Chapter 3
Calderón–Zygmund Theory for Boundary Layers in UR Domains

In [25], A.P. Calderón has initiated a breakthrough by proving the L^p-boundedness of the principal-value Cauchy integral operator on Lipschitz curves with small Lipschitz constant. Subsequently, R. Coifman, A. McIntosh, and Y. Meyer have successfully extended Calderón's estimate on Cauchy integrals to general Lipschitz curves in [32] and used this to establish the boundedness of higher-dimensional singular integral operators (such as the harmonic double layer K_Δ) on Lebesgue spaces $L^p(\Sigma, \mathcal{H}^{n-1})$ with $p \in (1, \infty)$, whenever Σ is a strongly Lipschitz surface in $\mathbb{R}^n$. This gave the impetus for studying such singular integral operators on surfaces more general than the boundaries of Lipschitz domains. Works of G. David [37, 38], G. David and D. Jerison [39], G. David and S. Semmes [40, 41], and S. Semmes [122] yield such boundedness when the $\Sigma \subseteq \mathbb{R}^n$ is a UR set, i.e., Σ is a closed Ahlfors regular set which contains "big pieces" of Lipschitz images in a quantitative, uniform, scale-invariant fashion (cf. Definition 2.5).

This body of results, which interfaced tightly with geometric measure theory, has been applied to problems in PDEs for the first time by S. Hofmann, M. Mitrea, and M. Taylor in [61] (see also [109] for PDEs in the setting of Riemannian manifolds). Here we continue this line of work with two specific goals in mind. First, we consider singular integral operators (SIOs) acting on a larger variety of function spaces and, second, we seek finer bounds on the operator norm of the singular integrals of double layer type. We begin by discussing the general setup.

3.1 Boundary Layer Potentials: The Setup

Fix $n \in \mathbb{N}$ with $n \geq 2$ along with some $M \in \mathbb{N}$, and denote by $\mathfrak{L}$ the collection of all homogeneous constant complex coefficient second-order $M \times M$ systems L in $\mathbb{R}^n$. Hence, any element L in $\mathfrak{L}$ may be written as a matrix of differential operators of the

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form $L = \left(a_{jk}^{\alpha\beta}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}$ for some complex numbers $a_{jk}^{\alpha\beta}$ (here and elsewhere, we shall use the usual convention of summation over repeated indices). In particular, the action of L on any given vector-valued distribution $u = (u_\beta)_{1\leq\beta\leq M}$ may be described as

$$Lu = \left(a_{jk}^{\alpha\beta}\partial_j\partial_k u_\beta\right)_{1\leq\alpha\leq M}, \tag{3.1}$$

and we denote by $L^\top := \left(a_{kj}^{\beta\alpha}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}$ the (real) transpose of L. We also define the characteristic matrix of L as

$$L(\xi) := \left[\left(-a_{jk}^{\alpha\beta}\xi_j\xi_k\right)_{1\leq\alpha,\beta\leq M}\right] \quad \text{for each} \quad \xi = (\xi_i)_{1\leq i\leq n} \in \mathbb{R}^n \tag{3.2}$$

and introduce

$$\mathfrak{L}_* := \left\{L \in \mathfrak{L} : \det[L(\xi)] \neq 0 \quad \text{for each} \quad \xi \in \mathbb{R}^n \setminus \{0\}\right\}. \tag{3.3}$$

We shall refer to a system $L \in \mathfrak{L}$ as being `weakly elliptic` if actually $L \in \mathfrak{L}_*$. This should be contrasted with the more stringent `Legendre-Hadamard (strong) ellipticity` condition which asks for the existence of some $c > 0$ such that

$$\mathrm{Re}\left\langle -L(\xi)\zeta, \overline{\zeta}\right\rangle \geq c\,|\xi|^2\,|\zeta|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad \zeta \in \mathbb{C}^M. \tag{3.4}$$

Next, let us consider

$$\mathfrak{A} := \left\{A = \left(a_{jk}^{\alpha\beta}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} : \text{ each } a_{jk}^{\alpha\beta} \text{ belongs to } \mathbb{C}\right\}, \tag{3.5}$$

the collection of coefficient tensors with constant complex entries. Adopting natural operations (i.e., componentwise addition and multiplication by scalars), this becomes a finite dimensional vector space (over $\mathbb{C}$), which we endow with the norm

$$\|A\| := \sum_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \left|a_{jk}^{\alpha\beta}\right| \quad \text{for each} \quad A = \left(a_{jk}^{\alpha\beta}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}. \tag{3.6}$$

Hence, if the transpose of each given $A = \left(a_{jk}^{\alpha\beta}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}$ is the coefficient tensor $A^\top := \left(a_{kj}^{\beta\alpha}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$, then $\mathfrak{A} \ni A \mapsto A^\top \in \mathfrak{A}$ is an isometry. With each coefficient tensor $A = \left(a_{jk}^{\alpha\beta}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}$ associate the system $L_A \in \mathfrak{L}$ according to

$$L_A := \left(a_{jk}^{\alpha\beta}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}. \tag{3.7}$$

Then the map

$$\mathfrak{A} \ni A \longmapsto L_A \in \mathfrak{L} \tag{3.8}$$

is linear and surjective, though it fails to be injective. Specifically, if we introduce

$$\mathfrak{A}^{\mathrm{ant}} := \Big\{B = \big(b_{jk}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A} : b_{jk}^{\alpha\beta} = -b_{kj}^{\alpha\beta} \text{ whenever } 1 \leq j,k \leq n \text{ and } 1 \leq \alpha,\beta \leq M\Big\}, \tag{3.9}$$

the collection of all coefficient tensors that are antisymmetric in the lower indices, then $\mathfrak{A}^{\mathrm{ant}}$ is a closed linear subspace of $\mathfrak{A}$, and for each $A, \widetilde{A} \in \mathfrak{A}$, we have

$$L_A = L_{\widetilde{A}} \iff A - \widetilde{A} \in \mathfrak{A}^{\mathrm{ant}}. \tag{3.10}$$

If we now define

$$\mathfrak{A}_L := \big\{A \in \mathfrak{A} : L = L_A\big\} \text{ for each } L \in \mathfrak{L}, \tag{3.11}$$

and for each $L \in \mathfrak{L}$, we set (with the distance considered in the normed vector space $\mathfrak{A}$)

$$\|L\| := \operatorname{dist}\big(A, \mathfrak{A}^{\mathrm{ant}}\big) \text{ for each/some } A \in \mathfrak{A}_L, \tag{3.12}$$

then $\mathfrak{L} \ni L \mapsto \|L\|$ is an unambiguously defined norm on the vector space $\mathfrak{L}$. In the topology induced by this norm, $\mathfrak{L}_*$ from (3.3) is an open subset of $\mathfrak{L}$, the mapping (3.8) is continuous, and $\mathfrak{L} \ni L \mapsto L^\top \in \mathfrak{L}$ is an isometry.

Finally, we denote by $\mathfrak{A}_{\mathrm{WE}}$ the collection of all coefficient tensors A with the property that the $M \times M$ homogeneous second-order system L_A associated with A in $\mathbb{R}^n$ as in (3.7) is weakly elliptic, i.e.,

$$\mathfrak{A}_{\mathrm{WE}} := \big\{A \in \mathfrak{A} : L_A \in \mathfrak{L}_*\big\}. \tag{3.13}$$

Then $\mathfrak{A}_{\mathrm{WE}}$ is an open subset of $\mathfrak{A}$.

The following theorem, itself a special case of [102, Theorem 11.1, p. 393], summarizes some of the main properties of a certain type of fundamental solution canonically associated with any given homogeneous, constant complex coefficient, weakly elliptic second-order system in $\mathbb{R}^n$.

Theorem 3.1 *Let L be a homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$, which is weakly elliptic (cf.* (1.2)*). Then there exists an $M \times M$*

matrix-valued function $E = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$, canonically associated with the given system L, such that the following properties are true:

1. *For any two indices $\alpha, \beta \in \{1, \dots, M\}$, one has $E_{\alpha\beta} \in \mathscr{C}^\infty\big(\mathbb{R}^n \setminus \{0\}\big)$ as well as $E_{\alpha\beta}(x) = E_{\alpha\beta}(-x)$ for every $x \in \mathbb{R}^n \setminus \{0\}$.*
2. *For each fixed point $y \in \mathbb{R}^n$, one has $L\big[E(\cdot - y)\big] = \delta_y I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$, where $I_{M\times M}$ is the $M \times M$ identity matrix and δ_y denotes the Dirac distribution with mass at y in $\mathbb{R}^n$. That is, using the standard Kronecker delta notation,*

$$a^{\alpha\beta}_{jk}\partial_{x_j}\partial_{x_k}\big[E_{\beta\gamma}(x-y)\big] = \delta_{\alpha\gamma}\delta_y(x), \qquad x \in \mathbb{R}^n, \tag{3.14}$$

   in the sense of distributions, for every $\alpha, \gamma \in \{1, \dots, M\}$.
3. *The transpose of E, i.e., $E^\top = (E_{\beta\alpha})_{1\leq\alpha,\beta\leq M}$, is a fundamental solution for the transpose system $L^\top$. In other words, for each fixed point $y \in \mathbb{R}^n$, one has $L^\top\big[E^\top(\cdot - y)\big] = \delta_y I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$, i.e.,*

$$a^{\beta\alpha}_{kj}\partial_{x_j}\partial_{x_k}\big[E_{\gamma\beta}(x-y)\big] = \delta_{\alpha\gamma}\delta_y(x), \qquad x \in \mathbb{R}^n, \tag{3.15}$$

   in the sense of distributions, for every $\alpha, \gamma \in \{1, \dots, M\}$.
4. *For every multi-index $\alpha \in \mathbb{N}_0^n$ with $n + |\alpha| > 2$, the function $\partial^\alpha E$ is positive homogeneous of degree $2 - n - |\alpha|$ and there exists a constant $C_\alpha \in (0, \infty)$ with the property that*

$$\big|(\partial^\alpha E)(x)\big| \leq C_\alpha |x|^{2-n-|\alpha|} \;\; \textit{for all} \;\; x \in \mathbb{R}^n \setminus \{0\}. \tag{3.16}$$

   Finally, corresponding to $n = 2$ and $\alpha = (0, \dots, 0)$, there exists $C \in (0, \infty)$ such that $|E(x)| \leq C\big(1 + \big|\ln|x|\big|\big)$ for every $x \in \mathbb{R}^2 \setminus \{0\}$.
5. *Let 'hat' denote the Fourier transform in $\mathbb{R}^n$ (originally defined on Schwartz functions and then extended to tempered distributions via duality). Then $\widehat{E}$ is a tempered distribution in $\mathbb{R}^n$ (which is positive homogeneous of degree -2 if $n \geq 3$), whose restriction to $\mathbb{R}^n \setminus \{0\}$ is a (matrix-valued) function of class $\mathscr{C}^\infty$. In fact,*

$$\widehat{E}(\xi) = \big[L(\xi)\big]^{-1} \;\; \textit{for every} \;\; \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3.17}$$

   More generally, given any $\gamma \in \mathbb{N}_0^n$, it follows that the tempered distribution $\widehat{\partial^\gamma E}$ is a function of class $\mathscr{C}^\infty$ when restricted to $\mathbb{R}^n \setminus \{0\}$, which, regarded as such, satisfies

$$\widehat{\partial^\gamma E}(\xi) = \mathrm{i}^{|\gamma|}\,\xi^\gamma\,\big[L(\xi)\big]^{-1} \;\; \textit{for every} \;\; \xi \in \mathbb{R}^n \setminus \{0\}, \tag{3.18}$$

   and

if $\gamma \in \mathbb{N}_0^n$, *then* $\widehat{\partial^\gamma E} = \mathrm{i}^{|\gamma|}\,\xi^\gamma\,\big[L(\xi)\big]^{-1}$ *as tempered distributions in* $\mathbb{R}^n$ *when either* $|\gamma| > 0$ *or* $n \geq 3$. (3.19)

6. *Writing* E_L *in place of* E *to emphasize the dependence on* L, *matters may be arranged so that*

$$\begin{aligned} &\big(E_L\big)^\top = E_{L^\top}, \quad \overline{\big(E_L\big)} = E_{\overline{L}}, \quad \big(E_L\big)^* = E_{L^*}, \\ &\textit{as well as } \ E_{\lambda L} = \lambda^{-1} E_L \ \textit{ for each } \lambda \in \mathbb{C} \setminus \{0\}, \end{aligned} \tag{3.20}$$

where $\top$, $\bar{\cdot}$, *and* $*$ *denote, respectively, transposition, complex conjugation, and complex (or Hermitian) adjunction.*

Moving on, assume $\Omega \subseteq \mathbb{R}^n$ is a given UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. In addition, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, and consider the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with L as in Theorem 3.1. Finally, fix a coefficient tensor $A = \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$, and pick an arbitrary function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.21}$$

In this setting, define the action of the boundary-to-domain double layer potential operator $\mathcal{D}_A$ on f as

$$\mathcal{D}_A f(x) := \left(-\int_{\partial\Omega} \nu_k(y) a_{jk}^{\beta\alpha}\,\big(\partial_j E_{\gamma\beta}\big)\,(x-y)\,f_\alpha(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\gamma\leq M}, \tag{3.22}$$

at each point $x \in \Omega$. From (3.16), we see that (3.21) is the most general functional analytic setting in which the integral in (3.22) is absolutely convergent. The double layer operator $\mathcal{D}$ may be regarded as a mechanism for generating lots of null-solutions for the given system L in Ω since, as is apparent from (3.22) and Theorem 3.1,

$$\begin{aligned} &\text{for each function } f \text{ as in (3.21), we have} \\ &\mathcal{D}_A f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \text{ and } \ L(\mathcal{D}_A f) = 0 \ \text{ in } \ \Omega. \end{aligned} \tag{3.23}$$

Going further, let us define the action of the boundary-to-boundary double layer potential operator K_A on f as in (3.21) by setting

$$K_A f(x) := \Bigg(-\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_k(y) a_{jk}^{\beta\alpha} \big(\partial_j E_{\gamma\beta}\big)(x-y) f_\alpha(y)\, d\sigma(y) \Bigg)_{1\le\gamma\le M}, \tag{3.24}$$

at σ-a.e. point $x \in \partial\Omega$. Another singular integral operator that is closely related to (3.24) is the so-called transpose double layer operator $K_A^{\#}$ defined by setting

$$K_A^{\#} f(x) := \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_k(x) a_{jk}^{\beta\alpha} (\partial_j E_{\gamma\beta})(x-y) f_\gamma(y)\, d\sigma(y) \Bigg)_{1\le\alpha\le M} \tag{3.25}$$

at σ-a.e. $x \in \partial\Omega$, for each function f as in (3.21). Since we are presently assuming that Ω is a UR domain, work in [114, Chapter 1] guarantees that the above singular integral operators are well defined in a σ-a.e. pointwise fashion for each function as in (3.21). Also, it is clear from definitions and the last line in (3.20) that

$$\begin{gathered} \mathcal{D}_{\lambda A} = \mathcal{D}_A, \quad K_{\lambda A} = K_A, \quad K_{\lambda A}^{\#} = K_A^{\#} \\ \text{for each } \lambda \in \mathbb{C} \text{ with } \lambda \neq 0. \end{gathered} \tag{3.26}$$

Example 3.1 The standard fundamental solution for the Laplacian in $\mathbb{R}^n$ is defined for $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E_\Delta(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)} \dfrac{1}{|x|^{n-2}}, & \text{if } n \geq 3, \\[2ex] \dfrac{1}{2\pi} \ln|x|, & \text{if } n = 2, \end{cases} \tag{3.27}$$

where, as usual, ω_{n-1} denotes the surface area of the unit sphere in $\mathbb{R}^n$ (cf. [102, Section 7.1]). Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Set $a_{jk}^{\alpha\beta} := a_{jk} := \delta_{jk}$ in (3.1) so that $L = \Delta$, and refer to $\mathcal{D}_\Delta$, K_Δ (constructed as in (3.22) and (3.24)) for this choice of coefficient tensor, i.e., for $A := I_{n\times n}$, the identity matrix) as being the (classical) harmonic double layer potentials. Concretely, for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, we have (writing, in this case, $\mathcal{D}_\Delta$, K_Δ, $K_\Delta^{\#}$ in place of $\mathcal{D}_{I_{n\times n}}$, $K_{I_{n\times n}}$, $K_{I_{n\times n}}^{\#}$)

$$\mathcal{D}_\Delta f(x) = \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega, \tag{3.28}$$

and, at σ-a.e. point $x \in \partial\Omega$,

$$K_{\Delta} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y), \tag{3.29}$$

$$K^{\#}_{\Delta} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(x), x-y\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y). \tag{3.30}$$

Returning to the mainstream discussion, continue to assume that $\Omega \subseteq \mathbb{R}^n$ is a UR domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, as before, continue to assume that L is a homogeneous constant complex coefficient weakly elliptic second-order $M \times M$ system in $\mathbb{R}^n$. Then, for each coefficient tensor $A \in \mathfrak{A}_L$, a basic identity relating the boundary-to-domain double layer potential operator $\mathcal{D}_A$ to the boundary-to-boundary double layer potential operator K_A is the jump-formula (proved in [114, §1.5]), to the effect that if I denotes the identity operator and $\kappa > 0$ is an arbitrary aperture parameter, then

$$\begin{aligned} &\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac12 I + K_A\big) f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \\ &\text{for each given function } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \end{aligned} \tag{3.31}$$

Another fundamental property of the boundary-to-domain double layer potential operator is the ability of absorbing an arbitrary spacial derivative and eventually relocating it, via integration by parts on the boundary, all the way to the function on which this was applied to begin with. This is made precise in the following basic proposition, proved in [114, §1.3].

Proposition 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some $M \in \mathbb{N}$, consider a weakly elliptic, homogeneous, constant (complex) coefficient, second-order, $M \times M$ system L in $\mathbb{R}^n$, written as in (3.1) for some choice of a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n \\ 1\le\alpha,\beta\le M}}$. Finally, associate with A and Ω the double layer potential operator $\mathcal{D}_A$ as in (3.22), and consider a function*

$$\begin{aligned} &f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \quad \textit{with the property that} \\ &\partial_{\tau_{jk}} f_\alpha \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \; \textit{for } 1\le j,k\le n \textit{ and } 1\le\alpha\le M. \end{aligned} \tag{3.32}$$

Then, for each index $\ell \in \{1, \dots, n\}$ and each point $x \in \Omega$, one has

$$\partial_\ell\big(\mathcal{D}_A f\big)(x) = \left(\int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\gamma\leq M}. \tag{3.33}$$

As a consequence, if Ω is actually a UR *domain then for each aperture parameter* $\kappa > 0$*, the nontangential boundary trace*

$$\big(\nabla\mathcal{D}_A f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \textit{exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \ \partial\Omega. \tag{3.34}$$

We next recall the following result from [114, §1.5], which identifies the commutator between the double layer potential operator K_A from (3.24) and the first-order tangential derivative operators $\partial_{\tau_{jk}}$ from (2.582) as being yet another commutator, of the sort considered in detail later, in Theorem 4.3 (with the function b a scalar component of the outward unit normal ν).

Proposition 3.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ and consider the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with L as in Theorem 3.1. Also, pick a coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$ and bring in K_A the boundary-to-boundary double layer potential operator associated with Ω and A as in (3.24). In addition, for each $j,k \in \{1,\dots,n\}$, define the singular integral operator U_{jk} acting on each given matrix-valued function $F = (F_{\alpha s})_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}$ with entries belonging to $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ as $U_{jk}F = \big((U_{jk}F)_\gamma\big)_{1\leq\gamma\leq M}$ where, for each index $\gamma \in \{1,\dots,M\}$,*

$$\begin{aligned}
&(U_{jk}F)_\gamma(x)\\
&\quad := -\lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(x)-\nu_k(y)]\nu_j(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha s}(y)\,\mathrm{d}\sigma(y)\\
&\qquad + \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(x)-\nu_j(y)]\nu_k(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha s}(y)\,\mathrm{d}\sigma(y)\\
&\qquad + \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(y)-\nu_k(x)]\nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha j}(y)\,\mathrm{d}\sigma(y)
\end{aligned}$$

$$- \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} [\nu_j(y) - \nu_j(x)]\nu_s(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) F_{\alpha k}(y)\, d\sigma(y) \tag{3.35}$$

at σ-a.e. point $x \in \partial\Omega$. Finally, fix some integrability exponents $p, q \in (1, \infty]$ and consider a function

$$\begin{aligned} f &\in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\mathrm{loc}}(\partial\Omega, \sigma)\Big]^M \ \text{ with the property that} \\ \partial_{\tau_{jk}} f &\in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^q_{\mathrm{loc}}(\partial\Omega, \sigma)\Big]^M \ \text{ for all } \ j, k \in \{1, \dots, n\}. \end{aligned} \tag{3.36}$$

Then, for each $j, k \in \{1, \dots, n\}$, one has

$$\partial_{\tau_{jk}}(K_A f) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \tag{3.37}$$

where $\nabla_{\tan} f$ is regarded as the $M \times n$ matrix-valued function $F = (F_{\alpha s})_{\substack{1\le\alpha\le M \\ 1\le s\le n}}$ whose entry $F_{\alpha s}$ is the s-th component of $\nabla_{\tan} f_\alpha$.

Once again, assume $\Omega \subseteq \mathbb{R}^n$ is a UR domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, as before, continue to assume that L is a homogeneous constant complex coefficient weakly elliptic second-order $M \times M$ system in $\mathbb{R}^n$. In general, different choices of the coefficient tensor $A \in \mathfrak{A}_L$ yield different double layer potential operators, so it makes sense to use the subscript A to highlight the dependence on the choice of the coefficient tensor A. One integral operator of layer potential variety which is intrinsically associated with the given system L is the so-called single layer potential operator $\mathscr{S}$, whose integral kernel is the matrix-valued function $E(x-y)$, for all points $x, y \in \partial\Omega$. In order to make sense of the action of such an operator on any function as in (3.21), it is necessary to alter said integral kernel and consider the following modified single layer potential operator

$$\begin{aligned} \mathscr{S}_{\mathrm{mod}} f(x) &:= \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\, d\sigma(y) \ \text{ for each } \ x \in \Omega, \\ &\text{for each } \ f \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M, \quad \text{where } \ E_* := E \cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}. \end{aligned} \tag{3.38}$$

In this regard, it is worth noting that for each $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ the function $\mathscr{S}_{\mathrm{mod}} f$ is well defined, belongs to the space $\big[\mathscr{C}^\infty(\Omega)\big]^M$, and for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$, one has

$$\partial^\alpha(\mathscr{S}_{\mathrm{mod}} f)(x) = \int_{\partial\Omega} (\partial^\alpha E)(x-y) f(y)\, d\sigma(y) \ \text{ for each } \ x \in \Omega. \tag{3.39}$$

In particular,

$$L\big(\mathscr{S}_{\text{mod}} f\big) = 0 \text{ in } \Omega \text{ for each } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.40}$$

As noted in [114, §1.5], if $n \geq 3$ then for each aperture parameter $\kappa > 0$ and each truncation parameter $\varepsilon > 0$ we have

$$\mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\text{mod}} f) \in \bigcap_{0<p<\frac{n-1}{n-2}} L^p_{\text{loc}}(\partial\Omega, \sigma) \;\text{ for each }\; f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.41}$$

Analogously to (3.38), let us now define the following modified version of the boundary-to-boundary single layer operator

$$\begin{gathered} S_{\text{mod}} f(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\, \mathrm{d}\sigma(y) \;\text{ at } \sigma\text{-a.e. }\; x \in \partial\Omega, \\ \text{for each }\; f \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M, \quad \text{where }\; E_* := E \cdot \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}. \end{gathered} \tag{3.42}$$

Then this operator is meaningfully defined, via an absolutely convergent integral, at σ-a.e. point in $\partial\Omega$, and it has been shown in [114, §1.5] that for each $\varepsilon > 0$ the operator

$$S_{\text{mod}} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\big)\Big]^M \longrightarrow \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \tag{3.43}$$

is well defined, linear, and bounded. In particular, from (3.43) and the embedding in (2.573) we see that for each weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ the following mapping is well defined, linear, and bounded:

$$S_{\text{mod}} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M. \tag{3.44}$$

In addition, it has been shown in [114, §1.5] that

$$\begin{gathered} S_{\text{mod}} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\text{loc}}(\partial\Omega, w)\Big]^M \longrightarrow \big[L^p_{\text{loc}}(\partial\Omega, w)\big]^M \\ \text{is a well-defined, linear, and continuous mapping} \\ \text{for each weight }\; w \in A_p(\partial\Omega, \sigma) \;\text{ with }\; p \in (1, \infty), \end{gathered} \tag{3.45}$$

and (with $\mathrm{Lip}(\partial\Omega)$ denoting the space of scalar-valued Lipschitz functions on $\partial\Omega$)

given an arbitrary Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, it follows that for each sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[L^p(\partial\Omega, w)\big]^M$ which is weak-$*$ convergent to some function $f \in \big[L^p(\partial\Omega, w)\big]^M$, one has that the limit $\lim\limits_{j\to\infty} \int_{\partial\Omega} \langle S_{\rm mod} f_j, \phi\rangle \, d\sigma = \int_{\partial\Omega} \langle S_{\rm mod} f, \phi\rangle \, d\sigma$ holds for each test function $\phi \in \big[\mathrm{Lip}(\partial\Omega)\big]^M$ with compact support. (3.46)

Also, with the modified boundary-to-domain single layer operator $\mathscr{S}_{\rm mod}$ as in (3.38), for each aperture parameter $\kappa > 0$ and each $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M$, one has

$$\Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = (S_{\rm mod} f)(x) \ \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega. \tag{3.47}$$

See [114, §1.5] for proofs of all these claims, and for a more in-depth discussion on this topic.

