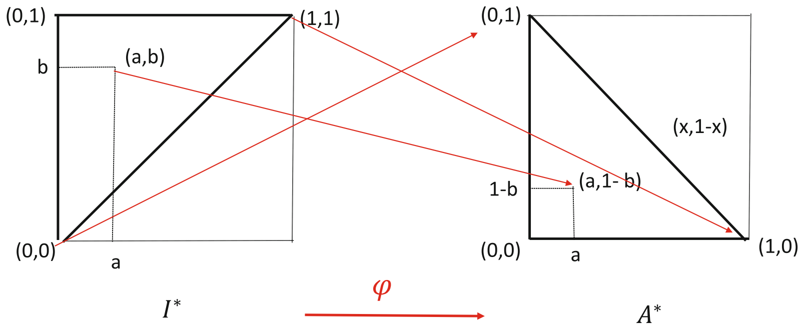

Fig. 4. Order Isomorphism between $(I^*, \leq_{I^*})$ and $(A^*, \leq_{A^*})$

Figure 4 represents the isomorphism φ, where the image of (a, b) is $(a, 1-b)$. Now, the proof of the following Lemma becomes easy.

Lemma 1. *Given an involution $\mathcal{N}$ in $(A^*, \leq_{A^*})$, and the function $\varphi : I^* \to A^*$, with $\varphi(a, b) = (a, 1-b)$, the map $\mathcal{N}^* : I^* \to I^*$, defined for all $(a, b) \in I^*$, by $\mathcal{N}^*(a, b) = \varphi^{-1}(\mathcal{N}(\varphi(a, b)))$ is an involution on $(I^*, \leq_{I^*})$.*

Proof.

$$\mathcal{N}^*(0,0) = \varphi^{-1}(\mathcal{N}(\varphi(0,0))) = \varphi^{-1}(\mathcal{N}(0,1)) = \varphi^{-1}(1,0) = (1,1)$$

$$\mathcal{N}^*(1,1) = \varphi^{-1}(\mathcal{N}(\varphi(1,1))) = \varphi^{-1}(\mathcal{N}(1,0)) = \varphi^{-1}(0,1)) = (0,0)$$

If $(a, b) \leq_{I^*} (c, d)$, we have that $\varphi(a, b) \leq_{A^*} \varphi(c, d)$; then $\mathcal{N}(\varphi(c, d)) \leq_{A^*} \mathcal{N}(\varphi(a, b))$, and $\varphi^{-1}(\mathcal{N}(\varphi(c, d))) \leq_{I^*} \varphi^{-1}(\mathcal{N}(\varphi(a, b)))$; so, we have that $\mathcal{N}^*(c, d) \leq_{I^*} \mathcal{N}^*(a, b)$.

Finally, for all $(a, b) \in I^*$, $\mathcal{N}^*(\mathcal{N}^*(a, b)) = \mathcal{N}^*(\varphi^{-1}(\mathcal{N}(\varphi(a, b))) = \varphi^{-1}(\mathcal{N}(\varphi(\varphi^{-1}(\mathcal{N}(\varphi(a, b)))))) = \varphi^{-1}(\mathcal{N}(\mathcal{N}(\varphi(a, b)))) = \varphi^{-1}(\varphi(a, b)) = (a, b)$.

That is, for each involution on $(A^*, \leq_{A^*})$ there is an involution on $(I^*, \leq_{I^*})$. Conversely, for each involution $\mathcal{N}^*$ on $(I^*, \leq_{I^*})$, it is easy to prove that there is an involution $\mathcal{N}$ on $(A^*, \leq_{A^*})$, defined as $\mathcal{N}(a, b) = \varphi(\mathcal{N}^*(\varphi^{-1}(a, b)))$.

As a consequence, it is possible to obtain the characterization of the involutions on $(I^*, \leq_{I^*})$.

In fact, $\mathcal{N}^*$ is an involution on $(I^*, \leq_{I^*})$ if and only if there exists an involution $\mathcal{N}$ on $(A^*, \leq_{A^*})$ such that for all $(a, b) \in I^*$, $\mathcal{N}^*(x, y) = \varphi^{-1}(\mathcal{N}(\varphi(x, y))) = \varphi^{-1}(\mathcal{N}(x, 1-y))$, and by the characterization of involutions on this set, if and only if there exists an involution n in $([0,1], \leq)$ such that $\mathcal{N}^*(x, y) = \varphi^{-1}(n(y), 1-n(x)) = (n(y), n(x))$.

