Chapter 7
Singular Integrals and Boundary Problems in Morrey and Block Spaces

The spaces which bear the name of Morrey have been introduced by C. Morrey in 1930s in relation to regularity problems for solutions to partial differential equations in the Euclidean setting. Membership of a function to a Morrey space amounts to a bound on the size of the L^p-integral average of said function over an arbitrary ball in terms of a fixed power of its radius. Since these are all measure-metric considerations, this brand of space naturally adapts to the more general setting of spaces of homogeneous type. Here we are concerned with the scale of Morrey spaces when the ambient is the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$. We make use of the Calderón–Zygmund theory for singular integral operators acting on Morrey spaces in such a setting as a platform that allows us to build in the direction of solving boundary value problems for weakly elliptic systems in δ-AR domains with boundary data in Morrey spaces (and their pre-duals).

7.1 Boundary Layer Potentials on Morrey and Block Spaces

The material in this section closely follows [113, §2.6]. We begin by discussing the scale of Morrey spaces on Ahlfors regular sets. To set the stage, assume $\Sigma \subseteq \mathbb{R}^n$ (where, as in the past, $n \in \mathbb{N}$ with $n \geq 2$) is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Given $p \in (0, \infty)$ and $\lambda \in (0, n-1)$, we then define the `Morrey space` $M^{p,\lambda}(\Sigma, \sigma)$ as

$$M^{p,\lambda}(\Sigma, \sigma) := \Big\{ f : \Sigma \to \mathbb{C} : \ f \text{ is } \sigma\text{-measurable and } \ \|f\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty \Big\}, \tag{7.1}$$

where, for each σ-measurable function f on Σ, we have set

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$$\|f\|_{M^{p,\lambda}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ 0<R<2\,\mathrm{diam}(\Sigma)}} \left\{ R^{\frac{n-1-\lambda}{p}} \left(⨍_{\Sigma\cap B(x,R)} |f|^p \, d\sigma \right)^{\frac{1}{p}} \right\}. \tag{7.2}$$

The space $M^{p,\lambda}(\Sigma,\sigma)$ is complete, hence Banach (though not separable) when equipped with the norm (7.2), and (cf. [112, §6.2] for a proof)

$$\begin{aligned} &M^{p,\lambda}(\Sigma,\sigma) \hookrightarrow L^p_{\mathrm{loc}}(\Sigma,\sigma) \cap L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\Big) \\ &\text{if } p \in [1,\infty),\ \lambda \in (0, n-1), \text{ and } 0 \le \varepsilon < \tfrac{n-1-\lambda}{p}. \end{aligned} \tag{7.3}$$

As may be seen from (7.1)–(7.2) and Hölder's inequality, we also have

$$L^s(\Sigma,\sigma) \hookrightarrow M^{p,\lambda}(\Sigma,\sigma) \ \text{ continuously, with } \ s := \tfrac{p(n-1)}{n-1-\lambda} \in (p,\infty). \tag{7.4}$$

In particular, there exists some $C \in (0,\infty)$ which depends only on n, p, λ, and the Ahlfors regularity constant of Σ, with the property that for each σ-measurable set $E \subseteq \Sigma$ we have

$$\big\|\mathbf{1}_E\big\|_{M^{p,\lambda}(\Sigma,\sigma)} \le C\big\|\mathbf{1}_E\big\|_{L^s(\Sigma,\sigma)} = C \cdot \sigma(E)^{(n-1-\lambda)/[p(n-1)]}. \tag{7.5}$$

As a consequence, $\mathbf{1}_E$ belongs to $M^{p,\lambda}(\Sigma,\sigma)$ whenever $E \subseteq \Sigma$ is a σ-measurable set with $\sigma(E) < +\infty$. Other examples of functions belonging to Morrey spaces are presented below (see [112, §6.2]).

Example 7.1 Let Σ, σ be as above, and for each fixed point $x_o \in \Sigma$ consider the function $f_{x_o} : \Sigma \to \mathbb{R}$ defined for each $x \in \Sigma\setminus\{x_o\}$ as $f_{x_o}(x) := |x - x_o|^{-(n-1-\lambda)/p}$. Then each f_{x_o} belongs to the Morrey space $M^{p,\lambda}(\Sigma,\sigma)$ and, in fact,

$$\sup_{x_o\in\Sigma} \|f_{x_o}\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty. \tag{7.6}$$

This being said, each f_{x_o} fails to be in $L^s(\Sigma,\sigma)$ with $s := \frac{p(n-1)}{n-1-\lambda}$, so the inclusion in (7.4) is strict.

In view of (7.4) it is of interest to define the space

$$\mathring{M}^{p,\lambda}(\Sigma,\sigma) := \text{ the closure of } L^s(\Sigma,\sigma) \text{ with } s := \tfrac{p(n-1)}{n-1-\lambda} \text{ in } M^{p,\lambda}(\Sigma,\sigma). \tag{7.7}$$

Hence, by design,

$$\begin{aligned} &\mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ is a closed linear subspace of } M^{p,\lambda}(\Sigma,\sigma) \\ &\text{such that } L^s(\Sigma,\sigma) \hookrightarrow \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ continuously and} \\ &\text{densely.} \end{aligned} \tag{7.8}$$

Thus, when equipped with the norm inherited from the larger ambient $M^{p,\lambda}(\Sigma,\sigma)$, the space $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ is complete (hence Banach). As a consequence of (7.8) and (2.508) we also see that

$$\text{the space } \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ is separable.} \tag{7.9}$$

As noted in [112, §6.2],

$$\begin{array}{l}\text{the operator of pointwise multiplication by any given function } b \in L^\infty(\Sigma,\sigma) \text{ is a bounded mapping from the space} \\ \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ into itself, with operator norm } \leq \|b\|_{L^\infty(\Sigma,\sigma)},\end{array} \tag{7.10}$$

and

$$\begin{array}{l}\text{if } f,g:\Sigma\to\mathbb{C} \text{ are two } \sigma\text{-measurable functions with the property that } |g|\leq|f| \text{ at } \sigma\text{-a.e. point on } \Sigma \text{ and } f\in\mathring{M}^{p,\lambda}(\Sigma,\sigma), \\ \text{then } g \text{ also belongs to the space } \mathring{M}^{p,\lambda}(\Sigma,\sigma).\end{array} \tag{7.11}$$

In relation to the space introduced in (7.7), we also wish to remark that since $\mathrm{Lip}_{\mathrm{comp}}(\Sigma)$ (the space of Lipschitz functions with compact support on Σ) is dense in $L^s(\Sigma,\sigma)$ and since, according to (7.4), the latter space embeds continuously into $M^{p,\lambda}(\Sigma,\sigma)$, we have

$$\mathring{M}^{p,\lambda}(\Sigma,\sigma) = \text{ the closure of } \mathrm{Lip}_{\mathrm{comp}}(\Sigma) \text{ in } M^{p,\lambda}(\Sigma,\sigma). \tag{7.12}$$

An immediate corollary of the latter description of the space $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ worth mentioning is that functions f belonging to $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ enjoy the "vanishing" property

$$\lim_{\rho\to 0^+}\ \sup_{\substack{x\in\Sigma \text{ and} \\ R\in(0,\rho)}} \left\{ R^{\frac{n-1-\lambda}{p}} \left(⨍_{\Sigma\cap B(x,R)} |f|^p \, d\sigma \right)^{\frac{1}{p}} \right\} = 0. \tag{7.13}$$

As such, it is natural to refer to $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ as being a `vanishing Morrey space`.

The topic addressed next pertains to the pre-duals of Morrey spaces, and the duals of vanishing Morrey spaces. Continue to assume that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. To set the stage, given an integrability exponent $q \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$, a function $b \in L^q(\Sigma,\sigma)$ is said to be a $\mathcal{B}^{q,\lambda}$-`block` on Σ (or, simply, a block) provided there exist some point $x_o \in \Sigma$ and some radius $R \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ such that

$$\mathrm{supp}\, b \subseteq B(x_o,R)\cap\Sigma \quad \text{and} \quad \|b\|_{L^q(\Sigma,\sigma)} \leq R^{\lambda\left(\frac{1}{q}-1\right)}. \tag{7.14}$$

With $r := \frac{q(n-1)}{n-1+\lambda(q-1)} \in (1, q)$ we then define the `block space`

$$\mathcal{B}^{q,\lambda}(\Sigma, \sigma) := \Big\{ f \in L^r(\Sigma, \sigma) : \text{ there exist a numerical sequence } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and a family } \{b_j\}_{j\in\mathbb{N}} \text{ of } \mathcal{B}^{q,\lambda}\text{-blocks on } \Sigma \text{ with } f = \sum_{j=1}^{\infty} \lambda_j b_j \text{ in } L^r(\Sigma, \sigma) \Big\}, \tag{7.15}$$

and for each $f \in \mathcal{B}^{q,\lambda}(\Sigma, \sigma)$ define

$$\|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} := \inf \Big\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j b_j \text{ in } L^r(\Sigma, \sigma) \text{ with } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and each } b_j \text{ a } \mathcal{B}^{q,\lambda}\text{-block on } \Sigma \Big\}. \tag{7.16}$$

Work in [112, §6.2] gives that

$$\Big(\mathcal{B}^{q,\lambda}(\Sigma, \sigma) , \|\cdot\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} \Big) \text{ is a separable Banach space, and } \mathcal{B}^{q,\lambda}(\Sigma, \sigma) \hookrightarrow L^r(\Sigma, \sigma) \text{ with } r := \tfrac{q(n-1)}{n-1+\lambda(q-1)} \in (1, q) \tag{7.17}$$

and

the operator of pointwise multiplication by any given function $b \in L^\infty(\Sigma, \sigma)$ is a linear and bounded mapping from the space $\mathcal{B}^{q,\lambda}(\Sigma, \sigma)$ into itself, with operator norm $\leq \|b\|_{L^\infty(\Sigma,\sigma)}$. (7.18)

Note that the latter property further implies that

if $f, g : \Sigma \longrightarrow \mathbb{C}$ are two σ-measurable functions such that $|g| \leq |f|$ at σ-a.e. point on Σ and $f \in \mathcal{B}^{q,\lambda}(\Sigma, \sigma)$, then we have $g \in \mathcal{B}^{q,\lambda}(\Sigma, \sigma)$ as well as $\|g\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} \leq \|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}$. (7.19)

Examples of functions in the block space (7.15) may be produced using the following result from [112, §6.2].

Proposition 7.1 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix an exponent $q \in (1, \infty)$ along with $\lambda \in (0, n-1)$. Then for each $a > \lambda$ one has the continuous and dense embedding*

$$L^q\Big(\Sigma, (1 + |x|)^{a(q-1)}\sigma(x)\Big) \hookrightarrow \mathcal{B}^{q,\lambda}(\Sigma, \sigma). \tag{7.20}$$

In particular,

$$\begin{aligned} &\text{if } N > \tfrac{\lambda(q-1)+n-1}{q} \text{ and } f_N(x) := (1+|x|)^{-N} \text{ for } x \in \Sigma, \\ &\text{then the function } f_N \text{ belongs to the space } \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \end{aligned} \tag{7.21}$$

Our primary interest in the space (7.15) stems from the fact that this turns out to be the pre-dual of a Morrey space. In turn, vanishing Morrey spaces are pre-duals of block spaces. Specifically, we have the following result proved in [112, §6.2].

Proposition 7.2 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix two exponents $p, q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$. Then there exists $C \in (0,\infty)$ which depends only on the Ahlfors regularity constant of Σ, n, p, and λ, with the property that*

$$\begin{aligned} &\int_\Sigma |f||g|\,\mathrm{d}\sigma \le C\|f\|_{M^{p,\lambda}(\Sigma,\sigma)}\|g\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} \\ &\text{for all } \ f \in M^{p,\lambda}(\Sigma,\sigma) \ \text{ and } \ g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \end{aligned} \tag{7.22}$$

In addition, the mapping

$$\begin{aligned} &M^{p,\lambda}(\Sigma,\sigma) \ni f \longmapsto \Lambda_f \in \big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\big)^* \ \text{ given by} \\ &\Lambda_f(g) := \int_\Sigma fg\,\mathrm{d}\sigma \ \text{ for each } \ g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma) \end{aligned} \tag{7.23}$$

is a well-defined, linear, bounded isomorphism, with bounded inverse. Simply put, the integral pairing yields the quantitative identification

$$\big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\big)^* = M^{p,\lambda}(\Sigma,\sigma). \tag{7.24}$$

Furthermore, regarding $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ as a Banach space equipped with the norm inherited from $M^{p,\lambda}(\Sigma,\sigma)$, the mapping

$$\begin{aligned} &\mathcal{B}^{q,\lambda}(\Sigma,\sigma) \ni g \longmapsto \Lambda_g \in \big(\mathring{M}^{p,\lambda}(\Sigma,\sigma)\big)^* \ \text{ given by} \\ &\Lambda_g(f) := \int_\Sigma fg\,\mathrm{d}\sigma \ \text{ for each } \ f \in \mathring{M}^{p,\lambda}(\Sigma,\sigma) \end{aligned} \tag{7.25}$$

is a well-defined, linear, bounded isomorphism, with bounded inverse. As such, the integral pairing yields the identification

$$\big(\mathring{M}^{p,\lambda}(\Sigma,\sigma)\big)^* = \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{7.26}$$

In the setting of Proposition 7.2, from (7.24), (7.17), and the Sequential Banach–Alaoglu Theorem we conclude that

$$\text{any bounded sequence in } M^{p,\lambda}(\Sigma,\sigma) \text{ has a sub-sequence which is weak-}* \text{ convergent.} \tag{7.27}$$

A result in this spirit in which a stronger conclusion is reached, provided one assumes more than mere boundedness for said sequence, has been proved in [112, §6.2].

Proposition 7.3 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix two exponents $p, q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$. Suppose $\{f_j\}_{j\in\mathbb{N}} \subseteq M^{p,\lambda}(\Sigma,\sigma)$ is a sequence of functions with the property that*

$$\begin{aligned} &f(x) := \lim_{j\to\infty} f_j(x) \ \ \textit{exists for } \sigma\textit{-a.e. } \ x \in \Sigma, \quad \textit{and} \\ &\textit{there exists some } g \in M^{p,\lambda}(\Sigma,\sigma) \textit{ such that for each} \\ &j \in \mathbb{N} \ \ \textit{one has } \ |f_j(x)| \leq |g(x)| \ \textit{ for } \sigma\textit{-a.e. } \ x \in \Sigma. \end{aligned} \tag{7.28}$$

Then $f \in M^{p,\lambda}(\Sigma,\sigma)$ and $f_j \to f$ as $j \to \infty$ weak-$*$ in $M^{p,\lambda}(\Sigma,\sigma)$, i.e.,

$$\lim_{j\to\infty} \int_\Sigma f_j h \, d\sigma = \int_\Sigma f h \, d\sigma \ \ \textit{for each} \ \ h \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{7.29}$$

Remarkably, certain types of estimates on Muckenhoupt weighted Lebesgue space imply estimates on Morrey spaces. Here is a basic result of this flavor from [112, §6.2] (cf. also [43] for related results in the Euclidean setting).

Proposition 7.4 *Let $\Sigma \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) be a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix an integrability exponent $p \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$. Finally, let $\mathcal{F}$ be a family of pairs (f, g) of σ-measurable functions defined on Σ such that*

$$\begin{aligned} &\textit{for each Muckenhoupt weight } w \in A_1(\Sigma,\sigma) \textit{ there exists some} \\ &\textit{constant } C_w = C([w]_{A_1}) \in (0,\infty), \textit{ which stays bounded as} \\ &[w]_{A_1} \textit{ stays bounded, and with the property that for each pair} \\ &(f,g) \in \mathcal{F} \textit{ one has } \|f\|_{L^p(\Sigma,w)} \leq C_w \|g\|_{L^p(\Sigma,w)}. \end{aligned} \tag{7.30}$$

Then there exist two constants $C_{\Sigma,p} \in (0,\infty)$ (depending only on p and the Ahlfors regularity constant of Σ) and $Q_{n,\lambda} \in (0,\infty)$ (depending only on n and λ) such that, with

$$C := C_{\Sigma,p} \cdot \sup_{\substack{w\in A_1(\Sigma,\sigma) \\ [w]_{A_1} \leq Q_{n,\lambda}}} C_w, \tag{7.31}$$

one has

$$\|f\|_{M^{p,\lambda}(\Sigma,\sigma)} \leq C \|g\|_{M^{p,\lambda}(\Sigma,\sigma)} \ \ \textit{for each pair} \ \ (f,g) \in \mathcal{F}. \tag{7.32}$$

Based on Propositions 7.4, 3.4, 7.2 (as well as Coltar's inequality and boundedness results for the Hardy–Littlewood maximal operator on Morrey and block spaces), the following result has been established in [113, §2.6].

Proposition 7.5 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $\partial\Omega$ is a UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is odd and positive homogeneous of degree $1-n$. Also, fix two integrability exponents $p, q \in (1,\infty)$ with $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$, and pick an aperture parameter $\kappa > 0$. In this setting, for each f belonging to either $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ define*

$$T_\varepsilon f(x) := \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k(x-y)f(y)\,\mathrm{d}\sigma(y) \ \ \textit{for each} \ \ x \in \partial\Omega, \tag{7.33}$$

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \ \ \textit{for each} \ \ x \in \partial\Omega, \tag{7.34}$$

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \ \ \textit{for } \sigma\textit{-a.e.} \ \ x \in \partial\Omega, \tag{7.35}$$

$$\mathcal{T}f(x) := \int_{\partial\Omega} k(x-y)f(y)\,\mathrm{d}\sigma(y) \ \ \textit{for each} \ \ x \in \Omega. \tag{7.36}$$

Then there exists a constant $C \in (0,\infty)$ which depends exclusively on n, p, λ, and the UR constants of $\partial\Omega$ with the property that for each $f \in M^{p,\lambda}(\partial\Omega,\sigma)$ one has

$$\|T_* f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.37}$$

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.38}$$

for each $f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ one has

$$\|T_* f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.39}$$

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.40}$$

and for each $f \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *one has*

$$\|T_* f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}, \tag{7.41}$$

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.42}$$

Also, for each function f *belonging to either* $M^{p,\lambda}(\partial\Omega, \sigma)$*,* $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$*, or* $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *the limit defining* $Tf(x)$ *in* (7.35) *exists at* σ*-a.e.* $x \in \partial\Omega$ *and the operators*

$$T : M^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow M^{p,\lambda}(\partial\Omega, \sigma), \tag{7.43}$$

$$T : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma), \tag{7.44}$$

$$T : \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma), \tag{7.45}$$

are well defined, linear, and bounded. In addition,

the (*real*) *transpose of the operator* (7.44) *is the operator* $-T$ *with* T *as in* (7.45)*, and the* (*real*) *transpose of the operator* (7.45) *is the operator* $-T$ *with* T *as in* (7.43)*.* (7.46)

Thus, the results from Proposition 7.5 are applicable to the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ defined as in (4.297) on the boundary of a UR domain $\Omega \subseteq \mathbb{R}^n$. This proves that, in such a setting, for each $p, q \in (1, \infty)$ and $\lambda \in (0, n-1)$

the operators $\{R_j\}_{1\leq j\leq n}$ are well defined, linear, and bounded on the spaces $M^{p,\lambda}(\partial\Omega, \sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$, and $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$. (7.47)

In concert with Theorem 4.3, (7.7), and duality (cf. Proposition 7.2), Proposition 7.4 also yields the following version of the commutator theorem from [31], in Morrey and block spaces.