Theorem 3.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a UR domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L = \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Recall the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with L as in Theorem 3.1 and define*

$$\begin{aligned} &k_\varepsilon^{(r\gamma\beta)} := (\partial_r E_{\gamma\beta}) \cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}} \ \ \textit{for each } \ \varepsilon > 0, \\ &\textit{each } \ \gamma, \beta \in \{1, \dots, M\} \ \textit{ and } \ r \in \{1, \dots, n\}. \end{aligned} \tag{3.48}$$

In this setting, consider the following modified version of the double layer operator (3.22)

$$\begin{aligned} &\big(\mathcal{D}_{A,mod} f\big)(x) \\ &\quad := \Big(- \int_{\partial\Omega} \nu_s(y) a_{rs}^{\beta\alpha} \big\{(\partial_r E_{\gamma\beta})(x-y) - k_1^{(r\gamma\beta)}(-y)\big\} f_\alpha(y)\, d\sigma(y) \Big)_{1\leq\gamma\leq M} \\ &\quad \textit{for each } f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \textit{ and } x \in \Omega, \end{aligned} \tag{3.49}$$

and consider the following modified boundary-to-boundary double layer potential operator (3.24)

$$K_{A,mod}f(x)$$

$$:= \Big(- \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} \nu_s(y) a^{\beta\alpha}_{rs} \big\{ k^{(r\gamma\beta)}_{\varepsilon}(x-y) - k^{(r\gamma\beta)}_{1}(-y) \big\} f_\alpha(y)\, d\sigma(y) \Big)_{1\leq\gamma\leq M}$$

$$\textit{for each } f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \textit{ and } \sigma\textit{-a.e. } x \in \partial\Omega, \tag{3.50}$$

Then the following properties hold.

(1) The operator $\mathcal{D}_{A,mod}$ is meaningfully defined, and satisfies

$$\mathcal{D}_{A,mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \textit{ and } \ L(\mathcal{D}_{A,mod} f) = 0 \ \textit{ in } \ \Omega, \qquad \textit{for each } \ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M. \tag{3.51}$$

In addition, the operator $\mathcal{D}_{A,mod}$ is compatible with $\mathcal{D}_A$ from (3.22), in the sense that for each function f belonging to the smaller space $\big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ the difference

$$C_f := \mathcal{D}_{A,mod} f - \mathcal{D} f \quad \textit{is a constant (belonging to } \mathbb{C}^M\textit{) in } \Omega. \tag{3.52}$$

As a consequence,

$$\nabla \mathcal{D}_{A,mod} f = \nabla \mathcal{D} f \ \textit{ in } \Omega \textit{ for each } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.53}$$

Moreover,

$\mathcal{D}_{A,mod}$ *maps constant ($\mathbb{C}^M$-valued) functions on $\partial\Omega$ into constant ($\mathbb{C}^M$-valued) functions in Ω.* (3.54)

In addition, at each point $x \in \Omega$ one may express

$$\partial_j\big(\mathcal{D}_{A,mod} f\big)(x) = \Big(- \int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs} (\partial_j \partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\, d\sigma(y) \Big)_{1\leq\gamma\leq M} \qquad \textit{for each } j \in \{1,\dots,n\} \textit{ and } f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^M. \tag{3.55}$$

Finally, given any function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \text{ with the property that}$$
$$\partial_{\tau_{jk}} f_\alpha \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \text{ for all } j,k \in \{1,\dots,n\} \text{ and } \alpha \in \{1,\dots,M\}, \tag{3.56}$$

it follows that for each index $\ell \in \{1,\dots,n\}$ and each point $x \in \Omega$, one has

$$\partial_\ell\big(\mathcal{D}_{A,mod} f\big)(x) = \Big(\int_{\partial\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}. \tag{3.57}$$

(2) Fix an aperture parameter $\kappa \in (0,\infty)$, a truncation parameter $\varepsilon > 0$, and an integrability exponent $p \in (1,\infty)$. Then the nontangential boundary trace

$$\begin{gathered}\big(\partial_\ell \mathcal{D}_{A,mod} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{C}^M\text{) at } \sigma\text{-a.e. point on } \partial\Omega,\\ \text{for each function } f \text{ as in (3.56) and each index } \ell \in \{1,\dots,n\}.\end{gathered} \tag{3.58}$$

Also, one has

$$\begin{gathered}\mathcal{N}_\kappa^\varepsilon\big(\nabla(\mathcal{D}_{A,mod} f)\big) \in L^p_{\text{loc}}(\partial\Omega,\sigma) \text{ for each function}\\ f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \text{ such that}\\ \partial_{\tau_{jk}} f_\alpha \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\\ \text{for all } j,k \in \{1,\dots,n\} \text{ and all } \alpha \in \{1,\dots,M\}.\end{gathered} \tag{3.59}$$

In addition,

$$\begin{gathered}\mathcal{N}_\kappa^\varepsilon(\mathcal{D}_{A,mod} f) \in L^p_{\text{loc}}(\partial\Omega,\sigma) \text{ for each function}\\ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\Big]^M.\end{gathered} \tag{3.60}$$

Furthermore, the following jump-formula holds:

$$\begin{gathered}\big(\mathcal{D}_{A,mod} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_{A,mod}\big)f \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ \text{for each given function } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M,\end{gathered} \tag{3.61}$$

where, as usual, I is the identity operator. As a consequence of (3.61) and (3.54), *the operator* $K_{A,mod}$ *maps constant* ($\mathbb{C}^M$*-valued) functions on* $\partial\Omega$ *into constant* ($\mathbb{C}^M$*-valued) functions on* $\partial\Omega$.

$$(3.62)$$

Finally, the operator $K_{A,mod}$ *(from* (3.50)*) is compatible with* K *(acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (3.24)*) in the sense that*

$$\begin{aligned}&\text{for each function } f \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M \text{ the difference}\\ &c_f := K_{mod} f - Kf \text{ is a constant (belonging to } \mathbb{C}^M) \text{ on } \partial\Omega.\end{aligned} \tag{3.63}$$

Moving on, in view of (3.63) and the fact that tangential derivatives annihilate locally constant functions, the following result from [114, §1.8] may be regarded as a generalization of Proposition 3.2.

Proposition 3.3 *Assume* $\Omega \subseteq \mathbb{R}^n$ *is a UR domain and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Consider an* $M \times M$ *homogeneous, second-order, constant complex coefficient, weakly elliptic system* L *in* $\mathbb{R}^n$*, and pick some coefficient tensor* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\1\leq r,s\leq n}}$ *for which* $L_A = L$. *Let* K_A *be the boundary-to-boundary double layer potential operator associated with* Ω *and* A *as in* (3.24)*, and bring in its modified version* $K_{A,mod}$ *from* (3.50)*. Finally, recall the family of singular integral operators* U_{jk} *with* $j,k \in \{1,\dots,n\}$ *defined in* (3.35) *and fix some integrability exponent* $p \in (1,\infty)$. *Then for each function*

$$\begin{aligned}f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n}}\big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M \ \text{such that}\\ \partial_{\tau_{jk}} f_\alpha \ \text{belongs to} \ L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\\ \text{for all} \ \ j,k \in \{1,\dots,n\} \ \text{and} \ \alpha \in \{1,\dots,M\},\end{aligned} \tag{3.64}$$

and each pair of indices $j,k, \in \{1,\dots,n\}$*, one has*

$$\partial_{\tau_{jk}}\big(K_{A,mod} f\big) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \tag{3.65}$$

where, as in the case of (3.37)*,* $\nabla_{\tan} f$ *is regarded as the* $M \times n$ *matrix-valued function whose* (α, s) *entry is the* s*-th component of the tangential gradient* $\nabla_{\tan} f_\alpha$.

We next introduce (and briefly elaborate on) the notion of conormal derivative operator associated with a given domain and a given coefficient tensor. Specifically, suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal $\nu = (\nu_1,\dots,\nu_n)$ is defined σ-a.e. on $\partial\Omega$. Also, fix a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\1\leq\alpha,\beta\leq M}}$ along with some aperture parameter $\kappa > 0$. In such a setting, for any function $u = (u_\beta)_{1\leq\beta\leq M} \in \big[W^{1,1}_{\rm loc}(\Omega)\big]^M$ with the property that the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists (in $\mathbb{C}^{M\times n}$) at σ-a.e. point on $\partial\Omega$ define the `conormal derivative` $\partial_\nu^A u$ as the $\mathbb{C}^M$-valued function

$$\partial_\nu^A u := \Big(\nu_r a_{rs}^{\alpha\beta}\big(\partial_s u_\beta\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)_{1\leq\alpha\leq M} \quad \text{at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{3.66}$$

In relation to this, it has been proved in [114, §1.5] that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ then for each function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ the conormal derivative $\partial_\nu^A \mathscr{S}_{\mathrm{mod}} f$ may be meaningfully considered in the sense of (3.66), and

$$\partial_\nu^A \mathscr{S}_{\mathrm{mod}} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \quad \text{at } \sigma\text{-a.e. point in } \ \partial\Omega, \tag{3.67}$$

where I is the identity, and $K^{\#}_{A^\top}$ is the operator associated as in (3.25) with the UR domain Ω and the transpose coefficient tensor $A^\top$.

We shall also need the following basic integral representation formula, established in [114, §1.8], for null-solutions of weakly elliptic systems in Ahlfors regular domains, in terms of modified boundary-to-domain layer potential operators.

Theorem 3.3 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an Ahlfors regular domain which is either bounded, or has an unbounded boundary. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, consider $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M\times M$ system in $\mathbb{R}^n$. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,mod}$ from (3.49), and the modified version of the single layer operator $\mathscr{S}_{mod}$ from (3.38). Finally, fix an aperture parameter $\kappa \in (0,\infty)$, a truncation parameter $\varepsilon \in (0,\infty)$, and consider a function $u : \Omega \to \mathbb{C}^M$ satisfying*

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \textit{in} \ \Omega, \quad \mathcal{N}_\kappa^\varepsilon u \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \textit{and} \ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^M, \\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \ \partial\Omega \ \textit{and} \ \mathcal{N}_\kappa(\nabla u) \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big). \end{gathered} \tag{3.68}$$

Then there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{A,mod}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathscr{S}_{mod}\big(\partial_\nu^A u\big) + c_u \ \ \textit{in} \ \ \Omega. \tag{3.69}$$

We proceed by recalling the following Fatou-type theorem established in [113, §3.3].

Theorem 3.4 *Suppose $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is an arbitrary UR domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a homogeneous constant (complex) coefficient second-order $M \times M$ system L in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$) which is weakly elliptic, and assume $u \in \left[\mathscr{C}^\infty(\Omega)\right]^M$ is a vector-valued function which, for some aperture parameter $\kappa > 0$, satisfies*

$$\begin{aligned} \mathcal{N}_\kappa(\nabla u) \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \ \ &\textit{for some} \ \ p \in (1, \infty] \\ \textit{and} \ \ Lu = 0 \ \ &\textit{in} \ \ \Omega. \end{aligned} \tag{3.70}$$

Then the nontangential boundary trace $\Big((\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x)$ exists (in $\mathbb{C}^{M\times n}$) at σ-a.e. point $x \in \partial\Omega$,

$$\textit{the function } (\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ belongs to the space } \left[L^p_{\mathrm{loc}}(\partial\Omega, \sigma)\right]^{M\times n}, \tag{3.71}$$

and

$$\Big|(\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big| \leq \mathcal{N}_\kappa(\nabla u) \ \ \textit{at } \sigma\textit{-a.e. point on} \ \ \partial\Omega. \tag{3.72}$$

A combination of Theorems 3.3 and 3.4 gives the following basic result.

Corollary 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an NTA domain with an unbounded Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For $M \in \mathbb{N}$, consider $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$.*

Once again, recall the modified version of the double layer operator $\mathcal{D}_{A,mod}$ from (3.49), and the modified version of the single layer operator $\mathscr{S}_{mod}$ from (3.38). Finally, fix an aperture parameter $\kappa \in (0, \infty)$ along with an integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. In this setting, consider a function $u : \Omega \to \mathbb{C}^M$ satisfying

$$u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \quad Lu = 0 \ \ \textit{in} \ \ \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w). \tag{3.73}$$

Then

$$\begin{gathered} u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \ \textit{exists } \sigma\textit{-a.e. on} \ \ \partial\Omega \ \ \textit{and belongs to} \ \ \left[\dot{L}^p_1(\partial\Omega, w)\right]^M, \\ (\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \ \textit{exists } \sigma\textit{-a.e. on} \ \ \partial\Omega, \ \ \textit{and} \ \ \partial^A_\nu u \ \ \textit{belongs to} \ \ \left[L^p(\partial\Omega, w)\right]^M, \end{gathered} \tag{3.74}$$

and there exists some $c_u \in \mathbb{C}^M$ with the property that

$$u = \mathcal{D}_{A,mod}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) - \mathscr{S}_{mod}\big(\partial^A_\nu u\big) + c_u \ \ \textit{in} \ \ \Omega. \tag{3.75}$$

Proof From Proposition 2.24, we see that $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$. In concert, the membership in (3.73), (2.608), (2.11) (used with $\sigma := w$), and (2.576) also implies that $\mathcal{N}_\kappa^\varepsilon u \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ for each $\varepsilon > 0$. Next, the present hypotheses on Ω ensure (cf. (2.48)) that Ω is a UR domain. Keeping this in mind, the Fatou-type result from Theorem 3.4 guarantees that the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists (in $\mathbb{C}^{M\cdot n}$) at σ-a.e. point on $\partial\Omega$. In particular, $\partial_\nu^A u$ is well defined and belongs to the space $\big[L^p(\partial\Omega, w)\big]^M$ (cf. (3.66), (3.71)–(3.72)). Hence, all conditions in (3.68) are satisfied, and this permits us to invoke Theorem 3.3 to conclude that (3.75) holds (for some constant $c_u \in \mathbb{C}^M$, given that the hypotheses on Ω ensure that this set is connected). $\square$

3.2 SIOs on Muckenhoupt Weighted Lebesgue and Sobolev Spaces

We begin by considering garden variety Calderón–Zygmund singular integral operators (SIOs), i.e., operators of convolution-type with odd, homogeneous, sufficiently smooth kernels, which otherwise lack any particular algebraic characteristics. The goal is to obtain estimates in Muckenhoupt weighted Lebesgue spaces on UR sets in $\mathbb{R}^n$.

Proposition 3.4 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N\big(\mathbb{R}^n \setminus \{0\}\big)$ which is odd and positive homogeneous of degree $1-n$. Also, fix an integrability exponent $p \in (1, \infty)$, along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$. In this setting, for each $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, define*

$$T_\varepsilon f(x) := \int_{\substack{y\in\Sigma\\ |x-y|>\varepsilon}} k(x-y) f(y)\, \mathrm{d}\sigma(y) \ \textit{for all}\ x \in \Sigma \ \textit{and}\ \varepsilon > 0, \tag{3.76}$$

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \ \textit{ for each }\ x \in \Sigma, \tag{3.77}$$

$$T f(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \ \textit{ for } \sigma\textit{-a.e.}\ x \in \Sigma. \tag{3.78}$$

Then there exists a constant $C \in (0, \infty)$ which depends exclusively on n, p, $[w]_{A_p}$, and the UR constants of Σ (and which stays bounded as $[w]_{A_p}$ stays bounded) with the property that for each $f \in L^p(\Sigma, w)$, one has

$$\|T_* f\|_{L^p(\Sigma,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{L^p(\Sigma,w)}\,. \tag{3.79}$$

In particular,

> *the truncated integral operators $T_\varepsilon : L^p(\Sigma, w) \to L^p(\Sigma, w)$ are well defined, linear, and bounded in a uniform fashion with respect to the truncation parameter $\varepsilon > 0$.* (3.80)

Moreover, for each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, the limit defining $Tf(x)$ in (3.78) exists at σ-a.e. $x \in \Sigma$ and the operator

$$T : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{3.81}$$

is well defined, linear, and bounded. Let $p' \in (1, \infty)$ denote the Hölder conjugate exponent of p, and, with $w' := w^{1-p'} \in A_{p'}(\Sigma, \sigma)$, consider the natural identification

$$\big(L^p(\Sigma, w)\big)^* = L^{p'}(\Sigma, w'). \tag{3.82}$$

Then, under the canonical integral pairing $(f, g) \mapsto \int_\Sigma fg\,\mathrm{d}\sigma$, it follows that

> *the (real) transpose of the operator (3.81) is the operator $-T : L^{p'}(\Sigma, w') \to L^{p'}(\Sigma, w')$.* (3.83)

Finally, assume $\Omega \subseteq \mathbb{R}^n$ is an open set such that $\partial\Omega$ is a UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and pick an aperture parameter $\kappa > 0$. With the integral kernel k as before, for each $f \in L^p(\partial\Omega, w)$, define

$$\mathcal{T} f(x) := \int_{\partial\Omega} k(x - y) f(y)\,\mathrm{d}\sigma(y) \quad \textit{for each} \quad x \in \Omega. \tag{3.84}$$

Then there exists a constant $C \in (0, \infty)$ which depends exclusively on n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ with the property that for each $f \in L^p(\partial\Omega, w)$, one has

$$\|\mathcal{N}_\kappa(\mathcal{T} f)\|_{L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{L^p(\partial\Omega,w)}\,. \tag{3.85}$$

Also, for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, one has the jump-formula

$$\Big(\mathcal{T} f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) = \frac{1}{2\mathrm{i}}\widehat{k}\big(\nu(x)\big) f(x) + (Tf)(x) \quad \textit{at } \sigma\textit{-a.e.} \ \ x \in \partial_*\Omega, \tag{3.86}$$

where $\widehat{k}$ denotes the Fourier transform of k. In particular, the jump-formula (3.86) is valid for each function $f \in L^p(\partial\Omega, w)$.

The above proposition points to uniform rectifiability as being intimately connected with the boundedness of a large class of Calderón–Zygmund like operators on Muckenhoupt weighted Lebesgue spaces. From the work of G. David and S. Semmes (cf. [40, 41]) and F. Nazarov, X. Tolsa, and A. Volberg in [118] (see also [96] for similar results proved earlier in the plane), we know that UR sets make up the most general context in which convolution-like singular integral operators are bounded on ordinary Lebesgue spaces. Moreover, under the background assumption of Ahlfors regularity, uniform rectifiability is implied[1] by the simultaneous L^2 boundedness of all truncated integral convolution type operators T_ε on Σ (cf. (3.76)) uniformly with respect to the truncation $\varepsilon > 0$, whose kernels are smooth, odd, and positive homogeneous of degree $1 - n$ in $\mathbb{R}^n \setminus \{0\}$. In light of (3.80), the above discussion highlights the optimality of demanding that Σ is a UR set in the context of Proposition 3.4. One of the early works on the higher-dimensional theory of singular integral operators in rough geometric settings is [23]; see also the survey paper [97] for an informative account of the development of this topic.

Results like Proposition 3.4 have been recently established in [113, §2.3-§2.5]. Here we present an alternative approach that makes essential use of the Fefferman–Stein sharp maximal function, considered in the setting of spaces of homogeneous type (for the Euclidean context, see [69, p. 52], [52, Theorem 3.6, p. 161]).

Proof of Proposition 3.4 To set the stage, recall the Fefferman–Stein sharp maximal operator $M^\#$ on Σ, acting on each function $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$ according to

$$M^\# f(x) := \sup_{\Delta \ni x} ⨍_\Delta \left| f - ⨍_\Delta f \, \mathrm{d}\sigma \right| \mathrm{d}\sigma, \qquad \forall\, x \in \Sigma, \tag{3.87}$$

where the supremum is taken over all surface balls $\Delta \subseteq \Sigma$ containing the point $x \in \Sigma$. Clearly, for each $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$ and each $x \in \Sigma$, we have

$$\sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} ⨍_\Delta |f - a| \, \mathrm{d}\sigma \leq M^\# f(x) \leq 2 \sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} ⨍_\Delta |f - a| \, \mathrm{d}\sigma. \tag{3.88}$$

Also, given $\alpha \in (0, 1)$, for each $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$, set

$$M^\#_\alpha f(x) := M^\#(|f|^\alpha)(x)^{1/\alpha} \quad \text{for all} \quad x \in \Sigma. \tag{3.89}$$

[1] In [40], the authors have dealt with the class of truncated singular integral operators associated with kernels in $\mathbb{R}^n \setminus \{0\}$ which are smooth, odd, and satisfy $\sup_{x \in \mathbb{R}^n \setminus \{0\}} \Big[|x|^{(n-1)+|\alpha|} \big| (\partial^\alpha k)(x) \big| \Big] < +\infty$ for all $\alpha \in \mathbb{N}_0^n$. In [118], it was shown that the truncated Riesz transforms on Σ alone will do.

Since having $0 < \alpha < 1$ ensures that $|X^\alpha - Y^\alpha| \le |X - Y|^\alpha$ for all $X, Y \in [0, \infty)$, from (3.89) and the last inequality in (3.88), one may readily check that

$$M^\#_\alpha f(x) \le 2^{1/\alpha} \sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} \left(⨍_\Delta |f - a|^\alpha \, d\sigma \right)^{1/\alpha} \tag{3.90}$$

for each $f \in L^1_{\rm loc}(\Sigma, \sigma)$ and each $x \in \Sigma$. Finally, recall from (2.522) the (non-centered) Hardy–Littlewood maximal operator $\mathcal{M}$ on Σ.

From (3.76)–(3.78), it is clear that the maximal operator T_* and the principal-value singular integral operator T depend in a homogeneous fashion on the kernel function k. In view of this observation, by working with k/K (in the case when k is not identically zero) where $K := \sum_{|\alpha| \le N} \sup_{S^{n-1}} |\partial^\alpha k|$, there is no loss of generality in assuming that

$$\sum_{|\alpha| \le N} \sup_{S^{n-1}} |\partial^\alpha k| = 1. \tag{3.91}$$

The fact that for each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, the limit defining $Tf(x)$ in (3.78) exists at σ-a.e. $x \in \Sigma$ has been proved in [113, §2.3]. To proceed, denote by $L^\infty_{\rm comp}(\Sigma, \sigma)$ the subspace of $L^\infty(\Sigma, \sigma)$ consisting of functions with compact support. Also, fix a power $\alpha \in (0, 1)$. We will first show that there exists a constant $C = C(\Sigma, n, \alpha) \in (0, \infty)$ such that

$$\begin{gathered} M^\#_\alpha(Tf)(x) \le C \cdot \mathcal{M} f(x) \\ \text{for all } f \in L^\infty_{\rm comp}(\Sigma, \sigma) \text{ and } x \in \Sigma. \end{gathered} \tag{3.92}$$

To this end, fix a function $f \in L^\infty_{\rm comp}(\Sigma, \sigma)$ along with a point $x \in \Sigma$, and consider a surface ball $\Delta = \Delta(x_0, r_0)$, with center at $x_0 \in \Sigma$ and radius $r_0 > 0$, containing the point x. Decompose $f = f_1 + f_2$, where $f_1 := f\mathbf{1}_{2\Delta}$ and $f_2 := f\mathbf{1}_{\Sigma \setminus 2\Delta}$. Then $\big|Tf_2(x_0)\big| < +\infty$ and we abbreviate $a := Tf_2(x_0) \in \mathbb{C}$. Note that

$$⨍_\Delta |Tf - a|^\alpha \, d\sigma \le ⨍_\Delta |Tf_1|^\alpha \, d\sigma + ⨍_\Delta |Tf_2 - a|^\alpha \, d\sigma. \tag{3.93}$$

For the first term in the right-hand side of (3.93), using Kolmogorov's inequality, the fact that T is bounded from $L^1(\Sigma, \sigma)$ to $L^{1,\infty}(\Sigma, \sigma)$ (cf. [113, §2.3], [61, Proposition 3.19]), and the fact that Σ is an Ahlfors regular set to write

$$\begin{aligned} ⨍_\Delta |Tf_1|^\alpha \, d\sigma &\le \frac{C_\alpha}{\sigma(\Delta)^\alpha} \|Tf_1\|^\alpha_{L^{1,\infty}(\Sigma,\sigma)} \le \frac{C_\alpha}{\sigma(\Delta)^\alpha} \|f_1\|^\alpha_{L^1(\Sigma,\sigma)} \\ &\le C_\alpha \left(⨍_{2\Delta} |f| \, d\sigma \right)^\alpha \le C_\alpha \cdot \mathcal{M} f(x)^\alpha. \end{aligned} \tag{3.94}$$

For the second term in the right-hand side of (3.93), note that the properties of k and (3.91) entail

$$|(\nabla k)(z)| = \Big|(\nabla k)\Big(\frac{z}{|z|}|z|\Big)\Big| \leq |z|^{-n} \sup_{|\omega|=1} |(\nabla k)(\omega)| = C_n|z|^{-n}, \tag{3.95}$$

for each $z \in \mathbb{R}^n \setminus \{0\}$, where $C_n \in (0, \infty)$ is a purely dimensional constant. On account of (3.95) and the Mean Value Theorem, we see that there exists a dimensional constant $C_n \in (0, \infty)$ with the property that for each $y \in \Delta$ and $z \in \Sigma \setminus 2\Delta$ we have

$$|k(y - z) - k(x_0 - z)| \leq C_n \frac{|y - x_0|}{|x_0 - z|^n} \leq \frac{C_n r_0}{|x_0 - z|^n}. \tag{3.96}$$

Using this, for every $y \in \Delta$, we may write

$$\begin{aligned} |Tf_2(y) - a| &= |Tf_2(y) - Tf_2(x_0)| \\ &\leq \int_{\Sigma\setminus 2\Delta} |k(y - z) - k(x_0 - z)||f(z)|\,\mathrm{d}\sigma(z) \\ &\leq Cr_0 \sum_{j=1}^{\infty} \int_{2^j r_0 \leq |x_0 - z| < 2^{j+1} r_0} \frac{|f(z)|}{|x_0 - z|^n}\,\mathrm{d}\sigma(z) \\ &\leq C \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta} |f(z)|\,\mathrm{d}\sigma(z) \\ &\leq C \cdot \mathcal{M}f(x), \end{aligned} \tag{3.97}$$

where $C \in (0, \infty)$ depends only on dimension and the Ahlfors regularity constant of Σ. At this stage, the claim in (3.92) follows by combining (3.90), (3.93), (3.94), and (3.97).

