# Chapter 4
# Boundedness and Invertibility of Layer Potential Operators

The key result in this work is Theorem 4.2 which elaborates on the nature of the operator norm of a singular integral operator $T$ defined on the boundary of a UR domain $\Omega$ whose integral kernel has a special algebraic format, through the presence of the inner product between the outward unit normal $\nu$ to $\Omega$ and the chord, as a factor. Proving this theorem requires extensive preparations and takes quite a bit of effort, but the redeeming feature of Theorem 4.2 is that said operator norm estimate involves the BMO semi-norm of $\nu$ as a factor. This hallmark attribute (which is shared by the double layer operator $K_A$ associated with a distinguished coefficient tensor $A$) entails that the flatter $\partial\Omega$ is, the smaller $\|T\|$ is. In particular, having $\partial\Omega$ sufficiently flat ultimately allows us to invert $\frac{1}{2}I + K_A$ on Muckenhoupt weighted Lebesgue spaces via a Neumann series, and this is of paramount importance later on, when dealing with boundary value problems via the method of boundary layer potentials. Subsequently, via operator identities relating the single and double layers, we also succeed in inverting the single layer potential operator in a similar geometric and algebraic setting.

## 4.1 Estimates for Euclidean Singular Integral Operators

We begin with a few generalities of functional analytic nature. Given two normed vector spaces $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$, consider a positively homogeneous mapping $T : X \to Y$, i.e., a function $T$ sending $X$ into $Y$ and satisfying $T(\lambda u) = \lambda T(u)$ for each $u \in X$ and each $\lambda \in (0, \infty)$ (note that taking $u := 0 \in X$ and $\lambda := 2$ implies $T(0) = 0 \in Y$). We shall denote by

$$\|T\|_{X\to Y} := \sup\Big\{\|Tu\|_Y : u \in X,\ \|u\|_X = 1\Big\} \in [0, \infty] \tag{4.1}$$

the operator norm of such a mapping $T$; in particular,



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$$\|Tu\|_Y \leq \|T\|_{X\to Y}\|u\|_X \quad \text{for each} \quad u \in X. \tag{4.2}$$

It is then straightforward to see that a positively homogeneous mapping $T : X \to Y$ is continuous at $0 \in X$ if and only if $T$ is bounded (i.e., it maps bounded subsets of $X$ into bounded subsets of $Y$) if and only if $\|T\|_{X\to Y} < +\infty$.

Consider now the special case when $X, Y$ are Lebesgue spaces (associated with a generic measure space) and $T$ is a sub-linear mapping of $X$ into $Y$ (i.e., $T : X \to Y$ satisfies the property $T(\lambda u) = |\lambda| T(u)$ for each scalar $\lambda$ and each function $u \in X$, as well as $T(u + w) \leq Tu + Tw$ at a.e. point, for each $u, w \in X$). Then, for each $u, w \in X$ we have $|Tu - Tw| \leq |T(u - w)|$ at a.e. point, which further implies that $\|Tu - Tw\|_Y \leq \|T(u - w)\|_Y \leq \|T\|_{X\to Y}\|u - w\|_X$. Consequently,

$$\begin{gathered} \text{a sub-linear map } T : X \to Y \text{ is continuous} \\ \text{if and only if } \|T\|_{X\to Y} < +\infty. \end{gathered} \tag{4.3}$$

Let us now start in earnest. To facilitate dealing with Theorem 4.1 a little later, we first isolate a useful estimate in the lemma below.

**Lemma 4.1** *Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^n, \mathcal{L}^n)$. Then there exists a constant $C \in (0, \infty)$ which only depends on $n$, $p$, and $[w]_{A_p}$, with the property that for each point $x \in \mathbb{R}^n$, each radius $r \in (0, \infty)$, and real-valued function $A \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ with*

$$\nabla A \in \big[\text{BMO}(\mathbb{R}^n, \mathcal{L}^n)\big]^n \tag{4.4}$$

*one has*

$$\begin{aligned} \int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|>r}} & \frac{\big|A(x) - A(y) - \langle \nabla A(y), x - y\rangle\big|^p}{|x - y|^{(n+1)p}} \, dw(y) \\ & \leq C r^p \, w\big(B(x, r)\big) \big\|\nabla A\big\|^p_{[\text{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}. \end{aligned} \tag{4.5}$$

***Proof*** Fix a function $A$ as in the statement of the lemma. From Lemma 2.13 and (4.4) we see that

$$\nabla A \in \big[L^1_{\text{loc}}(\mathbb{R}^n, w)\big]^n. \tag{4.6}$$

Next, recall from (2.533) that there exists $\varepsilon \in (0, p - 1)$ which depends only on $p$, $n$, and $[w]_{A_p}$, such that

$$w \in A_{p-\varepsilon}(\mathbb{R}^n, \mathcal{L}^n). \tag{4.7}$$

Fix $x \in \mathbb{R}^n$ and $r \in (0, \infty)$. By breaking up the integral dyadically, estimating the denominator, and using the doubling property of $w \in A_{p-\varepsilon}(\mathbb{R}^n, \mathcal{L}^n)$ (cf. item *(5)* of

Proposition 2.20) we may dominate

$$\int\limits_{\substack{y\in\mathbb{R}^n\\|x-y|>r}} \frac{\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p}{|x-y|^{(n+1)p}}\,\mathrm{d}w(y)$$
$$\leq C_{n,p}\sum_{j=1}^{\infty}\frac{w(B(x,2^j r))}{2^{j(n+1)p}}\cdot \mathrm{I}_j \leq C_{n,p,w}\sum_{j=1}^{\infty}\frac{2^{jn(p-\varepsilon)}w(B(x,r))}{2^{j(n+1)p}}\cdot \mathrm{I}_j, \tag{4.8}$$

where, for each $j\in\mathbb{N}$,

$$\mathrm{I}_j := \frac{1}{w(B(x,2^j r))}\int\limits_{2^{j-1}r<|x-y|\leq 2^j r}\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p\,\mathrm{d}w(y). \tag{4.9}$$

To proceed, for each $j\in\mathbb{N}$ introduce

$$A_j(z) := A(z)-\Big(⨍_{B(x,2^j r)}\nabla A\,\mathrm{d}w\Big)\cdot z \quad \text{for each} \quad z\in\mathbb{R}^n \tag{4.10}$$

(making use of (4.6) to ensure that this is meaningful), and observe that $\mathrm{I}_j$, originally defined in (4.9), does not change if the function $A$ is replaced by $A_j$. Consequently, for each $j\in\mathbb{N}$ we have

$$\mathrm{I}_j \leq C_p\cdot \mathrm{II}_j + C_p\cdot \mathrm{III}_j, \tag{4.11}$$

where

$$\mathrm{II}_j := \frac{1}{w(B(x,2^j r))}\int\limits_{2^{j-1}r<|x-y|\leq 2^j r}\big|A_j(x)-A_j(y)|^p\,\mathrm{d}w(y), \tag{4.12}$$

and

$$\mathrm{III}_j := \frac{2^{jp}r^p}{w(B(x,2^j r))}\int\limits_{2^{j-1}r<|x-y|\leq 2^j r}\big|\nabla A_j(y)\big|^p\,\mathrm{d}w(y). \tag{4.13}$$

Fix an integrability exponent $q\in(n,\infty)$ and pick $j\in\mathbb{N}$ arbitrary. Then for each $y\in\mathbb{R}^n$ such that $2^{j-1}r<|x-y|\leq 2^j r$ we may estimate

$$|A_j(x)-A_j(y)|\leq C_{q,n}|x-y|\Big(⨍_{|x-z|\leq 2|x-y|}|\nabla A_j(z)|^q\,\mathrm{d}z\Big)^{1/q}$$

$$
\begin{aligned}
&\leq C_{q,n,w}\cdot 2^j r\Big(\fint_{B(x,2|x-y|)}|\nabla A_j|^{pq}\,\mathrm{d}w\Big)^{1/(pq)}\\
&\leq C_{q,n,w}\cdot 2^j r\Big(\fint_{B(x,2|x-y|)}\Big|\nabla A-\fint_{B(x,2|x-y|)}\nabla A\,\mathrm{d}w\Big|^{pq}\,\mathrm{d}w\Big)^{1/(pq)}\\
&\qquad + C_{q,n,w}\cdot 2^j r\Big|\fint_{B(x,2^j r)}\nabla A\,\mathrm{d}w-\fint_{B(x,2|x-y|)}\nabla A\,\mathrm{d}w\Big|\\
&\leq C_{q,n,w}\cdot 2^j r\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,w)]^n}\\
&\leq C_{q,n,w}\cdot 2^j r\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}.
\end{aligned}
\tag{4.14}
$$

Above, the first estimate is provided by Mary Weiss' Lemma (cf. [24, Lemma 1.4, p. 144], or [58, Lemma 2.10, p. 477]), the second estimate uses the fact that we have $|x-y|\leq 2^j r$ and Lemma 2.12, the third estimate is implied by (4.10) which gives $\nabla A_j=\nabla A-\fint_{B(x,2^j r)}\nabla A\,\mathrm{d}w$, the penultimate estimate is a consequence of the John-Nirenberg inequality, (2.103) (written with $w$ in place of $\sigma$), and the doubling property of $w$, while the final estimate in (4.14) comes from Lemma 2.14. In turn, (4.12) and (4.14) yield

$$
\mathrm{II}_j\leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}. \tag{4.15}
$$

By combining (4.13) and (4.10) we also see that

$$
\begin{aligned}
\mathrm{III}_j&\leq 2^{jp}r^p\fint_{B(x,2^j r)}\Big|\nabla A-\fint_{B(x,2^j r)}\nabla A\,\mathrm{d}w\Big|^p\,\mathrm{d}w\\
&\leq C\cdot 2^{jp}r^p\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,w)]^n}\leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n},
\end{aligned}
\tag{4.16}
$$

where the last inequality is once again provided by Lemma 2.14. From (4.15)–(4.16) and (4.11) we then conclude that

$$
\mathrm{I}_j\leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}\quad\text{for each}\quad j\in\mathbb{N}. \tag{4.17}
$$

Using this back in (4.8) now readily yields (4.5), since $\sum_{j=1}^{\infty}2^{-jn\varepsilon}<\infty$. □

The next result, dealing with boundedness for certain type of singular integral operators in the Euclidean context, refines work in [61, Theorem 4.34, p. 2725].

**Theorem 4.1** *Pick an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. Denote by $p'\in(1,\infty)$ the Hölder conjugate exponent of $p$ and by $w'$ the dual weight $w':=w^{1-p'}\in A_{p'}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ of $w$. Next, fix three numbers $n,m,d\in\mathbb{N}$ with $n\geq 2$, and let $N=N(n,m)\in\mathbb{N}$ be a sufficiently large integer. Let $A\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ be a complex-valued function with* *the property that*

$$\nabla A \in \big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}. \tag{4.18}$$

*Also, for each* $j \in \{1, \dots, m\}$ *consider a real-valued function* $B_j \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ *with the property that*

$$\nabla B_j \in \big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}, \tag{4.19}$$

*and set* $B := (B_1, \dots, B_m)$*. In addition, consider a function* $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^d$ *for which there exists* $c \in (0, 1]$ *such that*

$$c|x' - y'| \leq |\Phi(x') - \Phi(y')| \leq c^{-1}|x' - y'| \ \text{ for all } \ x', y' \in \mathbb{R}^{n-1}; \tag{4.20}$$

*hence,* $\Phi$ *is bi-Lipschitz. Going further, suppose* $F \in \mathscr{C}^{N+2}(\mathbb{R}^m)$ *is a complex-valued function which is even, has the property that* $\partial^\alpha F$ *belongs to* $L^1(\mathbb{R}^m, \mathcal{L}^m)$ *for every multi-index* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \leq N + 2$*, and*

$$\sup_{X \in \mathbb{R}^m} \big[(1 + |X|)|F(X)|\big] < +\infty. \tag{4.21}$$

*Finally, for each function* $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ *and each point* $x' \in \mathbb{R}^{n-1}$ *define*

$$T^{A,B}_{\Phi,*} g(x') := \sup_{\varepsilon > 0} \Bigg| \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |\Phi(x') - \Phi(y')| > \varepsilon}} \frac{A(x') - A(y') - \langle \nabla A(y'), x' - y' \rangle}{|x' - y'|^n} \times$$

$$\times F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y')\,\mathrm{d}y' \Bigg|. \tag{4.22}$$

*Then* $T^{A,B}_{\Phi,*}$ *is a well-defined, continuous, sub-linear mapping of the Muckenhoupt weighted Lebesgue space* $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ *into itself, and there exists some constant* $C(n, p, w) \in (0, \infty)$ *which depends only on* $n$*,* $p$*, and* $[w]_{A_p}$ *with the property that*

$$\big\| T^{A,B}_{\Phi,*} \big\|_{L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \to L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})} \tag{4.23}$$

$$\leq C(n, p, w) \cdot c^{-3n} \Big( \sum_{|\alpha| \leq N+2} \big\| \partial^\alpha F \big\|_{L^1(\mathbb{R}^m, \mathcal{L}^m)} + \sup_{X \in \mathbb{R}^m} (1 + |X|)|F(X)| \Big)$$

$$\times \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big( 1 + \sum_{j=1}^m \|\nabla B_j\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big)^N.$$

Theorem 4.1 is an intricate piece of machinery allowing us to estimate, in a rather detailed and specific manner, the maximal operator associated with integral kernels that exhibit a certain type of algebraic structure. We shall put this to good use in Lemma 4.2 which, in turn, is a basic ingredient in the proof of Theorem 4.2 (the main result in this section). This being said, Theorem 4.1 is useful for a variety of other purposes.

To give a significant example in this regard, work in the one-dimensional setting and recall the Hilbert transform $H$ on the real line from (1.24). Consider a complex-valued function $A \in W^{1,1}_{\rm loc}(\mathbb{R})$ with the property that $A' \in {\rm BMO}(\mathbb{R}, \mathcal{L}^1)$. Let $M_A$ stand for the operator of pointwise multiplication by $A$, and denote by $D$ the one-dimensional derivative operator $f \mapsto {\rm d}f/{\rm d}x$ on the real line. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. Then the commutator $[H, M_A D]$, originally defined on functions from $\mathscr{C}^\infty_0(\mathbb{R})$, extends to a bounded linear mapping on $L^p(\mathbb{R}, w)$ with operator norm $\leq C\,\|A'\|_{{\rm BMO}(\mathbb{R},\mathcal{L}^1)}$ where $C \in (0, \infty)$ is an absolute constant. Indeed, given any function $f \in \mathscr{C}^\infty_0(\mathbb{R})$, at $\mathcal{L}^1$-a.e. differentiability point $x \in \mathbb{R}$ for $A$ (hence, at $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$) we may write (keeping in mind that, since the Hilbert transform is a multiplier, $H$ commutes with differentiation):

$$
\begin{aligned}
[H, M_A D]f(x) &= H(Af')(x) - A(x)\frac{{\rm d}}{{\rm d}x}(Hf(x)) = H(Af')(x) - A(x)(Hf')(x) \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R} \\ |x-y|>\varepsilon}} \frac{A(y)-A(x)}{x-y} f'(y)\,{\rm d}y \\
&= -\lim_{\varepsilon\to 0^+} \Big(\frac{A(y)-A(x)}{x-y} f(y)\Big|_{y=x-\varepsilon}^{y=x+\varepsilon}\Big) \\
&\quad - \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R} \\ |x-y|>\varepsilon}} \frac{{\rm d}}{{\rm d}y}\Big(\frac{A(y)-A(x)}{x-y}\Big) f(y)\,{\rm d}y \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R} \\ |x-y|>\varepsilon}} \frac{A(x)-A(y)-A'(y)(x-y)}{(x-y)^2} f(y)\,{\rm d}y.
\end{aligned}
\tag{4.24}
$$

(The fact that the limit in the third line of (4.24) vanishes is ensured by the differentiability of $A$ at $x$, and the continuity of $f$ at $x$.) Granted this formula, Theorem 4.1 applies with $n = 2$, $m = 1$, $\Phi$ the identity, $B \equiv 0$, and taking $F \in \mathscr{C}^\infty_0(\mathbb{R})$ to be an even function with $F(0) = 1$. The desired conclusion then follows from (4.23).

To offer another example where Theorem 4.1 plays a decisive role, fix some $\varkappa \in (0, \infty)$ and suppose $\Sigma$ is a $\varkappa$-CAC passing through infinity in $\mathbb{C}$. Recall the Cauchy integral operator on the chord-arc curve $\Sigma$ acts on $f \in L^1\big(\Sigma, \frac{d\mathcal{H}^1(\zeta)}{1+|\zeta|}\big)$ according to

$$(C_\Sigma f)(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta\in\Sigma \\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \Sigma. \tag{4.25}$$

Since from Proposition 2.10 we know that $\Sigma$ is the topological boundary of a UR domain, Proposition 3.4 guarantees that $C_\Sigma$ is a well-defined, linear, and bounded operator on the space $L^p(\Sigma, w)$ whenever $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$, where $\sigma := \mathcal{H}^1 \lfloor \Sigma$. Let us indicate how Theorem 4.1 may be used to show that

the flatter the chord-arc curve $\Sigma$ becomes, the closer the corresponding Cauchy operator becomes (with proximity measured in the operator norm on Muckenhoupt weighted Lebesgue spaces) to the (suitably normalized) Hilbert transform on the real line. (4.26)

A brief discussion on this topic may be found in [33, pp. 138-139]. In order to facilitate a direct comparison between the two singular integral operators mentioned in (4.26), it is natural to consider the pull-back of $C_\Sigma$ to $\mathbb{R}$ under the arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ of $\Sigma$. After natural adjustments in notation, this corresponds to the mapping sending each $f \in L^p(\mathbb{R}, w)$ into

$$(C_{\mathbb{R}} f)(t) := \lim_{\varepsilon\to 0^+} \frac{i}{2\pi} \int_{\substack{s\in\mathbb{R} \\ |z(t)-z(s)|>\varepsilon}} \frac{z'(s)}{z(t) - z(s)} f(s)\, ds \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}, \tag{4.27}$$

where $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. Recall from (2.219) that the function $z(\cdot)$ is bi-Lipschitz, specifically,

$$(1 + \varkappa)^{-1}|t - s| \le |z(t) - z(s)| \le |t - s| \quad \text{for all } t, s \in \mathbb{R}. \tag{4.28}$$

Keeping this in mind, a suitable application[1] of [62, Proposition B.2] allows to change the truncation in (4.27) to

$$(C_{\mathbb{R}} f)(t) = \lim_{\varepsilon\to 0^+} \frac{i}{2\pi} \int_{\substack{s\in\mathbb{R} \\ |t-s|>\varepsilon}} \frac{z'(s)}{z(t) - z(s)} f(s)\, ds \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}, \tag{4.29}$$

[1] While [62, Proposition B.2] is stated for ordinary Lebesgue spaces, the same type of result holds in the class of Muckenhoupt weighted Lebesgue spaces (thanks to the fact that the phenomenon in question is local in nature, and (2.576)).

for each $f \in L^p(\mathbb{R}, w)$ with $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. We wish to compare the operator written in this form with the (suitably normalized) Hilbert transform on the real line, acting on arbitrary functions $f \in L^p(\mathbb{R}, w)$, where $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, according to

$$(Hf)(t) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{s\in\mathbb{R}\\ |t-s|>\varepsilon}} \frac{f(s)}{t-s}\,\mathrm{d}s \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}. \tag{4.30}$$

Fix $p \in (1, \infty)$, $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and $f \in L^p(\mathbb{R}, w)$. Then at $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we may express

$$\begin{aligned}\big(C_{\mathbb{R}} - (\mathrm{i}/2)H\big)f(t) &= \lim_{\varepsilon\to 0^+} \frac{\mathrm{i}}{2\pi} \int\limits_{\substack{s\in\mathbb{R}\\ |t-s|>\varepsilon}} \Big(\frac{z'(s)}{z(t)-z(s)} - \frac{1}{t-s}\Big) f(s)\,\mathrm{d}s \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}} \int\limits_{\substack{s\in\mathbb{R}\\ |t-s|>\varepsilon}} \frac{z(t)-z(s)-z'(s)(t-s)}{(z(t)-z(s))(t-s)} f(s)\,\mathrm{d}s.\end{aligned} \tag{4.31}$$

Pick an even function $\phi \in \mathscr{C}_0^\infty(\mathbb{C})$ satisfying (with $\varkappa$ as in (4.28))

$$\begin{gathered} 0 \le \phi \le 1 \quad \text{and} \quad \operatorname{supp}\phi \subseteq B(0,2), \\ \phi \equiv 1 \quad \text{on} \quad B(0,1) \setminus B\big(0, (1+\varkappa)^{-1}\big), \\ \phi \equiv 0 \quad \text{on} \quad B\big(0, (2+2\varkappa)^{-1}\big), \end{gathered} \tag{4.32}$$

along with a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ which is even and satisfies

$$0 \le \psi \le 1, \quad \operatorname{supp}\psi \subseteq [-4,4], \quad \text{and} \quad \psi \equiv 1 \quad \text{on} \quad [-2,2] \setminus \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]. \tag{4.33}$$

We may then invoke Theorem 4.1 with $n := 2$, $m := 3$, and

$$\begin{gathered} \Phi(t) := t, \quad A(t) := z(t), \quad B(t) := \big(\operatorname{Re} z(t), \operatorname{Im} z(t), t\big) \quad \text{for all} \quad t \in \mathbb{R}, \\ F(a,b,c) := \frac{c}{a+\mathrm{i}b}\phi(a+\mathrm{i}b)\psi(c) \quad \text{for all} \quad (a,b,c) \in \mathbb{R}^3, \end{gathered} \tag{4.34}$$

and conclude from (4.23) and (2.228) that there exist some integer $\widetilde{N} \in \mathbb{N}$ and some constant $C_{p,w} \in (0, \infty)$ such that, with $\varkappa$ as in (4.28), we have

$$\big\| C_{\mathbb{R}} - (\mathrm{i}/2)H \big\|_{L^p(\mathbb{R},w)\to L^p(\mathbb{R},w)} \le C_{p,w}(1+\varkappa)^{\widetilde{N}}\sqrt{\varkappa}. \tag{4.35}$$

This lends credence to (4.26) since it implies

$$\big\|C_{\mathbb{R}} - (\mathrm{i}/2)H\big\|_{L^p(\mathbb{R},w)\to L^p(\mathbb{R},w)} = O(\sqrt{\varkappa}\,) \ \text{ as } \ \varkappa \to 0^+. \tag{4.36}$$

After this preamble, we are ready to present the proof of Theorem 4.1.

***Proof of Theorem 4.1*** Throughout, let us abbreviate

$$K(x',y') := \frac{A(x') - A(y') - \langle \nabla A(y'), x'-y'\rangle}{|x'-y'|^n} F\Big(\frac{B(x')-B(y')}{|x'-y'|}\Big), \tag{4.37}$$

for each $x' \in \mathbb{R}^{n-1}$ and $\mathcal{L}^{n-1}$-a.e. $y' \in \mathbb{R}^{n-1}$. Having $T_*^{A,B} g(x')$ in (4.56) well defined for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and each $x' \in \mathbb{R}^{n-1}$ is ensured by observing that

$$K(\cdot,\cdot) \text{ is an } \mathcal{L}^{n-1}\otimes\mathcal{L}^{n-1}\text{-measurable function on } \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \tag{4.38}$$

which is clear from (4.37), and

$$\begin{aligned} &\text{for each } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}),\ \varepsilon > 0,\ x' \in \mathbb{R}^{n-1}, \\ &\text{one has } \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} |K(x',y')||g(y')|\,\mathrm{d}y' < +\infty. \end{aligned} \tag{4.39}$$

The finiteness property in (4.39) is a consequence of Hölder's inequality, (4.37), the fact that $F$ is bounded, and Lemma 4.1 (used with $n$ replaced by $n-1$, $p'$ in place of $p$, and with $w'$ in place of $w$). In concert, these give that for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, each $\varepsilon > 0$, and each $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} |K(x',y')||g(y')|\,\mathrm{d}y' &\le C\varepsilon\big[w'\big(B(x',\varepsilon)\big)\big]^{1/p'}\Big(\sup_{X\in\mathbb{R}^m}|F(X)|\Big)\times \\ &\quad \times \|g\|_{L^p(\mathbb{R}^{n-1},w\mathcal{L}^{n-1})}\big\|\nabla A\big\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \infty. \end{aligned} \tag{4.40}$$

To proceed, for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, each truncation parameter $\varepsilon > 0$, and each point $x' \in \mathbb{R}^{n-1}$ define

$$T_{\Phi,\varepsilon}^{A,B} g(x') := \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |\Phi(x')-\Phi(y')|>\varepsilon}} K(x',y')g(y')\,\mathrm{d}y'. \tag{4.41}$$

Thanks to (4.20) and (4.38)–(4.39), the above integral is absolutely convergent, which means that $T_{\Phi,\varepsilon}^{A,B} g(x')$ is a well-defined number. If $\mathbb{Q}_+$ denotes the collection of all positive rational numbers, we next make the claim that for each arbitrary function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we have

$$\big(T^{A,B}_{\Phi,*}g\big)(x') = \sup_{\varepsilon\in\mathbb{Q}_+} \big|\big(T^{A,B}_{\Phi,\varepsilon}g\big)(x')\big| \ \text{ for every } \ x' \in \mathbb{R}^{n-1}. \tag{4.42}$$

To justify this, pick some $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. The idea is to show that if the point $x' \in \mathbb{R}^{n-1}$ is arbitrary and fixed then for every $\varepsilon \in (0,\infty)$ and for every sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$ we have

$$\lim_{j\to\infty} \big(T^{A,B}_{\Phi,\varepsilon_j}g\big)(x') = \big(T^{A,B}_{\Phi,\varepsilon}g\big)(x'). \tag{4.43}$$

To justify (4.43) note that

$$\{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\} \nearrow \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\} \tag{4.44}$$

as $j \to \infty$, in the sense that

$$\begin{aligned}\{y' \in \mathbb{R}^{n-1} : &|\Phi(x') - \Phi(y')| > \varepsilon\} \\ &= \bigcup_{j\in\mathbb{N}} \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\}\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}\{y' \in \mathbb{R}^{n-1} : &|\Phi(x') - \Phi(y')| > \varepsilon_j\} \\ &\subseteq \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_{j+1}\}\end{aligned} \tag{4.46}$$

for every $j \in \mathbb{N}$. Then (4.43) follows from (4.44) and Lebesgue's Dominated Convergence Theorem (whose applicability is ensured by (4.38)–(4.39)). Having established this, (4.42) readily follows on account of the density of $\mathbb{Q}_+$ in $(0,\infty)$.

Moving on, we claim that

$$\begin{gathered}\text{for each fixed threshold } \varepsilon > 0, \text{ the function} \\ \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \ni (x', y') \longmapsto \big(\mathbf{1}_{\{y'\in\mathbb{R}^{n-1},\, |\Phi(x')-\Phi(y')|>\varepsilon\}}\big)(y') \in \mathbb{R} \\ \text{is lower-semicontinuous, hence } \mathcal{L}^{n-1}\otimes\mathcal{L}^{n-1}\text{-measurable.}\end{gathered} \tag{4.47}$$

To justify this claim, observe that for every number $\lambda \in \mathbb{R}$ the set of points in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ where the given function is $> \lambda$ may be described as

$$\begin{cases} \varnothing \ \text{ if } \ \lambda \geq 1, \\ \big\{(x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\big\} \ \text{ if } \ \lambda \in [0, 1), \\ \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \ \text{ if } \ \lambda < 0. \end{cases} \tag{4.48}$$

Thanks to the fact that $\Phi$ is a continuous function, all sets appearing in (4.48) are open in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. This proves that the function (4.47) is indeed lower-semicontinuous.

We next claim that

$$\text{given any } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}), \text{ the function } T^{A,B}_{\Phi,*} g \text{ is } \mathcal{L}^{n-1}\text{-measurable.} \tag{4.49}$$

To see that this is the case, granted (4.42) and since the supremum of some countable family of $\mathcal{L}^{n-1}$-measurable functions is itself a $\mathcal{L}^{n-1}$-measurable function, it suffices to show that

$$T^{A,B}_{\Phi,\varepsilon} g \text{ is a } \mathcal{L}^{n-1}\text{-measurable function, for each fixed } \varepsilon \in (0, \infty) \text{ and each fixed } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}). \tag{4.50}$$

With this goal in mind, fix $\varepsilon \in (0, \infty)$ along with $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, and for each $j \in \mathbb{N}$ define

$$\begin{aligned} &G_j : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \ \text{ given at every } (x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \text{ by} \\ &G_j(x', y') := \big(\mathbf{1}_{B(0',j)}\big)(x') K(x', y') g(y') \big(\mathbf{1}_{\{y' \in \mathbb{R}^{n-1},\, |\Phi(x') - \Phi(y')| > \varepsilon\}}\big)(y'). \end{aligned} \tag{4.51}$$

Then, thanks to (4.38) and (4.47), it follows that $G_j$ is an $\mathcal{L}^{n-1} \otimes \mathcal{L}^{n-1}$-measurable function for each $j \in \mathbb{N}$. In addition, from (4.51), (4.39), and since balls have finite measure, we see that

$$\int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} |G_j(x', y')| \, \mathrm{d}x' \mathrm{d}y' < +\infty. \tag{4.52}$$

Granted these properties, Fubini's Theorem (whose applicability is ensured by the fact that $\big(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}\big)$ is a sigma-finite measure space) then guarantees that

$$\begin{aligned} &g_j : \mathbb{R}^{n-1} \to \mathbb{R}, \quad g_j(x') := \int_{\mathbb{R}^{n-1}} G_j(x', y') \, \mathrm{d}y', \quad \forall\, x' \in \mathbb{R}^{n-1}, \\ &\text{is an } \mathcal{L}^{n-1}\text{-measurable function, for each integer } j \in \mathbb{N}. \end{aligned} \tag{4.53}$$

On the other hand, from (4.51), (4.53), and (4.41) it is apparent that for each $j \in \mathbb{N}$ we have

$$g_j = \mathbf{1}_{B(0',j)}\, T^{A,B}_{\Phi,\varepsilon} g \ \text{ everywhere in } \ \mathbb{R}^{n-1}. \tag{4.54}$$

In particular, this implies

$$\lim_{j\to\infty} g_j = T^{A,B}_{\Phi,\varepsilon} g \ \text{ pointwise everywhere in } \ \mathbb{R}^{n-1}. \tag{4.55}$$

At this stage, the fact that $T^{A,B}_{\Phi,\varepsilon} g$ is an $\mathcal{L}^{n-1}$-measurable function follows from (4.55) and (4.53). The claim in (4.49) is therefore established.

