Chapter 2
Geometric Measure Theory

We begin with a quick review of notational conventions used in the monograph. Throughout, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ with $n \geq 2$, and $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure in $\mathbb{R}^n$. Also, we shall denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. It is a well-known fact (cf. [47, Theorem 1, p. 61]) that the $(n-1)$-dimensional Hausdorff outer-measure is a Borel-regular outer-measure in $\mathbb{R}^n$. Since the measure induced by an arbitrary outer-measure (as in Carathéodory's theorem) is automatically complete, it follows that

$$\mathcal{H}^{n-1} \text{ is a complete Borel-regular measure in } \mathbb{R}^n. \tag{2.1}$$

Next, for each set $E \subseteq \mathbb{R}^n$, we let $\mathbf{1}_E$ denote the characteristic function of E (defined as $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \in \mathbb{R}^n \setminus E$). Also, δ_{jk} is the Kronecker symbol (i.e., $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$). By $\{\mathbf{e}_j\}_{1\leq j\leq n}$ we shall denote the standard orthonormal basis in $\mathbb{R}^n$, i.e., $\mathbf{e}_j := (\delta_{jk})_{1\leq k\leq n}$ for each $j \in \{1, \dots, n\}$. For each $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ set $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The dot product of two vectors $u, v \in \mathbb{R}^n$ is denoted by $u \cdot v = \langle u, v\rangle$, and for each vector $v \in \mathbb{R}^n$ we set $\langle v\rangle^{\perp} := \{u \in \mathbb{R}^n : u \cdot v = 0\}$. Next, $\mathbb{R}^n_{\pm} := \{x \in \mathbb{R}^n : \pm\langle x, \mathbf{e}_n\rangle > 0\}$ denote, respectively, the upper half-space and the lower half-space in $\mathbb{R}^n$.

Given an arbitrary set $\Omega \subseteq \mathbb{R}^n$, we shall denote by $\mathscr{C}^0(\Omega)$ the space of continuous scalar-valued functions defined on Ω. Assuming now that $\Omega \subseteq \mathbb{R}^n$ is actually open, for each $k \in \mathbb{N} \cup \{0\}$ we shall denote by $\mathscr{C}^k(\Omega)$ the space of scalar-valued functions which have continuous partial derivatives of order $\leq k$ in Ω. Also, $\mathscr{C}^\infty_0(\Omega)$ stands for the space of compactly supported functions from $\mathscr{C}^\infty(\Omega)$. We shall let $\mathcal{D}'(\Omega)$ stand for the space of distributions in the set Ω and, for each integrability exponent $p \in [1, \infty]$ and integer $k \in \mathbb{N}$, we shall define the local L^p-based Sobolev space of order k in Ω as $W^{k,p}_{\rm loc}(\Omega) := \big\{u \in \mathcal{D}'(\Omega) : \partial^\alpha u \in L^p_{\rm loc}(\Omega, \mathcal{L}^n),\ |\alpha| \leq k\big\}$. The Jacobian matrix of a differentiable $\mathbb{C}^M$-valued function $u = (u_\alpha)_{1\leq\alpha\leq M}$ defined in an open subset of $\mathbb{R}^n$ is the $\mathbb{C}^{M\cdot n}$-valued function

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$$\nabla u := \big(\partial_j u_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq j\leq n}} = \begin{bmatrix} \partial_1 u_1 & \cdots & \partial_n u_1 \\ \vdots & \vdots & \vdots \\ \partial_1 u_M & \cdots & \partial_n u_M \end{bmatrix}. \tag{2.2}$$

We shall retain the same symbol ∇u when the components of u are actually distributions. Next, we agree to denote by $S^{n-1} := \partial B(0,1)$ the unit sphere in $\mathbb{R}^n$, and use $\omega_{n-1} := \mathcal{H}^{n-1}(S^{n-1})$ for the surface area of S^{n-1}. In addition, we shall let υ_{n-1} denote the volume of the unit ball in $\mathbb{R}^{n-1}$. Given any $x, y \in \mathbb{R}^n$, by $[x, y]$ we shall denote the line segment with endpoints x, y. We shall also need $\operatorname{dist}(x, E) := \inf\{|x-y| : y \in E\}$, the distance from a given point $x \in \mathbb{R}^n$ to a nonempty set $E \subseteq \mathbb{R}^n$. If (X, μ) is a given measure space, for each $p \in (0, \infty]$ we shall denote by $L^p(X, \mu)$ the Lebesgue space of μ-measurable functions which are p-th power integrable on X with respect to μ. Also, by $L^{p,q}(X, \mu)$ with $p, q \in (0, \infty]$ we shall denote the scale of Lorentz spaces on X with respect to the measure μ. In the same setting, for each μ-measurable set $E \subseteq X$ with $0 < \mu(E) < \infty$ and each function f which is absolutely integrable on E we set $⨍_E f\,\mathrm{d}\mu := \mu(E)^{-1}\int_E f\,\mathrm{d}\mu$. For two operators T and S, the symbol $[T, S] := T\circ S - S\circ T$ denotes the commutator of T and S. For a measurable function b, we let M_b be the pointwise multiplication by b, that is, $M_b(f)(x) := b(x)\cdot f(x)$. Given $N, M \in \mathbb{N}$, for any $a = (a_1, \dots, a_N) \in \mathbb{C}^N$ and $b = (b_1, \dots, b_M) \in \mathbb{C}^M$, we agree to define $a \otimes b$ to be the $N \times M$ matrix

$$a \otimes b := \big(a_j b_k\big)_{\substack{1\leq j\leq N\\ 1\leq k\leq M}} \in \mathbb{C}^{N\times M}. \tag{2.3}$$

Finally, we adopt the common convention of writing $A \approx B$ if there exists a constant $C \in (1, \infty)$ with the property that $A/C \leq B \leq CA$ for all values of the relevant parameters entering the definitions of A, B (something that is self-evident in each context we employ this notation).

2.1 Classes of Euclidean Sets of Locally Finite Perimeter

Given an open set $\Omega \subseteq \mathbb{R}^n$ and an aperture parameter $\kappa \in (0, \infty)$, define the nontangential approach regions

$$\Gamma_\kappa(x) := \big\{y \in \Omega : |y - x| < (1+\kappa)\operatorname{dist}(y, \partial\Omega)\big\} \quad \text{for each } x \in \partial\Omega. \tag{2.4}$$

In turn, these are used to define the nontangential maximal operator $\mathcal{N}_\kappa$, acting on each $\mathcal{L}^n$-measurable function u defined in Ω according to

$$\big(\mathcal{N}_\kappa u\big)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x), \mathcal{L}^n)} \quad \text{for each } x \in \partial\Omega, \tag{2.5}$$

with the convention that $\big(\mathcal{N}_\kappa u\big)(x) := 0$ whenever $x \in \partial\Omega$ is such that $\Gamma_\kappa(x) = \varnothing$. Note that, if we work (as one usually does) with equivalence classes, obtained by identifying functions which coincide $\mathcal{L}^n$-a.e., the nontangential maximal operator is independent of the specific choice of a representative in a given equivalence class. It turns out that (see [111, §8.2] for a proof)

$$\mathcal{N}_\kappa u : \partial\Omega \to [0, +\infty] \text{ is a lower-semicontinuous function.} \tag{2.6}$$

Also, it is apparent from definitions that

$$\begin{gathered}\text{whenever } u \in \mathscr{C}^0(\Omega) \text{ one actually has}\\ \big(\mathcal{N}_\kappa u\big)(x) = \sup_{y\in\Gamma_\kappa(x)} |u(y)| \;\text{ for all }\; x \in \partial\Omega.\end{gathered} \tag{2.7}$$

More generally, if $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function and $E \subseteq \Omega$ is a $\mathcal{L}^n$-measurable set, we denote by $\mathcal{N}_\kappa^E u$ the nontangential maximal function of u restricted to E, i.e.,

$$\begin{gathered}\mathcal{N}_\kappa^E u : \partial\Omega \longrightarrow [0, +\infty] \;\text{ defined as}\\ (\mathcal{N}_\kappa^E u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x)\cap E, \mathcal{L}^n)} \;\text{ for each }\; x \in \partial\Omega.\end{gathered} \tag{2.8}$$

Hence, $\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa(u \cdot \mathbf{1}_E)$. Throughout, we agree to use the simpler notation $\mathcal{N}_\kappa^\delta$ in the case when $E = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $\delta \in (0, \infty)$, i.e.,

$$\mathcal{N}_\kappa^\delta u := \mathcal{N}_\kappa\big(u\mathbf{1}_{O_\delta}\big) \;\text{ where }\; O_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}. \tag{2.9}$$

It turns out that, when the background measure is doubling, membership of the nontangential maximal function to Lorentz spaces is not contingent on the size of the aperture parameter. This is made precise in the proposition below (see [111, §8.4] for a proof).

