Application and Comparison of CC-Integrals in Business Group Decision Making

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Abstract. Optimized decisions is required by businesses (analysts) if they want to stay open. Even thought some of these are from the know-how of the managers/executives, most of them can be described mathematically and solved (semi)-optimally by computers. The Group Modular Choquet Random Technique for Order of Preference by Similarity to Ideal Solution (GMC-RTOPSIS) is a Multi-Criteria Decision Making (MCDM) that was developed as a method to optimize the later types of problems, by being able to work with multiple heterogeneous data types and interaction among different criteria. On the other hand the Choquet integral is widely used in various fields, such as brain-computer interfaces and classification problems. With the introduction of the CC-integrals, this study presents the GMC-RTOPSIS method with CC-integrals. We applied 30 different CC-integrals in the method and analyzed its results using 3 different methods. We found that by modifying the decision-making method we allow for more flexibility and certainty in the choosing process.

Keywords: CC-integral · Decision making · Generalized choquet integral · GMC-RTOPSIS

1 Introduction

Business managers rely on the right decisions to keep their business competitive. Many times a decision has to be made by multiple analysts and consid-

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ering various criteria. This is a time consuming and expensive task. Although, most of the time, it can be solved by an algorithm or mathematical model, like route, supplier chain, and location problems [1,7,24], releasing the pressure of the decision from the managers, and allow them to work on other processes of the company/industry.

The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [12] is one of the multi-criteria decision making (MCDM) methods that ranks the best possible solution among a set of alternatives. This approach is based on pre-defined criteria, using the alternative's distance to the best and worst possible solutions for the problems, Positive and Negative Ideal Solutions (PIS and NIS), respectively.

In 2017, the Group Modular Choquet Random TOPSIS (GMC-RTOPSIS) [15] was introduced. The method generalized the original TOPSIS allowing it to deal with multiple and heterogeneous data types. The approach models the interaction among the criteria by using the discrete Choquet integral [6]. The Choquet integral allows a function to be integrated by using non-additive fuzzy measures [5,6], which means that it can consider the interaction among the elements that are being integrated [9,21]. The GMC-RTOPSIS learns the fuzzy measure associated with the criteria with a Particle Swarm Optimization (PSO) algorithm [26].

The C_T-integrals [19] is a generalization of the Choquet integral that replaces the product operation by triangular norm (t-norm) functions [13]. The C_T-integrals are a family of integrals that are pre-aggregation functions [19]. Additionally, C_T-integrals are averaging functions, i.e., the result is always between the minimum and maximum of the input.

The T-separation measure [27] was introduced and applied in the GMC-RTOPSIS instead of the Choquet integral. In this study, the authors considered five different T-separation measures to tackle Case Study 2 from [15]. The problem consists of choosing a new supplier for a company by asking various decision-makers to give their opinions with different criteria. The problem is posed with a variety of data types, such as probability distributions, fuzzy numbers, and interval numbers. The paper also proposed to use the t-norm that better discriminates the first ranked alternative to the second one by calculating the difference of the rankings. The approach presented good results when using the Łukasiewicz t-norm ($T_{\text{Ł}}$), giving a better separation between the ranked alternatives than the standard Choquet integral.

After introducing the C_T-integrals, Lucca et al. have proposed the CC-integrals [18]. CC-integrals are a generalization of the Choquet integral in its expanded form, satisfying some properties, such as averaging, idempotency, and aggregation [11]. The authors applied the CC-integral in classification problems, showing that the function based on the minimum is the one that produced the highest performance of the classifier. The CC-integrals have been studied in the literature by Dimuro et al., where the properties of CMin integrals [10,16,20] were analyzed.

In this paper, we expand the analysis of the CC-separation measure study [28] by increasing the number of CC-integrals analyzed, elevating the 11 from the previews article to 30 in this one. We, again, apply the CC-integrals in

an application as an example, the same used in [15,27,28]. To better visualize the analysis by using the $\Delta_{R1,R2}$ difference we plotted it for each of the 30 different CC-integrals. Thereafter, in addition to using the $\Delta_{R1,R2}$ difference, we also analyze the results using the mode functions to find the alternative which most appears as first in the ranks. Finally, we introduce a new way to compare the ranks produced by different copula functions by using a mix of the $\Delta_{R1,R2}$ difference and the mode function.

The paper is organized as follows: Sect. 2 introduces the basic concepts about the fuzzy set theory and TOPSIS decision making, in addition to reviewing the definition of CC-separation measure. In Sect. 3 we detail our experiment, the required definitions of the decision-making problem and also introduce an alternative approach to compare the results from different CC-integrals. Lastly, the conclusion is in Sect. 4.

2 Background Theory

In this section, we recall the preliminary concepts necessary to develop the paper.

2.1 Fuzzy Set Theory

A Fuzzy Set [29] is defined on a universe X by a membership function $\mu_a : X \to [0,1]$, denoted by

$$a = \{\langle x,\ \mu_a(x)\rangle \ \mid x \in X\}.$$

We call a trapezoidal fuzzy number (**TFN**) the fuzzy set denoted by $a = (a_1, a_2, a_3, a_4)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, if the membership function μ_a is defined on $\mathbb{R}$ as:

$$\mu_a(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leq x < a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise.} \end{cases}$$

A measure of the distance between two TFNs $a = (a_1,\ a_2,\ a_3,\ a_4)$ and $b = (b_1,\ b_2,\ b_3,\ b_4)$ is defined as:

$$d(a,b) = \sqrt{\frac{1}{4}\sum_{i=1}^{4}(a_i - b_i)^2}.$$

The defuzzified value of a TFN $a = (a_1, a_2, a_3, a_4)$ is given by:

$$m(a) = \frac{a_1 + a_2 + a_3 + a_4}{4}.$$

An intuitionistic fuzzy set (**IFS**) A is defined on a universe X by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ such that $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, that is:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}.$$

Let $\tilde{\mu}_A$ and $\tilde{\nu}_A$ be the maximum membership degree and the minimum non-membership degree, respectively, of an IFS A.

An IFS A is an intuitionistic trapezoidal fuzzy number (**ITFN**), denoted by

$$A = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_A, \tilde{\nu}_A \rangle$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$, if μ_A and v_A are given, for all $x \in \mathbb{R}$, by

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}\tilde{\mu}_A, & \text{if } a_1 \leq x < a_2 \\ \tilde{\mu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}\tilde{\mu}_A, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \frac{1-\tilde{\nu}_A}{a_1-a_2}(x-a_1)+1, & \text{if } a_1 \leq x < a_2 \\ \tilde{\nu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{1-\tilde{\nu}_A}{a_4-a_3}(x-a_4)+1, & \text{if } a_3 < x \leq a_4 \\ 1, & \text{otherwise.} \end{cases}$$

The distance between two ITFNs $A = \langle (a_1,\ a_2,\ a_3,\ a_4),\ \tilde{\mu}_A,\ \tilde{\nu}_A \rangle$ and $B = \langle (b_1,\ b_2,\ b_3,\ b_4),\ \tilde{\mu}_B,\ \tilde{\nu}_B \rangle$ is:

$$d(A,B) = \frac{1}{2}\left[d_{\tilde{\mu}}(A,B) + d_{\tilde{\nu}}(A,B)\right]$$

where

$$d_\kappa(A,B) = \left\{ \frac{1}{4}\Big[(a_1-b_1)^2 + (1+(\kappa_A-\kappa_B)^2) \right.$$
$$(1+(a_2-b_2)^2 + (a_3-b_3)^2)$$
$$\left. -1+(a_4-b_4)^2 \Big] \right\}^{1/2}$$

for $\kappa_A = \tilde{\mu}_A$ and $\kappa_B = \tilde{\mu}_B$ when $\kappa = \mu$; and for $\kappa_A = \tilde{\nu}_A$ and $\kappa_B = \tilde{\nu}_B$ when $\kappa = \nu$.