Theorem 7.1 *Make the assumption that* $\Sigma \subseteq \mathbb{R}^n$ *be a closed Ahlfors regular set, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$*. Fix* $p_0 \in (1, \infty)$ *along with some non-decreasing function* $\Phi : (0, \infty) \to (0, \infty)$ *and let* T *be a linear operator which is bounded on* $L^{p_0}(\Sigma, w)$ *for every* $w \in A_{p_0}(\Sigma, \sigma)$*, with operator norm* $\leq \Phi\big([w]_{A_{p_0}}\big)$*.*

Then for each exponent $p \in (1, \infty)$ *and each parameter* $\lambda \in (0, n-1)$ *the operator* T *induces well-defined, linear, and bounded mappings in the contexts*

$$T : M^{p,\lambda}(\Sigma, \sigma) \longrightarrow M^{p,\lambda}(\Sigma, \sigma), \tag{7.48}$$

$$T : \mathring{M}^{p,\lambda}(\Sigma, \sigma) \longrightarrow \mathring{M}^{p,\lambda}(\Sigma, \sigma). \tag{7.49}$$

In addition, given any integrability exponent $p \in (1, \infty)$ *along with some parameter* $\lambda \in (0, n-1)$*, there exist two constants,* $C_1 = C_1(\Sigma, n, p_0, p, \lambda) \in (0, \infty)$ *and* $C_2 = C_2(\Sigma, n, p_0, p, \lambda) \in (0, \infty)$*, with the property that for every complex-valued function* $b \in L^\infty(\Sigma, \sigma)$ *one has*

$$\begin{aligned} \|[M_b, T]\|_{\mathring{M}^{p,\lambda}(\Sigma,\sigma)\to\mathring{M}^{p,\lambda}(\Sigma,\sigma)} &\leq \|[M_b, T]\|_{M^{p,\lambda}(\Sigma,\sigma)\to M^{p,\lambda}(\Sigma,\sigma)} \\ &\leq C_1\Phi(C_2)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,, \end{aligned} \tag{7.50}$$

where $[M_b, T] := bT(\cdot) - T(b\,\cdot)$ *is the commutator of* T (*considered either as in* (7.48) *or as in* (7.49)) *and the operator* M_b *of pointwise multiplication* (*either on* $M^{p,\lambda}(\Sigma, \sigma)$ *or on* $\mathring{M}^{p,\lambda}(\Sigma, \sigma)$) *by the function* b.

Moreover, if $T^\top$ *denotes the* (*real*) *transpose of the original operator* T*, then for each* $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *the operator* $T^\top$ *induces a well-defined, linear, and bounded mapping*

$$T^\top : \mathcal{B}^{q,\lambda}(\Sigma, \sigma) \longrightarrow \mathcal{B}^{q,\lambda}(\Sigma, \sigma). \tag{7.51}$$

Finally, for each $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *there exist two positive finite constants,* $C_1 = C_1(\Sigma, n, p_0, q, \lambda)$ *and* $C_2 = C_2(\Sigma, n, p_0, q, \lambda)$*, with the property that for every complex-valued function* $b \in L^\infty(\Sigma, \sigma)$ *one has*

$$\Big\|[M_b, T^\top]\Big\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\to\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} \leq C_1\Phi(C_2)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{7.52}$$

For example, if $\Omega \subseteq \mathbb{R}^n$ is a UR domain then, for each complex-valued function $k \in \mathscr{C}^N\big(\mathbb{R}^n \setminus \{0\}\big)$ (where $N = N(n) \in \mathbb{N}$ is sufficiently large) which is odd and positive homogeneous of degree $1-n$, Theorem 7.1 applies with $\Sigma := \partial\Omega$ and T as in (7.35). In such a scenario, from (7.52) and (7.46) we see that for each $b \in L^\infty(\partial\Omega, \sigma)$, $q \in (1, \infty)$, and $\lambda \in (0, n-1)$, the following commutator estimate holds:

$$\|[M_b, T]\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\, \|b\|_{\mathrm{BMO}(\partial\Omega,\sigma)}\,, \tag{7.53}$$

where $C \in (0, \infty)$ depends exclusively on n, q, λ, and the UR constants of $\partial\Omega$.

Following [112, §11.7], we may also consider `Morrey-based Sobolev spaces` on the boundaries of Ahlfors regular domains. Specifically, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then for each $p \in (1, \infty)$ and $\lambda \in (0, n-1)$ we define

$$M_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in M^{p,\lambda}(\partial\Omega,\sigma) \cap L^1_{1,\mathrm{loc}}(\partial\Omega,\sigma) : \qquad \partial_{\tau_{jk}} f \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ for each } \ j,k \in \{1,\dots,n\} \Big\}, \tag{7.54}$$

equipped with the natural norm

$$M_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.55}$$

A significant closed subspace of $M_1^{p,\lambda}(\partial\Omega,\sigma)$ is the `vanishing Morrey-based Sobolev space`

$$\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } \ j,k \in \{1,\dots,n\} \qquad \text{one has } \ \partial_{\tau_{jk}} f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \Big\}. \tag{7.56}$$

In the same vein, for each $q \in (1,\infty)$ let us also define the `block-based Sobolev space`

$$\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) : \text{ for each } \ j,k \in \{1,\dots,n\} \qquad \text{one has } \ \partial_{\tau_{jk}} f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \Big\}, \tag{7.57}$$

and endowed with the norm

$$\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.58}$$

It has been noted in [114, §3.3] that by combining the extrapolation result from Proposition 7.4 with Proposition 3.5 (while also keeping in mind Proposition 3.2, (7.3), (7.8), (7.17), Proposition 7.5, and (7.18)) one obtains the following result pertaining to the action of boundary layer potentials associated with weakly elliptic second-order systems in UR domains, on the scales of spaces discussed earlier.

Theorem 7.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let L be a homogeneous, weakly elliptic, constant complex coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Pick a coefficient tensor $A \in \mathfrak{A}_L$ and consider the double layer potential operators $\mathcal{D}_A$, K_A, $K_A^{\#}$ associated with the coefficient tensor A and the set Ω as in* (3.22), (3.24)*, and* (3.25)*, respectively. Finally, select $p \in (1,\infty)$ along with $\lambda \in (0, n-1)$ and some aperture parameter $\kappa > 0$.*

Then the operators

$$K_A, K_A^{\#} : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{7.59}$$

are well defined, linear, and bounded. Additionally, the operators $K_A, K_A^{\#}$ *in the context of* (7.59) *depend continuously on the underlying coefficient tensor* A. *Specifically, with the piece of notation introduced in* (3.13), *the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M\Big), \tag{7.60}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A^{\#} \in \mathrm{Bd}\Big(\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M\Big). \tag{7.61}$$

Furthermore, there exists a constant $C \in (0,\infty)$, *depending only on the* UR *constants of* $\partial\Omega$, L, n, κ, p, *and* λ, *with the property that*

$$\begin{aligned}&\big\|\mathcal{N}_\kappa\big(\mathcal{D}_A f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\\ &\quad \text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{aligned} \tag{7.62}$$

Moreover, for each given function f *in the Morrey space* $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *the following nontangential boundary trace formula holds* (*with* I *denoting the identity operator*)

$$\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_A\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.63}$$

In addition, for each function f *belonging to the Morrey-based Sobolev space* $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *it follows that*

$$\begin{aligned}&\text{the nontangential boundary trace } \big(\partial_\ell \mathcal{D}_A f\big)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{C}^M)\\ &\text{at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } \ell \in \{1,\dots,n\},\end{aligned} \tag{7.64}$$

and there exits some finite constant $C > 0$, *depending only on* $\partial\Omega$, L, n, κ, p, λ, *such that*

$$\begin{aligned}\big\|\mathcal{N}_\kappa(\mathcal{D}_A f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} &+ \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}_A f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C\|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{7.65}$$

In fact, similar results are valid with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced throughout by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ (*defined as in* (7.7) *with*

$\Sigma := \partial\Omega$)*, or by the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *with* $q \in (1, \infty)$ (*defined as in* (7.15)–(7.16) *with* $\Sigma := \partial\Omega$).

Next, the operators

$$K_A : \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.66}$$

$$K_A : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.67}$$

are well defined, linear, bounded and, for each $q \in (1, \infty)$*, so is*

$$K_A : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M. \tag{7.68}$$

Also, much as in (7.60)–(7.61)*, the operator* K_A *in the context of* (7.66)–(7.68) *depends in a continuous fashion on the underlying coefficient tensor* A.

Next we introduce the `homogeneous Morrey-based Sobolev spaces`. Consider an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Given an integrability exponent $p \in (1, \infty)$ and $\lambda \in (0, n-1)$ let us define the space

$$\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma) := \Big\{ f \in L^1\big(\partial\Omega\,,\, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma) : \\ \partial_{\tau_{jk}} f \in M^{p,\lambda}(\partial\Omega, \sigma) \text{ for each } \ j, k \in \{1, \dots, n\}\Big\} \tag{7.69}$$

and equip it with the semi-norm

$$\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma) \ni f \longmapsto \|f\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.70}$$

Then (7.3) ensures that we have the following continuous embedding

$$M_1^{p,\lambda}(\partial\Omega, \sigma) \hookrightarrow \dot{M}_1^{p,\lambda}(\partial\Omega, \sigma). \tag{7.71}$$

It is also clear that constant functions on $\partial\Omega$ belong to $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$ and have vanishing semi-norm. We shall occasionally work with $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$, the quotient space of classes $[\cdot]$ of equivalence modulo constants of functions in $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$, equipped with the semi-norm

$$\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim \ni [f] \mapsto \big\|[f]\big\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.72}$$

To proceed, choose a scalar-valued function $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ in $B(0, 1)$ and $\operatorname{supp}\phi \subseteq B(0, 2)$. Having fixed a reference point $x_0 \in \partial\Omega$, for each scale $r \in (0, \infty)$ define

$$\phi_r(x) := \phi\Big(\frac{x - x_0}{r}\Big) \ \text{ for each } \ x \in \mathbb{R}^n, \tag{7.73}$$

and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. For each $r \in (0, \infty)$ set $\Delta_r := \partial\Omega \cap B(x_0, r)$. Given any $f \in L^1_{\text{loc}}(\partial\Omega, \sigma)$, define

$$f_r := \phi_r \cdot \big(f - f_{\Delta_{2r}}\big) \ \text{ on } \ \partial\Omega, \quad \text{where } \ f_{\Delta_{2r}} := \fint_{\Delta_{2r}} f \, d\sigma. \tag{7.74}$$

Lemma 7.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain with the property that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix some reference point $x_0 \in \partial\Omega$, along with some integrability exponent $p \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Finally, pick a function f which belongs to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\text{loc}}(\partial\Omega, \sigma)$ and, for each radius $r \in (0, \infty)$, define the surface ball $\Delta_r := B(x_0, r) \cap \partial\Omega$ and $f_{\Delta_r} := \fint_{\Delta_r} f \, d\sigma$. Then the following statements are true.*

(i) There exists a constant $C = C(\Omega, p, \lambda, x_0) \in (0, \infty)$, independent of the function f, such that

$$\sup_{r>0} \frac{1}{r} \big\| |f - f_{\Delta_r}| \cdot \mathbf{1}_{\Delta_r} \big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C \sum_{j,k=1}^{n} \big\| \partial_{\tau_{jk}} f \big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.75}$$

(ii) For each $r \in (0, \infty)$ there exists a constant $C_r \in (0, \infty)$ which depends on Ω, p, λ, x_0, and r, but is independent of f, such that

$$\int_{\partial\Omega} \frac{|f(x) - f_{\Delta_r}|}{1 + |x|^n} \, d\sigma(x) \leq \frac{C_r}{\|\mathbf{1}_{\Delta_r}\|_{M^{p,\lambda}(\partial\Omega,\sigma)}} \sum_{j,k=1}^{n} \big\| \partial_{\tau_{jk}} f \big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.76}$$

(iii) There exists a constant $C = C(\Omega, p, \lambda, x_0) \in (0, \infty)$, independent of the function f, such that with the notation introduced in (7.74) *one has*

$$\sup_{r>0} \big\| \nabla_{\tan} f_r \big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^n} \leq C \big\| \nabla_{\tan} f \big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^n}. \tag{7.77}$$

Proof We shall prove all claims using extrapolation (cf. Proposition 7.4). Consider first the task of establishing *(i)*. Recall (2.585) and define

$$\mathcal{F}_1 := \Big\{ \Big(\tfrac{|f - f_{\Delta_r}|}{r} \mathbf{1}_{\Delta_r}, |\nabla_{\tan} f| \Big) : \\ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\text{loc}}(\partial\Omega, \sigma), \ r > 0 \Big\}. \tag{7.78}$$

We claim that for the given integrability exponent $p \in (1, \infty)$ and for every weight $w \in A_p(\partial\Omega, \sigma)$ there exists a constant $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0, \infty)$ such that

$$\|F_1\|_{L^p(\partial\Omega,w)} \leq C\|F_2\|_{L^p(\partial\Omega,w)} \tag{7.79}$$

for all $(F_1, F_2) \in \mathcal{F}_1$. Indeed, this inequality is trivial if $\|F_2\|_{L^p(\partial\Omega,w)} = \infty$, whereas if $\|F_2\|_{L^p(\partial\Omega,w)} < \infty$ we may rely on (7.78) and (2.586) to invoke Proposition 2.25 to obtain (2.618). This, in turn, gives (7.79) on account of (2.586). Moreover, the intervening constant C stays bounded if $[w]_{A_p}$ stays bounded. In particular, in view of item *(2)* from Proposition 2.20, the argument so far shows that (7.79) holds for every $w \in A_1(\partial\Omega, \sigma)$ and that the intervening constant stays bounded if $[w]_{A_1}$ stays bounded. We may then invoke Theorem 7.4 to conclude that for each given number $\lambda \in (0, n-1)$ we have $\|F_1\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|F_2\|_{M^{p,\lambda}(\partial\Omega,\sigma)}$ for each $(F_1, F_2) \in \mathcal{F}_1$. This and (2.585) then imply (7.75), finishing the proof of *(i)*.

Let us now address the claim made in item *(ii)*. Fix $r \in (0, \infty)$ and define

$$\mathcal{F}_2 := \Big\{\big(\|f - f_{\Delta_r}\|_{L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\right)} \mathbf{1}_{\Delta_r}, |\nabla_{\tan} f|\big) : \\ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)\Big\}. \tag{7.80}$$

As before, the goal is to check that (7.79) holds for all weights $w \in A_p(\partial\Omega, \sigma)$ and all pairs $(F_1, F_2) \in \mathcal{F}_2$ (where now the constant C is allowed to depend on the scale r, which has been fixed). This may be seen reasoning much as before, applying Proposition 2.25, but this time the relevant estimate is (2.620). Granted (7.79), we may then apply Theorem 7.4 to the family $\mathcal{F}_2$ and, as desired, conclude that (7.76) holds.

To justify the claim made in item *(iii)*, we introduce

$$\mathcal{F}_3 := \Big\{\big(|\nabla_{\tan} f_r|, |\nabla_{\tan} f|\big) : f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma),\ r > 0\Big\}. \tag{7.81}$$

In line with what we have done in the previous cases, we now wish to show that (7.79) holds for all weights $w \in A_p(\partial\Omega, \sigma)$ and all pairs $(F_1, F_2) \in \mathcal{F}_3$. Again, it suffices to consider the case when $\|F_2\|_{L^p(\partial\Omega,w)} < \infty$. By definition, we have $(F_1, F_2) = \big(|\nabla_{\tan} g_r|, |\nabla_{\tan} g|\big)$ for some $g \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ and some $r > 0$. This, the assumption $\|F_2\|_{L^p(\partial\Omega,w)} < \infty$, (2.586), and Proposition 2.25 then guarantee that $g \in \dot{L}^p_1(\partial\Omega, w)$. We may therefore proceed as in (4.370)–(4.377) in the proof of Theorem 4.11 to conclude that (4.377) holds. Equivalently, this proves (7.79) for the given choice of (F_1, F_2). Moreover, a careful examination of the proof shows that the intervening constant $C \in (0, \infty)$ stays bounded if $[w]_{A_p}$

stays bounded. We have therefore shown that (7.79) holds for each $(F_1, F_2) \in \mathcal{F}_3$ and each $w \in A_p(\partial\Omega, \sigma)$. In particular (cf. item *(2)* in Proposition 2.20), this is the case for every $w \in A_1(\partial\Omega, \sigma)$ and the intervening constant $C \in (0, \infty)$ stays bounded if $[w]_{A_1}$ stays bounded. As such, we may avail ourselves of Theorem 7.4 to conclude that, given any $\lambda \in (0, n-1)$, one has $\|F_1\|_{M_1^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|F_2\|_{M_1^{p,\lambda}(\partial\Omega,\sigma)}$ for every $(F_1, F_2) \in \mathcal{F}_3$. Hence, there exists $C = C(\Omega, p, \lambda, x_0) \in (0, \infty)$ such that

$$\big\||\nabla_{\tan} f_r|\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\big\||\nabla_{\tan} f|\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{7.82}$$

for every $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ and every $r > 0$. This completes the proof of (7.77). □

It turns out that, when considered on the boundaries of two-sided NTA domains, the quotient space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ is actually a Banach space.