Proposition 2.1 *Assume that Ω is an open nonempty proper subset of $\mathbb{R}^n$ and consider a doubling Borel measure σ on $\partial\Omega$. Fix two integrability exponents $p, q \in (0, \infty]$. Then for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$ and any two aperture parameters $\kappa_1, \kappa_2 \in (0, \infty)$ one has, in a quantitative sense,*

$$\mathcal{N}_{\kappa_1} u \in L^{p,q}(\partial\Omega, \sigma) \;\text{ if and only if }\; \mathcal{N}_{\kappa_2} u \in L^{p,q}(\partial\Omega, \sigma), \tag{2.10}$$

and, for each truncation parameter $\delta \in (0, \infty)$,

$$\mathcal{N}_{\kappa_1}^\delta u \in L^p_{loc}(\partial\Omega, \sigma) \;\text{ if and only if }\; \mathcal{N}_{\kappa_2}^\delta u \in L^p_{loc}(\partial\Omega, \sigma). \tag{2.11}$$

Continue to assume that Ω is an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$ and suppose u is some vector-valued $\mathcal{L}^n$-measurable function defined in Ω. Also, fix an aperture parameter $\kappa > 0$ and consider a point $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$ (i.e., x is an accumulation point for the nontangential approach region $\Gamma_\kappa(x)$). In this context, we shall say that the nontangential limit of u at x from within $\Gamma_\kappa(x)$ exists, and its value is the vector $a \in \mathbb{C}^M$, provided

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ there exists } r > 0 \text{ with the property}\\ &|u(y) - a| < \varepsilon \text{ for } \mathcal{L}^n\text{-a.e. point } y \in \Gamma_\kappa(x) \cap B(x,r).\end{aligned} \tag{2.12}$$

Whenever the nontangential limit of u at x from within $\Gamma_\kappa(x)$ exists, we agree to denote its value by the symbol $\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)$. It is then clear from definitions that whenever the latter exists we have

$$\Big|\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)\Big| \le \big(\mathcal{N}^\delta_\kappa u\big)(x) \le \big(\mathcal{N}_\kappa u\big)(x), \quad \text{for all } \delta > 0. \tag{2.13}$$

Moving on, recall that an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has `locally finite perimeter` if its `measure theoretic boundary`, i.e.,

$$\partial_*\Omega := \Big\{x \in \partial\Omega : \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap\Omega)}{r^n} > 0,\ \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus\Omega)}{r^n} > 0\Big\}, \tag{2.14}$$

satisfies

$$\mathcal{H}^{n-1}\big(\partial_*\Omega \cap K\big) < +\infty \quad \text{for each compact } K \subseteq \mathbb{R}^n \tag{2.15}$$

(cf. [47, Sections 5.7 and 5.11]). Alternatively, an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter if, with the gradient taken in the sense of distributions in $\mathbb{R}^n$,

$$\mu_\Omega := -\nabla \mathbf{1}_\Omega \tag{2.16}$$

is an $\mathbb{R}^n$-valued Borel measure in $\mathbb{R}^n$ of locally finite total variation. Occasionally, μ_Ω is referred to as the `Gauss-Green measure` of Ω (see, e.g., [89, Remark 12.2, p. 122]). Fundamental work of De Giorgi-Federer (cf., e.g., [47], [89] for modern accounts) then gives the following Polar Decomposition of the Radon measure μ_Ω:

$$\mu_\Omega = -\nabla\mathbf{1}_\Omega = \nu\,|\nabla\mathbf{1}_\Omega|, \tag{2.17}$$

where $|\nabla\mathbf{1}_\Omega|$, the total variation measure of the measure $\nabla\mathbf{1}_\Omega$, is given by

$$|\nabla\mathbf{1}_\Omega| = \mathcal{H}^{n-1}\lfloor\partial_*\Omega, \tag{2.18}$$

and where

$$\begin{aligned}&\nu \in \big[L^\infty(\partial_*\Omega, \mathcal{H}^{n-1})\big]^n \text{ is an } \mathbb{R}^n\text{-valued function}\\ &\text{satisfying } |\nu(x)| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial_*\Omega.\end{aligned} \tag{2.19}$$

We shall refer to ν above as the `geometric measure theoretic outward unit normal` to Ω. Note here that by simply eliminating the distribution theory jargon implicit in the interpretation of (2.17) (and using a straightforward limiting argument involving a mollifier) one already arrives at the formula

$$\int_\Omega \operatorname{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\big|_{\partial\Omega}\big)\,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for each vector field } \vec{F}\in\big[\mathscr{C}^1_0(\mathbb{R}^n)\big]^n. \tag{2.20}$$

For a set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter, we let $\partial^*\Omega$ denote the `reduced boundary` of Ω, that is,

$\partial^*\Omega$ consists of all points $x\in\partial\Omega$ satisfying the following two properties: $0<\mathcal{H}^{n-1}\big(B(x,r)\cap\partial_*\Omega\big)<+\infty$ for each radius $r\in(0,\infty)$, and $\lim\limits_{r\to 0^+} ⨍_{B(x,r)\cap\partial_*\Omega}\nu\,\mathrm{d}\mathcal{H}^{n-1}=\nu(x)\in S^{n-1}$. (2.21)

From [47, Lemma 2, p. 222] we know that

any $\mathcal{L}^n$-measurable set $\Omega\subseteq\mathbb{R}^n$ has the property that $\partial_*\Omega$ is a Borel set in $\mathbb{R}^n$ (in particular, $\partial_*\Omega$ is $\mathcal{H}^{n-1}$-measurable). (2.22)

In addition, given any set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter, from the structure theorem for sets of locally finite perimeter (cf. [47, Theorem 2, p. 205]) it follows that

$$\partial^*\Omega \ \text{ is countably rectifiable, of dimension } n-1 \quad \text{(hence, the set } \partial^*\Omega \text{ is also } \mathcal{H}^{n-1}\text{-measurable).} \tag{2.23}$$

Moreover, for any set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter we have (cf. [47, p. 208])

$$\partial^*\Omega\subseteq\partial_*\Omega\subseteq\partial\Omega \ \text{ and } \ \mathcal{H}^{n-1}\big(\partial_*\Omega\setminus\partial^*\Omega\big)=0. \tag{2.24}$$

It is also useful to note that, as remarked in [111, §5.6],

if $\Omega\subseteq\mathbb{R}^n$ is a set of locally finite perimeter and $m\in\mathbb{N}$, then $\widetilde{\Omega}:=\mathbb{R}^m\times\Omega\subseteq\mathbb{R}^{m+n}$ is a set of locally finite perimeter, satisfying $\partial_*\widetilde{\Omega}=\mathbb{R}^m\times\partial_*\Omega$, and whose geometric measure theoretic outward unit normal $\widetilde{\nu}$ is $\widetilde{\nu}(x,y)=(0,\nu(y))$ for $(\mathcal{L}^m\otimes\mathcal{H}^{n-1})$-a.e. point $(x,y)\in\partial_*\widetilde{\Omega}=\mathbb{R}^m\times\partial_*\Omega$, where $0\in\mathbb{R}^m$ and ν is the geometric measure theoretic outward unit normal to the set Ω. (2.25)

The following result, comparing the geometric measure theoretic outward unit normals of two sets of locally finite perimeter (on the intersection of their reduced boundaries), is going to be relevant for us later on, in Theorem 2.6 (and, by extension, in the proof of Theorem 4.2).