We next turn our attention to the main claim made in (4.23). The special case when $d := n-1$ and $\Phi(x') := x'$ for each $x' \in \mathbb{R}^{n-1}$ has been treated in [61], following basic work in [58]. Specifically, from [61, Theorem 4.34, p. 2725] we know that if for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we define

$$T^{A,B}_{*} g(x') := \sup_{\varepsilon>0} \Bigg| \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} K(x',y') g(y')\,\mathrm{d}y' \Bigg| \ \text{ at each } \ x' \in \mathbb{R}^{n-1}, \tag{4.56}$$

then

$$\begin{array}{l} T^{A,B}_{*} \ \text{is a well-defined sub-linear operator} \\ \text{from the space } L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \text{ into itself} \end{array} \tag{4.57}$$

and there exists a constant $C(n,p,w) \in (0,\infty)$ with the property that

$$\begin{aligned} &\big\| T^{A,B}_{*} \big\|_{L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \to L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})} \\ &\quad \le C(n,p,w) \Big( \sum_{|\alpha|\le N+2} \big\| \partial^\alpha F \big\|_{L^1(\mathbb{R}^m, \mathcal{L}^m)} + \sup_{X\in\mathbb{R}^m} (1+|X|)|F(X)| \Big) \\ &\qquad \times \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big( 1 + \sum_{j=1}^{m} \|\nabla B_j\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big)^N. \end{aligned} \tag{4.58}$$

To deal with the present case, in which the truncation is performed in the more general fashion described in (4.22), for each $\varepsilon > 0$ and each $x' \in \mathbb{R}^{n-1}$ abbreviate

$$\begin{aligned} D_\varepsilon(x') := &\big\{ y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon \ \text{ and } \ |x'-y'| \le \varepsilon \big\} \\ &\bigcup \big\{ y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| \le \varepsilon \ \text{ and } \ |x'-y'| > \varepsilon \big\}. \end{aligned} \tag{4.59}$$

Fix an arbitrary $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and define

$$Rg(x') := \sup_{\varepsilon>0} \int_{D_\varepsilon(x')} \left| \frac{A(x') - A(y') - \langle \nabla A(y'), x' - y' \rangle}{|x' - y'|^n} \times \right.$$
$$\left. \times\, F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| \mathrm{d}y' \tag{4.60}$$

at each point $x' \in \mathbb{R}^{n-1}$. The above definitions now imply that for each given function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we have

$$T^{A,B}_{\Phi,*} g(x') \leq T^{A,B}_{*} g(x') + Rg(x') \ \text{ for every } \ x' \in \mathbb{R}^{n-1}. \tag{4.61}$$

To estimate the last term appearing in the right-hand side of (4.61), pick some

$$\gamma \in (0, p-1) \text{ such that } w \in A_{p/(1+\gamma)}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{4.62}$$

fix an arbitrary point $x' \in \mathbb{R}^{n-1}$, consider an arbitrary threshold $\varepsilon > 0$, and select a function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. Also, abbreviate

$$Q := Q_{x',\varepsilon} := \big\{ y' \in \mathbb{R}^{n-1} : |x' - y'| < \varepsilon \big\} \tag{4.63}$$

and introduce

$$A_Q(z') := A(z') - \Big( \fint_Q \nabla A \, \mathrm{d}\mathcal{L}^{n-1} \Big) \cdot z' \ \text{ for each } \ z' \in \mathbb{R}^{n-1}. \tag{4.64}$$

Observe that the number $Rg(x')$, originally defined in (4.60), does not change if the function $A$ is replaced by $A_Q$. Consequently,

$$Rg(x') \leq R_1 g(x') + R_2 g(x'), \tag{4.65}$$

where

$$R_1 g(x') := \sup_{\varepsilon>0} \int_{D_\varepsilon(x')} \left| \frac{A_Q(x') - A_Q(y')}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| \mathrm{d}y' \tag{4.66}$$

and

$$R_2 g(x') := \sup_{\varepsilon>0} \int_{D_\varepsilon(x')} \left| \frac{\langle \nabla A_Q(y'), x' - y' \rangle}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| \mathrm{d}y'. \tag{4.67}$$

Note that, thanks to (4.20) and (4.59), we have

$$c\,\varepsilon \leq |x'-y'| \leq c^{-1}\varepsilon \quad \text{for each} \quad y' \in D_\varepsilon(x'). \tag{4.68}$$

Having fixed an integrability exponent $q \in (n-1, \infty)$, for each $y' \in D_\varepsilon(x')$ we may rely on Mary Weiss' Lemma (cf. [24, Lemma 1.4, p. 144]) in concert with (2.102), (2.103), (4.63), and (4.68) to estimate

$$\begin{aligned}
\frac{|A_Q(x')-A_Q(y')|}{|x'-y'|} &\leq C_{q,n}\Big(\fint_{|x'-z'|\leq 2|x'-y'|}|\nabla A_Q(z')|^q\,\mathrm{d}z'\Big)^{1/q}\\
&\leq C_{q,n}\Big(\fint_{|x'-z'|\leq 2|x'-y'|}\Big|\nabla A(z')-\fint_{|x'-\zeta'|\leq 2|x'-y'|}\nabla A(\zeta')\,\mathrm{d}\zeta'\Big|^q\,\mathrm{d}z'\Big)^{1/q}\\
&\quad + C_{q,n}\Big|\fint_Q \nabla A\,\mathrm{d}\mathcal{L}^{n-1}-\fint_{|x'-\zeta'|\leq 2|x'-y'|}\nabla A(\zeta')\,\mathrm{d}\zeta'\Big|\\
&\leq C_{q,n}\cdot c^{-2(n-1)/q}\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}.
\end{aligned} \tag{4.69}$$

Choosing $q := 2(n-1)$ it follows that there exists a constant $C_n \in (0,\infty)$, which depends only on $n$, such that

$$\begin{aligned}
|A_Q(x')-A_Q(y')| &\leq (C_n/c)|x'-y'|\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\\
&\quad \text{for each point} \quad y' \in D_\varepsilon(x').
\end{aligned} \tag{4.70}$$

In concert, (4.66), (4.68), and (4.70) allow us to conclude that

$$\begin{aligned}
R_1 g(x') &\leq C_n \cdot c^{1-2n}\big(\sup_{X\in\mathbb{R}^m}|F(X)|\big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\times\\
&\qquad\times \sup_{\varepsilon>0}\Big(\fint_{|x'-y'|<c^{-1}\varepsilon}|g(y')|\,\mathrm{d}y'\Big).
\end{aligned} \tag{4.71}$$

To estimate $R_2 g(x')$, bring in a brand of the Hardy–Littlewood maximal operator which associates to each $\mathcal{L}^{n-1}$-measurable function $f$ on $\mathbb{R}^{n-1}$ the function $\mathcal{M}_\gamma f$ defined as

$$\mathcal{M}_\gamma f(x') := \sup_{r>0}\Big(\fint_{|x'-y'|<r}|f(y')|^{1+\gamma}\,\mathrm{d}y'\Big)^{1/(1+\gamma)} \quad \text{for each} \quad x' \in \mathbb{R}^{n-1}. \tag{4.72}$$

Then, using (4.67), (4.64), Hölder's inequality, and (2.103) we may write

$$\begin{aligned}
R_2 g(x') &\leq C_n \cdot c^{2-2n}\big(\sup_{X\in\mathbb{R}^m}|F(X)|\big)\times\\
&\quad\times \sup_{\varepsilon>0}\Big(\fint_{|x'-y'|<c^{-1}\varepsilon}\Big|\nabla A(y')-\fint_Q \nabla A\,\mathrm{d}\mathcal{L}^{n-1}\Big||g(y')|\,\mathrm{d}y'\Big)
\end{aligned}$$

$$\le C_n \cdot c^{2-2n}\big(\sup_{X\in\mathbb{R}^m}|F(X)|\big)\mathcal{M}_\gamma g(x')\times$$

$$\times \sup_{\varepsilon>0}\Big(⨍_{|x'-y'|<c^{-1}\varepsilon}\Big|\nabla A(y') - ⨍_Q \nabla A\,\mathrm{d}\mathcal{L}^{n-1}\Big|^{(1+\gamma)/\gamma}\,\mathrm{d}y'\Big)^{\gamma/(1+\gamma)}$$

$$\le C_{n,\gamma}\cdot c^{3-3n}\big(\sup_{X\in\mathbb{R}^m}|F(X)|\big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\mathcal{M}_\gamma g(x'). \tag{4.73}$$

Collectively, (4.65), (4.71), and (4.73), and Hölder's inequality imply

$$Rg(x') \le C_{n,\gamma}\cdot c^{-3n}\big(\sup_{X\in\mathbb{R}^m}|F(X)|\big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\mathcal{M}_\gamma g(x'). \tag{4.74}$$

In turn, from (4.74) and (4.61) we conclude that for every $x'\in\mathbb{R}^{n-1}$ we have

$$\begin{aligned} 0 \le & T^{A,B}_{\Phi,*}g(x') \qquad (4.75)\\ & \le T^{A,B}_{*}g(x') + C_{n,\gamma}\cdot c^{-3n}\big(\sup_{X\in\mathbb{R}^m}|F(X)|\big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\mathcal{M}_\gamma g(x'). \end{aligned}$$

Granted (4.62), the maximal operator $\mathcal{M}_\gamma$ is a well-defined sub-linear bounded mapping from $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself. Bearing this in mind, from (4.75), (4.57), (4.58), and (2.575), and the fact that the space $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ is a lattice, the estimate claimed in (4.23) now follows. As a consequence, $T^{A,B}_{\Phi,*}$ is a sub-linear mapping of finite operator norm on $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. Hence, as remarked in (4.3), the operator $T^{A,B}_{\Phi,*}$ is continuous from $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself. □

The next step is to transfer the Euclidean result from Theorem 4.1 to singular integral operators on Lipschitz graphs, a task accomplished in the following lemma.

**Lemma 4.2** *Having fixed an arbitrary unit vector $\vec{n}\in S^{n-1}$, consider the hyperplane $H := \langle\vec{n}\rangle^{\perp}\subseteq\mathbb{R}^{n-1}$ and suppose $h: H\to\mathbb{R}$ is a function satisfying*

$$M := \sup_{\substack{x,y\in H\\ x\neq y}}\frac{|h(x)-h(y)|}{|x-y|} < +\infty. \tag{4.76}$$

*Fix an arbitrary point $x_0\in\mathbb{R}^n$ and let*

$$\mathcal{G} := \big\{x_0 + x + h(x)\vec{n} \,:\, x\in H\big\}\subseteq\mathbb{R}^n \tag{4.77}$$

*denote the graph of $h$ in the coordinate system $X=(x,t)\Leftrightarrow X = x_0+x+t\vec{n}$, with $x\in H$ and $t\in\mathbb{R}$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\mathcal{G}$ and denote by $\nu$ the unique unit normal to $\mathcal{G}$ satisfying $\nu\cdot\vec{n}<0$ at $\sigma$-a.e. point on $\mathcal{G}$. Also, fix some integrability exponent*

$p \in (1, \infty)$. *Given a complex-valued function* $k \in \mathscr{C}^{N+2}(\mathbb{R}^n \setminus \{0\})$*, for some sufficiently large integer* $N = N(n) \in \mathbb{N}$*, which is even and positive homogeneous of degree* $-n$*, consider the maximal singular integral operator* $T$ *acting on each* $f \in L^p(\mathcal{G}, \sigma)$ *as*

$$T_* f(x) := \sup_{\varepsilon>0} \left| \int_{\substack{y\in\mathcal{G} \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, \mathrm{d}\sigma(y) \right|, \quad \forall\, x \in \mathcal{G}. \tag{4.78}$$

*Then* $T_*$ *is a well-defined continuous sub-linear mapping from the space* $L^p(\mathcal{G}, \sigma)$ *into itself and there exists a constant* $C(n, p) \in (0, \infty)$*, which depends only on* $n$, $p$*, with the property that*

$$\|T_*\|_{L^p(\mathcal{G},\sigma)\to L^p(\mathcal{G},\sigma)} \leq C(n, p) M(1+M)^{4n+N} \Big( \sum_{|\alpha|\leq N+2} \sup_{S^{n-1}} \big|\partial^\alpha k\big| \Big). \tag{4.79}$$

*Moreover, corresponding to the end-point case* $p = 1$*, the operator* $T_*$ *induces a well-defined continuous sub-linear mapping from the space* $L^1(\mathcal{G}, \sigma)$ *into the space* $L^{1,\infty}(\mathcal{G}, \sigma)$ *and there exists a constant* $C_n \in (0, \infty)$ *along with some large exponent* $N_n \in \mathbb{N}$*, which depend only on n, with the property that*

$$\|T_*\|_{L^1(\mathcal{G},\sigma)\to L^{1,\infty}(\mathcal{G},\sigma)} \leq C_n (1+M)^{N_n} \Big( \sum_{|\alpha|\leq N_n} \sup_{S^{n-1}} \big|\partial^\alpha k\big| \Big). \tag{4.80}$$

***Proof*** Recall that $\{\mathbf{e}_j\}_{1\leq j\leq n}$ stands for the standard orthonormal basis in $\mathbb{R}^n$. Let us first treat the case when $x_0 = 0 \in \mathbb{R}^n$ and $\vec{n} := \mathbf{e}_n$, a scenario in which $H = \langle \mathbf{e}_n \rangle^\perp$ may be canonically identified with $\mathbb{R}^{n-1}$. Assume this is the case, and consider an even function $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that

$$\begin{aligned} &0 \leq \psi \leq 1, \quad \psi \text{ vanishes identically in } \mathbb{R}^n \setminus B\big(0, 2\sqrt{1+M^2}\big), \\ &\psi \equiv 1 \text{ on } \overline{B(0, \sqrt{1+M^2})} \setminus B(0,1), \quad \psi \equiv 0 \text{ on } B(0, 1/2), \\ &\text{and for each } \alpha \in \mathbb{N}_0^n \text{ there exists } C_\alpha \in (0,\infty), \text{ depending only} \\ &\text{on the given multi-index } \alpha, \text{ so that } \sup\nolimits_{x\in\mathbb{R}^n} |(\partial^\alpha \psi)(x)| \leq C_\alpha. \end{aligned} \tag{4.81}$$

Then $F := \psi k$ is an even function belonging to $\mathscr{C}^{N+2}(\mathbb{R}^n)$, and satisfying

$$\begin{aligned} \sum_{|\alpha|\leq N+2} \big\|\partial^\alpha F\big\|_{L^1(\mathbb{R}^n,\mathcal{L}^n)} &+ \sup_{x\in\mathbb{R}^n} (1+|x|)|F(x)| \\ &\leq C_n (1+M)^n \Big( \sum_{|\alpha|\leq N+2} \sup_{S^{n-1}} \big|\partial^\alpha k\big| \Big), \end{aligned} \tag{4.82}$$

for some purely dimensional constant $C_n \in (0, \infty)$. Moreover, if for each point $x' \in \mathbb{R}^{n-1}$ we set $\Phi(x') := (x', h(x'))$ then $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a bi-Lipschitz function and (4.81) implies that

$$k\left(\frac{\Phi(x') - \Phi(y')}{|x' - y'|}\right) = F\left(\frac{\Phi(x') - \Phi(y')}{|x' - y'|}\right) \quad \text{for each } x', y' \in \mathbb{R}^{n-1} \text{ with } x' \neq y'. \tag{4.83}$$

To proceed, note that for each $\sigma$-measurable set $E \subseteq \mathcal{G}$ and each $g \in L^1(E, \sigma)$ we have

$$\int_E g \, \mathrm{d}\sigma = \int_{\{y' \in \mathbb{R}^{n-1} : (y', h(y')) \in E\}} g(y', h(y')) \sqrt{1 + |(\nabla h)(y')|^2} \, \mathrm{d}y', \tag{4.84}$$

(cf., e.g., [136, Proposition 12.9, p. 164]) and

$$\nu(y', h(y')) = \frac{((\nabla h)(y'), -1)}{\sqrt{1 + |(\nabla h)(y')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } y' \in \mathbb{R}^{n-1}. \tag{4.85}$$

Also, fix $f \in L^p(\mathcal{G}, \sigma)$ and define $\widetilde{f}(x') := f(x', h(x'))$ for each $x' \in \mathbb{R}^{n-1}$. In particular, from (4.84) we conclude that

$$\widetilde{f} \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \quad \text{and} \quad \|\widetilde{f}\|_{L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})} \leq \|f\|_{L^p(\mathcal{G}, \sigma)}. \tag{4.86}$$

Then based on (4.78), (4.84), (4.85), the homogeneity of $k$, and (4.83) we may write

$$\begin{aligned}
&(T_* f)(x', h(x')) \\
&= \sup_{\varepsilon > 0} \Bigg| \int_{\substack{y' \in \mathbb{R}^{n-1} \text{ with} \\ \sqrt{|x'-y'|^2 + (h(x')-h(y'))^2} > \varepsilon}} \big(\langle \nabla h(y'), x' - y' \rangle + h(y') - h(x')\big) \times \\
&\qquad \times k\big(x' - y', h(x') - h(y')\big) \widetilde{f}(y') \, \mathrm{d}y' \Bigg| \\
&= \sup_{\varepsilon > 0} \Bigg| \int_{\substack{y' \in \mathbb{R}^{n-1} \\ |\Phi(x') - \Phi(y')| > \varepsilon}} \frac{h(x') - h(y') - \langle \nabla h(y'), x' - y' \rangle}{|x' - y'|^n} \times
\end{aligned}$$

$$\times F\Big(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\Big)\widetilde{f}(y')\,\mathrm{d}y'\Big|. \tag{4.87}$$

From (4.87), Theorem 4.1 (used with $m := n$, $d := n$, $A := h$, $B := \Phi$, and $w \equiv 1$), (4.82), and (4.84) we then conclude that (4.79) holds in this case.

To treat the case when $x_0 = 0$ but $\vec{n} \in S^{n-1}$ is arbitrary, pick an orthonormal basis $\{v_j\}_{1\le j\le n-1}$ in $H$ and consider the unitary transformation in $\mathbb{R}^n$ uniquely defined by the demand that $Uv_j = \mathbf{e}_j$ for $j \in \{1,\dots,n-1\}$ and $U\vec{n} = \mathbf{e}_n$. Then $\widetilde{\mathcal{G}} := U\mathcal{G}$ becomes the graph of $\widetilde{h} := h \circ U^{-1} : \mathbb{R}^{n-1} \to \mathbb{R}$, which is a Lipschitz function with the same Lipschitz constant $M$ as the original function $h$. Since the Hausdorff measure is rotation invariant, for each $g \in L^1(\mathcal{G}, \sigma)$ we have

$$\int_{y\in\mathcal{G}} g(y)\,\mathrm{d}\sigma(y) = \int_{\widetilde{y}\in\widetilde{\mathcal{G}}} (g\circ U^{-1})(\widetilde{y})\,\mathrm{d}\widetilde{\sigma}(\widetilde{y}), \tag{4.88}$$

where $\widetilde{\sigma} := \mathcal{H}^{n-1}\lfloor\widetilde{\mathcal{G}}$. Moreover, the unique unit normal $\widetilde{\nu}$ to $\widetilde{\mathcal{G}}$ satisfying $\widetilde{\nu}\cdot\mathbf{e}_n < 0$ at $\mathcal{H}^{n-1}$-a.e. point on $\widetilde{\mathcal{G}}$ is $\widetilde{\nu} = U(\nu\circ U^{-1})$. Consider $\widetilde{k} := k \circ U^{-1}$ and note that this is a complex-valued function of class $\mathscr{C}^{N+2}\big(\mathbb{R}^n \setminus \{0\}\big)$, which is even and positive homogeneous of degree $-n$. Finally, fix some function $f \in L^p(\mathcal{G},\sigma)$ and abbreviate $\widetilde{f} := f \circ U^{-1}$. Bearing in mind the fact that $U$ is a linear isometry satisfying $U^{-1} = U^{\top}$, from (4.78) and (4.88) we see that if $x \in \mathcal{G}$ and $\widetilde{x} := Ux$ then

$$T_* f(x) = \sup_{\varepsilon>0}\Big|\int_{\substack{\widetilde{y}\in\widetilde{\mathcal{G}}\\ |\widetilde{x}-\widetilde{y}|>\varepsilon}} \langle\widetilde{x}-\widetilde{y}, \widetilde{\nu}(\widetilde{y})\rangle\widetilde{k}(\widetilde{x}-\widetilde{y})\widetilde{f}(\widetilde{y})\,\mathrm{d}\widetilde{\sigma}(\widetilde{y})\Big|. \tag{4.89}$$

Hence,

$$T_* f(x) = \widetilde{T}_*\widetilde{f}(\widetilde{x}) \quad\text{whenever}\quad x\in\mathcal{G} \quad\text{and}\quad \widetilde{x} = Ux, \tag{4.90}$$

where $\widetilde{T}_*$ is the maximal operator associated as in (4.78) with the Lipschitz graph $\widetilde{\mathcal{G}}$ and the kernel $\widetilde{k}$. In particular, given that (4.90) and (4.88) imply

$$\int_{\mathcal{G}} (T_* f)(x)^p\,\mathrm{d}\sigma(x) = \int_{\widetilde{\mathcal{G}}} (\widetilde{T}_*\widetilde{f})(\widetilde{x})^p\,\mathrm{d}\widetilde{\sigma}(\widetilde{x}), \tag{4.91}$$

the estimate claimed in (4.79) becomes a consequence of the corresponding estimate for the maximal operator $\widetilde{T}_*$ established in the first part of the current proof.

The case when both $x_0 \in \mathbb{R}^n$ and $\vec{n} \in S^{n-1}$ are arbitrary follows from what we have proved so far using the natural invariance of the maximal operator (4.78) to translations.

Finally, the estimate claimed in (4.80) becomes a consequence of (4.79) (with, say, the choice $p = 2$), and standard Calderón–Zygmund theory (based on the classical Calderón–Zygmund Lemma, and Cotlar's inequality). See, for example, [56, Theorem 8.2.1, p. 584] for more details in the standard Euclidean setting. □

## 4.2 Estimates for Certain Classes of Singular Integrals on UR Sets

Theorem 4.2, which is central for the present work, is the main result regarding the size of the operator norm of certain maximal integral operators acting on Muckenhoupt weighted Lebesgue spaces on the boundary of UR domains. In turn, this is going to be the key ingredient in obtaining invertibility results for the brand of boundary double layer potential operators considered in this work.

To facilitate stating Theorem 4.2 we first introduce some notation and make some remarks. Specifically, with e denoting the base of natural logarithms, for each number $m \in \mathbb{N}_0$ and $t \in [0, \infty)$ let us define

$$t^{\langle 0 \rangle} := 1 \tag{4.92}$$

and, if $m \geq 1$,

$$t^{\langle m \rangle} := \begin{cases} 0 & \text{if } \; t = 0, \\ t \cdot \underbrace{\ln\Big(\cdots \ln\big(\ln(1/t)\big)\cdots\Big)}_{m \text{ natural logarithms}} & \text{if } \; 0 < t \leq ({}^{m}\mathrm{e})^{-1}, \\ ({}^{m}\mathrm{e})^{-1} & \text{if } \; t > ({}^{m}\mathrm{e})^{-1}, \end{cases} \tag{4.93}$$

where ${}^{m}\mathrm{e}$ is the $m$-th tetration of e (involving $m$ copies of e, combined by exponentiation), i.e.,

$${}^{m}\mathrm{e} := \underbrace{\mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot^{\mathrm{e}}}}}}}_{m \text{ copies of e}}, \quad \text{the } m\text{-th fold exponentiation of e.} \tag{4.94}$$

We also agree to set ${}^{0}\mathrm{e} := 1$. Hence, inductively, for each integer $m \in \mathbb{N}_0$ and each $t \in [0, \infty)$ we have

$$t^{\langle m+1\rangle} = \begin{cases} 0 & \text{if } t = 0, \\ t \cdot \ln\big(t^{\langle m\rangle}/t\big) & \text{if } 0 < t \le ({}^{m+1}\mathrm{e})^{-1}, \\ ({}^{m+1}\mathrm{e})^{-1} & \text{if } t > ({}^{m+1}\mathrm{e})^{-1}. \end{cases} \tag{4.95}$$

For further reference, it is useful to note that elementary calculus gives that this function enjoys the following properties:

$$[0,\infty) \ni t \longmapsto t^{\langle m\rangle} \in [0,\infty) \ \text{ is continuous, non-decreasing,} \tag{4.96}$$

$$\begin{gathered} t^{\langle m\rangle} \le t^{\langle m-1\rangle} \le \cdots \le t^{\langle 1\rangle} \le \big(\mathrm{e}^{\varepsilon-1}/\varepsilon\big) \cdot t^{1-\varepsilon} \\ \text{for each } \ t \in [0,\infty), \ m \in \mathbb{N}, \ \varepsilon \in (0,1), \end{gathered} \tag{4.97}$$

$$t \le \max\{1, ({}^{m}\mathrm{e})t\} \cdot t^{\langle m\rangle} \ \text{ for all } \ t \in [0,\infty) \ \text{ and } \ m \in \mathbb{N}_0, \tag{4.98}$$

$$(\lambda t)^{\langle m\rangle} \le \lambda t^{\langle m\rangle} \ \text{ for all } \ t \in [0,\infty), \ m \in \mathbb{N}_0, \ \text{ and } \ \lambda \in [1,\infty), \tag{4.99}$$

$$\begin{gathered} (t^{\alpha})^{\langle m\rangle} \le t^{\alpha} \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln(1/\min\{t, ({}^{m}\mathrm{e})^{-1}\})\big)\cdots\Big) \\ \text{for all } \ t \in [0,\infty), \ m \in \mathbb{N}, \ \text{ and } \ \alpha \in (0,1] \end{gathered} \tag{4.100}$$

(with the convention that the value at $t = 0$ for the function in the right-hand side of the inequality in (4.100) is its limit as $t \to 0^{+}$). In particular,

$$t^{\langle m\rangle} \le t \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/t)\big)\cdots\Big) \ \text{ for all } \ t \in [0,1], \ m \in \mathbb{N}. \tag{4.101}$$

In fact, up to a multiplicative constant, the opposite inequality in (4.101) is true as well. Specifically,

$$({}^{m}\mathrm{e})^{-1} \cdot t \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/t)\big)\cdots\Big) \le t^{\langle m\rangle} \ \text{ for all } \ t \in [0,1], \ m \in \mathbb{N}, \tag{4.102}$$

hence for each fixed $m \in \mathbb{N}$ we have

$$t^{\langle m\rangle} \approx t \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/t)\big)\cdots\Big), \ \text{ uniformly for } \ t \in [0,1]. \tag{4.103}$$

Here is the basic result mentioned earlier. Its proof is inspired by that of [61, Theorem 4.36, pp. 2728-2729].

**Theorem 4.2** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same symbol $w$ for the measure associated with the given weight $w$ as in* (2.509).

*Next, consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, where $N = N(n) \in \mathbb{N}$ is a sufficiently large integer. Associate with this function and the set $\Omega$ the maximal operator $T_*$ whose action on each given function $f \in L^p(\partial\Omega, w)$ is defined as*

$$T_* f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big| \ \text{ for each } \ x \in \partial\Omega, \tag{4.104}$$

*where, for each $\varepsilon > 0$,*

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, \mathrm{d}\sigma(y) \ \text{ for all } \ x \in \partial\Omega. \tag{4.105}$$

*Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0,\infty)$, which depends only on $m$, $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.106}$$

*Moreover, when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in* (4.106) *to depend itself only on said entities (i.e., $n$, $p$, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$) and $m$.*

Before presenting the proof of this theorem, several comments are in order.

*Remark 4.1* It is of interest to compare the estimate in the above theorem with the corresponding estimate from Proposition 3.4. Specifically, estimate (3.79) applied with $\Sigma := \partial\Omega$ gives that for $T_*$ as in (4.104) we have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C(\partial\Omega, p, [w]_{A_p})\big\|k\big|_{S^{n-1}}\big\|_{\mathscr{C}^N(S^{n-1})}, \tag{4.107}$$

where $C(\partial\Omega, p, [w]_{A_p}) \in (0,\infty)$ depends on $\partial\Omega$ solely through its UR constants. We observe that, in sharp contrast to this estimate, (4.106) features in the right-hand side $\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ as a multiplicative factor, something which the UR constants of $\partial\Omega$ cannot control.

Indeed, for (3.79) no provisions are in place to take advantage of the specific algebraic format of the present integral kernel $\langle x-y, \nu(y)\rangle k(x-y)$. For Proposition 3.4 to apply, this integral kernel needs to be dismantled into its most primordial building blocks, i.e., as $\sum_{j=1}^{n} k_j(x-y)\nu_j(y)$ with $k_j(z) := z_j k(z)$ for each point $z \in \mathbb{R}^n \setminus \{0\}$ and $j \in \{1, \dots, n\}$. Since multiplication by $\nu_j$ may be absorbed with the function $f$ (without changing its membership, or increasing its size, in the Muckenhoupt weighted Lebesgue space $L^p(\partial\Omega, w)$), Proposition 3.4 may then finally be invoked in relation to each maximal operator associated with the kernel $k_j$. Estimate (3.79), the end-product of such an approach, is then rendered insensitive to the flatness of $\partial\Omega$.

As an example, consider the scenario in which $\Omega$ is a half-space in $\mathbb{R}^n$. While is apparent from (4.104)–(4.105) that in this case $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} = 0$, estimate (3.79) only gives $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} < +\infty$. By way of contrast, since in this case $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 0$ given that $\nu$ is a constant vector, (4.106) accurately predicts $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} = 0$.

*Remark 4.2* In view of (2.118) and (4.103), in the estimate recorded in (4.106) we could use

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \cdot \underbrace{\ln\Big(\cdots\ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n})\big)\cdots\Big) \tag{4.108}$$

in place of $\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$. In particular, if we abbreviate

$$\|\nu\|_* := \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.109}$$

then corresponding to $m = 1$ we thus obtain

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_\Omega\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_* \ln\big(\mathrm{e}/\|\nu\|_*\big), \tag{4.110}$$

corresponding to $m = 2$ we have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_\Omega\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_* \ln\Big(\ln\big(\mathrm{e}^{\mathrm{e}}/\|\nu\|_*\big)\Big), \tag{4.111}$$

etc., where in each case $C_\Omega \in (0, \infty)$ depends only on $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$. In particular, all the aforementioned operator norms have at most linear growth in $\|\nu\|_*$, up to arbitrarily many iterated logarithms.

In the same vein, we may rely on the property recorded in (4.97) and we deduce from (4.106) that for each $\varepsilon \in (0, 1)$ we have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq \big(e^{\varepsilon-1}/\varepsilon\big)\cdot C_\Omega\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{1-\varepsilon}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.112}$$

where $C_\Omega \in (0,\infty)$ depends only on $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$.

*Remark 4.3* In the context of Theorem 4.2, estimate (4.106) continues to hold with a fixed constant $C_m \in (0,\infty)$ when the integrability exponent and the Muckenhoupt weight are allowed to vary with control. Specifically, an inspection of the proof of Theorem 4.2 given below shows that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a constant $C_m \in (0,\infty)$, which depends only on $m$, $n$, $I$, $W$, and the UR constants of $\partial\Omega$, with the property that (4.106) holds for each $p \in I$ and each $w \in A_p(\partial\Omega,\sigma)$ with $[w]_{A_p} \leq W$.

*Remark 4.4* From Proposition 3.4 we already know that $T_*$ is bounded on $L^p(\partial\Omega, w)$, with norm controlled in terms of $n$, $k$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$. The crux of the matter here is the more refined version of the estimate of the operator norm of $T_*$ given in (4.106).

*Remark 4.5* We focus on establishing the estimate claimed in (4.106) in the class of operators whose integral kernel factors as the product of $\langle x-y, \nu(y)\rangle$, i.e., the inner product between the unit normal $\nu(y)$ and the "chord" $x-y$, with some matrix-valued function $k \in \mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ which is even and positive homogeneous of degree $-n$, since it has been noted in (1.50) that this is the only type of kernel (in the class of double layer-like integral operators) for which said estimate has a chance of materializing.

*Remark 4.6* The class of domains to which Theorem 4.2 applies includes all NTA domains with an Ahlfors regular boundary.

*Remark 4.7* In the unweighted case, i.e., for $w \equiv 1$ (or, equivalently, when the measure $w$ coincides with $\sigma$), estimate (4.106) simply reads

$$\|T_*\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.113}$$

It turns out that whenever (4.113) is available one may produce a weighted version of such an estimate via interpolation. Specifically, recall the interpolation theorem of Stein-Weiss (cf. [14, Theorem 5.4.1, p. 115]) according to which for any two $\sigma$-measurable functions $w_0, w_1 : \partial\Omega \to [0,\infty]$ and any $\theta \in (0,1)$ we have

$$\big(L^p(\partial\Omega, w_0\sigma)\,,\,L^p(\partial\Omega, w_1\sigma)\big)_{\theta,p} = L^p(\partial\Omega, \widetilde{w}\sigma) \;\text{ where }\; \widetilde{w} := w_0^{1-\theta}\cdot w_1^{\theta}. \tag{4.114}$$

Now, given a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, from (2.533) we know that there exists some $\tau \in (1, \infty)$ (which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) such that $w^\tau \in A_p(\partial\Omega, \sigma)$. Upon specializing (4.114) to the case when $\theta := 1 - \tau^{-1} \in (0, 1)$, $w_0 := w^\tau$, and $w_1 := 1$ we therefore obtain

$$\big(L^p(\partial\Omega, w^\tau\sigma)\,,\ L^p(\partial\Omega, \sigma)\big)_{\theta,p} = L^p(\partial\Omega, w). \tag{4.115}$$

As a result, since $T_*$ is a sub-linear operator which is bounded both on $L^p(\partial\Omega, w^\tau\sigma)$ (given that $w^\tau \in A_p(\partial\Omega, \sigma)$), and on $L^p(\partial\Omega, \sigma)$ we may write

$$\begin{aligned}
&\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \\
&\quad \le \|T_*\|^{1-\theta}_{L^p(\partial\Omega,w^\tau\sigma)\to L^p(\partial\Omega,w^\tau\sigma)}\,\|T_*\|^{\theta}_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \\
&\quad \le C_{\Omega,m,n,p,k,[w]_{A_p}}\Big(\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^{\theta},
\end{aligned} \tag{4.116}$$

with the last inequality provided by (4.113).

While the weighted norm inequality established in (4.116) is in the spirit of (4.106), the manner in which the BMO semi-norm of the outward unit normal vector $\nu$ is involved is less optimal, as the small exponent $\theta$ tempers the rate at which the right-hand side of (4.116) vanishes as $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \to 0^+$ (indeed, we have $\lim_{t\to 0^+} (t^{\langle m\rangle})^\theta / t^{\langle m\rangle} = +\infty$ for each fixed $\theta \in (0, 1)$). Hence, a two-step approach consisting first of proving the plain estimate (4.113) and, second, deriving a weighted version based on the procedure based on interpolation described above, only yields a weaker result than the one advertised in (4.106). Given this, in the proof of (4.106) presented below we shall devise an alternative approach, which deals with the weighted case directly, incorporating the weight in all relevant intermediary steps.

We are ready to proceed to the task of providing the proof of Theorem 4.2.

***Proof of Theorem 4.2*** We shall write the proof of Theorem 4.2 using an approach designed to shed light on the specific manner in which the right-hand side of (4.106) depends on the BMO semi-norm of the geometric measure theoretic outward unit normal vector $\nu$ to the set $\Omega$.

The bulk of the proof is occupied by the justification of the following result (strongly reminiscent of an induction step, that allows us to boot-strap a weaker bound on $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ to a stronger one): knowing that there exists a function

$$\psi : [0, \infty) \longrightarrow [0, \infty) \tag{4.117}$$

which is quasi-increasing near the origin, i.e.,

$$\begin{aligned}&\text{there exist } t_* > 0 \text{ and } C \in [1, \infty) \text{ such that}\\&\psi(t_0) \leq C\psi(t_1) \text{ whenever } 0 \leq t_0 < t_1 < t_*,\end{aligned} \tag{4.118}$$

such that for each exponent $p \in (1, \infty)$ and each weight $w \in A_p(\partial\Omega, \sigma)$ there exists a constant $C \in (0, \infty)$, depending only on $n$, $p$, $[w]_{A_p}$, the UR constants of $\partial\Omega$, and $\psi$, with the property that

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\psi\big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\big), \tag{4.119}$$

implies that for each given integrability exponent $p \in (1, \infty)$, each Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and each function

$$\phi : [0, \infty) \longrightarrow [0, \infty) \tag{4.120}$$

satisfying

$$\begin{aligned}&\inf\{\phi(t) : t \geq \widetilde{t}\} > 0 \quad \text{for each } \widetilde{t} > 0,\\&\phi(\widetilde{t}) \geq \liminf_{t\searrow\widetilde{t}} \phi(t) \quad \text{for each } \widetilde{t} > 0,\\&\phi(0) = \lim_{t\to 0^+} \phi(t) = 0, \quad \phi'(0) := \lim_{t\to 0^+} \phi(t)/t = \infty,\\&\text{and } \psi(t)\cdot\phi(t)^{-1}\cdot \mathrm{e}^{-\phi(t)/t} = O(1) \text{ as } t \to 0^+,\end{aligned} \tag{4.121}$$

there exists a constant $C \in (0, \infty)$ depending only on $n$, $p$, $[w]_{A_p}$, the UR constants of $\partial\Omega$, $\psi$, and $\phi$, such that we also have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\phi\big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\big). \tag{4.122}$$

Henceforth we shall summarize the above claim by simply saying that "(4.119) implies (4.122)."