We shall now analyze two cases, depending on whether Σ is bounded or not. Consider first the case when Σ is unbounded. In such a setting, the A_∞-weighted version of the Fefferman–Stein inequality for spaces of homogeneous type (cf., e.g., [8, Sections 3.2 and 5]) gives that for every $q \in (0, \infty)$ there exists some constant $C_w \in (0, \infty)$, which depends on the weight $w \in A_p(\Sigma, \sigma) \subseteq A_\infty(\Sigma, \sigma)$ only through its characteristic $[w]_{A_p}$ (indeed, it can be expressed as an increasing function of $[w]_{A_p}$), such that

$$\begin{aligned} &\|\mathcal{M}g\|_{L^q(\Sigma,w)} \leq C_w \left\|M^{\#}g\right\|_{L^q(\Sigma,w)} \quad \text{for each} \\ &g \in L^1_{\mathrm{loc}}(\Sigma, \sigma) \ \text{ such that } \ \mathcal{M}g \in L^q(\Sigma, w). \end{aligned} \tag{3.98}$$

To proceed, fix $\alpha \in (0,1)$ and $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$. Let us momentarily work under the additional assumption that the weight w belongs to $L^\infty(\Sigma,\sigma)$. This permits us to estimate

$$\begin{aligned}\big\|\mathcal{M}(|Tf|^\alpha)\big\|_{L^{p/\alpha}(\Sigma,w)} &\leq \|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)}\,\big\|\mathcal{M}(|Tf|^\alpha)\big\|_{L^{p/\alpha}(\Sigma,\sigma)}\\ &\leq C\,\|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)}\,\|Tf\|^\alpha_{L^p(\Sigma,\sigma)}\\ &\leq C\,\|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)}\,\|f\|^\alpha_{L^p(\Sigma,\sigma)} < +\infty,\end{aligned} \tag{3.99}$$

where we have used the boundedness of $\mathcal{M}$ on $L^{p/\alpha}(\Sigma,\sigma)$ and the boundedness of T on $L^p(\Sigma,\sigma)$ (cf. [61, Proposition 3.18]). This allows us to use (3.98) (with $g := |Tf|^\alpha$ and $q := p/\alpha$) to obtain, for some constant $C_w \in (0,\infty)$ (again, depending in an increasing fashion on $[w]_{A_p}$),

$$\begin{aligned}\|Tf\|_{L^p(\Sigma,w)} &\leq \Big\|\mathcal{M}(|Tf|^\alpha)^{1/\alpha}\Big\|_{L^p(\Sigma,w)} = \big\|\mathcal{M}(|Tf|^\alpha)\big\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)}\\ &\leq C_w\,\Big\|M^{\#}(|Tf|^\alpha)\Big\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} = C_w\,\Big\|M^{\#}_\alpha(Tf)\Big\|_{L^p(\Sigma,w)}\\ &\leq C_w\,\|\mathcal{M}f\|_{L^p(\Sigma,w)} \leq C_w\,\|f\|_{L^p(\Sigma,w)},\end{aligned} \tag{3.100}$$

where the first inequality follows from Lebesgue's Differentiation Theorem (cf. [7]), the last equality is a consequence of (3.89), the penultimate inequality comes from (3.92), and the last inequality is implied by the boundedness of the Hardy–Littlewood operator $\mathcal{M}$ on $L^p(\Sigma,w)$.

To remove the restriction $w \in L^\infty(\Sigma,\sigma)$, we proceed as follows. For each integer $j \in \mathbb{N}$, let $w_j := \min\{w, j\} \in L^\infty(\Sigma,\sigma)$. Moreover, as in [57, Ex. 9.1.9], we have

$$[w_j]_{A_p} \leq C_p(1 + [w]_{A_p}) \tag{3.101}$$

for some $C_p \in (0,\infty)$ independent of $j \in \mathbb{N}$. As such, we may invoke (3.100) written for each w_j (which now involves a constant whose dependence of w_j may be expressed in terms of a non-decreasing function acting on $[w_j]_{A_p}$) to conclude that

$$\|Tf\|_{L^p(\Sigma,w_j)} \leq C\,\|f\|_{L^p(\Sigma,w_j)} \leq C\,\|f\|_{L^p(\Sigma,w)}, \tag{3.102}$$

for some constant $C \in (0,\infty)$ independent of $j \in \mathbb{N}$. Upon letting $j \to \infty$ and relying on Lebesgue's Monotone Convergence Theorem, we arrive at the conclusion that $\|Tf\|_{L^p(\Sigma,w)} \leq C\,\|f\|_{L^p(\Sigma,w)}$ for every $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$. Given that $L^\infty_{\rm comp}(\Sigma,\sigma)$ is dense in $L^p(\Sigma,w)$, this ultimately establishes the boundedness of the operator T in the context of (3.81) when Σ is unbounded.

Let us now consider the case when Σ is bounded. In this case, compared to (3.98), the A_∞-weighted version of the Fefferman–Stein inequality includes an extra term; namely, it now reads (cf. [8, Sections 3.2 and 5])

$$\begin{aligned}\|\mathcal{M}g\|_{L^q(\Sigma,w)} \leq C_w \left\| M^{\#}g \right\|_{L^q(\Sigma,w)} \\ + C\sigma(\Sigma)^{-1}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/q}\|g\|_{L^1(\Sigma,\sigma)} \qquad (3.103)\\ \text{for all } g \in L^1(\Sigma,\sigma) \text{ with } \mathcal{M}g \in L^q(\Sigma,w),\end{aligned}$$

where $C_w \in (0,\infty)$ is as before and $C \in (0,\infty)$ is a purely geometric constant. Fix $\alpha \in (0,1)$ and $f \in L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$. Assume first that $w \in L^\infty(\Sigma,\sigma)$ and note that (3.99) holds in the same way. This permits us to invoke (3.103) (with $g := |Tf|^\alpha$ and $q := p/\alpha$), so in place of (3.100), we now get

$$\begin{aligned}\|Tf\|_{L^p(\Sigma,w)} &\leq \left\|\mathcal{M}(|Tf|^\alpha)\right\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)}\\ &\leq C_w\left\|M^{\#}(|Tf|^\alpha)\right\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} + C\sigma(\Sigma)^{-1/\alpha}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/p}\left\||Tf|^\alpha\right\|^{1/\alpha}_{L^1(\Sigma,\sigma)}\\ &\leq C_w\,\|f\|_{L^p(\Sigma,w)} + C\sigma(\Sigma)^{-1/\alpha}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/p}\|Tf\|_{L^\alpha(\Sigma,\sigma)}, \qquad (3.104)\end{aligned}$$

where the first and last estimates follow as before. Here, the constant $C_w \in (0,\infty)$ depends on w only through its characteristic $[w]_{A_p}$ (again, this may be expressed as an increasing function of $[w]_{A_p}$), while $C \in (0,\infty)$ depends just on p, α, n, and the Ahlfors regularity constant of Σ.

It remains to estimate $\|Tf\|_{L^\alpha(\Sigma,\sigma)}$ in a satisfactory manner. Using Kolmogorov's inequality and the fact that T is bounded from $L^1(\Sigma,\sigma)$ into $L^{1,\infty}(\Sigma,\sigma)$ (cf. [113, §2.3], [61, Proposition 3.19]) and Hölder's inequality, we obtain

$$\begin{aligned}\|Tf\|_{L^\alpha(\Sigma,\sigma)} &\leq (1-\alpha)^{-1/\alpha}\sigma(\Sigma)^{(1-\alpha)/\alpha}\,\|Tf\|_{L^{1,\infty}(\Sigma,\sigma)}\\ &\leq C\sigma(\Sigma)^{(1-\alpha)/\alpha}\,\|f\|_{L^1(\Sigma,\sigma)}\\ &\leq C\sigma(\Sigma)^{(1-\alpha)/\alpha}\Big(\int_\Sigma w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}\|f\|_{L^p(\Sigma,w)}. \qquad (3.105)\end{aligned}$$

Let us record our progress. The argument so far proves that, if Σ is bounded, then for each $f \in L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$ we have

$$\|Tf\|_{L^p(\Sigma,w)} \leq \Big(C_w + C\,\sigma(\Sigma)^{-1}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/p}\Big(\int_\Sigma w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}\Big)\|f\|_{L^p(\Sigma,w)}$$
$$\leq \big(C_w + C\,[w]_{A_p}^{1/p}\big)\,\|f\|_{L^p(\Sigma,w)}\,, \tag{3.106}$$

where $C_w \in (0,\infty)$ is as above. As before, to remove the restriction $w \in L^\infty(\Sigma,\sigma)$, we work with $w_j := \min\{w, j\}$ for $j \in \mathbb{N}$. Thanks to (3.101) the constant in the right-hand side of (3.106) may be controlled uniformly in j. After passing to limit $j \to \infty$ and once again relying on the density $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$ into $L^p(\Sigma,w)$, we eventually conclude that the operator T is bounded in the context of (3.81) in this case as well. Moreover,

$$\|T\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C, \tag{3.107}$$

where $C \in (0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the UR constants of Σ. This finishes the proof of (3.81).

Next, recall Cotlar's inequality, to the effect that there exists some $C \in (0,\infty)$ which depends only on n, and the Ahlfors regularity constant of Σ, with the property that for every function $f \in L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$, we have

$$(T_* f)(x) \leq C\cdot\mathcal{M}(Tf)(x) + C\cdot\mathcal{M}f(x) \quad \text{for each} \quad x \in \Sigma. \tag{3.108}$$

Then (3.79) follows from (3.81), (3.108), the boundedness of the Hardy–Littlewood operator $\mathcal{M}$ on $L^p(\Sigma,w)$, and a density argument. Going further, (3.83) may be justified by first establishing a similar claim for the truncated operators (3.76) using Fubini's theorem and then invoking Lebesgue's Dominated Convergence Theorem (whose applicability is guaranteed by (3.79)) to pass to limit as $\varepsilon \to 0^+$.

Consider next the claims made in the last part of the statement. It is apparent from (3.84) that the boundary-to-domain operator $\mathcal{T}$ depends in a homogeneous fashion on the kernel function k. Much as before, this permits us to work under the additional assumption that (3.91) holds. Granted this, the estimate claimed in (3.85) is a direct consequence of inequality (3.79) and the formula (cf. [61, eq. (3.2.22)])

$$\mathcal{N}_\kappa\big(\mathcal{T}f\big)(x) \leq C\cdot T_* f(x) + C\cdot\mathcal{M}f(x) \quad \text{for each} \quad x \in \Sigma, \tag{3.109}$$

where $C \in (0,\infty)$ depends only on n and the Ahlfors regularity constant of Σ and where the maximal operator T_* and the Hardy–Littlewood maximal function $\mathcal{M}$ are now associated with the UR set $\Sigma := \partial\Omega$.

That the jump-formula (3.86) holds for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ has been established in [113, §2.5]. With this in hand, the very last claim in the statement of Proposition 3.4 is implied by (2.575). □

The stage has been set for considering the action of the boundary layer potentials associated with a given weakly elliptic system L and a given UR domain Ω in $\mathbb{R}^n$ as in (3.22)–(3.25) and (3.38) on Muckenhoupt weighted Lebesgue and Sobolev spaces on $\partial\Omega$. To state our main result in this regard, given any two Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, denote

$$\mathrm{Bd}\big(X \to Y\big) := \big\{T : X \to Y : T \text{ linear and bounded}\big\}, \tag{3.110}$$

and equip it with the standard operator norm $\mathrm{Bd}\big(X \to Y\big) \ni T \mapsto \|T\|_{X\to Y}$ (cf. (4.1)). Finally, corresponding to the case when $Y = X$, we agree to abbreviate

$$\mathrm{Bd}(X) := \mathrm{Bd}\big(X \to X\big). \tag{3.111}$$

Proposition 3.5 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a UR domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Pick $A \in \mathfrak{A}_L$ and consider the boundary layer potential operators $\mathcal{D}_A$, K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in (3.22), (3.24), and (3.25). Also, recall the modified single layer potential operator $\mathscr{S}_{mod}$ associated with Ω and L as in (3.38). Finally, fix an integrability exponent $p \in (1,\infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and an aperture parameter $\kappa > 0$.*

1. *The following operators are well defined, sub-linear, and bounded:*

$$\big[L^p(\partial\Omega, w)\big]^M \ni f \longmapsto \mathcal{N}_\kappa\big(\mathcal{D}_A f\big) \in L^p(\partial\Omega, w), \tag{3.112}$$

$$\big[L_1^p(\partial\Omega, w)\big]^M \ni f \longmapsto \mathcal{N}_\kappa\,(\nabla\mathcal{D}_A f) \in L^p(\partial\Omega, w). \tag{3.113}$$

Also,

$$\begin{aligned} &\textit{for each } f \in \big[L_1^p(\partial\Omega, w)\big]^M \textit{ the nontangential trace} \\ &\big(\nabla\mathcal{D}_A f\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \textit{ exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega. \end{aligned} \tag{3.114}$$

As a consequence of (3.114), (3.33), (3.66), (2.586), *and Proposition 3.4,*

$$\begin{aligned} &\textit{the map } \big[L_1^p(\partial\Omega, w)\big]^M \ni f \longmapsto \partial_\nu^A\big(\mathcal{D}_A f\big) \in \big[L^p(\partial\Omega, w)\big]^M \textit{ is} \\ &\textit{well defined, linear, and bounded, and there exists } C \in (0,\infty) \textit{ so} \\ &\textit{that } \big\|\partial_\nu^A\big(\mathcal{D}_A f\big)\big\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \textit{ for each} \\ &f \textit{ in the Muckenhoupt weighted Sobolev space } \big[L_1^p(\partial\Omega, w)\big]^M. \end{aligned} \tag{3.115}$$

2. *For every $f \in \big[L^p(\partial\Omega, w)\big]^M$, the limits in* (3.24) *and* (3.25) *exist at σ-a.e. point on $\partial\Omega$. Moreover, the operators K_A and $K_A^{\#}$ are well defined, linear, and bounded in the following contexts:*

$$K_A : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M, \tag{3.116}$$

$$K_A : \big[L^p_1(\partial\Omega, w)\big]^M \longrightarrow \big[L^p_1(\partial\Omega, w)\big]^M, \tag{3.117}$$

$$K^\#_A : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M. \tag{3.118}$$

Moreover, under the canonical integral pairing $(f, g) \mapsto \int_{\partial\Omega} \langle f, g\rangle \, d\sigma$, *it follows that*

the (real) transpose of the operator K_A *acting on the space* $\big[L^p(\partial\Omega, w)\big]^M$ *is the operator* $K^\#_A$ *acting on the space* $\big[L^{p'}(\partial\Omega, w')\big]^M$ *where* $p' \in (1, \infty)$ *is the Hölder conjugate exponent of* p *and* $w' := w^{1-p'} \in A_{p'}(\Sigma, \sigma)$. (3.119)

Additionally, the operators K_A, $K^\#_A$ *in* (3.116)–(3.118) *depend continuously on the underlying coefficient tensor* A. *More specifically, with the piece of notation introduced in* (3.13), *the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega, w)\big]^M\Big), \tag{3.120}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[L^p_1(\partial\Omega, w)\big]^M\Big), \tag{3.121}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K^\#_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega, w)\big]^M\Big). \tag{3.122}$$

Furthermore, the nontangential boundary trace of the boundary-to-domain double layer is related to the boundary-to-boundary double layer via a jump-formula, to the effect that for every $f \in \big[L^p(\partial\Omega, w)\big]^M$ *and* σ*-a.e. in* $\partial\Omega$, *one has*

$$\mathcal{D}_A f\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_A\big) f, \tag{3.123}$$

where I *is the identity operator.*

3. *For each* $f \in \big[L^p(\partial\Omega, w)\big]^M$, *one has*

$$\mathscr{S}_{mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{mod} f\big) = 0 \ \textit{ in } \ \Omega. \tag{3.124}$$

In addition, the trace

$$\big(\nabla \mathscr{S}_{mod} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \textit{ exists (in } \mathbb{C}^{M\cdot n}\textit{) at } \sigma\textit{-a.e. point on } \ \partial\Omega, \tag{3.125}$$

and the conormal derivative of the modified boundary-to-domain single layer satisfies the following jump-formula:

$$\partial_\nu^A \mathscr{S}_{mod} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) f \ \text{ at } \sigma\text{-a.e. point in } \ \partial\Omega, \tag{3.126}$$

where I is the identity, and $K^{\#}_{A^\top}$ is the operator associated as in (3.25) with the UR domain Ω and the transpose coefficient tensor $A^\top$. Also, there exists some constant $C = C(\Omega, p, w, L, \kappa) \in (0, \infty)$ independent of f such that

$$\big\|\mathcal{N}_\kappa(\nabla \mathscr{S}_{mod} f)\big\|_{L^p(\partial\Omega, w)} \le C\|f\|_{[L^p(\partial\Omega,w)]^M}. \tag{3.127}$$

4. *For each function $f \in \big[L^p(\partial\Omega, w)\big]^M$ and σ-a.e. point $x \in \partial\Omega$, one has*

$$\partial_{\tau_{jk}}\big(S_{mod} f\big)(x) = \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k E)(x-y) - \nu_k(x)(\partial_j E)(x-y)\big\} f(y)\,\mathrm{d}\sigma(y) \tag{3.128}$$

for each $j, k \in \{1, \ldots, n\}$, and

$$\begin{aligned}&\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) f\Big)(x) \\ &= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{js}}\big(S_{mod} f\big)_\alpha(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\le\mu\le M}.\end{aligned} \tag{3.129}$$

5. *For each $f \in \big[L^p_1(\partial\Omega, w)\big]^M$, there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$, one has*

$$\big(\tfrac{1}{2}I + K_A\big)\Big(\big(-\tfrac{1}{2}I + K_A\big) f\Big) = S_{mod}\Big(\partial^A_\nu\big(\mathscr{D}_A f\big)\Big) + c_f. \tag{3.130}$$

6. *The operator*

$$S_{mod} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \tag{3.131}$$

is well defined, linear, and bounded, when the target space is endowed with the semi-norm introduced in (2.599)*. As a consequence, if $\big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M$ denotes the M-th power of the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, equipped with the semi-norm* (2.601)*, then the operator*

$$\begin{aligned}&\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \ \text{ defined as} \\ &\big[S_{mod}\big] f := \big[S_{mod} f\big] \in \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M, \quad \forall\, f \in \big[L^p(\partial\Omega, w)\big]^M\end{aligned} \tag{3.132}$$

is well defined, linear, and bounded.

Proof With the exception of (3.120)–(3.122) and (3.128)–(3.131), all claims may be justified based on (3.22)–(3.40), Lemma 2.15, Proposition 3.1, Proposition 2.22, Proposition 3.4, and Theorem 3.1. The continuity properties of the operator-valued maps in (3.120)–(3.122), as well as formulas (3.128), (3.129), (3.130) have been proved in [114, §1.5]. Finally, (3.131) is a consequence of (2.598)–(2.599), (3.44)–(3.45), (3.128), and (3.81) in Proposition 3.4. □

Our next theorem contains fundamental properties of modified double layer potential operators acting on homogeneous Muckenhoupt weighted Sobolev spaces, considered on boundaries of uniformly rectifiable domains.

Theorem 3.5 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a UR domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In addition, for some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that $L := L_A$ as in (3.7) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Also, let $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ be the matrix-valued fundamental solution associated with L as in Theorem 3.1. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,mod}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (3.49). Finally, fix some aperture parameter $\kappa \in (0,\infty)$ along with an integrability exponent $p \in (1,\infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$.*

Then there exists some constant $C = C(\Omega, n, p, \kappa) \in (0,\infty)$ with the property that for each function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ it follows that

$$\begin{gathered}\mathcal{D}_{A,mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,mod} f\big) = 0 \ \text{ in } \ \Omega,\\ \big(\mathcal{D}_{A,mod} f\big)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}, \ \big(\nabla\mathcal{D}_{A,mod} f\big)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ exist } \sigma\text{-a.e. on } \ \partial\Omega,\\ \mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,mod} f\big) \ \text{ belongs to } \ L^p(\partial\Omega, w) \ \text{ and}\\ \big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,mod} f\big)\big\|_{L^p(\partial\Omega,w)} \leq C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}.\end{gathered} \tag{3.133}$$

In fact, for each function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, one has

$$(\mathcal{D}_{A,mod} f)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{A,mod}\big) f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \tag{3.134}$$

where I is the identity operator on $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, and $K_{A,mod}$ is the modified boundary-to-boundary double layer potential operator from (3.50).

Moreover, given any function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the homogeneous boundary Sobolev space $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, at σ-a.e. point $x \in \partial\Omega$, one has

$$\begin{aligned}&\big(\partial_\nu^A(\mathcal{D}_{A,mod}f)\big)(x) \\ &\quad=\Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\nu_i(x)a_{ij}^{\mu\gamma}a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}}f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\le\mu\le M},\end{aligned}\tag{3.135}$$

where the conormal derivative is considered as in (3.66).

Furthermore, the operator

$$\begin{aligned}&\partial_\nu^A\mathcal{D}_{A,mod}:\big[\dot{L}_1^p(\partial\Omega,w)\big]^M\longrightarrow\big[L^p(\partial\Omega,w)\big]^M \ \textit{defined as}\\ &\big(\partial_\nu^A\mathcal{D}_{A,mod}\big)f:=\partial_\nu^A(\mathcal{D}_{A,mod}f)\ \textit{for each}\ \ f\in\big[\dot{L}_1^p(\partial\Omega,w)\big]^M\end{aligned}\tag{3.136}$$

is well defined, linear, bounded (when the domain space is equipped with the seminorm (2.599)*), and*

$$\begin{aligned}&\partial_\nu^A\mathcal{D}_{A,mod}\ \ \textit{annihilates constant}\\ &(\mathbb{C}^M\textit{-valued) functions on}\ \ \partial\Omega.\end{aligned}\tag{3.137}$$

As a consequence of (3.136) *and* (3.137)*, the following operator is well defined and linear:*

$$\begin{aligned}&\big[\partial_\nu^A\mathcal{D}_{A,mod}\big]:\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\longrightarrow\big[L^p(\partial\Omega,w)\big]^M\ \ \textit{defined as}\\ &\big[\partial_\nu^A\mathcal{D}_{A,mod}\big][f]:=\partial_\nu^A(\mathcal{D}_{A,mod}f)\ \ \textit{for each}\ \ f\in\big[\dot{L}_1^p(\partial\Omega,w)\big]^M.\end{aligned}\tag{3.138}$$

Finally, if $\Omega\subseteq\mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator (3.138) *is also bounded, when the quotient space is equipped with the norm* (2.601)*. Moreover, in this setting the operator $\big[\partial_\nu^A\mathcal{D}_{A,mod}\big]$ in* (3.138) *depends continuously on the underlying coefficient tensor A, in the sense that (with the piece of notation introduced in* (3.13)*) the following operator-valued assignment is continuous:*

$$\mathfrak{A}_{\mathrm{WE}}\ni A\longmapsto\big[\partial_\nu^A\mathcal{D}_{A,mod}\big]\in\mathrm{Bd}\Big(\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\to\big[L^p(\partial\Omega,w)\big]^M\Big).\tag{3.139}$$

Proof For each function $f\in\big[\dot{L}_1^p(\partial\Omega,w)\big]^M$, the jump-formula (3.134) is seen from (3.61) (keeping in mind (2.598)). The claims in (3.133) are consequences of (2.598), (3.51), (3.61), (3.58), (3.57), (3.85), and Theorem 3.1. In particular, given an arbitrary function $f=(f_\alpha)_{1\le\alpha\le M}\in\big[\dot{L}_1^p(\partial\Omega,w)\big]^M$, the conormal derivative $\partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}}f)$ may be meaningfully defined, as in (3.66). Specifically, at σ-a.e. point $x\in\partial\Omega$, we have

$$
\begin{aligned}
\big(\partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}}f)\big)(x) &= \Big(\nu_i(x)\big(a_{ij}^{\mu\gamma}\partial_j(\mathcal{D}_{A,\mathrm{mod}}f)_\gamma\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x)\Big)_{1\le\mu\le M} \qquad (3.140)\\
&= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x)a_{ij}^{\mu\gamma}a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}}f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\le\mu\le M},
\end{aligned}
$$

where the first equality comes from (3.66) and the second equality is a consequence of (3.57) and the jump-formula (3.86). Having established (3.135), the claims made in relation to (3.136) follow with the help of Proposition 3.4 and Theorem 3.1. Note that (3.137) is also a consequence of (3.135). Next, the claims pertaining to (3.138) are consequences of what we have proved so far and (3.137). Finally, the continuity of the operator-valued assignment (3.139) follows from (3.135), Theorem 3.1, and work in [114, §1.8]. □

The modified boundary-to-boundary double layer potential operator on homogeneous Muckenhoupt weighted Sobolev spaces is studied next.

Theorem 3.6 *Let $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\ge 2$) be an NTA domain such that $\partial\Omega$ is an Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, with $M\in\mathbb{N}$, let $A=\big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ be a complex coefficient tensor with the property that $L:=L_A$ as in (3.7) is a weakly elliptic $M\times M$ system in $\mathbb{R}^n$. In this context, consider the modified boundary-to-boundary double layer potential operator $K_{A,mod}$ from (3.50). Finally, select an integrability exponent $p\in(1,\infty)$ along with some Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$.*

Then the operator

$$
K_{A,mod}:\big[\dot{L}_1^p(\partial\Omega,w)\big]^M\longrightarrow\big[\dot{L}_1^p(\partial\Omega,w)\big]^M \tag{3.141}
$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (2.599).

As a consequence of (3.141) *and* (3.62)*, the following is a well-defined and linear operator:*

$$
\begin{gathered}
\big[K_{A,mod}\big]:\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\longrightarrow\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\ \ \textit{defined as}\\
\big[K_{A,mod}\big][f]:=\big[K_{A,mod}f\big]\in\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M,\quad\forall\,f\in\big[\dot{L}_1^p(\partial\Omega,w)\big]^M
\end{gathered}
\tag{3.142}
$$

Finally, if $\Omega\subseteq\mathbb{R}^n$ is actually a two-sided NTA domain whose boundary is an unbounded Ahlfors regular set, then the operator (3.142) *is also bounded when all quotient spaces are endowed with the norm introduced in* (2.601)*. Moreover, in this setting, the operator $\big[K_{A,mod}\big]$ in* (3.142) *depends continuously on the underlying coefficient tensor A, in the sense that (with the piece of notation introduced in* (3.13)*) the following operator-valued assignment is continuous:*

$$
\mathfrak{A}_{\mathrm{WE}}\ni A\longmapsto\big[K_{A,mod}\big]\in\mathrm{Bd}\Big(\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\Big). \tag{3.143}
$$

Proof The present hypotheses guarantee (cf. (2.48)) that Ω is a UR domain. Pick an integrability exponent $p \in (1, \infty)$ and fix an aperture parameter $\kappa \in (0, \infty)$. Next, consider a function $f = (f_\alpha)_{1 \le \alpha \le M} \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ and define $u := \mathcal{D}_{A,\mathrm{mod}} f$ in Ω. Then $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ (cf. (3.51)), and the jump-formula (3.134) gives

$$u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.144}$$

From (3.133), we also know that

$$\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w) \ \text{ and } \ \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,w)} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \tag{3.145}$$

for some constant $C \in (0, \infty)$ independent of f. Granted these properties, we may invoke Proposition 2.24 to conclude that

$$\begin{aligned} &u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{belongs to the space } \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \\ &\text{and } \Big\|u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}. \end{aligned} \tag{3.146}$$

Collectively, (3.144) and (3.146) then prove that

$$\begin{aligned} &K_{A,\mathrm{mod}} f \ \text{belongs to the space } \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \\ &\text{and } \big\|K_{A,\mathrm{mod}} f\big\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}, \end{aligned} \tag{3.147}$$

from which the claims pertaining to (3.141) follow. Next, the claims regarding the operator (3.142) are readily seen from what we have just proved and definitions. Finally, the fact that the operator-valued assignment (3.143) is continuous is seen from (2.598), (2.601), (3.65), (2.576), (3.35), (3.120), Theorem 3.1, and work in [114, §1.8]. □

We shall now use Corollary 3.1 to derive some useful operator identities, involving boundary layer potentials, of the sort described below.