Proposition 2.2 *Let E, F be two sets of locally finite perimeter in $\mathbb{R}^n$. If ν_E and ν_F denote the geometric measure theoretic outward unit normal vectors to E and F, respectively, then at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$ one has either $\nu_E(x) = \nu_F(x)$ or $\nu_E(x) = -\nu_F(x)$.*

Proof This is a consequence of [89, Proposition 10.5, p. 101] according to which

any two locally $\mathcal{H}^{n-1}$-rectifiable sets $M_1, M_2 \subseteq \mathbb{R}^n$ have identical approximate tangent planes at $\mathcal{H}^{n-1}$-a.e. point in $M_1 \cap M_2$, (2.26)

and [129, Theorem 14.3, (1), pp. 72-73] where it has been shown that

given any set of locally finite perimeter $\Omega \subseteq \mathbb{R}^n$, its approximate tangent plane exists at each point $x \in \partial^*\Omega$ and is equal to $\langle \nu(x) \rangle^{\perp}$ (where ν denotes the geometric measure theoretic outward unit normal vector to Ω). (2.27)

Indeed, (2.15) and (2.24) tell us that $\partial^* E$, $\partial^* F$ are locally $\mathcal{H}^{n-1}$-rectifiable sets (cf. [89, p. 96]), so (2.26) (used with $M_1 := \partial^* E$ and $M_2 := \partial^* F$) together with (2.27) imply that $\langle \nu_E(x) \rangle^{\perp} = \langle \nu_F(x) \rangle^{\perp}$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$, from which the desired conclusion follows. □

Given a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, another piece of notation commonly used (cf., e.g., [47, p. 169]) is

$$\|\partial\Omega\| := \mathcal{H}^{n-1} \lfloor \partial^*\Omega. \tag{2.28}$$

From (2.28), (2.24), and (2.18) (cf. also [89, (15.10), p. 170]) we then see that

$$\begin{gathered} \|\partial\Omega\| \text{ agrees with the total variation of } \mu_\Omega, \\ \text{the Gauss-Green measure of } \Omega, \end{gathered} \tag{2.29}$$

and we also claim that[1]

$$\operatorname{supp} \|\partial\Omega\| = \overline{\partial^*\Omega}. \tag{2.30}$$

Indeed, from (2.28), (2.24), (2.21) we see that $\partial^*\Omega \subseteq \operatorname{supp} \|\partial\Omega\|$ and, as a consequence, $\overline{\partial^*\Omega} \subseteq \operatorname{supp} \|\partial\Omega\|$ since the latter set is closed. This proves the right-

[1] Given a topological space X along with some (non-negative) Borel measure μ on X, the support of μ is denoted by $\operatorname{supp} \mu$ and is defined as the set of all points $x \in X$ with the property that $\mu(O) > 0$ for each open set $O \subseteq X$ containing x.

to-left inclusion in (2.30). As for the opposite inclusion, if $x \in \mathbb{R}^n \setminus \overline{\partial^*\Omega}$, then there exists $r > 0$ with the property that $B(x,r) \cap \partial^*\Omega = \varnothing$. In concert with (2.28), this implies $\|\partial\Omega\|\big(B(x,r)\big) = 0$, hence $x \notin \operatorname{supp}\|\partial\Omega\|$. The proof of (2.30) is therefore complete. As a consequence of this, (2.29), and definitions[2] we therefore have

$$\operatorname{supp}\mu_\Omega = \operatorname{supp}\|\partial\Omega\| = \overline{\partial^*\Omega}. \tag{2.31}$$

See also [89, p. 168] in this regard.

Definition 2.1 A closed set $\Sigma \subseteq \mathbb{R}^n$ is called an `Ahlfors regular set` (or an Ahlfors-David regular set) if there exists a constant $C \in [1,\infty)$ such that

$$r^{n-1}/C \leq \mathcal{H}^{n-1}\big(B(x,r)\cap\Sigma\big) \leq Cr^{n-1}, \quad \forall\, r \in \big(0, 2\operatorname{diam}(\Sigma)\big), \quad \forall\, x \in \Sigma. \tag{2.32}$$

Also, given a closed set $\Sigma \subseteq \mathbb{R}^n$ and some $R \in (0,\infty]$, say that Σ is `Ahlfors regular up to scale` R, `with constant` $C \in [1,\infty)$, provided the double inequality in (2.32) is valid for each $r \in (0,R)$.

Finally, the labels `lower Ahlfors regular` and `upper Ahlfors regular` are employed when only the lower, respectively, upper, inequality in (2.32) is required to hold.

For a given closed set $\Sigma \subseteq \mathbb{R}^n$, the quality of being Ahlfors regular is not a regularity condition in a traditional analytic sense, but rather a property guaranteeing that, at all locations, Σ behaves (in a quantitative, scale-invariant fashion) like an $(n-1)$-dimensional "surface," with respect to the Hausdorff measure $\mathcal{H}^{n-1}$. For example, the classical four-corner Cantor set in the plane is an Ahlfors regular set (cf., e.g., [108, Proposition 4.79, p. 238]). Let us also observe that

if $\Omega \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set whose boundary is upper Ahlfors regular up to some scale $R \in (0,\infty]$ with some constant $C \in [1,\infty)$ then necessarily Ω is of locally finite perimeter. (2.33)

Indeed, this follows from (2.15) (bearing in mind that $\partial_*\Omega \subseteq \partial\Omega$; cf. (2.14)) and Definition 2.1.

Lemma 2.1 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed set which is lower Ahlfors regular with some constant $C \in [1,\infty)$ up to some scale $R \in (0,\infty]$. Then any set $E \subseteq \Sigma$ satisfying $\mathcal{H}^{n-1}(\Sigma \setminus E) = 0$ is necessarily dense in Σ, i.e., $\overline{E} = \Sigma$.*

Proof Seeking a contradiction, assume E is not dense in Σ. Then $\Sigma \setminus \overline{E} \neq \varnothing$. This means that there exist $x \in \Sigma$ and $r > 0$ such that $B(x,r) \cap E = \varnothing$. Without loss of generality we may assume that $r \in (0,R)$. We may then use the lower Ahlfors regularity property of Σ and the fact that $B(x,r) \cap \Sigma \subseteq \Sigma \setminus E$ to write

[2] Recall that the support of a vector measure μ is defined as the support of its total variation, i.e., $\operatorname{supp}\mu := \operatorname{supp}|\mu|$.

$$r^{n-1}/C \leq \mathcal{H}^{n-1}\big(B(x,r)\cap\Sigma\big) \leq \mathcal{H}^{n-1}(\Sigma\setminus E) = 0, \tag{2.34}$$

a contradiction. □

In analogy with Definition 2.1 we introduce the notion of Ahlfors regularity for measures:

Definition 2.2 A (non-negative) Borel measure μ in $\mathbb{R}^n$ is said to be `Ahlfors regular up to scale` $R \in (0,\infty]$, `with constant` $C \in [1,\infty)$, provided

$$r^{n-1}/C \leq \mu\big(B(x,r)\big) \leq Cr^{n-1}, \quad \forall\, r \in (0,R), \quad \forall\, x \in \operatorname{supp}\mu. \tag{2.35}$$

Also, say that μ is `lower Ahlfors regular`, or `upper Ahlfors regular`, when only the lower, respectively, upper, inequality in (2.35) is required to hold.

One may check straight from definitions that if $\Sigma \subseteq \mathbb{R}^n$ is a closed set, then Σ is an Ahlfors regular set up to scale $R \in (0,\infty]$ with constant $C \in [1,\infty)$ if and only if $\mu := \mathcal{H}^{n-1}\lfloor\Sigma$ is an Ahlfors regular measure up to scale $R \in (0,\infty]$ with constant $C \in [1,\infty)$. Moreover, similar considerations apply to lower/upper Ahlfors regularity.

Next, we recall the notion of Radon measure:

Definition 2.3 Let (X,τ) be a topological space, and let $\mathfrak{M}$ be a sigma-algebra of subsets of X containing all Borel sets in X. Call a measure $\mu : \mathfrak{M} \to [0,+\infty]$ `Radon` provided μ is locally finite (i.e., $\mu(K) < +\infty$ for every compact $K \subseteq X$), every open set is inner-regular, i.e.,

$$\mu(O) = \sup_{\substack{K \text{ compact} \\ K\subseteq O}} \mu(K) \quad \text{for each open set } O \subseteq X, \tag{2.36}$$

and every Borel set is outer-regular, i.e.,

$$\mu(E) = \inf_{\substack{O \text{ open} \\ E\subseteq O}} \mu(O) \quad \text{for all Borel sets } E \subseteq X. \tag{2.37}$$

We have the following well-known regularity result (cf., e.g., [51, Theorem 7.8, p. 217]).