Proposition 7.6 *Suppose* $\Omega \subseteq \mathbb{R}^n$ *is a two-sided* NTA *domain with an unbounded Ahlfors regular boundary and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Pick some integrability exponent* $p \in (1, \infty)$ *along with a parameter* $\lambda \in (0, n-1)$. *Finally, recall that* $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ *denotes the quotient space of classes* $[\,\cdot\,]$ *of equivalence modulo constants of functions in* $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$*, equipped with the semi-norm* (7.72)*.*

Then (7.72) *is a genuine norm on* $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$*, and* $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ *is a Banach space when equipped with the norm* (7.72)*.*

Proof Let us first observe from (7.76) that the semi-norm (7.72) is indeed a norm on the space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$. We shall next show that $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ is complete when equipped with the norm (7.72). With this goal in mind, let $\{f_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$ be such that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the quotient space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$. This means that $\big\{\partial_{\tau_{jk}} f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $M^{p,\lambda}(\partial\Omega, \sigma)$, for any two fixed indices $j, k \in \{1, \dots, n\}$. Using the fact that $M^{p,\lambda}(\partial\Omega, \sigma)$ is a Banach space, we then conclude that for each $j, k \in \{1, \dots, n\}$ there exists $g_{jk} \in M^{p,\lambda}(\partial\Omega, \sigma)$ such that

$$\partial_{\tau_{jk}} f_\alpha \to g_{jk} \ \text{ in } \ M^{p,\lambda}(\partial\Omega, \sigma) \ \text{ as } \ \alpha \to \infty. \tag{7.83}$$

Fix a reference point $x_0 \in \partial\Omega$ and, for each $r \in (0, \infty)$, set $\Delta_r := B(x_0, r) \cap \partial\Omega$. Also, set $f_{\alpha,\Delta_r} := ⨍_{\Delta_r} f_\alpha \, d\sigma$ for each $r \in (0, \infty)$ and each $\alpha \in \mathbb{N}$. Applying (7.76) to $f := f_\alpha - f_\beta$ we obtain that for any radius $r \in (0, \infty)$ there exists some constant $C_r \in (0, \infty)$ which depends only on Ω, p, λ, r, and x_0, such that that for all indices $\alpha, \beta \in \mathbb{N}$ we have

$$\big\|\big(f_\alpha - f_{\alpha,\Delta_r}\big) - \big(f_\beta - f_{\beta,\Delta_r}\big)\big\|_{L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\right)}$$

$$\leq \frac{C_r}{\|\mathbf{1}_{\Delta_r}\|_{M^{p,\lambda}(\partial\Omega,\sigma)}} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.84}$$

Since $\{\partial_{\tau_{jk}} f_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $M^{p,\lambda}(\partial\Omega,\sigma)$, it then follows that for each fixed $r \in (0,\infty)$ the sequence $\{f_\alpha - f_{\alpha,\Delta_r}\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Hence, for each fixed $r \in (0,\infty)$ there exists $h_r \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ such that

$$f_\alpha - f_{\alpha,\Delta_r} \to h_r \text{ in } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \text{ as } \alpha\to\infty. \tag{7.85}$$

On the other hand, by (7.75) (applied to the difference $f := f_\alpha - f_\beta$), there exists some constant $C = C(\Omega, p, \lambda, x_0) \in (0,\infty)$ such that for each fixed $r \in (0,\infty)$ we have

$$\begin{aligned}\big\||(f_\alpha - f_{\alpha,\Delta_r}) &- (f_\beta - f_{\beta,\Delta_r})| \cdot \mathbf{1}_{\Delta_r}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \\ &\leq C\,r \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\alpha\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}.\end{aligned} \tag{7.86}$$

Hence, the sequence $\{(f_\alpha - f_{\alpha,\Delta_r})\mathbf{1}_{\Delta_r}\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $M^{p,\lambda}(\partial\Omega,\sigma)$ for each fixed $r \in (0,\infty)$. As a result, for each fixed $r \in (0,\infty)$ it follows that

$$\begin{gathered}\text{there exists a function } k_r \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ such that} \\ (f_\alpha - f_{\alpha,\Delta_r})\mathbf{1}_{\Delta_r} \to k_r \text{ in } M^{p,\lambda}(\partial\Omega,\sigma) \text{ as } \alpha\to\infty.\end{gathered} \tag{7.87}$$

Note that convergence in $M_1^{p,\lambda}(\partial\Omega,\sigma)$ implies convergence in $L^p(\Delta_r,\sigma)$ and, after eventually passing to a sub-sequence, pointwise a.e. convergence. Thus (7.85) and (7.87) immediately give

$$h_r\big|_{\Delta_r} = k_r \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ for each } r \in (0,\infty). \tag{7.88}$$

Additionally, for each fixed $r_1, r_2 \in (0,\infty)$ the convergence recorded in (7.85) also yields

$$f_{\alpha,\Delta_{r_2}} - f_{\alpha,\Delta_{r_1},w} \to h_{r_1} - h_{r_2} \text{ in } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \text{ as } \alpha\to\infty. \tag{7.89}$$

Thus $h_{r_1} - h_{r_2}$ must be constant. This, (7.85), (7.88), and (7.3) eventually lead to

$$h_r \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each } r \in (0,\infty). \tag{7.90}$$

To continue we simply write h for h_r with $r = 1$, and c_α for f_{α,Δ_r} with $r = 1$. Then, as seen from (7.90),

$$h \;\text{ belongs to }\; L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma), \tag{7.91}$$

and the sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \mathbb{C}$ is such that

$$f_\alpha - c_\alpha \to h \;\text{ in }\; L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \;\text{ as }\; \alpha \to \infty. \tag{7.92}$$

For each $j, k \in \{1, \dots, n\}$ and each test function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ we may then write

$$\begin{aligned}\int_{\partial\Omega} h(\partial_{\tau_{jk}}\varphi)\,{\rm d}\sigma &= \lim_{\alpha\to\infty}\int_{\partial\Omega}(f_\alpha - c_\alpha)(\partial_{\tau_{jk}}\varphi)\,{\rm d}\sigma\\ &= -\lim_{\alpha\to\infty}\int_{\partial\Omega}\partial_{\tau_{jk}}(f_\alpha - c_\alpha)\varphi\,{\rm d}\sigma = -\lim_{\alpha\to\infty}\int_{\partial\Omega}(\partial_{\tau_{jk}}f_\alpha)\varphi\,{\rm d}\sigma\\ &= \int_{\partial\Omega} g_{jk}\varphi\,{\rm d}\sigma, \end{aligned}\tag{7.93}$$

thanks to (7.92), (2.583), (7.83), and (7.3). From this and (2.581)–(2.582) we then conclude that

$$\partial_{\tau_{jk}}h = g_{jk} \in M^{p,\lambda}(\partial\Omega, \sigma) \;\text{ for each }\; j, k \in \{1, \dots, n\}. \tag{7.94}$$

Collectively, (7.91) and (7.94) prove that $h \in \dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$. Finally, from (7.83), (7.94), and (7.72) we conclude that the sequence $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ converges to $[h]$, the class of h, in the quotient space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$. □

We continue by making the following definition, which should be compared with (7.69).

Definition 7.1 Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and pick an exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. In this context, define the `vanishing Morrey-based homogeneous Sobolev space` of order one on $\partial\Omega$ as

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega, \sigma) := \Big\{ f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega, \sigma) : \\ \partial_{\tau_{jk}}f \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \text{ for each } \; j, k \in \{1, \dots, n\}\Big\} \tag{7.95}$$

and equip this space with the semi-norm

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.96}$$

As seen from of Definition 7.1, all constant functions on $\partial\Omega$ belong to $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ and their respective semi-norms vanish. It is also apparent from (7.95)–(7.96) and (7.69)–(7.70) that

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) = \Big\{ f \in \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \\ \text{for all } \ j,k \in \{1,\dots,n\} \Big\} \tag{7.97}$$

and

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \text{ is a closed subspace of } \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma). \tag{7.98}$$

Moreover, we have the continuous embedding

$$\mathring{\dot{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \hookrightarrow \dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{7.99}$$

Much as in Proposition 7.6, if $\Omega \subseteq \mathbb{R}^n$ is a two-sided NTA domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, then

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim \ni [f] \longmapsto \big\|[f]\big\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{7.100}$$

is a genuine norm on $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim$, and $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim$ is a Banach space when equipped with the norm (7.100).

In a similar fashion, we introduce the following brand of block-based homogeneous Sobolev spaces:

Definition 7.2 Suppose that $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an integrability exponent $q \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$. Also, introduce

$$q_\lambda := \frac{q(n-1)}{n-1+\lambda(q-1)} \in (1,q). \tag{7.101}$$

In this context, define the `block-based homogeneous Sobolev space` of order one on $\partial\Omega$ as

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^{q_\lambda}_{\rm loc}(\partial\Omega,\sigma) : \tag{7.102}$$

$$\partial_{\tau_{jk}} f \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \text{ for each } \; j, k \in \{1, \dots, n\}\Big\}$$

and equip this space with the semi-norm

$$\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma) \ni f \mapsto \|f\|_{\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.103}$$

It turns out that we have the continuous embeddings

$$\mathcal{B}^{q,\lambda}_1(\partial\Omega, \sigma) \hookrightarrow \dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big), \tag{7.104}$$

and

$$\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma) \hookrightarrow \dot{\boldsymbol{L}}^{q_\lambda}_1(\partial\Omega, \sigma). \tag{7.105}$$

In the context of Definition 7.2 it follows that all constant functions on $\partial\Omega$ belong to $\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)$ and their respective semi-norms vanish. We shall occasionally work with the space $\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big/\sim$, the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)$, which we equip with the semi-norm

$$\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big/\sim\; \ni [f] \mapsto \big\|[f]\big\|_{\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.106}$$

Analogously to Proposition 7.6, we have the following completeness result (see [112, §11.13] for a proof).

Proposition 7.7 *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and pick an integrability exponent $q \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Then (7.106) is a genuine norm on $\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big/\sim$, and $\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big/\sim$ is a Banach space when equipped with the norm (7.106).*

We continue by recording the following remarkable trace result proved in [112, §11.13].

Proposition 7.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an* NTA *domain such that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa \in (0, \infty)$ along with some integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. In this setting, the following statements are true.*

(1) For each function $u : \Omega \to \mathbb{C}$ satisfying

$$u \in \mathscr{C}^1(\Omega) \;\; \textit{and} \;\; \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \tag{7.107}$$

the nontangential trace

$$
\begin{aligned}
&u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \partial\Omega, \ \textit{belongs to } \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma), \\
&\textit{and } \Big\| u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)} \leq C \big\| \mathcal{N}_\kappa(\nabla u) \big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}
\end{aligned}
\tag{7.108}
$$

for some constant $C \in (0,\infty)$ *independent of* u.

(2) For each function $u : \Omega \to \mathbb{C}$ *satisfying*

$$
u \in \mathscr{C}^1(\Omega) \ \textit{ and } \ \mathcal{N}_\kappa(\nabla u) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \tag{7.109}
$$

the nontangential trace

$$
\begin{aligned}
&u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \partial\Omega, \ \textit{belongs to } \dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma), \\
&\textit{and } \Big\| u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\|_{\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)} \leq C \big\| \mathcal{N}_\kappa(\nabla u) \big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}
\end{aligned}
\tag{7.110}
$$

for some constant $C \in (0,\infty)$ *independent of* u.

(3) For each function $u \in \mathscr{C}^1(\Omega)$ *satisfying*

$$
\mathcal{N}_\kappa(\nabla u) \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \ \textit{ and } \ (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \partial\Omega, \tag{7.111}
$$

the nontangential trace

$$
\begin{aligned}
&u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \partial\Omega, \ \textit{belongs to } \dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma), \\
&\textit{and } \Big\| u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)} \leq C \big\| \mathcal{N}_\kappa(\nabla u) \big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}
\end{aligned}
\tag{7.112}
$$

for some constant $C \in (0,\infty)$ *independent of* u.

It has also been noted in [114, §3.3] that Theorems 3.3, 3.4, and Proposition 7.8 imply the following Fatou-type results and integral representation formulas.

Theorem 7.3 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an* NTA *domain such that* $\partial\Omega$ *is an unbounded Ahlfors regular set. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Let* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *(where* $M \in \mathbb{N}$*) be a complex coefficient tensor with the property that* $L := L_A$ *is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$. *In this setting, recall the modified version of the double layer operator* $\mathcal{D}_{A,\mathrm{mod}}$ *from* (3.49)*, and the modified version of the single layer operator* $\mathscr{S}_{\mathrm{mod}}$ *from* (3.38). *Fix an aperture parameter* $\kappa \in (0,\infty)$ *along with some integrability exponents* $p, q \in (1,\infty)$ *and a number* $\lambda \in (0, n-1)$. *Finally, consider a function* $u : \Omega \to \mathbb{C}^M$ *satisfying*

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \ \textit{and} \ \ Lu = 0 \ \ \textit{in} \ \ \Omega. \tag{7.113}$$

(1) If $\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma)$ *then*

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \ \textit{and belongs to} \ \ \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \ \textit{and} \ \ \partial_\nu^A u \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \end{aligned} \tag{7.114}$$

and there exists some $\mathbb{C}^M$*-valued locally constant function* c_u *in* Ω *with the property that*

$$u = \mathcal{D}_{A,\mathrm{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathscr{S}_{\mathrm{mod}}\big(\partial_\nu^A u\big) + c_u \ \ \textit{in} \ \ \Omega. \tag{7.115}$$

(2) If $\mathcal{N}_\kappa(\nabla u) \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *then*

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \ \textit{and belongs to} \ \ \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \ \textit{and} \ \ \partial_\nu^A u \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \end{aligned} \tag{7.116}$$

and (7.115) *continues to hold.*

(3) If $\mathcal{N}_\kappa(\nabla u) \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ *then*

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \ \textit{and belongs to} \ \ \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \ \textit{and} \ \ \partial_\nu^A u \in \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \end{aligned} \tag{7.117}$$

and (7.115) *once again continues to hold.*

We wish to augment Theorem 7.2 with a series of results dealing with modified boundary layer potentials.

Theorem 7.4 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Denote by* $\nu = (\nu_1, \dots, \nu_n)$ *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, for some* $M \in \mathbb{N}$, *let* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ *be a complex coefficient tensor with the property that* $L := L_A$ *is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$. *Recall the modified boundary-to-boundary single layer operator* S_{mod} *associated with* L *and* Ω *as in* (3.42). *Finally, fix two exponents* $p, q \in (1, \infty)$ *along with a parameter* $\lambda \in (0, n-1)$. *Then the following properties are true.*

(1) The modified boundary-to-boundary single layer operator induces a mapping

$$S_{\mathrm{mod}} : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \tag{7.118}$$

which is well defined, linear, and bounded, when the target space is endowed with the semi-norm (7.70). *In particular,*

$$\text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and pair of indices } j,k \in \{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}}\big(S_{\rm mod} f\big) \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.119}$$

Also, for each function $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$*, at* σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big)(x) \tag{7.120}$$

$$= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) \partial_{\tau_{js}}\big(S_{\rm mod} f\big)_\alpha(y)\,{\rm d}\sigma(y)\Bigg)_{1\leq\mu\leq M},$$

where $K^{\#}_{A^\top}$ *is the singular integral operator associated as in* (3.25) *with the set* Ω *and the transpose coefficient tensor* $A^\top$*. Finally,*

$$\text{for each sequence of functions } \{f_j\}_{j\in\mathbb{N}} \subseteq \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ which is weak-}*\text{ convergent to some } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and for each test function } \phi \in \big[{\rm Lip}(\partial\Omega)\big]^M \text{ with compact support one has } \lim_{j\to\infty} \int_{\partial\Omega} \big\langle S_{\rm mod} f_j, \phi\big\rangle\,{\rm d}\sigma = \int_{\partial\Omega} \big\langle S_{\rm mod} f, \phi\big\rangle\,{\rm d}\sigma. \tag{7.121}$$

(2) As a consequence of (7.118)*, the following is a well-defined linear operator:*

$$\big[S_{\rm mod}\big] : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M \text{ defined as}$$
$$\big[S_{\rm mod}\big]f := \big[S_{\rm mod} f\big] \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M, \tag{7.122}$$
$$\text{for all } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M.$$

Moreover, if actually $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator* (7.122) *is also bounded when the quotient space is endowed with the norm introduced in* (7.72).

(3) With $\mathscr{S}_{\rm mod}$ *denoting the modified version of the single layer operator acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (3.38)*, for each given aperture parameter* $\kappa > 0$ *there exists some constant* $C = C(\Omega, L, n, p, \lambda, \kappa) \in (0,\infty)$ *with the property that for each given function* $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\begin{gathered}\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \textit{ in } \ \Omega, \\ \mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big) \ \textit{ belongs to } \ M^{p,\lambda}(\partial\Omega, \sigma) \ \textit{ and} \\ \big\| \mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}, \\ \Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)(x) = (S_{\rm mod} f)(x) \ \textit{ at } \sigma\textit{-a.e. point } \ x \in \partial\Omega.\end{gathered} \tag{7.123}$$

Moreover, for each given function f in the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ the following jump formula holds (*with I denoting the identity operator*)

$$\partial^A_\nu \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2} I + K^{\#}_{A^\top}\big) f \ \textit{ at } \sigma\textit{-a.e. point in } \ \partial\Omega, \tag{7.124}$$

where $K^{\#}_{A^\top}$ is the singular integral operator associated as in (3.25) *with the set Ω and the transpose coefficient tensor $A^\top$.*

(4) Similar properties to those described in items (1)–(3) are valid for block spaces (*and block-based homogeneous Sobolev spaces*) *in place of Morrey spaces* (*and homogeneous Morrey-based Sobolev spaces*)*. More specifically, the operator*

$$S_{\rm mod} : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M \tag{7.125}$$

is well defined, linear, and bounded, when the target space is endowed with the semi-norm (7.103)*. Also,*

$$\begin{gathered}\big[S_{\rm mod}\big] : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big/\sim\big]^M \ \textit{ defined as} \\ \big[S_{\rm mod}\big] f := \big[S_{\rm mod} f\big] \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big/\sim\big]^M, \\ \textit{for all } f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M\end{gathered} \tag{7.126}$$

is a well-defined linear operator, which is also bounded in the case when $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set (*assuming the quotient space is endowed with the norm introduced in* (7.106))*. Finally, for each aperture parameter $\kappa > 0$ there exists $C = C(\Omega, L, n, q, \lambda, \kappa) \in (0, \infty)$ with the property that for each function $f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ one has*

$$\begin{gathered}\mathscr{S}_{\text{mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\text{mod}} f\big) = 0 \ \ in \ \ \Omega,\\ \mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\text{mod}} f\big) \ \ belongs\ to \ \ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma),\\ \big\|\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\text{mod}} f\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M},\\ \Big(\big(\mathscr{S}_{\text{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = (S_{\text{mod}} f)(x) \ \ at\ \sigma\text{-}a.e.\ point \ \ x \in \partial\Omega,\\ and \ \ \partial_\nu^A \mathscr{S}_{\text{mod}} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) f \ \ at\ \sigma\text{-}a.e.\ point\ in \ \ \partial\Omega.\end{gathered} \tag{7.127}$$

(5) Analogous properties to those presented in items (1)–(3) above are also valid for vanishing Morrey spaces $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ *(cf.* (7.7)*) and homogeneous vanishing Morrey-based Sobolev spaces* $\mathring{\dot{\mathbb{M}}}{}_1^{p,\lambda}(\partial\Omega,\sigma)$ *(cf. Definition 7.1) in place of Morrey spaces and homogeneous Morrey-based Sobolev spaces, respectively.*

This theorem has been established in [114, §3.3, §1.5]. Here we wish to note that an alternative argument may be given along the lines of the proof of item *(2)* in Theorem 8.5 (where a more general result of this flavor is obtained).