Theorem 3.7 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \ge 2$) is an NTA domain whose boundary is an unbounded Ahlfors regular set. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Next, for some $M \in \mathbb{N}$, let $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ be a complex coefficient tensor with the property that $L := L_A$ as in (3.7) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Having fixed some integrability exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, recall the operators S_{mod} from (3.131), $\partial^A_\nu \mathcal{D}_{A,mod}$ from (3.136), and $K_{A,mod}$ from (3.141). Finally, let $K^\#_{A^\top}$ be the operator associated with the coefficient tensor $A^\top$ and the set Ω as in (3.25). Then the following statements are true.*

(1) For each $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$*, there exists some* $c_f \in \mathbb{C}^M$ *with the property that at* σ*-a.e. point on* $\partial\Omega$*, one has*

$$\big(\tfrac{1}{2}I + K_{A,mod}\big)\Big(\big(-\tfrac{1}{2}I + K_{A,mod}\big)f\Big) = S_{mod}\Big(\big(\partial^A_\nu \mathcal{D}_{A,mod}\big)f\Big) + c_f. \tag{3.148}$$

In particular,

$$\begin{gathered}\big(\tfrac{1}{2}I + \big[K_{A,mod}\big]\big)\big(-\tfrac{1}{2}I + \big[K_{A,mod}\big]\big) = \big[S_{mod}\big]\big[\partial^A_\nu \mathcal{D}_{A,mod}\big] \\ \text{as operators acting from } \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M.\end{gathered} \tag{3.149}$$

(2) For each function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$*, one has*

$$\big(\partial^A_\nu \mathcal{D}_{A,mod}\big)\big(K_{A,mod} f\big) = K^{\#}_{A^\top}\big(\partial^A_\nu \mathcal{D}_{A,mod}\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.150}$$

(3) For each $f \in \big[L^p(\partial\Omega, w)\big]^M$*, there exists some* $c_f \in \mathbb{C}^M$ *with the property that*

$$S_{mod}\big(K^{\#}_{A^\top} f\big) = K_{A,mod}\big(S_{mod} f\big) + c_f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.151}$$

In particular,

$$\begin{gathered}\big[S_{mod}\big]K^{\#}_{A^\top} = \big[K_{A,mod}\big]\big[S_{mod}\big] \\ \text{as operators acting from } \big[L^p(\partial\Omega, w)\big]^M.\end{gathered} \tag{3.152}$$

(4) For each $f \in \big[L^p(\partial\Omega, w)\big]^M$*, at* σ*-a.e. point on* $\partial\Omega$*, one has*

$$\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) = \big(\partial^A_\nu \mathcal{D}_{A,mod}\big)\big(S_{mod} f\big). \tag{3.153}$$

Proof The present hypotheses imply that Ω is a connected UR domain (see (2.48)). Select an aperture parameter $\kappa \in (0, \infty)$. To justify the claims made in items *(1)*–*(2)*, pick an arbitrary function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ and define $u := \mathcal{D}_{A,\mathrm{mod}} f$ in Ω. Then, from (3.133) and (3.134), we know that

$$\begin{gathered}u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \text{the boundary traces } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \ (\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f \text{ and } \partial^A_\nu u = \big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)f.\end{gathered} \tag{3.154}$$

Then Corollary 3.1 applies and gives that $\partial_\nu^A u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$, the trace $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, and there exists some $c_f \in \mathbb{C}^M$ with the property that

$$
\begin{aligned}
u &= \mathcal{D}_{A,\mathrm{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathscr{S}_{\mathrm{mod}}\big(\partial_\nu^A u\big) + c_u \\
&= \mathcal{D}_{A,\mathrm{mod}}\Big(\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f\Big) - \mathscr{S}_{\mathrm{mod}}\Big(\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)f\Big) + c_f \quad \text{in } \Omega.
\end{aligned}
\tag{3.155}
$$

Going nontangentially to the boundary in (3.155) then yields, on account of (3.154), (3.134), (3.47), (3.141), and (3.136),

$$
\begin{aligned}
\big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)\Big(\big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f\Big) \\
- S_{\mathrm{mod}}\Big(\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)f\Big) + c_f
\end{aligned}
\tag{3.156}
$$

at σ-a.e. point on $\partial\Omega$. From this, (3.148) readily follows. This takes care of the claim in item *(1)*.

To deal with the claim in item *(2)*, take the conormal derivative ∂_ν^A of the most extreme sides of (3.155) and use (3.67), (3.136), and the fact that $\partial_\nu^A c_u = 0$ (cf. (3.66)) to arrive at the conclusion that

$$
\begin{aligned}
\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)f = \big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)\Big(\big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f\Big) \\
- \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)\Big(\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)f\Big)
\end{aligned}
\tag{3.157}
$$

at σ-a.e. point on $\partial\Omega$, from which (3.150) readily follows.

Let us now turn our attention to the claims made in items *(3)-(4)*. Start with an arbitrary function $f \in \big[L^p(\partial\Omega, w)\big]^M$, and then consider $u := \mathscr{S}_{\mathrm{mod}} f$ in Ω. From (2.575), item *(c)* of Proposition 3.5, and (3.47), we see that

$$
\begin{gathered}
u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\
\text{the traces } u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \ (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\
u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = S_{\mathrm{mod}} f \quad \text{and} \quad \partial_\nu^A u = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f.
\end{gathered}
\tag{3.158}
$$

Again, Corollary 3.1 applies and gives that $\partial_\nu^A u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$, the trace $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, and there exists some $c_f \in \mathbb{C}^M$ such that

$$
u = \mathcal{D}_{A,\mathrm{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathscr{S}_{\mathrm{mod}}\big(\partial_\nu^A u\big) + c_u
$$

$$= \mathcal{D}_{A,\mathrm{mod}}\big(S_{\mathrm{mod}} f\big) - \mathscr{S}_{\mathrm{mod}}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big) + c_f \ \text{ in } \ \Omega. \tag{3.159}$$

Taking nontangential boundary traces in (3.159) then gives, thanks to (3.158), (3.47), (3.134), (3.131), and (3.118),

$$S_{\mathrm{mod}} f = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)\big(S_{\mathrm{mod}} f\big) - S_{\mathrm{mod}}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big) + c_f \tag{3.160}$$

at σ-a.e. point on $\partial\Omega$. With this in hand, (3.151) follows after simple algebra. This justifies the claim made in item *(3)*.

As regards item *(4)*, take the conormal derivative ∂^A_ν of the most extreme sides of (3.159) and rely on (3.158), (3.136), (3.126), (3.131), (3.118), and the fact that $\partial^A_\nu c_u = 0$ (cf. (3.66)) to conclude that

$$\begin{aligned}\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f &= \big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)\big(S_{\mathrm{mod}} f\big) \\ &\quad - \big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big)\end{aligned} \tag{3.161}$$

at σ-a.e. point on $\partial\Omega$, from which (3.153) readily follows. □

There are direct links between the layer potential operators discussed so far in this section and boundary value problems. To elaborate on this, we introduce a piece of notation. Given two vector spaces X, Y, for linear operator $T : X \to Y$ denote by

$$\mathrm{Im}\big(T : X \to Y\big) := \big\{Tx : x \in X\big\} \tag{3.162}$$

the image of T. Moreover, corresponding to the special case when $X = Y$, we agree to abbreviate $\mathrm{Im}\big(T; X\big) := \mathrm{Im}\big(T : X \to X\big)$.

Proposition 3.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an NTA domain with the property that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick some integrability exponent $p \in (1,\infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Finally, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, and fix a coefficient tensor $A \in \mathfrak{A}_L$.*

Then for each given function f belonging to $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, the homogeneous Muckenhoupt weighted boundary Sobolev space defined in (2.598)*, the following statements are equivalent:*

(a) The boundary value problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \ \ at\ \sigma\text{-}a.e.\ point\ on \ \ \partial\Omega \end{cases} \tag{3.163}$$

has a solution.

(b) The equivalence class of the function f modulo constants, denoted by $[f]$, belongs to the space

$$\begin{aligned} &\mathrm{Im}\Big(\tfrac{1}{2}I + \big[K_{A,mod}\big];\ \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big) \\ &\qquad + \mathrm{Im}\Big(\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big). \end{aligned} \tag{3.164}$$

(c) Again, with $[f]$ denoting the equivalence class of the function f modulo constants,

$$\begin{aligned} &\Big(-\tfrac{1}{2}I + \big[K_{A,mod}\big]\Big)[f] \ \ \textit{belongs to the space} \\ &\mathrm{Im}\Big(\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big). \end{aligned} \tag{3.165}$$

Proof Assume u solves (3.163). Then Corollary 3.1 guarantees that $\partial^A_\nu u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$ and that there exists some $c_u \in \mathbb{C}^M$ such that (3.75) holds. Going nontangentially to the boundary then yields, on account of (3.134), (3.47), and (2.575),

$$f = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f - S_{\mathrm{mod}}\big(\partial^A_\nu u\big) + c_u \ \ \text{on} \ \ \partial\Omega. \tag{3.166}$$

Taking equivalence classes modulo constants and keeping in mind (3.142), (3.132), we may recast (3.166) as

$$\big(-\tfrac{1}{2}I + \big[K_{A,\mathrm{mod}}\big]\big)[f] = \big[S_{\mathrm{mod}}\big]\big(\partial^A_\nu u\big). \tag{3.167}$$

From this, we conclude that (3.165) holds, hence *(a)* $\Rightarrow$ *(c)*.

Next, assume (3.165) holds. Since

$$[f] = \Big(\tfrac{1}{2}I + \big[K_{A,\mathrm{mod}}\big]\Big)[f] - \Big(-\tfrac{1}{2}I + \big[K_{A,\mathrm{mod}}\big]\Big)[f], \tag{3.168}$$

this implies that $[f]$ belongs to the space in (3.164). Thus, *(c)* $\Rightarrow$ *(b)*.

Finally, if $[f]$ belongs to the space in (3.164), it follows from (3.142) and (3.132) that

$$f = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)g + S_{\mathrm{mod}}h + c \tag{3.169}$$

for some

$$g \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M, \quad h \in \big[L^p(\partial\Omega, w)\big]^M, \quad c \in \mathbb{C}^M. \tag{3.170}$$

In view of this, (3.133), (3.134), (3.124), (3.127), (3.47), and (2.575), we then see that the function

$$u := \mathcal{D}_{A,\mathrm{mod}}g + \mathscr{S}_{\mathrm{mod}}h + c \ \text{ in } \ \Omega \tag{3.171}$$

solves the boundary value problem (3.163). Hence, *(b)* $\Rightarrow$ *(a)*, finishing the proof of the proposition. □

Here is a companion result to Proposition 3.6 for a Neumann type boundary value problem.

Proposition 3.7 *Let $\Omega \subseteq \mathbb{R}^n$ be an NTA domain such that its topological boundary, $\partial\Omega$, is an unbounded Ahlfors regular set. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an aperture parameter $\kappa \in (0, \infty)$, pick some integrability exponent $p \in (1, \infty)$, and consider some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and fix a coefficient tensor $A \in \mathfrak{A}_L$.*

Then, for each function $f \in \big[L^p(\partial\Omega, w)\big]^M$, the following statements are equivalent:

(a) The boundary value problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \textit{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial^A_\nu u = f \ \ \textit{at } \sigma\textit{-a.e. point on } \ \partial\Omega \end{cases} \tag{3.172}$$

has a solution.

(b) The function f belongs to the space

$$\begin{aligned} &\mathrm{Im}\Big(\big[\partial^A_\nu \mathcal{D}_{A,mod}\big] : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M\Big) \\ &\quad + \mathrm{Im}\Big(-\tfrac{1}{2}I + K^{\#}_{A^\top}; \big[L^p(\partial\Omega, w)\big]^M\Big). \end{aligned} \tag{3.173}$$

(c) One has

$$\begin{aligned}&\big(\tfrac{1}{2}I+K^{\#}_{A^{\top}}\big)f \ \ \textit{belongs to the space}\\ \mathrm{Im}\Big(&\big[\partial^A_\nu \mathcal{D}_{A,mod}\big]:\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M\longrightarrow\big[L^p(\partial\Omega,w)\big]^M\Big).\end{aligned}\tag{3.174}$$

Proof Suppose u solves (3.172). Then Corollary 3.1 gives that the nontangential boundary trace $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$ and belongs to $\big[\dot{L}^p_1(\partial\Omega,w)\big]^M$ and that the integral representation formula in (3.75) holds for some $c_u\in\mathbb{C}^M$. Taking the conormal derivative of both sides then yields, in view of (3.126),

$$f=\partial^A_\nu\Big(\mathcal{D}_{A,\text{mod}}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\Big)-\big(-\tfrac{1}{2}I+K^{\#}_{A^{\top}}\big)f \ \text{ on } \ \partial\Omega.\tag{3.175}$$

From this and (3.138), we then conclude that

$$\begin{aligned}&\big(\tfrac{1}{2}I+K^{\#}_{A^{\top}}\big)f=\partial^A_\nu\Big(\mathcal{D}_{A,\text{mod}}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\Big) \ \text{belongs to the space}\\ \mathrm{Im}\Big(&\big[\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big]:\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M\longrightarrow\big[L^p(\partial\Omega,w)\big]^M\Big),\end{aligned}\tag{3.176}$$

hence *(a)* $\Rightarrow$ *(c)*. Going further, assume (3.174) holds. Since

$$f=\big(\tfrac{1}{2}I+K^{\#}_{A^{\top}}\big)f-\big(-\tfrac{1}{2}I+K^{\#}_{A^{\top}}\big)f,\tag{3.177}$$

this implies that f belongs to the space in (3.173). As such, *(c)* $\Rightarrow$ *(b)*.

Finally, suppose the function f belongs to the space in (3.173), say

$$f=\big[\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big][g]+\big(-\tfrac{1}{2}I+K^{\#}_{A^{\top}}\big)h\tag{3.178}$$

for some

$$g\in\big[\dot{L}^p_1(\partial\Omega,w)\big]^M \ \text{ and } \ h\in\big[L^p(\partial\Omega,w)\big]^M.\tag{3.179}$$

Then (3.178), (3.179), (3.138), (3.133), (3.124), (3.126), and (3.127) collectively imply that the function

$$u:=\mathcal{D}_{A,\text{mod}}\,g+\mathscr{S}_{\text{mod}}\,h \ \text{ in } \ \Omega\tag{3.180}$$

solves the boundary value problem (3.172). Thus, *(b)* $\Rightarrow$ *(a)*, and the proof of the proposition is complete. □

3.3 Distinguished Coefficient Tensors

To each weakly elliptic system L, we may canonically associate a fundamental solution E as in Theorem 3.1. Having fixed a UR domain, this is then used to create a variety of double layer potential operators K_A, in relation to each choice of a coefficient tensor $A \in \mathfrak{A}_L$. While any such double layer K_A has a rich Calderón–Zygmund theory (as discussed in Proposition 3.5), seeking more specialized properties requires placing additional demands on the coefficient tensor A. We begin by recording a result proved in [115, §1.2] describing said demands phrased in several equivalent forms.

Proposition 3.8 *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and consider the matrix-valued function defined for each $\xi \in \mathbb{R}^n \setminus \{0\}$ as*

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\le\gamma,\beta\le M} := \big[L(\xi)\big]^{-1} \in \mathbb{C}^{M\times M} \tag{3.181}$$

(recall that the characteristic matrix $L(\xi)$ of L has been defined in (3.2)). Also, let $E = \big(E_{\alpha\beta}\big)_{1\le\alpha,\beta\le M}$ be the fundamental solution associated with the given system L as in Theorem 3.1.

Then, for each coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_L$ (cf. (3.11)), the following conditions are equivalent:

(a) For each $k, k' \in \{1,\dots,n\}$ and each $\alpha,\gamma \in \{1,\dots,M\}$, there holds

$$\big(x_{k'}a^{\beta\alpha}_{jk} - x_k a^{\beta\alpha}_{jk'}\big)(\partial_j E_{\gamma\beta})(x) = 0 \ \text{ for all } \ x = (x_i)_{1\le i\le n} \in \mathbb{R}^n \setminus \{0\}. \tag{3.182}$$

(b) For each $s, s' \in \{1,\dots,n\}$ and each $\alpha,\gamma \in \{1,\dots,M\}$, in the sense of tempered distributions in $\mathbb{R}^n$, one has

$$\Big[a^{\beta\alpha}_{rs}\partial_{\xi_{s'}} - a^{\beta\alpha}_{rs'}\partial_{\xi_s}\Big]\big[\xi_r \mathcal{E}_{\gamma\beta}(\xi)\big] = 0. \tag{3.183}$$

(c) For each $k, k' \in \{1,\dots,n\}$ and each $\alpha,\gamma \in \{1,\dots,M\}$, one has

$$\big(a^{\beta\alpha}_{k'k} - a^{\beta\alpha}_{kk'} + \xi_j a^{\beta\alpha}_{jk}\partial_{\xi_{k'}} - \xi_j a^{\beta\alpha}_{jk'}\partial_{\xi_k}\big)\mathcal{E}_{\gamma\beta}(\xi) = 0 \text{ for all } \xi \in \mathbb{R}^n\setminus\{0\} \tag{3.184}$$

and also

$$\int_{S^1} \big(a^{\beta\alpha}_{jk}\xi_{k'} - a^{\beta\alpha}_{jk'}\xi_k\big)\xi_j\, \mathcal{E}_{\gamma\beta}(\xi)\, d\mathcal{H}^1(\xi) = 0 \ \text{ if } \ n = 2. \tag{3.185}$$

(d) One has

$$\xi_r \xi_j \Big[a^{\beta\alpha}_{rs'} \big(a^{\lambda\mu}_{sj} + a^{\lambda\mu}_{js} \big) - a^{\beta\alpha}_{rs} \big(a^{\lambda\mu}_{s'j} + a^{\lambda\mu}_{js'} \big) \Big] \mathcal{E}_{\mu\beta}(\xi) + a^{\lambda\alpha}_{ss'} - a^{\lambda\alpha}_{s's} = 0$$
$$\text{for all } \xi \in S^{n-1}, \text{ all } s, s' \in \{1, \dots, n\}, \text{ and all } \alpha, \lambda \in \{1, \dots, M\}, \tag{3.186}$$

with the cancellation condition

$$\int_{S^1} \Big(a^{\beta\alpha}_{rs} \xi_{s'} - a^{\beta\alpha}_{rs'} \xi_s \Big) \xi_r \mathcal{E}_{\lambda\beta}(\xi)\, \mathrm{d}\mathcal{H}^1(\xi) = 0$$
$$\text{for all } s, s' \in \{1, \dots, n\} \text{ and } \alpha, \lambda \in \{1, \dots, M\}, \tag{3.187}$$

additionally imposed in the case when $n = 2$.

(e) *For each $\xi \in S^{n-1}$ and each $\alpha, \lambda \in \{1, \dots, M\}$,*

$$\text{the expression } \big(a^{\lambda\mu}_{sj} + a^{\lambda\mu}_{js} \big) \mathcal{E}_{\mu\beta}(\xi) \xi_j \xi_r a^{\beta\alpha}_{rs'} - a^{\lambda\alpha}_{s's}$$
$$\text{is symmetric in the indices } s, s' \in \{1, \dots, n\}, \tag{3.188}$$

with the condition that for each $\alpha, \lambda \in \{1, \dots, M\}$

$$\text{the expression } \int_{S^1} a^{\beta\alpha}_{rs} \xi_{s'} \xi_r \mathcal{E}_{\lambda\beta}(\xi)\, \mathrm{d}\mathcal{H}^1(\xi)$$
$$\text{is symmetric in the indices } s, s' \in \{1, 2\}, \tag{3.189}$$

also imposed in the case when $n = 2$.

(f) *There exists a matrix-valued function*

$$k = \big\{ k_{\gamma\alpha} \big\}_{1 \le \gamma, \alpha \le M} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}^{M \times M} \tag{3.190}$$

with the property that for each $\gamma, \alpha \in \{1, \dots, M\}$ and $s \in \{1, \dots, n\}$, one has

$$a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x) = x_s k_{\gamma\alpha}(x) \ \text{ for all } \ x \in \mathbb{R}^n \setminus \{0\}. \tag{3.191}$$

It is worth noting that the conditions in items *(a)–(f)* above are intrinsically formulated in terms of the given weakly elliptic system L. Observe that for each $x_* \in \mathbb{R}^n \setminus \{0\}$, we may find an open neighborhood O of the point x_* and an index $s \in \{1, \dots, n\}$ with the property that $x_s \neq 0$ for each $x \in O$. From this observation, (3.191), and Theorem 3.1, it follows that

$$\text{all entries of the matrix-valued function } k \text{ from (3.190) belong to } \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}), \text{ are even, as well as positive homogeneous of degree } -n. \tag{3.192}$$

Definition 3.1 Given a second-order, weakly elliptic, homogeneous, $M\times M$ system L in $\mathbb{R}^n$, with constant complex coefficients, call

$$A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}} \in \mathfrak{A}_L \tag{3.193}$$

a `distinguished coefficient tensor` for the system L provided any of the conditions *(a)-(f)* in Proposition 3.8 holds. Also, denote by $\mathfrak{A}_L^{\rm dis}$ the family of such distinguished coefficient tensors for L, say,

$$\mathfrak{A}_L^{\rm dis} := \Big\{A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}} \in \mathfrak{A}_L : \text{ conditions (3.184)–(3.185)} \tag{3.194}$$
$$\text{hold for each } \ k,k'\in\{1,\dots,n\} \ \text{ and } \ \alpha,\gamma\in\{1,\dots,M\}\Big\}.$$

Finally, introduce the class of weakly elliptic systems which posses a distinguished coefficient tensor, by setting

$$\mathfrak{L}^{\rm dis} := \big\{L\in\mathfrak{L}_* : \mathfrak{A}_L^{\rm dis}\neq\varnothing\big\}. \tag{3.195}$$

For example, from Proposition 3.8 and the second line in (3.20), we see that

for any weakly elliptic, homogeneous, second-order, constant complex coefficient, $M\times M$ system L in $\mathbb{R}^n$, any coefficient tensor $A\in\mathfrak{A}_L$, and any complex number $\lambda\in\mathbb{C}\setminus\{0\}$, it follows that $A\in\mathfrak{A}_L^{\rm dis}$ if and only if $\lambda A\in\mathfrak{A}_{\lambda L}^{\rm dis}$. (3.196)

The relevance of the distinguished coefficient tensors is most apparent from the following result proved in [115, §1.3].

Proposition 3.9 *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$, and suppose $A\in\mathfrak{A}_L$. Then the following statements are equivalent:*

- *(i) The coefficient tensor A belongs to $\mathfrak{A}_L^{\rm dis}$.*
- *(ii) Whenever Ω is a half-space in $\mathbb{R}^n$, the boundary-to-boundary double layer potential K_A associated with A and Ω as in (3.24) is the zero operator.*
- *(iii) Whenever Ω is a half-space in $\mathbb{R}^n$ with the property that $0\in\partial\Omega$, the modified boundary-to-boundary double layer operator $K_{A,mod}$ associated as in (3.50) with the set Ω and the given coefficient tensor A is actually the zero operator.*
- *(iii') Whenever Ω is a half-space in $\mathbb{R}^n$, the modified boundary-to-boundary double layer operator $K_{A,mod}$ associated as in (3.50) with the set Ω and the given coefficient tensor A maps each function from $\big[\mathscr{C}_c^\infty(\partial\Omega)\big]^M$ into a constant in $\mathbb{C}^M$.*
- *(iv) There exists a matrix-valued function $k\in\big[\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\big]^{M\times M}$ which is even, positive homogeneous of degree $-n$, and with the property that for each*

UR domain $\Omega \subseteq \mathbb{R}^n$*, the (matrix-valued) integral kernel of the double layer potential operator* K_A *associated with* A *and* Ω *as in* (3.24) *has the form*

$$\begin{aligned} &\langle \nu(y), x-y\rangle k(x-y) \\ &\textit{for each } x \in \partial\Omega \textit{ and } \mathcal{H}^{n-1}\textit{-a.e. } y \in \partial\Omega, \end{aligned} \tag{3.197}$$

where ν *is the geometric measure theoretic outward unit normal to* Ω*.*

(v) Whenever Ω *is a half-space in* $\mathbb{R}^n$*, the "transpose" double layer potential* $K_A^{\#}$ *associated with* A *and* Ω *as in* (3.25) *is the zero operator.*

(vi) There exists a matrix-valued function $k^{\#} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ *which is even, positive homogeneous of degree* $-n$*, and with the property that for each UR domain* $\Omega \subseteq \mathbb{R}^n$*, the (matrix-valued) integral kernel of the "transpose" double layer potential operator* $K_A^{\#}$ *associated with* A *and* Ω *as in* (3.25) *has the form*

$$\begin{aligned} &\langle \nu(x), y-x\rangle k^{\#}(x-y) \\ &\textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \partial\Omega \textit{ and each } y \in \partial\Omega, \end{aligned} \tag{3.198}$$

where ν *is the geometric measure theoretic outward unit normal to* Ω*.*

Moreover, whenever either (hence all) of the above conditions materializes, the matrices $k, k^{\#}$ *in items (iv), (vi) above are related to each other via* $k^{\#} = k^{\top}$*, where the superscript* $\top$ *indicates transposition.*

In light of Proposition 3.9 and (1.50), we are particularly interested in the class of weakly elliptic homogeneous constant complex coefficient second-order systems L with $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$. The following example shows that the latter condition is always satisfied by strongly elliptic scalar operators.