Aggregation functions (**AF**) [11] are used to unify inputs into a single value representing them all and are defined as a function that maps $n > 1$ arguments onto the unit interval, that is, a function $f : [0,1]^n \to [0,1]$ such that the boundaries, $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$, with $\mathbf{0}, \mathbf{1} \in [0,1]^n$, and the monotonicity properties, $\boldsymbol{x} \leq \boldsymbol{y} \implies f(\boldsymbol{x}) \leq f(\boldsymbol{y})$, $\forall \boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, hold.

A triangular norm (t-norm) is an aggregation function $T : [0,1]^2 \to [0,1]$ that satisfies, for any $x, y, z \in [0,1]$: the commutative and associative properties and the boundary condition.

An overlap function [3] $O : [0,1]^2 \to [0,1]$ is a function that satisfies the following conditions:

- O is commutative;
- $O(x, y) = 0 \iff xy = 0$;
- $O(x, y) = 1 \iff xy = 1$;
- O is increasing;
- O is continuous.

A bivariate function $Co : [0, 1]^2 \to [0, 1]$ is called a copula [22] if, for all $x, x', y, y' \in [0, 1]$ with $x \leq x'$ and $y \leq y'$, the following conditions hold:

- $Co(x, y) + Co(x', y') \geq Co(x, y') + Co(x', y)$;
- $Co(x, 0) = Co(0, x) = 0$;
- $Co(x, 1) = Co(1, x) = x$.

The Choquet integral is defined based on a fuzzy measure [25], that is, a function m from the power set of N to the unit interval, $m : 2^N \to [0, 1]$, that for all $X, Y \subset N$ holds the conditions:

(1) $m(\emptyset) = 0$ and $m(N) = 1$;
(2) if $X \subset Y$, then $m(X) \leq m(Y)$.

From this, Choquet defined the integral as: Let m be a fuzzy measure. The Choquet integral [6] of $\boldsymbol{x} \in [0, 1]^n$ with respect to m is defined as:

$$\begin{aligned} \mathfrak{C}_m : [0, 1]^n &\to [0, 1] \\ \boldsymbol{x} &\to \sum_{i=1}^{n} \left(x_{(i)} - x_{(i-1)}\right)\ m(A_{(i)}) \end{aligned}$$

where (i) is a permutation on 2^N such that $x_{(i-1)} \leq x_{(i)}$ for all $i = 1, \ldots, n$, with $x_{(0)} = 0$ and $A_{(i)} = \{(1), \ldots, (i)\}$.

Notice that one can use the distributive law to expand the Choquet integral into:

$$\mathfrak{C}_m = \sum_{i=1}^{n} \left(x_{(i)} m(A_{(i)}) - x_{(i-1)} m(A_{(i)})\right) \tag{1}$$

Recently, the Choquet integral was generalized by copula functions. By substituting the product operator by copulas in the expanded form of the Choquet integral (Eq. 1), CC-Integrals [18] were introduced.

Let m be a fuzzy measure and Co be a bivariate copula. The Choquet-like integral based on copula with respect to m is defined as a function $\mathfrak{C}_m^{Co} : [0, 1]^n \to [0, 1]$, for all $\boldsymbol{x} \in [0, 1]^n$, by

$$\mathfrak{C}_m^{Co} = \sum_{i=1}^{n} Co\left(x_{(i)},\ m(A_{(i)})\right) - Co\left(x_{(i-1)},\ m(A_{(i)})\right) \tag{2}$$

where (i), $x_{(i)}$ and $A_{(i)}$ is defined as the Choquet integral.

It is important to note that the Choquet integral, the C_T-integrals, and the CC-integrals are averaging functions, i.e., the results from them are always bounded by the minimum and maximum of their input.

2.2 Decision Making

The GMC-RTOPSIS [15] is a decision making algorithm that improved the classic TOPSIS [12] by allowing groups of decision-makers, modularity in the input, multiple input types and, by using the Choquet integral, the ability to measure the interaction among different criteria.

Figure 1 shows an overview of the decision making process with the Choquet integral. Here three different decision-makers give their ratings for three products based on three criteria. These ratings are then processed and inserted in the Choquet integral, where the interaction between the criteria is calculated. After, the results are ranked according to their highest classiness coefficient value.

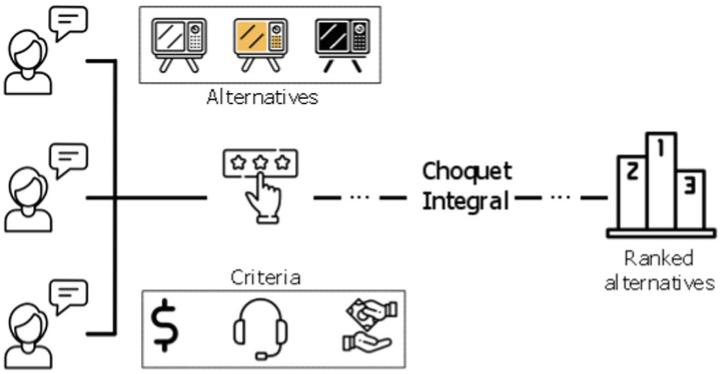

Fig. 1. Image description of the decision making process using the Choquet integral. Source: the authors [28].