Proposition 2.3 *Let (X,τ) be a locally compact Hausdorff topological space in which every open set is sigma-compact (recall that the latter condition automatically holds if (X,τ) is second countable hence, in particular, if (X,τ) is metrizable and separable). Then every locally finite Borel measure μ on X is a Radon measure.*

Let μ be a locally finite Borel measure in $\mathbb{R}^n$. In particular, Proposition 2.3 guarantees that μ is a Radon measure. If μ is also assumed to be lower Ahlfors regular up to scale $R \in (0,\infty]$ with constant $C \in [1,\infty)$, we may invoke [95, Theorem 6.9(2), p. 95] to conclude that

$$\mathcal{H}^{n-1}(A) \leq 2^{n-1} C\mu(A) \quad \text{for each } \mu\text{-measurable set } A \subseteq \operatorname{supp}\mu. \tag{2.38}$$

In particular,

$$\begin{aligned} &\mathcal{H}^{n-1}(A) = 0 \quad \text{whenever} \quad A \subseteq \operatorname{supp}\mu \\ &\text{is a } \mu\text{-measurable set with } \quad \mu(A) = 0. \end{aligned} \tag{2.39}$$

Given a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, we are interested when the measure $\|\partial\Omega\|$ is Ahlfors regular.

Proposition 2.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a set of locally finite perimeter, and fix some scale $R \in (0, \infty]$ along with a constant $C \in [1, \infty)$. Then the measure $\|\partial\Omega\|$ is lower Ahlfors regular with constant C up to scale R if and only if*

$$\mathcal{H}^{n-1}\big(\overline{\partial^*\Omega} \setminus \partial^*\Omega\big) = 0 \tag{2.40}$$

and the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R.

Furthermore, the measure $\|\partial\Omega\|$ is actually Ahlfors regular with constant C up to scale R if and only if (2.40) holds and the set $\overline{\partial^*\Omega}$ is Ahlfors regular with constant C up to scale R.

Proof Since $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, it follows that $\mu := \|\partial\Omega\|$ is a locally finite Borel measure in $\mathbb{R}^n$ (cf. (2.15), (2.24), and (2.28)). In addition, $A := \overline{\partial^*\Omega} \setminus \partial^*\Omega$ is a μ-measurable set contained in $\operatorname{supp}\mu$ with $\mu(A) = 0$ (cf. (2.22), (2.23), (2.30), (2.28)). Let us also note that, as apparent from (2.28), we have

$$\begin{aligned} &\|\partial\Omega\|\big(B(x, r)\big) = \mathcal{H}^{n-1}\big(\partial^*\Omega \cap B(x, r)\big) \\ &\text{for each } \ x \in \mathbb{R}^n \ \text{ and each } \ r \in (0, \infty). \end{aligned} \tag{2.41}$$

In one direction, assume the measure $\|\partial\Omega\|$ is lower Ahlfors regular with constant C up to scale R. Then (2.39) (used with μ and A as above) implies (2.40). Also, from Definition 2.2, (2.30), (2.41), and (2.40) we see that the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R. In the opposite direction, if (2.40) holds and the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R, we conclude from (2.41), Definition 2.2, and (2.30) that the measure $\|\partial\Omega\|$ is lower Ahlfors regular with constant C up to scale R. This finishes the proof of the first equivalence claimed in the statement of the proposition.

As regards the equivalence in the last part of the statement, assume the measure $\|\partial\Omega\|$ is in fact Ahlfors regular with constant C up to scale R. Then, from what we have proved already, the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R and (2.40) holds. Since now the measure $\|\partial\Omega\|$ is additionally assumed to be upper Ahlfors regular with constant C up to scale R, we deduce from Definition 2.2, (2.30), (2.40), and (2.41) that the set $\overline{\partial^*\Omega}$ is also upper Ahlfors regular with constant

C up to scale R. This establishes one implication. Finally, the opposite implication is seen from Definition 2.2, (2.30), (2.41), and (2.40). □

For future use, let us record here the following off-diagonal Carleson measure estimate of reverse Hölder type, proved in [111, §8.6].

Proposition 2.5 *Let Ω be an open subset of $\mathbb{R}^n$ with an unbounded Ahlfors regular boundary and define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix $\kappa \in (0,\infty)$, and pick $\theta \in (0,1)$ along with $p \in (0,\infty)$, all arbitrary. Then there exists $C \in (0,\infty)$ with the property that for every $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{R}$, every point $x \in \partial\Omega$, and every radius $r \in (0,\infty)$ one has*

$$\Big(\int_{\Omega\cap B(x,r)} |u|^{\frac{np}{n-1}}\,d\mathcal{L}^n\Big)^{\frac{n-1}{np}} \leq C\Big(\int_{\partial\Omega\cap B(x,Cr)} \big(\mathcal{N}_\kappa^{Cr} u\big)^p\,d\sigma\Big)^{\frac{1}{p}}, \tag{2.42}$$

where $\mathcal{N}_\kappa^{Cr}$ is the truncated nontangential maximal operator (defined as in (2.9) with $\delta := Cr$).

Following [61] we now introduce the class of Ahlfors regular domains.

Definition 2.4 An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is called an `Ahlfors regular domain` provided $\partial\Omega$ is an Ahlfors regular set and $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big) = 0$.

If $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, then the upper Ahlfors regularity condition satisfied by $\partial\Omega$ (i.e., the second inequality in (2.32) with $\Sigma := \partial\Omega$) guarantees that (2.15) holds, hence Ω is a set of locally finite perimeter. Also, the fact that the measure theoretic boundary $\partial_*\Omega$ is presently assumed to have full measure (with respect to $\mathcal{H}^{n-1}$) in the topological boundary $\partial\Omega$ ensures that the geometric measure theoretic outward unit normal ν to Ω (cf. (2.19)) is actually well defined at $\mathcal{H}^{n-1}$-a.e. point on $\partial\Omega$. Ultimately,

$$\begin{aligned}&\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an Ahlfors regular domain then}\\ &\nu \in \big[L^\infty(\partial\Omega,\mathcal{H}^{n-1})\big]^n \text{ is an } \mathbb{R}^n\text{-valued function}\\ &\text{satisfying } |\nu(x)| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial\Omega.\end{aligned} \tag{2.43}$$

From [61, Proposition 2.9, p. 2588] we also know that

$$\begin{aligned}&\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an Ahlfors regular domain, and if } \kappa \in (0,\infty) \text{ is}\\ &\text{an arbitrary aperture parameter, then } x \in \overline{\Gamma_\kappa(x)} \text{ (that is, } x \text{ is an}\\ &\text{accumulation point for the nontangential approach region } \Gamma_\kappa(x))\\ &\text{for } \mathcal{H}^{n-1}\text{-a.e. point } x \text{ in the topological boundary } \partial\Omega.\end{aligned} \tag{2.44}$$

In particular, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and u is an $\mathcal{L}^n$-measurable function defined in Ω, then for any fixed aperture parameter $\kappa > 0$ it is meaningful to attempt to define the nontangential boundary trace $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial\Omega$.

It turns out that the class of Ahlfors regular domains is bi-Lipschitz invariant.

Lemma 2.2 *Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, and $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$. Then for any given bi-Lipschitz mapping $F : \mathcal{O} \to \mathbb{R}^n$ the set $\widetilde{\Omega} := F(\Omega)$ is also an Ahlfors regular domain, with the Ahlfors regularity constant of $\partial\widetilde{\Omega}$ controlled in terms of the Ahlfors regularity constant of $\partial\Omega$ and the bi-Lipschitz constants of F.*

Proof This is a consequence of [59, Proposition 3.1, p. 610] and the proof of [59, Proposition 3.7, (3.88), p. 621]. □

We shall also need the following result, appearing in [111, §5.10].

Lemma 2.3 *If $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain (in the sense of Definition 2.4) then $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ is also an Ahlfors regular domain, whose topological boundary coincides with that of Ω, and whose geometric measure theoretic boundary agrees with that of Ω, i.e.,*

$$\partial(\Omega_-) = \partial\Omega \quad and \quad \partial_*(\Omega_-) = \partial_*\Omega. \tag{2.45}$$

Moreover, the geometric measure theoretic outward unit normal to Ω_- is $-\nu$ at σ-a.e. point on $\partial\Omega$.

The following definition is due to G. David and S. Semmes (cf. [41]).

Definition 2.5 A closed set $\Sigma \subseteq \mathbb{R}^n$ is said to be a `uniformly rectifiable` set (or simply a UR set) if Σ is an Ahlfors regular set and there exist $\varepsilon, M \in (0, \infty)$ such that for each location $x \in \Sigma$ and each scale $R \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ it is possible to find a Lipschitz map $\varphi : B_R^{n-1} \to \mathbb{R}^n$ (where B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$ and such that

$$\mathcal{H}^{n-1}\big(\Sigma \cap B(x, R) \cap \varphi(B_R^{n-1})\big) \geq \varepsilon R^{n-1}. \tag{2.46}$$

Collectively, ε, M are referred to as the UR `constants of` Σ.