Example 3.2 Assume L is a second-order, homogeneous, constant complex coefficient, scalar differential operator in $\mathbb{R}^n$ (i.e., as in (3.1) with $M = 1$), which is `strongly elliptic`. Specifically, suppose $L = a_{jk}\partial_j\partial_k$ with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1, \ldots, n\}$ having the property that there exists a constant $c \in (0, \infty)$ such that

$$\mathrm{Re}\Big[\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k\Big] \geq c|\xi|^2, \quad \forall\, \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{3.199}$$

Introduce $A := (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$ and then define

$$(\widetilde{a}_{jk})_{1\leq j,k\leq n} := \mathrm{sym}\, A := \frac{A + A^{\top}}{2}, \quad (b_{jk})_{1\leq j,k\leq n} := (\mathrm{sym}\, A)^{-1}. \tag{3.200}$$

In particular, $L = L_{\mathrm{sym}\,A} := \widetilde{a}_{jk}\partial_j\partial_k$, i.e., the coefficient matrix $\mathrm{sym}\,A$ may be used to represent the given differential operator L. In this case, it turns out that the fundamental solution E canonically associated with the operator L as in Theorem 3.1 may be explicitly identified (cf. [102, Theorem 7.68, pp. 314-315]) as the function $E \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)$ given at each point $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E(x) = \begin{cases} -\dfrac{1}{(n-2)\,\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n-2}{2}} & \text{if } n \geq 3, \\[2ex] \dfrac{1}{4\pi\sqrt{\det(\mathrm{sym}\,A)}}\log(\langle(\mathrm{sym}\,A)^{-1}x, x\rangle) + c_A & \text{if } n = 2, \end{cases} \tag{3.201}$$

where log denotes the principal branch of the complex logarithm (defined for complex numbers $z \in \mathbb{C} \setminus (-\infty, 0]$ so that $z^a = \mathrm{e}^{a\log z}$ for each $a \in \mathbb{R}$), and c_A is a complex constant which depends solely on A. As both $\mathrm{sym}\,A$ and $(\mathrm{sym}\,A)^{-1}$ are symmetric matrices, for each index $j \in \{1, \dots, n\}$ and each point $x = (x_i)_{1\leq i\leq n} \in \mathbb{R}^n \setminus \{0\}$, we therefore have (in all dimensions $n \geq 2$)

$$\begin{aligned}(\partial_j E)(x) &= \frac{\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}(\delta_{rj}b_{rs}x_s + \delta_{sj}b_{rs}x_r)}{2\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}} \\ &= \frac{\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}b_{rj}x_r}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}. \end{aligned} \tag{3.202}$$

Thus, with $C_{A,n}$ abbreviating $\big(\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}\big)^{-1} \in \mathbb{C}$, for each pair of integers $k, k' \in \{1, \dots, n\}$, we may compute

$$\begin{aligned}\big(x_{k'}\widetilde{a}_{jk} - x_k\widetilde{a}_{jk'}\big)(\partial_j E)(x) &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}\big(x_{k'}\widetilde{a}_{kj} - x_k\widetilde{a}_{k'j}\big)\big(b_{jr}x_r\big) \\ &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}\big(x_{k'}\delta_{kr} - x_k\delta_{k'r}\big)x_r \\ &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}\big(x_{k'}x_k - x_kx_{k'}\big) = 0. \end{aligned} \tag{3.203}$$

This shows that condition (3.182) is presently verified for the choice of coefficient tensor $\mathrm{sym}\,A$ in the representation of the given differential operator L. Hence, $\mathrm{sym}\,A \in \mathfrak{A}^{\mathrm{dis}}_L$, which proves that, in the case when $M = 1$, we have

$$\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing \text{ for every scalar, strongly elliptic, homogeneous, second-order, constant complex coefficient operator } L \text{ in } \mathbb{R}^n. \tag{3.204}$$

Consequently, Proposition 3.9 guarantees that for each UR domain $\Omega \subseteq \mathbb{R}^n$ the integral kernel of the double layer potential operator $K_{\mathrm{sym}\,A}$ associated with $\mathrm{sym}\,A$ and Ω as in (3.24) has the form (3.197). This being said, it is actually of interest to identify said integral kernel explicitly. Based on (3.200)–(3.202) and (3.24), we see that the kernel of if $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to Ω, then the integral kernel of the double layer potential operator $K_{\mathrm{sym}\,A}$ is

$$-\nu_k(y)\widetilde{a}_{jk}\left(\partial_j E\right)(x-y) = -\frac{\langle(\mathrm{sym}\,A)^{-1}(x-y), x-y\rangle^{-\frac{n}{2}}\nu_k(y)b_{rj}\widetilde{a}_{jk}(x-y)_r}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}$$

$$= -\frac{\langle(\mathrm{sym}\,A)^{-1}(x-y), x-y\rangle^{-\frac{n}{2}}\langle\nu(y), x-y\rangle}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}} \tag{3.205}$$

for each $x \in \partial\Omega$ and $\mathcal{H}^{n-1}$-a.e. $y \in \partial\Omega$,

which, as already anticipated, is of the form (3.197) with

$$k(z) := -\frac{\langle(\mathrm{sym}\,A)^{-1}z, z\rangle^{-\frac{n}{2}}}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}, \qquad \forall\, z \in \mathbb{R}^n \setminus \{0\}. \tag{3.206}$$

In the same scenario as above, we also wish to elaborate on the nature of $\mathfrak{A}_L^{\mathrm{dis}}$ (see the conclusion reached in (3.218) below). To set the stage, recall that any given matrix $A = (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$ may be decompose into its symmetric and antisymmetric parts, i.e.,

$$A = \mathrm{sym}\,A + \mathrm{asym}\,A \quad \text{where} \quad \mathrm{asym}\,A := A - \mathrm{sym}\,A = \frac{A - A^{\top}}{2}. \tag{3.207}$$

Consequently, for each UR domain $\Omega \subseteq \mathbb{R}^n$ with geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$, the integral kernel of the double layer potential operator K_A is given by

$$-\nu_k(y)\widetilde{a}_{jk}\left(\partial_j E\right)(x-y) - \nu_k(y)\widehat{a}_{jk}\left(\partial_j E\right)(x-y), \tag{3.208}$$

where E is as in (3.201), the entries $(\widetilde{a}_{jk})_{1\leq j,k\leq n}$ are as in (3.200), and

$$(\widehat{a}_{jk})_{1\leq j,k\leq n} := \mathrm{asym}\,A. \tag{3.209}$$

If Ω is a half-space, then, as seen from (3.205) and (3.208), the integral kernel of the double layer potential operator K_A reduces to

$$-\nu_k(y)\widehat{a}_{jk}\left(\partial_j E\right)(x-y). \tag{3.210}$$

From this and Proposition 3.9, we then conclude that $A \in \mathfrak{A}_L^{\mathrm{dis}}$ if and only if

$$-\nu_k(y)\widehat{a}_{jk}\left(\partial_j E\right)(x-y)=0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x,y\in\partial\Omega \\ \text{whenever } \Omega \text{ is a half-space in } \mathbb{R}^n. \tag{3.211}$$

The same type of argument which, starting with (1.44), has produced (1.47) now shows that (3.211) implies the existence of a function $k\in\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})$ which is even, positive homogeneous of degree $-n$, and such that

$$\widehat{a}_{rs}\,(\partial_r E)\,(x)=x_s k(x) \quad \text{for each} \quad x\in\mathbb{R}^n\setminus\{0\} \quad \text{and each} \quad s\in\{1,\dots,n\}. \tag{3.212}$$

Multiply this equality by $(\partial_s E)(x)$, and summing up in $s\in\{1,\dots,n\}$ yields, on account of the antisymmetry of $(\widehat{a}_{rs})_{1\le r,s\le n}=\operatorname{asym} A$,

$$x_s(\partial_s E)(x)k(x)=0 \quad \text{for each} \quad x\in\mathbb{R}^n\setminus\{0\}. \tag{3.213}$$

On the other hand, if $n\ge 3$, it follows that $E\in\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})$ is positive homogeneous of degree $2-n$ (cf. (3.201)), so Euler's formula gives in this case

$$x_s(\partial_s E)(x)=(2-n)E(x) \quad \text{for each} \quad x\in\mathbb{R}^n\setminus\{0\}. \tag{3.214}$$

By combining (3.213)–(3.214), we therefore arrive at the conclusion that

$$\text{if } n\ge 3 \text{ then } E(x)k(x)=0 \text{ for each } x\in\mathbb{R}^n\setminus\{0\}. \tag{3.215}$$

Since as is apparent from (3.201), at each point $x\in\mathbb{R}^n\setminus\{0\}$, we have $E(x)\neq 0$, this ultimately forces $k(x)=0$ for each $x\in\mathbb{R}^n\setminus\{0\}$. When used back in (3.212), this permits us to conclude (assuming $n\ge 3$) that

$$\widehat{a}_{rs}\,(\partial_r E)\,(x)=0 \quad \text{for each} \quad x\in\mathbb{R}^n\setminus\{0\} \quad \text{and each} \quad s\in\{1,\dots,n\}. \tag{3.216}$$

Together, (3.216) and (3.202) prove (again, assuming $n\ge 3$) that for each index $s\in\{1,\dots,n\}$, we have

$$\frac{\langle(\operatorname{sym} A)^{-1}x,x\rangle^{-\frac{n}{2}}\widehat{a}_{rs}b_{kr}x_k}{\omega_{n-1}\sqrt{\det(\operatorname{sym} A)}}=0 \quad \text{for all} \quad x=(x_k)_{1\le k\le n}\in\mathbb{R}^n\setminus\{0\}, \tag{3.217}$$

where $(b_{jk})_{1\le j,k\le n}:=(\operatorname{sym} A)^{-1}$ (cf. (3.200)). Thus, assuming $n\ge 3$, we deduce from (3.217) that in fact $(\operatorname{asym} A)(\operatorname{sym} A)^{-1}=0$. This is equivalent to having $\operatorname{asym} A=0$, i.e., the matrix $A\in\mathfrak{A}_L^{\mathrm{dis}}$ is necessarily symmetric. In concert with (3.204) and its proof, the above argument shows that

$$\begin{array}{l}\text{assuming } n\ge 3, \text{ it follows that for each given strongly elliptic,}\\ \text{scalar, homogeneous, second-order operator } L=\operatorname{div} A\nabla \text{ in}\\ \mathbb{R}^n \text{ with constant complex coefficients, the class } \mathfrak{A}_L^{\mathrm{dis}} \text{ consists}\\ \text{precisely of one matrix, namely } \operatorname{sym} A:=(A+A^\top)/2.\end{array} \tag{3.218}$$

Our next example shows that, for scalar operators in dimensions $n \geq 3$, weak ellipticity itself guarantees the existence of a unique distinguished coefficient tensor.

Example 3.3 Suppose $n \geq 3$, and consider an arbitrary second-order, homogeneous, constant complex coefficient, scalar differential operator L in $\mathbb{R}^n$ (i.e., as in (3.1) with $M = 1$), which is merely *weakly elliptic*. Recall (cf. (1.2)) that this means that we may express $L = a_{jk}\partial_j\partial_k$ with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1, \dots, n\}$ having the property that

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \neq 0, \qquad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{3.219}$$

Introduce $A := (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$. It has been shown in [113, §1.4] that (here is where $n \geq 3$ is used)

> there exists an angle $\theta \in [0, 2\pi)$ such that if we set $A_\theta := \mathrm{e}^{\mathrm{i}\theta} A$ then the matrix $\mathrm{sym} A_\theta := (A_\theta + A_\theta^\top)/2 \in \mathbb{C}^{n\times n}$ is strongly elliptic, in the sense that there exists some $c \in (0, \infty)$ such that $\mathrm{Re}\,\langle(\mathrm{sym}\, A_\theta)\xi, \xi\rangle \geq c|\xi|^2$ for each $\xi \in \mathbb{R}^n$ (cf. (3.199)). (3.220)

From this and (3.201), we conclude that the fundamental solution $E \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)$ canonically associated as in Theorem 3.1 with the operator

$$L := \mathrm{e}^{-\mathrm{i}\theta} L_{A_\theta} = \mathrm{e}^{-\mathrm{i}\theta} L_{\mathrm{sym} A_\theta} \tag{3.221}$$

presently may be expressed at each point $x \in \mathbb{R}^n \setminus \{0\}$ as

$$E(x) = -\frac{\mathrm{e}^{\mathrm{i}\theta}}{(n-2)\omega_{n-1}\sqrt{\det(\mathrm{sym}\, A_\theta)}}\Big\langle(\mathrm{sym}\, A_\theta)^{-1}x, x\Big\rangle^{\frac{2-n}{2}}. \tag{3.222}$$

In view of this formula and the fact that $\mathrm{sym}\, A := (A + A^\top)/2$ is related to $\mathrm{sym}\, A_\theta$ via $\mathrm{sym}\, A_\theta = \mathrm{e}^{\mathrm{i}\theta}\mathrm{sym}\, A$, we conclude from (3.201)–(3.203) that condition (3.182) currently holds for the choice of coefficient matrix $\mathrm{sym}\, A$ in the representation of the given differential operator L. Thus, $\mathrm{sym}\, A \in \mathfrak{A}_L^{\mathrm{dis}}$. In concert with (3.196) and (3.218), this goes to show that the following sharper version of (3.218) holds:

> if $n \geq 3$ then for each weakly elliptic, scalar, homogeneous, second-order operator $L = \mathrm{div}\, A\nabla$ in $\mathbb{R}^n$ with constant complex coefficients, the class $\mathfrak{A}_L^{\mathrm{dis}}$ consists precisely of one matrix, namely $\mathrm{sym}\, A := (A + A^\top)/2$. (3.223)

Turning our attention to genuine systems, below we pay special attention to the Lamé system of elasticity.

Example 3.4 Consider the following complexified version of the Lamé system (originally arising in the study of linear elasticity), defined for any two parameters $\mu, \lambda \in \mathbb{C}$ (referred to as Lamé moduli) as

$$L := L_{\mu,\lambda} := \mu\Delta + (\mu + \lambda)\nabla\mathrm{div}, \tag{3.224}$$

acting on vector fields $u = (u_\beta)_{1\leq\beta\leq n}$ defined in (open subsets of) $\mathbb{R}^n$, with the Laplacian applied componentwise. Hence, $L = L^\top$, and one may check (cf. [102, Proposition 10.14, p. 366]) that

$$\begin{array}{l}\text{the complex Lamé system (3.224) is weakly elliptic}\\ \text{if and only if one has } \mu \neq 0 \text{ as well as } 2\mu + \lambda \neq 0.\end{array} \tag{3.225}$$

We may express the complex Lamé system L as in (3.1) (with $M := n$) using a variety of coefficient tensors, such as those belonging to the one-parameter family

$$\begin{gathered} A(\zeta) = \big(a_{jk}^{\alpha\beta}(\zeta)\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq n}} \text{ defined for each } \zeta \in \mathbb{C} \text{ according to} \\ a_{jk}^{\alpha\beta}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu + \lambda - \zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \quad 1 \leq j, k, \alpha, \beta \leq n. \end{gathered} \tag{3.226}$$

In other words, for each vector field $u = (u_\beta)_{1\leq\beta\leq n} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ and each parameter $\zeta \in \mathbb{C}$, the Lamé system (3.224) satisfies

$$Lu = \Big(a_{jk}^{\alpha\beta}(\zeta)\partial_j\partial_k u_\beta\Big)_{1\leq\alpha\leq n} \text{ in } \big[\mathcal{D}'(\mathbb{R}^n)\big]^n. \tag{3.227}$$

In relation to the coefficient tensor (3.226), it turns out that, for any $\mu, \lambda, \zeta \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$, if L is as in (3.224), then we have (cf. [61] for specific details)

$$A(\zeta) \in \mathfrak{A}_L^{\mathrm{dis}} \iff 3\mu + \lambda \neq 0 \ \text{ and } \ \zeta = \frac{\mu(\mu + \lambda)}{3\mu + \lambda}. \tag{3.228}$$

This ultimately shows that

$$\begin{array}{l}\text{whenever the Lamé moduli } \mu, \lambda \in \mathbb{C} \text{ satisfy } \mu \neq 0,\ 2\mu + \lambda \neq 0,\\ \text{and } 3\mu + \lambda \neq 0, \text{ the Lamé operator } L \text{ defined as in (3.227) has}\\ \text{the property that } \mathfrak{A}_L^{\mathrm{dis}} = \mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing.\end{array} \tag{3.229}$$

It is of interest to concretely identify the format of the double layer potential operators associated with the complex Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ in $\mathbb{R}^n$, associated as in (1.52) with the Lamé moduli $\mu, \lambda \in \mathbb{C}$ satisfying

$$\mu \neq 0 \ \text{ and } \ 2\mu + \lambda \neq 0 \tag{3.230}$$

(thus ensuring the weak ellipticity of $L_{\mu,\lambda}$; cf. (3.225)). For this system, the fundamental solution E of $L_{\mu,\lambda}$ from Theorem 3.1 has the explicit form $E = (E_{jk})_{1\leq j,k\leq n}$, a matrix whose (j,k) entry is defined at each point $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ according to

$$E_{jk}(x) = \begin{cases} \dfrac{-1}{2\mu(2\mu+\lambda)\omega_{n-1}}\left[\dfrac{\delta_{jk}(3\mu+\lambda)}{(n-2)|x|^{n-2}} + \dfrac{(\mu+\lambda)x_j x_k}{|x|^n}\right] & \text{if } n \geq 3, \\[2ex] \dfrac{1}{4\pi\mu(2\mu+\lambda)}\left[\delta_{jk}(3\mu+\lambda)\ln|x| - \dfrac{(\mu+\lambda)x_j x_k}{|x|^2}\right] + c_{\mu,\lambda}\delta_{jk} & \text{if } n = 2, \end{cases} \tag{3.231}$$

for every $j, k \in \{1, \dots, n\}$, where $c_{\mu,\lambda} \in \mathbb{C}$ is the constant given by

$$c_{\mu,\lambda} := \frac{(1+\ln 4)(\lambda+\mu)}{8\pi\mu(\lambda+2\mu)} - \frac{\ln 2}{2\pi\mu}. \tag{3.232}$$

Let us now fix an arbitrary UR domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. In such a setting, with each choice of $\zeta \in \mathbb{C}$, associate a double layer potential operator $K_{A(\zeta)}$ as in (3.24). A direct computation based on (3.231), (3.226), and (3.24) then shows that the integral kernel $\Theta^{\zeta}(x, y)$ of the principal-value double layer potential operator $K_{A(\zeta)}$ is an $n \times n$ matrix whose (j,k) entry, $1 \leq j, k \leq n$, is explicitly given by

$$\begin{aligned} \Theta^{\zeta}_{jk}(x,y) = &-C_1(\zeta)\frac{\delta_{jk}}{\omega_{n-1}}\frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} \\ &- (1-C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle(x_j-y_j)(x_k-y_k)}{|x-y|^{n+2}} \\ &- C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x_j-y_j)\nu_k(y) - (x_k-y_k)\nu_j(y)}{|x-y|^n}, \end{aligned} \tag{3.233}$$

for σ-a.e. $x, y \in \partial\Omega$, where the constants $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ are defined as

$$C_1(\zeta) := \frac{\mu(3\mu+\lambda) - \zeta(\mu+\lambda)}{2\mu(2\mu+\lambda)}, \qquad C_2(\zeta) := \frac{\mu(\mu+\lambda) - \zeta(3\mu+\lambda)}{2\mu(2\mu+\lambda)}. \tag{3.234}$$

Thus, with notation introduced in (2.3), for each $\zeta \in \mathbb{C}$, the integral kernel $\Theta^{\zeta}(x, y)$ of $K_{A(\zeta)}$ may be recast as

$$\Theta^{\zeta}(x,y) = -C_1(\zeta)\frac{1}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle}{|x-y|^n} I_{n\times n}$$

$$
\begin{aligned}
&- (1 - C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y, \nu(y)\rangle (x-y)\otimes(x-y)}{|x-y|^{n+2}} \\
&- C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x-y)\otimes\nu(y) - \nu(y)\otimes(x-y)}{|x-y|^{n}}, \qquad (3.235)
\end{aligned}
$$

for σ-a.e. $x, y \in \partial\Omega$, where $I_{n\times n}$ is the $n \times n$ identity matrix. The penultimate term above suggests that for each function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$, we define

$$
\begin{aligned}
Qf(x) &:= \lim_{\varepsilon\to 0^+}\frac{n}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle x-y, \nu(y)\rangle (x-y)\otimes(x-y)}{|x-y|^{n+2}} f(y)\,\mathrm{d}\sigma(y) \\
&= \lim_{\varepsilon\to 0^+}\frac{n}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle x-y, \nu(y)\rangle\langle x-y, f(y)\rangle}{|x-y|^{n+2}}(x-y)\,\mathrm{d}\sigma(y),
\end{aligned}
\qquad (3.236)
$$

at σ-a.e. point $x \in \partial\Omega$. Then, if

$$
3\mu + \lambda \neq 0 \quad \text{and} \quad \zeta_* := \frac{\mu(\mu+\lambda)}{3\mu+\lambda}, \qquad (3.237)
$$

from (3.234), we see that $C_2(\zeta_*) = 0$, so the last term in (3.235) drops out and the principal-value double layer potential operator $K_{A(\zeta_*)}$ becomes

$$
\begin{aligned}
K_{A(\zeta_*)} &= C_1(\zeta_*)K_\Delta I_{n\times n} - (1 - C_1(\zeta_*))Q \\
&= \frac{2\mu}{3\mu+\lambda}K_\Delta I_{n\times n} - \frac{\mu+\lambda}{3\mu+\lambda}Q, \qquad (3.238)
\end{aligned}
$$

where K_Δ is the harmonic double layer potential operator (cf. (3.29)). In view of (3.29) and (3.236), this is in agreement with the prediction made in item *(iv)* of Proposition 3.9.

Traditionally, the singular integral operator $K_{A(\zeta_*)}$ from (3.238) has been called the (boundary-to-boundary) `pseudo-stress` double layer potential operator for the Lamé system, and the alternative notation K_Ψ has been occasionally employed.

We conclude this series of examples by discussing a case of a second-order, homogeneous, real constant coefficient, and weakly elliptic system which does *not* possess a distinguished coefficient tensor.

Example 3.5 Work in the plane $\mathbb{R}^2 \equiv \mathbb{C}$, and consider the second-order, homogeneous, real constant coefficient, 2×2 system

$$L = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}. \tag{3.239}$$

An example of a coefficient tensor in $\mathfrak{A}_L$ is given by $A = \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq 2\\ 1\leq\alpha,\beta\leq 2}}$ with

$$\begin{aligned} &a_{11}^{11} = a_{11}^{22} = \tfrac{1}{4}, \quad a_{22}^{11} = a_{22}^{22} = -\tfrac{1}{4}, \quad a_{12}^{11} = a_{21}^{11} = a_{12}^{22} = a_{21}^{22} = 0, \\ &a_{12}^{12} = a_{21}^{12} = -\tfrac{1}{4}, \quad a_{12}^{21} = a_{21}^{21} = \tfrac{1}{4}, \quad a_{11}^{21} = a_{22}^{21} = a_{22}^{12} = a_{11}^{12} = 0. \end{aligned} \tag{3.240}$$

The characteristic matrix of the system L is given by (cf. (3.2))

$$L(\xi) = \frac{-1}{4}\begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix} \text{ at each } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{3.241}$$

Hence, at each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, we have

$$\det\,[L(\xi)] = \tfrac{1}{16}\big[(\xi_1^2 - \xi_2^2)^2 + (2\xi_1\xi_2)^2\big] = \tfrac{1}{16}(\xi_1^2 + \xi_2^2)^2 = \tfrac{1}{16}|\xi|^4 \neq 0, \tag{3.242}$$

which goes to show that

$$\text{the system } L \text{ from (3.239) is weakly elliptic.} \tag{3.243}$$

In particular, L has a fundamental solution as in Theorem 3.1, which, once a UR domain in the plane has been fixed, may then be used to associate double layer potential operators K_A with any coefficient tensor $A \in \mathfrak{A}_L$ as in (3.24), and all these singular integral operators enjoy the properties discussed in Proposition 3.5.

This being said, since with $\eta := (1, 0) \in \mathbb{C}^2$, we have

$$\langle -L(\xi)\eta, \overline{\eta}\rangle = \tfrac{1}{4}(\xi_1^2 - \xi_2^2) \text{ for each } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{3.244}$$

and since the last expression above vanishes identically on the diagonal of $\mathbb{R}^2$, it follows that the system L from (3.239) *fails* to satisfy the Legendre–Hadamard strong ellipticity condition (cf. (3.4)).

To better understand this system, observe that its transpose is

$$L^\top = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & 2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}, \tag{3.245}$$

and, if $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ are the canonical coordinate projections, defined as

$$\pi_1(z_1, z_2) := z_1 \text{ and } \pi_2(z_1, z_2) = z_2 \text{ for each } (z_1, z_2) \in \mathbb{C}^2, \tag{3.246}$$

then

$$L(u_1, u_2) = \Big(\pi_1 L^\top(u_1, -u_2), -\pi_2 L^\top(u_1, -u_2)\Big)$$
$$\text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \tag{3.247}$$
$$\text{complex-valued functions } \ u_1, u_2 \in \mathscr{C}^2(\Omega).$$

As a consequence,

$$L(u_1, u_2) = 0 \Longleftrightarrow L^\top(u_1, -u_2) = 0$$
$$\text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any} \tag{3.248}$$
$$\text{complex-valued functions } \ u_1, u_2 \in \mathscr{C}^2(\Omega).$$

Pressing on, recall the Cauchy–Riemann operator and its conjugate

$$\partial_{\bar z} := \tfrac{1}{2}\big(\partial_x + \mathrm{i}\partial_y\big), \quad \partial_z := \tfrac{1}{2}\big(\partial_x - \mathrm{i}\partial_y\big), \quad \text{where } \ z = x + \mathrm{i}y, \tag{3.249}$$

then bring in Bitsadze's operator (cf. [16, 17]), which is simply the square of $\partial_{\bar z}$, i.e.,

$$\mathbb{L} := \partial_{\bar z}^2 = \tfrac{1}{4}\partial_x^2 + \tfrac{\mathrm{i}}{2}\partial_x\partial_y - \tfrac{1}{4}\partial_y^2, \qquad z = x + \mathrm{i}y. \tag{3.250}$$

To place things into a broader perspective, recall that there are three basic prototypes of scalar, constant coefficient, second-order, elliptic operators in the plane: the Laplacian $4\partial_z\partial_{\bar z}$, plus Bitsadze's operator $\partial_{\bar z}^2$ and its complex conjugate ∂_z^2. With $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ the canonical coordinate projections from (3.246), the system L introduced in (3.239) is related to Bitsadze's operator $\mathbb{L} = \partial_{\bar z}^2$ via

$$\mathbb{L}(u_1 + \mathrm{i}u_2) = \pi_1 L(u_1, u_2) + \mathrm{i}\pi_2 L(u_1, u_2)$$
$$\text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \tag{3.251}$$
$$\text{complex-valued functions } \ u_1, u_2 \in \mathscr{C}^2(\Omega).$$

In particular,

$$L\big(\operatorname{Re} U, \operatorname{Im} U\big) = \big(\operatorname{Re}(\mathbb{L}U), \operatorname{Im}(\mathbb{L}U)\big)$$
$$\text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any} \tag{3.252}$$
$$\text{complex-valued function } \ U \in \mathscr{C}^2(\Omega).$$

On the other hand, given any open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ along with any complex-valued function $U \in \mathscr{C}^2(\Omega)$, we have $\partial_{\bar z}^2 U = 0$ if and only if $f := -\partial_{\bar z} U$ is holomorphic in Ω, and the latter condition is further equivalent to the demand that $g(z) := U(z) + \bar z f(z)$ for each $z \in \Omega$ is a holomorphic function in Ω. As such, the general format of null-solution of $\partial_{\bar{z}}^2$ in an open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is

$$\begin{gathered} U(z) = g(z) - \bar{z} f(z) \ \text{ for all } \ z \in \Omega, \ \text{ where} \\ f \text{ and } g \text{ are holomorphic functions in } \Omega. \end{gathered} \tag{3.253}$$

This is akin to the description of affine functions on the real line as null-solutions of the one-dimensional Laplacian $\mathrm{d}^2/\mathrm{d}x^2$, with the role of $\mathrm{d}/\mathrm{d}x$ now played by the Cauchy–Riemann operator $\partial_{\bar{z}}$, with $\bar{z}$ now playing the role of the variable x, and with holomorphic functions playing the role of constants.