To describe the GMC-RTOPSIS method let q represent the q-th decision maker in a collection of $Q \in \mathbb{N} = \{1, 2, 3, \ldots\}$ ones. Let $\boldsymbol{A} = \{A_1, \ldots, A_m\}$ be the set of alternatives for the problem and $\boldsymbol{C}_q = \{C_1, \ldots, C_{n_q}\}$ represent the criteria set for decision maker q. With $\boldsymbol{C} = \{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_Q\} = \{C_1, \ldots, C_n\}$, where $n = \sum_{q=1}^{Q} n_q$, representing the criteria set of all the decision makers. From these notations we can represent each of the q-th decision maker by the matrix below (Eq. (3)), called decision matrix DM:

$$DM^q = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_{n_q} \end{array} \\ \begin{pmatrix} s^q_{11}(\boldsymbol{Y}^q) & s^q_{12}(\boldsymbol{Y}^q) & \cdots & s^q_{1n_q}(\boldsymbol{Y}^q) \\ s^q_{21}(\boldsymbol{Y}^q) & s^q_{22}(\boldsymbol{Y}^q) & \cdots & s^q_{2n_q}(\boldsymbol{Y}^q) \\ \vdots & \vdots & \ddots & \vdots \\ s^q_{m1}(\boldsymbol{Y}^q) & s^q_{m2}(\boldsymbol{Y}^q) & \cdots & s^q_{mn_q}(\boldsymbol{Y}^q) \end{pmatrix} \end{array} \tag{3}$$

Each matrix cell $s^q_{ij}(\boldsymbol{Y}^q)$, with $1 \leq i \leq m,\ 1 \leq j \leq n_q$, is called the rating of the criterion j for alternative i. Also, notice that the rating is a function of $\boldsymbol{Y} = (\boldsymbol{Y}_{rand},\ \boldsymbol{Y}_{det})$, which are factors that model random and deterministic

events. Random events are modeled by stochastic processes, and deterministic are events which are not random, like time, location or a parameter of a random event. A fixed value x of the deterministic vector is called a state, and the set of all states is represented by $\mathcal{X}$.

In possession of all decision matrices from all decision-makers Q, the algorithm can be applied. The process is quite similar to the original TOPSIS, presented in 1981. It uses the same definition of Positive Ideal Solution (**PIS**) and Negative Ideal Solution (**NIS**) that are, respectively, the one that is closer to the best possible solution and the one that is distant from the best possible solution, see Eq. (4). The most significant difference is that each criterion may use a different distance measure since each may have its own type. So, the distances of each criterion are calculated separately and aggregated afterward in the separation measure step of the algorithm (see Fig. 2).

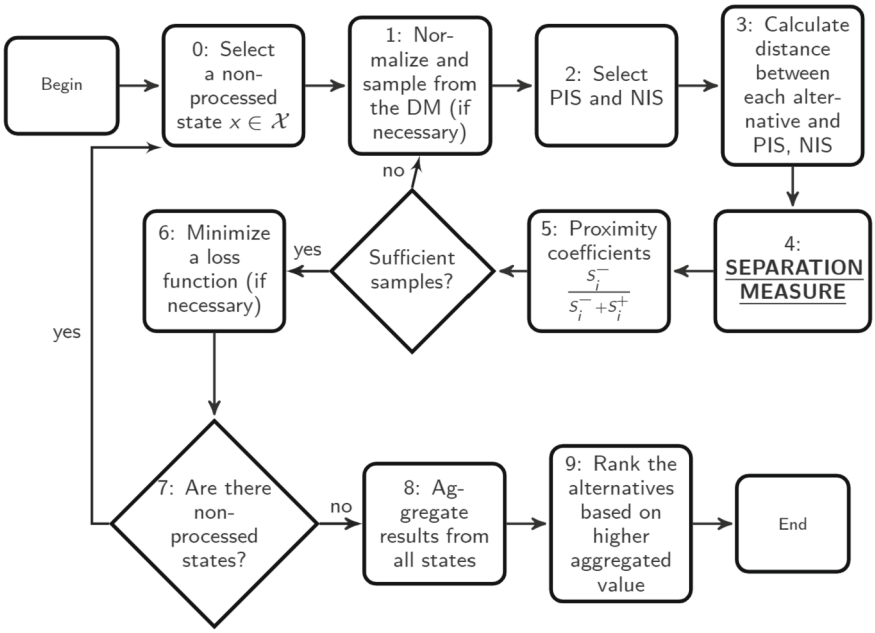

Fig. 2. Diagram of the GMC-RTOPSIS process. The separation measure step is where the CC-separation measure is used. Source: The authors [28].

In order to ease the comprehension of our approach, we present in Fig. 2 the steps of the GMC-RTOPSIS, where:

Step 0. Select a state $x \in \mathcal{X}$ not yet processed;
Step 1. Normalize all matrices;
Step 2. Select the PIS, denoted by $s_j^+(\boldsymbol{Y})$, and the NIS, denoted by $s_j^-(\boldsymbol{Y})$, considering, for each $j \in \{1, \ldots, n\}$, respectively:

$$s_j^+(\boldsymbol{Y}) = \begin{cases} \max\limits_{1\leq i\leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \min\limits_{1\leq i\leq m} s_{ij}, & \text{if it is a cost/loss criterion,} \end{cases}$$
$$s_j^-(\boldsymbol{Y}) = \begin{cases} \min\limits_{1\leq i\leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \max\limits_{1\leq i\leq m} s_{ij}, & \text{if it is a cost/loss criterion;} \end{cases} \tag{4}$$

Step 3. Calculate the distance measure for each criterion C_j, with $j \in \{1, \ldots, n\}$, to the PIS and NIS solutions, that is,

$$d_{ij}^+ = d(s_j^+(\boldsymbol{Y}), s_{ij}(\boldsymbol{Y})),$$
$$d_{ij}^- = d(s_j^-(\boldsymbol{Y}), s_{ij}(\boldsymbol{Y})),$$

where $i \in \{1, \ldots, m\}$ and d is a distance measure associated with the criteria data type;
Step 4. Calculate the separation measure, for each $i \in \{1, \ldots, m\}$, using the Choquet integral as follows:

$$S_i^+(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} \left(\left(d_{i(j)}^+\right)^2 - \left(d_{i(j-1)}^+\right)^2 \right) m_{\boldsymbol{Y}}(\boldsymbol{C}_{(j)}^+)}$$

$$S_i^-(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} \left(\left(d_{i(j)}^-\right)^2 - \left(d_{i(j-1)}^-\right)^2 \right) m_{\boldsymbol{Y}}(\boldsymbol{C}_{(j)}^-)}$$

where $d_{i(1)}^+ \leq \ldots \leq d_{i(n)}^+$, $d_{i(1)}^- \leq \ldots \leq d_{i(n)}^-$, for each $j \in \{1, \ldots, n\}$, $C_{(j)}^+$ is the criterion correspondent to $d_{i(j)}^+$, $C_{(j)}^-$ is the criterion correspondent to $d_{i(j)}^-$, $\boldsymbol{C}_{(j)}^+ = \{C_{(j)}^+, C_{(j+1)}^+, \ldots, C_{(n)}^+\}$, $\boldsymbol{C}_{(j)}^- = \{C_{(j)}^-, C_{(j+1)}^-, \ldots, C_{(n)}^-\}$, $\boldsymbol{C}_{(n+1)}^+ = \boldsymbol{C}_{(n+1)}^- = \emptyset$, $d_{i(0)}^+ = d_{i(0)}^- = 0$ and m_Y is the learned fuzzy measure by a particle swarm optimization algorithm [26].
Here, the separation measure is the square root of the Choquet integral of squared distances, and this means that it is the square root of a d-Choquet integral [4]. Also, for each state, we may have a different fuzzy measure, which means that the fuzzy measure is dependent on $\boldsymbol{Y}_{det}$
Step 5. For each $i \in \{1, \ldots, m\}$, calculate the relative closeness coefficient to the ideal solution with:

$$CC_i(\boldsymbol{Y}) = \frac{S_i^-(\boldsymbol{Y})}{S_i^-(\boldsymbol{Y}) + S_i^+(\boldsymbol{Y})};$$