Some of the main properties of the modified boundary-to-domain double layer potential operators and their conormal derivatives acting on homogeneous Morrey-based and block-based Sobolev spaces on boundaries of UR domains are collected in the next theorem from [114, §3.3].

Theorem 7.5 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Denote by* $\nu = (\nu_1,\dots,\nu_n)$ *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. In addition, for some* $M \in \mathbb{N}$*, let* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ *be a complex coefficient tensor with the property that* $L := L_A$ *is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$*. Also, let* $E = (E_{\gamma\beta})_{1\le\gamma,\beta\le M}$ *be the matrix-valued fundamental solution associated with* L *as in Theorem 3.1. In this setting, recall the modified version of the double layer operator* $\mathcal{D}_{A,\text{mod}}$ *acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (3.49)*. Finally, fix some integrability exponents* $p, q \in (1,\infty)$ *along with a number* $\lambda \in (0, n-1)$*, and an aperture parameter* $\kappa \in (0,\infty)$*. Then the following statements are true.*

(1) There exists some constant $C = C(\Omega, A, n, p, \lambda, \kappa) \in (0,\infty)$ *with the property that for each function* $f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *it follows that*

$$\begin{gathered}\mathcal{D}_{A,\text{mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,\text{mod}} f\big) = 0 \ \ in \ \ \Omega,\\ \big(\mathcal{D}_{A,\text{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \ \ \big(\nabla\mathcal{D}_{A,\text{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exist\ \sigma\text{-}a.e.\ on \ \ \partial\Omega,\\ \mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\text{mod}} f\big) \ \ belongs\ to \ \ M^{p,\lambda}(\partial\Omega,\sigma) \ \ and\\ \big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\text{mod}} f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M}.\end{gathered} \tag{7.128}$$

In fact, for each function $f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *one has*

$$(\mathcal{D}_{A,\text{mod}} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f \quad \textit{at } \sigma\textit{-a.e. point on } \partial\Omega, \tag{7.129}$$

where I is the identity operator on $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, and $K_{A,\text{mod}}$ is the modified boundary-to-boundary double layer potential operator from (3.50) *and* (3.48).

(2) Given any function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the homogeneous Morrey-based Sobolev space $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, at σ-a.e. point $x \in \partial\Omega$ one has

$$\begin{aligned}\big(\partial_\nu^A(\mathcal{D}_{A,\text{mod}} f)\big)(x) = \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a_{ij}^{\mu\gamma} a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x-y) \times \\ \times \big(\partial_{\tau_{js}} f_\alpha\big)(y)\, d\sigma(y)\Bigg)_{1\leq\mu\leq M},\end{aligned} \tag{7.130}$$

where the conormal derivative is considered as in (3.66).

(3) The operator

$$\begin{aligned}&\partial_\nu^A \mathcal{D}_{A,\text{mod}} : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \quad \textit{defined as} \\ &\big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big) f := \partial_\nu^A(\mathcal{D}_{A,\text{mod}} f) \quad \textit{for each} \quad f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M\end{aligned} \tag{7.131}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm (7.70). *As a consequence of* (7.131)*, the following is a well-defined linear operator:*

$$\begin{aligned}\big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big] : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ \textit{given by} \quad \big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big][f] := \partial_\nu^A(\mathcal{D}_{A,\text{mod}} f) \\ \textit{for each function} \quad f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{aligned} \tag{7.132}$$

If, in fact, $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator (7.132) *is also bounded when the quotient space is equipped with the norm* (7.72).

(4) With $K_{A^\top}^\#$ denoting the singular integral operator associated as in (3.25) *with the set Ω and the transpose coefficient tensor $A^\top$, one has*

$$\begin{aligned}\big(\tfrac{1}{2}I + K_{A^\top}^\#\big)\big(-\tfrac{1}{2}I + K_{A^\top}^\#\big) = \big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big]\big[S_{\text{mod}}\big] \\ \textit{as mappings acting from} \quad \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M,\end{aligned} \tag{7.133}$$

and

$$\begin{aligned}&\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big]\big[K_{A,\mathrm{mod}}\big] = K^{\#}_{A^\top}\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big]\\ &\textit{as mappings acting from } \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M.\end{aligned}\tag{7.134}$$

Moreover, if $\partial\Omega$ is connected then also

$$\begin{aligned}&\big(\tfrac12 I + \big[K_{A,\mathrm{mod}}\big]\big)\big(-\tfrac12 I + \big[K_{A,\mathrm{mod}}\big]\big) = \big[S_{\mathrm{mod}}\big]\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big]\\ &\textit{as mappings acting from } \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M,\end{aligned}\tag{7.135}$$

and

$$\begin{aligned}&\big[S_{\mathrm{mod}}\big]K^{\#}_{A^\top} = \big[K_{A,\mathrm{mod}}\big]\big[S_{\mathrm{mod}}\big]\\ &\textit{as mappings acting from } \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{aligned}\tag{7.136}$$

(5) *Similar properties to those described in items (1)–(4) above are also valid for block spaces (and block-based homogeneous Sobolev spaces) in place of Morrey spaces (and homogeneous Morrey-based Sobolev spaces). Concretely, there exists a constant $C = C(\Omega, A, n, q, \lambda, \kappa) \in (0,\infty)$ with the property that for each function $f \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$ one has*

$$\begin{aligned}&\mathcal{D}_{A,\mathrm{mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,\mathrm{mod}} f\big) = 0 \ \textit{ in } \ \Omega,\\ &\big(\mathcal{D}_{A,\mathrm{mod}} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac12 I + K_{A,\mathrm{mod}}\big) f \ \textit{ at } \sigma\textit{-a.e. point on } \ \partial\Omega,\\ &\big(\nabla \mathcal{D}_{A,\mathrm{mod}} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{exist at } \sigma\textit{-a.e. point on } \ \partial\Omega,\\ &\mathcal{N}_\kappa\big(\nabla \mathcal{D}_{A,\mathrm{mod}} f\big) \ \textit{belongs to } \ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \ \textit{ and}\\ &\big\|\mathcal{N}_\kappa\big(\nabla \mathcal{D}_{A,\mathrm{mod}} f\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)]^M}.\end{aligned}\tag{7.137}$$

Also, formula (7.130) *remains true for each function $f = (f_\alpha)_{1\le\alpha\le M}$ belonging to the space $\big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$, and the operator*

$$\begin{aligned}&\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}} : \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \ \textit{ defined as}\\ &\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big) f := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \textit{ for each } \ f \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M\end{aligned}\tag{7.138}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm (7.103)*. Furthermore,*

$$\begin{aligned}&\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big] : \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M\\ &\textit{defined as } \big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big][f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f)\\ &\textit{for each } \ f \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M\end{aligned}\tag{7.139}$$

is a well-defined linear operator, which is also bounded when the quotient space is equipped with the norm (7.106) *if, in fact,* $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set. Finally, the operator identities in* (7.133)–(7.135) *are valid for functions in* $\big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M$.

(6) *Analogous properties to those presented in items (1)–(4) above are also valid for homogeneous vanishing Morrey-based Sobolev spaces* $\mathring{\dot{\mathbb{M}}}^{p,\lambda}_1(\partial\Omega,\sigma)$ (*cf. Definition 7.1*) *in place of homogeneous Morrey-based Sobolev spaces.*

We next study mapping properties for modified boundary-to-boundary double layer potential operators acting on homogeneous Morrey-based and block-based Sobolev spaces on boundaries of UR domains.

Theorem 7.6 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$ *satisfies* $n \geq 2$) *be an NTA domain such that* $\partial\Omega$ *is an Ahlfors regular set, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, let* $L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ *be a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order* $M \times M$ *system in* $\mathbb{R}^n$ (*for some integer* $M \in \mathbb{N}$). *In this context, consider the modified boundary-to-boundary double layer potential operator* $K_{A,\mathrm{mod}}$ *from* (3.50). *Finally, select some exponents* $p, q \in (1,\infty)$ *along with a parameter* $\lambda \in (0, n-1)$. *Then the following statements are valid.*

(1) *The modified boundary-to-boundary double layer potential operator induces a mapping*

$$K_{A,\mathrm{mod}} : \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \tag{7.140}$$

which is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (7.70). *As a corollary of* (7.140), *the following operator is well defined and linear:*

$$\begin{aligned} &\big[K_{A,\mathrm{mod}}\big] : \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M \\ &\textit{given by } \big[K_{A,\mathrm{mod}}\big][f] := \big[K_{A,\mathrm{mod}} f\big] \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M, \\ &\qquad \textit{for each function } f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.141}$$

Moreover, if actually $\Omega \subseteq \mathbb{R}^n$ *is a two-sided* NTA *domain whose boundary is an unbounded Ahlfors regular set then the operator* (7.141) *is also bounded when all quotient spaces are endowed with the norm introduced in* (7.72).

(2) *If* U_{jk} *with* $j, k \in \{1,\dots,n\}$ *is the family of singular integral operators defined in* (3.35), *then*

$$\partial_{\tau_{jk}}\big(K_{A,\text{mod}} f\big) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \ \ \textit{at } \sigma\textit{-a.e. point on } \ \partial\Omega$$
$$\textit{for each } f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \textit{ and each } j,k \in \{1,\dots,n\}. \tag{7.142}$$

(3) Similar properties to those described in items (1)–(2) are valid for block-based homogeneous Sobolev spaces in place of homogeneous Morrey-based Sobolev spaces. Specifically,

$$K_{A,\text{mod}} : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \tag{7.143}$$

is a well-defined, linear, and bounded operator when the spaces involved are endowed with the semi-norm (7.103). Also,

$$\big[K_{A,\text{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M$$
$$\textit{given by } \ \big[K_{A,\text{mod}}\big][f] := \big[K_{\text{mod}} f\big] \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \tag{7.144}$$
$$\textit{for each function } \ f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$$

is a well-defined linear mapping, which is also bounded when all quotient spaces are endowed with the norm introduced in (7.106) if in fact $\Omega \subseteq \mathbb{R}^n$ is a two-sided NTA *domain whose boundary is an unbounded Ahlfors regular set. Finally,*

$$\partial_{\tau_{jk}}\big(K_{A,\text{mod}} f\big) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \ \ \textit{at } \sigma\textit{-a.e. point on } \ \partial\Omega$$
$$\textit{for each } f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \textit{ and each } j,k \in \{1,\dots,n\}. \tag{7.145}$$

(4) Analogous properties to those presented in items (1)–(2) above are also valid for homogeneous vanishing Morrey-based Sobolev spaces $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ (cf. Definition 7.1) in place of homogeneous Morrey-based Sobolev spaces.

7.2 Inverting Double Layer Operators on Morrey and Block Spaces

The starting point is deriving estimates for the operator norms of singular integral operators whose integral kernels contain, as a factor, the crucial inner product between the unit normal and the "chord" (cf. (7.146), (7.147)), of the sort obtained earlier in Theorem 4.2 and Corollary 4.2 in the context of Muckenhoupt weighted Lebesgue spaces, but now working in the framework of Morrey spaces, vanishing Morrey spaces, and block spaces. We carry out this task in Theorem 7.7 below.

Theorem 7.7 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix an arbitrary integrability exponent $p \in (1, \infty)$ along with some parameter $\lambda \in (0, n-1)$. Also, consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ (for some sufficiently large integer $N = N(n) \in \mathbb{N}$) which is even and positive homogeneous of degree $-n$. In this setting consider the principal-value singular integral operators T, $T^\#$ acting on each given function $f \in M^{p,\lambda}(\partial\Omega, \sigma)$ according to*

$$Tf(x) := \lim_{\varepsilon \to 0^+} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x - y, \nu(y) \rangle k(x-y) f(y)\, \mathrm{d}\sigma(y), \tag{7.146}$$

and

$$T^\# f(x) := \lim_{\varepsilon \to 0^+} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle y - x, \nu(x) \rangle k(x-y) f(y)\, \mathrm{d}\sigma(y), \tag{7.147}$$

at σ-a.e. point $x \in \partial\Omega$. Also, define the action of the maximal operator T_* on each given function $f \in M^{p,\lambda}(\partial\Omega, \sigma)$ as

$$T_* f(x) := \sup_{\varepsilon > 0} \Bigg| \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x - y, \nu(y) \rangle k(x-y) f(y)\, \mathrm{d}\sigma(y) \Bigg| \quad \textit{for each} \quad x \in \partial\Omega, \tag{7.148}$$

and its companion

$$T_*^\# f(x) := \sup_{\varepsilon > 0} \Bigg| \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x - y, \nu(x) \rangle k(x-y) f(y)\, \mathrm{d}\sigma(y) \Bigg| \quad \textit{for each} \quad x \in \partial\Omega. \tag{7.149}$$

Then the following are well-defined, bounded operators

$$T_*, T_*^\#, T, T^\# : M^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow M^{p,\lambda}(\partial\Omega, \sigma), \tag{7.150}$$

$$T_*, T_*^\#, T, T^\# : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma), \tag{7.151}$$

and for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$, which depends only on m, n, p, λ, and the UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|T_*\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \leq \|T_*\|_{M^{p,\lambda}(\partial\Omega,\sigma) \to M^{p,\lambda}(\partial\Omega,\sigma)}$$

$$\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.152}$$

$$\begin{aligned}\big\|T^\#_*\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} &\leq \big\|T^\#_*\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{7.153}$$

$$\begin{aligned}\|T\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} &\leq \|T\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{7.154}$$

$$\begin{aligned}\big\|T^\#\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} &\leq \big\|T^\#\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}.\end{aligned} \tag{7.155}$$

Furthermore, for each $q\in(1,\infty)$ *the operators*

$$T,\,T^\#:\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\longrightarrow\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \tag{7.156}$$

are well defined, linear, bounded, and for each $m\in\mathbb{N}$ *there exists some constant* $C_m\in(0,\infty)$*, which depends only on* m*,* n*,* q*,* λ*, and the* UR *constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\begin{aligned}\max\Big\{\|T\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\,,\,\big\|T^\#\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\Big\}&\\ \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}.&\end{aligned} \tag{7.157}$$

Proof The claims made in (7.150)–(7.155) follow from Theorem 4.2, Corollary 4.2, and Proposition 7.4 (also keeping in mind (7.3) and (7.7)). Then the claims in (7.156)–(7.157) become consequences of what we have just proved and duality (cf. Proposition 7.2 and (7.46)). □

In concert with the commutator estimates discussed earlier (cf. Theorem 7.1), Theorem 7.7 implies the following result, which is the Morrey space (respectively, vanishing Morrey space, and block space) counterpart of Theorem 4.6.

Corollary 7.1 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu = (\nu_k)_{1\leq k\leq n}$ *the geometric measure theoretic outward unit normal to* Ω. *Also, fix two arbitrary integrability exponents* $p, q \in (1, \infty)$ *and a parameter* $\lambda \in (0, n-1)$. *Finally, recall the boundary-to-boundary harmonic double layer potential operator* K_Δ *from* (3.29)*, the Riesz transforms* $\{R_j\}_{1\leq j\leq n}$ *from* (4.297)*, and for each index* $k \in \{1, \dots, n\}$ *denote by* M_{ν_k} *the operator of pointwise multiplication by the* k*-th scalar component of* ν.

Then for each $m \in \mathbb{N}$ *there exists some* $C_m \in (0, \infty)$ *which depends only on* m, n, p, q, λ, *and the* UR *constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_\Delta\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.158}$$

$$\|K_\Delta\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.159}$$

and

$$\|K_\Delta\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.160}$$

Proof The estimates claimed in (7.158)–(7.160) are implied by (3.29), Theorem 7.7, (4.297), Proposition 3.4, and Theorem 7.1. □

We shall revisit Corollary 7.1 later, in Theorem 7.15, which contains estimates in the opposite direction to those obtained in (7.158)–(7.160).

For the time being, we take up the task of establishing estimates akin to those obtained in Theorem 4.7 for Muckenhoupt weighted Lebesgue and Sobolev spaces, now working in the setting of Morrey spaces, vanishing Morrey spaces, block spaces, as well as the brands of Sobolev spaces naturally associated with these scales.

Theorem 7.8 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Also, let* L *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$

system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in (3.24) *and* (3.25)*, respectively. Finally, fix two integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$.*

Then for each $m \in \mathbb{N}$ there exists some constant $C_m \in (0, \infty)$ which depends only on m, n, A, p, q, λ, and the UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_A\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.161}$$

$$\|K_A\|_{[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.162}$$

$$\|K_A\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.163}$$

$$\|K_A\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.164}$$

$$\|K_A\|_{[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.165}$$

$$\|K_A\|_{[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.166}$$

as well as

$$\big\|K_A^{\#}\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.167}$$

$$\big\|K_A^{\#}\big\|_{[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.168}$$

$$\big\|K_A^{\#}\big\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.169}$$

Proof All claims are justified as in the proof of Theorem 4.7, now making use of Theorem 7.7, Proposition 3.2, Theorem 7.1, (7.54)–(7.58), as well as (7.3), (7.8), (7.10), (7.17), (7.18). □

Remark 7.1 Similar estimates to those established in Theorem 7.8 are valid for the double layer operators acting on sums of Morrey spaces, vanishing Morrey spaces, and block spaces (cf. (4.332)).