In connection with (4.121) we wish to make two remarks. Our first remark pertains to the case when we assume

$$\lim_{t\to 0^+} \psi(t)/t = \infty. \tag{4.123}$$

In particular,

$$t_e := \sup\big\{t_o \in (0, \infty) : \psi(t)/t > \mathrm{e} \text{ for all } t \in (0, t_o)\big\} \in (0, \infty] \tag{4.124}$$

is well defined and

$$\psi(t)/t > \mathrm{e} \quad \text{for all} \quad t \in (0, t_e). \tag{4.125}$$

Then among all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the last property in (4.121) the smallest (up to multiplicative constants) in terms of behavior near the origin is actually the function

$$\begin{aligned} &\widehat{\psi} : [0, \infty) \longrightarrow [0, \infty) \text{ given for each } t \geq 0 \text{ by} \\ &\widehat{\psi}(0) := 0, \quad \widehat{\psi}(t) := t \ln(\psi(t)/t) \quad \text{if} \quad t \in (0, t_e), \\ &\text{and} \quad \widehat{\psi}(t) := t_e \ln(\psi(t_e)/t_e) \quad \text{for all} \quad t \in [t_e, \infty). \end{aligned} \tag{4.126}$$

To justify the minimality of (4.126), observe that the property in the last line of (4.121) implies that there exist $t_b, M \in (0, \infty)$ such that

$$\psi(t) \leq M\phi(t) \cdot \mathrm{e}^{\phi(t)/t} \quad \text{for each} \quad t \in (0, t_b). \tag{4.127}$$

Elementary calculus gives

$$x\mathrm{e}^x \leq \mathrm{e}^{2x-1} \quad \text{for each} \quad x \in [0, \infty). \tag{4.128}$$

From this used with $x := \phi(t)/t$ and (4.127) we then obtain

$$\psi(t)/t \leq M\mathrm{e}^{2\phi(t)/t-1} \quad \text{for each} \quad t \in (0, t_b). \tag{4.129}$$

In turn, this forces

$$\tfrac{1}{2}t \ln\big(\mathrm{e}\,\psi(t)/Mt\big) \leq \phi(t) \quad \text{for each} \quad t \in (0, t_b), \tag{4.130}$$

and since thanks to (4.123) we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{\tfrac{1}{2}t \ln\big(\mathrm{e}\,\psi(t)/Mt\big)}{t\ln(\psi(t)/t)} &= \lim_{t\to 0^+} \frac{\tfrac{1}{2}\ln(\mathrm{e}/M) + \tfrac{1}{2}\ln(\psi(t)/t)}{\ln(\psi(t)/t)} \\ &= \frac{1}{2} + \frac{1}{2}\ln(\mathrm{e}/M) \lim_{t\to 0^+} \frac{1}{\ln(\psi(t)/t)} = \frac{1}{2}, \end{aligned} \tag{4.131}$$

we ultimately conclude that

given any $\phi : [0, \infty) \to [0, \infty)$ satisfying the last property in (4.121) it follows that $\phi(t)$ dominates, up to a multiplicative constant, $\widehat{\psi}(t)$ for all $t \geq 0$ sufficiently close to 0. (4.132)

This justifies the claim about the minimality of $\widehat{\psi}$ made in the previous paragraph.

The second remark we wish to make in connection with (4.121) is that if in addition to (4.118) and (4.123) we also assume that

$$\psi \text{ is continuous and } \lim_{t\to 0^+} t\ln(\psi(t)/t) = 0, \tag{4.133}$$

then

$$\begin{array}{l}\text{the function } \widehat{\psi} \text{ defined in (4.126) is continuous, quasi-increasing} \\ \text{near the origin (in the sense of (4.118)), } \lim_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) = 0, \\ \text{and the function } \phi := \widehat{\psi} \text{ satisfies all properties listed in (4.121).}\end{array} \tag{4.134}$$

That $\widehat{\psi}$ is continuous is clear from (4.126) and (4.133). In particular, $\phi := \widehat{\psi}$ satisfies the second property listed in (4.121). To check the second claim made in (4.134), observe that

$$\begin{array}{l}(0,\infty) \ni y \longmapsto x\ln(y/x) \text{ is a strictly increasing function for} \\ \text{each fixed } x \in (0,\infty)\text{, and each fixed } y \in (0,\infty) \text{ the function} \\ (0, y/\mathrm{e}) \ni x \longmapsto x\ln(y/x) \text{ is also strictly increasing.}\end{array} \tag{4.135}$$

If $t_* > 0$ and $C \in (0,\infty)$ are as in (4.118), if $t_e \in (0,\infty)$ is as in (4.125), and if $t^* > 0$ is small enough such that

$$\max\{C, \mathrm{e}/C\} \leq \psi(t)/t \quad \text{for each} \quad t \in (0, t^*), \tag{4.136}$$

(something we may always arrange, thanks to the property assumed in (4.123)) then whenever $0 \leq t_0 < t_1 < \min\{t_*, t^*, t_e\}$ we may write (using (4.126), (4.118), (4.125), (4.135), and (4.136))

$$\begin{aligned}\widehat{\psi}(t_0) &= t_0\ln(\psi(t_0)/t_0) \leq t_0\ln(C\psi(t_1)/t_0) \leq t_1\ln(C\psi(t_1)/t_1) \\ &= t_1\ln(C) + t_1\ln(\psi(t_1)/t_1) \leq 2t_1\ln(\psi(t_1)/t_1) = 2\widehat{\psi}(t_1),\end{aligned} \tag{4.137}$$

ultimately proving that $\widehat{\psi}$ is, as claimed, quasi-increasing near the origin. In fact, the same type of argument as in (4.137) (with $C := 1$) shows that

$$\begin{array}{l}\text{if the original function } \psi \text{ is genuinely non-decreasing, then the} \\ \text{function } \widehat{\psi} \text{ associated with } \psi \text{ as in (4.126) is strictly increasing} \\ \text{on } (0, t_e) \text{ and constant thereafter.}\end{array} \tag{4.138}$$

Next, (4.125) and (4.126) readily imply (bearing in mind that the function $\psi$ is continuous) that $\inf\{\widehat{\psi}(t) : t \geq \widetilde{t}\} > 0$ for each $\widetilde{t} > 0$. The fact that $\widehat{\psi}$ is continuous at the origin is seen from (4.126) and (4.133). Furthermore, (4.123) implies

$$\lim_{t\to 0^+} \widehat{\psi}(t)/t = \lim_{t\to 0^+} \ln(\psi(t)/t) = \infty. \tag{4.139}$$

Let us also note here that (4.139), (4.126), the fact that $\ln(\ln x) \leq \ln x$ for each $x > 1$, and (4.133) allow us to write

$$
\begin{aligned}
0 \le \liminf_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) &\le \limsup_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) = \limsup_{t\to 0^+} t\ln\big(\ln(\psi(t)/t)\big) \\
&\le \limsup_{t\to 0^+} t\ln(\psi(t)/t) = 0,
\end{aligned} \tag{4.140}
$$

ultimately proving that, as claimed, $\lim_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) = 0$. Finally, (4.126) and (4.123) give

$$
\begin{aligned}
\psi(t)\cdot\widehat{\psi}(t)^{-1}\cdot \mathrm{e}^{-\widehat{\psi}(t)/t} &= \psi(t)\cdot\frac{1}{t\ln(\psi(t)/t)}\cdot \mathrm{e}^{-\ln(\psi(t)/t)} \\
&= \frac{1}{\ln(\psi(t)/t)} = o(1) \quad \text{as} \quad t\to 0^+.
\end{aligned} \tag{4.141}
$$

This completes the proof of (4.134).

Assuming for the time being that (4.119) implies (4.122), let us explain how this inductive step may be used to establish (4.106). From Proposition 3.4 (which guarantees that the maximal operator $T_*$ is bounded in $L^p(\partial\Omega, w)$ for each exponent $p\in(1,\infty)$ and each weight $w\in A_p(\partial\Omega,\sigma)$ with norm controlled solely in terms of $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$) we conclude that (4.119) holds for the constant function

$$
\psi_0(t) := 1 \quad \text{for each} \quad t\in[0,\infty). \tag{4.142}
$$

Incidentally, we may recast this as $\psi_0(t) = t^{\langle 0\rangle}$ for each $t\in[0,\infty)$ (cf. (4.92)). This choice of function satisfies (4.118) (in fact, $\psi_0$ is non-decreasing), as well as (4.123) and (4.133). Granted these, we may then conclude from (4.134) and the working hypothesis, according to which (4.119) implies (4.122), that (4.122) holds with

$$
\psi_1 := \widehat{\psi_0} \tag{4.143}
$$

playing the role of the function $\phi$. This selection of the function $\phi$ is actually optimal, since $\widehat{\psi_0}$ enjoys the minimality property described in (4.132). Specifically, given any $\phi : [0,\infty) \to [0,\infty)$ satisfying the last property in (4.121) with $\psi := \psi_0$ it follows that $\phi(t)$ dominates, up to a multiplicative constant, the quantity $\psi_1(t) = \widehat{\psi_0}(t)$ for all $t\ge 0$ sufficiently close to 0.

In addition, from (4.134) and (4.138) we see that

the function $\psi_1$ is continuous, strictly increasing near the origin, globally nondecreasing, and satisfies $\lim_{t\to 0^+}\psi_1(t)/t = \infty$ as well as $\lim_{t\to 0^+} t\ln(\psi_1(t)/t) = 0$. (4.144)

In fact, according to (4.124)–(4.126), we have

$$\begin{aligned}&\psi_1:[0,\infty)\longrightarrow[0,\infty)\text{ is given for each }t\geq 0\text{ by}\\ &\psi_1(0):=0,\quad \psi_1(t):=t\ln(1/t)\ \text{ if }\ t\in(0,1/\mathrm{e}),\\ &\quad\text{and }\ \psi_1(t):=1/\mathrm{e}\ \text{ for all }\ t\in[1/\mathrm{e},\infty),\end{aligned}\tag{4.145}$$

hence (cf. (4.93))

$$\psi_1(t)=t^{\langle 1\rangle}\ \text{ for each }\ t\in[0,\infty).\tag{4.146}$$

In view of the aforementioned properties of $\psi_1$ and the fact that (4.122) holds with $\psi_1$ playing the role of the function $\phi$, the present working hypothesis (according to which (4.119) implies (4.122)) shows that (4.122) also holds with $\psi_2:=\widehat{\psi_1}$ playing the role of the function $\phi$, and that $\psi_2$ satisfies similar properties to those listed in (4.144). Actually, (4.145) and (4.124)–(4.126) yield a concrete description of $\psi_2$, namely:

$$\begin{aligned}&\psi_2:[0,\infty)\longrightarrow[0,\infty)\text{ is given for each }t\geq 0\text{ by}\\ &\psi_2(0):=0,\quad \psi_2(t):=t\ln\big(\ln(1/t)\big)\ \text{ if }\ t\in(0,1/\mathrm{e}^{\mathrm{e}}),\\ &\quad\text{and }\ \psi_2(t):=1/\mathrm{e}^{\mathrm{e}}\ \text{ for all }\ t\in[1/\mathrm{e}^{\mathrm{e}},\infty).\end{aligned}\tag{4.147}$$

Equivalently (cf. (4.93)),

$$\psi_2(t)=t^{\langle 2\rangle}\ \text{ for each }\ t\in[0,\infty).\tag{4.148}$$

Iterating this scheme $m$ times then proves (see (4.95)) that (4.122) holds with $\phi$ replaced by the function described (using notation introduced in (4.93)–(4.94)) as

$$\begin{aligned}&\psi_m:[0,\infty)\longrightarrow[0,\infty)\text{ given by}\\ &\psi_m(t)=t^{\langle m\rangle}\text{ for each }t\in[0,\infty).\end{aligned}\tag{4.149}$$

This induction establishes (4.106), modulo the proof of the fact that (4.119) implies (4.122) (which we shall deal with momentarily). The above line of reasoning explains the format of the conclusion in (4.106), while it also makes it clear that (4.106) is the best outcome one can produce working under the assumption that (4.119) implies (4.122).

On to the proof of the fact that (4.119) implies (4.122). Our working hypothesis is that there exists some function $\psi:[0,\infty)\to[0,\infty)$ which is quasi-increasing near the origin (in the sense of (4.118)) such that for each exponent $p\in(1,\infty)$ and each weight $w\in A_p(\partial\Omega,\sigma)$ the estimate recorded in (4.119) holds for some constant $C\in(0,\infty)$ depending only on $n$, $p$, $[w]_{A_p}$, the UR constants of $\partial\Omega$, and $\psi$. Having fixed a function $\phi$ as in (4.120)–(4.121), the goal is to prove (4.122).

To get started, it is visible from (4.104)–(4.105) that the maximal operator $T_*$ depends in a homogeneous fashion on the kernel function $k$. As such, by working with $k/K$ (in the case when $k$ is not identically zero) for the choice $K := \sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|$, matters are reduced to proving that whenever (4.118) holds for any $p \in (1, \infty)$ and, in addition, we have

$$\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \leq 1 \tag{4.150}$$

then for each integrability exponent $p \in (1, \infty)$ and each Muckenhoupt weight $w$ in $A_p(\partial\Omega, \sigma)$ it is possible to find a constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the UR constants of $\partial\Omega$ such that

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\phi\big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\big). \tag{4.151}$$

Henceforth, assume (4.150).

To proceed, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Pick a parameter $\delta_* \in (0, 1)$. Along the way, we will impose further restrictions on the size of $\delta_*$, depending only on $n$, $p$, $[w]_{A_p}$, the UR constants of $\partial\Omega$, and the functions $\psi$, $\phi$. In the case when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \delta_*$, the estimate claimed in (4.151) follows directly (simply by adjusting constants) from the first line in (4.121) and Proposition 3.4, which ensures that the maximal operator $T_*$ is bounded in $L^p(\partial\Omega, w)$. Therefore, there remains to consider the case when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_*$. Assume this is the case and pick some $\delta$ such that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta < \delta_*. \tag{4.152}$$

Recall that our long-term goal is to prove (4.151) for some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the UR constants of $\partial\Omega$. Since we may assume that $\delta_*$ is sufficiently small relative to the Ahlfors regularity constant of $\partial\Omega$ and the dimension $n$, we may invoke Theorem 2.3 which guarantees that

> the set $\partial\Omega$ is unbounded and $\Omega$ satisfies a two-sided local John condition with constants which depend only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension $n$; in particular, the UR constants of $\partial\Omega$ are also controlled solely in terms of the dimension $n$ and the Ahlfors regularity constant of $\partial\Omega$. (4.153)

In addition, Proposition 2.15 ensures that there exists some constant $C_\Omega \in (0, \infty)$, which depends only on $n$ and the Ahlfors regularity constant of $\partial\Omega$, such that for each dilation factor $\mu \in [1, \infty)$ we have

$$\sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(x,\mu R)} R^{-1}\big|\langle x-y, \nu_{\Delta(z,R)}\rangle\big| \leq C_\Omega \cdot \mu(1+\log_2 \mu)\delta. \tag{4.154}$$

For reasons which are going to be clear momentarily, in addition to the truncated operators $T_\varepsilon$ from (4.105) we shall need a version in which the truncation is performed using a smooth cutoff function (rather than a characteristic function). Specifically, fix a function $\zeta \in \mathscr{C}^\infty(\mathbb{R})$ satisfying $0 \leq \zeta \leq 1$ on $\mathbb{R}$ and with the property that $\zeta \equiv 0$ in $(-\infty, 1]$ and $\zeta \equiv 1$ in $[2, \infty)$. For each $\varepsilon > 0$ then define the action of the smoothly truncated operator $T_{(\varepsilon)}$ on each $f \in L^p(\partial\Omega, w)$ by setting

$$T_{(\varepsilon)} f(x) := \int_{\partial\Omega} \zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\langle x-y, \nu(y)\rangle\, k(x-y) f(y)\, \mathrm{d}\sigma(y) \tag{4.155}$$

for each $x \in \partial\Omega$. Let us also define a smoothly truncated version of the maximal operator (4.104) by setting, for each $f \in L^p(\partial\Omega, w)$,

$$T_{(*)} f(x) := \sup_{\varepsilon>0} \big|T_{(\varepsilon)} f(x)\big| \quad \text{at every point } \; x \in \partial\Omega. \tag{4.156}$$

For the time being, the goal is to compare roughly truncated singular integral operators with their smoothly truncated counterparts. To accomplish this task, for each fixed $\gamma \geq 0$ bring in a brand of Hardy–Littlewood maximal operator which associates to each $\sigma$-measurable function $f$ on $\partial\Omega$ the function $\mathcal{M}_\gamma f$ defined as

$$\mathcal{M}_\gamma f(x) := \sup_{\Delta \ni x} \Big(\fint_\Delta |f|^{1+\gamma}\, \mathrm{d}\sigma\Big)^{1/(1+\gamma)} \quad \text{for each } \; x \in \partial\Omega, \tag{4.157}$$

where the supremum is taken over all surface balls $\Delta \subseteq \partial\Omega$ containing the point $x$. On to the task at hand, having fixed some $\varepsilon > 0$, for each $f \in L^p(\partial\Omega, w)$ and each $x \in \partial\Omega$ we may estimate

$$\begin{aligned}
\big|(T_\varepsilon f - T_{(\varepsilon)} f)(x)\big| &\leq \int_{\Delta(x,2\varepsilon)\setminus\overline{\Delta(x,\varepsilon)}} \big|\langle x-y, \nu(y)\rangle\big|\, |k(x-y)||f(y)|\, \mathrm{d}\sigma(y) \\
&\leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} \big|\langle x-y, \nu(y)\rangle\big|\, |f(y)|\, \mathrm{d}\sigma(y) \\
&\leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} \big|\big\langle x-y, \nu(y) - \nu_{\Delta(x,2\varepsilon)}\big\rangle\big| |f(y)|\, \mathrm{d}\sigma(y) \\
&\quad + C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} \big|\big\langle x-y, \nu_{\Delta(x,2\varepsilon)}\big\rangle\big| |f(y)|\, \mathrm{d}\sigma(y) \\
&\leq C \Big(\fint_{\Delta(x,2\varepsilon)} \big|\nu(y) - \nu_{\Delta(x,2\varepsilon)}\big|^{\frac{\gamma+1}{\gamma}}\, \mathrm{d}\sigma(y)\Big)^{\frac{\gamma}{1+\gamma}} \Big(\fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\, \mathrm{d}\sigma(y)\Big)^{\frac{1}{1+\gamma}}
\end{aligned}$$

$$+ C\Big(\sup_{y\in\Delta(x,2\varepsilon)} \varepsilon^{-1}\big|\langle x-y, \nu_{\Delta(x,2\varepsilon)}\rangle\big|\Big)\Big(\fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\,d\sigma(y)\Big)^{\frac{1}{1+\gamma}}$$

$$\leq C\delta\cdot \inf_{\Delta(x,2\varepsilon)} \mathcal{M}_\gamma f, \tag{4.158}$$

using Hölder's inequality, (2.102), (4.152), (4.154), and (4.157). Ultimately, the estimate recorded in (4.158) implies that there exists some $C \in (0,\infty)$, which depends only on $\gamma$, $n$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that for each function $f \in L^p(\partial\Omega, w)$ we have

$$\big|T_* f(x) - T_{(*)} f(x)\big| \leq C\delta\cdot\mathcal{M}_\gamma f(x) \ \text{ for each } \ x\in\partial\Omega. \tag{4.159}$$

Henceforth we agree to fix $\gamma \in (0, p-1)$, which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that $w \in A_{p/(1+\gamma)}(\partial\Omega,\sigma)$, with $[w]_{A_{p/(1+\gamma)}}$ controlled in terms of $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. From (2.533) we know that such a choice is possible.

To proceed, consider a dyadic grid $\mathbb{D}(\partial\Omega)$ on the Ahlfors regular set $\partial\Omega$ (as in Proposition 2.19, presently used with $\Sigma := \partial\Omega$). Also, choose a compactly supported function $f \in L^p(\partial\Omega, w)$. Note that for each $\varepsilon > 0$ the function $T_{(\varepsilon)} f$ is continuous on $\partial\Omega$, by Lebesgue's Dominated Convergence Theorem (whose applicability in the present setting is ensured by Lemma 2.15). Since the pointwise supremum of any collection of continuous functions is lower-semicontinuous, we conclude that for each $\lambda > 0$ the set

$$\big\{x\in\partial\Omega:\ T_{(*)} f(x) > \lambda\big\} \ \text{ is relatively open in } \ \partial\Omega. \tag{4.160}$$

Next, fix a reference point $x_0 \in \partial\Omega$ and abbreviate $\Delta_0 := \Delta(x_0, 2^{-m})$ for some $m \in \mathbb{Z}$ chosen so that

$$\operatorname{supp} f \subseteq 2\Delta_0. \tag{4.161}$$

We emphasize that all subsequent constants are going to be independent of the function $f$, the point $x_0$, and the integer $m$. Upon recalling (2.500), define

$$\mathcal{Q}_0 := \big\{Q\in\mathbb{D}_m(\partial\Omega):\ Q\cap 2\Delta_0 \neq \varnothing\big\} \tag{4.162}$$

then introduce

$$I_0 := \bigcup_{Q\in\mathcal{Q}_0} Q. \tag{4.163}$$

By design, $I_0$ is a relatively open subset of $\partial\Omega$. Recall the parameter $a_1 > 0$ appearing in (2.502) of Proposition 2.19. We claim that

$$I_0 \subseteq a\Delta_0 \quad \text{where} \quad a := 2(1 + a_1) > 2. \tag{4.164}$$

Indeed, if $x \in I_0$ then $x \in Q$ for some $Q \in \mathcal{Q}_0$. In particular, $Q \cap 2\Delta_0 \neq \varnothing$ so we may pick some $y \in Q \cap 2\Delta_0$. Then $x, y \in Q \subseteq \Delta(x_Q, a_1 2^{-m})$ by (2.502), where $x_Q$ denotes the "center" of the dyadic cube $Q$. Consequently, $|x - y| < a_1 2^{-m+1}$ which permits us to estimate $|x - x_0| \leq |x - y| + |y - x_0| < a_1 2^{-m+1} + 2^{-m+1} = a \cdot 2^{-m}$. Thus $x \in B(x_0, a \cdot 2^{-m}) \cap \partial\Omega = a\Delta_0$, proving the inclusion in (4.164).

We also claim that

$$\text{there exists a } \sigma\text{-measurable set } N \subseteq \partial\Omega \text{ with the property that } \sigma(N) = 0 \text{ and } 2\Delta_0 \setminus N \subseteq I_0. \tag{4.165}$$

To justify this, recall from (2.504) that

$$N := \partial\Omega \setminus \Big(\bigcup\nolimits_{Q \in \mathbb{D}_m(\partial\Omega)} Q\Big) \text{ is a } \sigma\text{-measurable set satisfying } \sigma(N) = 0 \text{ and } \partial\Omega \setminus N = \bigcup\nolimits_{Q \in \mathbb{D}_m(\partial\Omega)} Q. \tag{4.166}$$

Intersecting both sides of the last equality in (4.166) with $2\Delta_0$ while bearing in mind (4.162)–(4.163) then yields

$$2\Delta_0 \setminus N = \bigcup_{Q \in \mathbb{D}_m(\partial\Omega)} \big(Q \cap 2\Delta_0\big) = \bigcup_{Q \in \mathcal{Q}_0} \big(Q \cap 2\Delta_0\big) \subseteq \bigcup_{Q \in \mathcal{Q}_0} Q = I_0, \tag{4.167}$$

ultimately proving (4.165).

Let us now define

$$A := \theta \cdot \phi(\delta)^{-1} \in (0, \infty) \quad \text{for some fixed small} \quad \theta \in (0, 1). \tag{4.168}$$

At various stages in the proof we shall make specific demands on the size of $\theta$, though always in relation to the background geometric parameters, the weight, and the function $\phi$, namely $n$, $p$, $[w]_{A_p}$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$ (the final demand of this nature is made in connection with (4.240)). We find it convenient to abbreviate

$$\eta(\theta, \delta) \tag{4.169}$$
$$:= C\Big\{\theta^{1+\gamma} + \theta^{1+\gamma/2}\Big(\frac{\psi(\delta)}{\phi(\delta)} \cdot \mathrm{e}^{-\phi(\delta)/\delta}\Big)^{1+\gamma/2} + \mathrm{e}^{-(3+\gamma+2/\gamma)\phi(\delta)/\delta}\Big\},$$

where $C \in (0, \infty)$ is a constant which depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$. We agree to retain the notation $\eta(\theta, \delta)$ even when $C \in (0, \infty)$ may occasionally change in size (while retaining the same nature, however).

Since $w \in A_p(\partial\Omega, \sigma) \subseteq A_\infty(\partial\Omega, \sigma)$, there exists some small number $\tau > 0$ such that (2.537) holds. Our long-term goal is to obtain the following type of good-

$\lambda$ inequality: there exists $C \in (0, \infty)$ as above (entering the makeup of the entity $\eta(\theta, \delta)$ defined in (4.169)) such that for each $\lambda > 0$ we have

$$w\Big(\big\{x \in I_0 : T_* f(x) > 4\lambda \text{ and } \mathcal{M}_\gamma f(x) \le A\lambda\big\}\Big) \\ \le \eta(\theta, \delta)^\tau \cdot w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}\Big). \tag{4.170}$$

Here and elsewhere, we employ our earlier convention of using the same symbol $w$ for the measure associated with the given weight $w$ as in (2.509). The reader is also alerted to the fact that the maximal operator appearing in the right-hand side of (4.170) employs smooth truncations (as in (4.156)).

To prove (4.170), fix an arbitrary $\lambda > 0$ and abbreviate

$$\mathcal{F}_\lambda := \big\{x \in I_0 : T_* f(x) > 4\lambda \text{ and } \mathcal{M}_\gamma f(x) \le A\lambda\big\}. \tag{4.171}$$

Proposition 3.4 implies that $T_* f$ is a $\sigma$-measurable function. Since so is $\mathcal{M}_\gamma f$ (cf. [7] or [111, §7.6] for a proof), it follows that $\mathcal{F}_\lambda$ is necessarily a $\sigma$-measurable set. From (4.160) and the fact that $I_0$ is a relatively open subset of $\partial\Omega$ we also conclude that $\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}$ is a relatively open subset of $\partial\Omega$ (hence, $\sigma$-measurable). As such, the good-$\lambda$ inequality is meaningfully formulated in (4.170).

Clearly, it is enough to consider the case $\mathcal{F}_\lambda \ne \varnothing$ since otherwise (4.170) is trivially satisfied by any choice of $C \in (0, \infty)$. For the remainder of the proof, assume this is the case. Since $\mathcal{F}_\lambda \subseteq I_0$ and $I_0 \subseteq a\Delta_0$, we conclude that

$$\mathcal{F}_\lambda \subseteq I_0 \subseteq a\Delta_0 \text{ and } \sup_{\mathcal{F}_\lambda} \mathcal{M}_\gamma f \le A\lambda. \tag{4.172}$$

To proceed, decompose $I_0 = \mathcal{P}_\lambda \cup \mathcal{S}_\lambda$ (disjoint union) where, with the smoothly truncated maximal operator $T_{(*)}$ as in (4.156),

$$\mathcal{P}_\lambda := \big\{x \in I_0 : T_{(*)} f(x) \le \lambda\big\} \text{ and } \mathcal{S}_\lambda := \big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}. \tag{4.173}$$

As a consequence of (4.160) and the fact that $I_0$ is a relatively open subset of $\partial\Omega$, the set $\mathcal{S}_\lambda$ is itself a relatively open subset of $\partial\Omega$. Moreover, using (4.159) and (4.172), for each point $x \in \mathcal{F}_\lambda$ we may estimate

$$4\lambda < T_* f(x) \le T_{(*)} f(x) + C\delta \cdot \mathcal{M}_\gamma f(x) \le T_{(*)} f(x) + C\delta A\lambda \\ = T_{(*)} f(x) + C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda < T_{(*)} f(x) + 3\lambda, \tag{4.174}$$

by our choice of $A$ in (4.168), the fact that $\theta \in (0, 1)$, and taking $\delta_*$ small enough to begin with (while keeping in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)). From

(4.174) we see that $T_{(*)}f(x) > \lambda$, hence $x \in \mathcal{S}_\lambda$ which ultimately goes to show that $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$. Thus,

$$\begin{array}{l}\mathcal{S}_\lambda \text{ is a nonempty relatively open subset of}\\ \partial\Omega, \text{ with the property that } \mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda \subseteq I_0.\end{array} \tag{4.175}$$

We first treat the case in which there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \varnothing$ or, equivalently,

$$Q_0 \subseteq \mathcal{S}_\lambda. \tag{4.176}$$

Apply Theorem 2.6 to the (center and radius of the) surface ball $a\Delta_0$. This guarantees the existence of three constants $C_0, C_1, C_2 \in (0, \infty)$ of a purely geometric nature (i.e., depending only on $n$ and the Ahlfors regularity constant of $\partial\Omega$) with the following significance. Take

$$\widetilde{\phi} := \frac{(1+\gamma)(1+\gamma/2)}{C_2(\gamma/2)}\phi = \frac{3+\gamma+2/\gamma}{C_2}\phi \tag{4.177}$$

to play the role of the function in (2.360)–(2.361)). Assuming $\delta_* \in (0, 1)$ to be sufficiently small to begin with, we then have the decomposition

$$a\Delta_0 \subseteq G \cup E, \tag{4.178}$$

where $G$ and $E$ are disjoint $\sigma$-measurable subsets of $\partial\Omega$ satisfying properties implied by (2.363)–(2.368) (relative to $x_0$ and the scale $r := a2^{-m}$) in the present setting. Also, $G$ is contained in the graph $\mathcal{G} = \big\{x_0 + x + h(x)\vec{n} \,:\, x \in H\big\}$ of a Lipschitz function $h : H \to \mathbb{R}$ (where $\vec{n} \in S^{n-1}$ is a unit vector and $H = \langle\vec{n}\rangle^\perp$ is the hyperplane in $\mathbb{R}^n$ orthogonal to $\vec{n}$) such that

$$\sup_{\substack{x,y\in H\\ x\neq y}} \frac{|h(x)-h(y)|}{|x-y|} \leq C_0\widetilde{\phi}(\delta), \tag{4.179}$$

whereas $E$ satisfies

$$\sigma(E) \leq C_1 \mathrm{e}^{-C_2\widetilde{\phi}(\delta)/\delta}\sigma(a\Delta_0). \tag{4.180}$$

Since $\operatorname{supp} f \subseteq 2\Delta_0$ and $a > 2$ it follows that $f = f\mathbf{1}_{a\Delta_0}$. Based on this observation and the fact that $I_0 \subseteq a\Delta_0$ (cf. (4.172)), we may then estimate

$$\sigma(\mathcal{F}_\lambda) \leq \sigma\Big(\big\{x \in a\Delta_0 : T_*\big(f\mathbf{1}_{a\Delta_0}\big)(x) > 4\lambda\big\}\Big). \tag{4.181}$$

By further decomposing $f\mathbf{1}_{a\Delta_0} = f\mathbf{1}_G + f\mathbf{1}_E$ (cf. (4.178) and the fact that we have $f = f\mathbf{1}_{a\Delta_0}$), then using the sub-linearity of $T_*$, as well as (4.178), (4.180), and

(4.177) we obtain

$$\begin{aligned}
\sigma\Big(\big\{x \in a\Delta_0 : T_*\big(f\mathbf{1}_{a\Delta_0}\big)(x) > 4\lambda\big\}\Big) \\
&\leq \sigma\Big(\big\{x \in G:\ T_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) \\
&\quad + \sigma\Big(\big\{x \in G:\ T_*\big(f\mathbf{1}_E\big)(x) > 2\lambda\big\}\Big) \\
&\quad + C_1 \mathrm{e}^{-(3+\gamma+2/\gamma)\phi(\delta)/\delta}\sigma(a\Delta_0).
\end{aligned} \tag{4.182}$$

To bound the first term in the right-hand side of (4.182), the idea is to use the fact that $G$ is contained in the graph $\mathcal{G}$ of the function $h$, then employ Lemma 4.2 while taking advantage of (4.179). Turning to specifics, denote by $\widetilde{\sigma}$ the surface measure on $\mathcal{G}$, and by $\widetilde{T}_*$ the maximal operator associated with $\mathcal{G}$ as in (4.78) (much as $T_*$ in (4.104)–(4.105) is associated with $\partial\Omega$). That is, for each $\widetilde{f} \in L^p(\mathcal{G}, \widetilde{\sigma})$ set

$$\widetilde{T}_*\widetilde{f}(x) := \sup_{\varepsilon>0}\big|\widetilde{T}_\varepsilon\widetilde{f}(x)\big|, \quad \forall\, x \in \mathcal{G}, \tag{4.183}$$

where for each $\varepsilon > 0$ we have set

$$\widetilde{T}_\varepsilon\widetilde{f}(x) := \int\limits_{\substack{y\in\mathcal{G}\\ |x-y|>\varepsilon}} \langle x-y, \widetilde{\nu}(y)\rangle k(x-y)\widetilde{f}(y)\,\mathrm{d}\widetilde{\sigma}(y), \quad \forall\, x \in \mathcal{G}, \tag{4.184}$$

with $\widetilde{\nu}$ denoting the unit normal vector to the Lipschitz graph $\mathcal{G}$, pointing toward the upper-graph of the function $h$. From (2.377) we know that

$$\nu(x) = \widetilde{\nu}(x) \ \text{ at } \sigma\text{-a.e. point } \ x \in G. \tag{4.185}$$

We continue by fixing a point $\widetilde{x} \in \mathcal{F}_\lambda$ (which, according to (4.172), also places $\widetilde{x}$ into $a\Delta_0$). As regards the first term in the right-hand side of (4.182), we may rely on (4.185), the fact that the measures $\sigma$ and $\widetilde{\sigma}$ agree on $\partial\Omega \cap \mathcal{G}$ (as they are both manifestations of $\mathcal{H}^{n-1}$), (4.183)–(4.184), (4.104)–(4.105), Chebyshev's inequality, Lemma 4.2, (4.177), (4.161) (and the fact that $a > 2$), (4.178), (4.157), (4.172), and (4.168) to estimate

$$\begin{aligned}
\sigma\Big(\big\{x \in G : T_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) &= \widetilde{\sigma}\Big(\big\{x \in G:\ \widetilde{T}_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) \\
&\leq \widetilde{\sigma}\Big(\big\{x \in \mathcal{G}:\ \widetilde{T}_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) \\
&\leq \frac{1}{(2\lambda)^{1+\gamma}}\int_{\mathcal{G}} |\widetilde{T}_*(f\mathbf{1}_G)|^{1+\gamma}\,\mathrm{d}\widetilde{\sigma} \leq C\frac{\widetilde{\phi}(\delta)^{1+\gamma}}{\lambda^{1+\gamma}}\int_{\mathcal{G}} |f\mathbf{1}_G|^{1+\gamma}\,\mathrm{d}\widetilde{\sigma}
\end{aligned}$$

$$= C\frac{\phi(\delta)^{1+\gamma}}{\lambda^{1+\gamma}}\int_G |f|^{1+\gamma}\,\mathrm{d}\sigma \le C\phi(\delta)^{1+\gamma}\,\frac{\sigma(a\Delta_0)}{\lambda^{1+\gamma}}\fint_{a\Delta_0}|f|^{1+\gamma}\,\mathrm{d}\sigma$$

$$\le C\phi(\delta)^{1+\gamma}\,\frac{\sigma(a\Delta_0)}{\lambda^{1+\gamma}}\Big[\mathcal{M}_\gamma f(\widetilde{x})\Big]^{1+\gamma} \le C\,(A\phi(\delta))^{1+\gamma}\,\sigma(a\Delta_0)$$

$$= C\,\theta^{1+\gamma}\,\sigma(a\Delta_0), \tag{4.186}$$

for some constant $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$.