Step 6. By using probability distributions in the DM, it is introduced a bootstrapped probability distribution in the CC_i values, so as a point representation for this distribution we minimize a pre-defined risk function:

$$cc_i = \arg\min_c R(c)$$
$$= \arg\min_c \int_{\mathbb{R}} L(c,\ CC_i(\boldsymbol{Y}))\ dF(CC_i(\boldsymbol{Y})); \tag{5}$$

Step 7. If there is at least one non-processed state x, return to Step 0;
Step 8. Aggregate the cc_i values from all the states with $\widehat{cc_i} = f_{x\in\mathcal{X}}(cc_i(x))$, where f is an aggregation function.
Step 9. Finally, rank the alternatives from the highest to the lowest $\widehat{cc_i}$ values.

2.3 Generalization of the GMC-RTOPSIS by Using CC-Integrals

Using the Choquet integral in the separation measure, the GMC-RTOPSIS method allows for interaction among different criteria. This is the step where this study incorporates the CC-integrals in place of the Choquet integral.

We remind the definition of the CC-separation measure by:

Definition 1 (CC-separation measure [28]). *Let Co be a bivariate copula and m a fuzzy measure. A CC-separation measure $S^*:[0,1]^2\to[0,1]$ is defined, for all $i\in\{1,\ldots,m\}$, by the functions:*

$$S_i^+(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} Co\left(\left(d_{i(j)}^+\right)^2,\ m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^+\right)\right) - Co\left(\left(d_{i(j-1)}^+\right)^2,\ m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^+\right)\right)}$$

$$S_i^-(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} Co\left(\left(d_{i(j)}^-\right)^2,\ m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^-\right)\right) - Co\left(\left(d_{i(j-1)}^-\right)^2,\ m_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^-\right)\right)}$$

where $d_{i(j)}^+$, $d_{i(j)}^-$, $\boldsymbol{C}_{(j)}^+$, $\boldsymbol{C}_{(j)}^-$ and $m_{\boldsymbol{Y}}$ are defined as in Step 4 of the GMC-RTOPSIS algorithm. Note that the separation measure is the squared root of the CC-integral, which is an aggregation function as shown in [18].

3 Experiments

In this section, we present the application of the CC-separations in the GMC-RTOPSIS. To do so, we start describing the methodology adopted in the study; after that, the example in which we apply our approach is described, and lastly, the obtained results are presented and discussed.

3.1 Methodology

In this study, we will apply the proposed CC-separation measure to the Case Study 2 introduced in [15] and used in [27] to ease the comparison between the different CC-integrals.

To perform the simulation, we used 10,000 samples from the DM. We also applied a particle swarm optimization to learn the fuzzy measure using 30 particles and 100 interactions. The PSO is used since the original method had good outcomes with the method.

Table 1. Examples of Copulas [28].

(I) T-norms	
Definition	Name/Description
$T_M(x,y) = \min\{x,y\}$	Minimum
$T_P(x,y) = xy$	Algebraic Product
$T_L(x,y) = \max\{0, x+y-1\}$	Łukasiewicz
$T_{NM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y>1 \\ 0 & \text{otherwise} \end{cases}$	Nilpotent Minimum
$T_{HP}(x,y) = \begin{cases} 0 & \text{if } x=y=0 \\ \frac{xy}{x+y-xy} & \text{otherwise} \end{cases}$	Hamacher Product
(II) Non-associative overlap functions	
Definition	Reference/Description
$O_B(x,y) = \min\{x\sqrt{y}, y\sqrt{x}\}$	Cuadras-Augé family of copulas [22]
$O_{mM}(x,y) = \min\{x,y\}\max\{x^2,y^2\}$	[8,23]
$O_\alpha(x,y) = xy(1+\alpha(1-x)(1-y))$, where $\alpha \in [-1,0[\cup]0,1]$	[2,17]
(III) Non-associative copulas, which are neither t-norms nor overlap functions	
Definition	Reference/Description
$C_F(x,y) = xy + x^2y(1-x)(1-y)$	[13]
$C_L(x,y) = \max\{\min\{x, \frac{y}{2}\}, x+y-1\}$	[2]
$C_{Div}(x,y) = \frac{xy+\min\{x,y\}}{2}$	[2]

We highlight that we used 20 different values for the α parameter, varying it by 0.1 from -1.0 to 1.0 excluding 0.0, as the function O_α is not defined for this value.

For the risk function, given in Eq. (5), we used the squared loss (Table 1):

$$L(cc,\ CC_i) = (cc - CC_i)^2\ .$$

This results in the mean function being the point estimator for the process.

Also, we used the Weighted Arithmetic Mean aggregation function for Step 8 of the algorithm:

$$WAM_i = w(S_1) \cdot cc_i(S_1) + w(S_2) \cdot cc_i(S_2).$$

For the analysis of the results from the different copula functions, we use two different approaches. The first one is by using the Big Delta [28], defined bellow, to see which copula function gives the biggest difference between rankings first and second.

$$\Delta_{R1,R2} = \max(\hat{c}_1) - \max(\hat{c}_2)$$

where $\hat{c}_1 = \{\widehat{cc_i} \mid i \in \{1,\ldots,m\}\}$ and $\hat{c}_2 = \hat{c}_1 - \{\max(\hat{c}_1)\}$.

The latter is by using the mode function in the first of the ranks, which gives the most appeared alternative.

Lastly, it is important to notice that since we are only changing the Choquet function in the method, it maintains the original complexity described in Lourenzutti et al. [14].

3.2 The Decision-making Problem

This section describes the investigated problem to which we apply the GMC-RTOPSIS with the CC-integrals.

A company needs a new supplier for a provision and is evaluating four different suppliers, namely A_1, A_2, A_3 and A_4. The company called three of its managers to analyze the suppliers and give their ratings based on their criteria.

The first manager is a budget manager. He considered the price per batch (in thousands) as $C_1^{(1)}$, warranty (in days) as $C_2^{(1)}$ and payment conditions (in days) as $C_3^{(1)}$. Also, it was considered that the demand for the product is higher in December. He modeled it by using a binary variable τ, that is $\tau = 0$ when the month is between January and November, and $\tau = 1$ when it is December. Finally, he assigned a weight for each of his criterion with a weighting vector: $\boldsymbol{w}^{(1)} = (0.5,\ 0.25,\ 0.25)$.