The following definition appears in [61].

Definition 2.6 An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is called a `UR domain` (short for uniformly rectifiable domain) provided $\partial\Omega$ is a UR set (in the sense of Definition 2.5) and $\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0$.

By design, any UR domain is an Ahlfors regular domain. A basic subclass of UR domains has been identified by G. David and D. Jerison in [39]. To state (a version of) their result, we first recall the following definition.

Definition 2.7 Fix $R \in (0, \infty]$ and $c \in (0, 1)$. A nonempty proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, c)-`corkscrew condition` (or, simply, a `corkscrew condition` if the particular values of R, c are not important) if for each location

$x \in \partial\Omega$ and each scale $r \in (0, R)$ there exists a point $z \in \Omega$ (called a corkscrew point relative to x and r) with the property that $B(z, c\,r) \subseteq B(x, r) \cap \Omega$.

Also, a nonempty proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, c)-`two-sided corkscrew condition` provided both Ω and $\mathbb{R}^n \setminus \Omega$ satisfy the (R, c)-corkscrew condition (with the same convention regarding the omission of R, c).

It is then clear from definitions that we have

$$\partial_*\Omega = \partial\Omega \text{ for any } \mathcal{L}^n\text{-measurable set } \Omega \subseteq \mathbb{R}^n \text{ satisfying a two-sided corkscrew condition.} \tag{2.47}$$

Also, [39, Theorem 1, p. 840] implies that, in a quantitative fashion,

> if Ω is a nonempty proper open subset of $\mathbb{R}^n$ satisfying a two-sided corkscrew condition and whose boundary is an Ahlfors regular set, then Ω is a UR domain. (2.48)

Following [66], we define the class of nontangentially accessible domains as those open sets satisfying a two-sided corkscrew condition and the following Harnack chain condition.

Definition 2.8 Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open set $\Omega \subseteq \mathbb{R}^n$ is said to satisfy the (R, N)-`Harnack chain condition` (or, simply, a `Harnack chain condition` if the particular values of R, N are irrelevant) provided whenever $\varepsilon > 0$, $k \in \mathbb{N}$, $z \in \partial\Omega$, and $x, y \in \Omega$ with $\max\{|x - z|, |y - z|\} < R/4$ as well as $|x - y| \leq 2^k\varepsilon$ and $\min\{\text{dist}(x, \partial\Omega), \text{dist}(y, \partial\Omega)\} \geq \varepsilon$, one may find a chain of balls $B_1, B_2, \dots, B_K$ with $K \leq Nk$, such that $x \in B_1$, $y \in B_K$, $B_i \cap B_{i+1} \neq \varnothing$ for every $i \in \{1, \dots, K-1\}$, and

$$N^{-1} \cdot \text{diam}(B_i) \leq \text{dist}(B_i, \partial\Omega) \leq N \cdot \text{diam}(B_i), \tag{2.49}$$

$$\text{diam}(B_i) \geq N^{-1} \cdot \min\{\text{dist}(x, B_i)\,,\ \text{dist}(y, B_i)\}, \tag{2.50}$$

for every $i \in \{1, \dots, K\}$.

Note that, in the context of Definition 2.8, consecutive balls must have comparable radii. The "nontangentiality" condition (2.49) further implies that

$$\lambda B_i \subseteq \Omega \quad \text{for each} \quad \lambda \in (0, 2N^{-1} + 1] \quad \text{and} \quad i \in \{1, \dots, K\}. \tag{2.51}$$

The Harnack chain condition described in Definition 2.8 should be thought of as a quantitative local connectivity condition. In particular,

> any open set $\Omega \subseteq \mathbb{R}^n$ satisfying an (∞, N)-Harnack chain condition (for some $N \in \mathbb{N}$) is pathwise connected (hence also connected) in a quantitative fashion. (2.52)

To elaborate on the latter aspect, we find it convenient to eliminate the parameter $\varepsilon > 0$ and also relabel 2^k simply as k in Definition 2.8. Assuming $R = \infty$, this implies that for each $k \geq 2$ there exists $L_k \in \mathbb{N}$ (which is bounded by $N \cdot \log_2 k$) with the property that for each

$$x_1, x_2 \in \Omega \quad \text{with} \quad |x_1 - x_2| \leq k \cdot \min\big\{\text{dist}(x_1, \partial\Omega)\,,\ \text{dist}(x_2, \partial\Omega)\big\} \tag{2.53}$$

one can find a sequence of balls

$$\begin{aligned}&B_j := B(y_j, r_j) \text{ with } 1 \leq j \leq \ell, \text{ where } \ell \in \mathbb{N} \text{ satisfies } \ell \leq L_k,\\ &\text{such that } B\big(y_j, (2N^{-1} + 1)r_j\big) \subseteq \Omega \text{ for every } j \in \{1, \dots, \ell\},\\ &x_1 \in B(y_1, r_1),\, x_2 \in B(y_\ell, r_\ell), \text{ and such that there exists a point}\\ &z_j \in B(y_j, r_j) \cap B(y_{j+1}, r_{j+1}) \text{ for each } j \in \{1, \dots, \ell - 1\}.\end{aligned} \tag{2.54}$$

The fact that $L_k = O(\log_2 k)$ as $k \to \infty$ quantifies the intuitive idea that the closer to the boundary the points x_1, x_2 are, and the further apart from each other they happen to be, the larger the numbers of balls in the Harnack chain joining them. To proceed, we agree to abbreviate

$$\delta_{\partial\Omega}(x) := \text{dist}(x, \partial\Omega) \quad \text{for each} \quad x \in \Omega. \tag{2.55}$$

Then the first property in (2.54) implies that we have

$$\delta_{\partial\Omega}(x) \geq 2N^{-1} r_j \quad \text{for all} \quad j \in \{1, \dots, \ell\} \quad \text{and all} \quad x \in B(y_j, r_j). \tag{2.56}$$

In concert with the second inequality in (2.49) this further permits us to estimate

$$\delta_{\partial\Omega}(a) \leq (N + 1) \cdot \delta_{\partial\Omega}(b) \ \text{ for all } \ j \in \{1, \dots, \ell\} \ \text{ and } \ a, b \in B(y_j, r_j). \tag{2.57}$$

Indeed, whenever $a, b \in B(y_j, r_j)$ with $j \in \{1, \dots, \ell\}$ we may use (2.56) to write

$$\begin{aligned}\delta_{\partial\Omega}(a) &\leq |a - b| + \delta_{\partial\Omega}(b) \leq 2r_j + \delta_{\partial\Omega}(b)\\ &\leq N \cdot \delta_{\partial\Omega}(b) + \delta_{\partial\Omega}(b) = (N + 1) \cdot \delta_{\partial\Omega}(b),\end{aligned} \tag{2.58}$$

proving (2.57). In particular, for each index $j \in \{1, \dots, \ell - 1\}$ we have

$$(N + 1)^{-1} \cdot \delta_{\partial\Omega}(z_j) \leq \delta_{\partial\Omega}(z_{j+1}) \leq (N + 1) \cdot \delta_{\partial\Omega}(z_j). \tag{2.59}$$

Joining $x_1, y_1, z_1, y_2, z_2, y_3, \dots, y_{\ell-1}, z_{\ell-1}, y_\ell, x_2$ with line segments yields a polygonal arc γ joining x_1 with x_2 in Ω, whose length may be estimated as follows:

$$\text{length}(\gamma) \leq \sum_{j=1}^{\ell} 2r_j \leq N \cdot \delta_{\partial\Omega}(x_1) + N \sum_{j=1}^{\ell-1} \delta_{\partial\Omega}(z_j)$$

$$\leq N \cdot \delta_{\partial\Omega}(x_1) + N \sum_{j=1}^{\ell-1} (N+1)^{j-1} \cdot \delta_{\partial\Omega}(z_1)$$

$$\leq N \cdot \delta_{\partial\Omega}(x_1) + N \sum_{j=1}^{\ell-1} (N+1)^{j} \cdot \delta_{\partial\Omega}(x_1)$$

$$= N \sum_{j=0}^{\ell-1} (N+1)^{j} \cdot \delta_{\partial\Omega}(x_1) = N \frac{(N+1)^{\ell} - 1}{N} \delta_{\partial\Omega}(x_1)$$

$$\leq (N+1)^{L_k} \cdot \delta_{\partial\Omega}(x_1), \tag{2.60}$$

thanks to (2.56) (used with x replaced by $x_1, z_1, \dots, z_{\ell-1}$), iterations of (2.59), and (2.57) (with $j := 1$, $a := z_1$, $b := x_1$), while also keeping in mind that $\ell \leq L_k$. In a similar fashion, emphasizing x_2 in place of x_1 yields $\mathrm{length}(\gamma) \leq (N+1)^{L_k} \cdot \delta_{\partial\Omega}(x_2)$ hence, ultimately,

$$\mathrm{length}(\gamma) \leq (N+1)^{L_k} \cdot \min\big\{\delta_{\partial\Omega}(x_1)\,,\ \delta_{\partial\Omega}(x_2)\big\}. \tag{2.61}$$