4 Involutions on $(I^*, \leq_{I^*})$ and on $(I^2, \leq_{I^2})$

In this section, we introduce a new class of Goguen L-fuzzy sets. Let $(I^2, \leq_{I^2})$ be the bounded lattice $I^2 = [0,1]^2$, where $(a, b) \leq_{I^2} (c, d)$ if and only if $a \leq c$

and $b \leq d$. The minimum and maximum elements are, respectively, $0_{I^2} = (0,0)$ and $1_{I^2} = (1,1)$. This bounded lattice is used, for example, in measures with two fuzzy sets as arguments (see [5]).

In the previous section, we have obtained a characterization of the involutions on $(I^*, \leq_{I^*})$ from the characterization of involutions on $(A^*, \leq_{A^*})$. That has been possible, thanks to the order isomorphism between these two sets. That reasoning suggests us that, if we can find an order isomorphism between the sets $(A^*, \leq_{A^*})$ and $(I^2, \leq_{I^2})$, it would be easy to obtain a characterization of the involutions on the last one, independently of the characterization given in [5]. Nevertheless, the following theorem, shows that there is no a such isomorphism.

Theorem 1. *There is no order isomorphism between* $(I^*, \leq_{I^*})$ *and* $(I^2, \leq_{I^2})$.

Proof. If such an isomorphism $\varphi : I^* \to I^2$ exists, it would have to be $\varphi(0,0) = (0,0)$ and $\varphi(1,1) = (1,1)$.

What would be the image of $(0,1)$?

1. $\varphi(0,1) \neq (0,0)$ and $\varphi(0,1) \neq (1,1)$.
2. If $\varphi(0,1) = (a,b) \neq (1,0)$ and $(a,b) \neq (0,1)$ (see Fig. 5), there would be two non-comparable elements $(x,y), (x',y')$, satisfying at least one of the two following cases:
   (a) $(0,0) < (x,y), (x',y') < (a,b)$. In this case,
   $$\varphi^{-1}(0,0) = (0,0) < \varphi^{-1}(x,y)\ ,\ \varphi^{-1}(x',y') < (0,1).$$
   But then $\varphi^{-1}(x,y)$ and $\varphi^{-1}(x',y')$ are comparable, contradicting the fact that φ is an order isomorphism.
   (b) $(a,b) < (x,y), (x',y') < (1,1)$. Then
   $$\varphi^{-1}(a,b) = (0,1) < \varphi^{-1}(x,y)\ ,\ \varphi^{-1}(x',y') < \varphi^{-1}(1,1) = (1,1).$$
   Hence, $\varphi^{-1}(x,y)$ and $\varphi^{-1}(x',y')$ have to be comparable, again obtaining a contradiction.

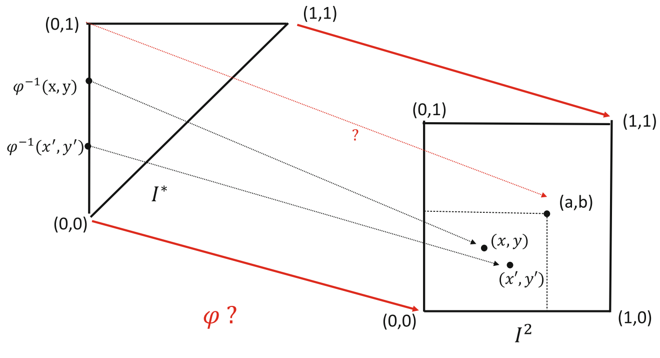

Fig. 5. It is not possible that $\varphi(0,1) \neq (1,0)$ and $\varphi(0,1) \neq (0,1)$.

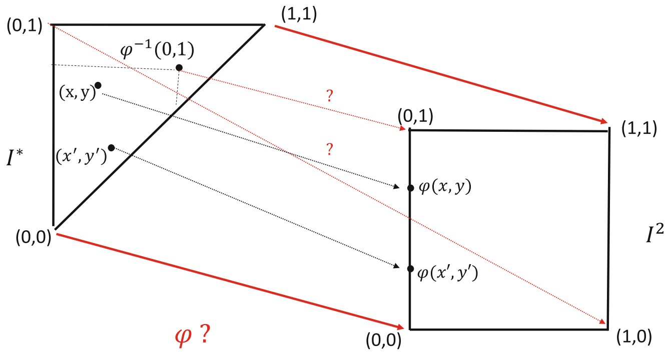

Fig. 6. It is not possible that $\varphi(0, 1) = (1, 0)$.