The second manager, a product manager, considered the price as $C_1^{(2)}$, delivery time (in hours) as $C_2^{(2)}$, production capacity $C_3^{(2)}$, product quality $C_4^{(2)}$ and the time to respond to a support request (in hours) as $C_5^{(2)}$. Additionally, to account for the reliability in the production process and what a failure in the process could cause to the supplier's production capacity, he let P_i be a random variable such that $P_i = 0$ occurs when there are no failures in the production process of the supplier A_i, and $P_i = 1$ when there are failures. Also, in December, the production is accelerated, so the chance of failure is higher, so he modeled a stochastic process with the help of the function:

$$f_i(x, y) = x\big(1 + y(P_i + \tau)^2\big).$$

Lastly, the production capacity was modeled by using ITFNs:

$$\begin{aligned}
s_{13}^2 &= \big((0.8^{1+P_1}, 0.9^{1+P_1}, 1.0^{1+P_1}, 1.0^{1+P_1}),\ 1.0,\ 0.0\big)\\
s_{23}^2 &= \big((0.8^{1+4P_2}, 0.9^{1+4P_2}, 1.0^{1+4P_2}, 1.0^{1+4P_2}),\ 0.7,\ 0.1\big)\\
s_{33}^2 &= \big((0.6^{1+2P_3}, 0.7^{1+2P_3}, 0.8^{1+2P_3}, 1.0^{1+2P_3}),\ 0.8,\ 0.0\big)\\
s_{43}^2 &= \big((0.5^{1+3P_4}, 0.6^{1+3P_4}, 0.8^{1+3P_4}, 0.9^{1+3P_4}),\ 0.8,\ 0.1\big).
\end{aligned}$$

This manager selected the same weight for all criteria, i.e., $\boldsymbol{w}^{(2)} = (0.2,\ 0.2,\ 0.2,\ 0.2,\ 0.2)$.

The commercial manager was the third. He considered the product lifespan (in years) as $C_1^{(3)}$, social and environmental responsibility as $C_2^{(3)}$, the quantity of quality certifications as $C_3^{(3)}$and the price as $C_4^{(3)}$. The weighting vector provided by this manager is $\boldsymbol{w}^{(3)} = (0.25,\ 0.12,\ 0.23,\ 0.4)$.

The P_i distribution was determined by historical data of each supplier and it is given as follows:

Table 2. Decision matrices for the managers [28].

(a) Budget manager

Alternatives	$C_1^{(1)}$	$C_2^{(1)}$	$C_3^{(1)}$	
			$\tau = 0$	$\tau = 1$
A_1	$260.00(1 + 0.15\tau)$	90	G	G
A_2	$250.00(1 + 0.25\tau)$	90	P	W
A_3	$350.00(1 + 0.20\tau)$	180	G	I
A_4	$550.00(1 + 0.10\tau)$	365	I	W

(b) Production manager

Alternatives	$C_1^{(2)}$	$C_2^{(2)}$	$C_3^{(2)}$	$C_4^{(2)}$	$C_5^{(2)}$
A_1	260.00	U($f_1(48,\ 0.10)$, $f_1(96,\ 0.10)$)	s_{13}^2	I	[24, 48]
A_2	250.00	U($f_2(72,\ 0.20)$, $f_2(120,\ 0.20)$)	s_{23}^2	P	[24, 48]
A_3	350.00	U($f_3(36,\ 0.15)$, $f_3(72,\ 0.15)$)	s_{33}^2	G	[12, 36]
A_4	550.00	U($f_4(48,\ 0.25)$, $f_4(96,\ 0.25)$)	s_{34}^2	E	[0, 24]

(c) Commercial manager

Alternatives	$C_1^{(3)}$	$C_2^{(3)}$	$C_3^{(3)}$	$C_4^{(3)}$
A_1	Exp(3.5)	W	1	260.00
A_2	Exp(3.0)	W	0	250.00
A_3	Exp(4.5)	P	3	350.00
A_4	Exp(5.0)	I	5	550.00

Table 3. Linguistic variables and their respective trapezoidal fuzzy numbers [28].

Linguistic variables	Trapezoidal fuzzy numbers
Worst (W)	(0, 0, 0.2, 0.3)
Poor (P)	(0.2, 0.3, 0.4, 0.5)
Intermediate (I)	(0.4, 0.5, 0.6, 0.7)
Good (G)	(0.6, 0.7, 0.8, 1)
Excellent (E)	(0.8, 0.9, 1, 1)

For $\tau = 0$:

$$p(P_1 = 0|S_1) = 0.98,$$
$$p(P_2 = 0|S_1) = 0.96,$$
$$p(P_3 = 0|S_1) = 0.97,$$
$$p(P_4 = 0|S_1) = 0.95.$$

For $\tau = 1$:

$$p(P_1 = 0|S_2) = 0.96,$$
$$p(P_2 = 0|S_2) = 0.92,$$
$$p(P_3 = 0|S_2) = 0.96,$$
$$p(P_4 = 0|S_2) = 0.90.$$

Considering all the DMs, we have the following underlying factors: a random component $\boldsymbol{Y}_{rand} = (P_1, P_2, P_3, P_4)$ and a deterministic component $Y_{det} = \tau$ that has two states: S_1 when $\tau = 0$ and S_2 when $\tau = 1$. The underlying factors can be represented by $\boldsymbol{Y} = (\boldsymbol{Y}_{rand},\ \boldsymbol{Y}_{det})$. The managers agreed that the state S_2 was more important, since the production is higher, so they gave it a higher weight for it in the aggregation step (Step 8 of the method) by setting $w(S_1) = 0.4$ and $w(S_2) = 0.6$.

The DMs of all managers are presented in Table 2, where the linguistic variables (W, P, I, G and E) are defined as in Table 3.

The company, considering the opinion of manager 2 more important, assigned a weighting vector for the managers represented by $\boldsymbol{w} = (0.3,\ 0.4,\ 0.3)$. Furthermore, they wanted to include some interaction between the criteria, so a variation of 30% was allowed for each fuzzy measure in relation to the coefficient in the additive fuzzy measure. This measure is calculated computationally by means of the PSO algorithm [15,26].

3.3 Results

The aggregated ranked results are presented in Table 4 (mean and standard deviations shown in Table 5). The table shows for each copula function Co, the rank of alternatives from columns 2 to 5, with each alternative's aggregated value inside parenthesis. Column $\Delta_{R1,R2}$ shows the difference between the aggregate values between the alternative ranked first and the second.

To ease the comprehension of the results, we provide in Fig. 3, for each considered CC-integral, the difference between the first (A_3) and second (A_4) ranked alternative. Also, in that Figure, we sort the ranks from the biggest to the smallest values of $\Delta_{R1,R2}$. The functions are presented in the X axis, where the value adopted by the function is provided. The Y axis are the values related to the difference value. Finally, for each function, we provide the value of the difference above each line.

From Fig. 3, one can observe that the biggest difference is achieved by the Łukazievicz t-norm. On the other hand, the smallest difference is achieved by the O_α, with the parameter set as -1.