As regards the second term in the right-hand side of (4.182), once again fix a point $\widetilde{x} \in \mathcal{F}_\lambda$ (which then also belongs to $a\Delta_0$). Also, assume that $\delta_* \in (0, t_*)$. We may then use Chebyshev's inequality, the hypothesis made in (4.119) (used with $p := 1+\gamma/2$ and $w := 1$), the assumption (4.150), (4.118), (4.152), the fact that $0 < \delta_* < t_*$, Hölder's inequality, (4.180), (4.157), (4.177), (4.172), and (4.168) to obtain[2]

$$\sigma\Big(\big\{x \in G : T_*\big(f\mathbf{1}_E\big)(x) > 2\lambda\big\}\Big)$$

$$\le \sigma\Big(\big\{x \in \partial\Omega :\ T_*\big(f\mathbf{1}_E\big)(x) > 2\lambda\big\}\Big)$$

$$\le \frac{1}{(2\lambda)^{1+\gamma/2}}\int_{\partial\Omega}\big(T_*\big(f\mathbf{1}_E\big)\big)^{1+\gamma/2}\,\mathrm{d}\sigma$$

$$\le \frac{\big(\,\|T_*\|_{L^{1+\gamma/2}(\partial\Omega,\sigma)\to L^{1+\gamma/2}(\partial\Omega,\sigma)}\,\big)^{1+\gamma/2}}{(2\lambda)^{1+\gamma/2}}\int_{\partial\Omega}\big(|f|\,\mathbf{1}_E\big)^{1+\gamma/2}\,\mathrm{d}\sigma$$

$$\le \frac{(C\psi(\delta))^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\int_{a\Delta_0}|f|^{1+\gamma/2}\,\mathbf{1}_E\,\mathrm{d}\sigma$$

$$\le \frac{(C\psi(\delta))^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\,\sigma(E)^{\frac{\gamma/2}{1+\gamma}}\left(\int_{a\Delta_0}|f|^{1+\gamma}\,\mathrm{d}\sigma\right)^{\frac{1+\gamma/2}{1+\gamma}}$$

$$= \frac{(C\psi(\delta))^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\left(\frac{\sigma(E)}{\sigma(a\Delta_0)}\right)^{\frac{\gamma/2}{1+\gamma}}\left(\fint_{a\Delta_0}|f|^{1+\gamma}\,\mathrm{d}\sigma\right)^{\frac{1+\gamma/2}{1+\gamma}}\sigma(a\Delta_0)$$

[2] It is from the format of (4.187) that the value of having the last property in (4.121) is most apparent. Indeed, since the left-most side of (4.187) is obviously dominated by $\sigma(G) \le \sigma(a\Delta_0)$ (cf. (4.178)), the estimate derived in (4.187) is only useful if $\psi(\delta)\phi(\delta)^{-1}\cdot\exp\Big\{-\frac{\phi(\delta)}{\delta}\Big\}$ stays bounded for $\delta$ close to 0.

$$\leq C\frac{\psi(\delta)^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\exp\Big\{-\frac{C_2(\gamma/2)\widetilde{\phi}(\delta)}{(1+\gamma)\delta}\Big\}\Big[\mathcal{M}_\gamma f(\widetilde{x})\Big]^{1+\gamma/2}\sigma(a\Delta_0)$$

$$\leq C\big(A\psi(\delta)\big)^{1+\gamma/2}\cdot\exp\Big\{-\frac{(1+\gamma/2)\phi(\delta)}{\delta}\Big\}\sigma(a\Delta_0)$$

$$= C\theta^{1+\gamma/2}\Big[\psi(\delta)\phi(\delta)^{-1}\cdot\exp\Big\{-\frac{\phi(\delta)}{\delta}\Big\}\Big]^{1+\gamma/2}\sigma(a\Delta_0), \tag{4.187}$$

where $C \in (0,\infty)$ depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$. Gathering (4.182), (4.186), and (4.187) then yields

$$\sigma\Big(\big\{x\in a\Delta_0 : T_*\big(f\mathbf{1}_{a\Delta_0}\big)(x) > 4\lambda\big\}\Big)$$

$$\leq C\Big\{\theta^{1+\gamma}+\theta^{1+\gamma/2}\Big(\frac{\psi(\delta)}{\phi(\delta)}\cdot e^{-\phi(\delta)/\delta}\Big)^{1+\gamma/2}+e^{-(3+\gamma+2/\gamma)\phi(\delta)/\delta}\Big\}\sigma(a\Delta_0)$$

$$=\eta(\theta,\delta)\sigma(a\Delta_0), \tag{4.188}$$

where $\eta(\theta,\delta) \in (0,\infty)$ is as in (4.169). Finally, from (4.188) and (4.181) we see that

$$\sigma(\mathcal{F}_\lambda) \leq \eta(\theta,\delta)\sigma(a\Delta_0), \tag{4.189}$$

where $\eta(\theta,\delta) \in (0,\infty)$ is as in (4.169).

Moving on, observe that (2.502) implies that there exists a point $x_{Q_0} \in \partial\Omega$ with the property that

$$\Delta(x_{Q_0}, a_0 2^{-m}) \subseteq Q_0 \subseteq \Delta(x_{Q_0}, a_1 2^{-m}). \tag{4.190}$$

From this inclusion and (4.162) we then conclude that there exists some $c \in (0,\infty)$, which only depends on the Ahlfors regularity constant of $\partial\Omega$, with the property that $a\Delta_0 \subseteq c\Delta(x_{Q_0}, a_1 2^{-m})$. As a consequence of this inclusion we may write (for some $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$)

$$w(a\Delta_0) \leq w\big(c\Delta(x_{Q_0}, a_1 2^{-m})\big) \leq C\,w\big(\Delta(x_{Q_0}, a_0 2^{-m})\big) \leq Cw(Q_0), \tag{4.191}$$

where we have also used the fact that $w$ is a doubling measure (cf. (2.535)) and (4.190). With this in hand, we may now estimate

$$w(\mathcal{F}_\lambda) \leq \eta(\theta,\delta)^\tau\cdot w(a\Delta_0) \leq \eta(\theta,\delta)^\tau\cdot w(Q_0)$$

$$\leq \eta(\theta,\delta)^\tau\cdot w(\mathcal{S}_\lambda), \tag{4.192}$$

where the first inequality uses (2.537), the fact that $\mathcal{F}_\lambda \subseteq a\Delta_0$ (cf. (4.172)), and (4.189), the second inequality is based on (4.191), while the last inequality is a consequence of (4.176). Therefore (4.170) holds whenever there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \varnothing$.

To complete the proof of (4.170), it remains to consider the case $\mathcal{P}_\lambda \cap Q \neq \varnothing$ for each $Q \in \mathcal{Q}_0$. In this scenario, consider an arbitrary dyadic cube $Q \in \mathcal{Q}_0$. From (4.163) we know that $Q \subseteq I_0$. Subdivide $Q$ dyadically and stop when $\mathcal{P}_\lambda \cap Q' = \varnothing$. This process produces a family of pairwise disjoint (stopping time) dyadic cubes $\{Q^j\}_{j\in J_Q} \subset \mathbb{D}(\partial\Omega)$ such that $Q^j \cap \mathcal{P}_\lambda = \varnothing$, $Q^j \subseteq Q$ but $Q^j \neq Q$ (since we have $Q^j \cap \mathcal{P}_\lambda = \varnothing$ but $Q \cap \mathcal{P}_\lambda \neq \varnothing$), and $Q' \cap \mathcal{P}_\lambda \neq \varnothing$ for all $Q' \in \mathbb{D}(\partial\Omega)$ such that $Q^j \subsetneq Q' \subseteq Q$. In particular $Q^j \subsetneq Q$ for every $j \in J_Q$ and $\widetilde{Q}^j$, the dyadic parent of $Q^j$, satisfies $\widetilde{Q}^j \subseteq Q$. With the $\sigma$-nullset $N$ as in (2.505), we now claim that

$$\bigcup_{j\in J_Q} Q^j \subseteq \mathcal{S}_\lambda \cap Q \subseteq \Big(\bigcup_{j\in J_Q} Q^j\Big) \cup N. \tag{4.193}$$

To justify the first inclusion above, observe that if $j \in J_Q$ then $Q^j \subseteq \mathcal{S}_\lambda \cap Q$, since $Q^j \subseteq Q \subseteq I_0$ and $Q^j \cap \mathcal{P}_\lambda = \varnothing$ imply that $Q_j \subseteq Q \setminus \mathcal{P}_\lambda = Q \cap \mathcal{S}_\lambda$. This establishes the first inclusion in (4.193). As regards the second inclusion claimed in (4.193), consider an arbitrary point $x \in \big(\mathcal{S}_\lambda \cap Q\big) \setminus N$. Then $T_{(*)} f(x) > \lambda$ which, in view of (4.160), ensures that we may find a surface ball $\Delta_x := \Delta(x, r_x)$ such that $T_{(*)} f(y) > \lambda$ for every $y \in \Delta_x$. Thanks to (2.502) and (2.504) we may then choose a dyadic cube $Q_x \in \mathbb{D}(\partial\Omega)$ such that $x \in Q_x$ and $Q_x \subseteq \Delta_x \cap Q \subseteq I_0$. This forces $Q_x \subseteq \mathcal{S}_\lambda \cap Q$, hence $Q_x \cap \mathcal{P}_\lambda = \varnothing$. By the maximality of the family chosen above, $Q_x \subseteq Q^j$ for some $j \in J_Q$ which goes to show that $x \in Q^j$. Ultimately, this proves the second inclusion in (4.193).

Going further, the idea is to carry out the stopping time argument just described for each dyadic cube $Q \in \mathcal{Q}_0$. For ease of reference, organize the resulting collection of dyadic cubes $\big\{Q^j : Q \in \mathcal{Q}_0 \text{ and } j \in J_Q\big\}$ (which is an at most countable set) as a single-index family $\big\{Q_\ell\big\}_{\ell\in\mathcal{I}}$ of mutually disjoint dyadic cubes; in particular,

$$\bigcup_{Q\in\mathcal{Q}_0} \bigcup_{j\in J_Q} Q^j = \bigcup_{\ell\in\mathcal{I}} Q_\ell, \tag{4.194}$$

with the latter union comprised of pairwise disjoint dyadic cubes in $\partial\Omega$. Note that $\mathcal{S}_\lambda \cap Q$ might be empty for some $Q \in \mathcal{Q}_0$ and in this case $J_Q = \varnothing$ (i.e., the family of cubes $\{Q^j\}_{j\in J_Q}$ is empty, since there are no stopping time dyadic cubes produced in this case). However, (4.163) and (4.175) imply that $\mathcal{S}_\lambda \cap Q$ cannot be empty for every $Q \in \mathcal{Q}_0$ and, as a consequence, $\mathcal{I} \neq \varnothing$. Going further, using (4.163) and the fact that $\mathcal{S}_\lambda \subseteq I_0$ (cf. (4.173)) we may write

$$\bigcup_{Q\in\mathcal{Q}_0} (\mathcal{S}_\lambda \cap Q) = \mathcal{S}_\lambda \tag{4.195}$$

which further entails, on account of (4.194) and (4.193), that

$$\bigcup_{\ell\in\mathcal{I}} Q_\ell \subseteq \mathcal{S}_\lambda \subseteq \Big(\bigcup_{\ell\in\mathcal{I}} Q_\ell\Big) \cup N. \tag{4.196}$$

By construction, for each index $\ell \in \mathcal{I}$ there exists a point $x_\ell^*$ such that

$$x_\ell^* \in \widetilde{Q}_\ell \cap \mathcal{P}_\lambda = \widetilde{Q}_\ell \cap \big(I_0 \setminus \mathcal{S}_\lambda\big), \tag{4.197}$$

where $\widetilde{Q}_\ell$ denotes the dyadic parent of $Q_\ell$ (cf. item *(4)* in Proposition 2.19). For each $\ell \in \mathcal{I}$ we let $\Delta_\ell := \Delta_{Q_\ell}$ and $\widetilde{\Delta}_\ell := \Delta_{\widetilde{Q}_\ell}$ be as in (2.502). Pressing on, split the collection $\{\Delta_\ell\}_{\ell\in\mathcal{I}}$ into two sub-classes. Specifically, bring in

$$\begin{gathered}\mathcal{I}_1 := \big\{\ell \in \mathcal{I} :\ \text{there exists } x_\ell^{**} \in \Delta_\ell \text{ such that } \mathcal{M}_\gamma f(x_\ell^{**}) \leq A\lambda\big\} \\ \text{and } \ \mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1.\end{gathered} \tag{4.198}$$

Hence, by design, $\mathcal{F}_\lambda \cap \Delta_\ell = \varnothing$ for each $\ell \in \mathcal{I}_2$. Recall now from (4.175) that $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$. From this, (4.196), and (2.502) we then obtain (bearing in mind that $\sigma(N) = 0$; cf. (2.505))

$$w(\mathcal{F}_\lambda) = \sum_{\ell\in\mathcal{I}} w(\mathcal{F}_\lambda \cap Q_\ell) \leq \sum_{\ell\in\mathcal{I}_1} w(\mathcal{F}_\lambda \cap \Delta_\ell). \tag{4.199}$$

Let us also consider

$$F_\ell := \big\{x \in \Delta_\ell : T_* f(x) > 4\lambda\big\} \ \text{ for each } \ \ell \in \mathcal{I}_1, \tag{4.200}$$

and observe that this entails

$$\mathcal{F}_\lambda \cap \Delta_\ell \subseteq F_\ell \ \text{ for each } \ \ell \in \mathcal{I}_1. \tag{4.201}$$

Our next goal is to prove that

$$\sigma(F_\ell) \leq \eta(\theta,\delta)\cdot\sigma(\Delta_\ell) \ \text{ for each } \ \ell \in \mathcal{I}_1. \tag{4.202}$$

Granted this, using (2.537) it would follow that

$$w(F_\ell) \leq \eta(\theta,\delta)^\tau \cdot w(\Delta_\ell) \ \text{ for each } \ \ell \in \mathcal{I}_1 \tag{4.203}$$

which, in concert with (4.199), (4.201), (2.502) plus the fact that $w$ is a doubling measure, and (4.196), would then imply

$$w(\mathcal{F}_\lambda) \leq \sum_{\ell\in\mathcal{I}_1} w(\mathcal{F}_\lambda \cap \Delta_\ell) \leq \sum_{\ell\in\mathcal{I}_1} w(F_\ell) \leq \eta(\theta,\delta)^\tau \cdot \sum_{\ell\in\mathcal{I}_1} w(\Delta_\ell)$$

$$\leq \eta(\theta,\delta)^{\tau}\cdot\sum_{\ell\in\mathcal{I}_1} w(Q_\ell) \leq \eta(\theta,\delta)^{\tau}\cdot\sum_{\ell\in\mathcal{I}} w(Q_\ell)$$

$$= \eta(\theta,\delta)^{\tau}\cdot w(\mathcal{S}_\lambda), \tag{4.204}$$

finishing the justification of (4.170).

We now turn to the proof of (4.202). Fix $\ell\in\mathcal{I}_1$ and, in order to lighten notation, in the sequel we agree to suppress the dependence of $\Delta_\ell$, $\widetilde{\Delta}_\ell$, $F_\ell$, $x^*_\ell$, and $x^{**}_\ell$ on the index $\ell$, and simply write $\Delta$, $\widetilde{\Delta}$, $F$, $x^*$, and $x^{**}$, respectively. With this convention in mind, observe first that

$$\Delta\subseteq 2\widetilde{\Delta}. \tag{4.205}$$

To justify this inclusion, recall from (2.502) that we may write $\Delta = B(x_Q, r_Q)\cap\partial\Omega$ and $\widetilde{\Delta} = B(x_{\widetilde{Q}}, r_{\widetilde{Q}})\cap\partial\Omega$; moreover, since $\widetilde{Q}$ is the parent of $Q$, we have $r_{\widetilde{Q}} = 2r_Q$. Then for each $x\in\Delta$ we have

$$|x - x_{\widetilde{Q}}| \leq |x - x_Q| + |x_Q - x_{\widetilde{Q}}| < r_Q + r_{\widetilde{Q}} = (3/2)r_{\widetilde{Q}} < 2r_{\widetilde{Q}} \tag{4.206}$$

which ultimately proves (4.205). Going forward, let us also denote by $\Delta^*$ the surface ball of center $x^*$ and radius $R := \Lambda\cdot r_Q$, for a sufficiently large constant $\Lambda\in(2,\infty)$ (depending only on the implicit constants in the dyadic grid construction, which in turn depend only on the Ahlfors regularity constant of $\partial\Omega$) chosen so that

$$2\widetilde{\Delta}\subseteq\Delta^*. \tag{4.207}$$

We then decompose

$$f = f_1 + f_2 \quad\text{where}\quad f_1 := f\mathbf{1}_{\overline{2\Delta^*}} \quad\text{and}\quad f_2 := f\mathbf{1}_{\partial\Omega\setminus\overline{2\Delta^*}}. \tag{4.208}$$

By virtue of the sub-linearity of $T_*$ and the fact that $\Delta\subseteq\Delta^*\subseteq 4\Delta^*$ (cf. (4.205)–(4.207)) this implies

$$\sigma(F) \leq \sigma\Big(\big\{x\in\Delta:\ T_*f_1(x) > 2\lambda\big\}\Big) + \sigma\Big(\big\{x\in\Delta:\ T_*f_2(x) > 2\lambda\big\}\Big)$$

$$\leq \sigma\Big(\big\{x\in 4\Delta^*:\ T_*f_1(x) > 2\lambda\big\}\Big) + \sigma\Big(\big\{x\in\Delta:\ T_*f_2(x) > 2\lambda\big\}\Big). \tag{4.209}$$

The contribution from $f_1$ in the last line above is handled as in (4.178)–(4.180), (4.182)–(4.188) by performing a decomposition of $4\Delta^*$ as in Theorem 2.6. Indeed, $a\Delta_0$, $\widetilde{x}$, $f$, and $\lambda$ are replaced by $4\Delta^*$, $x^{**}$, $f_1$, and $\frac{1}{2}\lambda$, respectively, and we use the fact that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (4.198)), supp $f_1\subseteq\overline{2\Delta^*}\subseteq 4\Delta^*$ (cf. (4.208)), and $\sigma(4\Delta^*)\leq c\cdot\sigma(\Delta)$ for some $c\in(0,\infty)$ depending only on the Ahlfors regularity constant of $\partial\Omega$ (since $\partial\Omega$ is Ahlfors regular and the surface balls $4\Delta^*$, $\Delta$ have comparable radii) to run the same proof as before. The conclusion is that

$$\sigma\Big(\big\{x \in 4\Delta^* : T_* f_1(x) > 2\lambda\big\}\Big) \le \eta(\theta, \delta) \cdot \sigma(\Delta). \tag{4.210}$$

In view of the conclusion we seek (cf. (4.202)), this suits our purposes.

As for $f_2$, recall that $R$ is the radius of the surface ball $\Delta^*$, and for each $\varepsilon > 0$ set $\varepsilon' := \max\{\varepsilon, 2R\}$. Based on this choice of $\varepsilon'$, the definition of the truncated singular integral operators in (4.105), the truncation in the definition of the function $f_2$, the estimate in (4.158) (presently used with $x^*$ in place of $x$ and $\varepsilon'$ in place of $\varepsilon$), the fact that $x^{**} \in \Delta \subseteq \Delta^* \subseteq \Delta(x^*, 2\varepsilon')$ (cf. (4.198) and (4.205)–(4.207)), the fact that $\mathcal{M}_\gamma f(x^{**}) \le A\lambda$ (cf. (4.198)), the definition of $T_{(*)} f(x^*)$ (cf. (4.156)), the membership of $x^*$ to $\mathcal{P}_\lambda$ (cf. (4.197)), and the first formula in (4.173), we may write

$$\begin{aligned}
\big|T_\varepsilon f_2(x^*)\big| &= \big|T_{\varepsilon'} f(x^*)\big| \le |T_{\varepsilon'} f(x^*) - T_{(\varepsilon')} f(x^*)\big| + \big|T_{(\varepsilon')} f(x^*)\big| \\
&\le C\delta \cdot \mathcal{M}_\gamma f(x^{**}) + T_{(*)} f(x^*) \le C\delta A\lambda + \lambda \\
&= C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda + \lambda \le \tfrac{3}{2}\lambda,
\end{aligned} \tag{4.211}$$

with the last line a consequence of our choice of $A$ in (4.168), the fact that $\theta \in (0, 1)$, and the ability of taking $\delta_* \in (0, 1)$ small enough to begin with (while bearing in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)). With $\varepsilon > 0$ momentarily fixed, consider now an arbitrary point $x \in \Delta$ and bound

$$\big|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)\big| \le \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{4.212}$$

where

$$\begin{aligned}
\mathrm{I} := \int_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x-y|>\varepsilon,\ |x^*-y|>\varepsilon}} &\Big|\langle x-y, \nu(y)\rangle k(x-y) \\
&\quad - \langle x^*-y, \nu(y)\rangle k(x^*-y)\Big| |f(y)|\, \mathrm{d}\sigma(y), \\
\mathrm{II} := \int_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x-y|>\varepsilon,\ |x^*-y|\le\varepsilon}} &|\langle x-y, \nu(y)\rangle||k(x-y)||f(y)|\, \mathrm{d}\sigma(y),
\end{aligned}$$

$$\mathrm{III} := \int\limits_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x^*-y|>\varepsilon,\ |x-y|\leq\varepsilon}} |\langle x^*-y, \nu(y)\rangle||k(x^*-y)||f(y)|\,\mathrm{d}\sigma(y). \tag{4.213}$$

In preparation for estimating the term I, we will first analyze the difference between I and a similar expression in which $\nu(y)$ has been replaced by the integral average $\nu_{\Delta^*} := ⨍_{\Delta^*} \nu\,\mathrm{d}\sigma$. To set the stage, for each fixed $y \in \partial\Omega \setminus \overline{2\Delta^*}$ consider the function

$$F_y(z) := \langle z-y, \nu(y)-\nu_{\Delta^*}\rangle k(z-y) \quad \text{for each} \quad z \in B(x^*, R). \tag{4.214}$$

Then

$$|(\nabla F_y)(z)| \leq \Big(\sum_{|\alpha|\leq 1} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \frac{|\nu(y)-\nu_{\Delta^*}|}{|z-y|^n} \quad \text{for each} \quad z \in B(x^*, R). \tag{4.215}$$

Keeping in mind that $x \in \Delta \subseteq \Delta^* = B(x^*, R) \cap \partial\Omega$ (cf. (4.205)–(4.207)), we have

$$|x - x^*| < R. \tag{4.216}$$

Also, (recall that $[x, x^*]$ denotes the line segment with endpoints $x, x^*$),

$$|x^*-y| \leq 2|\xi - y| \quad \text{for each} \quad y \in \partial\Omega \setminus \overline{2\Delta^*} \quad \text{and each} \quad \xi \in [x, x^*]. \tag{4.217}$$

Hence, by (4.214)–(4.215), the Mean Value Theorem (bearing in mind (4.150)), (4.216)–(4.217), and Hölder's inequality it follows that

$$\begin{aligned}
&\int_{\partial\Omega\setminus\overline{2\Delta^*}} \Big|\langle x-y, \nu(y)-\nu_{\Delta^*}\rangle k(x-y) - \langle x^*-y, \nu(y)-\nu_{\Delta^*}\rangle k(x^*-y)\Big||f(y)|\,\mathrm{d}\sigma(y) \\
&= \int_{\partial\Omega\setminus\overline{2\Delta^*}} \big|F_y(x)-F_y(x^*)\big||f(y)|\,\mathrm{d}\sigma(y) \\
&\leq \int_{\partial\Omega\setminus\overline{2\Delta^*}} |x-x^*| \cdot \sup_{\xi\in[x,x^*]} \big|(\nabla F_y)(\xi)\big||f(y)|\,\mathrm{d}\sigma(y) \\
&\leq C \int_{\partial\Omega\setminus 2\Delta^*} \frac{R}{|x^*-y|^n} \big|\nu(y)-\nu_{\Delta^*}\big||f(y)|\,\mathrm{d}\sigma(y) \\
&\leq C \sum_{j=1}^{\infty} 2^{-j} ⨍_{2^{j+1}\Delta^*\setminus 2^j\Delta^*} \big|\nu(y)-\nu_{\Delta^*}\big||f(y)|\,\mathrm{d}\sigma(y)
\end{aligned}$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} \left( \fint_{2^{j+1}\Delta^*} \left( \left|\nu(y) - \nu_{2^{j+1}\Delta^*}\right| + \left|\nu_{2^{j+1}\Delta^*} - \nu_{\Delta^*}\right| \right)^{\frac{1+\gamma}{\gamma}} \mathrm{d}\sigma(y) \right)^{\frac{\gamma}{1+\gamma}} \times$$

$$\times \left( \fint_{2^{j+1}\Delta^*} |f(y)|^{1+\gamma} \, \mathrm{d}\sigma(y) \right)^{\frac{1}{1+\gamma}}$$

$$\leq C \Big( \sum_{j=1}^{\infty} (j+2)\, 2^{-j} \Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \mathcal{M}_\gamma f(x^{**})$$

$$\leq C A\, \delta\lambda, \tag{4.218}$$

for some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Above, the fifth inequality relies on (2.102) and the fact that

$$\left|\nu_{2^{j+1}\Delta^*} - \nu_{\Delta^*}\right| \leq C\,(j+1)\; \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \quad \text{for each} \quad j \in \mathbb{N} \tag{4.219}$$

for some $C \in (0, \infty)$ depending only on $n$ and the Ahlfors regular constant of $\partial\Omega$, which is a direct consequence of (2.105). The fifth inequality in (4.218) also uses the fact that $x^{**} \in \Delta \subseteq \Delta^* \subseteq 2^{j+1}\Delta^*$ for each integer $j \in \mathbb{N}$. The last inequality in (4.218) is a consequence of the fact that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (4.198)).

On the other hand, from the properties of the kernel $k$ and the Mean Value Theorem we obtain

$$\int_{\partial\Omega \setminus \overline{2\Delta^*}} \Big| \langle x - y, \nu_{\Delta^*}\rangle k(x-y) - \langle x^* - y, \nu_{\Delta^*}\rangle k(x^* - y) \Big| |f(y)| \, \mathrm{d}\sigma(y)$$

$$= \int_{\partial\Omega \setminus \overline{2\Delta^*}} \Big| \big( \langle x - y, \nu_{\Delta^*}\rangle - \langle x^* - y, \nu_{\Delta^*}\rangle \big) k(x^* - y)$$

$$+ \langle x - y, \nu_{\Delta^*}\rangle \big( k(x-y) - k(x^* - y) \big) \Big| |f(y)| \, \mathrm{d}\sigma(y)$$

$$\leq C_n \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j\Delta^*} \Big( \frac{|\langle x - x^*, \nu_{\Delta^*}\rangle|}{|x^* - y|^n} + R \frac{|\langle x - y, \nu_{\Delta^*}\rangle|}{|x^* - y|^{n+1}} \Big) |f(y)| \, \mathrm{d}\sigma(y)$$

$$\leq C_n \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j\Delta^*} \frac{|\langle x - x^*, \nu_{\Delta^*}\rangle|}{|x^* - y|^n} |f(y)| \, \mathrm{d}\sigma(y)$$

$$+ C_n R \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j\Delta^*} \frac{|\langle x - y, \nu_{\Delta^*} - \nu_{2^{j+1}\Delta^*}\rangle|}{|x^* - y|^{n+1}} |f(y)| \, \mathrm{d}\sigma(y)$$

$$+ C_n R \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j \Delta^*} \frac{|\langle x - y, \nu_{2^{j+1}\Delta^*}\rangle|}{|x^* - y|^{n+1}} |f(y)| \, d\sigma(y)$$

$$=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \tag{4.220}$$

To estimate $\mathrm{I}_1$, write

$$\begin{aligned}
\mathrm{I}_1 &\leq C_n R^{-1} |\langle x - x^*, \nu_{\Delta^*}\rangle| \sum_{j=1}^{\infty} 2^{-j} ⨍_{2^{j+1}\Delta^*} |f(y)| \, d\sigma(y) \\
&\leq C\delta \sum_{j=1}^{\infty} 2^{-j} \mathcal{M}_\gamma f(x^{**}) \leq C\delta \mathcal{M}_\gamma f(x^{**}) \\
&\leq C A \, \delta\lambda,
\end{aligned} \tag{4.221}$$

where $C \in (0, \infty)$ depends only on $n$, and the Ahlfors regularity constant of $\partial\Omega$. The second inequality above is a consequence of (4.154) used here with $z := x^*$, $y := x^*$, $\mu := 2$ (a valid choice given that $x \in \Delta(x^*, 2R)$ since, as seen from (4.205)–(4.207), we have $x \in \Delta \subseteq \Delta^* = \Delta(x^*, R)$) and $x^{**} \in \Delta \subseteq \Delta^* \subseteq 2^{j+1}\Delta^*$ for each $j \in \mathbb{N}$. The last inequality (4.221) uses $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (4.198)).

To treat $\mathrm{I}_2$, we write (for some $C \in (0, \infty)$ which depends only on $n$, and the Ahlfors regularity constant of $\partial\Omega$),

$$\begin{aligned}
\mathrm{I}_2 &\leq C R \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j \Delta^*} \frac{|\nu_{\Delta^*} - \nu_{2^{j+1}\Delta^*}|}{|x^* - y|^n} |f(y)| \, d\sigma(y) \\
&\leq C \sum_{j=1}^{\infty} (j+1) \, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, 2^{-j} ⨍_{2^{j+1}\Delta^*} |f(y)| \, d\sigma(y) \\
&\leq C\delta \, \mathcal{M}_\gamma f(x^{**}) \leq C A \, \delta\lambda,
\end{aligned} \tag{4.222}$$

where the first inequality uses the definition of $\mathrm{I}_2$ (given in (4.220)) as well as the estimate $|x - y| \leq (3/2)|x^* - y|$ valid for each $y \in \partial\Omega \setminus 2\Delta^*$, the second inequality takes into account (4.219) and the Ahlfors regularity of $\partial\Omega$, while the remaining inequalities are justified as in (4.221).

As regards $\mathrm{I}_3$, write (again, with $C \in (0, \infty)$ depending only on $n$, and the Ahlfors regularity constant of $\partial\Omega$)

$$\mathrm{I}_3 \leq C \sum_{j=1}^{\infty} 2^{-j} ⨍_{2^{j+1}\Delta^*} \frac{|\langle x - y, \nu_{2^{j+1}\Delta^*}\rangle|}{2^{j+1}R} |f(y)| \, d\sigma(y)$$

$$\leq C\delta \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta^*} |f(y)| \, d\sigma(y) \leq C\delta \mathcal{M}_\gamma f(x^{**})$$

$$\leq CA\,\delta\lambda. \tag{4.223}$$

The second inequality in (4.223) is based on (4.154) used with $z := x^*$ and $R$ replaced by $2^{j+1}R$. The remaining inequalities in (4.223) are then justified much as in (4.221).

At this stage, by combining (4.218) and (4.220)–(4.223) we conclude that there exists some $C \in (0, \infty)$ which depends only on $n$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{I} \leq CA\,\delta\lambda. \tag{4.224}$$

To bound II in (4.213), recall that $x, x^{**} \in \Delta$ and assume $y \in \partial\Omega \setminus \overline{2\Delta^*}$ is such that $|x^* - y| \leq \varepsilon$ and $|x - y| > \varepsilon$. Then, $2R < |x^* - y| \leq \varepsilon$ and since $x, x^{**} \in \Delta \subseteq B(x_Q, r_Q)$ (where $x_Q$ and $r_Q$ are, respectively, the center and radius of the surface ball $\Delta$) and $R = \Lambda \cdot r_Q$ with $\Lambda > 2$, we have $|x - x^{**}| < 2r_Q < R < \varepsilon/2$. Hence, the point $x^{**}$ belongs to the surface ball $\Delta(x, \varepsilon/2)$. Moreover, on account of (4.216) we may write $|x - y| \leq |x - x^*| + |x^* - y| < R + \varepsilon < (3/2)\varepsilon$ which, in particular, guarantees that $y \in \Delta(x, 2\varepsilon)$. Consequently, $\varepsilon < |x - y| < 2\varepsilon$ hence $|k(x - y)| \leq \varepsilon^{-n}$ and (for some $C \in (0, \infty)$ which depends only on depends only on $n$ and the Ahlfors regularity constant of $\partial\Omega$),

$$\mathrm{II} \leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x - y, \nu(y)\rangle| \, |f(y)| \, d\sigma(y)$$

$$\leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x - y, \nu(y) - \nu_{\Delta(x,2\varepsilon)}\rangle| \, |f(y)| \, d\sigma(y)$$

$$+ C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x - y, \nu_{\Delta(x,2\varepsilon)}\rangle| \, |f(y)| \, d\sigma(y)$$

$$=: \mathrm{II}_1 + \mathrm{II}_2. \tag{4.225}$$

Using Hölder's inequality, (2.102), (4.198), and (4.152) we obtain that there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{II}_1 \leq C \left( \fint_{\Delta(x,2\varepsilon)} |\nu(y) - \nu_{\Delta(x,2\varepsilon)}|^{\frac{1+\gamma}{\gamma}} \, d\sigma(y) \right)^{\frac{\gamma}{1+\gamma}} \left( \fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma} \, d\sigma(y) \right)^{\frac{1}{1+\gamma}}$$

$$\leq C \, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, \mathcal{M}_\gamma f(x^{**}) \leq CA\,\delta\lambda, \tag{4.226}$$

since $x^{**}$ is contained in $\Delta(x, \varepsilon/2) \subseteq \Delta(x, 2\varepsilon)$ and $\mathcal{M}_\gamma f(x^{**}) \leq CA\lambda$, as already noted earlier. As for $\mathrm{II}_2$, invoking (4.154), Hölder's inequality, and (4.198), it follows that (with $C \in (0, \infty)$ as above)

$$\begin{aligned}
\mathrm{II}_2 &\leq C\Big(\sup_{y\in\Delta(x,2\varepsilon)} \varepsilon^{-1}|\langle x-y, \nu_{\Delta(x,2\varepsilon)}\rangle|\Big) ⨍_{\Delta(x,2\varepsilon)} |f(y)|\,\mathrm{d}\sigma(y) \\
&\leq C\delta\Big(⨍_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\,\mathrm{d}\sigma(y)\Big)^{\frac{1}{1+\gamma}} \\
&\leq C\delta\cdot\mathcal{M}_\gamma f(x^{**}) \leq CA\,\delta\lambda. \qquad (4.227)
\end{aligned}$$

From (4.225)–(4.227) we see that there exists $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{II} \leq CA\,\delta\lambda. \qquad (4.228)$$

Turning our attention to III, recall that $x, x^{**} \in \Delta$ and suppose $y \in \partial\Omega \setminus \overline{2\Delta^*}$ is such that $|x^* - y| > \varepsilon$ and $|x - y| \leq \varepsilon$. Then $|x^* - y| > 2R > R + |x - x^*|$ by (4.216) which further entails $\varepsilon \geq |x - y| \geq |x^* - y| - |x - x^*| > R$. In particular, $R < \varepsilon$. If we now abbreviate $\widetilde{R} := R + \varepsilon$ then, on the one hand, we may write the estimate $|x^* - y| \leq |x^* - x| + |x - y| < R + \varepsilon = \widetilde{R}$, while on the other hand having $|x^* - y| > \varepsilon$ and $|x^* - y| > 2R$ implies $|x^* - y| > R + (\varepsilon/2) > \frac{1}{2}\widetilde{R}$. As such, $|k(x^* - y)| \leq \widetilde{R}^{-n}$ and

$$\mathrm{III} \leq C_n \widetilde{R}^{-1} ⨍_{\Delta(x^*,\widetilde{R})} |\langle x^* - y, \nu(y)\rangle|\,|f(y)|\,\mathrm{d}\sigma(y). \qquad (4.229)$$

Granted this, the same type of argument which, starting with the first line in (4.225) has produced (4.228) (reasoning with $\widetilde{R}/2$ replacing $\varepsilon$ and with $x^*$ replacing $x$) will now yield (for some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$)

$$\mathrm{III} \leq CA\,\delta\lambda, \qquad (4.230)$$

as soon as we show that $x^{**} \in \Delta(x^*, \widetilde{R})$. To justify this membership, start by recalling that $|x - x^{**}| < 2r_Q < R$ and then use (4.216), the triangle inequality, and the fact that $R < \varepsilon$ to estimate $|x^* - x^{**}| \leq |x - x^*| + |x - x^{**}| < 2R < \widetilde{R}$. The proof of (4.230) is therefore complete.