In addition, for each $x \in \gamma$ there exists $j_x \in \{1, \dots, \ell\}$ such that $x \in B(y_{j_x}, r_{j_x})$. If $j_x \geq 2$ we write

$$\delta_{\partial\Omega}(x) \geq (N+1)^{-1} \cdot \delta_{\partial\Omega}(z_{j_x-1}) \geq (N+1)^{1-j_x} \cdot \delta_{\partial\Omega}(z_1)$$

$$\geq (N+1)^{-j_x} \cdot \delta_{\partial\Omega}(x_1) \geq (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1), \tag{2.62}$$

by (2.57) with $b := x$ and $a := z_{j_x-1}$, iterations of (2.59), and (2.57) applied with $b := z_1$ and $a := x_1$. If $j_x = 1$ we simply have

$$\delta_{\partial\Omega}(x) \geq (N+1)^{-1} \cdot \delta_{\partial\Omega}(x_1) \geq (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1). \tag{2.63}$$

Thus, in all cases we reach the conclusion that $\delta_{\partial\Omega}(x) \geq (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1)$. Analogously, $\delta_{\partial\Omega}(x) \geq (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_2)$ which goes to show that

$$\delta_{\partial\Omega}(x) \geq (N+1)^{-L_k} \cdot \max\big\{\delta_{\partial\Omega}(x_1)\,,\ \delta_{\partial\Omega}(x_2)\big\} \ \text{ for each } \ x \in \gamma. \tag{2.64}$$

The existence of such a path γ is going to be used in Lemma 2.4 and Lemma 2.5 which, in turn, play a significant role in the proof of Theorem 2.7. For now, following [66, pp. 93-94] (cf. also [75, Definition 2.1, p. 3]), we introduce the class of NTA domains.

Definition 2.9 Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is said to be an (R, N)-`nontangentially accessible domain` (or simply an NTA domain if the particular values of R, N are not important) if Ω satisfies both the (R, N^{-1})-two-sided corkscrew condition and the (R, N)-Harnack chain condition.

Call Ω a (R, N)-two-sided nontangentially accessible domain (or, simply, a two-sided NTA domain if the particular values of R, N are not relevant) provided Ω is an open, nonempty, proper subset of $\mathbb{R}^n$ with the property that both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are (R, N)-nontangentially accessible domains.

A set $\Omega \subseteq \mathbb{R}^n$ is said to be an (R, N)-one-sided NTA domain provided Ω satisfies the (R, N)-Harnack chain condition and the (R, N^{-1})-corkscrew condition (once again, with the convention that the parameters R, N are dropped if theirs values are not relevant).

Finally, it is agreed that, in all cases, one takes $R = \infty$ if and only if $\partial\Omega$ is unbounded.

For example, the complement of the classical four-corner Cantor set in the plane is a one-sided NTA domain with an Ahlfors regular boundary. Also, from the last convention in Definition 2.9 and (2.52) we see that

any NTA domain with an unbounded boundary (or, equivalently, any (∞, N)-nontangentially accessible domain for some number $N \in \mathbb{N}$) is pathwise connected, hence also connected. (2.65)

It turns out that from any point in a given one-sided NTA domain one may proceed along a path toward to the interior of said domain, which progressively distances itself from the boundary. This is made precise in the lemma below.

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^n$ be an (∞, N)-one-sided* NTA *domain for some $N \in \mathbb{N}$. Then there exists a constant $C_N \in (1, \infty)$ with the following significance. For each location $x \in \Omega$ and each scale $r \in (0, \infty)$ there exists a point $x_* \in \Omega$ and a polygonal arc γ joining x with x_* in Ω such that*

$$\begin{aligned} &|x - x_*| < 2r, \quad \delta_{\partial\Omega}(x_*) \geq r/N^2, \quad \mathit{length}(\gamma) \leq C_N \cdot r, \\ &\text{and } \ \mathit{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \text{ for each point } \ y \in \gamma, \end{aligned} \tag{2.66}$$

where $\gamma_{x,y}$ is the sub-arc of γ joining x with y.

Proof Without loss of generality assume $N \geq 2$. In the case when $\delta_{\partial\Omega}(x) \geq r/N$, we shall simply take $x_* := x$ and $\gamma := \{x\}$. If $\delta_{\partial\Omega}(x) < r/N$, there exists $m \in \mathbb{N}$ such that $r/N^{m+1} \leq \delta_{\partial\Omega}(x) < r/N^m$. Pick some $z \in \partial\Omega$ so that $\delta_{\partial\Omega}(x) = |x - z|$ and define $r_j := N^j \cdot \delta_{\partial\Omega}(x) \in (0, \infty)$ for each $j \in \{1, \dots, m\}$. The fact that Ω satisfies (∞, N^{-1})-corkscrew condition guarantees that for each $j \in \{1, \dots, m\}$ there exists a corkscrew point $x_j \in \Omega$ relative to the location z and scale r_j. Hence, for each $j \in \{1, \dots, m\}$ we have $B(x_j, r_j/N) \subseteq B(z, r_j) \cap \Omega$ which entails

$$
\begin{aligned}
&N^j \cdot \delta_{\partial\Omega}(x) = r_j > \delta_{\partial\Omega}(x_j) > r_j/N = N^{j-1} \cdot \delta_{\partial\Omega}(x) \\
&\text{and} \ \ |x_j - z| < r_j = N^j \cdot \delta_{\partial\Omega}(x) \ \ \text{for each} \ \ j \in \{1, \dots, m\}.
\end{aligned}
\tag{2.67}
$$

Denote $x_0 := x$ and observe that for each $j \in \{1, \dots, m\}$ we have that the points $x_{j-1}, x_j \in B(z, r_j)$. Together with (2.67), for each $j \in \{1, \dots, m\}$ this permits us to estimate

$$
|x_{j-1} - x_j| < 2r_j = 2N^j \cdot \delta_{\partial\Omega}(x) \leq 2N^2 \cdot \min\big\{\delta_{\partial\Omega}(x_{j-1})\,,\ \delta_{\partial\Omega}(x_j)\big\}. \tag{2.68}
$$

Hence, we are in the scenario described in (2.53) with x_{j-1}, x_j playing the roles of x_1, x_2, and with $k := 2N^2$. From (2.61), (2.64) we then conclude that there exists $\widetilde{C}_N \in (1, \infty)$ with the property that for each $j \in \{1, \dots, m\}$ we may find a polygonal arc γ_j joining x_{j-1} with x_j in Ω such that

$$
\text{length}(\gamma_j) \leq \widetilde{C}_N \cdot \min\big\{\delta_{\partial\Omega}(x_{j-1})\,,\ \delta_{\partial\Omega}(x_j)\big\} \leq \widetilde{C}_N \cdot N^j \cdot \delta_{\partial\Omega}(x), \tag{2.69}
$$

and

$$
\widetilde{C}_N \cdot \delta_{\partial\Omega}(y) \geq \max\big\{\delta_{\partial\Omega}(x_{j-1})\,,\ \delta_{\partial\Omega}(x_j)\big\} \geq N^{j-1} \cdot \delta_{\partial\Omega}(x) \ \ \text{for each} \ \ y \in \gamma_j. \tag{2.70}
$$