Our first analysis used the $\Delta_{R1,R2}$ as the criterion to choose which rank one should consider when using multiple CC-integrals. From that we can see that for the t-norms the values are proportional to the ones presented in the study that used C_T-integral instead of the Choquet integral [27]. As in that paper, here the $T_{\text{Ł}}$ t-norm has the biggest difference, with $\Delta_{R1,\ R2} = 0.0700$. Although the $T_{\text{Ł}}$ presented such a big difference, the other t-norms did not do so well. One can see that only the T_{MN} t-norm performs well compared with the copulas, such as O_α and C_F.

The second biggest difference was achieved by using the copula O_α with α parameter set to 0.6, with $\Delta_{R1,\ R2} = 0.0502$. The next of this family tested was the one with $\alpha = -0.2$, where it resulted in a quite lower difference value, with only $\Delta_{R1,\ R2} = 0.0425$. Among the other tested overlap functions from the α

Table 4. Rank of the alternatives with each of the Co, ordered by the biggest $\Delta_{R1,R2}$ value.

Co	Ranked 1st	Ranked 2nd	Ranked 3rd	Ranked 4th	$\Delta_{R1,R2}$
T_L	$A_3(0.6462)$	$A_4(0.5762)$	$A_1(0.4616)$	$A_2(0.3782)$	0.0700
$O_{0.6}$	$A_3(0.5897)$	$A_4(0.5395)$	$A_1(0.4716)$	$A_2(0.4282)$	0.0502
C_F	$A_3(0.5991)$	$A_4(0.5525)$	$A_1(0.4453)$	$A_2(0.4194)$	0.0466
$O_{-0.2}$	$A_3(0.6034)$	$A_4(0.5609)$	$A_1(0.4498)$	$A_2(0.4034)$	0.0425
T_{NM}	$A_3(0.5919)$	$A_4(0.5493)$	$A_1(0.4713)$	$A_2(0.3910)$	0.0425
$O_{0.3}$	$A_3(0.5959)$	$A_4(0.5574)$	$A_1(0.4584)$	$A_2(0.4104)$	0.0385
$O_{-0.4}$	$A_3(0.6002)$	$A_4(0.5621)$	$A_1(0.4441)$	$A_2(0.4012)$	0.0382
$O_{-0.3}$	$A_3(0.5989)$	$A_4(0.5616)$	$A_1(0.4436)$	$A_2(0.4059)$	0.0373
$O_{-0.8}$	$A_3(0.6089)$	$A_4(0.5735)$	$A_1(0.4368)$	$A_2(0.3936)$	0.0354
$O_{0.5}$	$A_3(0.5910)$	$A_4(0.5563)$	$A_1(0.4573)$	$A_2(0.4191)$	0.0347
$O_{0.8}$	$A_3(0.5847)$	$A_4(0.5502)$	$A_1(0.4546)$	$A_2(0.4082)$	0.0345
$O_{-0.9}$	$A_3(0.6096)$	$A_4(0.5752)$	$A_1(0.4252)$	$A_2(0.3930)$	0.0344
$O_{0.2}$	$A_3(0.5917)$	$A_4(0.5578)$	$A_1(0.4658)$	$A_2(0.4066)$	0.0339
$O_{-0.5}$	$A_3(0.6020)$	$A_4(0.5706)$	$A_1(0.4390)$	$A_2(0.3989)$	0.0314
$O_{0.1}$	$A_3(0.5953)$	$A_4(0.5659)$	$A_1(0.4453)$	$A_2(0.3962)$	0.0294
O_{mM}	$A_3(0.5995)$	$A_4(0.5715)$	$A_1(0.4454)$	$A_2(0.3927)$	0.0280
$O_{-0.6}$	$A_3(0.5960)$	$A_4(0.5695)$	$A_1(0.4344)$	$A_2(0.3955)$	0.0265
$O_{-0.1}$	$A_3(0.5934)$	$A_4(0.5676)$	$A_1(0.4545)$	$A_2(0.3990)$	0.0258
$O_{0.4}$	$A_3(0.5821)$	$A_4(0.5575)$	$A_1(0.4648)$	$A_2(0.4157)$	0.0246
C_{Div}	$A_4(0.5234)$	$A_3(0.5016)$	$A_1(0.4868)$	$A_2(0.4250)$	0.0218
$O_{0.9}$	$A_3(0.5775)$	$A_4(0.5558)$	$A_1(0.4498)$	$A_2(0.4039)$	0.0217
$O_{0.7}$	$A_3(0.5797)$	$A_4(0.5590)$	$A_1(0.4425)$	$A_2(0.4048)$	0.0207
$O_{-0.7}$	$A_3(0.5959)$	$A_4(0.5766)$	$A_1(0.4256)$	$A_2(0.3876)$	0.0193
$O_{1.0}$	$A_3(0.5764)$	$A_4(0.5578)$	$A_1(0.4370)$	$A_2(0.4061)$	0.0187
C_L	$A_4(0.5273)$	$A_3(0.5097)$	$A_1(0.4914)$	$A_2(0.4361)$	0.0176
T_{HP}	$A_3(0.5351)$	$A_4(0.5221)$	$A_1(0.5049)$	$A_2(0.4308)$	0.0131
T_P	$A_3(0.5821)$	$A_4(0.5701)$	$A_1(0.4346)$	$A_2(0.3977)$	0.0120
O_B	$A_3(0.5511)$	$A_4(0.5395)$	$A_1(0.4713)$	$A_2(0.4133)$	0.0116
T_M	$A_4(0.5229)$	$A_3(0.5118)$	$A_1(0.4737)$	$A_2(0.4386)$	0.0110
$O_{-1.0}$	$A_3(0.5980)$	$A_4(0.5894)$	$A_1(0.4234)$	$A_2(0.3765)$	0.0086

family the $\Delta_{R1,R2}$ differences ranged from as low as 0.0086 to as high as 0.0385, for $\alpha = -1.0$ and $\alpha = 0.3$ respectively.

The copula C_F had the third biggest $\Delta_{R1,R2}$, difference achieving 0.0466. On the other hand, the C_{Div} had less than half of the C_F difference with only $\Delta_{R1,R2} = 0.0218$. And lower was the C_L with a difference of $\Delta_{R1,R2} = 0.0176$.

Table 5. Mean and standard deviation of the alternatives for State 1 and State 2. The highest mean for each function and state is in **boldface** and the alternative with highest mean for the criterion has an asterisk*.