3. If $\varphi(0, 1) = (1, 0)$, what could be $\varphi^{-1}(0, 1)$? (see Fig. 6)

$$\varphi^{-1}(0,1) \neq (1,0) \text{ (as } (1,0) \notin I^*), \quad \varphi^{-1}(0,1) \neq (0,0),$$
$$\varphi^{-1}(0,1) \neq (1,1) \quad \text{and} \quad \varphi^{-1}(0,1) \neq (0,1).$$

   Then, there exist two non-comparable elements $(x, y), (x', y')$, satisfying at least one of the two following conditions:
   (a) $(0, 0) < (x, y), (x', y') < \varphi^{-1}(0, 1)$,
   (b) $\varphi^{-1}(0, 1) < (x, y), (x', y') < (1, 1)$.
   In a similar way to the reasoning of the previous point, we again obtain a contradiction.
4. Finally, if $\varphi(0, 1) = (0, 1)$, what could be the $\varphi^{-1}(1, 0)$? (see Fig. 7). As before, in this case we will attain a contradiction.

Then, there is no order isomorphism between $(I^*, \leq_{I^*})$ and $(I^2, \leq_{I^2})$.

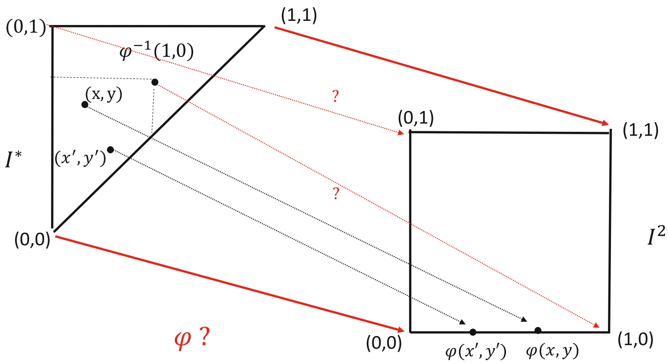

Fig. 7. It is not possible $\varphi(0, 1) = (0, 1)$

Nevertheless, is it possible to find a relationship between the involutions on $(I^*, \leq_{I^*})$ and the involutions on $(I^2, \leq_{I^2})$?

Lemma 2. *Every involution on* $(I^*, \leq_{I^*})$ *can be extended to an involution on* $(I^2, \leq_{I^2})$.

Proof. If $\mathcal{N}^* : I^* \to I^*$ is an involution, according to the previous Section, it can be expressed as $\mathcal{N}^*(a,b) = (n(b), n(a))$ for some involution n on $[0,1]$.

If we define $\mathfrak{N} : I^2 \to I^2$, as $\mathfrak{N}(a,b) = (n(b), n(a))$, it is easy to prove that $\mathfrak{N}$ is an involution on $(I^2, \leq_{I^2})$ (see [5]), and it trivially is an extension of $\mathcal{N}^*$, being the trivial inclusion $i : I^* \to I^2$ such that $i(a,b) = (a,b)$.

Now we can address the reverse problem: What involutions on I^2, are such that their restriction to I^* is also an involution?

Lemma 3. *The only involutions on* $(I^2, \leq_{I^2})$ *whose restrictions to* $(I^* \leq_{I^*})$ *are also involutions, are those of the form* $\mathfrak{N}(a,b) = (n(b), n(a))$*, with* n *involution on* $[0,1]$.

Proof. Let us remember that if $\mathfrak{N}$ is an involution on $(I^2, \leq_{I^2})$ it should be [5] either:

1. $\mathfrak{N}(a,b) = (n_1(a), n_2(b))$, with n_1 and n_2 involutions on $[0,1]$, or
2. $\mathfrak{N}(a,b) = (n_1(b), n_1^{-1}(a))$, with n_1 bijective negation on $[0,1]$.

Let's now consider both cases.

1. If $\mathfrak{N}(a,b) = (n_1(a), n_2(b))$, in order that its restriction to I^*, $\mathcal{N}^*$, be an involution, for all $(a,b) \in I^2$, with $a \leq b$ (that is, $(a,b) \in I^*$), it should be $\mathfrak{N}(a,b) = (n(b), n(a))$, for some involution n. But, in this case, for all $x \in [0,1]$, it must be
$$\mathcal{N}^*(x,x) = (n_1(x), n_2(x)) = (n(x), n(x))$$
Hence, for all $x \in [0,1]$, it is $n_1(x) = n(x) = n_2(x)$, inferring that $n_1 = n = n_2$, and $\mathfrak{N}(a,b) = (n(a), n(b))$. But if $a < b$, $(a,b) \in I^*$, $n(b) < n(a)$ and $\mathfrak{N}(a,b) = (n(a), n(b)) \notin I^*$. Then we attain a contradiction.
2. If $\mathfrak{N}(a,b) = (n_1(b), n_1^{-1}(a))$, for all $a \leq b$, $\mathcal{N}^*(a,b) = (n(b), n(a))$ for some involution n.
Then for all $x \in [0,1]$, $\mathcal{N}^*(x,x) = (n_1(x), n_1^{-1}(x)) = (n(x), n(x))$, and so $n_1(x) = n(x) = n_1^{-1}(x)$, n is an involution and $\mathfrak{N}(a,b) = (n(b), n(a))$.