Let us summarize our progress. From (4.212), (4.224), (4.228), (4.230), and our choice of $A$ in (4.168) we conclude that there exists some $C \in (0, \infty)$, which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\left|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)\right| \leq C\,A\,\delta\lambda = C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda, \qquad \forall\, x \in \Delta, \quad \forall\, \varepsilon > 0. \tag{4.231}$$

In view of the fact that $\theta \in (0, 1)$, and taking $\delta_* \in (0, 1)$ small enough to begin with (again, keeping in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)), from (4.231) we conclude that

$$\left|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)\right| \leq \tfrac{1}{2}\lambda, \qquad \forall\, x \in \Delta, \quad \forall\, \varepsilon > 0. \tag{4.232}$$

By combining (4.211), (4.232), and (4.104) we thus obtain

$$T_* f_2(x) \leq 2\lambda \ \text{ for all } \ x \in \Delta, \tag{4.233}$$

whenever $\delta_* \in (0, 1)$ is small enough. Therefore, for this choice of $\delta_*$, we conclude that

$$\sigma\Big(\big\{x \in \Delta : \, T_* f_2(x) > 2\lambda\big\}\Big) = 0 \tag{4.234}$$

which, in concert with (4.209) and (4.210), establishes (4.202). This finishes the proof of the good-λ inequality (4.170).

Once (4.170) has been established, we proceed to prove (4.151). First, using (4.159), by our definition of $A$, and by possibly choosing a smaller $\delta_* \in (0, 1)$ (again, bearing in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)), for each point $x \in I_0$ with $T_{(*)} f(x) > \lambda$ and $\mathcal{M}_\gamma f(x) \leq A\lambda$ we may write

$$\begin{aligned} \lambda < T_{(*)} f(x) &\leq T_* f(x) + C\delta \cdot \mathcal{M}_\gamma f(x) \\ &\leq T_* f(x) + C\delta A\lambda = T_* f(x) + C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda \\ &< T_* f(x) + \tfrac{1}{2}\lambda. \end{aligned} \tag{4.235}$$

Hence, for such a choice of $\delta_* \in (0, 1)$ we have

$$\tfrac{1}{2}\lambda < T_* f(x) \text{ whenever the point } x \in I_0 \text{ is such that } T_{(*)} f(x) > \lambda \text{ and } \mathcal{M}_\gamma f(x) \leq A\lambda. \tag{4.236}$$

Consequently,

$$\big\{x \in I_0 : \, T_{(*)} f(x) > \lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\big\} \subseteq \big\{x \in I_0 : \, T_* f(x) > \tfrac{\lambda}{2}\big\} \tag{4.237}$$

which, in turn, permits us to estimate

$$
\begin{aligned}
w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}\Big) &\le w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \le A\lambda\big\}\Big) \\
&\quad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big) \\
&\le w\Big(\big\{x \in I_0 : T_* f(x) > \tfrac{\lambda}{2}\big\}\Big) \\
&\quad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big). \qquad (4.238)
\end{aligned}
$$

From (4.169) and (4.121) it is clear that for each fixed $\theta$ we have

$$\eta(\theta, \delta) = C\big(\theta^{1+\gamma} + \theta^{1+\gamma/2} \cdot O(1) + o(1)\big) \ \text{ as } \ \delta \to 0^+. \qquad (4.239)$$

This makes it is possible to first choose the threshold $\delta_* \in (0, 1)$, then pick the coefficient $\theta \in (0, 1)$ small enough depending only on $n$, $p$, $[w]_{A_p}$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$, so that

$$\eta(\theta, \delta)^\tau < (2 \cdot 8^p)^{-1}. \qquad (4.240)$$

This is the last demand imposed on $\delta_*, \theta$, and the totality of all these size specifications imply that the final choice of these parameters ultimately depends only on $n$, $p$, $[w]_{A_p}$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$. Combining (4.238) with (4.170) and keeping (4.240) in mind we then get

$$
\begin{aligned}
&w\Big(\big\{x \in I_0 : T_* f(x) > 4\lambda\big\}\Big) \\
&\qquad \le w\Big(\big\{x \in I_0 : T_* f(x) > 4\lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \le A\lambda\big\}\Big) \\
&\qquad\quad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big) \\
&\qquad \le \eta(\theta, \delta)^\tau \cdot w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}\Big) \\
&\qquad\quad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big) \\
&\qquad < (2 \cdot 8^p)^{-1}\, w\Big(\big\{x \in I_0 : T_* f(x) > \tfrac{\lambda}{2}\big\}\Big) \\
&\qquad\quad + \big(1 + (2 \cdot 8^p)^{-1}\big)\, w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big). \qquad (4.241)
\end{aligned}
$$

Recall that $\gamma \in (0, p-1)$ has been chosen so that $w \in A_{p/(1+\gamma)}(\partial\Omega, \sigma)$, hence $\mathcal{M}_\gamma$ is bounded on $L^p(\partial\Omega, w)$. Multiply the most extreme sides of (4.241) by $p\lambda^{p-1}$ and integrate over $\lambda \in (0, \infty)$. Bearing in mind that $A = \theta \cdot \phi(\delta)^{-1}$, after three natural changes of variables (namely, $\widetilde{\lambda} := 4\lambda$ in the first integral, $\widetilde{\lambda} := \frac{1}{2}\lambda$ in the second integral, and $\widetilde{\lambda} := \theta\phi(\delta)^{-1}\lambda$ in the third integral) we therefore obtain

$$\begin{aligned}\int_{I_0} |T_* f|^p \, \mathrm{d}w &\leq \frac{1}{2}\int_{I_0} |T_* f|^p \, \mathrm{d}w + \phi(\delta)^p\theta^{-p}\big(2^{2p} + 2^{-p-1}\big)\int_{I_0} (\mathcal{M}_\gamma f)^p \, \mathrm{d}w \\ &\leq \frac{1}{2}\int_{I_0} |T_* f|^p \, \mathrm{d}w + C\,\phi(\delta)^p \int_{\partial\Omega} |f|^p \, \mathrm{d}w, \end{aligned} \tag{4.242}$$

for some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$ (hence, in particular, independent of the function $f$, the quantity $\delta$, as well as the parameters $x_0, m$ defining the set $I_0$). Since $f \in L^p(\partial\Omega, w)$ and the operator $T_*$ maps the space $L^p(\partial\Omega, w)$ into itself (cf. Proposition 3.4), it follows that $\int_{I_0} |T_* f|^p \, \mathrm{d}w \leq \|T_* f\|^p_{L^p(\partial\Omega,w)} < \infty$. Hence, the first integral in the right-most side of (4.242) may be absorbed in the left-most side. By also taking into account (4.165), we therefore obtain

$$\int_{2\Delta_0} |T_* f|^p \, \mathrm{d}w \leq \int_{I_0} |T_* f|^p \, \mathrm{d}w \leq C\phi(\delta)^p \int_{\partial\Omega} |f|^p \, \mathrm{d}w. \tag{4.243}$$

Recall that $2\Delta_0 = \Delta(x_0, 2^{-m+1})$ and the only constraint on the integer $m \in \mathbb{Z}$ has been that $\operatorname{supp} f \subseteq 2\Delta_0$. Upon letting $m \to -\infty$ and invoking Lebesgue's Monotone Convergence Theorem we arrive at the conclusion that, for some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$, we have the estimate

$$\int_{\partial\Omega} |T_* f|^p \, \mathrm{d}w \leq C\phi(\delta)^p \int_{\partial\Omega} |f|^p \, \mathrm{d}w, \quad \text{for every } f \in L^p(\partial\Omega, w) \text{ with compact support.} \tag{4.244}$$

To treat the case when the function $f \in L^p(\partial\Omega, w)$ is now arbitrary, for each $j \in \mathbb{N}$ define $f_j := \mathbf{1}_{\Delta(x_0,j)} f$. Then Lebesgue's Dominated Convergence Theorem implies that $f_j \to f$ in $L^p(\partial\Omega, w)$ as $j \to \infty$, and since $T_*$ is continuous on $L^p(\partial\Omega, w)$ we also have $T_* f_j \to T_* f$ in $L^p(\partial\Omega, w)$ as $j \to \infty$. Writing the estimate in (4.244) for $f_j$ in place of $f$ and passing to limit $j \to \infty$ then yields

$$\int_{\partial\Omega} |T_* f|^p \, \mathrm{d}w \leq C\phi(\delta)^p \int_{\partial\Omega} |f|^p \, \mathrm{d}w \quad \text{for each} \quad f \in L^p(\partial\Omega, w), \tag{4.245}$$

where $C \in (0, \infty)$ depends only on $n$, $p$, $[w]_{A_p}$, $\psi$, $\phi$, and the Ahlfors regularity constant of $\partial\Omega$. Sending $\delta \searrow \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ (cf. (4.152) and the second line in (4.121)), then finishes the proof of (4.151).

Finally, the very last claim in the statement of Theorem 4.2 follows from (4.153). The proof of Theorem 4.2 is therefore complete. □

Recall the notion of chord-arc domain introduced, in the two-dimensional setting, in Definition 2.16.

**Corollary 4.1** *Fix $\varkappa_* \in (0, \infty)$ and let $\Omega \subseteq \mathbb{R}^2$ be a $\varkappa$-CAD for some $\varkappa \in [0, \varkappa_*)$. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. In addition, select some integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Consider next a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^2 \setminus \{0\})$, for a sufficiently large integer $N \in \mathbb{N}$, which is even and positive homogeneous of degree $-2$, and define the maximal operator $T_*$ acting on each function $f \in L^p(\partial\Omega, w)$ according to*

$$T_* f(x) := \sup_{\varepsilon>0} \left| T_\varepsilon f(x) \right| \text{ for each } x \in \partial\Omega, \tag{4.246}$$

*where, for each $\varepsilon > 0$,*

$$T_\varepsilon f(x) := \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x - y, \nu(y) \rangle k(x - y) f(y) \, d\sigma(y) \text{ for all } x \in \partial\Omega. \tag{4.247}$$

*Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$, which depends only on $m$, $\varkappa_*$, $p$, and $[w]_{A_p}$ such that*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \Big( \sum_{|\alpha|\leq N} \sup_{S^1} |\partial^\alpha k| \Big) \times \tag{4.248}$$
$$\times \sqrt{\varkappa} \cdot \underbrace{\ln \Big( \cdots \ln \big( \ln}_{m \text{ natural logarithms}} (1/\min\{\varkappa, ({}^m e)^{-1}\}) \big) \cdots \Big).$$

Of course, the crux of the matter is the presence of $\sqrt{\varkappa}$ as a multiplicative factor in the right-hand side of (4.248). As a consequence, $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ is as small as we please if $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD whose constant $\varkappa \in (0, 1)$ is sufficiently small (relative to the integral exponent $p$, the characteristic $[w]_{A_p}$ of the Muckenhoupt weight, and the integral kernel $k$).

***Proof of Corollary 4.1*** From (2.229) and (2.118) we deduce that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq \min\left\{1, 2\sqrt{\varkappa(2+\varkappa)}\right\} \leq \sqrt{4 + \sqrt{20}} \cdot \sqrt{\varkappa}. \tag{4.249}$$

Also, Proposition 2.10 implies that $\Omega$ is a UR domain, with the UR constants of $\partial\Omega$ controlled in terms of $\varkappa_*$. Granted these properties, Theorem 4.2 applies and (4.106) together with (4.100) give (4.248). □

Theorem 4.2 readily implies similar operator norm estimates for principal-value singular integral operators whose integral kernel has a special algebraic format, in that it involves the inner product between the outward unit normal and the chord, as a factor. This is made precise later on (see Theorem 4.7). Specifically, for a given second-order, homogeneous, constant complex coefficient system $L$ with $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, and a given UR domain $\Omega \subseteq \mathbb{R}^n$, we shall employ Corollary 4.2 below with $T$ either the boundary-to-boundary double layer potential operator $K_A$ associated with a coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$ or its "transpose" version $K_A^{\#}$, acting on Muckenhoupt weighted Lebesgue spaces on $\partial\Omega$.

**Corollary 4.2** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w$ in $A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same symbol $w$ for the measure associated with the given weight $w$ as in* (2.509). *Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$ and suppose $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ is a complex-valued function which is even and positive homogeneous of degree $-n$. In this setting consider the principal-value singular integral operators $T$, $T^{\#}$ acting on each function $f \in L^p(\partial\Omega, w)$ according to*

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y), \tag{4.250}$$

*and*

$$T^{\#}f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\, d\sigma(y), \tag{4.251}$$

*at $\sigma$-a.e. point $x \in \partial\Omega$. Then for each $m \in \mathbb{N}$ there exists a constant $C_m \in (0,\infty)$, which depends only on $m$, $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|T\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \tag{4.252}$$

*and*

$$\Big\|T^{\#}\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.253}$$

*Also, if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in* (4.252)–(4.253) *to depend only on said entities (i.e., $n$, $p$, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$) and $m$.*

*In addition, with* $p' \in (1, \infty)$ *denoting the Hölder conjugate exponent of* $p$ *and with* $w' := w^{1-p'} \in A_{p'}(\partial\Omega, \sigma)$*, it follows that*

$$\begin{aligned}&\textit{the (real) transpose of } T : L^p(\partial\Omega, w) \to L^p(\partial\Omega, w)\\ &\textit{is the operator } T^\# : L^{p'}(\partial\Omega, w') \to L^{p'}(\partial\Omega, w').\end{aligned} \tag{4.254}$$

***Proof*** Fix $m \in \mathbb{N}$. In view of the fact that

$$\|T\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq \|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}, \tag{4.255}$$

the estimate claimed in (4.252) follows directly from (4.106). The claim in the subsequent paragraph in the statement follows from Theorem 2.3. Next, observe that (4.254) is implied by (4.250)–(4.251) and (3.83). To justify the claim made in (4.253), we write

$$\begin{aligned}\left\|T^\#\right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} &= \|T\|_{L^{p'}(\partial\Omega,w')\to L^{p'}(\partial\Omega,w')}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{4.256}$$

thanks to (4.252) used with $p'$, $w'$ in place of $p$, $w$. □

*Remark 4.8* Of course, in the special case when $w \equiv 1$, Theorem 4.2 and Corollary 4.2 yield estimates on ordinary Lebesgue spaces, $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. Via real interpolation, these further imply similar estimates on the scale of Lorentz spaces on $\partial\Omega$. Specifically, from (4.106), (4.252)–(4.253), and real interpolation (for sub-linear operators) we conclude that for each $m \in \mathbb{N}$, $p \in (1, \infty)$, and $q \in (0, \infty]$ there exists a constant $C_m \in (0, \infty)$, which depends only on $m$, $n$, $p$, $q$, and the UR constants of $\partial\Omega$, with the property that

$$\|T_*\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.257}$$

$$\|T\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.258}$$

and

$$\left\|T^\#\right\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.259}$$

More general results of this type are discussed later, in Theorem 8.8 (cf. also Examples 8.2 and 8.6).

*Remark 4.9* In the context of Corollary 4.2, estimates (4.252)–(4.253) remain valid with a fixed constant $C_m \in (0, \infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are allowed to vary while retaining control. Concretely, Remark 4.3 implies that for each compact interval $I \subset (0, \infty)$ and each number $W \in (0, \infty)$ there exists a constant $C_m \in (0, \infty)$, which depends only on $m, n, I, W$, and the UR constants of $\partial\Omega$, with the property that (4.252)–(4.253) hold for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

Similar considerations apply to the estimates in (4.257)–(4.259).

## 4.3 Norm Estimates and Invertibility Results for Double Layers

We first recall a result (cf. [61, Theorem 2.16, p. 2603]) which is a combination of the extrapolation theorem of Rubio de Francia and the commutator theorem of Coifman et al., [31], suitably adapted to the setting of spaces of homogeneous type.

**Theorem 4.3** *Make the assumption that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0 \in (1, \infty)$ along with some non-decreasing function $\Phi : (0, \infty) \to (0, \infty)$ and let $T$ be a linear operator which is bounded on $L^{p_0}(\Sigma, w)$ for every $w \in A_{p_0}(\Sigma, \sigma)$, with operator norm $\leq \Phi\big([w]_{A_{p_0}}\big)$.*

*Then for each integrability exponent $p \in (1, \infty)$ there exist $C_1, C_2 \in (0, \infty)$ which depend exclusively on the dimension $n$, the exponents $p_0$, $p$, and the Ahlfors regularity constant of $\Sigma$, such that for any Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ the operator*

$$T : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{4.260}$$

*is well defined, linear, and bounded, with operator norm*

$$\|T\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi\Big(C_2 \cdot [w]_{A_p}^{1+(p_0-1)/(p-1)}\Big). \tag{4.261}$$

*In addition, given any $p \in (1, \infty)$ along with some $w \in A_p(\Sigma, \sigma)$, there exists a constant $C = C(\Sigma, n, p_0, p, [w]_{A_p}) \in (0, \infty)$ with the property that for every complex-valued function $b \in L^\infty(\Sigma, \sigma)$ one has (with $C_1$ as before)*

$$\|[M_b, T]\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi(C)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,, \tag{4.262}$$

*where $[M_b, T]$ is the commutator of $T$ considered as in (4.260) and the operator $M_b$ of pointwise multiplication on $L^p(\Sigma, w)$ by the function $b$, i.e.,*

$$[M_b, T]f := b(Tf) - T(bf) \ \ \textit{for each} \ \ f \in L^p(\Sigma, w). \tag{4.263}$$

*In particular, from* (4.262) *with* $w \equiv 1$ *and real interpolation it follows that, for any* $p \in (1, \infty)$ *and* $q \in (0, \infty]$*, there exists some* $C = C(\Sigma, n, p, q) \in (0, \infty)$ *with the property that for every complex-valued function* $b \in L^\infty(\Sigma, \sigma)$ *one has*

$$\|[M_b, T]\|_{L^{p,q}(\Sigma,\sigma)\to L^{p,q}(\Sigma,\sigma)} \le C_1 \cdot \Phi(C)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{4.264}$$

Theorem 4.3 is a particular case of a more general result proved in Theorem 4.4, stated just after the following remark.

*Remark 4.10* Even though Theorem 4.3 suffices for the purposes we have in mind, it is worth noting that there is a version of (4.262) in which the pointwise multiplier $b$ is allowed to belong to the larger space $\mathrm{BMO}(\Sigma, \sigma)$. The price to pay is that we now no longer may regard $[M_b, T]$ as in (4.263) and, instead, have to interpret this as an abstract extension (by density) of a genuine commutator. Specifically, given a real-valued function $b \in \mathrm{BMO}(\Sigma, \sigma)$, for each $N \in \mathbb{N}$ define

$$b_N := \min\Big\{\max\{b, -N\}, N\Big\} = \max\Big\{\min\{b, N\}, -N\Big\}, \tag{4.265}$$

and note that there exists $C \in (0, \infty)$ such that

$$\begin{aligned}
&b_N \in L^\infty(\Sigma, \sigma),\ \text{ thus }\ b_N \in \mathrm{BMO}(\Sigma, \sigma),\ \text{ and}\\
&\|b_N\|_{\mathrm{BMO}(\Sigma,\sigma)} \le 2\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\ \text{ for all }\ N \in \mathbb{N},\\
&|b_N(x)| \le |b(x)|\ \text{ for all }\ x \in \Sigma\ \text{ and }\ N \in \mathbb{N},\\
&\lim_{N\to\infty} b_N(x) = b(x)\ \text{ for each } x \text{ belonging to }\ \Sigma.
\end{aligned} \tag{4.266}$$

Fix an exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ and pick a function $f \in L^p(\Sigma, w)$ with the property that $bf \in L^p(\Sigma, w)$. Then Lebesgue's Dominated Convergence Theorem implies $b_N f \to bf$ in $L^p(\Sigma, w)$ as $N \to \infty$, hence also $T(b_N f) \to T(bf)$ in $L^p(\Sigma, w)$ as $N \to \infty$ by (4.260). Since we also have $b_N T(f) \to bT(f)$ at each point in $\Sigma$ as $N \to \infty$, we ultimately conclude that

$$\begin{aligned}
&\text{for each function } f \in L^p(\Sigma, w) \text{ such that } bf \in L^p(\Sigma, w) \text{ there}\\
&\text{exists a strictly increasing sequence } \{N_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N} \text{ for which}\\
&\big[M_{b_{N_j}}, T\big]f \to \big[M_b, T\big]f \text{ at } \sigma\text{-a.e. point in } \Sigma \text{ as } j \to \infty.
\end{aligned} \tag{4.267}$$

For example, the fact that we have $\mathrm{BMO}(\Sigma, \sigma) \subseteq L^p_{\mathrm{loc}}(\Sigma, w)$ (cf. Lemma 2.13) means that the pointwise convergence result in (4.267) is valid for each function $f$ belonging to $L^\infty_{\mathrm{comp}}(\Sigma, w) = L^\infty_{\mathrm{comp}}(\Sigma, \sigma)$, the space of essentially bounded functions with compact support in $\Sigma$.

Granted (4.267), for each such function $f \in L^p(\Sigma, w)$ such that $bf \in L^p(\Sigma, w)$ we may now write (bearing in mind that $w$ and $\sigma$ have the same nullsets)

$$\begin{aligned}\int_\Sigma \big|[M_b,T]f\big|^p\,\mathrm{d}w &= \int_\Sigma \liminf_{j\to\infty}\Big|\big[M_{b_{N_j}},T\big]f\Big|^p\,\mathrm{d}w\\ &\leq \liminf_{j\to\infty}\int_\Sigma \Big|\big[M_{b_{N_j}},T\big]f\Big|^p\,\mathrm{d}w\\ &\leq \liminf_{j\to\infty}\Big(C_1\cdot\Phi(C)\,\|b_{N_j}\|_{\mathrm{BMO}(\Sigma,\sigma)}\Big)^p\|f\|^p_{L^p(\Sigma,w)}\\ &\leq \Big(2C_1\cdot\Phi(C)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\Big)^p\|f\|^p_{L^p(\Sigma,w)},\end{aligned}\tag{4.268}$$

where the equality comes from (4.267), the first inequality is implied by Fatou's Lemma, the second inequality is a consequence of (4.262) (bearing in mind the first property in (4.266)), and the last inequality follows from the second line of (4.266).

In turn, (4.268) proves that

$$\begin{aligned}&\text{the operator } [M_b,T]:=b(T\cdot)-T(b\cdot) \text{ maps the linear space}\\ &\big\{f\in L^p(\Sigma,w): bf\in L^p(\Sigma,w)\big\} \text{ boundedly into } L^p(\Sigma,w).\end{aligned}\tag{4.269}$$

Given that $\big\{f \in L^p(\Sigma,w) : bf \in L^p(\Sigma,w)\big\}$ is dense in $L^p(\Sigma,w)$ (since, as already noted, this contains $L^\infty_{\mathrm{comp}}(\Sigma,w)$ which is itself dense in $L^p(\Sigma,w)$), we finally conclude that $[M_b,T]$, originally acting as a commutator in the manner described in (4.269), extends by density to a linear and bounded mapping from $L^p(\Sigma,w)$ into itself.

Here is a generalization of Theorem 4.3, involving the "maximal commutator" associated with a given family of linear and bounded operators.

**Theorem 4.4** *Suppose $\Sigma\subseteq\mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0\in(1,\infty)$ and let $\{T_j\}_{j\in\mathbb{N}}$ be a family of linear operators which are bounded on $L^{p_0}(\Sigma,w)$ for every $w\in A_{p_0}(\Sigma,\sigma)$. Define the action of the maximal operator associated with this family on each function $f\in L^{p_0}(\Sigma,w)$ with $w\in A_{p_0}(\Sigma,\sigma)$ as*

$$T_{\max}f(x):=\sup_{j\in\mathbb{N}}|T_jf(x)| \ \textit{ for each } \ x\in\Sigma. \tag{4.270}$$

*Assume that for each $w\in A_{p_0}(\Sigma,\sigma)$ the sub-linear operator $T_{\max}$ maps $L^{p_0}(\Sigma,w)$ into itself, and that there exists some non-decreasing function $\Phi:(0,\infty)\to(0,\infty)$ with the property that*

$$\|T_{\max}\|_{L^{p_0}(\Sigma,w)\to L^{p_0}(\Sigma,w)}\leq\Phi\big([w]_{A_{p_0}}\big) \ \textit{ for each } \ w\in A_{p_0}(\Sigma,\sigma). \tag{4.271}$$

*Then the following statements are true.*

*(i) For each integrability exponent* $p \in (1, \infty)$ *there exist* $C_1, C_2 \in (0, \infty)$ *which depend exclusively on the dimension n, the exponents* $p_0$, $p$, *and the Ahlfors regularity constant of* $\Sigma$, *with the property that for any Muckenhoupt weight* $w \in A_p(\Sigma, \sigma)$ *the operator*

$$T_{\max} : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{4.272}$$

*is well defined, sub-linear, and bounded, with operator norm*

$$\|T_{\max}\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi\Big(C_2 \cdot [w]_{A_p}^{1+(p_0-1)/(p-1)}\Big). \tag{4.273}$$

*In particular, for each* $j \in \mathbb{N}$ *the operator* $T_j$ *is a well-defined, linear, and bounded mapping on* $L^p(\Sigma, w)$ *with operator norm satisfying a similar estimate to* (4.273).

*(ii) Pick an arbitrary* $p \in (1, \infty)$ *along with some* $w \in A_p(\Sigma, \sigma)$, *and fix an arbitrary complex-valued function* $b \in L^\infty(\Sigma, \sigma)$. *Define the action of the "maximal commutator" (associated with the given function b and the family* $\{T_j\}_{j\in\mathbb{N}}$*) on each function* $f \in L^p(\Sigma, w)$ *as*

$$C_{\max} f(x) := \sup_{j\in\mathbb{N}} \big|[M_b, T_j]f(x)\big| \ \text{ for each } \ x \in \Sigma, \tag{4.274}$$

*where* $M_b$ *denotes the operator of pointwise multiplication by the function b. Then there exist two constants* $C_i = C_i(\Sigma, n, p_0, p, [w]_{A_p}) \in (0, \infty)$, $i \in \{1, 2\}$, *independent of the function b and the family* $\{T_j\}_{j\in\mathbb{N}}$, *with the property that*

$$\|C_{\max}\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi(C_2)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{4.275}$$

The particular case when all operators in the family $\{T_j\}_{j\in\mathbb{N}}$ are identical to one another corresponds to Theorem 4.3.

***Proof of Theorem 4.4*** The fact that for each $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$ the sub-linear operator $T_{\max}$ induces a bounded mapping on $L^p(\Sigma, w)$ whose operator norm may be estimated as in (4.273) follows from Rubio de Francia's extrapolation theorem, in the format presented in [111, §7.7] (this is responsible for the specific format of the constant in (4.273); see also [34, Theorem 3.22, p.40] and [42, Theorem 3.2] for the Euclidean setting). This takes care of item *(i)*.

To deal with item *(ii)*, we shall adapt the argument in [31], [69], [61], [13]. First, from simple linearity and homogeneity considerations, there is no loss of generality in assuming that $b \in L^\infty(\Sigma, \sigma)$ is actually real-valued and satisfies $\|b\|_{\mathrm{BMO}(\Sigma,\sigma)} = 1$ (the case when $b$ is constant is trivial). Fix now $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$. From item *(8)* of Proposition 2.20 we know that there exists some small $\varepsilon = \varepsilon(\Sigma, p, [w]_{A_p}) > 0$ with the property that for each complex number $z$ with $|z| \leq \varepsilon$ we have

$$w \cdot \mathrm{e}^{(\mathrm{Re}\, z)b} \in A_p(\Sigma, \sigma) \quad \text{with} \quad \big[w \cdot \mathrm{e}^{(\mathrm{Re}\, z)b}\big]_{A_p} \leq C, \tag{4.276}$$

where the constant $C = C(\Sigma, p, [w]_{A_p}) \in (0, \infty)$ is independent of $z$.

To proceed, denote by $\mathscr{L}(L^p_w)$ the space of all linear and bounded operators from $L^p(\Sigma, w)$ into itself, equipped with the operator norm $\|\cdot\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)}$. The idea is now to observe that, for each $j \in \mathbb{N}$,

$$\begin{aligned} &\Phi_j : \big\{z \in \mathbb{C} : \, |z| < \varepsilon/2\big\} \longrightarrow \mathscr{L}(L^p_w) \ \text{ defined as} \\ &\Phi_j(z) := M_{\mathrm{e}^{zb}} T_j M_{\mathrm{e}^{-zb}} \ \text{ for each } \ z \in \mathbb{C} \ \text{ with } \ |z| < \varepsilon/2 \end{aligned} \tag{4.277}$$

is an analytic map which, for each $z \in \mathbb{C}$ with $|z| < \varepsilon/2$ and each $f \in L^p(\Sigma, w)$, satisfies

$$\begin{aligned} \int_\Sigma & \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, w(x)\, \mathrm{d}\sigma(x) \\ &= \int_\Sigma \sup_{j\in\mathbb{N}} \big|T_j(\mathrm{e}^{-zb} f)(x)\big|^p \, w(x) \cdot \mathrm{e}^{(\mathrm{Re}\, z)b(x)}\, \mathrm{d}\sigma(x) \\ &= \int_\Sigma \big|T_{\max}(\mathrm{e}^{-zb} f)(x)\big|^p \, w(x) \cdot \mathrm{e}^{(\mathrm{Re}\, z)b(x)}\, \mathrm{d}\sigma(x) \\ &\leq \|T_{\max}\|^p_{L^p(\Sigma, w\cdot \mathrm{e}^{(\mathrm{Re}\, z)b})\to L^p(\Sigma, w\cdot \mathrm{e}^{(\mathrm{Re}\, z)b})} \times \\ &\qquad \times \int_\Sigma \big|\mathrm{e}^{-zb(x)} f(x)\big|^p \, w(x) \cdot \mathrm{e}^{(\mathrm{Re}\, z)b(x)}\, \mathrm{d}\sigma(x) \\ &\leq C_1^p \cdot \Phi\Big(C_2 \cdot C^{1+(p_0-1)/(p-1)}\Big)^p \|f\|^p_{L^p(\Sigma,w)}, \end{aligned} \tag{4.278}$$

thanks to (4.277), (4.270), (4.276), and (4.273). In addition, from (4.277) and Cauchy's reproducing formula for analytic functions we see that for each $j \in \mathbb{N}$ we have

$$[M_b, T_j] = \Phi_j'(0) = \frac{1}{2\pi \mathrm{i}} \int_{|z|=\varepsilon/4} \frac{\Phi_j(z)}{z^2}\, \mathrm{d}z. \tag{4.279}$$

Consequently, for each $f \in L^p(\Sigma, w)$ and $x \in \Sigma$, we have

$$C_{\max} f(x) = \sup_{j\in\mathbb{N}} \big|[M_b, T_j] f(x)\big| \le \frac{8}{\pi\varepsilon^2} \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big| \, d\mathcal{H}^1(z), \tag{4.280}$$

hence

$$\big|C_{\max} f(x)\big|^p \le \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, d\mathcal{H}^1(z). \tag{4.281}$$

From the last property in item *(i)* and (4.274) we see that for each $f \in L^p(\Sigma, w)$ the function $C_{\max} f$ is $\sigma$-measurable. In concert with (4.281) and (4.278), this permits us to estimate

$$\begin{aligned}
&\int_\Sigma \big|C_{\max} f(x)\big|^p \, dw(x) \\
&\quad\le \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_\Sigma \Big( \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, d\mathcal{H}^1(z) \Big) \, dw(x) \\
&\quad= \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{|z|=\varepsilon/4} \Big( \int_\Sigma \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, dw(x) \Big) \, d\mathcal{H}^1(z) \\
&\quad\le \Big(\frac{2^{3p-1}}{\pi^{p-1}\varepsilon^{2p-1}}\Big) C_1^p \cdot \Phi\Big(C_2 \cdot C^{1+(p_0-1)/(p-1)}\Big)^p \|f\|^p_{L^p(\Sigma,w)},
\end{aligned} \tag{4.282}$$

and (4.275) readily follows from this. □

We next discuss a companion result to Theorem 4.2, the novelty being the consideration of a maximal "transpose" operator as defined below in (4.283).