State	State 1 (S_1, $\tau = 0$)								State 2 (S_2, $\tau = 1$)							
A_i	A_1		A_2		A_3		A_4		A_1		A_2		A_3		A_4	
Co	Mean	Std.Dev	Mean	Std.Dev	Mean	Std.Dev	Mean	Std.Dev	Mean	Std.Dev	Mean	Std.Dev	Mean	Std.Dev	Mean	Std.Dev
C_{Div}	0.5100	0.0101	0.4654	0.0187	0.5139	0.0202	**0.5221**	0.0191	0.4713	0.0293	0.3980	0.0088	0.4934	0.0242	**0.5242**	0.0307
C_F	0.4774	0.0181	0.4604	0.0173	**0.6074**	0.0858	0.5270	0.0135	0.4239	0.0194	0.3921	0.0208	**0.5935**	0.0710	0.5695	0.0174
C_L	0.5251*	0.0082	0.4670	0.0144	**0.5335**	0.0058	0.5242	0.0126	0.4690	0.0427	0.4155	0.0165	0.4938	0.0186	**0.5293**	0.0314
$O_{1.0}$	0.4339	0.0356	0.4248	0.0151	**0.5802**	0.0584	0.5673	0.0155	0.4390	0.0293	0.3937	0.0139	**0.5739**	0.0539	0.5514	0.0263
$O_{0.9}$	0.4679	0.0126	0.4350	0.0100	**0.5900**	0.0592	0.5597	0.0088	0.4377	0.0297	0.3831	0.0156	**0.5691**	0.0544	0.5532	0.0269
$O_{0.8}$	0.4704	0.0130	0.4357	0.0112	**0.5882**	0.0664	0.5581	0.0100	0.4441	0.0383	0.3898	0.0186	**0.5824**	0.0544	0.5449	0.0337
$O_{0.7}$	0.4579	0.0203	0.4318	0.0113	**0.5938**	0.0625	0.5619	0.0111	0.4323	0.0382	0.3868	0.0156	**0.5703**	0.0564	0.5570	0.0329
$O_{0.6}$	0.4852	0.0134	0.4791*	0.0232	**0.5742**	0.0731	0.5078	0.0108	0.4625	0.0266	0.3943	0.0165	**0.6001**	0.0593	0.5607	0.0214
$O_{0.5}$	0.4856	0.0142	0.4466	0.0124	**0.5918**	0.0851	0.5455	0.0119	0.4385	0.0303	0.4008	0.0139	**0.5904**	0.0597	0.5635	0.0194
$O_{0.4}$	0.4788	0.0130	0.4427	0.0133	**0.5630**	0.0678	0.5493	0.0135	0.4554	0.0284	0.3977	0.0159	**0.5948**	0.0708	0.5629	0.0201
$O_{0.3}$	0.4688	0.0173	0.4379	0.0154	**0.5952**	0.0733	0.5518	0.0139	0.4515	0.0224	0.3921	0.0186	**0.5964**	0.0570	0.5611	0.0205
$O_{0.2}$	0.4686	0.0167	0.4292	0.0152	**0.5932**	0.0689	0.5630	0.0133	0.4639	0.0382	0.3916	0.0213	**0.5907**	0.0613	0.5543	0.0308
$O_{0.1}$	0.4568	0.0127	0.4282	0.0117	**0.6013**	0.0791	0.5656	0.0098	0.4377	0.0275	0.3749	0.0223	**0.5913**	0.0611	0.5661	0.0232
$O_{-0.1}$	0.4585	0.0152	0.4256	0.0121	**0.5993**	0.0672	0.5683	0.0103	0.4518	0.0210	0.3813	0.0162	**0.5895**	0.0678	0.5671	0.0188
$O_{-0.2}$	0.4721	0.0134	0.4311	0.0113	**0.6046**	0.0773	0.5631	0.0088	0.4349	0.0345	0.3849	0.0210	**0.6026**	0.0615	0.5595	0.0302
$O_{-0.3}$	0.4708	0.0142	0.4328	0.0157	**0.6056**	0.0738	0.5571	0.0145	0.4254	0.0353	0.3879	0.0184	**0.5944**	0.0611	0.5646	0.0318
$O_{-0.4}$	0.4616	0.0145	0.4256	0.0158	**0.6069**	0.0787	0.5649	0.0144	0.4325	0.0411	0.3850	0.0167	**0.5958**	0.0665	0.5602	0.0360
$O_{-0.5}$	0.4637	0.0150	0.4228	0.0138	**0.6218**	0.0731	0.5702	0.0121	0.4225	0.0353	0.3830	0.0139	**0.5888**	0.0662	0.5709	0.0318
$O_{-0.6}$	0.4564	0.0131	0.4228	0.0138	**0.6108**	0.0865	0.5686	0.0117	0.4197	0.0408	0.3773	0.0159	**0.5862**	0.0679	0.5701	0.0362
$O_{-0.7}$	0.4291	0.0111	0.4121	0.0135	**0.5998**	0.0743	0.5796	0.0127	0.4232	0.0298	0.3713	0.0172	**0.5933**	0.0635	0.5746	0.0272
$O_{-0.8}$	0.4546	0.0148	0.4174	0.0151	**0.6272**	0.0767	0.5739	0.0138	0.4250	0.0379	0.3777	0.0155	**0.5967**	0.0720	0.5733	0.0322
$O_{-0.9}$	0.4347	0.0115	0.4183	0.0153	**0.6194**	0.0818	0.5713	0.0158	0.4189	0.0308	0.3762	0.0124	**0.6030**	0.0710	0.5778	0.0273
$O_{-1.0}$	0.4290	0.0108	0.4030	0.0101	**0.6123**	0.0716	0.5921*	0.0079	0.4196	0.0354	0.3588	0.0193	**0.5885**	0.0664	0.5876	0.0281
O_B	0.4843	0.0148	0.4443	0.0138	**0.5584**	0.0422	0.5487	0.0101	0.4626	0.0360	0.3926	0.0156	**0.5462**	0.0388	0.5334	0.0306
O_{mM}	0.4519	0.0134	0.4218	0.0171	**0.6119**	0.0772	0.5668	0.0156	0.4411	0.0308	0.3733	0.0150	**0.5912**	0.0789	0.5746	0.0271
T_{HP}	0.4976	0.0083	0.4648	0.0141	**0.5328**	0.0169	0.5253	0.0125	0.5097*	0.0198	0.4081	0.0114	**0.5367**	0.0207	0.5199	0.0157
T_L	0.4702	0.0482	0.4360	0.0116	**0.6438***	0.0367	0.5588	0.0119	0.4558	0.0506	0.3397	0.0334	**0.6478***	0.0477	0.5878*	0.0250
T_M	0.4701	0.0478	0.4615	0.0217	**0.5326**	0.0018	0.5279	0.0231	0.4761	0.0270	0.4234*	0.0232	0.4980	0.0087	**0.5195**	0.0269
T_P	0.4567	0.0120	0.4270	0.0093	**0.5976**	0.0655	0.5674	0.0098	0.4198	0.0396	0.3782	0.0190	0.5718	0.0605	**0.5719**	0.0360

Additionally, one can see that the T_P t-norm resulted in one of the smallest $\Delta_{R1,R2}$ differences. This may consequently introduce a doubt on which of the alternatives is the better one, since their aggregated values are close. Moreover, notice that when using C_{div}, C_L and T_M copulas the alternatives A_3 and A_4 change position. This is from the influence of the state 2 result, where these functions may have weighted higher criteria for alternative A_4. Furthermore, the relative small difference $\Delta_{R1,R2}$ make the top of the rank prone to invert positions.

Last, it is observable in the obtained results that the copulas T_{HP} , T_P, O_B and T_M obtained a similar performance in the lowest part of the table, with the smallest separations.