Specializing the expression of U in (3.253) to the case when $g(z) := zf(z)$ for each $z \in \Omega$, we obtain the following particular family of null-solutions for Bitsadze's operator $\mathbb{L}$ in any given open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$:

$$\begin{gathered} U(z) = (z - \bar{z}) f(z), \ \text{ where } f \\ \text{is any holomorphic function in } \Omega. \end{gathered} \tag{3.254}$$

From this and (3.252), we then conclude that

> given any holomorphic function f in an open set $\Omega \subseteq \mathbb{C}$, the vector-valued function $u = (u_1, u_2)$ with components given by $u_1(z) := \operatorname{Re}\big[(z - \bar{z}) f(z)\big]$ and $u_2(z) := \operatorname{Im}\big[(z - \bar{z}) f(z)\big]$ for each $z \in \Omega$ is a null-solution of the system L from (3.239). (3.255)

In particular, by further specializing this property to the case when $\Omega := \mathbb{R}^2_+ \equiv \mathbb{C}_+$ and the holomorphic function $f(z) := (z + \mathrm{i})^{-m}$ for $z \in \mathbb{C}_+$, where the integer $m \in \mathbb{N}$ is arbitrary, shows that the vector-valued function $u^{(m)} = \big(u_1^{(m)}, u_2^{(m)}\big)$ with components defined at each $z \in \mathbb{C}_+$ as

$$u_1^{(m)}(z) := \operatorname{Re}\big[(z - \bar{z})(z + \mathrm{i})^{-m}\big] \ \text{ and } \ u_2^{(m)}(z) := \operatorname{Im}\big[(z - \bar{z})(z + \mathrm{i})^{-m}\big] \tag{3.256}$$

is a null-solution of the system L from (3.239). Note that each function $u^{(m)}$ belongs to $\big[\mathscr{C}^\infty(\overline{\mathbb{R}^2_+})\big]^2$ and vanishes identically on $\partial\mathbb{R}^2_+ \equiv \mathbb{R}$ (since $z - \bar{z} = 0$ for each $z \in \mathbb{R}$), and for each multi-index $\alpha \in \mathbb{N}_0^2$, there exists some $C_\alpha \in (0, \infty)$ with the property that

$$\big|\partial^\alpha u^{(m)}(z)\big| \le C_\alpha (1 + |z|)^{1-m-|\alpha|} \ \text{ for all } \ z \in \mathbb{R}^2_+. \tag{3.257}$$

The estimate above implies that, having fixed an aperture parameter $\kappa > 0$, for each multi-index $\alpha \in \mathbb{N}_0^2$ there exists some $C_\alpha \in (0, \infty)$ such that

$$\mathcal{N}_\kappa\big(\partial^\alpha u^{(m)}\big)(x) \le C_\alpha (1 + |x|)^{1-m-|\alpha|} \ \text{ for all } \ x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{3.258}$$

As such, for any given $p \in (1, \infty)$, any Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and any multi-index $\alpha \in \mathbb{N}_0^2$, we have $\mathcal{N}_\kappa\big(\partial^\alpha u^{(m)}\big) \in L^p(\mathbb{R}, w)$ as long as $m + |\alpha| \geq 2$ (cf. (2.572)). Let us also observe that for each $m \in \mathbb{N}$, we have

$$u_2^{(2m)}(\mathrm{i}y) = 2(-1)^m y(y+1)^{-2m} \quad \text{for each} \quad y \in (0, \infty) \tag{3.259}$$

and that the functions

$$\big\{y(y+1)^{-2m}\big\}_{m\in\mathbb{N}}, \text{ for } 0 < y < \infty, \text{ are linearly independent.} \tag{3.260}$$

Indeed, suppose that for some family of positive integers $m_1 < m_2 < \cdots < m_N$ and nonzero constants $c_1, \dots, c_N$, we have $\sum_{j=1}^N c_j y(y+1)^{-2m_j} = 0$ for each $y > 0$. Divide by $y(y+1)^{-2m_1}$ to obtain $c_1 + \sum_{j=2}^N c_j (y+1)^{-2(m_j - m_1)} = 0$ for each $y \in (0, \infty)$. Sending $y \to \infty$ yields $c_1 = 0$, a contradiction that establishes (3.260). Ultimately, this proves that the linear space of all vector-valued functions u satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \\ \mathcal{N}_\kappa\big(\partial^\alpha u\big) \in L^p(\mathbb{R}, w) \ \text{ for all } \ \alpha \in \mathbb{N}_0^2, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R} \end{cases} \tag{3.261}$$

is infinite dimensional, i.e.,

the null-space of the Infinite-Order Regularity Problem for the system L (from (3.239)) in $\mathbb{R}^2_+$ is infinite dimensional. (3.262)

In particular,

the space of null-solutions of the corresponding Dirichlet Problem for the system L in $\mathbb{R}^2_+$ (formulated as in (1.76) with $n = 2$, $M = 2$, L as in (3.239), and $\Omega := \mathbb{R}^2_+$) is infinite dimensional. (3.263)

Since in item *(d)* of Theorem 6.2 we shall learn that this cannot happen if $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$, we then conclude that we necessarily have $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$ in this case. In other words, $L^\top$ from (3.245) is a weakly elliptic, second-order, homogeneous, real constant coefficient, 2×2 system in $\mathbb{R}^2$ which does not possess any distinguished coefficient tensor.

We may also run a variant of this argument, in which we now start with $L^\top$ instead of L. If

$$\overline{\mathbb{L}} = \partial_z^2 = \tfrac{1}{4}\partial_x^2 - \tfrac{\mathrm{i}}{2}\partial_x\partial_y - \tfrac{1}{4}\partial_y^2 \tag{3.264}$$

is the complex conjugate of Bitsadze's operator $\mathbb{L}$ from (3.250), then in place of (3.251)–(3.252) we now have

$$\begin{gathered}\overline{\mathbb{L}}(u_1+\mathrm{i}u_2)=\pi_1L^\top(u_1,u_2)+\mathrm{i}\pi_2L^\top(u_1,u_2)\\ \text{for any open set }\Omega\subseteq\mathbb{R}^2\equiv\mathbb{C}\text{ and any two}\\ \text{complex-valued functions }\ u_1,u_2\in\mathscr{C}^2(\Omega),\end{gathered}\tag{3.265}$$

and, respectively,

$$\begin{gathered}L^\top\big(\mathrm{Re}\,U,\mathrm{Im}\,U\big)=\big(\mathrm{Re}(\overline{\mathbb{L}}U),\mathrm{Im}\,(\overline{\mathbb{L}}U)\big)\\ \text{for any open set }\Omega\subseteq\mathbb{R}^2\equiv\mathbb{C}\text{ and any}\\ \text{complex-valued function }\ U\in\mathscr{C}^2(\Omega).\end{gathered}\tag{3.266}$$

Keeping in mind that U is a null-solution of $\overline{\mathbb{L}}$ if and only if $\overline{U}$ is a null-solution of $\mathbb{L}$ and reasoning as before, we conclude that, for each $m\in\mathbb{N}$, the vector-valued function $v^{(m)}=\big(v_1^{(m)},v_2^{(m)}\big)$ with components defined at each $z\in\mathbb{C}_+$ as

$$v_1^{(m)}(z):=\mathrm{Re}\big[(\bar z-z)(\bar z-\mathrm{i})^{-m}\big]\ \text{ and }\ v_2^{(m)}(z):=\mathrm{Im}\big[(\bar z-z)(\bar z-\mathrm{i})^{-m}\big]\tag{3.267}$$

is a null-solution of the system $L^\top$ from (3.245). In turn, this goes to show that the null-space of the Infinite-Order Regularity Problem for the system $L^\top$ in $\mathbb{R}^2_+$ (formulated as in (3.261) with $L^\top$ now replacing L) is infinite dimensional. Once this has been established, from item *(c)* in Theorem 6.2 we conclude that $\mathfrak{A}_L^{\mathrm{dis}}=\varnothing$. The bottom line is that

$$\begin{gathered}L\text{ in (3.239) is an example of a weakly elliptic, second-order,}\\ \text{homogeneous, real constant coefficient, }2\times 2\text{ system in }\mathbb{R}^2\text{, with}\\ \text{the property that }\mathfrak{A}_L^{\mathrm{dis}}=\varnothing\text{ and }\mathfrak{A}_{L^\top}^{\mathrm{dis}}=\varnothing.\end{gathered}\tag{3.268}$$

In particular, this goes to show that not every weakly elliptic, second-order, homogeneous, real constant coefficient, system has a distinguished coefficient tensor.

Remark 3.1 There is yet another proof of (3.268) which is not based on well-posedness results, but instead uses directly the algebraic characterization of distinguished coefficient tensors in Proposition 3.8. Specifically, to conclude that $\mathfrak{A}_L^{\mathrm{dis}}=\varnothing$, from (3.185), it suffices to show that for every coefficient tensor $B=\big(b_{jk}^{\alpha\beta}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}}$ such that $L=L_B$ there exist indices $k,k'\in\{1,\dots,n\}$ as well as $\alpha,\gamma\in\{1,\dots,M\}$ such that

$$\int_{S^1}\big(b_{jk}^{\beta\alpha}\xi_{k'}-b_{jk'}^{\beta\alpha}\xi_k\big)\xi_j\,\mathcal{E}_{\gamma\beta}(\xi)\,\mathrm{d}\mathcal{H}^1(\xi)\neq 0.\tag{3.269}$$

To this end, we first note that, using (3.181), (3.241), and (3.242), we have

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\leq\gamma,\beta\leq M} = \big[L(\xi)\big]^{-1} = \left[\frac{-1}{4}\begin{pmatrix}\xi_1^2-\xi_2^2 & -2\xi_1\xi_2\\ 2\xi_1\xi_2 & \xi_1^2-\xi_2^2\end{pmatrix}\right]^{-1}$$

$$= \frac{-4}{|\xi|^4}\begin{pmatrix}\xi_1^2-\xi_2^2 & 2\xi_1\xi_2\\ -2\xi_1\xi_2 & \xi_1^2-\xi_2^2\end{pmatrix}. \tag{3.270}$$

In particular, if $\xi \in S^1$, then $\xi = (\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi)$ and hence

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\leq\gamma,\beta\leq M} = -4\begin{pmatrix}\cos^2(\theta)-\sin^2(\theta) & 2\cos(\theta)\sin(\theta)\\ -2\cos(\theta)\sin(\theta) & \cos^2(\theta)-\sin^2(\theta)\end{pmatrix}$$

$$= -4\begin{pmatrix}\cos(2\theta) & \sin(2\theta)\\ -\sin(2\theta) & \cos(2\theta)\end{pmatrix}. \tag{3.271}$$

In order to facilitate the presentation, for each $j, k, \gamma, \beta \in \{1, 2\}$ introduce

$$\mathrm{I}_{jk}^{\gamma\beta} := \int_{S^1} \xi_j\,\xi_k\,\mathcal{E}_{\gamma\beta}\,d\mathcal{H}^1(\xi). \tag{3.272}$$

Then using elementary trigonometric formulas, we obtain

$$\mathrm{I}_{11}^{11} = -4\int_0^{2\pi}\cos^2(\theta)\cos(2\theta)\,d\theta = -\int_0^{2\pi}\big(2\cos(2\theta)+\cos(4\theta)+1\big)\,d\theta = -2\pi, \tag{3.273}$$

$$\mathrm{I}_{12}^{11} = -4\int_0^{2\pi}\sin(\theta)\cos(\theta)\cos(2\theta)\,d\theta = -\int_0^{2\pi}\sin(4\theta)\,d\theta = 0, \tag{3.274}$$

$$\mathrm{I}_{22}^{11} = -4\int_0^{2\pi}\sin^2(\theta)\cos(2\theta)\,d\theta = -4\int_0^{2\pi}\cos(2\theta)\,d\theta + \mathrm{I}_{11}^{11} = 2\pi, \tag{3.275}$$

$$\mathrm{I}_{22}^{12} = -8\int_0^{2\pi}\sin^3(\theta)\cos(\theta)\,d\theta = -2(\sin^4(2\pi)-\sin^4(0)) = 0, \tag{3.276}$$

$$\mathrm{I}_{12}^{12} = -8\int_0^{2\pi}\sin^2(\theta)\cos^2(\theta)\,d\theta = -\int_0^{2\pi}\big(1-\cos(4\theta)\big)\,d\theta = -2\pi, \tag{3.277}$$

$$\mathrm{I}_{11}^{12} = -8\int_0^{2\pi}\sin(\theta)\cos^3(\theta)\,d\theta = 2(\cos^4(2\pi)-\cos^4(0)) = 0. \tag{3.278}$$

Finally, from (3.271)–(3.272), it follows that

$$
\begin{aligned}
&\mathrm{I}^{22}_{11} = \mathrm{I}^{11}_{11} = -2\pi, \qquad \mathrm{I}^{22}_{12} = \mathrm{I}^{11}_{12} = 0, \qquad \mathrm{I}^{22}_{22} = \mathrm{I}^{11}_{22} = 2\pi,\\
&\mathrm{I}^{21}_{11} = -\mathrm{I}^{12}_{11} = 0, \qquad \mathrm{I}^{21}_{12} = -\mathrm{I}^{12}_{12} = 2\pi, \qquad \mathrm{I}^{21}_{22} = -\mathrm{I}^{12}_{22} = 0.
\end{aligned}
\tag{3.279}
$$

We are now ready to compute the integral in (3.269) with $k = \alpha = 1$ and $k' = \gamma = 2$:

$$
\begin{aligned}
\int_{S^1} \big(b^{\beta 1}_{j1}\xi_2 - b^{\beta 1}_{j2}\xi_1\big)\xi_j\, \mathcal{E}_{2\beta}(\xi)\, d\mathcal{H}^1(\xi) &= b^{11}_{11}\cdot \mathrm{I}^{21}_{12} + b^{11}_{21}\cdot \mathrm{I}^{21}_{22} + b^{21}_{11}\cdot \mathrm{I}^{22}_{12}\\
&\quad + b^{21}_{21}\cdot \mathrm{I}^{22}_{22} - b^{11}_{12}\cdot \mathrm{I}^{21}_{11} - b^{11}_{22}\cdot \mathrm{I}^{21}_{12} - b^{21}_{12}\cdot \mathrm{I}^{22}_{11} - b^{21}_{22}\cdot \mathrm{I}^{22}_{12}\\
&= 2\pi\big(b^{11}_{11} + b^{21}_{21} - b^{11}_{22} + b^{21}_{12}\big).
\end{aligned}
\tag{3.280}
$$

Next, we use the fact that B may be expressed as $B = A + C$, where A is a fixed coefficient tensor such that $L = L_A$ and C is a coefficient tensor which is antisymmetric in the lower indices. In particular, taking A as in (3.240), we conclude from (3.280) that

$$
\begin{aligned}
&\int_{S^1} \big(b^{\beta 1}_{j1}\xi_2 - b^{\beta 1}_{j2}\xi_1\big)\xi_j\, \mathcal{E}_{2\beta}(\xi)\, d\mathcal{H}^1(\xi)\\
&\qquad = 2\pi\big(a^{11}_{11} + a^{21}_{21} - a^{11}_{22} + a^{21}_{12} + c^{11}_{11} + c^{21}_{21} - c^{11}_{22} + c^{21}_{12}\big)\\
&\qquad = 2\pi\Big(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 0 + c^{21}_{21} - 0 + c^{21}_{12}\Big)\\
&\qquad = 2\pi \neq 0.
\end{aligned}
\tag{3.281}
$$

This justifies (3.269) and ultimately proves that $\mathfrak{A}^{\mathrm{dis}}_{L} = \varnothing$. The same argument as above works for $L^{\top}$, so we also conclude that $\mathfrak{A}^{\mathrm{dis}}_{L^{\top}} = \varnothing$.

In relation to the system L from (3.239), it is of interest to identify the space of boundary traces of its null-solutions in the upper half-plane whose nontangential maximal function belongs to a Muckenhoupt weighted Lebesgue space.

Proposition 3.10 *Fix an integrability index $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and choose an aperture parameter $\kappa > 0$. Also, recall the 2×2 system L defined in the plane as in (3.239).*

Then if $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function satisfying

$$
Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w),
\tag{3.282}
$$

and such that the nontangential boundary trace

$$f := u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists (in } \mathbb{C}^2\text{) at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \tag{3.283}$$

it follows that the function f belongs to $\big[L^p(\mathbb{R}, w)\big]^2$ and, if $f_1, f_2 \in L^p(\mathbb{R}, w)$ are the scalar components of f (i.e., $f = (f_1, f_2)$), then with H denoting the Hilbert transform on the real line (cf. (1.24)*) one has*

$$Hf_1 = f_2 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.284}$$

In the converse direction, for any given $f \in L^p(\mathbb{R}, w)$, there exists a vector-valued function $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying

$$\begin{aligned} &Lu = 0 \text{ in } \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \quad \text{and} \\ &u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (f, Hf) \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \end{aligned} \tag{3.285}$$

Altogether, the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L in the upper half-plane, i.e.,

$$\begin{aligned} \Big\{u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \ Lu = 0 \text{ in } \mathbb{R}^2_+, \ \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ \text{and } u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}\Big\}, \end{aligned} \tag{3.286}$$

is precisely

$$\big\{(f, Hf) : f \in L^p(\mathbb{R}, w)\big\}. \tag{3.287}$$

As a consequence of this and (3.248)*, one also has*

$$\begin{aligned} \Big\{u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \ L^\top u = 0 \text{ in } \mathbb{R}^2_+, \ \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ \text{and } u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}\Big\} \\ = \big\{(f, -Hf) : f \in L^p(\mathbb{R}, w)\big\}. \end{aligned} \tag{3.288}$$

Proof That the function f belongs to $\big[L^p(\mathbb{R}, w)\big]^2$ is clear from $\Big|u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big| \le \mathcal{N}_\kappa u$, the fact that $u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ is $\mathcal{L}^1$-measurable (cf. [111, §8.9]), and the last property in (3.282).

To proceed, fix a function $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying (3.282)–(3.283) and denote by $u_1, u_2 \in \mathscr{C}^\infty(\mathbb{R}^2_+)$ its scalar components. Hence, $u = (u_1, u_2)$ in $\mathbb{R}^2_+$. Also, pick an arbitrary $\varepsilon > 0$ and define

$$U_\varepsilon(z) := u_1(z+\varepsilon \mathrm{i}) + \mathrm{i}u_2(z+\varepsilon \mathrm{i}) \quad \text{for each} \quad z \in \big(\mathbb{R}^2_+ - \varepsilon \mathrm{i}\big). \tag{3.289}$$

Then $U_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^2_+ - \varepsilon \mathrm{i})$ and, as seen from (3.251), the fact that $Lu = 0$ in $\mathbb{R}^2_+$ translates into $\partial^2_{\bar z} U_\varepsilon = 0$ in $\mathbb{R}^2_+ - \varepsilon \mathrm{i}$. Granted this, (3.253) then guarantees the existence of two holomorphic functions $f_\varepsilon, g_\varepsilon$ in $\mathbb{R}^2_+ - \varepsilon \mathrm{i}$ with the property that

$$U_\varepsilon(z) = g_\varepsilon(z) - \bar z f_\varepsilon(z) \quad \text{for each} \quad z \in \big(\mathbb{R}^2_+ - \varepsilon \mathrm{i}\big). \tag{3.290}$$

More specifically, the unique holomorphic functions $f_\varepsilon, g_\varepsilon$ which do the job in (3.290) are

$$f_\varepsilon(z) := -\partial_{\bar z} U_\varepsilon(z) \quad \text{and} \quad g_\varepsilon(z) := U_\varepsilon(z) + \bar z f_\varepsilon(z) \quad \text{for each} \quad z \in \big(\mathbb{R}^2_+ - \varepsilon \mathrm{i}\big). \tag{3.291}$$

Henceforth, we agree to restrict $U_\varepsilon, f_\varepsilon, g_\varepsilon$ to $\mathbb{R}^2_+$. With this interpretation, introduce

$$W_\varepsilon(z) := g_\varepsilon(z) - z f_\varepsilon(z) \quad \text{for each} \quad z \in \mathbb{R}^2_+. \tag{3.292}$$

Hence, W_ε is holomorphic in $\mathbb{R}^2_+$ and extends continuously to $\overline{\mathbb{R}^2_+}$, and

$$\begin{aligned} U_\varepsilon(z) - W_\varepsilon(z) &= 2\mathrm{i}y f_\varepsilon(z) = -2\mathrm{i}y\big(\partial_{\bar z} U_\varepsilon\big)(z) \\ &\text{for each} \quad z = x + \mathrm{i}y \in \mathbb{R}^2_+. \end{aligned} \tag{3.293}$$

From the fact that $\partial^2_{\bar z} U_\varepsilon = 0$ in $\mathbb{R}^2_+$, we also conclude that $0 = \partial^2_z \partial^2_{\bar z} U_\varepsilon = \frac{1}{16}\Delta^2 U_\varepsilon$, i.e., the function U_ε is bi-harmonic in $\mathbb{R}^2_+$. Select $\theta \in (0,1)$ and $\widetilde\kappa \in (0,\kappa)$ both small so that

$$\frac{1+\theta+\widetilde\kappa}{1-\theta} < 1+\kappa. \tag{3.294}$$

Fix an arbitrary point $x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+$ and pick some $z_o = x_o + \mathrm{i}y_o \in \Gamma_{\widetilde\kappa}(x)$. The inequality demanded in (3.294) ensures that

$$B(z_o, \theta y_o) \subseteq \Gamma_\kappa(x). \tag{3.295}$$

Based on interior estimates for bi-harmonic functions (cf. [102, Theorem 11.12, p. 415]), (3.293), and (3.295), we may then write

$$\begin{aligned} \big|U_\varepsilon(z_o) - W_\varepsilon(z_o)\big| &= 2y_o\big|\big(\partial_{\bar z} U_\varepsilon\big)(z_o)\big| \leq \sqrt{2}y_o|(\nabla U_\varepsilon)(z_o)| \\ &\leq C \fint_{B(z_o,\theta y_o)} |U_\varepsilon| \,\mathrm{d}\mathcal{L}^2 \leq C\big(\mathcal{N}_\kappa U_\varepsilon\big)(x), \end{aligned} \tag{3.296}$$

for some constant $C = C(\theta) \in (0, \infty)$. Taking the supremum over all $z_o \in \Gamma_{\widetilde{\kappa}}(x)$ this ultimately yields

$$\big(\mathcal{N}_{\widetilde{\kappa}}(U_\varepsilon - W_\varepsilon)\big)(x) \leq C\big(\mathcal{N}_\kappa U_\varepsilon\big)(x) \ \text{ for each } \ x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{3.297}$$

In turn, (3.297) implies

$$\begin{aligned}\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon &\leq \mathcal{N}_{\widetilde{\kappa}} U_\varepsilon + \mathcal{N}_{\widetilde{\kappa}}(U_\varepsilon - W_\varepsilon) \leq \mathcal{N}_\kappa U_\varepsilon + C\mathcal{N}_\kappa U_\varepsilon \\ &= (1+C)\mathcal{N}_\kappa U_\varepsilon \leq (1+C)\mathcal{N}_\kappa u \ \text{ on } \ \mathbb{R} \equiv \partial\mathbb{R}^2_+. \end{aligned}\tag{3.298}$$

Upon recalling that the nontangential maximal function $\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon$ is non-negative and lower-semicontinuous, we then conclude from (3.298), the last property in (3.282), and (2.575) that

$$\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon \in L^1\Big(\mathbb{R}, \tfrac{\mathcal{L}^1(x)}{1+|x|}\Big). \tag{3.299}$$

Let us record our progress. The argument so far shows that the function W_ε is holomorphic in $\mathbb{R}^2_+$ and extends continuously to $\overline{\mathbb{R}^2_+}$, and there exists some aperture parameter $\widetilde{\kappa} > 0$ such that $\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon$ belongs to $L^1\Big(\mathbb{R}, \tfrac{\mathcal{L}^1(x)}{1+|x|}\Big)$. These properties allow us to invoke the Cauchy reproducing formula (proved in [113, §1.1] in much more general geometric settings) which asserts that

$$W_\varepsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\big(W_\varepsilon\big|_{\mathbb{R}}\big)(y)}{y - z}\, dy, \quad \text{for each } \ z \in \mathbb{R}^2_+. \tag{3.300}$$

Since $f_\varepsilon, g_\varepsilon$ extend continuously to $\overline{\mathbb{R}^2_+}$, from (3.290), (3.292), and the fact that $z = \bar{z}$ on $\mathbb{R} \equiv \partial\mathbb{R}^2_+$, we conclude that

$$W_\varepsilon\big|_{\mathbb{R}} = U_\varepsilon\big|_{\mathbb{R}} \ \text{ on } \ \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{3.301}$$

As such, if we abbreviate

$$h_\varepsilon := U_\varepsilon\big|_{\mathbb{R}} \ \text{ on } \ \mathbb{R} \equiv \partial\mathbb{R}^2_+, \tag{3.302}$$

after taking the nontangential boundary traces of both sides in (3.300) and using the Plemelj jump-formula for the Cauchy operator (which continues to be valid in this setting; see [114, §1.6]), we arrive at

$$h_\varepsilon = \big(\tfrac{1}{2}I + \tfrac{\mathrm{i}}{2}H\big)h_\varepsilon \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}, \tag{3.303}$$

where I is the identity and H is the Hilbert transform on $\mathbb{R}$. Hence, on the one hand, we may rewrite (3.303) simply as

$$Hh_\varepsilon = -\mathrm{i}h_\varepsilon \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.304}$$

On the other hand, from (3.302) and (3.289), we see that

$$h_\varepsilon(x) = u_1(x + \varepsilon \mathrm{i}) + \mathrm{i}u_2(x + \varepsilon \mathrm{i}) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}. \tag{3.305}$$

In turn, this implies

$$|h_\varepsilon(x)| \leq \sqrt{2}\big(\mathcal{N}_\kappa u\big)(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R} \tag{3.306}$$

and, when used in concert with (3.283), that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} h_\varepsilon(x) &= \big(u_1\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\big)(x) + \mathrm{i}\big(u_2\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\big)(x) \\ &= f_1(x) + \mathrm{i}\, f_2(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}.\end{aligned} \tag{3.307}$$

Thanks to (3.306)–(3.307) and the last property in (3.282), we may invoke Lebesgue's Dominated Convergence Theorem to conclude that

$$\lim_{\varepsilon\to 0^+} h_\varepsilon = f_1 + \mathrm{i}\, f_2 \quad \text{in } L^p(\mathbb{R}, w). \tag{3.308}$$

Having established this, on account of (3.304) and the continuity of the Hilbert transform H on the Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R}, w)$, we obtain

$$H(f_1 + \mathrm{i}\, f_2) = -\mathrm{i}(f_1 + \mathrm{i}\, f_2) \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.309}$$

The idea is now write $u = \operatorname{Re} u + \mathrm{i}\operatorname{Im} u$ and observe that, since the coefficients of the system L are real, $\operatorname{Re} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ and $\operatorname{Im} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ enjoy the same properties as the function u in (3.282)–(3.283). Granted what we have proved already, it follows that if ϕ_1, ϕ_2 are the scalar components of $(\operatorname{Re} u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ and if ψ_1, ψ_2 are the scalar components of $(\operatorname{Im} u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ then $\phi_1, \phi_2, \psi_1, \psi_2$ are real-valued functions belonging to $L^p(\mathbb{R}, w)$, and the conclusion in (3.309) written separately for $\operatorname{Re} u$ and $\operatorname{Im} u$ gives

$$H(\phi_1 + \mathrm{i}\phi_2) = -\mathrm{i}(\phi_1 + \mathrm{i}\phi_2) \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \tag{3.310}$$

and, respectively,

$$H(\psi_1 + \mathrm{i}\psi_2) = -\mathrm{i}(\psi_1 + \mathrm{i}\psi_2) \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.311}$$

In particular, taking the real parts in (3.310)–(3.311) (keeping in mind that H maps real-valued functions into real-valued functions) leads to the conclusion that

$$H\phi_1 = \phi_2 \quad \text{and} \quad H\psi_1 = \psi_2. \tag{3.312}$$

Upon observing that $u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = (\operatorname{Re} u)\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} + \mathrm{i}(\operatorname{Im} u)\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}$ implies $f_1 = \phi_1 + \mathrm{i}\psi_1$ and $f_2 = \phi_2 + \mathrm{i}\psi_2$, from (3.312), we readily obtain the formula claimed in (3.284).