The stage is now set for obtaining invertibility results for certain types of double layer potential operators acting on Morrey spaces, vanishing Morrey spaces, block spaces, as well as on the brands of Sobolev spaces naturally associated with these scales.

Theorem 7.9 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \emptyset$. Pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^\#$ associated with Ω and the coefficient tensor A as in* (3.24) *and* (3.25)*, respectively. Finally, fix two integrability exponents $p, q \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$, and some number $\varepsilon \in (0, \infty)$.*

Then there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, q, λ, A, ε, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if Ω is a δ-AR domain; cf. Definition* 2.15) *it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following operators are invertible:*

$$zI + K_A : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.170}$$

$$zI + K_A : \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.171}$$

$$zI + K_A : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.172}$$

$$zI + K_A : \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.173}$$

$$zI + K_A : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.174}$$

$$zI + K_A : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.175}$$

$$zI + K_A^\# : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.176}$$

$$zI + K_A^\# : \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.177}$$

$$zI + K_A^\# : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M. \tag{7.178}$$

In addition, the inverses in (7.170)–(7.175) *are compatible with one another and also with the inverses of* (4.309)–(4.310)*. Also, the inverses in* (7.176)–(7.178) *are compatible with one another and also with the inverse of* (4.311)*.*

Proof All claims are consequence of Theorem 7.8, reasoning as in the proof of Theorem 4.8 and Proposition 4.2. □

Remark 7.2 The conclusions in Theorem 7.9 may fail when $A \notin \mathfrak{A}_L^{\rm dis}$ even when Ω is a half-space. For example, from Proposition 3.13 and Theorem 7.2 we see that in such a scenario it may happen that $\frac{1}{2}I + K_A$ has an infinite dimensional cokernel when acting on Morrey and block spaces.

The operators in Remarks 4.14-4.15 (now considered on Morrey and block spaces) also offer counter-examples for the conclusions in Theorem 7.9 in the case when $A \notin \mathfrak{A}_L^{\rm dis}$ even when Ω is a half-space.

Remark 7.3 In the context of Theorem 7.9, if the threshold $\delta \in (0, 1)$ is taken sufficiently small in such a way that the operator $zI + K_A$ is invertible on the space $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ we also claim that there exists some constant $C \in (0, \infty)$ with the property that

$$\begin{aligned} &\text{whenever} \quad f \in \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \\ &\text{and} \quad g := (zI + K_A)^{-1} f \in \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \\ &\text{then} \quad \|\nabla_{\tan} g\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \leq C\, \|\nabla_{\tan} f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}. \end{aligned} \tag{7.179}$$

To justify this, use (3.37) to write, for each $j, k \in \{1, \dots, n\}$,

$$\begin{aligned} \partial_{\tau_{jk}} f &= \partial_{\tau_{jk}}\big[\big(zI + K_A\big)g\big] = \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}(\nabla_{\tan} g) \\ &= \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}\Big(\big(\nu_r \partial_{\tau_{rs}} g_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}\Big) \end{aligned} \tag{7.180}$$

at σ-a.e. point on $\partial\Omega$, where $\nu = (\nu_r)_{1\leq r\leq n}$ is the geometric measure theoretic outward unit normal to Ω. Using the abbreviations introduced in (4.345), the formulas in (7.180), corresponding to all indices $j, k \in \{1, \dots, n\}$, may be collectively re-fashioned as

$$\nabla_\tau f = \big(zI + R\big)(\nabla_\tau g), \tag{7.181}$$

where I is the identity and R is the operator acting from $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^{M\cdot n^2}$ into itself much as in (4.347)–(4.348). From these, (7.161), (3.35), Theorem 7.1, and (3.81), we then conclude that for each $m \in \mathbb{N}$ we have

$$\|R\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{M\cdot n^2} \to [M^{p,\lambda}(\partial\Omega,\sigma)]^{M\cdot n^2}} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle} \tag{7.182}$$

for some $C_m \in (0, \infty)$ which depends only on m, n, A, p, λ, and the UR constants of $\partial\Omega$. As a consequence of this, if we assume $\delta > 0$ to be sufficiently small to begin with, a Neumann series argument gives that

$$zI + R \ \text{ is invertible on } \ \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^{M\cdot n^2} \tag{7.183}$$

and provides an estimate for the norm of the inverse. At this stage, the estimate claimed in (7.179) follows from (7.181), (7.183), (4.345), and (2.585)–(2.586).

We may be further enhance the invertibility results from Theorem 7.9 by allowing the coefficient tensor to be a small perturbation of any distinguished coefficient tensor of the given system. Specifically, Theorem 7.8 in concert with the continuity of the operator-valued assignments $\mathfrak{A}_{\rm WE} \ni A \mapsto K_A$ and $\mathfrak{A}_{\rm WE} \ni A \mapsto K_A^{\#}$, considered in all contexts discussed in Theorem 7.2, yield the following result.

Theorem 7.10 *Retain the original background assumptions on the set Ω from Theorem 7.9 and, as before, fix some integrability exponents $p, q \in (1, \infty)$, a parameter $\lambda \in (0, n-1)$, and some number $\varepsilon \in (0, \infty)$. Consider $L \in \mathfrak{L}^{\rm dis}$ (cf. (3.195)) and pick an arbitrary $A_o \in \mathfrak{A}_L^{\rm dis}$. Then there exist some small threshold $\delta \in (0, 1)$ along with some open neighborhood $\mathcal{O}$ of A_o in $\mathfrak{A}_{\rm WE}$, both of which depend only on n, p, q, λ, A_o, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence, Ω is a δ-AR domain; cf. Definition 2.15) then for each $A \in \mathcal{O}$ and each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$, the operators (7.170)–(7.178) are invertible.*

We close this section with the following remark.

Remark 7.4 In the two-dimensional setting, more can be said about the Lamé system. Specifically, the versions of Theorem 4.14 and Corollary 4.3 naturally formulated in terms of Morrey spaces, vanishing Morrey spaces, block spaces, as well as their associated Sobolev spaces, continue to hold, virtually with the same proofs (now making use of Proposition 7.5, Theorems 7.1, 7.2, and 7.7).

7.3 Invertibility on Morrey/Block-Based Homogeneous Sobolev Spaces

The starting point in this section is the following counterpart of Theorem 4.10 containing operator norm estimates for double layer potentials associated with distinguished coefficient tensors on Morrey-based and block-based Sobolev spaces. As in the past, the key feature of said estimates is the explicit dependence on the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain.

Theorem 7.11 *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided* NTA *domain whose boundary is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix some integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Next, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Finally, pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the modified boundary-to-boundary double layer potential operator $\big[K_{A,{\rm mod}}\big]$ associated with Ω and the coefficient tensor A as in Theorem 7.6.*

Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$ which depends only on m, n, A, p, q, λ, the two-sided NTA *constants of Ω, and the Ahlfors regularity constant* *of* $\partial\Omega$*, such that, with the piece of notation introduced in* (4.93)*, one has*

$$\big\|\big[K_{A,\mathrm{mod}}\big]\big\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M\to[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.184}$$

$$\big\|\big[K_{A,\mathrm{mod}}\big]\big\|_{[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim]^M\to[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.185}$$

$$\big\|\big[K_{A,\mathrm{mod}}\big]\big\|_{[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M\to[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.186}$$

Proof The estimate claimed in (7.184) is justified much as in the proof of Theorem 4.10, making use of (7.141), (7.142), Theorem 7.7, and Theorem 7.1. For the estimate in (7.185), use (7.144), (7.145), Theorem 7.7, and Theorem 7.1. Finally, the estimate in (7.186) is dealt with similarly, relying on item *(4)* in Theorem 7.6. □

Having established Theorem 7.11, we now arrive at the first main result in this section concerning invertibility properties of boundary-to-boundary double layer potential operators associated with distinguished coefficient tensors on Morrey-based and block-based Sobolev spaces.

Theorem 7.12 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. Assume* L *is a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *and consider the modified boundary-to-boundary double layer potential operator* $\big[K_{A,\mathrm{mod}}\big]$ *associated with* Ω *and the coefficient tensor* A *as in Theorem 7.6. Finally, fix some integrability exponents* $p, q \in (1,\infty)$*, a parameter* $\lambda \in (0, n-1)$*, and some number* $\varepsilon \in (0,\infty)$*.*

Then there exists some small threshold $\delta \in (0,1)$ *which depends only on* n*,* p*,* q λ*,* A*,* ε*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*hence* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *it follows that for each spectral parameter* $z \in \mathbb{C}$ *with* $|z| \geq \varepsilon$ *the operators*

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \tag{7.187}$$

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \tag{7.188}$$

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \tag{7.189}$$

are all invertible.

Proof Pick $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then Ω is a two-sided NTA domain with an unbounded boundary. That this is possible is guaranteed by Theorem 2.3. Then all desired invertibility result follow (via a Neumann series argument) from Theorem 7.11. □

Remark 7.5 The conclusions in Theorem 7.12 may fail when $A \notin \mathfrak{A}_L^{\text{dis}}$ even when Ω is a half-space. For example, Proposition 3.13 and Theorem 7.5 imply that in such a case it may happen that $\frac{1}{2}I + \big[K_{A,\text{mod}}\big]$ has an infinite dimensional cokernel when acting on homogeneous Morrey-based and block-based Sobolev spaces.

Our next main result in this section, concerning the invertibility of S_{mod} in quotient Morrey/block spaces, reads as follows:

Theorem 7.13 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider the modified boundary-to-boundary single layer potential operator S_{mod} associated with Ω and the system L as in* (3.42)*. Fix some exponent $p \in (1,\infty)$ and $\lambda \in (0, n-1)$. Finally, use $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M$ to denote the M-th power of the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)$, equipped with the semi-norm* (7.72)*.*

Then the following statements are valid.

(1) [Boundedness] *If Ω satisfying a two-sided local John condition then the operator*

$$\begin{gathered}\big[S_{\text{mod}}\big] : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \\ \textit{defined as } \big[S_{\text{mod}}\big]f := \big[S_{\text{mod}}f\big] \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M, \\ \textit{for all } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M\end{gathered} \tag{7.190}$$

is well defined, linear, and bounded.

(2) [Surjectivity] *Whenever $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain; cf. Definition 2.15) it follows that* (7.72) *is a genuine norm and the operator* (7.190) *is surjective.*

(3) [Injectivity] *Whenever $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain) it follows that the operator* (7.190) *is injective.*

(4) [Isomorphism] *Whenever both $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence the domain Ω is a δ-*AR *domain) it follows that $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M$ is a Banach space when equipped with the norm* (7.72) *and the operator* (7.190) *is an isomorphism.*

(5) [Other spaces] *For each given $q \in (1,\infty)$, similar results to those described in items (1)–(4) are valid for the operator*

$$\begin{gathered}\big[S_{\rm mod}\big]:\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M\\ \textit{defined as }\big[S_{\rm mod}\big]f:=\big[S_{\rm mod}f\big]\in\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M,\\ \textit{for all }f\in\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M,\end{gathered}\tag{7.191}$$

as well as the operator

$$\begin{gathered}\big[S_{\rm mod}\big]:\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M\\ \textit{defined as }\big[S_{\rm mod}\big]f:=\big[S_{\rm mod}f\big]\in\big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M,\\ \textit{for all }f\in\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{gathered}\tag{7.192}$$

(6) [Optimality] *If* $\mathfrak{A}_L^{\rm dis}=\varnothing$ *then the operator* $\big[S_{\rm mod}\big]$ *may fail to be surjective (in fact, may have an infinite dimensional cokernel) in all settings considered above even in the case when* Ω *is a half-space, and if* $\mathfrak{A}_{L^\top}^{\rm dis}=\varnothing$ *then the operator* $\big[S_{\rm mod}\big]$ *may fail to be injective (in fact, may have an infinite dimensional kernel) in all settings considered above even in the case when* Ω *is a half-space.*

Proof That the operator (7.190) is well defined, linear, and bounded follows from item *(2)* in Theorem 7.4, bearing in mind (2.87) and (2.48). This takes care of item *(1)*.

To deal with the claims in item *(2)*, pick a coefficient tensor $A\in\mathfrak{A}_L^{\rm dis}$. Together, Theorems 2.3, 7.9, and 4.8 guarantee that we may choose a threshold $\delta\in(0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}<\delta$ (a condition which we shall henceforth assume) then

$$\Omega\text{ is a two-sided NTA domain with an unbounded boundary,}\tag{7.193}$$

and

$$\begin{gathered}\text{the operators }\pm\tfrac{1}{2}I+K_A\text{ are invertible on}\\ \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M\ \text{ and on }\ \big[L_1^p(\partial\Omega,\sigma)\big]^M.\end{gathered}\tag{7.194}$$

To proceed, choose a scalar-valued function $\phi\in\mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\phi\equiv1$ on $B(0,1)$ and $\operatorname{supp}\phi\subseteq B(0,2)$. Having fixed a reference point $x_0\in\partial\Omega$, for each scale $r\in(0,\infty)$ define ϕ_r as in (7.73) and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. Suppose now some arbitrary function $g\in\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ has been given, and for each $r\in(0,\infty)$ define g_r as in (7.74). Thanks to (7.69) we may invoke item *(iii)* in Lemma 7.1 which gives

$$\big\|\nabla_{\tan}g_r\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}\le C\big\|\nabla_{\tan}g\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}\tag{7.195}$$

for some $C \in (0, \infty)$ independent of g and r. For each $r \in (0, \infty)$ let us now define h_r as in (4.378) (here it helps to note that $\pm\frac{1}{2}I + K_A$ are invertible both on $\left[M_1^{p,\lambda}(\partial\Omega, \sigma)\right]^M$ and on $\left[L_1^p(\partial\Omega, \sigma)\right]^M$, and the two inverses are compatible). Using the formula $\partial_{\tau_{jk}} g_r = \left(\partial_{\tau_{jk}} \phi_r\right) \cdot \left(g - g_{\Delta_{2r}}\right) + \phi_r \cdot \partial_{\tau_{jk}} g$, the fact that the function g belongs to the space $\left[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\right]^M$, and (7.69) it is straightforward to show that $g_r \in \left[M_1^{p,\lambda}(\partial\Omega, \sigma) \cap L_1^p(\partial\Omega, \sigma)\right]^M$. Hence, h_r is a meaningfully defined function which belongs to $\left[M_1^{p,\lambda}(\partial\Omega, \sigma) \cap L_1^p(\partial\Omega, \sigma)\right]^M$. Moreover, from the definition of h_r (cf. (4.378)), (7.179), and (7.195) we conclude that there exists a constant $C \in (0, \infty)$, independent of g and r, such that

$$\|\nabla_{\tan} h_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \leq C\, \|\nabla_{\tan} g_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \quad \text{for each } \ r \in (0, \infty). \tag{7.196}$$

Going further, for each $r \in (0, \infty)$ abbreviate

$$f_r := \partial_\nu^A\left(\mathcal{D}_A h_r\right) \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{7.197}$$

Since $h_r \in \left[M_1^{p,\lambda}(\partial\Omega, \sigma) \cap L_1^p(\partial\Omega, \sigma)\right]^M$, the boundedness result recorded in (3.115) implies that $f_r \in \left[L^p(\partial\Omega, w)\right]^M$ and for each $r \in (0, \infty)$ we have

$$\|f_r\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\nabla_{\tan} h_r\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}, \tag{7.198}$$

where $C \in (0, \infty)$ is independent of g and r. Moreover, (7.64), (3.33), (3.66), (2.586), Proposition 7.5, and (7.196) permit us to write

$$\begin{aligned}\|f_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} &\leq C\|\nabla_{\tan} h_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \\ &\leq C\|\nabla_{\tan} g\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}.\end{aligned} \tag{7.199}$$

We use next that $h_r \in \left[L_1^p(\partial\Omega, \sigma)\right]^M$, (3.130), (4.378), (7.197), and Theorem 2.4 to ensure that for each $r \in (0, \infty)$ there exists some constant $c_r \in \mathbb{C}^M$ such that

$$S_{\text{mod}} f_r = g_r + c_r \ \text{ on } \ \partial\Omega. \tag{7.200}$$

Select now a sequence $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, \infty)$ which converges to infinity. Since from (7.199) we know that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\left[M^{p,\lambda}(\partial\Omega, \sigma)\right]^M$, we can rely on the Banach–Alaoglu Theorem (cf. (7.27)) and (7.24) to assume, without loss of generality, that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is actually weak-$*$ convergent to some function $f \in \left[M^{p,\lambda}(\partial\Omega, \sigma)\right]^M$. On account of (7.121), (7.200), and the definition of g_r given in (7.74), for each test function $\psi \in \left[\text{Lip}(\partial\Omega)\right]^M$ with compact support we may write

$$\begin{aligned}\int_{\partial\Omega}\big\langle S_{\rm mod}f,\psi\big\rangle\,{\rm d}\sigma &= \lim_{j\to\infty}\int_{\partial\Omega}\big\langle S_{\rm mod}f_{r_j},\psi\big\rangle\,{\rm d}\sigma = \lim_{j\to\infty}\int_{\partial\Omega}\big\langle g_{r_j}+c_{r_j},\psi\big\rangle\,{\rm d}\sigma\\ &= \lim_{j\to\infty}\int_{\partial\Omega}\big\langle \phi_{r_j}\cdot\big(g-g_{\Delta_{2r_j}}\big)+c_{r_j},\psi\big\rangle\,{\rm d}\sigma\\ &= \lim_{j\to\infty}\int_{\partial\Omega}\big\langle g-g_{\Delta_{2r_j}}+c_{r_j},\psi\big\rangle\,{\rm d}\sigma\\ &= \int_{\partial\Omega}\langle g,\psi\rangle\,{\rm d}\sigma+\lim_{j\to\infty}\Big\langle c_{r_j}-g_{\Delta_{2r_j}},\int_{\partial\Omega}\psi\,{\rm d}\sigma\Big\rangle. \end{aligned} \tag{7.201}$$

Since ψ is arbitrary, we conclude that the sequence $\big\{c_{r_j}-g_{\Delta_{2r_j}}\big\}_{j\in\mathbb{N}}\subseteq\mathbb{C}^M$ converges to some constant $c\in\mathbb{C}^M$. Hence, we may then conclude from (7.201) that

$$\int_{\partial\Omega}\big\langle S_{\rm mod}f,\psi\big\rangle\,{\rm d}\sigma = \int_{\partial\Omega}\langle g+c,\psi\rangle\,{\rm d}\sigma \tag{7.202}$$

for each function $\psi\in\big[{\rm Lip}(\partial\Omega)\big]^M$ with compact support. Eventually, from (7.202) we obtain (see [111, §3.7] for a general measure theoretic result of this nature)

$$S_{\rm mod}f = g+c \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.203}$$

Hence, $\big[S_{\rm mod}\big]f=\big[S_{\rm mod}f\big]=[g]$ and since $[g]\in\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M$ is arbitrary, it follows that the operator (7.190) is surjective. Moreover, from (7.199) we see that

$$\begin{aligned}\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} &\leq \limsup_{j\to\infty}\|f_{r_j}\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\leq C\,\|\nabla_{\tan}g\|_{M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}\\ &\leq C\,\|[g]\|_{[\dot{M}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M},\end{aligned} \tag{7.204}$$

for some constant $C\in(0,\infty)$ independent of g, so the surjectivity of the operator in (7.190) comes with quantitative control.