Our second analysis considers the mode function applied to the ranked first alternatives. From the 30 mix of Co functions and parameters (when necessary), 27 of them ranked first the Alternative 3 (A_3) and only 3 ranks have Alternative 4 (A_4) as the first one. Additionally to the alternative A_3 appearing much more in first, one can notice that the $\Delta_{R1,R2}$ difference generally achieves much high degrees, being up to 3.2 times the difference to when the alternative A_4 is ranked first.

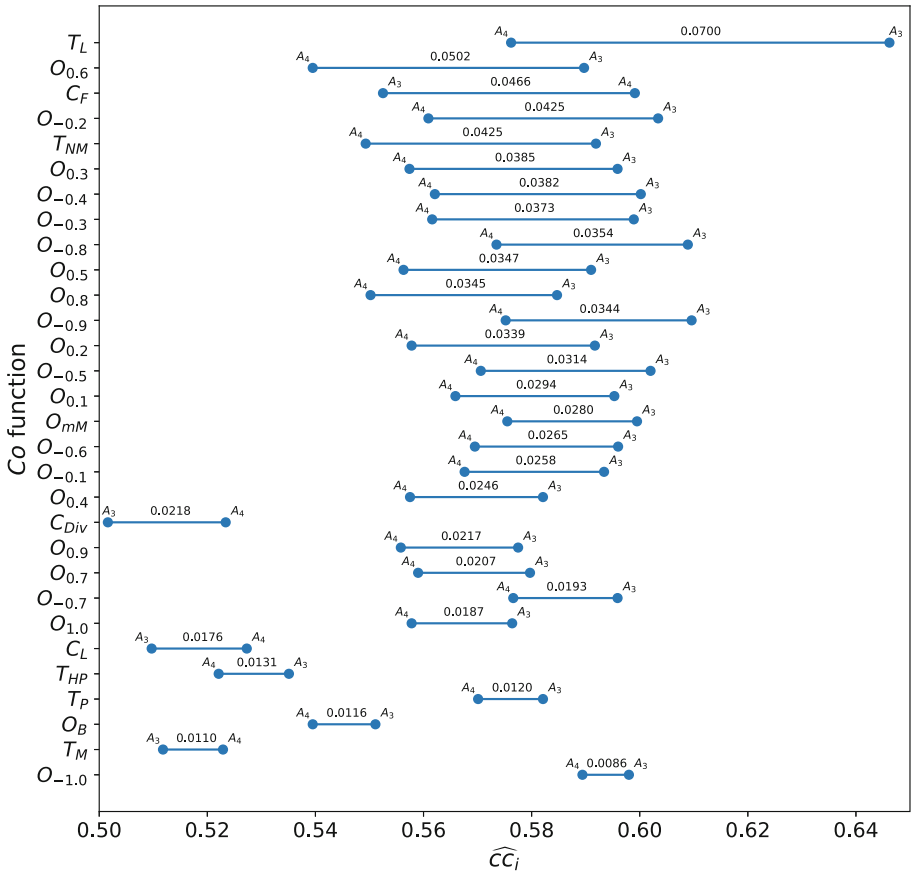

Fig. 3. $\Delta_{R1,R2}$ differences between the 2nd and 1st ranked alternatives, ordered by biggest to lowest.

3.4 An Alternative Approach to Multiple Ranks Resulted by CC-Integrals

From the results one can see that the alternative A_3 was much more preferable to rank as first one when compared to alternative A_4 because of both, the biggest $\Delta_{R1,R2}$ differences and also the fact that this alternative (A_3) appears much more in the first place, when multiple Co functions are used in the CC-separation measure. But this is not always the case, when we have alternatives much more close together this may give agglomerate both ranks, that is, half + 1 of the results may give alternative A_u as the first one and the other half - 1 may give alternative A_v as the first in the rank. Additionally, the $\Delta_{R1,R2} = \Delta_{A_u,A_v}$ may be too similar to $\Delta_{R1,R2} = \Delta_{A_v,A_u}$ for some Co functions.

To overcome this little issue, we suggest the use of $\Delta_{R1,R2}$ differences' mean for each alternative ranked first. That is:

$$\overline{\Delta}_{A_i,R1,R2} = \frac{|AR_i|}{|AR|} \cdot \sum_{\Delta_{R1,R2} \in AR_i} \Delta_{R1,R2}$$

where AR_i is the set of $\Delta_{R1,R2}$ values that has the alternative A_i as ranked first and AR is the set of all ranked alternatives.

By calculating this for each alternative that achieved first rank we can compare and use the one with the biggest $\overline{\Delta}_{A_i,R1,R2}$ value.

For example, take the problem described early in this article. There we have 27 of the 30 CC-integrals ranking the alternative A_3 as the first one and only 3 ranked alternative A_4 as the first. Therefore we can use the above formula to see which one to choose. For the alternative A_3 we have:

$$\overline{\Delta}_{A_3,R1,R2} = \frac{|AR_3|}{|AR|} \cdot \sum_{\Delta_{R1,R2} \in AR_3} \Delta_{R1,R2} = \frac{27}{30} \cdot 0.8301 = 0.7471.$$

And for alternative A_4:

$$\overline{\Delta}_{A_4,R1,R2} = \frac{|AR_4|}{|AR|} \cdot \sum_{\Delta_{R1,R2} \in AR_4} \Delta_{R1,R2} = \frac{27}{30} \cdot 0.0504 = 0.0454.$$

Therefore, since $0.7471 > 0.0454$ we should use the alternative A_3. Surely that for this problem it was not necessary to use this method since the $\Delta_{R1,R2}$ and the mode had already demonstrated clearly that the alternative A_3 should be the chosen one.

4 Conclusion

The GMC-RTOPSIS is a decision method that chooses the alternative that is closer to an ideal solution. It is capable of dealing with multiple data types as inputs and, also, through the Choquet integral, considers the interaction among different criteria.

In this paper, we extend the study of the CC-separation measure. That is a measure to be used in the GMC-RTOPSIS method that utilizes the CC-integrals instead of the Choquet integral. The CC-integrals is a generalization of the Choquet integral that presented good results when applied in classification problems.

By using an example from the literature, we tested the method with 30 different copula functions, with one of them using 20 distinct parameters. When analyzing by using the Big Delta function the results indicate that the Łukasiewicz t-norm is the best copula function to use in this example problem since it gives the greatest separation between the alternatives ranked first and second. Additionally, the Overlap alpha family, with $\alpha = 0.6$, the C_F and the T_{NM} also presented good separations.

Additionally, we demonstrated how to use the mode function as an alternative to the Big Delta. Moreover, we introduced a solution to when some alternatives may be too close together that both, the Big Delta difference and the mode function may have too similar results. The solution is to use the Big Delta means for each alternative ranked first and, then, compare its result.

By being able to verify the separation between the ranks, we can choose more confidently the alternative that better suits the problem. Therefore, by using multiple functions in the CC-separation measure, we can see how the problem behaves in different situations.

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