**Theorem 4.5** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and recall the earlier convention of using the same symbol $w$ for the measure associated with the given weight $w$ as in* (2.509). *Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$. Given a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, consider the maximal operator $T_*^\#$ whose action on each given function $f \in L^p(\partial\Omega, w)$ is defined as*

$$T_*^\# f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon^\# f(x)\big| \quad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega, \tag{4.283}$$

*where, for each $\varepsilon > 0$,*

$$T_\varepsilon^\# f(x) := \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y) \ \ \textit{for } \sigma\textit{-a.e. } x\in\partial\Omega. \tag{4.284}$$

*Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0,\infty)$, which depends only on $m$, $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has*

$$\left\| T_*^\# \right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \Big( \sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.285}$$

*Furthermore, when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in (4.285) to depend itself only on said entities (i.e., $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) and $m$.*

In particular, Theorem 4.5 may be used to give a direct proof of (4.253), without having to rely on duality.

***Proof of Theorem 4.5*** To get started, we observe that if $\mathbb{Q}_+$ denotes the collection of all positive rational numbers, then for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ we have

$$(T_*^\# f)(x) = \sup_{\varepsilon\in\mathbb{Q}_+} \big|(T_\varepsilon^\# f)(x)\big| \ \text{ for every } \ x\in\partial^*\Omega. \tag{4.286}$$

To justify this, pick some $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$. We claim that if $x \in \partial^*\Omega$ is arbitrary and fixed then for each $\varepsilon \in (0,\infty)$ and each sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\varepsilon_j \searrow \varepsilon$ as $j\to\infty$ we have

$$\lim_{j\to\infty} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon_j}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y) = \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y). \tag{4.287}$$

To justify (4.287) note that

$$\{y\in\partial\Omega : |x-y|>\varepsilon_j\} \nearrow \{y\in\partial\Omega : |x-y|>\varepsilon\} \ \text{ as } \ j\to\infty, \tag{4.288}$$

in the sense that

$$\{y \in \partial\Omega : |x-y| > \varepsilon\} = \bigcup_{j\in\mathbb{N}} \{y \in \partial\Omega : |x-y| > \varepsilon_j\} \text{ and}$$
$$\{y \in \partial\Omega : |x-y| > \varepsilon_j\} \subseteq \{y \in \partial\Omega : |x-y| > \varepsilon_{j+1}\} \text{ for every } j \in \mathbb{N}. \tag{4.289}$$

Then (4.287) follows from (4.288), the properties of $k$, and Lebesgue's Dominated Convergence Theorem. What we have just proved amounts to saying that for every function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ we have

$$\lim_{j\to\infty} (T^{\#}_{\varepsilon_j} f)(x) = (T^{\#}_{\varepsilon} f)(x) \text{ for every } x \in \partial^*\Omega, \tag{4.290}$$

whenever $\varepsilon \in (0,\infty)$ and $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ are such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$. Having established this, (4.286) readily follows on account of the density of $\mathbb{Q}_+$ in $(0,\infty)$.

To proceed, let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}_+$. Also, bring back the operators (4.105) and observe that for each $j \in \mathbb{N}$, each $f \in L^p(\partial\Omega, w)$, and each $x \in \partial^*\Omega$ we have

$$T^{\#}_{\varepsilon_j} f(x) + T_{\varepsilon_j} f(x) = \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon_j}} \langle y-x, \nu(x)-\nu(y)\rangle k(x-y) f(y)\, d\sigma(y). \tag{4.291}$$

Write $(\nu_i)_{1\leq i\leq n}$ for the scalar components of the geometric measure theoretic outward unit normal $\nu$ to $\Omega$ and, for every $i \in \{1,\dots,n\}$, every $j \in \mathbb{N}$, and every $f \in L^p(\partial\Omega, w)$ set

$$\mathbb{T}^{(i)}_j f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon_j}} (y_i - x_i) k(x-y) f(y)\, d\sigma(y) \text{ for each } x \in \partial\Omega. \tag{4.292}$$

Then, for each $j \in \mathbb{N}$ and each $f \in L^p(\partial\Omega, w)$ we may recast (4.291) as

$$T^{\#}_{\varepsilon_j} f(x) + T_{\varepsilon_j} f(x) = \sum_{i=1}^{n} \big[M_{\nu_i}, \mathbb{T}^{(i)}_j\big] f(x) \text{ for each } x \in \partial^*\Omega. \tag{4.293}$$

If for each $i \in \{1,\dots,n\}$ and each $f \in L^p(\partial\Omega, w)$ we now define

$$C^{(i)}_{\max} f(x) := \sup_{j\in\mathbb{N}} \Big|\big[M_{\nu_i}, \mathbb{T}^{(i)}_j\big] f(x)\Big| \text{ for each } x \in \partial^*\Omega, \tag{4.294}$$

then, thanks to Proposition 3.4, for each $i \in \{1,\dots,n\}$ we may invoke Theorem 4.4 for the family $\big\{\mathbb{T}^{(i)}_j\big\}_{j\in\mathbb{N}}$ to conclude that

$$\left\| C^{(i)}_{\max} \right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big( \sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.295}$$

where $C \in (0, \infty)$ depends only on $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$. Also, from (4.293), (4.294), (4.286), and (4.104) we deduce that for each $f \in L^p(\partial\Omega, w)$ we have

$$T^{\#}_* f(x) \leq T_* f(x) + \sum_{i=1}^{n} C^{(i)}_{\max} f(x) \ \text{ for each } \ x \in \partial^*\Omega. \tag{4.296}$$

At this stage, the estimate in (4.285) becomes a consequence of (4.296), (4.106), (4.295), (4.98), and (2.118), keeping in mind that, as is apparent from (4.286), the function $T^{\#}_* f$ is $\sigma$-measurable, and that we currently have $\sigma\big(\partial\Omega \setminus \partial^*\Omega\big) = 0$ (cf. Definition 2.4 and (2.24)). Finally, the very last claim in the statement is seen from Theorem 2.3. □

To discuss a significant application of Theorem 4.3 let us first formally introduce the family of Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on the boundary a UR domain $\Omega \subseteq \mathbb{R}^n$. Specifically, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, for each $j \in \{1, \dots, n\}$ the $j$-th Riesz transform $R_j$ acts on any given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y) \tag{4.297}$$

at $\sigma$-a.e. point $x \in \partial\Omega$.

**Theorem 4.6** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ from* (3.29)*, the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ from* (4.297)*, and for each index $k \in \{1, \dots, n\}$ denote by $M_{\nu_k}$ the operator of pointwise multiplication by $\nu_k$, the $k$-th scalar component of the vector $\nu$.*

*Then there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ and, for each $m \in \mathbb{N}$, there exists some $C_m \in (0, \infty)$ which depends only on $m$, $n$, $p$, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ with the property that, with the piece of notation introduced in* (4.93)*, one has*

$$\begin{aligned} &\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \ \text{ and} \\ &\max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \end{aligned} \tag{4.298}$$

*Also, when* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ *is sufficiently small relative to n, p,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$ *one may take the constants* $C, C_m \in (0,\infty)$ *appearing in* (4.298) *to depend only on said entities (i.e., n, p,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$*) and, in the case of* $C_m$*, also on m.*

***Proof*** The estimate claimed in (4.298) is implied by (3.29), Corollary 4.2, (4.297), Proposition 3.4, and Theorem 4.3. The very last claim in the statement is implied by Theorem 2.3. □

We shall, once again, see Theorem 4.3 in action shortly, in the proof of Theorem 4.7. In the latter result the focus is obtaining operator norm estimates for double layer potentials associated with distinguished coefficient tensors on Muckenhoupt weighted Lebesgue and Sobolev spaces, exhibiting explicit dependence on the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain.

**Theorem 4.7** *Let* $\Omega \subseteq \mathbb{R}^n$ *be a UR domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu$ *the geometric measure theoretic outward unit normal to* $\Omega$*. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *and consider the boundary-to-boundary double layer potential operators* $K_A, K_A^{\#}$ *associated with* $\Omega$ *and the coefficient tensor A as in* (3.24) *and* (3.25)*, respectively. Finally, fix an integrability exponent* $p \in (1,\infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$*.*

*Then for each* $m \in \mathbb{N}$ *there exists some* $C_m \in (0,\infty)$ *which depends only on m, n, A, p,* $[w]_{A_p}$*, and the UR constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.299}$$

$$\|K_A\|_{[L^p_1(\partial\Omega,w)]^M \to [L^p_1(\partial\Omega,w)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.300}$$

*and*

$$\left\|K_A^{\#}\right\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.301}$$

*In addition, when* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ *is sufficiently small relative to n, p,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$ *one may take* $C_m \in (0,\infty)$ *appearing in* (4.299)–(4.301) *to depend itself only on said entities (i.e., n, p,* $[w]_{A_p}$*, the Ahlfors regularity constant of* $\partial\Omega$*) and m.*

Note that the estimate in (4.299) implies that the operator $K_A$ becomes identically zero whenever $\Omega$ is a half-space in $\mathbb{R}^n$. From *(i)* ⇔ *(ii)* in Proposition 3.9 we know that this may only occur if $A \in \mathfrak{A}_L^{\mathrm{dis}}$. Hence, the assumption $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ is actually *necessary* in light of the conclusion in Theorem 4.7.

***Proof of Theorem 4.7*** The estimates claimed in (4.299) and (4.301) are direct consequences of Corollary 4.2 and Proposition 3.9, bearing in mind (3.24) and (3.25).

Turning to the task of proving (4.300), it is apparent from (3.35) that each $U_{jk}$ is a sum of operators of commutator type. Then, given any integer $m \in \mathbb{N}$ along with any function $f \in \big[L_1^p(\partial\Omega, w)\big]^M$, based on (3.37), (4.299), Theorem 4.3, and (4.98) we may write

$$
\begin{aligned}
\|K_A f\|_{[L_1^p(\partial\Omega,w)]^M} &= \|K_A f\|_{[L^p(\partial\Omega,w)]^M} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}}(K_A f)\big\|_{[L^p(\partial\Omega,w)]^M} \\
&= \|K_A f\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + \sum_{j,k=1}^{n} \Big(\big\|K_A(\partial_{\tau_{jk}} f)\big\|_{[L^p(\partial\Omega,w)]^M} + \big\|U_{jk}(\nabla_{\tan} f)\big\|_{[L^p(\partial\Omega,w)]^M}\Big) \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|f\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^M} \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|f\|_{[L_1^p(\partial\Omega,w)]^M}, \qquad (4.302)
\end{aligned}
$$

which establishes (4.300). The very last claim in the statement is a consequence of Theorem 2.3. □

*Remark 4.11* The unweighted case (i.e., the scenario in which $w \equiv 1$) of Theorem 4.7 gives norm estimates for the double layer operator and its formal transpose on ordinary Lebesgue and Sobolev spaces. By relying on (4.258)–(4.259), Proposition 3.2, (4.264), and (2.589) we may also obtain similar estimates on Lorentz spaces and Lorentz-based Sobolev spaces (cf. (2.590)–(2.591)). Specifically, in the same setting as Theorem 4.7, the aforementioned results imply that for each $m \in \mathbb{N}$, $p \in (1, \infty)$ and $q \in (0, \infty]$ there exists some $C_m \in (0, \infty)$ which depends only on $m$, $n$, $A$, $p$, $q$, and UR constants of $\partial\Omega$, such that

$$
\|K_A\|_{[L^{p,q}(\partial\Omega,\sigma)]^M \to [L^{p,q}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \qquad (4.303)
$$

$$
\|K_A\|_{[L_1^{p,q}(\partial\Omega,\sigma)]^M \to [L_1^{p,q}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \qquad (4.304)
$$

and

$$\left\| K_A^{\#} \right\|_{[L^{p,q}(\partial\Omega,\sigma)]^M \to [L^{p,q}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.305}$$

More general results of this type are discussed later, in Theorem 8.9 (see also Examples 8.2 and 8.6).

*Remark 4.12* By reasoning much as in the proof of Theorem 4.7, we may also obtain operator norm estimates for the double layer $K_A$ with $A \in \mathfrak{A}_L^{\mathrm{dis}}$ on *off-diagonal* weighted Sobolev spaces, i.e., when the integrability exponents and the weights for the Lebesgue spaces to which the actual function and its tangential derivatives belong to are allowed to be different. Specifically, given two integrability exponents $p_1, p_2 \in (1, \infty)$ along with two Muckenhoupt weights $w_1 \in A_{p_1}(\partial\Omega, \sigma)$ and $w_2 \in A_{p_2}(\partial\Omega, \sigma)$, define the off-diagonal weighted Sobolev space

$$\begin{aligned} L_1^{p_1;p_2}(\partial\Omega, w_1; w_2) := \big\{ f \in & L^{p_1}(\partial\Omega, w_1) : \\ & \partial_{\tau_{jk}} f \in L^{p_2}(\partial\Omega, w_2),\ \ 1 \leq j, k \leq n \big\}, \end{aligned} \tag{4.306}$$

equipped with the natural norm defined for each $f \in L_1^{p_1;p_2}(\partial\Omega, w_1; w_2)$ as

$$\|f\|_{L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)} := \|f\|_{L^{p_1}(\partial\Omega,w_1)} + \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^{p_2}(\partial\Omega,w_2)}. \tag{4.307}$$

Then much as in (4.302), for each $m \in \mathbb{N}$ we now obtain

$$\|K_A\|_{[L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)]^M \to [L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)]^M} \leq C_m \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.308}$$

for some $C_m \in (0, \infty)$ which depends only on $m$, $n$, $A$, $p_1$, $p_2$, $[w_1]_{A_{p_1}}$, $[w_2]_{A_{p_2}}$, and the UR constants of $\partial\Omega$.

*Remark 4.13* In the setting of Theorem 4.7, estimates (4.299)–(4.301) continue to hold with a fixed constant $C_m \in (0, \infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are permitted to vary with control. Specifically, from Remark 4.9 and the proof of Theorem 4.7 we see that for each $m \in \mathbb{N}$, each compact interval $I \subset (0, \infty)$, and each number $W \in (0, \infty)$ there exists a constant $C_m \in (0, \infty)$, which depends only on $n$, $I$, $W$, and the UR constants of $\partial\Omega$, with the property that (4.299)–(4.301) hold for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

Having proved Theorem 4.7, we may now establish invertibility results for boundary double layer potentials associated with distinguished coefficient tensors, assuming $\Omega$ is a $\delta$-flat Ahlfors regular domain with $\delta$ suitably small relative to $n$ and the Ahlfors regularity constant of $\partial\Omega$. By means of counterexamples we show that assuming that the double layer potentials are associated with distinguished coefficient tensors is a hypothesis one cannot simply omit. Also, as explained a little later, in Remark 4.19, the flatness condition imposed on the domain is actually in the nature of best possible as far as the invertibility results from Theorem 4.8 are concerned.

**Theorem 4.8** *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu$ *the geometric measure theoretic outward unit normal to* $\Omega$*. Also, let* $L$ *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\rm dis} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\rm dis}$ *and consider the boundary-to-boundary double layer potential operators* $K_A$, $K_A^{\#}$ *associated with* $\Omega$ *and the coefficient tensor* $A$ *as in* (3.24) *and* (3.25)*, respectively. Finally, fix an integrability exponent* $p \in (1, \infty)$*, a Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$*, and some number* $\varepsilon \in (0, \infty)$.

*Then there exists some small threshold* $\delta \in (0, 1)$ *which depends only on* $n$*,* $p$*,* $[w]_{A_p}$*,* $A$*,* $\varepsilon$*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *it follows that for each spectral parameter* $z \in \mathbb{C}$ *with* $|z| \geq \varepsilon$ *the following operators are linear, bounded, and invertible:*

$$zI + K_A : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M, \tag{4.309}$$

$$zI + K_A : \big[L_1^p(\partial\Omega, w)\big]^M \longrightarrow \big[L_1^p(\partial\Omega, w)\big]^M, \tag{4.310}$$

$$zI + K_A^{\#} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M. \tag{4.311}$$

*Furthermore, the above result is optimal in the sense that if* $A \notin \mathfrak{A}_L^{\rm dis}$ *then either of the operators* (4.309)–(4.311) *may fail to be invertible even when* $z = 1/2$ *and* $\Omega = \mathbb{R}^n_+$.

***Proof*** Let $C$ denote the constant appearing in estimates (4.299)–(4.301), for the choice $m := 1$, and choose $t_\varepsilon \in (0, 1/\mathrm{e})$ small enough so that $t_\varepsilon \cdot \ln(1/t_\varepsilon) < \varepsilon/C$. To get going, pick $\delta \in (0, t_\varepsilon)$. By decreasing $\delta$ if necessary, we may insure that $\Omega$ is a UR domain, with the UR constants of $\partial\Omega$ controlled solely in terms of the dimension $n$ and the Ahlfors regularity constant of $\partial\Omega$ (cf. Theorem 2.3). Granted this, Theorem 4.7 applies and gives that

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C\delta^{\langle 1\rangle} \leq C(t_\varepsilon)^{\langle 1\rangle} < \varepsilon. \tag{4.312}$$

Analogously,

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M \to [L_1^p(\partial\Omega,w)]^M} < \varepsilon, \tag{4.313}$$

$$\big\|K_A^{\#}\big\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} < \varepsilon. \tag{4.314}$$

In particular, the operators in (4.309)–(4.311) are invertible for each given $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ using a Neumann series, i.e.,

$$\big(zI + K_A\big)^{-1} = z^{-1}\sum_{m=0}^{\infty}\big(-z^{-1}K_A\big)^m \tag{4.315}$$

with convergence in the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$ as well as on $\big[L_1^p(\partial\Omega, w)\big]^M$, and

$$\big(zI + K_A^{\#}\big)^{-1} = z^{-1}\sum_{m=0}^{\infty}\big(-z^{-1}K_A^{\#}\big)^m \tag{4.316}$$

with convergence in the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$.

There remains to address the optimality claim in the last part of the statement. To this end, recall the second-order, weakly elliptic, constant (real) coefficient, symmetric, $n \times n$ system $L_D$ defined in (3.371). From (3.23), (3.31), (2.575), (3.112), and (3.377) we see that if $K_A$, $K_A^{\#}$ are the boundary layer potential operators associated as in (3.24), (3.25) with $\Omega := \mathbb{R}^n_+$ and any coefficient tensor $A \in \mathfrak{A}_{L_D}$ then

$$\Big\{\big(\tfrac{1}{2}I + K_A\big)f : f \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n\Big\} \subseteq \Big\{(f_1, \dots, f_n) \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n : f_n = \sum_{j=1}^{n-1} R_j f_j\Big\}. \tag{4.317}$$

Thus, $\Big\{(0, \dots, 0, f) : f \in L^p(\mathbb{R}^{n-1}, w)\Big\}$ is an infinite dimensional subspace of $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ whose intersection with $\Big\{\big(\tfrac{1}{2}I + K_A\big)f : f \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n\Big\}$ is $\{0\}$. Consequently, $\tfrac{1}{2}I + K_A$ acting on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ has an infinite dimensional cokernel for each $p \in (1, \infty)$ and each $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. By duality (cf. (3.119)), it follows that $\tfrac{1}{2}I + K_A^{\#}$ acting on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ has an infinite dimensional kernel for each $p \in (1, \infty)$ and each $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. In particular, the operators in (4.309), (4.311) corresponding to $z = 1/2$ and $\Omega = \mathbb{R}^n_+$ fail to be invertible in this case.

Likewise, from (3.23), (3.31), (2.575), (3.112), (3.113), and (3.378) it follows that if $K_A$ is the double layer potential operator associated as in (3.24) with $\Omega := \mathbb{R}^n_+$ and any coefficient tensor $A \in \mathfrak{A}_{L_D}$ then

$$\Big\{\big(\tfrac{1}{2}I + K_A\big)f : f \in \big[L_1^p(\mathbb{R}^{n-1}, w)\big]^n\Big\}$$

$$\subseteq \Big\{(f_1,\dots,f_n)\in\big[L^p_1(\mathbb{R}^{n-1},w)\big]^n:\ f_n=\sum_{j=1}^{n-1}R_jf_j\Big\}. \tag{4.318}$$

Much as before, this shows that $\frac{1}{2}I+K_A$ acting on $\big[L^p_1(\mathbb{R}^{n-1},w)\big]^n$ has an infinite dimensional cokernel for each $p\in(1,\infty)$ and each $w\in A_p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. In particular, the operator in (4.310) corresponding to $z=1/2$ and $\Omega=\mathbb{R}^n_+$ also fails to be invertible in this case.

In all cases, the source of the failure for invertibility is the fact that any coefficient tensor $A\in\mathfrak{A}_{L_D}$ fails to be distinguished (cf. (3.406)). □

In Remarks 4.14–4.15 we continue to elaborate on the nature of the optimality claim in the last portion of the statement of Theorem 4.8.

*Remark 4.14* Work with a scalar operator in the two-dimensional setting (i.e., when $n=2$ and $M=1$). Specifically, take $L:=\Delta$, the Laplacian in the plane, written as $\Delta=a_{jk}\partial_j\partial_k$ for the matrix $A=(a_{jk})_{1\leq j,k\leq 2}$ given by

$$A:=\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\in\mathbb{C}^{2\times 2}. \tag{4.319}$$

Then, as noted in (1.23)–(1.24), the boundary-to-boundary double layer potential operator $K_A$ associated as in (3.24) with this coefficient tensor and the domain $\Omega:=\mathbb{R}^2_+$ is $K_A=(\mathrm{i}/2)H$ where $H$ is the Hilbert transform on the real line. Fix an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\mathbb{R},\mathcal{L}^1)$. Since $-H^2=I$, the identity operator on $L^p(\mathbb{R},w)$, it follows that we currently have $\big(K_A\big)^2=4^{-1}I$ on $L^p(\mathbb{R},w)$. This further entails

$$\big(\tfrac{1}{2}I+K_A\big)\big(-\tfrac{1}{2}I+K_A\big)=0\ \text{ on }\ L^p(\mathbb{R},w) \tag{4.320}$$

which, in view of the fact that $K_A\neq\pm\frac{1}{2}I$, ultimately proves that the operator $\frac{1}{2}I+K_A$ is not invertible[3] on any Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R},w)$.

From what we have just proved and duality (cf. (3.119)) we then see that the operator $\frac{1}{2}I+K^{\#}_A$ fails to be invertible on any Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R}^{n-1},w)$ as well. Finally, given that (4.320) implies

$$\big(\tfrac{1}{2}I+K_A\big)\big(-\tfrac{1}{2}I+K_A\big)=0\ \text{ on }\ L^p_1(\mathbb{R},w), \tag{4.321}$$

we also infer that the operator $\frac{1}{2}I+K_A$ fails to be invertible when acting on any Muckenhoupt weighted Sobolev space $L^p_1(\mathbb{R},w)$.

[3] In fact, $\frac{1}{2}I+K_A$ acting on $L^p(\mathbb{R},w)$ has an infinite dimensional kernel and an infinite dimensional cokernel.

Since $A \neq I_{2\times 2}$ and $\mathfrak{A}^{\text{dis}}_{\Delta} = \{I_{2\times 2}\}$, the above analysis shows that for coefficient tensors $A \in \mathfrak{A}_{\Delta} \setminus \mathfrak{A}^{\text{dis}}_{\Delta}$ it may actually happen that the conclusions in Theorem 4.8 corresponding to $z := 1/2$ and $\Omega := \mathbb{R}^2_+$ fail.

The following is a higher-dimensional version of Remark 4.14.

*Remark 4.15* Fix $n \in \mathbb{N}$ with $n \geq 2$ and define $M := 2^n$. Bring back the $M \times M$ second-order system $L := \Delta \cdot I_{M\times M}$ in $\mathbb{R}^n$ (cf. (1.31)). In particular, from (3.396) and Proposition 3.9 we see that $\mathfrak{A}^{\text{dis}}_{L} = \{I_{M\times M}\}$. Consequently, the coefficient tensor $A := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$ with entries as in (1.33) satisfies

$$A \in \mathfrak{A}_L \setminus \mathfrak{A}^{\text{dis}}_{L}. \tag{4.322}$$

To proceed, let $K_A$ be the boundary-to-boundary double layer potential operator associated as in (3.24) with the coefficient tensor (4.322) and the domain $\Omega := \mathbb{R}^n_+$. Given some arbitrary integrability exponent $p \in (1, \infty)$ along with some arbitrary Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, the same type of argument as in (1.39) gives

$$\big(K_A\big)^2 = \tfrac{1}{4} I \quad \text{on} \quad \big[L^p(\mathbb{R}^{n-1}, w)\big]^M, \tag{4.323}$$

where $I$ is the identity operator on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$. Thus,

$$\big(\tfrac{1}{2} I + K_A\big)\big(-\tfrac{1}{2} I + K_A\big) = 0 \quad \text{on} \quad \big[L^p(\mathbb{R}^{n-1}, w)\big]^M. \tag{4.324}$$

In view of the fact that[4] $K_A \neq \pm\frac{1}{2} I$, the above identity ultimately proves that the operator $\frac{1}{2} I + K_A$ is not invertible[5] on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$.

Ultimately, this discussion shows that for coefficient tensors as in (4.322) it may well happen that the operator $\frac{1}{2} I + K_A$ is not invertible on any Muckenhoupt weighted Lebesgue space $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$. Via duality (cf. (3.119)) we conclude that the operator $\frac{1}{2} I + K^{\#}_A$ fails to be invertible on any Muckenhoupt weighted Lebesgue space $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$ as well. Finally, since (4.324) implies

$$\big(\tfrac{1}{2} I + K_A\big)\big(-\tfrac{1}{2} I + K_A\big) = 0 \quad \text{on} \quad \big[L^p_1(\mathbb{R}^{n-1}, w)\big]^M, \tag{4.325}$$

[4] Since $K_A$ is a Fourier multiplier operator with symbol $m(\xi') := \frac{\xi_j}{2|\xi'|} E_n E_j$ for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$.

[5] In fact, $\frac{1}{2} I + K_A$ acting on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$ has both an infinite dimensional kernel and an infinite dimensional cokernel.

we also conclude that the operator $\frac{1}{2}I + K_A$ is not invertible when acting on any Muckenhoupt weighted Sobolev space $\big[L^p_1(\mathbb{R}^{n-1}, w)\big]^M$. Hence, all conclusions in Theorem 4.8 corresponding to $z := 1/2$ and $\Omega := \mathbb{R}^2_+$ fail.

*Remark 4.16* In view of (4.303)–(4.305), and (4.308), invertibility results which are similar to those proved in Theorem 4.8 may be established on Lorentz spaces and Lorentz-based Sobolev spaces, as well as on the brand of off-diagonal Muckenhoupt weighted Sobolev spaces defined as in (4.306)–(4.307).

*Remark 4.17* It is of interest to contrast Theorem 4.8 with the precise invertibility results known in the particular case when $\Omega$ is an infinite sector in the plane, with opening angle $\theta \in (0, 2\pi)$ and when $L = \Delta$ (the two-dimensional Laplacian). In such a setting, it is known (cf. [48], [115, §4.2], [126, Theorem 5, p. 192]) that

given $p \in (1,\infty)$, the operators $\pm\frac{1}{2}I + K_\Delta$ are invertible on $L^p(\partial\Omega,\sigma)$ if and only if $p \neq 1 + |\pi - \theta|/\pi$ (which amounts to saying that necessarily $p \neq \frac{2\pi-\theta}{\pi}$ for $\theta \in (0,\pi)$ and $p \neq \frac{\theta}{\pi}$ for $\theta \in (\pi, 2\pi)$). (4.326)

When $\theta = \pi$ (i.e., when $\Omega$ is a half-plane) then, of course, any $p \in (1,\infty)$ will do. In this vein, see also [105, Lemma 4.5, p. 2042]. Consider next the case of the two-dimensional Lamé system in an infinite sector of aperture $\theta \in (0, 2\pi)$, and recall from the discussion at the end of Example 3.4 that the pseudo-stress double layer potential operator for the Lamé system is denoted by $K_\Psi$. Then there are two critical values of the integrability exponent $p \in (1,\infty)$, which depend on $\theta$ and a specific combination of the Lamé moduli, for which the invertibility of the operators $\pm\frac{1}{2}I + K_\Psi$ on $\big[L^p(\partial\Omega,\sigma)\big]^2$ fails. See [110, Theorem 1.1(A.2) on pp. 153-154, and Theorem 1.3 on pp. 157-158] for more precise information in this regard (including the location of these critical values, which are no longer as explicit as in the case of the Laplacian, and certain monotonicity properties with respect to the angle $\theta$ and the Lamé moduli). We shall revisit the case of the two-dimensional Lamé system in Sect. 4.5.

*Remark 4.18* In the context of Theorem 4.8, the operators in (4.309)–(4.311) continue to be invertible when the integrability exponent and the corresponding Muckenhoupt weight are permitted to vary while retaining control. More specifically, from Remark 4.13 and the proof of Theorem 4.8 it follows that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a threshold $\delta \in (0,1)$, which depends only on $n$, $I$, $W$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta \tag{4.327}$$

then the operators (4.309)–(4.311) are linear, bounded, and invertible for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

*Remark 4.19* The more general version of Theorem 4.8 from Remark 4.18 is in the nature of best possible, in the sense that the simultaneous invertibility result described in Remark 4.18 forces $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ to be small (relative to the other geometric characteristics of $\Omega$). To illustrate this, consider the case when $\Omega = \Omega_\theta$, an infinite sector in the plane with opening angle $\theta \in (0, 2\pi)$ (cf. (2.289)), and when $L = \Delta$, the two-dimensional Laplacian. We are interested in the geometric implications of having the operators $\pm\frac{1}{2}I + K_\Delta$ invertible on $L^p(\partial\Omega_\theta, \sigma_\theta)$ (where $\sigma_\theta := \mathcal{H}^1 \lfloor \partial\Omega_\theta$) for all $p$'s belonging to a compact sub-interval of $(1, \infty)$.

Specifically, suppose said operators are invertible whenever $p \in I_\eta := [1 + \eta, 2]$ for some fixed $\eta \in (0, 1)$. From (4.326) we see that this forces $\theta \neq \pi(2 - p)$ if $\theta \in (0, \pi)$ and $\theta \neq \pi p$ if $\theta \in (\pi, 2\pi)$. As $p$ swipes the interval $[1 + \eta, 2]$, the set of prohibited values for the aperture $\theta$ becomes $\big(0, (1-\eta)\pi\big] \cup \big[(1+\eta)\pi, 2\pi\big)$. Hence, we necessarily have $\theta \in \big((1 - \eta)\pi, (1 + \eta)\pi\big)$ which further entails

$$-\sin\big(\eta\tfrac{\pi}{2}\big) = \cos\big((1+\eta)\tfrac{\pi}{2}\big) < \cos(\theta/2) < \cos\big((1-\eta)\tfrac{\pi}{2}\big) = \sin\big(\eta\tfrac{\pi}{2}\big). \tag{4.328}$$

If $\nu$ denotes the outward unit normal vector to $\Omega_\theta$, then from (4.328) and (2.290) we conclude that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^2} = |\cos(\theta/2)| < \sin\big(\eta\tfrac{\pi}{2}\big) \longrightarrow 0^+ \ \text{ as } \ \eta \to 0^+. \tag{4.329}$$

This goes to show that, in general, the smallness of the BMO semi-norm of the geometric measure theoretic outward unit normal stipulated in (4.327) *cannot* be dispensed with, as far as the invertibility of the operator in (4.309) (in this case, with $z \in \{\pm\frac{1}{2}\}$, $L = \Delta$, $A$ the identity matrix, $M = 1$, and $w \equiv 1$) for each $p \in I_\eta$ is concerned.

The invertibility results from Theorem 4.8 may be further enhanced by allowing the coefficient tensor to be a small perturbation of any distinguished coefficient tensor of the given system. Concretely, by combining Theorem 4.7 with the continuity of the operator-valued assignments in (3.120)–(3.122), we obtain the following result.

**Theorem 4.9** *Retain the original background assumptions on the set $\Omega$ from Theorem 4.8 and, as before, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, 1)$. Consider $L \in \mathfrak{L}^{\mathrm{dis}}$ (cf.* (3.195)*) and pick an arbitrary $A_o \in \mathfrak{A}^{\mathrm{dis}}_L$. Then there exist some small threshold $\delta \in (0, 1)$ along with some open neighborhood $O$ of $A_o$ in $\mathfrak{A}_{\mathrm{WE}}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $A_o$, $\varepsilon$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for each $A \in O$ and each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$, the operators* (4.309)–(4.311) *are linear, bounded, and invertible.*

In the last portion of this section we discuss the issue of the compatibility of the inverses of the integral operators from Theorem 4.8 when simultaneously considered on different spaces. This requires that we briefly digress for the purpose of bringing in useful language and basic results of general functional analytic nature. Specifically, call two linear normed spaces, $X_0 = \big(X_0, \|\cdot\|_{X_0}\big)$ and $X_1 = \big(X_1, \|\cdot\|_{X_1}\big)$, `compatible` if there exists a Hausdorff topological vector space $\mathcal{X}$ such that

$$X_i \hookrightarrow \mathcal{X} \text{ continuously, } i \in \{0, 1\}. \tag{4.330}$$

Note that, in this scenario, we can talk about the algebraic sum $X_0 + X_1 (\subseteq \mathcal{X})$. This becomes a linear normed space when equipped with

$$\|x\|_{X_0+X_1} := \inf_{\substack{x=x_0+x_1\\ x_0\in X_0,\, x_1\in X_1}} \big(\|x_0\|_{X_0} + \|x_1\|_{X_1}\big), \qquad \forall\, x \in X_0 + X_1, \tag{4.331}$$

and $X_0 + X_1 \hookrightarrow \mathcal{X}$ continuously. Furthermore, $X_i \hookrightarrow X_0 + X_1$ continuously, for $i \in \{0, 1\}$. One may check that if $X_0, X_1$ are complete then so is $X_0 + X_1$ equipped with $\|\cdot\|_{X_0+X_1}$. Hence, $X_0+X_1$ turns out to be a Banach space if $X_0, X_1$ are Banach spaces to begin with.