If we now define $x_* := x_m$ and take $\gamma := \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_m$ then γ is a polygonal arc joining $x = x_0$ with $x_* = x_m$ in Ω whose length satisfies

$$
\begin{aligned}
\text{length}(\gamma) &= \sum_{j=1}^{m} \text{length}(\gamma_j) \leq \sum_{j=1}^{m} \widetilde{C}_N \cdot N^j \cdot \delta_{\partial\Omega}(x) \\
&\leq \frac{N \cdot \widetilde{C}_N}{N-1} N^m \cdot \delta_{\partial\Omega}(x) \leq \Big(\frac{N \cdot \widetilde{C}_N}{N-1}\Big) r,
\end{aligned}
\tag{2.71}
$$

thanks to (2.69) and our choice of m. Also, for each point $y \in \gamma$ there exists some $j_y \in \{1, \dots, m\}$ such that $y \in \gamma_{j_y}$, hence we may use (2.70) to bound the length of the sub-arc $\gamma_{x,y}$ of γ joining x with y by

$$
\begin{aligned}
\text{length}(\gamma_{x,y}) &\leq \sum_{j=1}^{j_y} \text{length}(\gamma_j) \leq \sum_{j=1}^{j_y} \widetilde{C}_N \cdot N^j \cdot \delta_{\partial\Omega}(x) \\
&\leq \frac{N^2 \cdot \widetilde{C}_N}{N-1} N^{j_y - 1} \cdot \delta_{\partial\Omega}(x) \leq \Big(\frac{N^2 \cdot \widetilde{C}_N^2}{N-1}\Big) \delta_{\partial\Omega}(y).
\end{aligned}
\tag{2.72}
$$

Our choice of x_*, the first line in (2.67), and our choice of m also permit us to conclude that

$$\delta_{\partial\Omega}(x_*) = \delta_{\partial\Omega}(x_m) > N^{m-1} \cdot \delta_{\partial\Omega}(x) \geq r/N^2. \tag{2.73}$$

Finally, since $x, x_* \in B(z, r_m)$ it follows that $|x - x_*| < 2r_m = 2N^m \cdot \delta_{\partial\Omega}(x) < 2r$, so all properties claimed in (2.66) are verified. □

Our next lemma shows that one-sided NTA domains satisfy a quantitative connectivity property of the sort considered by O. Martio and J. Sarvas in [93], where the class of uniform domains has been introduced. See also [10, Theorem 2.15] in this regard.

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^n$ be an (∞, N)-one-sided* NTA *domain for some $N \in \mathbb{N}$. Then there exists a constant $C_N \in (1, \infty)$ with the following significance. For any two points $x, \widetilde{x} \in \Omega$ and any scale $r \in (0, \infty)$ with $r \geq |x - \widetilde{x}|$ there exists a polygonal arc Γ joining x with $\widetilde{x}$ in Ω such that*

$$\begin{aligned} &\mathit{length}(\Gamma) \leq C_N \cdot r, \quad \textit{and for each point} \quad y \in \Gamma \\ &\min\big\{\mathit{length}(\Gamma_{x,y})\,,\ \mathit{length}(\Gamma_{y,\widetilde{x}})\big\} \leq C_N \cdot \delta_{\partial\Omega}(y), \end{aligned} \tag{2.74}$$

where $\Gamma_{x,y}$ and $\Gamma_{y,\widetilde{x}}$ are the sub-arcs of Γ joining x with y and, respectively, y with $\widetilde{x}$.

Proof Fix two points $x, \widetilde{x} \in \Omega$ and pick a scale $r \in (0, \infty)$ with $r \geq |x - \widetilde{x}|$. If $\delta_{\partial\Omega}(x) > 2r$ then $\widetilde{x} \in \overline{B(x, r)} \subseteq B(x, 2r) \subseteq \Omega$. In such a scenario, take Γ to be the line segment with endpoints $x, \widetilde{x}$ and all desired properties follow. There remains to treat the case when

$$\delta_{\partial\Omega}(x) \leq 2r. \tag{2.75}$$

To proceed, let $x_*, \widetilde{x}_*$ be associated with the given points $x, \widetilde{x}$ as in Lemma 2.4, and denote by $\gamma, \widetilde{\gamma}$ the polygonal arcs joining x with x_* and $\widetilde{x}$ with $\widetilde{x}_*$ in Ω, having the properties described in (2.66), for the current scale r. Specifically, for this choice of the scale, (2.66) gives

$$\begin{aligned} |x - x_*| < 2r, &\quad |\widetilde{x} - \widetilde{x}_*| < 2r, \\ \delta_{\partial\Omega}(x_*) \geq r/N^2, &\quad \delta_{\partial\Omega}(\widetilde{x}_*) \geq r/N^2, \\ \text{length}(\gamma) \leq C_N \cdot r, &\quad \text{length}(\widetilde{\gamma}) \leq C_N \cdot r, \\ \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) &\ \text{ for each } \ y \in \gamma, \\ \text{length}(\widetilde{\gamma}_{\widetilde{x},y}) \leq C_N \cdot \delta_{\partial\Omega}(y) &\ \text{ for each } \ y \in \widetilde{\gamma}. \end{aligned} \tag{2.76}$$

Note that

$$|x_* - \widetilde{x}_*| \leq |x_* - x| + |x - \widetilde{x}| + |\widetilde{x} - \widetilde{x}_*| < 2r + r + 2r = 5r. \tag{2.77}$$

From (2.77) and the second line in (2.76) we then see that

$$|x_* - \widetilde{x}_*| < 5r \leq 5N^2 \cdot \min\big\{\delta_{\partial\Omega}(x_*)\,,\ \delta_{\partial\Omega}(\widetilde{x}_*)\big\}. \tag{2.78}$$

Thus, we are in the scenario described in (2.53) with $x_1 := x_*$, $x_2 := \widetilde{x}_*$, and with $k := 5N^2$. From (2.61), (2.64) we then conclude that there exist a constant $C_N \in (1, \infty)$ along with a polygonal arc $\widehat{\gamma}$ joining x_* with $\widetilde{x}_*$ in Ω such that

$$\text{length}(\widehat{\gamma}) \leq C_N \cdot \min\big\{\delta_{\partial\Omega}(x_*)\,,\ \delta_{\partial\Omega}(\widetilde{x}_*)\big\} \leq 2C_N \cdot r, \tag{2.79}$$

where the last inequality comes from (2.75), and

$$C_N \cdot \delta_{\partial\Omega}(y) \geq \max\big\{\delta_{\partial\Omega}(x_*)\,,\ \delta_{\partial\Omega}(\widetilde{x}_*)\big\} \geq r/N^2 \ \text{ for each } \ y \in \widehat{\gamma}, \tag{2.80}$$

with the last inequality provided by the second line in (2.76).

If we now define

$$\Gamma := \gamma \cup \widehat{\gamma} \cup \widetilde{\gamma}, \tag{2.81}$$

then Γ is a polygonal arc joining x with $\widetilde{x}$ in Ω. Also, (2.76) and (2.79) allow us to estimate

$$\text{length}(\Gamma) = \text{length}(\gamma) + \text{length}(\widehat{\gamma}) + \text{length}(\widetilde{\gamma}) \leq C_N \cdot r, \tag{2.82}$$

proving the first estimate in (2.74). Fix now a point $y \in \Gamma$. If y belongs to γ, then $\Gamma_{x,y} = \gamma_{x,y}$ which further entails $\text{length}(\Gamma_{x,y}) = \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y)$ by (2.76). Thus, the last estimate in (2.74) holds in this case. Similarly, if $y \in \widetilde{\gamma}$, then $\text{length}(\Gamma_{y,\widetilde{x}}) = \text{length}(\widetilde{\gamma}_{\widetilde{x},y}) \leq C_N \cdot \delta_{\partial\Omega}(y)$ again by (2.76), so the last estimate in (2.74) holds in this case as well. Finally, in the case when $y \in \widehat{\gamma}$ we may write

$$\min\big\{\text{length}(\Gamma_{x,y})\,,\ \text{length}(\Gamma_{y,\widetilde{x}})\big\} \leq \text{length}(\Gamma) \leq C_N \cdot r \leq C_N \cdot \delta_{\partial\Omega}(y), \tag{2.83}$$

by (2.82) and (2.80). □

When its endpoints belong to a suitable neighborhood of infinity, the polygonal arc constructed in Lemma 2.5 may be chosen as to avoid any given bounded set. This property, established in the next lemma, is going to be relevant later on, in the course of the proof of Theorem 2.7.