Let us also observe that the fact that (7.72) is, as claimed, a genuine norm is clear from (7.193) and Proposition 7.6.

Moving on, let us now deal with item *(3)*. Pick a coefficient tensor $\widetilde{A}\in\mathfrak{A}_L$ such that $\widetilde{A}^\top\in\mathfrak{A}^{\rm dis}_{L^\top}$. By Theorem 7.9 we may then choose $\delta\in(0,1)$ small enough so that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n}<\delta$ (something we shall henceforth assume) then

$$\text{the operators } \pm\tfrac{1}{2}I+K^{\#}_{\widetilde{A}^\top}\text{ are invertible on } \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.205}$$

The goal is to show that the operator (7.190) is injective. To this end, suppose the function $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ is such that $\big[S_{\rm mod}\big]f = [0]$. Hence, $\big[S_{\rm mod} f\big] = [0]$ which implies that there exists some constant $c \in \mathbb{C}^M$ for which

$$S_{\rm mod} f = c \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.206}$$

This, together with (7.120), allows us to obtain

$$\big(\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^{\top}}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^{\top}}\big)f\Big) = 0 \;\text{ at } \sigma\text{-a.e. point on } \partial\Omega \tag{7.207}$$

which, by (7.205), leads to $f = 0$. Since the operator (7.190) is linear, it follows that this is indeed injective.

Next, to treat the claims in item *(4)*, assume that $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^{\top}} \neq \varnothing$. Then, by the previous items the operator (7.190) is a continuous bijection. Moreover, Proposition 7.6 and (7.193) imply that $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M$ is a Banach space, hence the operator (7.190) is a linear isomorphism.

Considered now the claims made in item *(5)*. First, the fact that the operator (7.191) is well defined, linear, and bounded is seen from item *(4)* in Theorem 7.4, keeping in mind (2.87) and (2.48). Second, that the operator (7.191) satisfies the properties described in items *(2)–(3)* of Theorem 7.13 is a consequence of the operator identities

$$\begin{gathered}\big(\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big) = \big[\partial^A_\nu \mathcal{D}_{A,\rm mod}\big]\big[S_{\rm mod}\big] \\ \text{as mappings acting from } \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M,\end{gathered} \tag{7.208}$$

and

$$\begin{gathered}\big(\tfrac{1}{2}I + \big[K_{A,\rm mod}\big]\big)\big(-\tfrac{1}{2}I + \big[K_{A,\rm mod}\big]\big) = \big[S_{\rm mod}\big]\big[\partial^A_\nu \mathcal{D}_{A,\rm mod}\big] \\ \text{as mappings acting from } \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M,\end{gathered} \tag{7.209}$$

both of which are contained in Theorem 7.5, (7.178) in Theorem 7.9 (specialized to $z = \pm\frac{1}{2}$), (7.188) in Theorem 7.12 (again with $z = \pm\frac{1}{2}$), as well as (7.139), Theorem 2.3, and Theorem 2.4. The case of the operator $\big[S_{\rm mod}\big]$ in (7.192) is handled analogously.

Finally, the optimality results in item *(6)* are seen from (3.406) and the natural version of Proposition 4.4 for Morrey and block spaces. □

Remark 7.6 Together, (7.133), Theorem 7.9 (with $z = \pm\frac{1}{2}$), (7.135), Theorem 7.12 (with $z = \pm\frac{1}{2}$), (7.132), Theorems 2.3, and 2.4 provide an alternative proof of items *(2)–(3)* in Theorem 7.13.

We conclude this section with the following theorem addressing the issue of invertibility for the conormal of the double layer operator acting from homogeneous Morrey-based and block-spaces Sobolev spaces.

Theorem 7.14 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Fix some exponent $p \in (1, \infty)$ along with some parameter $\lambda \in (0, n-1)$. Pick some coefficient tensor $A \in \mathfrak{A}_L$ and consider the modified conormal derivative of the modified double layer operator in the context of (7.132), i.e.,*

$$\begin{aligned} &\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}} : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \ \textit{ defined as} \\ &\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)[f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \textit{ for each } \ f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.210}$$

From Theorem 7.5 this is known to be a well-defined, linear, and bounded operator when the quotient space is equipped with the norm (7.72). In relation to this, the following statements are valid.

(1) [*Injectivity*] *Whenever $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and actually $A \in \mathfrak{A}_L^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-AR domain; cf. Definition 2.15) then the operator (7.210) is injective.*

(2) [*[*Surjectivity] *Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ and actually $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain) then the operator (7.210) is surjective.*

(3) [Isomorphism] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, and $A \in \mathfrak{A}_L^{\mathrm{dis}}$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain) then the operator (7.210) is an isomorphism.*

(4) [Other spaces] *For each $q \in (1,\infty)$, similar results to those described in items (1)–(3) above are valid for the modified conormal derivative of the modified double layer operator in the context of block and vanishing Morrey spaces, i.e.,*

$$\begin{aligned} &\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \ \textit{ given by} \\ &\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big][f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \textit{ for each } \ f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{7.211}$$

and

$$\begin{aligned} &\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}} : \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \ \textit{ given by} \\ &\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)[f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \textit{ for each } \ f \in \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.212}$$

Proof All claims may be established by arguing as in the proof of Theorem 4.13, now making use of Theorems 7.5, 7.9, and 7.12. □

7.4 Characterizing Flatness in Terms of Morrey and Block Spaces

How do the quantitative aspects of the analysis of a certain geometric environment affect the very geometric features of said environment? Here we address a specific aspect of this general question by characterizing the flatness of a "surface" in terms of the size of the norms of certain singular integral operators acting on Morrey and block spaces considered on this surface.

In order be able to elaborate on this topic, we need some notation. Given a UR domain $\Omega \subseteq \mathbb{R}^n$, denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. From Proposition 7.4 and (5.16)–(5.18) we then conclude that whenever $p \in (1, \infty)$ and $\lambda \in (0, n-1)$, the operators

$$\mathbf{C} : M^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n, \tag{7.213}$$

$$\mathbf{C} : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n, \tag{7.214}$$

and

$$\mathbf{C}^{\#} : M^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n, \tag{7.215}$$

$$\mathbf{C}^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \tag{7.216}$$

are all well defined, linear, and continuous, with

$$\begin{aligned} &\|\mathbf{C}\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \\ &\|\mathbf{C}^{\#}\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \\ &\|\mathbf{C}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \\ &\|\mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \end{aligned} \tag{7.217}$$

bounded exclusively in terms of n, p, λ, and the UR constants of $\partial\Omega$.

Granted these, via duality (cf. (5.19) and Proposition 7.2) we also obtain that for each $q \in (1, \infty)$ and $\lambda \in (0, n-1)$ the operators

$$\mathbf{C} : \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n, \tag{7.218}$$

$$\mathbf{C}^{\#} : \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n \longrightarrow \mathcal{B}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n \tag{7.219}$$

are all well defined, linear, and bounded, with

$$\begin{aligned} &\|\mathbf{C}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n}, \\ &\|\mathbf{C}^{\#}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \end{aligned} \tag{7.220}$$

controlled only in terms of n, q, λ, and the UR constants of $\partial\Omega$.

In addition, from (5.20) and duality (cf. (5.19) and Proposition 7.2) we conclude that, for each $p, q \in (1, \infty)$ and $\lambda \in (0, n-1)$,

the operator identities $\mathbf{C}^2 = \frac{1}{4}I$ and $\big(\mathbf{C}^{\#}\big)^2 = \frac{1}{4}I$ are valid on either of the spaces $M^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$, $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$, and $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$. (7.221)

More delicate estimates than (7.217), (7.220) turn out to hold for the antisymmetric part of the Cauchy–Clifford operator, i.e., for the difference $\mathbf{C} - \mathbf{C}^{\#}$, of the sort described in the proposition below.

Proposition 7.9 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix two integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Then for each $m \in \mathbb{N}$ there exists some constant $C_m \in (0, \infty)$ which depends only on m, n, p, q, λ, and the* UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.222}$$

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.223}$$

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.224}$$

Proof This is implied by the structural result from Lemma 5.1 (bearing in mind (7.3), (7.8), (7.17)), together with Theorems 7.1, 7.7, and (3.29). □

Remarkably, it is also possible to establish bounds from below for the operator norm of $\mathbf{C} - \mathbf{C}^{\#}$ on Morrey spaces and their pre-duals, considered on the boundary of a UR domain, in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal vector to the said domain.

Proposition 7.10 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain such that $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1, \infty)$ along* *with a parameter* $\lambda \in (0, n-1)$. *Then there exists some* $C \in (0, \infty)$ *which depends only on n, p,* λ*, and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}$$
$$\leq C \left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}. \tag{7.225}$$

Furthermore, for each $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *there exists some constant* $C \in (0, \infty)$ *which depends only on n, q,* λ*, and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}. \tag{7.226}$$

Proof The argument largely follows the proof of the unweighted version of Theorem 5.1 (i.e., when $w \equiv 1$), so we will only indicate the main changes. First, in place of (5.45) we now write (making use of (7.2), the fact that $\mathbf{1}_{\Delta(y_0,R)} \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$, and (7.5))

$$\begin{aligned}
&⨍_{\Delta(x_0,R)} |(\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|^p \, d\sigma(x)\\
&\quad \leq R^{-(n-1-\lambda)} \left\|(\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}\right\|^p_{M^{p,\lambda}(\partial\Omega,\sigma)}\\
&\quad \leq R^{-(n-1-\lambda)} \left\|\mathbf{1}_{\Delta(y_0,R)}\right\|^p_{M^{p,\lambda}(\partial\Omega,\sigma)} \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}\\
&\quad \leq C R^{-(n-1-\lambda)} \sigma\big(\Delta(y_0, R)\big)^{(n-1-\lambda)/(n-1)} \times\\
&\qquad \times \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}\\
&\quad \leq C \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},
\end{aligned} \tag{7.227}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$.

Second, thanks to (7.227), in place of (5.46) we have

$$\begin{aligned}
&⨍_{\Delta(x_0,R)} \left| \int_{\Delta(y_0,R)} \left\{ \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) + \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n} \right\} d\sigma(y) \right|^p d\sigma(x)\\
&\leq C(\Lambda^{-n} \ln \Lambda)^p \|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + C_{n,p} ⨍_{\Delta(x_0,R)} |(\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|^p \, d\sigma(x)
\end{aligned}$$

$$+ C\Lambda^{-np} \fint_{\Delta(x_0,R)} \big|\nu(x) - \nu_{\Delta(x_0,R)}\big|^p \, d\sigma(x)$$

$$\leq C(\Lambda^{-n} \ln \Lambda)^p \|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$$

$$+ C\|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \tag{7.228}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$.

Third, with (7.228) in hand, the same type of argument as in the end-game of the proof of Theorem 5.1 (cf. (5.47)–(5.54)) presently gives

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C(\Lambda^{-1} \ln \Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$$

$$+ C\Lambda^{n-1}\|\mathbf{C} - \mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \tag{7.229}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$. By eventually further increasing Λ as to ensure that $\Lambda^{-1} \ln \Lambda < 1/(2C)$, we finally conclude from (7.229) that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\big\|\mathbf{C} - \mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \tag{7.230}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$. This establishes the first estimate claimed in (7.225). The second estimate in (7.225) is a direct consequence of (7.8).

Finally, the estimate claimed in (7.226) follows from the first inequality in (7.225), plus the fact that whenever $p, q \in (1, \infty)$ are such that $1/p + 1/q = 1$ then the (real) transpose of

$$\mathbf{C} - \mathbf{C}^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \tag{7.231}$$

is the operator

$$\mathbf{C}^{\#} - \mathbf{C} : \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n. \tag{7.232}$$

See (5.19) and Proposition 7.2 in this regard. □

Our next result contains estimates in the opposite direction to those presented in Corollary 7.1.

Theorem 7.15 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix two arbitrary integrability exponents $p, q \in (1, \infty)$ along with some parameter $\lambda \in (0, n-1)$. Finally, recall the boundary-to-boundary harmonic double* *layer potential operator* K_Δ *from* (3.29)*, the Riesz transforms* $\{R_j\}_{1\le j\le n}$ *on* $\partial\Omega$ *from* (4.297)*, and for each index* $k \in \{1,\dots,n\}$ *denote by* M_{ν_k} *the operator of pointwise multiplication by the* k*-th scalar component of* ν*. Then there exists some* $C \in (0,\infty)$ *which depends only on* n*,* p*,* q*,* λ*, and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{ \|K_\Delta\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\le j,k\le n} \big\|[M_{\nu_k}, R_j]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \Big\}, \tag{7.233}$$

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{ \|K_\Delta\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\le j,k\le n} \big\|[M_{\nu_k}, R_j]\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \Big\}, \tag{7.234}$$

and

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{ \|K_\Delta\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \max_{1\le j,k\le n} \big\|[M_{\nu_k}, R_j]\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \Big\}. \tag{7.235}$$

Proof If $\partial\Omega$ is unbounded then all estimates are implied by Proposition 7.10 and the structural result from Lemma 5.1 (keeping in mind (7.3), (7.8), (7.17)). When $\partial\Omega$ is bounded, we have $K_\Delta 1 = \pm\frac{1}{2}$ (cf. [114, §1.5]) with the sign plus if Ω is bounded, and the sign minus if Ω is unbounded, hence the norm of K_Δ on either $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ or $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ is $\ge \frac{1}{2}$ in such a case. Given that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le 1$ (cf. (2.118)), the estimates claimed in (7.233)–(7.235) are valid in this case if we take $C \ge 2$. □

In turn, the results established in Theorem 7.15 may be used to characterize the class of δ-AR domains in $\mathbb{R}^n$, in the spirit of Corollary 5.2, using Morrey spaces and their pre-duals.

By way of contrast, Theorem 7.16 discussed next is a stability result stating that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain with an unbounded boundary for which the URTI (cf. (5.58)) are "almost" true in the context of either Morrey or block spaces, then $\partial\Omega$ is "almost" flat, in that the BMO semi-norm of the outward unit normal to Ω is small.

Theorem 7.16 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain with an unbounded boundary. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, fix* $p,q \in (1,\infty)$ *along with* $\lambda \in (0,n-1)$*, and recall the Riesz transforms* $\{R_j\}_{1\le j\le n}$ *on* $\partial\Omega$ *from* (4.297)*. Then there exists some* $C \in (0,\infty)$ *which depends only on* n*,* p*,* q*,* λ*, and the* UR *constants of* $\partial\Omega$

with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{\Big\|I+\sum_{j=1}^{n}R_j^2\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n}\big\|[R_j,R_k]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\Big\}, \tag{7.236}$$

plus similar estimates with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$*, or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$*.*

Proof A key ingredient is the fact that we have the operator identities

$$\mathbf{C}-\mathbf{C}^{\#} = \mathbf{C}\Big(I+\sum_{j=1}^{n}R_j^2\Big) + \sum_{1\leq j<k\leq n}\mathbf{C}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k \quad \text{on } M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n, \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n. \tag{7.237}$$

These are proved much like formula [61, (4.6.46), p. 2752], now making use of (7.221). Once (7.237) has been established, Proposition 7.10 and (7.213)–(7.220) to conclude (much as in the proof of Theorem 5.3) that the estimate claimed in (7.236) as well as its related versions on vanishing Morrey spaces and block spaces are all true. □

The last result in this section contains estimates in the opposite direction to those from Theorem 7.16. Together, Theorems 7.17 and 7.16 amount to saying that, under natural background geometric assumptions on the set Ω, the URTI are "almost" true on Morrey spaces or block spaces if and only if ∂Ω is "almost" flat (in that the BMO semi-norm of the outward unit normal to Ω is small).

Theorem 7.17 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, fix* $p, q \in (1,\infty)$ *along with* $\lambda \in (0, n-1)$*, and recall the Riesz transforms* $\{R_j\}_{1\leq j\leq n}$ *on* ∂Ω *from* (4.297)*.*

Then for each $m \in \mathbb{N}$ *there exists some constant* $C_m \in (0,\infty)$ *which depends only on* m*,* n*,* p*,* q*,* λ*, and the* UR *constants of* ∂Ω *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\Big\|I+\sum_{j=1}^{n}R_j^2\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.238}$$

$$\max_{1\leq j<k\leq n}\big\|[R_j,R_k]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.239}$$

plus similar estimates with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$*, or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$*.*

Proof The starting point is to observe that we have the operator identities

$$\begin{aligned} &\mathbf{C}(\mathbf{C}^{\#}-\mathbf{C}) = -\tfrac{1}{4}\Big(I+\sum_{j=1}^{n} R_j^2\Big) - \tfrac{1}{4}\sum_{1\leq j<k\leq n} [R_j, R_k]\mathbf{e}_j \odot \mathbf{e}_k, \\ &\text{on } M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n,\ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n, \end{aligned} \tag{7.240}$$

which are themselves consequences of (7.237) and (7.221). With (7.240) in hand, the estimates claimed in the statement of the theorem may then be justified via an estimate similar in spirit to (5.66), and also invoking Proposition 7.9 (as well as (7.217), (7.220)) in the process. □

7.5 Boundary Value Problems in Morrey and Block Spaces

We begin by discussing the Dirichlet Problem for weakly elliptic systems in δ-AR domains with boundary data in ordinary Morrey spaces, vanishing Morrey spaces, and block spaces.