We continue by recording two useful basic results of functional analytic nature. To state the first such result, suppose $X_0 = \big(X_0, \|\cdot\|_{X_0}\big)$ and $X_1 = \big(X_1, \|\cdot\|_{X_1}\big)$ on the one hand, and $Y_0 = \big(Y_0, \|\cdot\|_{Y_0}\big)$ and $Y_1 = \big(Y_1, \|\cdot\|_{Y_1}\big)$ on the other hand, are two pairs of compatible linear normed spaces. Then

having a linear mapping $T : X_0 + X_1 \to Y_0 + Y_1$ which satisfies $TX_i \subseteq Y_i$ for $i \in \{0, 1\}$ is equivalent to having two linear maps $T_i : X_i \to Y_i$ for $i \in \{0, 1\}$ that are compatible with one another, in the sense that $T_0\big|_{X_0\cap X_1} = T_1\big|_{X_0\cap X_1}$; in this case one has $\|T\|_{X_0+X_1\to Y_0+Y_1} \leq \max\big\{\|T_0\|_{X_0\to Y_0}, \|T_1\|_{X_1\to Y_1}\big\}$. (4.332)

To state our second result alluded to above, assume now that $X, Y$ are two Banach spaces with the property that $Y \subseteq X$. One may check without difficulty that

if $T : X \to X$ is a linear isomorphism with the property that $T(Y) \subseteq Y$ and $T\big|_Y : Y \to Y$ is also an isomorphism, then $T^{-1}(Y) \subseteq Y$ and $\big(T\big|_Y\big)^{-1} = T^{-1}\big|_Y$ as operators on $Y$. (4.333)

We are now ready to establish norm estimates for double layer operators acting on sums of Muckenhoupt weighted Lebesgue and Sobolev spaces.

**Proposition 4.1** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic*

*$M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$. Pick a coefficient tensor $A \in \mathfrak{A}_L^{\text{dis}}$ and consider the boundary-to-boundary double layer potential operators $K_A$, $K_A^{\#}$ associated with $\Omega$ and the coefficient tensor $A$ as in* (3.24) *and* (3.25)*, respectively. Finally, fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$.*

*Then for each $m \in \mathbb{N}$ there exists some constant $C \in (0,\infty)$ which depends only on $m$, $n$, $A$, $p_0$, $p_1$, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_A\|_{[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M\to[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.334}$$

$$\|K_A\|_{[L_1^{p_0}(\partial\Omega,w_0)+L_1^{p_1}(\partial\Omega,w_1)]^M\to[L_1^{p_0}(\partial\Omega,w_0)+L_1^{p_1}(\partial\Omega,w_1)]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.335}$$

$$\big\|K_A^{\#}\big\|_{[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M\to[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.336}$$

*Also, if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in* (4.334)*–*(4.336) *to depend only on said entities (i.e., $n$, $p$, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$) and $m$.*

***Proof*** This is a consequence of Theorems 4.7, (4.332), and 2.3. In the case of (4.334) and (4.336) take $X_i := Y_i := \big[L^{p_i}(\partial\Omega, w_i)\big]^M$ for $i \in \{0,1\}$, in which case (4.330) is satisfied if we choose $\mathcal{X} := \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ (cf. (2.575)). Finally, for the estimate claimed in (4.335), take $X_i := Y_i := \big[L_1^{p_i}(\partial\Omega, w_i)\big]^M$ for each $i \in \{0,1\}$, so now the inclusion in (4.330) holds if $\mathcal{X} := \Big[L_1^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ where

$$L_1^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) := \Big\{f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) : \partial_{\tau_{jk}} f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \text{ for each } j,k \in \{1,\dots,n\}\Big\}, \tag{4.337}$$

equipped with the norm

$$\|f\|_{L^1_1\left(\partial\Omega,\,\frac{\sigma(x)}{1+|x|^{n-1}}\right)} := \|f\|_{L^1\left(\partial\Omega,\,\frac{\sigma(x)}{1+|x|^{n-1}}\right)} + \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^1\left(\partial\Omega,\,\frac{\sigma(x)}{1+|x|^{n-1}}\right)} \tag{4.338}$$

for each $f \in L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$. □

Here are the compatibility results for the inverses of integral operators from Theorem 4.8 when simultaneously considered on different Muckenhoupt weighted Lebesgue and Sobolev spaces.

**Proposition 4.2** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}^{\rm dis}_L \neq \varnothing$. Pick $A \in \mathfrak{A}^{\rm dis}_L$ and consider the boundary-to-boundary double layer potential operators $K_A$, $K^\#_A$ associated with $\Omega$ and the coefficient tensor $A$ as in* (3.24) *and* (3.25)*, respectively. Finally, fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$, and some number $\varepsilon \in (0,1)$.*

*Then there exists some small threshold $\delta \in (0,1)$ which depends only on $n$, $p_0$, $p_1$, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, $A$, $\varepsilon$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following properties hold:*

*the operator $zI + K_A$ is invertible both as a mapping from the space $\big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself and also from the space $\big[L^{p_0}_1(\partial\Omega, w_0) + L^{p_1}_1(\partial\Omega, w_1)\big]^M$ onto itself;* (4.339)

*the operator $zI + K_A$ is invertible both as a mapping from $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ onto itself and also as a mapping from $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself, and the two inverses are in fact compatible with one another on the intersection;* (4.340)

*the operator $zI + K_A$ is invertible both as a mapping from $\big[L^{p_0}_1(\partial\Omega, w_0)\big]^M$ onto itself and also as a mapping from $\big[L^{p_1}_1(\partial\Omega, w_1)\big]^M$ onto itself, and the two inverses are in fact compatible with one another on the intersection;* (4.341)

*the operator $zI + K^\#_A$ is invertible both as a mapping from $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ onto itself and also as a mapping from $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself, and the two inverses are in fact compatible with one another on the intersection.* (4.342)

***Proof*** Bring in the constant $C$ appearing in estimates (4.334)–(4.336) (corresponding to $m := 1$), and denote by $t_\varepsilon \in (0, 1)$ the unique solution of the equation $t \cdot \ln(\mathrm{e}/t) = \varepsilon / \max\{C, 1\}$. Pick $\delta \in (0, t_\varepsilon)$ and, if necessary, further decrease $\delta$ as to insure that $\Omega$ is a UR domain, with the UR constants of $\partial\Omega$ controlled solely in terms of the dimension $n$ and the Ahlfors regularity constant of $\partial\Omega$ (cf. Theorem 2.3).

Then, via a Neumann series argument (much as in the proof of Theorem 4.8) it follows that $zI + K_A$ is invertible when considered from $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ onto itself, from $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself, from $\big[L^{p_0}(\partial\Omega, w_0)+L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself, and also from $\big[L_1^{p_0}(\partial\Omega, w_0) + L_1^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself. Invoking (4.333) with $X := \big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$ and with $Y$ either $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ or $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$, then proves that both the inverse of $zI + K_A$ on $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ and the inverse of $zI + K_A$ on $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$ arise as restrictions to these respective spaces of a common operator, namely the inverse of the operator $zI + K_A$ on the bigger space $\big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$. As such, they agree with one another so the conclusion in (4.340) follows. The claims in (4.341)–(4.342) are proved in a similar fashion. □

*Remark 4.20* Compatibility results similar in spirit to the ones proved in Proposition 4.2 are also valid for other spaces of interest. For example, in the context of Proposition 4.2, taking the threshold $\delta \in (0, 1)$ sufficiently small ensures that the operator $zI + K_A$ is invertible on the hybrid space $\big[L_1^{p_1;p_2}(\partial\Omega, w_1; w_2)\big]^M$ (cf. Remark 4.12) and its inverse continues to be compatible with the inverse of $zI + K_A$ on any other (a priori) given Muckenhoupt weighted Lebesgue space or Sobolev space on $\partial\Omega$. In this vein, we also claim that there exists some constant $C \in (0, \infty)$ with the property that

$$
\begin{aligned}
&\text{whenever } \ f \in \big[L_1^{p_1;p_2}(\partial\Omega, w_1; w_2)\big]^M \\
&\text{and } \ g := (zI + K_A)^{-1} f \in \big[L_1^{p_1;p_2}(\partial\Omega, w_1; w_2)\big]^M \\
&\text{then } \ \|\nabla_{\tan} g\|_{[L^{p_2}(\partial\Omega, w_2)]^{n\cdot M}} \leq C\, \|\nabla_{\tan} f\|_{[L^{p_2}(\partial\Omega, w_2)]^{n\cdot M}}\,.
\end{aligned}
\tag{4.343}
$$

To justify this, use (3.37) to write, for each $j, k \in \{1, \dots, n\}$,

$$
\begin{aligned}
\partial_{\tau_{jk}} f = \partial_{\tau_{jk}}\big[\big(zI + K_A\big)g\big] &= \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}(\nabla_{\tan} g) \\
&= \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}\Big(\big(\nu_r \partial_{\tau_{rs}} g_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}\Big)
\end{aligned}
\tag{4.344}
$$

at $\sigma$-a.e. point on $\partial\Omega$, where $\nu = (\nu_r)_{1\leq r\leq n}$ is the geometric measure theoretic outward unit normal to $\Omega$. Using the abbreviations

$$
\nabla_\tau f := \big(\partial_{\tau_{jk}} f\big)_{1\leq j,k\leq n}, \quad \nabla_\tau g := \big(\partial_{\tau_{jk}} g\big)_{1\leq j,k\leq n}, \tag{4.345}
$$

we find it convenient to recast the collection of all formulas as in (4.344), corresponding to all indices $j, k \in \{1, \dots, n\}$, simply as

$$\nabla_\tau f = \big(zI + R\big)(\nabla_\tau g), \tag{4.346}$$

where $I$ is the identity and $R$ is the operator acting from $\big[L^{p_2}(\partial\Omega, w_2)\big]^{M\cdot n^2}$ into itself according to

$$R := K_A + \Big(U_{jk} \circ \big(M_{\nu_r} \circ \pi^\alpha_{rs}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}\Big)_{1\leq j,k\leq n}. \tag{4.347}$$

Above, we let $K_A$ act on each $F = \big(F^\alpha_{rs}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq r,s\leq n}} \in \big[L^{p_2}(\partial\Omega, w_2)\big]^{M\cdot n^2}$ by setting

$$K_A F := \Big(K_A\big(F^\alpha_{rs}\big)_{1\leq\alpha\leq M}\Big)_{1\leq r,s\leq n}. \tag{4.348}$$

Also recall that, much as in the past, each $M_{\nu_r}$ denotes the operator of pointwise multiplication by $\nu_r$, the $r$-th scalar component of $\nu$. Finally, in (4.347) we let each $\pi^\alpha_{rs}$ be the "coordinate-projection" operator which acts as $\pi^\alpha_{rs}(X) := X^\alpha_{rs}$ for every $X = \big(X^\alpha_{rs}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq r,s\leq n}} \in \mathbb{C}^{M\cdot n^2}$. From (4.347), (4.299), (3.35), Theorem 4.3, and (3.81), we then conclude that

$$\|R\|_{[L^{p_2}(\partial\Omega,w_2)]^{M\cdot n^2}\to[L^{p_2}(\partial\Omega,w_2)]^{M\cdot n^2}} \leq C\|\nu\|^{\langle 1\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \tag{4.349}$$

for some $C \in (0,\infty)$ which depends only on $n$, $A$, $p_2$, $[w_2]_{A_{p_2}}$, and the Ahlfors regularity constant of $\partial\Omega$. As a consequence of this, if we assume $\delta > 0$ to be sufficiently small to begin with, a Neumann series argument gives that

$$zI + R \;\text{ is invertible on }\; \big[L^{p_2}(\partial\Omega, w_2)\big]^{M\cdot n^2} \tag{4.350}$$

and provides an estimate for the norm of the inverse. At this stage, the estimate claimed in (4.343) follows from (4.346), (4.350), (4.345), and (2.585)–(2.586).

We conclude this section by proving estimates for the operator norm of the modified boundary-to-boundary double layer operator acting on homogeneous Muckenhoupt weighted Sobolev spaces in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain, complementing results in Theorem 4.7.

**Theorem 4.10** *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided NTA domain whose boundary is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}^{\rm dis}_L \neq \varnothing$. Pick $A \in \mathfrak{A}^{\rm dis}_L$ and consider the modified* *boundary-to-boundary double layer potential operator* $\left[K_{A,mod}\right]$ *associated with* $\Omega$ *and the coefficient tensor* $A$ *as in* (3.142)*. Finally, fix an integrability exponent* $p \in (1, \infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$*.*

*Then for each* $m \in \mathbb{N}$ *there exists some* $C_m \in (0, \infty)$ *which depends only on* $m$*,* $n$*,* $A$*,* $p$*,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$*, such that*

$$\big\|\left[K_{A,mod}\right]\big\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M \to [\dot{L}^p_1(\partial\Omega,w)/\sim]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.351}$$

*Furthermore, the above result is optimal in the sense that, given any* $A \in \mathfrak{A}_L$*, having* (4.351) *valid for every half-space in* $\mathbb{R}^n$ *implies that actually* $A \in \mathfrak{A}^{\mathrm{dis}}_L$*.*

***Proof*** From (2.88) we know that $\Omega$ satisfies a two-sided local John condition. Pick an arbitrary function $f \in \left[\dot{L}^p_1(\partial\Omega, w)\right]^M$. In particular, from (2.598) and (2.576) we see that

$$f \in \left[L^q_{\mathrm{loc}}(\partial\Omega, \sigma)\right]^M \ \text{ for some } \ q \in (1, \infty). \tag{4.352}$$

Keeping this in mind, we may rely on (3.142), Propositions 2.26, 3.3, (4.299), Theorem 4.3, and (4.98) to write, for each given $m \in \mathbb{N}$,

$$\begin{aligned}
\big\|\left[K_{A,\mathrm{mod}}\right][f]\big\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M} &= \big\|\left[K_{A,\mathrm{mod}} f\right]\big\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M} \\
&= \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}}(K_{A,\mathrm{mod}} f)\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\leq \sum_{j,k=1}^{n} \Big(\big\|K_A(\partial_{\tau_{jk}} f)\big\|_{[L^p(\partial\Omega,w)]^M} + \big\|U_{jk}(\nabla_{\tan} f)\big\|_{[L^p(\partial\Omega,w)]^M}\Big) \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{nM}} \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|[f]\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M},
\end{aligned} \tag{4.353}$$

bearing in mind that each $U_{jk}$ is a sum of operators of commutator type (cf. (3.35)).

There remain to address the optimality claim made in the last portion of the statement of the theorem. To this end, suppose $A \in \mathfrak{A}_L$ is such that (4.351) is valid in every half-space $\Omega$ in $\mathbb{R}^n$. In view of the fact that the BMO semi-norm of the normal vanishes in such cases, this amounts to having the modified boundary-to-boundary double layer operator $K_{A,\mathrm{mod}}$ map each function from $\left[\mathscr{C}^\infty_c(\partial\Omega)\right]^M$ into a constant in $\mathbb{C}^M$. Granted this, the implication *(iii') ⇒ (i)* in Proposition 3.9 gives that actually $A \in \mathfrak{A}_L^{\rm dis}$. □

## 4.4 Invertibility on Muckenhoupt Weighted Homogeneous Sobolev Spaces

Earlier in (3.132), we have considered the boundary-to-boundary single layer operator $\big[S_{\rm mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \to \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M$. Its invertibility properties are going to be of basic importance in the context of boundary value problems for the system $L$ in $\Omega$. For example, under suitable geometric assumptions on the set $\Omega$, if $\big[S_{\rm mod}\big]$ is injective then the Homogeneous Regularity Problem for $L$ in $\Omega$ has at most one solution, and if $\big[S_{\rm mod}\big]$ is surjective then the Homogeneous Regularity Problem for $L$ in $\Omega$ is solvable. In particular, having $\big[S_{\rm mod}\big]$ bijective guarantees the well-posedness of the Homogeneous Regularity Problem for $L$ in $\Omega$. Lemma 4.3 and Proposition 4.3 elaborate on this topic.

**Lemma 4.3** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture parameter $\kappa > 0$, an integrability exponent $p \in (1,\infty)$, and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Also, suppose $L$ is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider the Homogeneous Regularity Problem for $L$ in $\Omega$, with boundary data prescribed in homogeneous Muckenhoupt weighted Sobolev spaces, i.e.,*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} = f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M, \end{cases} \tag{4.354}$$

*where $\dot{L}^p_1(\partial\Omega, w)$ is the homogeneous Muckenhoupt weighted boundary Sobolev space defined in* (2.598)*. Also, consider the operator (cf.* (3.132)*)*

$$\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M. \tag{4.355}$$

*Then the following statements are true:*

*(a) If $\big[S_{mod}\big]$ as in* (4.355) *is surjective then the Homogeneous Regularity Problem* (4.354) *has a solution.*

*(b) If $\Omega$ is actually an NTA domain with an unbounded Ahlfors regular boundary and if $\big[S_{mod}\big]$ as in* (4.355) *is injective then the Homogeneous Regularity Problem* (4.354) *has at most one solution modulo constants.*

***Proof*** Suppose the operator in (4.355) is surjective and let $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ be arbitrary. Then there exists $g \in \big[L^p(\partial\Omega, w)\big]^M$ such that $S_{\text{mod}}g = f + c$ for some $c \in \mathbb{C}^M$. If we now define $u := \mathscr{S}_{\text{mod}}g - c$ then item *(c)* in Proposition 3.5, (3.47), and (2.575) imply that this is a solution of (4.354) for the boundary datum $f$.

To deal with the claim in item *(b)*, strengthen the original hypotheses on $\Omega$ by assuming now that $\Omega$ is actually an NTA domain with an unbounded Ahlfors regular boundary (in particular, $\Omega$ is connected; see (2.65)). Also, suppose $\big[S_{\text{mod}}\big]$ defined as in (4.355) is an injective operator. To proceed, denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$ and pick an arbitrary coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}} \in \mathfrak{A}_L$. Let $u$ be a solution of (4.354) corresponding to $f := c \in \mathbb{C}^M$. From the current assumptions and the Fatou-type result recalled in Theorem 3.4 (whose present applicability is ensured by (2.576)) we conclude that

$$\text{the trace } (\nabla u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exists and belongs to } \big[L^p(\partial\Omega, w)\big]^{M\times n}. \tag{4.356}$$

In view of this and (3.66), the conormal derivative

$$\begin{aligned} \partial^A_\nu u := \Big(\nu_r a^{\alpha\beta}_{rs}\big(\partial_s u_\beta\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq M} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \\ \text{and belongs to } \big[L^p(\partial\Omega, w)\big]^M. \end{aligned} \tag{4.357}$$

Based on (4.354), (2.575), (3.54), Proposition 2.24, the fact that $u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = c$, the integral representation formula (3.69), and the fact that we are presently assuming that $\Omega$ is connected, we may write

$$u = -\mathscr{S}_{\text{mod}}\big(\partial^A_\nu u\big) + c_u \text{ in } \Omega, \tag{4.358}$$

for some constant $c_u \in \mathbb{C}^M$ (depending on $u$). By taking the nontangential trace to the boundary (recall (3.47)) the latter implies $c = -S_{\text{mod}}\big(\partial^A_\nu u\big) + c_u$, hence

$$\big[S_{\text{mod}}\big]\big(\partial^A_\nu u\big) = 0. \tag{4.359}$$

Since $\partial^A_\nu u \in \big[L^p(\partial\Omega, w)\big]^M$ and since we are assuming that the operator $\big[S_{\text{mod}}\big]$ is injective in the context of (4.355), this forces $\partial^A_\nu u = 0$. When used back in (4.358), this proves that $u$ is constant in $\Omega$. The claim in *(b)* is therefore established. □

Our next result builds on Lemma 4.3 by establishing a two-way street between invertibility of the single layer potential operator and the well-posedness of the Homogeneous Regularity Problem.

**Proposition 4.3** *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided NTA domain with an unbounded Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture* *parameter* $\kappa > 0$, *an integrability exponent* $p \in (1, \infty)$, *and a Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$. *Next, assume L is a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *and denote by* $(\mathrm{HRP}_+)$ *and* $(\mathrm{HRP}_-)$ *the Homogeneous Regularity Problems formulated as in* (4.354) *corresponding to* $\Omega_+ := \Omega$ *and to* $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$, *respectively. Finally, recall the operator* $\big[S_{mod}\big]$ *from* (4.355). *Then the following statements are true:*

*(a) The operator* $\big[S_{mod}\big]$ *is injective in the context of* (4.355) *if and only if* $(\mathrm{HRP}_+)$ *and* $(\mathrm{HRP}_-)$ *have at most one solution modulo constants.*
*(b) The operator* $\big[S_{mod}\big]$ *is surjective in the context of* (4.355) *if and only if* $(\mathrm{HRP}_+)$ *and* $(\mathrm{HRP}_-)$ *have a solution.*
*(c) The operator* $\big[S_{mod}\big]$ *is an isomorphism in the context of* (4.355) *if and only if* $(\mathrm{HRP}_+)$ *and* $(\mathrm{HRP}_-)$ *are well-posed.*

***Proof*** Suppose $(\mathrm{HRP}_+)$ and $(\mathrm{HRP}_-)$ have at most one solution modulo constants and let $f \in \big[L^p(\partial\Omega, w)\big]^M$ be such that $S_{\rm mod} f = c \in \mathbb{C}^M$. Then $u^+ := \mathscr{S}_{\rm mod} f$ in $\Omega_+$ and $u^- := \mathscr{S}_{\rm mod} f$ in $\Omega_-$ solve $(\mathrm{HRP}_+)$ and $(\mathrm{HRP}_-)$, respectively, for the boundary datum $c$ (see item *(c)* in Proposition 3.5, (3.47), and (2.575)). In view of the current working hypothesis, this forces $u^\pm$ to be constant functions in $\Omega_\pm$. Picking $A \in \mathfrak{A}_L$ and invoking (3.126) as well as (6.191)–(6.192), we obtain that $f = \partial_\nu^A u^- - \partial_\nu^A u^+ = 0$, where the last equality is implied by the fact that the functions $u^\pm$ are constant in $\Omega_\pm$ and (3.66). Hence, $\big[S_{\rm mod}\big]$ is injective in the context of (4.355). The converse implication stated in *(a)* is a consequence of item *(b)* in Lemma 4.3 (used both for $\Omega_+$ and $\Omega_-$).

Moving on to the claim made in item *(b)*, suppose $(\mathrm{HRP}_+)$ and $(\mathrm{HRP}_-)$ are solvable and pick $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ arbitrary. Denote by $u^+$ and $u^-$ a solution of $(\mathrm{HRP}_+)$ and of $(\mathrm{HRP}_-)$, respectively, for the boundary datum $f$. Also, fix a coefficient tensor $A \in \mathfrak{A}_L$. Collectively, the current assumptions, the Fatou-type result recalled in Theorem 3.4 (whose present applicability is ensured by (2.576)), (2.575), and Proposition 2.24 guarantee that the integral representation formula (3.69) holds both for $u^+$ in $\Omega_+$ and for $u^-$ in $\Omega_-$. Specifically,

$$
\begin{aligned}
u^+ &= \mathcal{D}_{A,\rm mod}\big(u^+\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) - \mathscr{S}_{\rm mod}\big(\partial_\nu^A u^+\big) + c_+ \ \text{ in } \ \Omega_+,\\
u^- &= -\mathcal{D}_{A,\rm mod}\big(u^-\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) + \mathscr{S}_{\rm mod}\big(\partial_\nu^A u^-\big) + c_- \ \text{ in } \ \Omega_-,
\end{aligned}
\tag{4.360}
$$

for some constants $c_\pm \in \mathbb{C}^M$ (keep in mind that both $\Omega_+$ and $\Omega_-$ are connected; cf. (2.65)). Taking nontangential boundary traces in (4.360) yields

$$
\begin{aligned}
f &= \big(\tfrac{1}{2}I + K_{A,\rm mod}\big)f - S_{\rm mod}\big(\partial_\nu^A u^+\big) + c_+ \ \text{ on } \ \partial\Omega,\\
f &= -\big(-\tfrac{1}{2}I + K_{A,\rm mod}\big)f + S_{\rm mod}\big(\partial_\nu^A u^-\big) + c_- \ \text{ on } \ \partial\Omega,
\end{aligned}
\tag{4.361}
$$

on account of (3.61) and (3.47). After adding the two equalities in (4.361) we arrive at

$$f = S_{\rm mod}\big( - \partial_\nu^A u^+ + \partial_\nu^A u^-\big) + c_+ + c_- \ \text{ on } \ \partial\Omega, \tag{4.362}$$

hence $[f] = \big[S_{\rm mod}\big]\big( - \partial_\nu^A u^+ + \partial_\nu^A u^-\big)$. The latter proves that the operator $S_{\rm mod}$ is surjective in the context of (4.355), since $-\partial_\nu^A u^+ + \partial_\nu^A u^- \in \big[L^p(\partial\Omega, w)\big]^M$. The converse implication stated in *(b)* is a consequence of Lemma 4.3 (used both for $\Omega_+$ and $\Omega_-$). Finally, the claim in item *(c)* follows from *(a)*-*(b)*, so the proof of the proposition is complete. □

We next turn our attention to the issue of invertibility (or lack thereof) for the operator $\big[S_{\rm mod}\big]$ in the context of (4.355). We begin with the following proposition, which offers an example of the failure of the operator (4.355) to be Fredholm (in every single respect: $\big[S_{\rm mod}\big]$ has an infinite dimensional kernel, as well as an infinite dimensional cokernel) even when the underlying domain is a half-space and when the system involved is symmetric. As we shall see a little later, in Theorem 4.11, the source of this failure is the lack of a distinguished coefficient tensor for said system.

**Proposition 4.4** *Consider the second-order $n \times n$ system $L_D := \Delta - 2\nabla{\rm div}$ in $\mathbb{R}^n$ with $n \geq 2$. Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. Then the single layer potential operator $\big[S_{mod}\big]$, associated as in* (3.42) *with the system $L_D$ and the domain $\Omega := \mathbb{R}^n_+$, acting in the context*

$$\big[S_{mod}\big] : \big[L^p(\mathbb{R}^{n-1}, w)\big]^n \longrightarrow \big[\dot{L}^p_1(\mathbb{R}^{n-1}, w)\big/\sim\big]^n \tag{4.363}$$

*has an infinite dimensional kernel and an infinite dimensional cokernel.*

***Proof*** Denote by $\mathrm{Ker}\big(\mathrm{HRP}_{L_D}\big)$ the space of null-solutions of the Homogeneous Regularity Problem for the system $L_D$ in the upper half-space, i.e., the space of functions $u$ satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \\ L_D u = 0 \ \text{ in } \ \mathbb{R}^n_+, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}^{n-1}, w), \\ u\big|^{\kappa-\mathrm{n.t.}}_{\mathbb{R}^{n-1}} = 0. \end{cases} \tag{4.364}$$

Also, denote by $\mathrm{Ker}[S_{\rm mod}]$ the kernel of the operator (4.363) and fix a coefficient tensor $A \in \mathfrak{A}_{L_D}$. Then, as seen from the proof of part *(b)* in Lemma 4.3 (see the reasoning leading up to (4.359)), the mapping

$$\mathrm{Ker}\big(\mathrm{HRP}_{L_D}\big) \ni u \longmapsto \partial_\nu^A u \in \mathrm{Ker}[S_{\rm mod}] \tag{4.365}$$

is well defined and injective. Being also linear, this entails

$$\dim\big(\mathrm{Ker}[S_{\rm mod}]\big) \geq \dim\big(\mathrm{Ker}\big(\mathrm{HRP}_{L_D}\big)\big). \tag{4.366}$$

The later when combined with (3.391) shows that $\dim\big(\mathrm{Ker}[S_{\text{mod}}]\big) = +\infty$.

Also, much as in the proof of item *(a)* in Lemma 4.3, from item *(c)* in Proposition 3.5, (3.47), and (2.575) we see that $\mathrm{Im}[S_{\text{mod}}]$, the image of the operator (4.363), is a subspace of

$$\Big\{ u\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, L_D u = 0 \text{ in } \mathbb{R}^n_+, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}^{n-1}, w)\Big\}. \tag{4.367}$$

Recalling (3.385), this proves that $\dim\big(\mathrm{CoKer}[S_{\text{mod}}]\big) = +\infty$, where $\mathrm{CoKer}[S_{\text{mod}}]$ denotes the cokernel of the operator (4.363). □

We now turn our attention to the issue of identifying concrete algebraic and geometric conditions guaranteeing the injectivity, surjectivity, and the eventual invertibility of the modified single layer potential operator in the context of (3.132).

**Theorem 4.11** *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider the modified boundary-to-boundary single layer potential operator $S_{mod}$ associated with $\Omega$ and the system $L$ as in* (3.42)*. Fix some exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$.*

*Finally, recall that $\big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ denotes the $M$-th power of the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, equipped with the semi-norm defined in* (2.601) *and, additionally, recall the operator $\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \to \big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ defined as in* (3.132)*. In relation to this, the following statements are valid.*

*(1) [Surjectivity] Whenever $\mathfrak{A}^{\text{dis}}_L \neq \varnothing$, there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that* (2.601) *is a genuine norm and the operator* (3.132) *is surjective.*

*(2) [Injectivity] Whenever $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$, there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that the operator* (3.132) *is injective.*

*(3) [Isomorphism] Whenever both $\mathfrak{A}^{\text{dis}}_L \neq \varnothing$ and $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$, there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that $\big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ is a Banach space when equipped with the norm* (2.601) *and the operator* (3.132) *is an isomorphism.*

*(4) [Optimality] If $\mathfrak{A}^{\text{dis}}_L = \varnothing$ then the operator* (3.132) *may fail to be surjective (in fact, may have an infinite dimensional cokernel) even in the case when $\Omega$ is a*

*half-space, and if* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$ *then the operator* (3.132) *may fail to be injective (in fact, may have an infinite dimensional kernel) even in the case when* $\Omega$ *is a half-space.*

We wish to note that, corresponding to the case when $\Omega$ is the upper-graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, the operator $L$ is the Laplacian $\Delta$ in $\mathbb{R}^n$ (hence, $M = 1$), and for the integrability exponent $p = 2$, the invertibility of the harmonic single layer has been treated in [35, Lemma 3.1, p. 451] using Rellich estimates.

***Proof of Theorem 4.11*** To deal with item *(1)*, assume $\mathfrak{A}^{\mathrm{dis}}_{L} \neq \varnothing$. Pick a coefficient tensor $A \in \mathfrak{A}^{\mathrm{dis}}_{L}$ then select some threshold $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a condition which we shall henceforth assume) then

$$\Omega \text{ is a two-sided NTA domain with an unbounded boundary,} \tag{4.368}$$

and

$$\text{the operators } \pm\tfrac{1}{2}I + K_A \text{ are invertible on } \big[L^p_1(\partial\Omega, w)\big]^M. \tag{4.369}$$

Theorem 2.3 together with Theorems 2.4 and 4.8 ensure that this is indeed possible. To proceed, choose a scalar-valued function $\phi \in \mathscr{C}^\infty_0(\mathbb{R}^n)$ with $\phi \equiv 1$ on $B(0,1)$ and $\operatorname{supp}\phi \subseteq B(0,2)$. Having fixed a reference point $x_0 \in \partial\Omega$, for each scale $r \in (0,\infty)$ define

$$\phi_r(x) := \phi\Big(\frac{x - x_0}{r}\Big) \quad \text{for each} \quad x \in \mathbb{R}^n, \tag{4.370}$$

and use the same notation to denote the restriction of $\phi_r$ to $\partial\Omega$. Suppose now some arbitrary function $g \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ has been given. Hence, from (2.598) we have

$$\begin{aligned} & g \in \big[L^p_{\mathrm{loc}}(\partial\Omega, w) \cap L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^M \quad \text{and} \\ & \partial_{\tau_{jk}} g \in \big[L^p(\partial\Omega, w)\big]^M \quad \text{for} \quad 1 \leq j,k \leq n. \end{aligned} \tag{4.371}$$

For each $r \in (0,\infty)$ set $\Delta_r := \partial\Omega \cap B(x_0, r)$ and define $g_{\Delta_r} := \fint_{\Delta_r} g \, d\sigma \in \mathbb{C}^M$ then set

$$g_r := \phi_r \cdot \big(g - g_{\Delta_{2r}}\big) \quad \text{on} \quad \partial\Omega. \tag{4.372}$$

From Proposition 2.25 (whose applicability in the current setting is ensured by (4.368) and (4.371)) we know that there exists $C = C(\Omega, p, w, x_0) \in (0,\infty)$, independent of the function $g$, with the property that

$$\sup_{r>0} \frac{1}{r}\Big(\int_{\Delta_r} \big|g - g_{\Delta_r}\big|^p \, dw\Big)^{1/p} \leq C \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} g\big\|_{[L^p(\partial\Omega, w)]^M}. \tag{4.373}$$

Also, from (4.371)–(4.372) we see that for each radius $r \in (0, \infty)$ and all indices $j, k \in \{1, \dots, n\}$ we have

$$g_r \in \big[L^p_1(\partial\Omega, w)\big]^M \quad \text{and} \quad \partial_{\tau_{jk}} g_r = \big(\partial_{\tau_{jk}}\phi_r\big) \cdot \big(g - g_{\Delta_{2r}}\big) + \phi_r \cdot \partial_{\tau_{jk}} g. \tag{4.374}$$

Since there exists a constant $C \in (0, \infty)$ such that for each $j, k \in \{1, \dots, n\}$ and each $r \in (0, \infty)$ we have

$$\operatorname{supp}\big(\partial_{\tau_{jk}}\phi_r\big) \subseteq \Delta_{2r} \quad \text{and} \quad \big|\partial_{\tau_{jk}}\phi_r\big| \leq C/r \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \tag{4.375}$$

it follows that for each $j, k \in \{1, \dots, n\}$ and each $r \in (0, \infty)$ we may estimate, making use of the version of the Poincaré inequality recorded in (4.373),

$$\begin{aligned} \Big\|\big(\partial_{\tau_{jk}}\phi_r\big) \cdot \big(g - g_{\Delta_{2r}}\big)\Big\|_{[L^p(\partial\Omega, w)]^M} &\leq C r^{-1}\Big(\int_{\Delta_{2r}} \big|g - g_{\Delta_{2r}}\big|^p \, dw\Big)^{1/p} \\ &\leq C \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} g\big\|_{[L^p(\partial\Omega, w)]^M}, \end{aligned} \tag{4.376}$$

for some constant $C \in (0, \infty)$ independent of $g$ and $r$. In turn, from (4.374), (4.376), (2.585)–(2.586), and (2.576) we conclude that

$$\big\|\nabla_{\tan} g_r\big\|_{[L^p(\partial\Omega, w)]^{n \cdot M}} \leq C \big\|\nabla_{\tan} g\big\|_{[L^p(\partial\Omega, w)]^{n \cdot M}} \tag{4.377}$$

for some $C \in (0, \infty)$ independent of $g$ and $r$. If for each $r \in (0, \infty)$ we now define

$$h_r := \big(\tfrac{1}{2}I + K_A\big)^{-1}\big(-\tfrac{1}{2}I + K_A\big)^{-1} g_r \tag{4.378}$$

then from the membership in (4.374) and the invertibility results in (4.369) it follows that $h_r$ is a meaningfully defined function which belongs to $\big[L^p_1(\partial\Omega, w)\big]^M$. In addition, from (4.378), (4.343), and (4.377) we conclude that there exists a constant $C \in (0, \infty)$, independent of $g$, such that

$$\|\nabla_{\tan} h_r\|_{[L^p(\partial\Omega, w)]^{n \cdot M}} \leq C \,\|\nabla_{\tan} g\|_{[L^p(\partial\Omega, w)]^{n \cdot M}} \quad \text{for each} \quad r \in (0, \infty). \tag{4.379}$$

Going further, for each $r \in (0, \infty)$ define

$$f_r := \partial^A_\nu\big(\mathcal{D}_A h_r\big) \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{4.380}$$

Since $h_r \in \big[L^p_1(\partial\Omega, w)\big]^M$, the boundedness result recorded in (3.115) together with (4.379) imply that $f_r \in \big[L^p(\partial\Omega, w)\big]^M$ and for each $r \in (0, \infty)$ we have

$$\|f_r\|_{[L^p(\partial\Omega,w)]^M} \le C\|\nabla_{\tan} h_r\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \le C\,\|\nabla_{\tan} g\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}\,, \tag{4.381}$$

where $C \in (0,\infty)$ is independent of $g$ and $r$. Collectively, (3.130), (4.378), (4.380), and Theorem 2.4 also ensure that for each $r \in (0,\infty)$ there exists some constant $c_r \in \mathbb{C}^M$ such that

$$S_{\rm mod} f_r = g_r + c_r \quad \text{on} \ \ \partial\Omega. \tag{4.382}$$

Select now a sequence $\{r_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ which converges to infinity. Since from (4.381) we know that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\big[L^p(\partial\Omega,w)\big]^M$, we may rely on the Banach–Alaoglu Theorem to assume, without loss of generality, that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is actually weak-$*$ convergent to some $f \in \big[L^p(\partial\Omega,w)\big]^M$. On account of (3.46), (4.382), and (4.372), for each test function $\psi \in \big[\mathrm{Lip}(\partial\Omega)\big]^M$ with compact support we may write

$$\begin{aligned}
\int_{\partial\Omega}\big\langle S_{\rm mod} f,\psi\big\rangle\,d\sigma &= \lim_{j\to\infty}\int_{\partial\Omega}\big\langle S_{\rm mod} f_{r_j},\psi\big\rangle\,d\sigma = \lim_{j\to\infty}\int_{\partial\Omega}\big\langle g_{r_j}+c_{r_j},\psi\big\rangle\,d\sigma\\
&= \lim_{j\to\infty}\int_{\partial\Omega}\big\langle \phi_{r_j}\cdot\big(g-g_{\Delta_{2r_j}}\big)+c_{r_j},\psi\big\rangle\,d\sigma\\
&= \lim_{j\to\infty}\int_{\partial\Omega}\big\langle g-g_{\Delta_{2r_j}}+c_{r_j},\psi\big\rangle\,d\sigma\\
&= \int_{\partial\Omega}\langle g,\psi\rangle\,d\sigma + \lim_{j\to\infty}\Big\langle c_{r_j}-g_{\Delta_{2r_j}},\int_{\partial\Omega}\psi\,d\sigma\Big\rangle.
\end{aligned} \tag{4.383}$$

In view of the arbitrariness of $\psi$, this forces the sequence $\big\{c_{r_j}-g_{\Delta_{2r_j}}\big\}_{j\in\mathbb{N}} \subseteq \mathbb{C}^M$ to converge to some constant $c \in \mathbb{C}^M$. Bearing this in mind, we may then conclude from (4.383) that

$$\int_{\partial\Omega}\big\langle S_{\rm mod} f,\psi\big\rangle\,d\sigma = \int_{\partial\Omega}\langle g+c,\psi\rangle\,d\sigma \tag{4.384}$$

for each function $\psi \in \big[\mathrm{Lip}(\partial\Omega)\big]^M$ with compact support. Ultimately, from (4.384) and (2.578) we obtain

$$S_{\rm mod} f = g + c \quad \text{at } \sigma\text{-a.e. point on} \ \ \partial\Omega. \tag{4.385}$$

Hence, $\big[S_{\rm mod}\big]f = \big[S_{\rm mod} f\big] = [g]$ and since $[g] \in \big[\dot{L}^p(\partial\Omega,w)/\sim\big]^M$ is arbitrary, it follows that the operator (3.132) is surjective. Moreover, from (4.381) we see that

$$\|f\|_{[L^p(\partial\Omega,w)]^M} \leq \limsup_{j\to\infty} \|f_{r_j}\|_{[L^p(\partial\Omega,w)]^M} \leq C\,\|\nabla_{\tan} g\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}$$

$$\leq C\,\|[g]\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M}, \tag{4.386}$$

for some constant $C \in (0,\infty)$ independent of $g$, so the surjectivity of (3.132) comes with quantitative control.