Then if $\mathfrak{N}$ is an involution on I^2, whose restriction to I^* is also an involution, it should be $\mathfrak{N}(a,b) = (n(b), n(a))$, for some n involution on $[0,1]$.

5 Involutions on $([0,1], \leq)$ and on $(I^2, \leq_{I^2})$

Newly, we address the task of obtaining a relationship between the involutions on $([0,1], \leq)$ and on $(I^2, \leq_{I^2})$. To do this, we define an inclusion $i : [0,1] \to I^2$, where $i(x) = (x, x)$.

If n is an involution on $[0,1]$, an extension $\mathfrak{N}$ to $(I^2, \leq_{I^2})$ should satisfy $\mathfrak{N}(i(x)) = i(n(x)) = (n(x), n(x))$.

What involutions $\mathfrak{N}$ on $(I^2, \leq_{I^2})$ are extensions of involutions on $([0,1], \leq)$?

Lemma 4. *The only involutions on $(I^2, \leq_{I^2})$ whose restrictions to $([0,1], \leq)$ are also involutions, are those of the form $\mathfrak{N}(a,b) = (n(b), n(a))$, or the form $\mathfrak{N}(a,b) = (n(a), n(b))$, with n involution on $[0,1]$.*

Proof. 1. If $\mathfrak{N}(a,b) = (n_1(a), n_2(b))$, with n_1 and n_2 involutions on $[0,1]$, we have $\mathfrak{N}(i(x)) = \mathfrak{N}(x,x) = (n_1(x), n_2(x)) = (n(x), n(x)) = i(n(x))$, and then $n_1(x) = n(x) = n_2(x)$, for all $x \in [0,1]$. Hence, the involutions n_1 and n_2 must be the same.

2. If $\mathfrak{N}(a,b) = (n_1(b), n_1^{-1}(a))$, with n_1 bijective negation on $[0,1]$, we have $\mathfrak{N}(i(x)) = \mathfrak{N}(x,x) = (n_1(x), n_1^{-1}(x)) = (n(x), n(x)) = i(n(x))$, and so $n_1(x) = n(x) = n_1^{-1}(x)$, for all $x \in [0,1]$. Hence, the negation n_1 must be an involution.

Then if an involution on $(I^2, \leq_{I^2})$ is such that its restriction to $([0,1], \leq)$ is also an involution, it should be $\mathfrak{N}(a,b) = (n(b), n(a))$ or $\mathfrak{N}(a,b) = (n(a), n(b))$, for some n involution on $[0,1]$.

6 Conclusions

This paper has been devoted to relate the involutions in some different bounded lattices. In particular, the lattices $([0,1], \leq)$, $(A^*, \leq_{A^*})$, $(I^*, \leq_{I^*})$, and $(I^2, \leq_{I^2})$, have been considered.

In a preliminary step we have reminded the definitions of involution, order homomorphism, order isomorphism and extension of a function, as tools to be used in the different discussions.

From that point we have shown that:

1. It is possible to extend any involution in $([0,1], \leq)$ to an involution in $(A^*, \leq_{A^*})$.
2. It is possible to characterize involutions in $(I^*, \leq_{I^*})$ from the characterization of the involutions in $(A^*, \leq_{A^*})$. To do so we have previously remembered an order isomorphism between the two sets.
3. There is no order isomorphism between $(I^*, \leq_{I^*})$ and $(I^2, \leq_{I^2})$.
   (a) Nevertheless, we have extended every involution on $(I^*, \leq_{I^*})$ to an involution on $(I^2, \leq_{I^2})$.
   (b) Moreover, we have determined the only involutions on $(I^2, \leq_{I^2})$ whose restriction to $(I^*, \leq_{I^*})$ is also an involution.

4. Finally, we have found the involutions on $(I^2, \leq_{I^2})$ whose restrictions to $([0,1], \leq)$ are also involutions.

In future research, new bounded lattices, sets of membership degrees of other L-fuzzy sets, will be considered, establishing relationships between the corresponding involutions.

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