Theorem 7.18 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa > 0$. Also, pick an exponent $p \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, consider the Dirichlet Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{cases} \tag{7.241}$$

The following claims are true:

(a) [Existence, Regularity, and Estimates] *If $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\text{dis}}$, then there exists $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-AR domain; cf. Definition 2.15) then $\frac{1}{2}I + K_A$ is an invertible operator on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \Big(\mathcal{D}_A\Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x) \ \ \textit{for all} \ \ x \in \Omega, \tag{7.242}$$

is a solution of the Dirichlet Problem (7.241). Moreover,

$$\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{7.243}$$

Furthermore, the function u defined in (7.242) *satisfies the following regularity result*

$$\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma) \Longleftrightarrow f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{7.244}$$

and if either of these conditions holds then

$$\begin{gathered}(\nabla u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists (in } \mathbb{C}^{n\cdot M}\text{) at } \sigma\text{-a.e. point on } \partial\Omega \text{ and}\\ \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}.\end{gathered} \tag{7.245}$$

(b) [Uniqueness] *Whenever* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$, *there exists some* $\delta \in (0,1)$ *which depends only on n, p,* $[w]_{A_p}$, *L,* η, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*hence* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the Dirichlet Problem* (7.241) *has at most one solution.*

(c) [Well-Posedness] *If* $\mathfrak{A}^{\mathrm{dis}}_{L} \neq \varnothing$ *and* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$ *then there exists some* $\delta \in (0,1)$ *which depends only on n, p,* $[w]_{A_p}$, *A,* η, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that whenever* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the Dirichlet Problem* (7.241) *is uniquely solvable and the solution satisfies* (7.243).

(d) [Sharpness] *If* $\mathfrak{A}^{\mathrm{dis}}_{L} = \varnothing$ *then the Dirichlet Problem* (7.241) *may fail to be solvable* (*actually for boundary data belonging to an infinite dimensional subspace of the corresponding Morrey space*)*. Also, if* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$ *then the Dirichlet Problem* (7.241) *may have more than one solution* (*in fact, the linear space of null-solutions may actually be infinite dimensional*).

(e) [Other Spaces of Boundary Data] *Similar results to those described in items (a)–(d) above hold with the Morrey space* $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, *or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$.

In addition, given any pair of integrability exponents $p_0, p_1 \in (1,\infty)$ *along with any pair of parameters* $\lambda_0, \lambda_1 \in (0, n-1)$, *similar results are valid for the Dirichlet Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa u \in M^{p_0,\lambda_0}(\partial\Omega,\sigma) + M^{p_1,\lambda_1}(\partial\Omega,\sigma), \\ u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[M^{p_0,\lambda_0}(\partial\Omega,\sigma) + M^{p_1,\lambda_1}(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{7.246}$$

as well as for its versions with the Morrey spaces replaced by vanishing Morrey space or block spaces.

To give an example, suppose $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain and fix an arbitrary aperture parameter $\kappa > 0$ along with some power $a \in (0, n-1)$. In addition, choose a number $\lambda \in (0, n-1-a)$ and define $p := (n-1-\lambda)/a \in (1, \infty)$. Then, if $\delta > 0$ is sufficiently small (relative to n, a, λ, and the Ahlfors regularity constant of $\partial\Omega$), it follows that for each point $x_o \in \partial\Omega$ the Dirichlet Problem

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega, \sigma), \\ \Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = |x - x_o|^{-a} \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega \end{cases} \tag{7.247}$$

has a unique solution. Moreover, there exists a constant $C(\Omega, n, \kappa, a, \lambda) \in (0, \infty)$ with the property that said solution satisfies $\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a, \lambda)$. The reason is that, as seen from Example 7.1, the function $f_{x_o}(x) := |x - x_o|^{-a}$ for σ-a.e. point $x \in \partial\Omega$ belongs to the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ and we have $\sup_{x_o \in \partial\Omega} \|f_{x_o}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty$. As such, the result in item *(c)* of Theorem 7.18 applies and yields the desired conclusion.

In addition, there is a naturally accompanying regularity result. To formulate it, assume $q \in (1, \infty)$ and $\mu \in (0, n-1)$ are such that $a + 1 = (n - 1 - \mu)/q$. Starting from the realization that the boundary datum f_{x_o} actually belongs to a suitably defined off-diagonal Morrey-based Sobolev space on $\partial\Omega$, from (6.37) and Example 7.1 we see that there exists $C(\Omega, n, \kappa, a, q, \mu) \in (0, \infty)$ independent of $x_o \in \partial\Omega$ such that, if $\delta > 0$ is sufficiently small to begin with, then the unique solution of the Dirichlet Problem (7.247) satisfies the following additional regularity properties

$$\begin{aligned} &(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{R}^n) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\ &\text{and } \|\mathcal{N}_\kappa(\nabla u)\|_{M^{q,\mu}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a, q, \mu). \end{aligned} \tag{7.248}$$

To wrap up the discussion about (7.247) we wish to note that since the inverse of $\frac{1}{2}I + K_\Delta$ on $M^{p,\lambda}(\partial\Omega, \sigma)$ is compatible with the inverse of $\frac{1}{2}I + K_\Delta$ on $L^{p,\infty}(\partial\Omega, \sigma)$ (as alluded to in Remark 4.20), we conclude (from the manner in which the solution is constructed; cf. (7.242)) that the solution u of the Dirichlet Problem (7.247) actually coincides with the solution u of the Dirichlet Problem (6.35).

In closing, let us also mention that boundary value problems in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with boundary data with components in the Morrey spaces $M^{2,\lambda}(\partial\Omega, \sigma)$ (with λ belonging to a certain sub-interval of $(0, n-1)$) for symmetric, homogeneous, second-order, systems with constant real coefficients satisfying the Legendre–Hadamard strong ellipticity condition have been considered in [127].

After this digression we turn to the task of giving the proof of Theorem 7.18.

Proof of Theorem 7.18 The argument parallels the proof of Theorem 6.2. First, Theorem 7.9 shows that there exists some number $\delta \in (0, 1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $\frac{1}{2}I + K_A$ is invertible on the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$. Hence, the function u in (7.242) is meaningfully defined, and according to (3.23), (7.3), and Theorem 7.2, we have $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, $Lu = 0$ in Ω, $\mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega, \sigma)$, and (7.243) holds. Concerning the equivalence claimed in (7.244), if $f \in \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ then Theorem 7.9 gives (assuming $\delta > 0$ is sufficiently small) that $\Big(\frac{1}{2}I + K_A\Big)^{-1} f$ belongs to $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$. With this in hand, (7.64)–(7.65) then imply that the function u defined as in (7.242) satisfies $\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma)$, the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $\partial\Omega$, and the left-pointing inequality in the equivalence claimed in (7.245) holds. In particular, this justifies the left-pointing implication in (7.244). The right-pointing implication in (7.244) together with the right-pointing inequality in the equivalence claimed in (7.245) are consequences of (7.3) and Proposition 2.22.

Turning our attention to the uniqueness result claimed in item *(b)*, make the assumption that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$. Also, denote by $q \in (1, \infty)$ the Hölder conjugate exponent of p. From Theorem 7.9, presently used with L replaced by $L^\top$, we know that there exists $\delta \in (0, 1)$, which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if Ω is a δ-AR domain then the following operator is invertible:

$$\tfrac{1}{2}I + K_{A^\top} : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M. \tag{7.249}$$

Also, decreasing the value of $\delta \in (0, 1)$ if necessary guarantees that Ω is an NTA domain with unbounded boundary (cf. Theorem 2.3). In such a case, (6.2) ensures that Ω is globally pathwise nontangentially accessible.

Moving on, recall the fundamental solution $E = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ associated with the system L as in Theorem 3.1. Pick $x_\star \in \mathbb{R}^n \setminus \overline{\Omega}$ along with $x_0 \in \Omega$, arbitrary. Also, fix $\rho \in \big(0, \frac{1}{4}\,\mathrm{dist}(x_0, \partial\Omega)\big)$ and define $K := \overline{B(x_0, \rho)}$. Finally, recall the aperture parameter $\widetilde{\kappa} > 0$ associated with Ω and κ as in Theorem 6.1. To proceed, for each fixed index $\beta \in \{1, \dots, M\}$, consider the $\mathbb{C}^M$-valued function

$$f^{(\beta)}(x) := \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big)_{1\leq\alpha\leq M}, \qquad \forall\, x \in \partial\Omega. \tag{7.250}$$

Based on (7.19), (7.250), (7.57), (2.579), (7.21), (3.16), and the Mean Value Theorem we then conclude that

$$f^{(\beta)} \in \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M. \tag{7.251}$$

Consequently, with $\Big(\frac{1}{2}I + K_{A^\top}\Big)^{-1}$ denoting the inverse of the operator in (7.249),

$$v_\beta := \big(v_{\beta\alpha}\big)_{1\leq\alpha\leq M} := \mathcal{D}_{A^\top}\Big(\Big(\tfrac{1}{2}I + K_{A^\top}\Big)^{-1} f^{(\beta)}\Big) \tag{7.252}$$

is a well-defined $\mathbb{C}^M$-valued function in Ω which, by virtue of Theorem 7.2, satisfies

$$\begin{gathered} v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_{\widetilde{\kappa}} v_\beta \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \quad \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \\ \text{and } \ v_\beta\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{gathered} \tag{7.253}$$

In addition, from (7.251)–(7.252) and (7.64) we see that

$$(\nabla v_\beta)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists (in } \mathbb{C}^{n\cdot M}) \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{7.254}$$

For each pair of indices $\alpha, \beta \in \{1, \dots, M\}$ let us now define

$$G_{\alpha\beta}(x) := v_{\beta\alpha}(x) - \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big), \qquad \forall\, x \in \Omega\setminus\{x_0\}. \tag{7.255}$$

Regarding $G := \big(G_{\alpha\beta}\big)_{1\le\alpha,\beta\le M}$ as a $\mathbb{C}^{M\times M}$-valued function defined $\mathcal{L}^n$-a.e. in Ω, from (7.255) and Theorem 3.1 we then see that $G \in \big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^{M\times M}$. Furthermore, by design,

$$\begin{gathered} L^\top G = -\delta_{x_0} I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^{M\times M} \ \text{ and} \\ G\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \big(\nabla G\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{gathered} \tag{7.256}$$

while if $v := \big(v_{\beta\alpha}\big)_{1\le\alpha,\beta\le M}$ then from (2.8), (3.16), and the Mean Value Theorem it follows that at each point $x \in \partial\Omega$ we have

$$\begin{gathered} \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K} G\big)(x) \le \big(\mathcal{N}_{\widetilde{\kappa}} v\big)(x) + C_{x_0,\rho}(1+|x|)^{1-n} \ \text{ and} \\ \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big)(x) \le \big(\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\big)(x) + C_{x_0,\rho}(1+|x|)^{-n}, \end{gathered} \tag{7.257}$$

where $C_{x_0,\rho} \in (0,\infty)$ is independent of x. From (7.253), (7.257), (7.21), and (7.19) we see that the conditions listed in (6.4) are presently satisfied and, in fact,

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma). \tag{7.258}$$

Assume now that $u = (u_\beta)_{1\le\beta\le M}$ is a $\mathbb{C}^M$-valued function in Ω satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \text{and } \ \mathcal{N}_\kappa u \ \text{ belongs to the space } \ M^{p,\lambda}(\partial\Omega,\sigma). \end{gathered} \tag{7.259}$$

Since (7.258) and (7.22) imply

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\,\mathrm{d}\sigma$$
$$\leq C\big\|\mathcal{N}_\kappa u\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}\big\|\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} < \infty, \tag{7.260}$$

we may rely on Theorem 6.1 to conclude that the Poisson integral representation formula (6.6) holds. In particular, said formula proves that whenever $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0$ at σ-a.e. point on $\partial\Omega$ we necessarily have $u(x_0) = 0$. Given that x_0 has been arbitrarily chosen in Ω, this ultimately shows such a function u is actually identically zero in Ω. This finishes the proof of the uniqueness claim made in item *(b)*. The well-posedness claim in item *(c)* is a consequence of what we have already proved in items *(a)–(b)*.

Going further, the first claim in item *(d)*, regarding the potential failure of solvability of the Dirichlet Problem (7.241), is a consequence of Proposition 3.10 formulated for Morrey spaces. Its proof goes through virtually unchanged, with one caveat. Specifically, to justify (3.308), instead of Lebesgue's Dominated Convergence Theorem on Muckenhoupt weighted Lebesgue spaces we now use the weak-$*$ convergence on Morrey spaces from Proposition 7.3 (bearing in mind the continuity and skew-symmetry of the Hilbert transform on Morrey and block spaces on the real line). For higher dimensions, see Proposition 3.13. Also, the second claim in item *(d)*, regarding the potential failure of uniqueness for the Dirichlet Problem (7.241), is a consequence of Example 3.5 (keeping in mind (3.258) and (7.4)). Again, for higher dimensions see Proposition 3.13.

Consider next the claim made in item *(e)*. When the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ in the formulation of (7.241), virtually the same proof goes through, given that matters may be arranged (by taking $\delta > 0$ sufficiently small) so that the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and $\big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 7.9). In the scenario in which the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ for some given $q \in (1,\infty)$ in the formulation of (7.241), the same line of reasoning applies, with a few notable changes. First, if p is the Hölder conjugate exponent of q, then taking δ sufficiently small we may ensure that the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, $\big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, and $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 7.9). Second, with $f^{(\beta)}$ as in (7.250), thanks to (7.4) in place of (7.251) we now have

$$f^{(\beta)} \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.261}$$

In place of (7.258), this eventually implies

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in M^{p,\lambda}(\partial\Omega,\sigma), \tag{7.262}$$

so in place of (7.260) we now have (again, thanks to (7.22))

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\,\mathrm{d}\sigma \leq C \big\|\mathcal{N}_\kappa u\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \big\|\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty. \tag{7.263}$$

As before, this allows us to invoke Theorem 6.1 to conclude that the Poisson integral representation formula (6.6) holds. Ultimately, this readily implies the uniqueness result we presently seek. The versions of the claims in item *(d)* for vanishing Morrey spaces and block spaces are dealt with much as before (for the former scale, use (7.8); in the case of block spaces, it is useful to observe that (7.17) and Lebesgue's Dominated Convergence Theorem yield, in place of (3.308), that $\lim_{\varepsilon\to 0^+} h_\varepsilon = f_1 + \mathrm{i} f_2$ in $L^r(\mathbb{R}, \mathcal{L}^1)$ where r is as in (7.17), and this suffices to conclude that (3.309) holds in this case). Once more, for higher dimensions see Proposition 3.13. Finally, one deals with (7.246) and its related versions along the lines of the proof of Theorem 6.3. The proof of Theorem 7.18 is therefore complete. □

It turns out that the solvability results established in Theorem 7.18 may be further enhanced, via perturbation arguments, as described in our next theorem.

Theorem 7.19 *Retain the original background assumptions on the set Ω from Theorem 7.18 and, as before, fix two integrability exponents $p, q \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Then the following statements are true.*

(a) [Existence] *For each given system $L_o \in \mathfrak{L}^{\mathrm{dis}}$ (cf. (3.195)) there exist some small threshold $\delta \in (0, 1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, q, λ, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (7.241), along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$, are all solvable.*

(b) [Uniqueness] *For each given system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\mathrm{dis}}$ there exist some small threshold $\delta \in (0, 1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, q, λ, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (7.241) along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$, have at most one solution.*

(c) [Well-Posedness] *For each given system $L_o \in \mathfrak{L}^{\mathrm{dis}}$ with $L_o^\top \in \mathfrak{L}^{\mathrm{dis}}$ there exist some small threshold $\delta \in (0, 1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, q, λ, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (7.241) along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$, are all well posed.*

Proof This may be justified by reasoning as in the proof of Theorem 6.4, now making use of the invertibility results from Theorem 7.10. □

We continue by discussing the Inhomogeneous Regularity Problem for weakly elliptic systems in δ-*AR* domains with boundary data in Morrey-based Sobolev spaces, vanishing Morrey-based Sobolev spaces, as well as block-based Sobolev spaces.

Theorem 7.20 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa > 0$. Also, pick an exponent $p \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, consider the Inhomogeneous Regularity Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{cases} \tag{7.264}$$

The following statements are true:

(a) [Existence and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, then there exists $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then $\frac{1}{2}I + K_A$ is an invertible operator on the Morrey-based Sobolev space $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and the function*

$$u(x) := \Big(\mathcal{D}_A\Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x), \qquad \forall\, x \in \Omega \tag{7.265}$$

is a solution of the Inhomogeneous Regularity Problem (7.264). In addition,

$$\begin{aligned} &\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \quad and \\ &\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}. \end{aligned} \tag{7.266}$$

(b) [Uniqueness] *Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, there exists $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then the Inhomogeneous Regularity Problem (7.264) has at most one solution.*

(c) [Well-Posedness] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ then there exists $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then*

the Inhomogeneous Regularity Problem (7.264) *is uniquely solvable and the solution satisfies* (7.266).

(d) [Other Spaces of Boundary Data] *Analogous results to those described in items (a)–(c) above are also valid for the Inhomogeneous Regularity Problem formulated with boundary data in the vanishing Morrey-based Sobolev space* $\big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, *or the block-based Sobolev space* $\big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ *with* $q \in (1,\infty)$.