Let us also observe that the fact that (2.601) is, as claimed, a genuine norm is clear from (4.368) and Proposition 2.26.

Moving on, we treat item *(2)*, now working under the assumption that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Select a coefficient tensor $\widetilde{A} \in \mathfrak{A}_L$ such that $\widetilde{A}^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$, then choose $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (something we shall henceforth assume) then

$$\text{the operators } \pm\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top} \text{ are invertible on } \big[L^p(\partial\Omega,w)\big]^M. \tag{4.387}$$

That this is indeed possible is guaranteed by Theorem 4.8. The goal is to show that the operator (3.132) is injective. To this end, suppose $f \in \big[L^p(\partial\Omega,w)\big]^M$ is such that $\big[S_{\rm mod}\big]f = [0]$. Hence, $\big[S_{\rm mod}f\big] = [0]$ which implies that there exists some constant $c \in \mathbb{C}^M$ for which

$$S_{\rm mod}f = c \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{4.388}$$

In concert with (3.129), this further implies

$$\big(\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top}\big)f\Big) = 0 \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega \tag{4.389}$$

which, in view of (4.387), forces $f = 0$. Since the operator (3.132) is linear, it follows that this is indeed injective.

As far as the claims in item *(3)* are concerned, assume that $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Results established earlier then guarantee that the operator (3.132) is a continuous bijection. Since $\big[\dot{L}^p_1(\partial\Omega,w)\big/\sim\big]^M$ is a Banach space (cf. Proposition 2.26 and (4.368)) it follows that the operator (3.132) is a linear isomorphism.

Finally, the claims in item *(4)* are clear from Proposition 4.4 and (3.406). The proof of Theorem 4.11 is therefore complete. □

Here is a useful variant of Theorem 4.11:

*Remark 4.21* Let $\Omega$, $L$, be as in Theorem 4.11 and assume $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$. Fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$. From (4.341) and the proof of Theorem 4.11 (cf. (4.378)) it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on $n$, $p_0$, $p_1$, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for every given function $g$ in $\big[\dot{L}_1^{p_0}(\partial\Omega, w_0) \cap \dot{L}_1^{p_1}(\partial\Omega, w_1)\big]^M$ there exist some function $f \in \big[L^{p_0}(\partial\Omega, w_0) \cap L^{p_1}(\partial\Omega, w_1)\big]^M$ and a constant $c \in \mathbb{C}^M$ such that $S_{\rm mod} f = g + c$.

As a consequence of Theorem 4.10 we shall prove the invertibility result contained in the next theorem, for modified boundary-to-boundary double layer operators associated with weakly elliptic systems possessing a distinguished coefficient tensor acting on homogeneous Muckenhoupt weighted Sobolev spaces on the boundary of sufficiently flat Ahlfors regular domains. Moreover, we show that this is optimal in the sense that in the absence of a distinguished coefficient tensor the modified boundary-to-boundary double layer operator may actually have an infinite dimensional cokernel, even when the underlying domain is a half-space.

**Theorem 4.12** *Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain. Denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the modified boundary-to-boundary double layer potential operator $\big[K_{A,mod}\big]$ associated with $\Omega$ and the coefficient tensor $A$ as in* (3.142). *Finally, fix an integrability exponent $p \in (1,\infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and some number $\varepsilon \in (0,\infty)$.*

*Then there exists some small threshold $\delta \in (0,1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $A$, $\varepsilon$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the operator*

$$zI + \big[K_{A,mod}\big] : \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \tag{4.390}$$

*is invertible. Moreover, this conclusion may fail when $\mathfrak{A}_L^{\rm dis} = \varnothing$ even when $\Omega$ is a half-space (in fact, in such a scenario it may happen that $\frac{1}{2}I + \big[K_{A,mod}\big]$ has an infinite dimensional cokernel when acting on the space $\big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M$).*

***Proof*** Theorems 2.3 and 2.4 imply that there exists some threshold $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then $\Omega$ is a two-sided NTA domain with an unbounded boundary. Granted this, the desired invertibility result pertaining to the operator (4.390) follows from Theorem 4.10, via a Neumann series argument.

In addition, from (3.133)–(3.134), (3.385), and (3.406) we conclude that the operator $\frac{1}{2}I + \big[K_{A,\rm mod}\big]$ associated with the $n \times n$ system $L_D$ defined in (3.371) and the set $\Omega := \mathbb{R}^n_+$ has an infinite dimensional cokernel when acting on the space $\big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^n$. □

Here is another useful version of Theorem 4.12:

*Remark 4.22* Let $\Omega$, $L$, be as in Theorem 4.12 and assume $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega, \sigma)$ and $w_1 \in A_{p_1}(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, 1)$. From the proof of Theorem 4.12 (which produces a Neumann series representation for the inverse) we see that there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p_0$, $p_1$, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, $L$, $\varepsilon$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ it follows that

$$\begin{aligned}&\text{the operator } zI + \big[K_{A,\text{mod}}\big] \text{ is invertible both as a mapping}\\ &\text{from } \big[\dot{L}_1^{p_0}(\partial\Omega, w_0)/\sim\big]^M \text{ onto itself and also as a mapping}\\ &\text{from } \big[\dot{L}_1^{p_1}(\partial\Omega, w_1)/\sim\big]^M \text{ onto itself, and the two inverses}\\ &\text{are in fact compatible with one another on the intersection.}\end{aligned} \tag{4.391}$$

See the proof of Proposition 4.2 for details in similar circumstances.

We next discuss invertibility results for the conormal of the double layer operator acting from homogeneous Muckenhoupt weighted Sobolev spaces.

**Theorem 4.13** *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Fix some exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Pick some coefficient tensor $A \in \mathfrak{A}_L$ and consider the modified conormal derivative of the modified double layer operator in the context of* (3.138)*, i.e.,*

$$\begin{aligned}&\big[\partial_\nu^A \mathcal{D}_{A,mod}\big] : \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M \ \textit{ defined as}\\ &\big[\partial_\nu^A \mathcal{D}_{A,mod}\big][f] := \partial_\nu^A(\mathcal{D}_{A,mod} f) \ \textit{ for each } \ f \in \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M.\end{aligned} \tag{4.392}$$

*From Theorem 3.5 this is known to be a well-defined, linear, and bounded operator when the quotient space is equipped with the norm* (2.601)*. In relation to this, the following statements are valid.*

*(1) [Injectivity] Whenever $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and actually $A \in \mathfrak{A}_L^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator* (4.392) *is injective.*

*(2) [Surjectivity] Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ and actually $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ it follows that there exists a small threshold $\delta \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator* (4.392) *is surjective.*

*(3) [Isomorphism] If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, it follows that there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator* (4.392) *is an isomorphism.*

***Proof*** To deal with the claim made in item *(1)*, assume $A \in \mathfrak{A}_L^{\rm dis}$. From Theorems 2.3, 2.4, and 4.12 we know that it is possible to pick some threshold $\delta \in (0, 1)$ small enough so that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ then

$$\Omega \text{ is a two-sided NTA domain with an unbounded connected boundary,} \tag{4.393}$$

and

$$\begin{gathered} \pm\tfrac{1}{2}I + \big[K_{A,{\rm mod}}\big] \text{ are invertible operators} \\ \text{on the Banach space } \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M. \end{gathered} \tag{4.394}$$

Granted these, (3.149) then implies that the operator (4.392) is injective.

To justify the claim made in item *(2)*, suppose next that $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$. By relying on Theorems 2.3 and 4.8 we may choose $\delta \in (0, 1)$ small enough such that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ then $\Omega$ is a two-sided NTA domain with an unbounded boundary and

$$\pm\tfrac{1}{2}I + K^{\#}_{A^\top} \text{ are invertible operators on } \big[L^p(\partial\Omega, w)\big]^M. \tag{4.395}$$

Once these properties are satisfied, we may invoke (3.153) to conclude that the operator (4.392) is surjective. Finally, the claim made in item *(3)* is a direct consequence of the current items *(1)*-*(2)* and Theorem 3.9. □

*Remark 4.23* Let $\Omega$, $L$, be as in Theorem 4.13. Also, assume $A \in \mathfrak{A}_L^{\rm dis}$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$. Finally, fix some pair of exponents $p_0, p_1 \in (1, \infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega, \sigma)$ and $w_1 \in A_{p_1}(\partial\Omega, \sigma)$. From the proof of Theorem 4.13 (cf. (4.394), (4.395), Remark 4.22, and Proposition 4.2) it follows that there exists some small threshold $\delta \in (0, 1)$ which depends only on $n$, $p_0$, $p_1$, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, $L$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ then

$$\text{the operator } \big[\partial_\nu^A \mathcal{D}_{A,{\rm mod}}\big] \text{ is invertible both as a mapping from } \big[\dot{L}_1^{p_0}(\partial\Omega, w_0)/\sim\big]^M \text{ onto } \big[L_1^{p_0}(\partial\Omega, w_0)\big]^M \text{ and as a mapping from } \big[\dot{L}_1^{p_1}(\partial\Omega, w_1)/\sim\big]^M \text{ onto } \big[L^{p_1}(\partial\Omega, w_1)\big]^M, \text{ and these two inverses are compatible with one another on the intersection.} \tag{4.396}$$

*Remark 4.24* An alternative proof of Theorem 4.11 can be obtained by taking collectively, (3.149), Theorem 4.12 (with $z = \pm\frac{1}{2}$), (3.153), Theorem 4.8 (with $z = \pm\frac{1}{2}$), (3.138), Theorems 2.3, and 2.4.

## 4.5 Another Look at Double Layers for the Two-Dimensional Lamé System

Throughout this section, we shall work in the two-dimensional case, i.e., in the case $n = 2$. As a preamble, we introduce a singular integral operator which is going to be relevant shortly. To set the stage, suppose $\Omega \subseteq \mathbb{R}^2$ is a UR domain, abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$, and denote by $\nu = (\nu_1, \nu_2)$ the geometric measure theoretic outward unit normal to $\Omega$. Then for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|}\big)$ define

$$R_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\nu_1(y)(y_2 - x_2) - \nu_2(y)(y_1 - x_1)}{|x-y|^2} f(y)\, d\sigma(y), \tag{4.397}$$

at $\sigma$-a.e. point $x \in \partial\Omega$. Let us fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. It has been proved in [113, §2.5] that the singular integral operator $R_\Delta$ introduced in (4.397) is bounded on $L^p(\partial\Omega, w)$ and satisfies

$$(R_\Delta)^2 = \big(\tfrac{1}{2}I + K_\Delta\big)\big(-\tfrac{1}{2}I + K_\Delta\big) \ \text{ on } \ L^p(\partial\Omega, w), \tag{4.398}$$

$$K_\Delta R_\Delta + R_\Delta K_\Delta = 0 \ \text{ on } \ L^p(\partial\Omega, w), \tag{4.399}$$

where $K_\Delta$ is the harmonic double layer potential operator in this setting (i.e., $K_\Delta$ is as in (3.29) with $n := 2$).

Our main result in this section is Theorem 4.14 below, which elaborates on the spectra of double layer potential operators, associated with the two-dimensional complex Lamé system, when acting on Muckenhoupt weighted Lebesgue and Sobolev spaces on the boundary of a $\delta$-AR unbounded domain in the plane.

**Theorem 4.14** *Let $\Omega \subseteq \mathbb{R}^2$ be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix two Lamé moduli $\mu, \lambda \in \mathbb{C}$ satisfying*

$$\mu \neq 0, \qquad 2\mu + \lambda \neq 0, \tag{4.400}$$

*and bring back the one-parameter family coefficient tensors from* (3.226) *(corresponding to $n = 2$), i.e.,*

$$A(\zeta) = \left(a_{jk}^{\alpha\beta}(\zeta)\right)_{\substack{1\leq j,k\leq 2\\ 1\leq\alpha,\beta\leq 2}} \text{ defined for each } \zeta \in \mathbb{C} \text{ according to}$$
$$a_{jk}^{\alpha\beta}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu+\lambda-\zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \tag{4.401}$$
$$\text{for } 1 \leq j,k,\alpha,\beta \leq 2,$$

*which allows to represent the* $2 \times 2$ *Lamé system* $L_{\mu,\lambda} = \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$ *in* $\mathbb{R}^2$ *as*

$$L_{\mu,\lambda} = \left(a_{jk}^{\alpha\beta}(\zeta)\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq 2} \text{ for each } \zeta \in \mathbb{C}. \tag{4.402}$$

*Fix some integrability exponent* $p \in (1,\infty)$ *along with some Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$. *Finally, suppose* $z, \zeta \in \mathbb{C}$ *are such that*

$$z \neq \pm\frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{4\mu(2\mu+\lambda)}, \tag{4.403}$$

*and associate the double layer potential operator* $K_{A(\zeta)}$ *with the coefficient tensor* $A(\zeta)$ *and the domain* $\Omega$ *as in* (3.24).

*Then there exists some small threshold* $\delta \in (0,1)$ *which depends only on* $\mu$, $\lambda$, $p$, $[w]_{A_p}$, $z$, $\zeta$, *and the Ahlfors regular constant of* $\partial\Omega$, *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta$ *it follows that*

$$\begin{gathered}\text{the operator } zI_{2\times 2} + K_{A(\zeta)} \text{ is invertible}\\ \text{both on } \left[L^p(\partial\Omega,w)\right]^2 \text{ and on } \left[L_1^p(\partial\Omega,w)\right]^2.\end{gathered} \tag{4.404}$$

Before presenting the proof of this theorem, a few clarifications are in order. From (4.309)–(4.310) in Theorem 4.8 and (3.228)–(3.229) we already know that, under suitable geometric assumptions, the conclusion in (4.404) holds (and this is true in all dimensions $n \geq 2$) when

$$3\mu+\lambda \neq 0 \text{ and } \zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}. \tag{4.405}$$

The point of Theorem 4.14 is that, for the two-dimensional Lamé system, the invertibility results from (4.309)–(4.310) holds with $A = A(\zeta)$ as in (3.226) for *a much larger range of* $\zeta$'s than the singleton in (4.405). (Parenthetically we wish to note that what is special about the scenario described in (4.405) is that this makes $\pm\frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{4\mu(2\mu+\lambda)}$ zero, so (4.403) simply reads $z \in \mathbb{C}\setminus\{0\}$ in this case, as was assumed in Theorem 4.8.) It should be also remarked that, in the setting on Theorem 4.14, the double layer $K_{A(\zeta)}$ does *not* necessarily have small operator norm, and this is in stark contrast with the case of the double layer operators considered in Theorem 4.8. References to other related results may be found in [82, Chapter 7]; in this vein, see also [99].

We are now ready to present the proof of Theorem 4.14.

***Proof of Theorem 4.14*** From Theorem 2.3 we know that it is possible to pick some threshold $\delta \in (0, 1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta$ then $\Omega$ is a UR domain, with the UR constants of $\partial\Omega$ controlled solely in terms of the Ahlfors regularity constant of $\partial\Omega$. Henceforth, assume this is the case.

Recall the numbers $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ associated with $\zeta, \mu, \lambda$ as in (3.234). From (3.29), (3.235), (3.236), and (4.397) we see that for each $\zeta \in \mathbb{C}$ we have

$$K_{A(\zeta)} = C_1(\zeta)K_\Delta I_{2\times 2} - (1 - C_1(\zeta))Q + C_2(\zeta)\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix} \tag{4.406}$$

as operators on $\big[L^p(\partial\Omega, w)\big]^2$. Note that (4.398) implies

$$\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix}^2 = \big(\tfrac{1}{4}I - (K_\Delta)^2\big)I_{2\times 2} \ \text{ on } \ \big[L^p(\partial\Omega, w)\big]^2. \tag{4.407}$$

Staring with (4.406) and then using (4.407), (4.399) we may write, with all operators acting on the space $\big[L^p(\partial\Omega, w)\big]^2$,

$$\begin{aligned} \big(zI_{2\times 2} + K_{A(\zeta)}\big)\big(-zI_{2\times 2} + K_{A(\zeta)}\big) &= (K_{A(\zeta)})^2 - z^2 I_{2\times 2} \\ &= \big[\tfrac{1}{4}C_2(\zeta)^2 - z^2\big]I_{2\times 2} + T_\zeta, \end{aligned} \tag{4.408}$$

for all $z, \zeta \in \mathbb{C}$, where $T_\zeta$ is the operator

$$\begin{aligned} T_\zeta = {}& \big(C_1(\zeta)^2 - C_2(\zeta)^2\big)K_\Delta^2 I_{2\times 2} + (1 - C_1(\zeta))^2 Q^2 \\ & - C_1(\zeta)(1 - C_1(\zeta))(K_\Delta I_{2\times 2})Q - C_1(\zeta)(1 - C_1(\zeta))Q(K_\Delta I_{2\times 2}) \\ & - C_2(\zeta)(1 - C_1(\zeta))Q\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix} - C_2(\zeta)(1 - C_1(\zeta))\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix}Q. \end{aligned} \tag{4.409}$$

Fix now $\zeta \in \mathbb{C}$ along with $\varepsilon > 0$ arbitrary. Note that $T_\zeta$ in (4.409) is a finite linear combination of compositions of pairs of singular integral operators such that, in each case, at least one of them falls under the scope of Corollary 4.2. As a consequence of this and Proposition 3.4, it follows that there exists $\delta \in (0, 1)$ small enough (relative to $\mu, \lambda, \zeta, \varepsilon, p, [w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$), matters may be arranged so that, under the additional assumption that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta, \tag{4.410}$$

we have

$$\|T_\zeta\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \leq \varepsilon^2/2. \tag{4.411}$$

Consider now

$$z \in \mathbb{C} \setminus \Big\{B\big(\tfrac{1}{2}C_2(\zeta),\varepsilon\big) \cup B\big(-\tfrac{1}{2}C_2(\zeta),\varepsilon\big)\Big\}, \tag{4.412}$$

which entails

$$\big|\tfrac{1}{4}C_2(\zeta)^2 - z^2\big| = \big|\tfrac{1}{2}C_2(\zeta) - z\big|\big|\tfrac{1}{2}C_2(\zeta) + z\big| \geq \varepsilon^2. \tag{4.413}$$

Then from (4.413), (4.411) it follows that

$$\begin{gathered}\big[\tfrac{1}{4}C_2(\zeta)^2 - z^2\big]I_{2\times 2} + T_\zeta \ \text{ is invertible on } \big[L^p(\partial\Omega,w)\big]^2 \\ \text{for each } z \text{ as in (4.412)},\end{gathered} \tag{4.414}$$

and

$$\begin{gathered}\Big\|\Big(\big[\tfrac{1}{4}C_2(\zeta)^2 - z^2\big]I_{2\times 2} + T_\zeta\Big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \leq (\varepsilon^2/2)^{-1} \\ \text{for each } z \text{ as in (4.412)}.\end{gathered} \tag{4.415}$$

Since the operators $zI_{2\times 2} + K_{A(\zeta)}$ and $-zI_{2\times 2} + K_{A(\zeta)}$ commute with one another, from (4.408) and (4.414) we ultimately conclude that

$$zI_{2\times 2} + K_{A(\zeta)} \ \text{ is invertible on } \big[L^p(\partial\Omega,w)\big]^2 \ \text{ for each } z \text{ as in (4.412)}. \tag{4.416}$$

In relation to (4.416) we also claim that there exists some small number

$$c := c\big(\Omega,\varepsilon,\zeta,p,[w]_{A_p}\big) \in (0,1], \tag{4.417}$$

where the dependence of $c$ on $\Omega$ manifests itself only through the Ahlfors regularity constant of $\partial\Omega$, with the property that

$$\begin{gathered}c\|f\|_{[L^p(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big\|_{[L^p(\partial\Omega,w)]^2} \\ \text{for each } z \text{ as in (4.412) and each } f \in \big[L^p(\partial\Omega,w)\big]^2.\end{gathered} \tag{4.418}$$

To prove this, first observe that

$$\begin{aligned}&\text{whenever } |z| > 1 + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} \text{ then}\\ &zI_{2\times 2} + K_{A(\zeta)} \text{ is invertible on } \big[L^p(\partial\Omega, w)\big]^2 \text{ and}\\ &\Big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} < 1.\end{aligned} \tag{4.419}$$

Hence, as long as $|z| > 1 + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}$, the estimate in (4.418) is true for any choice of $c \in (0, 1]$. As such, there remains to study the case in which

$$\begin{aligned}&z \text{ is as in (4.412) and also satisfies}\\ &|z| \le 1 + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}.\end{aligned} \tag{4.420}$$

Henceforth assume $z$ is as in (4.420). From (4.408) and (4.415) we know that

$$\Big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\big(-zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} \le (\varepsilon^2/2)^{-1}. \tag{4.421}$$

Write $\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}$ as

$$\Big[\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\big(-zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\Big]\big(-zI_{2\times 2} + K_{A(\zeta)}\big), \tag{4.422}$$

then use this formula and (4.421) to estimate

$$\begin{aligned}\Big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} &\le (\varepsilon^2/2)^{-1}\|-zI_{2\times 2} + K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}\\ &\le (\varepsilon^2/2)^{-1}\Big(|z| + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}\Big)\\ &\le C\big(\Omega, \varepsilon, \zeta, p, [w]_{A_p}\big),\end{aligned} \tag{4.423}$$

where the last inequality comes from (4.420), and

$$C\big(\Omega, \varepsilon, \zeta, p, [w]_{A_p}\big) := 2\varepsilon^{-2} + 4\varepsilon^{-2}\|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}. \tag{4.424}$$

Hence, if we define

$$c := c\big(\Omega, \varepsilon, \zeta, p, [w]_{A_p}\big) := \min\Big\{1, \big[C(\Omega, \varepsilon, \zeta, p, [w]_{A_p})\big]^{-1}\Big\} \in (0, 1], \tag{4.425}$$

we may rely on (4.423) to write

$$c\|f\|_{[L^p(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big\|_{[L^p(\partial\Omega,w)]^2}, \quad \text{for all } f \in \big[L^p(\partial\Omega, w)\big]^2, \tag{4.426}$$

finishing the proof of (4.418).

We next claim that, if the threshold $\delta \in (0, 1)$ appearing in (4.410) is taken sufficiently small to begin with, we also have

$$zI_{2\times 2} + K_{A(\zeta)} \text{ invertible on } \big[L^p_1(\partial\Omega, w)\big]^2 \text{ for each } z \text{ as in (4.412).} \tag{4.427}$$

For starters, observe that for each point $z \in \mathbb{C}$, and each $f \in \big[L^p_1(\partial\Omega, w)\big]^2$, Proposition 3.2 gives

$$\partial_{\tau_{12}}\big[\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big] = \big(zI_{2\times 2} + K_{A(\zeta)}\big)(\partial_{\tau_{12}} f) + U^{\zeta}_{12}(\nabla_{\tan} f), \tag{4.428}$$

where the commutator $U^{\zeta}_{12}$ is defined as in (3.35) with $n = 2$, $j = 1$, $k = 2$, and the coefficient tensor $A(\zeta)$ as in (4.401). If $z$ is as in (4.412) then, on account of (4.428), (4.418), and Theorem 4.3 (also keeping in mind Proposition 3.4) for each $f \in \big[L^p_1(\partial\Omega, w)\big]^2$ we may estimate

$$\begin{aligned} c\|\partial_{\tau_{12}} f\|_{[L^p(\partial\Omega,w)]^2} &\leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)(\partial_{\tau_{12}} f)\big\|_{[L^p(\partial\Omega,w)]^2} \\ &\leq \big\|\partial_{\tau_{12}}\big[\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big]\big\|_{[L^p(\partial\Omega,w)]^2} + \big\|U^{\zeta}_{12}(\nabla_{\tan} f)\big\|_{[L^p(\partial\Omega,w)]^2} \\ &\leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big\|_{[L^p_1(\partial\Omega,w)]^2} + C\delta\|\partial_{\tau_{12}} f\|_{[L^p(\partial\Omega,w)]^2}, \end{aligned} \tag{4.429}$$

(since we presently have $\partial_{\tau_{11}} = \partial_{\tau_{22}} = 0$ and $\partial_{\tau_{12}} = -\partial_{\tau_{21}}$), where $C \in (0, \infty)$ depends only on $\mu$, $\lambda$, $\zeta$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Assuming $\delta < c/(2C)$ to begin with, the very last term above may be absorbed in the left-most side of (4.429). By combining the resulting inequality with (4.418) we therefore arrive at the conclusion that if $\delta$ in (4.410) is small enough then we may find some small $\eta > 0$ with the property that

$$\eta\|f\|_{[L_1^p(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2}+K_{A(\zeta)}\big)f\big\|_{[L_1^p(\partial\Omega,w)]^2}$$
$$\text{for each } z \text{ as in (4.412) and each } \ f\in\big[L_1^p(\partial\Omega,w)\big]^2. \tag{4.430}$$

In such a scenario, (4.430) implies that the operator $zI_{2\times 2}+K_{A(\zeta)}$ acting on $\big[L_1^p(\partial\Omega,w)\big]^2$ is injective and has closed range for each $z$ as in (4.412). Consequently, the operator $zI_{2\times 2}+K_{A(\zeta)}$ acting on $\big[L_1^p(\partial\Omega,w)\big]^2$ is semi-Fredholm for each $z$ as in (4.412). Since this depends continuously on $z$, the homotopic invariance of the index on connected sets then ensures that the index of $zI_{2\times 2}+K_{A(\zeta)}$ on $\big[L_1^p(\partial\Omega,w)\big]^2$ is independent of $z$ in said range. Given that, via a Neumann series argument,

$$zI_{2\times 2}+K_{A(\zeta)} \ \text{ is invertible on } \ \big[L_1^p(\partial\Omega,w)\big]^2$$
$$\text{if } \ |z|>\big\|K_{A(\zeta)}\big\|_{[L_1^p(\partial\Omega,w)]^2\to[L_1^p(\partial\Omega,w)]^2}, \tag{4.431}$$

we may therefore conclude that the index of $zI_{2\times 2}+K_{A(\zeta)}$ on $\big[L_1^p(\partial\Omega,w)\big]^2$ is zero for each $z$ as in (4.412). In view of the fact that, as already noted from (4.430), the operator $zI_{2\times 2}+K_{A(\zeta)}$ is injective on $\big[L_1^p(\partial\Omega,w)\big]^2$ for each $z$ as in (4.412), this ultimately proves that $zI_{2\times 2}+K_{A(\zeta)}$ is invertible on $\big[L_1^p(\partial\Omega,w)\big]^2$ for each $z$ as in (4.412). Hence, the claim made in (4.427) is true. At this stage, the claim made in (4.404) readily follows from (4.416) and (4.427). □

It is of interest to single out the case $z=\pm\frac{1}{2}$ in (4.404), and in Corollary 4.3 stated next we do just that.

**Corollary 4.3** *Let $\Omega\subseteq\mathbb{R}^2$ be an Ahlfors regular domain. Abbreviate $\sigma:=\mathcal{H}^1\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix two Lamé moduli $\mu,\lambda\in\mathbb{C}$ satisfying*

$$\mu\neq 0,\qquad 2\mu+\lambda\neq 0,\qquad 3\mu+\lambda\neq 0, \tag{4.432}$$

*and recall the one-parameter family coefficient tensors $A(\zeta)$ defined for each $\zeta\in\mathbb{C}$ as in (4.401). Fix an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$. Finally, pick some*

$$\zeta\in\mathbb{C}\setminus\Big\{-\mu,\ \tfrac{\mu(5\mu+3\lambda)}{3\mu+\lambda}\Big\} \tag{4.433}$$

*and associate double layer potential operator $K_{A(\zeta)}$ with the coefficient tensor $A(\zeta)$ and the domain $\Omega$ as in* (3.24).

*Then there exists some small threshold $\delta\in(0,1)$ which depends only on $\mu$, $\lambda$, $p$, $[w]_{A_p}$, $\zeta$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2}<\delta$ it follows that*

$$\begin{aligned}&\textit{the operators}\quad \pm\tfrac{1}{2}I_{2\times 2}+K_{A(\zeta)}\quad \textit{are invertible}\\&\textit{both on}\quad \big[L^p(\partial\Omega,w)\big]^2\quad \textit{and on}\quad \big[L^p_1(\partial\Omega,w)\big]^2,\end{aligned}\tag{4.434}$$

*and*

$$\textit{the operators}\quad \pm\tfrac{1}{2}I_{2\times 2}+K^{\#}_{A(\zeta)}\quad \textit{are invertible on}\quad \big[L^p(\partial\Omega,w)\big]^2.\tag{4.435}$$

As seen from (4.433) (also keeping in mind (4.432)), under the additional assumption that $\mu+\lambda\neq 0$ the value $\zeta:=\mu$ becomes acceptable in the formulation of the conclusions in (4.434)–(4.435). This special choice leads to the conclusion that, if $\Omega$ is sufficiently flat (relative to $\mu$, $\lambda$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) then the operators

$$\pm\tfrac{1}{2}I_{2\times 2}+K_{A(\mu)}:\big[L^p(\partial\Omega,w)\big]^2\longrightarrow\big[L^p(\partial\Omega,w)\big]^2,\tag{4.436}$$

$$\pm\tfrac{1}{2}I_{2\times 2}+K_{A(\mu)}:\big[L^p_1(\partial\Omega,w)\big]^2\longrightarrow\big[L^p_1(\partial\Omega,w)\big]^2,\tag{4.437}$$

$$\pm\tfrac{1}{2}I_{2\times 2}+K^{\#}_{A(\mu)}:\big[L^p(\partial\Omega,w)\big]^2\longrightarrow\big[L^p(\partial\Omega,w)\big]^2,\tag{4.438}$$

are all invertible whenever

$$\mu\neq 0,\qquad \mu+\lambda\neq 0,\qquad 2\mu+\lambda\neq 0,\qquad 3\mu+\lambda\neq 0.\tag{4.439}$$

This is relevant in the context of Remark 6.10.

***Proof of Corollary 4.3*** The claim in (4.434) is a direct consequence of Theorem 4.14, upon observing that when $z=\pm 1/2$ the demand in (4.403) becomes equivalent to the condition stipulated in (4.433). The claim in (4.435) then follows from (4.434) and duality. □