In the converse direction, suppose first that the function $f \in L^p(\mathbb{R}, w)$ is real-valued. Then $Hf \in L^p(\mathbb{R}, w)$ and work in [114, §1.5-§1.6] ensures that

$$U(z) := \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{(f + \mathrm{i}Hf)(y)}{y - z}\, dy, \quad \text{for each } z \in \mathbb{R}^2_+, \tag{3.313}$$

is a holomorphic function in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ and

$$U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + \tfrac{\mathrm{i}}{2}H\big)(f + \mathrm{i}Hf) = f + \mathrm{i}Hf \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \tag{3.314}$$

since the Hilbert transform satisfies $H^2 = -I$ on $L^p(\mathbb{R}, w)$. If we now introduce $u_1 := \operatorname{Re} U$ and $u_2 := \operatorname{Im} U$, then $u := (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function with real-valued scalar components. Thanks to (3.252), we have

$$Lu = L\big(\operatorname{Re} U, \operatorname{Im} U\big) = \big(\operatorname{Re}(\partial_{\bar z}^2 U), \operatorname{Im}(\partial_{\bar z}^2 U)\big) = 0 \in \mathbb{C}^2 \quad \text{in } \mathbb{R}^2_+, \tag{3.315}$$

since $\partial_{\bar z} U = 0$ in $\mathbb{R}^2_+$ by the Cauchy–Riemann equations. In addition, we observe that $\mathcal{N}_\kappa u = \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ given that, by design, $|u| = |U|$. Finally, at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we have

$$u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = \Big(\operatorname{Re} U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}, \operatorname{Im} U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}\Big) = (f, Hf), \tag{3.316}$$

by virtue of (3.314) and the fact that f is real-valued. Thus, u satisfies all requirements in (3.285).

To deal with an arbitrary function $f \in L^p(\mathbb{R}, w)$, which is not necessarily real-valued, denote by ϕ and ψ its real and imaginary parts so that $f = \phi + \mathrm{i}\psi$. From what we have proved so far, there exist $v, \omega \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ as in (3.285) such that $v\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = (\phi, H\phi)$ and $\omega\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = (\psi, H\psi)$. Then it follows that the function $u := v + \mathrm{i}\omega \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is as in (3.285) and satisfies $u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = (f, Hf)$, as wanted. □

We continue by making four remarks in relation to Proposition 3.10 and its proof.

Remark 3.2 Suppose $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ for some exponent $p \in (1, \infty)$ and choose an aperture parameter $\kappa > 0$. Also, let L be the 2×2 system from (3.239), and assume $u : \mathbb{R}^2_+ \to \mathbb{C}^2$ is a function satisfying

$$\begin{aligned} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ \text{and } \ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \end{aligned} \tag{3.317}$$

In particular, u satisfies (3.282)–(3.283) with $f = (f_1, f_2) = (0, 0)$. Retaining notation introduced during the proof of Proposition 3.10, from (3.300), (3.301), and (3.302), we see that

$$W_\varepsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{h_\varepsilon(y)}{y - z}\, \mathrm{d}y, \quad \text{for each } \ z \in \mathbb{R}^2_+. \tag{3.318}$$

Let $U := u_1 + \mathrm{i}u_2$, where u_1 and u_2 are the two scalar components of the $\mathbb{C}^2$-valued function u. On the one hand, from (3.289)–(3.292), it is clear that

$$\lim_{\varepsilon \to 0^+} W_\varepsilon(z) = U(z) + (z - \bar{z})\big(\partial_{\bar{z}} U\big)(z) \ \text{ for fixed each } \ z \in \mathbb{R}^2_+. \tag{3.319}$$

On the other hand, for each fixed $z \in \mathbb{R}^2_+$, on account of (3.308) and the fact that we currently have $f_1 + \mathrm{i} f_2 = 0$, we conclude that the limit as $\varepsilon \to 0^+$ of the integral in (3.318) is zero. Based on these observations and (3.318), we ultimately conclude that

if u is as in (3.317) then the $\mathbb{C}$-valued function $U := u_1 + \mathrm{i}u_2$ (where u_1, u_2 are the two scalar components of the $\mathbb{C}^2$-valued function u) satisfies $U(z) = (\bar{z} - z)\big(\partial_{\bar{z}} U\big)(z)$ for each $z \in \mathbb{R}^2_+$. (3.320)

The same type of argument also shows that

$$\left.\begin{array}{r} U \in \mathscr{C}^\infty(\mathbb{R}^2_+) \\ \partial_{\bar{z}}^2 U = 0 \ \text{ in } \ \mathbb{R}^2_+ \\ \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w) \\ U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ on } \ \mathbb{R} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} U(z) = (\bar{z} - z)\big(\partial_{\bar{z}} U\big)(z) \\ \text{for all } \ z \in \mathbb{R}^2_+. \end{array}\right. \tag{3.321}$$

Bearing in mind that for any U as in the left side of (3.321) the function $f := -\partial_{\bar{z}} U$ is holomorphic in $\mathbb{R}^2_+$, we may recast the conclusion in (3.321) as saying that there exists some holomorphic function f in $\mathbb{R}^2_+$ such that $U(z) = (z - \bar{z}) f(z)$ for each $z \in \mathbb{R}^2_+$. In particular, this shows that the choice $g(z) := z f(z)$ which has led to the conclusion in (3.254) is actually canonical in the case when $\Omega = \mathbb{R}^2_+$, the nontangential trace of U vanishes, and the nontangential maximal function of U belongs to a Muckenhoupt weighted Lebesgue space.

Remark 3.3 A version of (3.321) which involves the nontangential maximal operator of the gradient of the function U goes as follows:

$$\begin{aligned}&\text{given any function } U \in \mathscr{C}^\infty(\mathbb{R}^2_+) \text{ with } \partial_{\bar z}^2 U = 0 \text{ in } \mathbb{R}^2_+ \text{ and}\\ &\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w) \text{ for some } p \in (1,\infty),\ w \in A_p(\mathbb{R}, \mathcal{L}^1),\\ &\text{and } \kappa \in (0,\infty) \text{ then } U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} = 0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R} \text{ if and}\\ &\text{only if } U(z) = (\bar z - z)\big(\partial_{\bar z} U\big)(z) \text{ for all } z \in \mathbb{R}^2_+.\end{aligned} \tag{3.322}$$

Indeed, the left-pointing implication is a consequence of the fact that (see the Fatou-type result recalled in Theorem 3.4) the nontangential boundary trace

$$(\partial_{\bar z} U)\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.323}$$

To justify the right-pointing implication in (3.322), define

$$W(z) := U(z) - (\bar z - z)\big(\partial_{\bar z} U\big)(z) \quad \text{for all} \quad z \in \mathbb{R}^2_+, \tag{3.324}$$

and note that, from assumptions and (3.323), we have

$$\begin{aligned}&W \in \mathscr{C}^\infty(\mathbb{R}^2_+), \ \ \partial_{\bar z} W = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \text{and}\\ &W\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}.\end{aligned} \tag{3.325}$$

In addition, based on assumptions and interior estimates, we conclude (by reasoning much as in (3.293)–(3.297)) that

$$\mathcal{N}_\kappa(\nabla W) \in L^p(\mathbb{R}, w). \tag{3.326}$$

To proceed, we find it useful to bring in a modified boundary-to-domain Cauchy integral operator for the upper half-plane acting on each $f \in \dot{L}^p_1(\mathbb{R}, w)$ according to

$$\mathcal{C}_{\mathrm{mod}} f(z) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \left\{ \frac{1}{y - z} - \frac{\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(y)}{y} \right\} f(y)\,\mathrm{d}y \quad \text{for all} \quad z \in \mathbb{R}^2_+. \tag{3.327}$$

Work in [114, §1.8] then shows that W may be recovered, up to an additive constant, from the action of this modified Cauchy operator on the boundary trace of W. In the present case, this guarantees the existence of some $c \in \mathbb{C}$ such that

$$W = \mathcal{C}_{\mathrm{mod}}\Big(W\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} \Big) + c \ \text{ in } \ \mathbb{R}^2_+, \tag{3.328}$$

hence $W \equiv c$ in $\mathbb{R}^2_+$, thanks to the last property recorded in (3.325). In turn, this forces $c = W\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} = 0$ hence, ultimately, $W = 0$ in $\mathbb{R}^2_+$. In view of the definition of W, this finishes the proof of the right-pointing implication in (3.322). In closing, we wish to note that, having fixed $p \in (1,\infty)$, $w \in A_p(\mathbb{R},\mathcal{L}^1)$, and $\kappa \in (0,\infty)$, from (3.322) and the fact that

$$\begin{aligned}&\text{for each holomorphic function } h \text{ in } \mathbb{R}^2_+ \text{ with } \mathcal{N}_\kappa h \in L^p(\mathbb{R},w),\\ &\text{the nontangential boundary trace } h\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists } \mathcal{L}^1\text{-a.e. on } \mathbb{R}\end{aligned} \tag{3.329}$$

(e.g., this is implied by the Fatou results proved in [113, §3.1]), we conclude that

$$\begin{aligned}&\Big\{U \in \mathscr{C}^\infty(\mathbb{R}^2_+) : \partial_{\bar z}^2 U = 0 \text{ in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R},w), \text{ and}\\ &\qquad\qquad U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}\Big\}\\ &\quad = \Big\{(\bar z - z)h(z) : h \text{ holomorphic in } \mathbb{R}^2_+ \text{ with } \mathcal{N}_\kappa h \in L^p(\mathbb{R},w)\Big\}.\end{aligned} \tag{3.330}$$

This provides an explicit description of the space of null-solutions of the Homogeneous Regularity Problem for the operator $\partial_{\bar z}^2$ in the upper half-plane. In turn, this readily implies that the space of null-solutions of the Inhomogeneous Regularity Problem for the operator $\partial_{\bar z}^2$ in the upper half-plane may be described as

$$\begin{aligned}&\Big\{U \in \mathscr{C}^\infty(\mathbb{R}^2_+) : \partial_{\bar z}^2 U = 0 \text{ in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa U,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R},w), \text{ and}\\ &\qquad\qquad U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}\Big\}\\ &\quad = \Big\{(\bar z - z)h(z) : h \text{ holomorphic in } \mathbb{R}^2_+ \text{ with } \mathcal{N}_\kappa h \in L^p(\mathbb{R},w)\\ &\qquad\qquad \text{and } \mathcal{N}_\kappa\big(\mathbb{R}^2_+ \ni z \mapsto (\bar z - z)h(z)\big) \in L^p(\mathbb{R},w)\Big\}.\end{aligned} \tag{3.331}$$

Finally, we wish to mention that it is also possible to describe the space of null-solutions of the Dirichlet Problem for the operator $\partial_{\bar z}^2$ in the upper half-plane, namely

$$\begin{aligned}&\Big\{U \in \mathscr{C}^\infty(\mathbb{R}^2_+) : \partial_{\bar z}^2 U = 0 \text{ in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa U \in L^p(\mathbb{R},w), \text{ and } U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0\Big\}\\ &\quad = \Big\{(\bar z - z)h(z) : h \text{ holomorphic in } \mathbb{R}^2_+ \text{ with}\\ &\qquad\qquad \big[\mathbb{R}^2_+ \ni z \mapsto (\bar z - z)h(z)\big]\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0\\ &\qquad\qquad \text{and } \mathcal{N}_\kappa\big(\mathbb{R}^2_+ \ni z \mapsto (\bar z - z)h(z)\big) \in L^p(\mathbb{R},w)\Big\}.\end{aligned} \tag{3.332}$$

See [115, Chapter 2] for this and other similar results in the higher-dimensional setting (some of which we will review a little further).

Remark 3.4 Bring in the complexified Cauchy–Riemann equations in the upper half-plane, i.e., consider

$$
\begin{aligned}
&A, B : \mathbb{R}^2_+ \to \mathbb{C} \ \text{ of class } \mathscr{C}^\infty, \ \text{ satisfying} \\
&\partial_x A = \partial_y B \ \text{ and } \ \partial_y A = -\partial_x B \ \text{ in } \ \mathbb{R}^2_+.
\end{aligned}
\tag{3.333}
$$

Write $(A, B) \in \mathrm{CR}(\mathbb{R}^2_+)$ whenever A, B are as in (3.333). Hence, $\mathrm{CR}(\mathbb{R}^2_+)$ is a complex vector space with the property that for each $(A, B) \in \mathrm{CR}(\mathbb{R}^2_+)$ we have

$$
\begin{aligned}
&\big(\mathrm{Re}\, A, \mathrm{Re}\, B\big) \in \mathrm{CR}(\mathbb{R}^2_+), \quad \big(\mathrm{Im}\, A, \mathrm{Im}\, B\big) \in \mathrm{CR}(\mathbb{R}^2_+), \\
&\text{and } A + \mathrm{i}B \text{ is a holomorphic function in } \mathbb{R}^2_+.
\end{aligned}
\tag{3.334}
$$

Also,

$$
\begin{aligned}
&\big(\mathrm{Re}\, U, \mathrm{Im}\, U\big) \in \mathrm{CR}(\mathbb{R}^2_+) \ \text{ for each} \\
&\text{holomorphic function } U \text{ in } \mathbb{R}^2_+.
\end{aligned}
\tag{3.335}
$$

Having fixed some $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ and an aperture parameter $\kappa > 0$, we claim that

$$
\begin{aligned}
&\big\{(f, Hf) :\ f \in L^p(\mathbb{R}, w)\big\} \\
&= \Big\{\Big(A\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}, B\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) :\ (A, B) \in \mathrm{CR}(\mathbb{R}^2_+),\ \mathcal{N}_\kappa A,\ \mathcal{N}_\kappa B \in L^p(\mathbb{R}, w)\Big\}.
\end{aligned}
\tag{3.336}
$$

Formula (3.336) carries special significance in the present context. Indeed, in view of Proposition 3.10, we conclude that

> the space (described in (3.286)) of admissible boundary data for the Dirichlet Problem in the upper half-plane formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L defined in (3.239) coincides with the space of nontangential boundary traces of pairs of functions satisfying the complexified Cauchy–Riemann equations (3.333) whose nontangential maximal functions belong to said Muckenhoupt weighted Lebesgue spaces. (3.337)

Hence, in the big picture, the space of admissible boundary data for the Dirichlet Problem for the *second-order* system L from (3.239) coincides with the space of boundary traces of null-solutions of a *first-order* system, namely the complexified Cauchy–Riemann equations (3.333).

To justify (3.336), observe that since both sets involved are actually vector spaces over the field of complex numbers and since (3.334)–(3.335) hold, it suffices to show that

$$\begin{aligned}&\big\{(f, Hf):\ f\in L^p(\mathbb{R},w)\ \text{ real-valued}\big\}\\&\quad=\Big\{\Big(\mathrm{Re}\,U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+},\ \mathrm{Im}\,U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big):\ U\ \text{ holomorphic in }\ \mathbb{R}^2_+,\ \ \mathcal{N}_\kappa U\in L^p(\mathbb{R},w)\Big\}.\end{aligned}\tag{3.338}$$

As far as the equality in (3.338) is concerned, work in [113, §3.1] implies that for any holomorphic function U in $\mathbb{R}^2_+$ with $\mathcal{N}_\kappa U\in L^p(\mathbb{R},w)$ the nontangential boundary trace $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ exists at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. Also, this trace belongs to $L^p(\mathbb{R},w)$ and the following Cauchy reproducing formula holds:

$$U(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)(y)}{y-z}\,\mathrm{d}y,\quad\text{for each }\ z\in\mathbb{R}^2_+.\tag{3.339}$$

Going nontangentially to the boundary then yields

$$U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}=\big(\tfrac12 I+\tfrac{\mathrm{i}}{2}H\big)\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big).\tag{3.340}$$

Hence $U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}=\mathrm{i}H\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)$, from which we deduce that

$$\mathrm{Im}\,U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}=H\big(\mathrm{Re}\,U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big).\tag{3.341}$$

This proves the right-to-left inclusion in (3.338). As regards the left-to-right inclusion in (3.338), given any real-valued function $f\in L^p(\mathbb{R},w)$, it follows that

$$U(z):=\frac{1}{\pi\mathrm{i}}\int_{\mathbb{R}}\frac{f(y)}{y-z}\,\mathrm{d}y,\quad\text{for each }\ z\in\mathbb{R}^2_+,\tag{3.342}$$

is holomorphic in $\mathbb{R}^2_+$, has $\mathcal{N}_\kappa U\in L^p(\mathbb{R},w)$, and satisfies $U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}=(I+\mathrm{i}H)f$. In particular, $\big(\mathrm{Re}\,U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+},\mathrm{Im}\,U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)=(f,Hf)$, finishing the proof of (3.338).

Remark 3.5 The space $\big\{(f,Hf):\ f\in L^p(\mathbb{R},w)\big\}$ appearing in (3.287) is the complexification of the space appearing in the first line of (3.338). In turn, via the identification $\mathbb{R}^2\ni(a,b)\equiv a+\mathrm{i}b\in\mathbb{C}$, the latter space may be viewed as

$$\big\{f+\mathrm{i}Hf:\ f\in L^p(\mathbb{R},w)\ \text{ real-valued}\big\},\tag{3.343}$$

which, by virtue of (3.338), is further equal to the Muckenhoupt weighted Hardy space

$$\Big\{U\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} : U \text{ holomorphic in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)\Big\}. \tag{3.344}$$

From Proposition 3.10, we then conclude that the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L in the upper half-plane (cf. (3.286)) is ultimately linked to the Muckenhoupt weighted Hardy space (3.344) in the manner detailed in the above discussion.

By further building on Proposition 3.10, below we identify the space of admissible boundary data for the Muckenhoupt weighted version of the Regularity Problem for the system L from (3.239) in the upper half-plane.

Proposition 3.11 *Fix an integrability index $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and choose an aperture parameter $\kappa > 0$. Also, recall the 2×2 system L defined in the plane as in (3.239). Then the space of admissible boundary data for the Muckenhoupt weighted version of the Regularity Problem for the system L in the upper half-plane, i.e.,*

$$\Big\{u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ Lu = 0 \ \textit{in}\ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)\Big\}, \tag{3.345}$$

coincides with

$$\big\{(f, Hf) : f \in L^p_1(\mathbb{R}, w)\big\}. \tag{3.346}$$

As a consequence of this and (3.248), one also has

$$\Big\{u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ L^\top u = 0 \ \textit{in}\ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)\Big\}$$

$$= \big\{(f, -Hf) : f \in L^p_1(\mathbb{R}, w)\big\}. \tag{3.347}$$

That the nontangential boundary traces exist in the context of (3.345), (3.347) is a consequence of Proposition 2.24.

Proof of Proposition 3.11 Consider some function $u = (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying $Lu = 0$ in $\mathbb{R}^2_+$, with $\mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$, and such that $u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}$ exists at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. Proposition 3.10 guarantees that $u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = (f, Hf)$ for some $f \in L^p(\mathbb{R}, w)$. Then actually $f = u_1\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} \in L^p_1(\mathbb{R}, w)$, thanks to Proposition 2.22 whose applicability with u_1 in place of u and with $\Omega := \mathbb{R}^2_+$ is ensured by Theorem 3.4. This proves that the nontangential boundary trace $u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}$ belongs to the space in (3.346).

Conversely, start with a function $f \in L^p_1(\mathbb{R}, w)$, which is first assumed to be real-valued. Work in [114, §1.6] (in more general settings) ensures that $Hf \in L^p_1(\mathbb{R}, w)$ and

$$U(z) := \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{(f + \mathrm{i}Hf)(y)}{y - z}\,\mathrm{d}y, \quad \text{for each } z \in \mathbb{R}^2_+, \tag{3.348}$$

is a holomorphic function in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa U,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$ and, much as in (3.314), $U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = f + \mathrm{i}Hf$. Then $u := \big(\operatorname{Re} U, \operatorname{Im} U\big) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function with real-valued scalar components, with the property that $\mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$ and

$$u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = \Big(\operatorname{Re} U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}, \operatorname{Im} U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}\Big) = (f, Hf), \tag{3.349}$$

since f is real-valued. Given that, much as for (3.315), we also have $Lu = 0$ in $\mathbb{R}^2_+$, it follows that (f, Hf) belongs to the space in (3.345). Finally, the general case when $f \in L^p_1(\mathbb{R}, w)$ is not necessarily real-valued is dealt with based on what we have just proved, decomposing f into its real and imaginary parts. This eventually shows that the space from (3.346) is contained in the space from (3.345). By double inclusion, we may therefore conclude that these spaces are in fact equal. □

There is also a version of Proposition 3.11 for the *Homogeneous Regularity Problem*, involving homogeneous Muckenhoupt weighted Sobolev spaces. To state this result, we shall need the homogeneous Muckenhoupt weighted Sobolev space $\dot{L}^p_1(\mathbb{R}, w)$ defined for each integrability exponent $p \in (1, \infty)$ and for each weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ as (compare with (2.598))

$$\dot{L}^p_1(\mathbb{R}, w) := \Big\{ f \in L^1\big(\mathbb{R}, \tfrac{dx}{1+|x|^2}\big) \cap L^p_{\text{loc}}(\mathbb{R}, w) :\ f' \in L^p(\mathbb{R}, w)\Big\}, \tag{3.350}$$

where the derivative is taken in the sense of distributions. We shall also need the operator H_{mod}, the modified version of the classical Hilbert transform H on the real line from (3.351), whose action on functions $f \in \dot{L}^p_1(\mathbb{R}, w)$ is given by

$$H_{\text{mod}} f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\mathbb{R}} \Big\{ \frac{\mathbf{1}_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}(y)}{x - y} - \frac{\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(y)}{-y} \Big\} f(y)\,\mathrm{d}y \tag{3.351}$$

at $\mathcal{L}^1$-a.e. point $x \in \mathbb{R}$.

Proposition 3.12 *Pick an integrability index $p \in (1, \infty)$, fix a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and choose an aperture parameter $\kappa > 0$. Then the space of admissible boundary data for the Muckenhoupt weighted version of the*

Homogeneous Regularity Problem in the upper half-plane for the 2×2 *system* L *from* (3.239)*, i.e.,*

$$\Big\{ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ Lu = 0 \ \ in \ \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w) \Big\} \tag{3.352}$$

is equal to

$$\Big\{ f = (f_1, f_2) \in \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2 : \ H(f_1') = f_2' \ \ at\ \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R} \Big\} \tag{3.353}$$

$$= \Big\{ (f, H_{mod} f + c) \in \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2 : \ f \in \dot{L}^p_1(\mathbb{R}, w) \ \ and \ \ c \in \mathbb{C} \Big\}.$$

The fact that the nontangential boundary traces exist in the context of (3.352) is a consequence of Proposition 2.24.

Proof of Proposition 3.12 Consider a vector-valued function

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2 \ \ \text{satisfying} \\ Lu = 0 \ \ \text{in} \ \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w). \end{gathered} \tag{3.354}$$

From Theorem 3.4, Proposition 2.24, and (2.576), we know that

$$\begin{gathered} u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \ \text{exists and belongs to} \ \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2, \text{ the nontangential} \\ \text{boundary trace } (\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \ \text{exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \text{ and} \\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \text{ belong to the space } L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}^1). \end{gathered} \tag{3.355}$$

In particular, if we set $\widetilde{u} := \partial_x u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$, then

$$\begin{gathered} L\widetilde{u} = 0 \ \ \text{in} \ \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa \widetilde{u} \in L^p(\mathbb{R}, w), \ \ \text{and} \\ \widetilde{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \ \text{exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \end{gathered} \tag{3.356}$$

In addition, if at $\mathcal{L}^1$-a.e. point $x \in \mathbb{R}$ we set

$$f(x) = \big(f_1(x), f_2(x)\big) := \Big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big)(x) \in \mathbb{C}^2, \tag{3.357}$$

then (3.355) gives $f \in \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2$, and Proposition 2.22 tells us that

$$f' = \partial_x \Big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = \Big((\partial_x u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = \widetilde{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \ \ \text{at } \mathcal{L}^1\text{-a.e. point on } \ x \in \mathbb{R}. \tag{3.358}$$

Granted these properties, Proposition 3.10 applies to $\widetilde{u}$ and, with H denoting the Hilbert transform on the real line (cf. (1.24)), implies that we necessarily have

$$H(f_1') = f_2' \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \tag{3.359}$$

This proves that the set from (3.352) is included in the set described in the first line of (3.353).