(e) [Perturbation Results] *In each of the cases considered in items (a)–(d), there are naturally accompanying perturbation results of the sort described in Theorem 7.19.*

(f) [Sharpness] *If* $\mathfrak{A}_L^{\rm dis} = \varnothing$ *the Regularity Problem* (7.264) (*and its variants involving vanishing Morrey-based Sobolev spaces, or block-based Sobolev spaces*) *may fail to be solvable, and if* $\mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$ *the Inhomogeneous Regularity Problem* (7.264) (*along with its aforementioned variants*) *may possess more than one solution.*

Proof The claims in items *(a)–(d)* are implied by Theorems 7.9 and 7.18, while the claim in item *(e)* may be justified by reasoning as in the proof of Theorem 6.4, now making use of the invertibility results from Theorem 7.10. Finally, the claims in item *(f)* are consequences of the versions of Example 3.5 and Proposition 3.11 formulated for Morrey spaces, as well as vanishing Morrey spaces and block spaces (whose proofs naturally adapt to these spaces; see the discussion in the proof of item *(d)* in Theorem 7.18). For higher dimensions see Proposition 3.13. □

Remark 7.7 Much as indicated in Remark 6.3, similar solvability and well-posedness results as in Theorem 7.20 hold for the versions of the Regularity Problem (7.264) formulated with boundary data belonging to suitably defined off-diagonal Morrey-based Sobolev spaces (as well as off-diagonal vanishing Morrey-based Sobolev spaces, and off-diagonal block-based Sobolev spaces).

The next goal is to formulate and solve the Homogeneous Regularity Problem with boundary data from homogeneous Morrey-based Sobolev spaces. This augments solvability results established earlier in Theorems 7.18 and 7.20.

Theorem 7.21 *Assume* $\Omega \subseteq \mathbb{R}^n$ *is an Ahlfors regular domain. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, fix an aperture parameter* $\kappa > 0$ *and pick some exponent* $p \in (1,\infty)$ *along with a number* $\lambda \in (0, n-1)$. *For a given homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system* L *in* $\mathbb{R}^n$, *consider the Homogeneous Regularity Problem*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \left[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\right]^M, \end{cases} \tag{7.267}$$

where $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$ *is the homogeneous Morrey-based boundary Sobolev space defined in* (7.69). *In relation to this, the following statements are valid:*

(a) [Existence, Estimates, and Integral Representations] *If* $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ *then there exists* $\delta \in (0, 1)$ *which depends only on n, p, λ, L, and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the following properties are true. First, the operator*

$$\left[S_{\text{mod}}\right] : \left[M^{p,\lambda}(\partial\Omega, \sigma)\right]^M \longrightarrow \left[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/ \sim\right]^M \tag{7.268}$$

is surjective and the Homogeneous Regularity Problem (7.267) *is solvable. More specifically, with* $[f] \in \left[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/ \sim\right]^M$ *denoting the equivalence class* (*modulo constants*) *of the boundary datum f, and with*

$$g \in \left[M^{p,\lambda}(\partial\Omega, \sigma)\right]^M \ \ \textit{selected so that} \ \ \left[S_{\text{mod}}\right]g = [f], \tag{7.269}$$

there exists a constant $c \in \mathbb{C}^M$ *such that the function*

$$u := \mathscr{S}_{\text{mod}}\, g + c \ \ in \ \ \Omega \tag{7.270}$$

is a solution of the Homogeneous Regularity Problem (7.267). *In addition, this solution satisfies* (*with implicit constants independent of f*)

$$\|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|\nabla_{\tan} f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}. \tag{7.271}$$

Second, for each coefficient tensor $A \in \mathfrak{A}_L^{\text{dis}}$ *the operator*

$$\tfrac{1}{2}I + \left[K_{A,\text{mod}}\right] : \left[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)/\sim\right]^M \longrightarrow \left[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)/\sim\right]^M \tag{7.272}$$

is an isomorphism, and the Homogeneous Regularity Problem (7.267) *may be solved as*

$$u := \mathcal{D}_{A,\text{mod}} h + c \ \ in \ \ \Omega, \tag{7.273}$$

for a suitable constant $c \in \mathbb{C}^M$ *and with*

$$h \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ such that } [h] = \Big(\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\Big)^{-1}[f]. \tag{7.274}$$

Moreover, this solution continues to satisfy (7.271).

(b) [Uniqueness] *Whenever* $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$, *there exists* $\delta \in (0,1)$ *which depends only on* n, p, λ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Homogeneous Regularity Problem* (7.267) *has at most one solution.*

(c) [Well-Posedness and Additional Integrability/Regularity] *Whenever* $\mathfrak{A}^{\text{dis}}_{L} \neq \varnothing$ *and* $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$ *it follows that there exists* $\delta \in (0,1)$ *which depends only on* n, p, λ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Homogeneous Regularity Problem* (7.267) *is uniquely solvable. Moreover, the unique solution* u *of* (7.267) *satisfies* (*in a quantitative fashion*)

$$\mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega,\sigma) \Longleftrightarrow f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.275}$$

In particular, the equivalence in (7.275) *proves that the unique solution of the Homogeneous Regularity Problem* (7.267) *for a boundary datum* f *belonging to* $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (*which is a subspace of* $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$; *cf.* (7.71)) *is actually the unique solution of the Inhomogeneous Regularity Problem* (7.264) *for the boundary datum* f.

(d) [Other Spaces of Boundary Data] *Analogous results to those described in items (a)–(c) above are also valid for the Homogeneous Regularity Problem formulated with boundary data in homogeneous vanishing Morrey-based Sobolev spaces, or homogeneous block-based Sobolev spaces.*

(e) [Perturbation Results] *In each of the scenarios considered in items (a)–(d), there are naturally accompanying perturbation results of the sort described in Theorem 7.19.*

(f) [Sharpness] *If* $\mathfrak{A}^{\text{dis}}_{L} = \varnothing$ *the Homogeneous Regularity Problem* (7.267) *may fail to be solvable* (*actually for boundary data belonging to an infinite dimensional subspace of the corresponding weighted homogeneous Sobolev space*), *and if* $\mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing$ *the Homogeneous Regularity Problem* (7.267) *may possess more than one solution* (*in fact, the linear space of null-solutions may actually be infinite dimensional*), *even in the case when* $\Omega = \mathbb{R}^n_+$. *In particular, if either* $\mathfrak{A}^{\text{dis}}_{L} = \varnothing$ *or* $\mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing$, *then the Homogeneous Regularity Problem* (7.267) *may fail to be well posed, again, even in the case when* $\Omega = \mathbb{R}^n_+$.

Proof All claims are established by reasoning along the lines of the proof of Theorem 6.8, now making use of Proposition 7.8, Theorems 7.4, 7.5, 7.9, 7.12, 7.13, and 7.3. □

We next treat the Neumann Problem for weakly elliptic systems in δ-AR domains with boundary data in Morrey spaces, vanishing Morrey spaces, and block spaces.

Theorem 7.22 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Next, suppose L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some coefficient tensor $A \in \mathfrak{A}_L$ and consider the Neumann Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M. \end{cases} \tag{7.276}$$

Then the following statements are valid:

(a) [Existence, Estimates, and Integral Representation] *If $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ then there exists $\delta \in (0,1)$, depending only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then the operator $-\frac{1}{2}I + K^\#_{A^\top}$ is invertible on the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \Big(\mathscr{S}_{\mathrm{mod}}\Big(-\tfrac{1}{2}I + K^\#_{A^\top}\Big)^{-1} f\Big)(x) \ \text{ for all } \ x \in \Omega, \tag{7.277}$$

is a solution of the Neumann Problem (7.276) which satisfies

$$\|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\,\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \tag{7.278}$$

for some constant $C \in (0,\infty)$ independent of f. Also, the operator (7.210) is surjective which implies that, for some constant $C \in (0,\infty)$,

$$\begin{array}{l} \text{there exists } g \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega, w)\big]^M \text{ with } \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}}\, g) = f \\ \text{and such that } \|g\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,w)]^M} \leq C\,\|f\|_{[M^{p,\lambda}(\partial\Omega,w)]^M}. \end{array} \tag{7.279}$$

Consequently, the function

$$u := \mathcal{D}_{A,\mathrm{mod}}\, g \ \text{ in } \ \Omega \tag{7.280}$$

is a solution of the Neumann Problem (7.276) which continues to satisfy (7.278).

(b) [Uniqueness (modulo constants)] *Whenever $A \in \mathfrak{A}_L^{\mathrm{dis}}$ there exists $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then any two solutions of the Neumann Problem (7.276) differ by a constant from $\mathbb{C}^M$.*

(c) [Well-Posedness] *Whenever* $A \in \mathfrak{A}_L^{\rm dis}$ *and* $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$ *then there exists* $\delta \in (0, 1)$ *which depends only on n, p, λ, A, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Neumann Problem* (7.276) *is solvable, the solution is unique modulo constants from* $\mathbb{C}^M$*, and each solution satisfies* (7.278).

(d) [Other Spaces of Boundary Data and Perturbation Results] *Similar results as in items (a)–(c) are valid with the Morrey space* $M^{p,\lambda}(\partial\Omega, \sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$*, or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *with* $q \in (1, \infty)$*. In each of these cases there are naturally accompanying perturbation results of the sort described in Theorem 7.19. Finally, given any pair of integrability exponents* $p_0, p_1 \in (1, \infty)$ *along with any pair of parameters* $\lambda_0, \lambda_1 \in (0, n-1)$*, similar results are valid for the Neumann Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p_0,\lambda_0}(\partial\Omega, \sigma) + M^{p_1,\lambda_1}(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \big[M^{p_0,\lambda_0}(\partial\Omega, \sigma) + M^{p_1,\lambda_1}(\partial\Omega, \sigma)\big]^M, \end{cases} \tag{7.281}$$

as well as for its versions with the Morrey spaces replaced by vanishing Morrey space or block spaces.

(e) [Sharpness] *If* $A^\top \notin \mathfrak{A}_{L^\top}^{\rm dis}$ *then the Neumann Problem* (7.276) *may not be solvable. In addition, if* $A \notin \mathfrak{A}_L^{\rm dis}$ *then the Neumann Problem* (7.276) *may have more than one solution. In fact, even the two-dimensional Laplacian may be written as* $\Delta = \operatorname{div} A\nabla$ *for some matrix* $A \in \mathbb{C}^{2\times 2}$ (*not belonging to* $\mathfrak{A}_\Delta^{\rm dis} = \{I_{2\times 2}\}$) *such that the Neumann Problem formulated for this as in* (7.276) *for this choice of A and with* $\Omega := \mathbb{R}^2_+$ *fails to have a solution for each non-zero boundary datum belonging to an infinite-dimensional linear subspace of the full space of boundary data, and the linear space of null-solutions for the Neumann Problem formulated as in* (7.276) *for this choice of A and with* $\Omega := \mathbb{R}^2_+$ *is actually infinite dimensional. The aforementioned lack of Fredholm solvability is also present for the Neumann Problem formulated in other function spaces, like those considered in item (d).*

Proof Theorem 7.9 guarantees the existence of some threshold $\delta \in (0, 1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $-\frac{1}{2}I + K^\#_{A^\top}$ is invertible on $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, $\big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, and $\big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ (assuming $q \in (1, \infty)$ has been fixed to begin with). Granted this, all conclusions, save for the very last claim in item *(d)*, follow from Theorems 7.4, 7.9, 7.10, and 7.14 by reasoning as in the proof of Theorem 6.11. The claims pertaining to the Neumann Problem (7.281) are dealt with much as in the proof of Theorem 6.14. Finally, the sharpness aspect highlighted in item *(e)* may be justified by reasoning much as in the proof of Theorem 6.11. □

In relation to Theorem 7.22, we wish to note that in the formulation of the Neumann Problem (7.276) for the two-dimensional Lamé system we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (4.401) for any ζ as in (6.155) (see Remark 7.4 and Remark 6.10 in this regard).

Finally, we formulate and solve the Transmission Problem for weakly elliptic systems in δ-AR domains with boundary data in Morrey spaces, vanishing Morrey spaces, and block spaces. In the formulation on this problem, the clarifications made right after the statement of Theorem 6.15 continue to remain relevant.

Theorem 7.23 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and set*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{7.282}$$

Also, pick an exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$, an aperture parameter $\kappa > 0$, and a transmission (or coupling) parameter $\eta \in \mathbb{C}$. Next, assume L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some $A \in \mathfrak{A}_L$ and consider the Transmission Problem

$$\begin{cases} u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in M^{p,\lambda}(\partial\Omega,\sigma), \\ u^+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = g \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,w)\big]^M, \\ \partial_\nu^A u^+ - \eta\cdot\partial_\nu^A u^- = f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{cases} \tag{7.283}$$

In relation to this, the following statements are valid:

(a) [Uniqueness (modulo constants)] *Suppose either*

$$A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}} \text{ and } \eta \in \mathbb{C} \setminus \{-1\}, \tag{7.284}$$

or

$$A \in \mathfrak{A}_L^{\mathrm{dis}} \text{ and } \eta \in \mathbb{C} \setminus \{0, -1\}. \tag{7.285}$$

*Then there exists $\delta \in (0,1)$ which depends only on n, η, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that whenever $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a scenario which renders Ω a δ-*AR *domain; cf. Definition 2.15) it follows any two solutions of the Transmission Problem (7.283) differ by a constant (from $\mathbb{C}^M$).*

(b) [Well-Posedness, Integral Representations, and Additional Regularity] *Assume*

$$A \in \mathfrak{A}_L^{\mathrm{dis}}, \ A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}, \text{ and } \eta \in \mathbb{C} \setminus \{-1\}. \tag{7.286}$$

Then there exists some small $\delta \in (0,1)$ *which depends only on* n, p, λ, A, η, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *it follows that the Transmission Problem* (7.283) *is solvable. Specifically, in the scenario described in* (7.286), *the operator* $-\frac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^\top}$ *is invertible on the Morrey space* $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, *the operator* $\big[S_{\rm mod}\big]$ *is invertible from* $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *onto the space* $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M$, *and the functions* $u^\pm : \Omega_\pm \to \mathbb{C}^M$ *defined as*

$$
\begin{aligned}
u^+ &:= \mathscr{S}^+_{\rm mod} h_0 + \mathscr{S}^+_{\rm mod} h_1 - c \ \text{ in } \ \Omega_+,\\
u^- &:= \mathscr{S}^-_{\rm mod} h_0 \ \text{ in } \ \Omega_-,
\end{aligned}
\tag{7.287}
$$

where the superscripts $\pm$ *indicate that the modified single layer potentials are associated with the sets* $\Omega_\pm$ *and*

$$
\begin{aligned}
&h_1 := \big[S_{\rm mod}\big]^{-1}[g] \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \quad c := S_{\rm mod} h_1 - g \in \mathbb{C}^M,\\
&h_0 := \Big(-\tfrac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^\top}\Big)^{-1}\Big(f - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)h_1\Big),
\end{aligned}
\tag{7.288}
$$

solve the Transmission Problem (7.283) *and satisfy, for a finite constant* $C > 0$ *independent of* f *and* g,

$$
\big\|\mathcal{N}_\kappa(\nabla u^\pm)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\Big(\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} + \|g\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}\Big).
\tag{7.289}
$$

Moreover, any two solutions of the Transmission Problem (7.283) *differ by a constant (from* $\mathbb{C}^M$*). In particular, any solution of the Transmission Problem* (7.283) *satisfies* (7.289).

Alternatively, under the conditions imposed in (7.286) *and, again, assuming* Ω *is a* δ-AR *domain with* $\delta \in (0,1)$ *sufficiently small, a solution of the Transmission Problem* (7.283) *may also be found in the form*

$$
\begin{aligned}
u^+ &:= \mathcal{D}^+_{A,{\rm mod}}\psi_0 + c \ \text{ in } \ \Omega_+,\\
u^- &:= \mathcal{D}^-_{A,{\rm mod}}\psi_1 \ \text{ in } \ \Omega_-,
\end{aligned}
\tag{7.290}
$$

where the superscripts $\pm$ *indicate that the modified double layer potentials are associated with the sets* $\Omega_\pm$, *where* $c \in \mathbb{C}^M$ *is a suitable constant, and where* $\psi_0, \psi_1 \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ *are two suitable functions satisfying*

$$
\begin{aligned}
&\|\psi_0\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M} + \|\psi_1\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}\\
&\qquad\qquad \le C\Big(\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} + \|g\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}\Big),
\end{aligned}
\tag{7.291}
$$

for some constant $C \in (0, \infty)$ *independent of* f *and* g. *In particular,* $u^{\pm}$ *in* (7.290) *also satisfy* (7.289).

(c) [Well-Posedness for $\eta = 1$] *In the case when*

$$\eta = 1 \textit{ and } \Omega \textit{ is a two-sided NTA domain with an unbounded Ahlfors regular boundary} \tag{7.292}$$

the Transmission Problem (7.283) *is solvable, and any two solutions of the Transmission Problem* (7.283) *differ by a constant. Any solution is given by*

$$\begin{aligned} u^+ &:= \mathcal{D}^+_{A,\mathrm{mod}} g - \mathscr{S}^+_{\mathrm{mod}} f + c \;\; \textit{in} \;\; \Omega_+, \\ u^- &:= -\mathcal{D}^-_{A,\mathrm{mod}} g - \mathscr{S}^-_{\mathrm{mod}} f + c \;\; \textit{in} \;\; \Omega_-, \end{aligned} \tag{7.293}$$

for some $c \in \mathbb{C}^M$, *where the superscripts* $\pm$ *indicate that the modified layer potentials are associated with the sets* $\Omega_\pm$ *introduced in* (7.282). *In addition, any solution satisfies* (7.289).

(d) [Other Spaces of Boundary Data and Perturbation Results] *Analogous results hold with the Morrey space* $M^{p,\lambda}(\partial\Omega, \sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$, *the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *with* $q \in (1, \infty)$, *or by sums of such spaces. In addition, in each of these cases there are naturally accompanying perturbation results of the sort described in Theorem 7.19.*

Proof For each fixed $\eta \in \mathbb{C} \setminus \{-1\}$, $p, q \in (1, \infty)$, and $\lambda \in (0, n-1)$, Theorem 7.9 guarantees that there exists some threshold $\delta \in (0, 1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $-\frac{\eta+1}{2} I + (1-\eta) K^{\#}_{A^\top}$ is invertible on $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, $\big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, and $\big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$. With this in hand, the same type of argument as in the proof of Theorem 6.15 (which now relies on Theorems 7.2, 7.4, 7.5, 7.9, 7.12, 7.13) and the proof of Theorem 6.4 (which now makes use of Theorem 7.10) yields all desired conclusions. □

We close by noting that, in the formulation of the Transmission Problem (7.283) for the two-dimensional Lamé system, we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (4.401) for any ζ as in (6.262) (see Remarks 7.4 and 6.16 in this regard).