To proceed, we need to recall some results from [114, Chapter 1]. First, H_{mod} maps $\dot{L}_1^p(\mathbb{R}, w)$ boundedly into itself, and for each $f \in \dot{L}_1^p(\mathbb{R}, w)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[H_{\text{mod}} f\big] = H(f') \ \text{ at } \mathcal{L}^1\text{-a.e. point in } \ \mathbb{R}. \tag{3.360}$$

In particular,

$$H_{\text{mod}} \text{ maps constants into constants.} \tag{3.361}$$

In addition, for each $f \in \dot{L}_1^p(\mathbb{R}, w)$, there exists some constant $c_f \in \mathbb{C}$ with the property that

$$H_{\text{mod}}\big(H_{\text{mod}} f\big) = -f + c_f. \tag{3.362}$$

Finally, recall the modified boundary-to-domain Cauchy integral operator for the upper half-plane from (3.327). Then, for each given function $f \in \dot{L}_1^p(\mathbb{R}, w)$, we have

$$\begin{aligned} &\mathcal{C}_{\text{mod}} f \text{ is holomorphic in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa\big(\nabla \mathcal{C}_{\text{mod}} f\big) \in L^p(\mathbb{R}, w), \text{ and} \\ &\text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R} \text{ we have } (\mathcal{C}_{\text{mod}} f)\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \big(\tfrac{1}{2}I + \tfrac{\mathrm{i}}{2}H_{\text{mod}}\big)f. \end{aligned} \tag{3.363}$$

Next, note that for each $f_1, f_2 \in \dot{L}_1^p(\mathbb{R}, w)$ having $H(f_1') = f_2'$ at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ amounts (cf. (3.360)) to having $\frac{\mathrm{d}}{\mathrm{d}x}\big(H_{\text{mod}} f_1 - f_2\big) = 0$ at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. Hence, in this case we have $f_2 = H_{\text{mod}} f_1 + c$ for some constant $c \in \mathbb{C}$, proving that the set in the first line of (3.353) is contained in the set in the second line of (3.353).

At this stage, there remains to show that the set from the second line of (3.353) is contained in (3.352). To deal with this inclusion, observe that both sets are actually vector spaces over the field of complex numbers. Moreover, the vector space in the second line of (3.353) is the linear span of pairs of the form $(f, H_{\text{mod}} f + c)$ with $f \in \dot{L}_1^p(\mathbb{R}, w)$ real-valued function and $c \in \mathbb{R}$. As such, it suffices to prove that for any real-valued function $f \in \dot{L}_1^p(\mathbb{R}, w)$ and any number $c \in \mathbb{R}$, there exists some vector-valued function u as in (3.354) such that

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (f, H_{\text{mod}} f + c). \tag{3.364}$$

To this end, define

$$U(z) := 2C_{\text{mod}} f(z) + \mathrm{i}c \quad \text{for each} \quad z \in \mathbb{R}^2_+. \tag{3.365}$$

Then (3.363) guarantees that U is a holomorphic function in $\mathbb{R}^2_+$, with the property that $\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$ and that at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we have

$$U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \big(I + \mathrm{i}H_{\text{mod}}\big) f + \mathrm{i}c. \tag{3.366}$$

If we now set $u_1 := \operatorname{Re} U$ and $u_2 := \operatorname{Im} U$, then $u := (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function, with real-valued scalar components, satisfying

$$Lu = L\big(\operatorname{Re} U, \operatorname{Im} U\big) = \big(\operatorname{Re}(\partial^2_{\bar z} U), \operatorname{Im}(\partial^2_{\bar z} U)\big) = 0 \in \mathbb{C}^2 \quad \text{in} \quad \mathbb{R}^2_+, \tag{3.367}$$

thanks to (3.252) and the fact that $\partial_{\bar z} U = 0$ in $\mathbb{R}^2_+$, by the Cauchy–Riemann equations. Also, $\mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$ given that $\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$. Finally, bearing in mind that f, $H_{\text{mod}} f$ are real-valued and that $c \in \mathbb{R}$, at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we may use (3.366) to compute

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \Big(\operatorname{Re} U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}, \operatorname{Im} U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = \big(f, H_{\text{mod}} f + c\big), \tag{3.368}$$

proving (3.364). □

A higher-dimensional version of the theory presented in connection with the planar 2×2 system L from (3.239) has been worked out in [115, Chapter 2], where analogous results to Proposition 3.10 have been established. In order to describe them, we need some notation in the n-dimensional Euclidean space, where $n \in \mathbb{N}$ with $n \geq 2$. First, recall the family of Riesz transforms $(R_j)_{1 \leq j \leq n-1}$ in the hyperplane $\mathbb{R}^{n-1} \times \{0\} \equiv \mathbb{R}^{n-1}$. Specifically, the j-th Riesz transform R_j on $\mathbb{R}^{n-1}$, with $j \in \{1, \dots, n-1\}$, is the singular integral operator acting on any given function $f \in L^1\Big(\mathbb{R}^{n-1}, \frac{\mathcal{L}^{n-1}(x')}{1+|x'|^{n-1}}\Big)$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ according to

$$R_j f(x') := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} \frac{x_j - y_j}{|x'-y'|^n} f(y') \, d\mathcal{L}^{n-1}(y'). \tag{3.369}$$

We shall also need the j-th modified Riesz transform R_j^{mod}, acting on each function $f \in L^1\Big(\mathbb{R}^{n-1}, \frac{\mathcal{L}^{n-1}(x')}{1+|x'|^{n}}\Big)$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ according to

$$R_j^{\text{mod}} f(x') := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\mathbb{R}^{n-1}} \Big\{ \frac{x_j - y_j}{|x'-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1}\setminus\overline{B((x',0),\varepsilon)}}(y') \tag{3.370}$$

$$- \frac{-y_j}{|-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1} \setminus \overline{B(0,1)}}(y') \Big\} f(y')\, \mathrm{d}\mathcal{L}^{n-1}(y').$$

Finally, following [115, Chapter 2], we shall consider a special system, namely the homogeneous, constant real coefficient, symmetric, $n \times n$ second-order system acting on each vector-valued distribution $\vec{u} = (u_1, \dots, u_n)$ (defined in an open subset of $\mathbb{R}^n$) according to

$$L_D \vec{u} := \Delta \vec{u} - 2\nabla \mathrm{div}\, \vec{u}. \tag{3.371}$$

That is,

$$\begin{gathered} L_D = \big(a_{jk}^{\alpha\beta} \partial_j \partial_k\big)_{1 \leq \alpha, \beta \leq n} \ \text{ with} \\ a_{jk}^{\alpha\beta} = \delta_{jk}\delta_{\alpha\beta} - 2\delta_{j\alpha}\delta_{k\beta} \ \text{ for all } \ \alpha, \beta, j, k \in \{1, \dots, n\}. \end{gathered} \tag{3.372}$$

Here is the result which amounts to a higher-dimensional version of Propositions 3.10, 3.11, and 3.12.

Proposition 3.13 *Fix $n \in \mathbb{N}$, with $n \geq 2$. Pick an integrability index $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, and choose some aperture parameter $\kappa > 0$. Also, recall the second-order, weakly elliptic, constant (real) coefficient, symmetric, $n \times n$ system L_D defined in (3.371).*

Then if $\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n$ is a vector-valued function satisfying

$$L_D \vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa \vec{u} \in L^p(\mathbb{R}^{n-1}, w), \tag{3.373}$$

and such that the nontangential boundary trace

$$\vec{f} = (f_1, \dots, f_n) := \vec{u}\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} \ \text{ exists (in } \mathbb{C}^n\text{) at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}, \tag{3.374}$$

it follows that the vector-valued function $\vec{f}$ belongs to $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ and

$$f_n = -\sum_{j=1}^{n-1} R_j f_j \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}. \tag{3.375}$$

In the converse direction, for any given $\vec{f} = (f_1, \dots, f_n) \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ satisfying (3.375), there exists a vector-valued function $\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n$ satisfying

$$\begin{gathered} L_D \vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa \vec{u} \in L^p(\mathbb{R}^{n-1}, w), \ \text{ and} \\ \vec{u}\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} = \vec{f} \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}. \end{gathered} \tag{3.376}$$

Altogether, the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L_D in the upper half-space may be described as follows:

$$\begin{aligned}\Big\{\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} : \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D\vec{u} = 0 \ \textit{in}\ \mathbb{R}^n_+,\ \mathcal{N}_\kappa\vec{u} \in L^p(\mathbb{R}^{n-1}, w),\\ \textit{and}\ \ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\ \textit{exists at}\ \mathcal{L}^{n-1}\textit{-a.e. point on}\ \mathbb{R}^{n-1}\Big\}\\ = \Big\{(f_1, \dots, f_n) \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n : f_n = -\sum_{j=1}^{n-1} R_j f_j\Big\}.\end{aligned} \tag{3.377}$$

Furthermore, the space of admissible boundary data for the Inhomogeneous Regularity Dirichlet Problem with boundary data in Muckenhoupt weighted Sobolev spaces for the system L_D in the upper half-space is given by[2]

$$\begin{aligned}\Big\{\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} : \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D\vec{u} = 0 \ \textit{in}\ \mathbb{R}^n_+,\ \mathcal{N}_\kappa\vec{u},\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w)\Big\}\\ = \Big\{(f_1, \dots, f_n) \in \big[L^p_1(\mathbb{R}^{n-1}, w)\big]^n : f_n = -\sum_{j=1}^{n-1} R_j f_j\Big\}.\end{aligned} \tag{3.378}$$

Also, the space of admissible boundary data for the Homogeneous Regularity Dirichlet Problem with boundary data in homogeneous Muckenhoupt weighted Sobolev spaces for the system L_D in the upper half-space may be characterized as follows:[3]

$$\begin{aligned}\Big\{\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} : \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D\vec{u} = 0 \ \textit{in}\ \mathbb{R}^n_+,\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w)\Big\}\\ = \Big\{(f_1, \dots, f_n) \in \big[\dot{L}^p_1(\mathbb{R}^{n-1}, w)\big]^n : f_n + \sum_{j=1}^{n-1} \overset{mod}{R_j} f_j\ \textit{is constant}\Big\}.\end{aligned} \tag{3.379}$$

Finally, similar results are valid on the scales of Morrey spaces and block spaces (cf. Sect. 7.1).

In particular, it is apparent from (3.377) that no nonzero vector-valued function from the space

[2] With the existence of the nontangential boundary traces guaranteed by Proposition 2.24.

[3] The existence of the nontangential boundary traces here being guaranteed by Proposition 2.24. Also, the homogeneous Muckenhoupt weighted Sobolev space $\dot{L}^p_1(\mathbb{R}^{n-1}, w)$ is defined as in (2.598) with $\Omega := \mathbb{R}^n_+$.

$$\Big\{(0,\dots,0,f):\ f\in L^p(\mathbb{R}^{n-1},w)\Big\} \tag{3.380}$$

can possibly be an admissible boundary datum for the Dirichlet Problem for system L_D in the upper half-space. As such,

the codimension of the admissible boundary data for the Dirichlet Problem for system L_D in the upper half-space (i.e., the space in the first line of (3.377)) into the full data space $\big[L^p(\mathbb{R}^{n-1},w)\big]^n$ is $+\infty$. (3.381)

Likewise, since no nonzero vector-valued function from the space

$$\Big\{(0,\dots,0,f):\ f\in L^p_1(\mathbb{R}^{n-1},w)\Big\} \tag{3.382}$$

can possibly be an admissible boundary datum for the Inhomogeneous Regularity Problem for system L_D in the upper half-space, it follows that

the codimension of the admissible boundary data for the Inhomogeneous Regularity Problem for system L_D in the upper half-space (i.e., the space in the first line of (3.378)) into the full data space $\big[L^p_1(\mathbb{R}^{n-1},w)\big]^n$ is $+\infty$. (3.383)

Finally, given that no nonzero vector-valued function from the space

$$\Big\{(0,\dots,0,f):\ f\in \dot{L}^p_1(\mathbb{R}^{n-1},w)\Big\} \tag{3.384}$$

can possibly be an admissible boundary datum for the Homogeneous Regularity Problem for system L_D in the upper half-space, we see that

the codimension of the admissible boundary data for the Homogeneous Regularity Problem for system L_D in the upper half-space (i.e., the space in the first line of (3.379)) into the full data space $\big[\dot{L}^p_1(\mathbb{R}^{n-1},w)\big]^n$ is $+\infty$. (3.385)

It has also been noted in [115, §2.6] that for each scalar function

$$\omega\in\mathscr{C}^\infty(\mathbb{R}^n_+)\ \text{ with }\ \Delta\omega=0\ \text{ in }\ \mathbb{R}^n_+\ \text{ and }\ \mathcal{N}_\kappa(\nabla\omega)\in L^p(\mathbb{R}^{n-1},w), \tag{3.386}$$

the vector-valued function

$$\begin{gathered}\vec{u}:\mathbb{R}^n_+\longrightarrow\mathbb{C}^n\ \text{ given by}\\ \vec{u}(x):=x_n(\nabla\omega)(x)\ \text{ for each }\ x=(x_1,\dots,x_n)\in\mathbb{R}^n_+\end{gathered} \tag{3.387}$$

satisfies

$$
\begin{gathered}
\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \quad L_D\vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w), \\
\text{and } \ \vec{u}\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}.
\end{gathered}
\tag{3.388}
$$

In the converse direction, each vector-valued function $\vec{u}$ as in (3.388) has the format described in the second line of (3.387) for some scalar function ω as in (3.386). Finally, it has been noted in [115, §2.6] that if in place of (3.386), one now assumes

$$
\begin{gathered}
\omega \in \mathscr{C}^\infty(\mathbb{R}^n_+) \ \text{ with } \ \Delta\omega = 0 \ \text{ in } \ \mathbb{R}^n_+ \ \text{ and} \\
\mathcal{N}_\kappa\omega \in L^p(\mathbb{R}^{n-1}, w), \quad \mathcal{N}_\kappa(\nabla\omega) \in L^p(\mathbb{R}^{n-1}, w),
\end{gathered}
\tag{3.389}
$$

then the vector-valued function $\vec{u}$ defined as in (3.387) for this choice of ω has the additional property that $\mathcal{N}_\kappa\vec{u} \in L^p(\mathbb{R}^{n-1}, w)$, i.e., satisfies

$$
\begin{gathered}
\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \quad L_D\vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa\vec{u},\, \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w), \\
\text{and } \ \vec{u}\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}.
\end{gathered}
\tag{3.390}
$$

In particular, these considerations readily imply that

the space of null-solutions for the Homogeneous Regularity Problem for the system L_D in the upper half-space (i.e., the space of functions as in (3.388)) is infinite dimensional (3.391)

and that

the space of null-solutions for the Inhomogeneous Regularity Problem for the system L_D in the upper half-space (i.e., the space of functions as in (3.390)) is infinite dimensional. (3.392)

As a corollary of (3.392), we also see that

the space of null-solutions for the Dirichlet Problem for the system L_D in the upper half-space is infinite dimensional. (3.393)

We next turn our attention to the issue of existence and uniqueness of distinguished coefficient tensors for a given weakly elliptic system and its transposed. The starting point is the following result, proved in [115, §1.5], for strongly elliptic systems.

Theorem 3.8 *Fix $M, n \in \mathbb{N}$ with $n \geq 2$. Let L be a homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$ which satisfies the strong Legendre–Hadamard ellipticity condition* (3.4). *Then either*

$$
\mathfrak{A}^{\text{dis}}_L = \varnothing \ \ \textit{and} \ \ \mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing,
\tag{3.394}
$$

or

$$\mathfrak{A}_L^{\mathrm{dis}} = \{A\} \textit{ and } \mathfrak{A}_{L^\top}^{\mathrm{dis}} = \{A^\top\} \textit{ for some } A \in \mathfrak{A}_L. \tag{3.395}$$

As a corollary, if $M, n \in \mathbb{N}$ with $n \geq 2$ and L is a homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$ satisfying the Legendre–Hadamard (strong) ellipticity condition, then

$$\mathfrak{A}_L^{\mathrm{dis}} \textit{ is either empty or a singleton.} \tag{3.396}$$

We next state a result, augmenting Theorem 3.8, pertaining to weakly elliptic systems, also established in [115, §1.5].

Theorem 3.9 *Let $M, n \in \mathbb{N}$ with $n \geq 2$ and consider a weakly elliptic, homogeneous, second-order, constant complex coefficient, $M \times M$ system L in $\mathbb{R}^n$ with the property that $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$. Then both $\mathfrak{A}_L^{\mathrm{dis}}$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}}$ are singletons. In fact, $\mathfrak{A}_L^{\mathrm{dis}} = \{A\}$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}} = \{A^\top\}$ for some $A \in \mathfrak{A}_L$.*

In particular, if $M, n \in \mathbb{N}$ with $n \geq 2$ and L is a symmetric, weakly elliptic, homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$, then $\mathfrak{A}_L^{\mathrm{dis}}$ is either empty or a singleton, and, in the latter case, one has $\mathfrak{A}_L^{\mathrm{dis}} = \{A\}$ for some $A \in \mathfrak{A}_L$ satisfying $A^\top = A$.

For example, from (3.223), we know that

$$\mathfrak{A}_\Delta^{\mathrm{dis}} = \{I_{n\times n}\} \text{ where } \Delta \text{ is the Laplacian in } \mathbb{R}^n \text{ with } n \geq 2, \tag{3.397}$$

$$\mathfrak{A}_{\mathrm{div}\, A\nabla}^{\mathrm{dis}} = \{(A + A^\top)/2\} \text{ if } n \geq 3 \text{ and } A \in \mathbb{C}^{n\times n} \text{ is invertible,} \tag{3.398}$$

while Theorem 3.9 and (3.228) imply that, for the complex Lamé system $L_{\mu,\lambda}$ defined in (3.224), we have

$$\begin{gathered} \mathfrak{A}_{L_{\mu,\lambda}}^{\mathrm{dis}} = \Big\{ \big(a_{jk}^{\alpha\beta} \big)_{\substack{1\leq j,k\leq n \\ 1\leq \alpha,\beta\leq n}} \Big\} \text{ if } \mu \neq 0,\, 2\mu+\lambda \neq 0, \text{ and } 3\mu+\lambda \neq 0, \text{ where} \\ a_{jk}^{\alpha\beta} := \mu\delta_{jk}\delta_{\alpha\beta} + \tfrac{(\mu+\lambda)(2\mu+\lambda)}{3\mu+\lambda}\delta_{j\alpha}\delta_{k\beta} + \tfrac{\mu(\mu+\lambda)}{3\mu+\lambda}\delta_{j\beta}\delta_{k\alpha}, \\ \text{for } 1 \leq j, k, \alpha, \beta \leq n. \end{gathered} \tag{3.399}$$

Here is an equivalent characterization of the existence of a distinguished coefficient tensor proved in [115, §1.6].

Theorem 3.10 *Fix $M, n \in \mathbb{N}$ with $n \geq 2$. Let L be an $M \times M$ second-order, homogeneous, constant complex coefficient, weakly elliptic system in $\mathbb{R}^n$. Then the following statements are equivalent:*

(i) The system L possesses a distinguished coefficient tensor, i.e., $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$.

(ii) There exists a matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ *which is positive homogeneous of degree* $-n$ *and satisfies*

$$\int_{S^{n-1}} k \, d\mathcal{H}^{n-1} = I_{M\times M} \tag{3.400}$$

(where $I_{M\times M}$ *is the* $M \times M$ *identity matrix), as well as*

$$L\big(x_s k(x)\big) = 0 \cdot I_{M\times M} \;\; in \;\; \mathbb{R}^n \setminus \{0\} \;\; for\ each \;\; s \in \{1, \dots, n\}. \tag{3.401}$$

Moreover, if L *has a unique distinguished coefficient tensor (i.e., if* $\#\mathfrak{A}_L^{\rm dis} = 1$*), then there is only one function* k *as in item (ii).*

It has been noted in [115, §1.6] that Theorem 3.10 has the following noteworthy consequence:

Corollary 3.2 *Fix* $M, n \in \mathbb{N}$ *with* $n \geq 2$*. Let* L *be an* $M \times M$ *second-order, homogeneous, constant complex coefficient, weakly elliptic system in* $\mathbb{R}^n$*. Assume that there exists a matrix-valued function* $k_* \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ *which is positive homogeneous of degree* $-n$*, is not identically zero, and satisfies*

$$\int_{S^{n-1}} k_* \, d\mathcal{H}^{n-1} = 0 \cdot I_{M\times M} \tag{3.402}$$

(where $I_{M\times M}$ *is the* $M \times M$ *identity matrix), as well as*

$$L\big(x_s k_*(x)\big) = 0 \cdot I_{M\times M} \;\; in \;\; \mathbb{R}^n \setminus \{0\} \;\; for\ each \;\; s \in \{1, \dots, n\}. \tag{3.403}$$

Then either $\mathfrak{A}_L^{\rm dis} = \varnothing$*, or* $\mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$*.*

To proceed, we revisit the special system L_D from (3.371) which turns out not to have any distinguished coefficient tensors. Indeed, it has been noted in [115, §1.6] that if E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, defined at each point $x \in \mathbb{R}^n \setminus \{0\}$ according to

$$E_\Delta(x) := \begin{cases} \dfrac{1}{(2-n)\omega_{n-1}} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\[2ex] \dfrac{1}{2\pi} \ln |x| & \text{if } n = 2, \end{cases} \tag{3.404}$$

and if k_* is the Hessian matrix of E_Δ, defined at each point $x \in \mathbb{R}^n \setminus \{0\}$ by

$$k_*(x) := \big((\partial_i \partial_j E_\Delta)(x)\big)_{1\le i,j\le n} = \Big(\frac{\delta_{ij}}{\omega_{n-1}}\frac{1}{|x|^n} - \frac{n}{\omega_{n-1}}\frac{x_i x_j}{|x|^{n+2}}\Big)_{1\le i,j\le n}, \tag{3.405}$$

then (3.403)–(3.402) hold for $L = L_D$, the special system L_D from (3.371). In view of the fact that L_D is symmetric, Corollary 3.2 then gives

for each $n \in \mathbb{N}$ with $n \ge 2$, the $n \times n$ system L_D in $\mathbb{R}^n$ from (3.371) is weakly elliptic, second-order, homogeneous, constant real coefficient, symmetric, and $\mathfrak{A}^{\rm dis}_{L_D} = \mathfrak{A}^{\rm dis}_{L_D^\top} = \varnothing$. (3.406)

Remark 3.6 Consider the complex Lamé system $L_{\mu,\lambda}$, defined earlier in (3.224), in the regime $\mu, \lambda \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$. From (3.225), we know that this is equivalent with the weak ellipticity of $L_{\mu,\lambda}$. Hence, this is the range in which we may consider the issue of whether $L_{\mu,\lambda}$ possesses distinguished coefficient tensors. In this regard, we wish to note that from (3.229) and Theorem 3.9, it follows that $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}}$ is a singleton when $3\mu+\lambda \neq 0$. In addition, from (3.406) and (3.371), we see that $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}}$ is empty when $3\mu + \lambda = 0$. Collectively, these observations prove that

given any $\mu, \lambda \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$, then $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}} \neq \varnothing$ if and only if $3\mu + \lambda \neq 0$ if and only if $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}}$ is a singleton (namely the coefficient tensor $A(\zeta)$ described in (3.226), corresponding to the choice $\zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}$). (3.407)

One final remark is as follows. Consider an arbitrary second-order, weakly elliptic, homogeneous, constant complex coefficient, $M \times M$ system L in $\mathbb{R}^n$, and pick a coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_L$. For each invertible matrix $C = (c_{jk})_{1\le j,k\le n} \in \mathbb{C}^{M\times M}$, define

$$AC := \big(a^{\alpha\beta}_{j\ell} c_{\ell k}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_{LC} \tag{3.408}$$

and

$$CA := \big(c_{j\ell} a^{\alpha\beta}_{\ell k}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_{CL}, \tag{3.409}$$

with the systems LC and CL naturally interpreted in the sense of multiplication of $M \times M$ matrices. With this notation, it has been noted in [115, §1.2] that for each invertible matrix $C \in \mathbb{C}^{M\times M}$ we have

$$A \in \mathfrak{A}^{\rm dis}_L \iff AC \in \mathfrak{A}^{\rm dis}_{LC}, \tag{3.410}$$

and

$$A \in \mathfrak{A}_L^{\text{dis}} \iff CA \in \mathfrak{A}_{CL}^{\text{dis}}. \tag{3.411}$$

A useful consequence of (3.410)–(3.411) and Corollary 3.2 is as follows. Bring back the second-order, homogeneous, real constant coefficient, 2×2 system in the plane

$$L_B = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}, \tag{3.412}$$

which is matrix representation of Bitsadze's operator $\mathbb{L}$ from (3.250). Also, recall the two-dimensional version of the special system L_D from (3.371), i.e.,

$$L_D = \begin{pmatrix} \partial_y^2 - \partial_x^2 & -2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}. \tag{3.413}$$

Hence, if we let

$$V := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.414}$$

then $V^\top = V = V^{-1}$ and

$$L_B = \tfrac{1}{4} L_D V \quad \text{and} \quad L_B^\top = \tfrac{1}{4} V L_D. \tag{3.415}$$

These together with (3.410) and (3.411) imply

$$A \in \mathfrak{A}_{L_D}^{\text{dis}} \iff A\,V \in \mathfrak{A}_{L_B}^{\text{dis}} \tag{3.416}$$

and

$$A \in \mathfrak{A}_{L_D}^{\text{dis}} \iff VA \in \mathfrak{A}_{L_B^\top}^{\text{dis}}. \tag{3.417}$$

Since we have proved that $\mathfrak{A}_{L_D}^{\text{dis}} = \varnothing$ (cf. (3.406)), the equivalences in (3.416)–(3.417) imply that

$$\mathfrak{A}_{L_B}^{\text{dis}} = \varnothing \quad \text{and} \quad \mathfrak{A}_{L_B^\top}^{\text{dis}} = \varnothing. \tag{3.418}$$