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^n$ be an (∞, N)-one-sided* NTA *domain for some $N \in \mathbb{N}$ such that $\mathbb{R}^n \setminus \overline{\Omega} \neq \varnothing$. Fix some point $z_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and some radius $R \in (0, \infty)$. Then there exist a large constant $C = C(N) \in (0, \infty)$ together with a small number $\varepsilon = \varepsilon(N) \in (0, 1)$ with the property that for any two points $x, \widetilde{x} \in \Omega \setminus B(z_0, R)$ and any scale $r \in (0, \infty)$ with $r \geq \max\big\{|x - \widetilde{x}|, C \cdot R\big\}$ the polygonal arc Γ joining x with $\widetilde{x}$ in Ω as in Lemma 2.5 is disjoint from $B(z_0, \varepsilon R)$.*

Proof Consider $\varepsilon \in (0, 1)$ and $C \in (0, \infty)$ to be specified momentarily. Recall formula (2.81). Assume there exists a point $y \in \gamma \cap B(z_0, \varepsilon R)$. Then $y \in \gamma \subseteq \Omega$ so the line segment with endpoints y and z_0 intersects $\partial\Omega$. As such, $\delta_{\partial\Omega}(y) \leq \varepsilon R$. Also, $\gamma_{x,y}$ joins the point $x \in \mathbb{R}^n \setminus B(z_0, R)$ with the point $y \in B(z_0, \varepsilon R)$, which forces $\text{length}(\gamma_{x,y}) \geq (1 - \varepsilon)R$. In concert with the last line in (2.66) this permits us to write

$$(1 - \varepsilon)R \leq \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \leq C_N \cdot \varepsilon R, \tag{2.84}$$

which leads to a contradiction if we choose $\varepsilon := 1/[2(C_N + 1)]$. Thus, for this choice of ε we have $\gamma \cap B(z_0, \varepsilon R) = \varnothing$. In a similar fashion, $\widetilde{\gamma} \cap B(z_0, \varepsilon R) = \varnothing$. Finally, if there exists a point $y \in \widehat{\gamma} \cap B(z_0, \varepsilon R)$ then based on (2.80) and the nature of the scale r we may estimate

$$\varepsilon R \geq \delta_{\partial\Omega}(y) \geq r/(N^2 \cdot C_N) \geq (C \cdot R)/(N^2 \cdot C_N) \tag{2.85}$$

which leads to a contradiction if $C = C(N) \in (0, \infty)$ is sufficiently large. □

The following definition of yet another brand of local path connectivity condition first appeared in [61].

Definition 2.10 An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is said to satisfy a `local John condition` if there exist $\theta \in (0, 1)$ and $R \in (0, \infty]$ (with the requirement that $R = \infty$ if $\partial\Omega$ is unbounded) such that for every point $x \in \partial\Omega$ and every scale $r \in (0, R)$ one may find $x_r \in B(x, r) \cap \Omega$ such that $B(x_r, \theta r) \subseteq \Omega$ and with the property that for each $y \in B(x, r) \cap \partial\Omega$ there exists a rectifiable path $\gamma_y : [0, 1] \to \overline{\Omega}$ whose length is $\leq \theta^{-1} r$ and such that

$$\gamma_y(0) = y, \quad \gamma_y(1) = x_r, \quad \text{dist}\big(\gamma_y(t), \partial\Omega\big) > \theta|\gamma_y(t) - y| \text{ for all } t \in (0, 1]. \tag{2.86}$$

Finally, a nonempty open set $\Omega \subseteq \mathbb{R}^n$ which is not dense in $\mathbb{R}^n$ is said to satisfy a `two-sided local John condition` if both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a local John condition.

It is clear from the definitions that, in a quantitative sense,

> any set satisfying a local John condition (respectively, a two-sided local John condition) also satisfies a corkscrew condition (respectively, a two-sided corkscrew condition). (2.87)

Moreover, given any $R \in (0, \infty]$ and $N \in \mathbb{N}$, from [61, Lemma 3.13, p. 2634] we know that

> any (R, N)-nontangentially accessible domain satisfies a local John condition, and any (R, N)-two-sided nontangentially accessible domain satisfies a two-sided local John condition. (2.88)

To be able to define the class of δ-flat Ahlfors regular domains we first need to formally introduce the John-Nirenberg space of functions of bounded mean oscillations on Ahlfors regular sets. Specifically, given a closed set $\Sigma \subseteq \mathbb{R}^n$, for each $x \in \Sigma$ and $r > 0$define the surface ball $\Delta := \Delta(x, r) := B(x, r) \cap \Sigma$. For any constant $\lambda > 0$ we also agree to define $\lambda\Delta := \Delta(x, \lambda r) := B(x, \lambda r) \cap \Sigma$. Make the assumption that Σ is Ahlfors regular and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. For each $f \in L^1_{\rm loc}(\Sigma, \sigma)$ introduce

$$f_\Delta := \fint_\Delta f \, d\sigma \quad \text{for each surface ball} \quad \Delta \subseteq \Sigma, \tag{2.89}$$

then consider the semi-norm

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} := \sup_{\Delta\subseteq\Sigma} \fint_\Delta \big|f - f_\Delta\big| \, d\sigma, \tag{2.90}$$

where the supremum in the right side of (2.90) is taken over all surface balls $\Delta \subseteq \Sigma$. We shall then denote by $\mathrm{BMO}(\Sigma, \sigma)$ the space of all functions $f \in L^1_{\rm loc}(\Sigma, \sigma)$ with the property that $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} < \infty$.

The above considerations may be naturally adapted to the case of vector-valued functions. Specifically, given $N \in \mathbb{N}$, for each $f : \Sigma \to \mathbb{C}^N$ with locally integrable scalar components, we define

$$\|f\|_{[\mathrm{BMO}(\Sigma,\sigma)]^N} := \sup_{\Delta\subseteq\Sigma} \fint_\Delta \big|f - f_\Delta\big| \, d\sigma, \tag{2.91}$$

where the supremum in the right side of (2.91) is taken over all surface balls $\Delta \subseteq \Sigma$, the integral average $f_\Delta \in \mathbb{C}^N$ is taken componentwise, and $|\cdot|$ is the standard Euclidean norm in $\mathbb{C}^N$. In an analogous fashion, we then define $\big[\mathrm{BMO}(\Sigma, \sigma)\big]^N$ as the space of all $\mathbb{C}^N$-valued functions $f \in \big[L^1_{\rm loc}(\Sigma, \sigma)\big]^N$ with the property that $\|f\|_{[\mathrm{BMO}(\Sigma,\sigma)]^N} < \infty$.

A natural version of the classical John-Nirenberg inequality concerning exponential integrability of functions of bounded mean oscillations remains valid in this setting. Specifically, [88, Theorem 1.4, p. 2000] (see also [5], [30], [135, Theorem 2, p. 33]) implies that there exists a small constant $c \in (0, \infty)$ and a large constant $C \in (0, \infty)$, both of which depend only on the doubling character of σ, with the property that

$$\fint_\Delta \exp\left\{ \frac{c\,|f - f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}} \right\} d\sigma \leq C \tag{2.92}$$

for each non-constant function $f \in \mathrm{BMO}(\Sigma, \sigma)$ and each surface ball $\Delta \subseteq \Sigma$. Note that, trivially, for each surface ball $\Delta \subseteq \Sigma$ and each $\lambda \in (0, \infty)$ we have

$$
\begin{gathered}
1 \leq \exp\Big\{-\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\cdot\exp\Big\{\tfrac{c\,|f(x)-f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\} \\
\text{for every } \ x\in\Delta \ \text{ with } \ |f(x)-f_\Delta|>\lambda.
\end{gathered}
\tag{2.93}
$$

This shows that (2.92) implies the following level set estimate with exponential decay:

$$
\begin{aligned}
\sigma\Big(\big\{x\in\Delta : |f(x)-f_\Delta|>\lambda\big\}\Big)
&\leq \exp\Big\{-\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\int_\Delta \exp\Big\{\frac{c\,|f-f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\,d\sigma \\
&\leq C\cdot\exp\Big\{-\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\sigma(\Delta)
\end{aligned}
\tag{2.94}
$$

for each non-constant function $f\in\mathrm{BMO}(\Sigma,\sigma)$, each surface ball $\Delta\subseteq\Sigma$, and each $\lambda\in(0,\infty)$. Conversely, (2.94) implies an estimate like (2.92), namely

$$
\fint_\Delta \exp\Big\{\frac{c_o\,|f-f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\,d\sigma \leq 1+\frac{C}{c/c_o-1},
\tag{2.95}
$$

for each non-constant function $f\in\mathrm{BMO}(\Sigma,\sigma)$ and each surface ball $\Delta\subseteq\Sigma$, as long as $c_o\in(0,c)$. See also [18, Theorem 3.15], [44, Theorem 3.1, p. 1397], [77, Lemma 2.4, p. 409], [94], and [135, Theorem 2, p. 33] in this regard. Here we wish to emphasize that only the doubling property of the underlying measure plays a role. In turn, the John-Nirenberg level set estimate (2.94) has many notable consequences. For one thing, (2.92) implies that $e^f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ if f is a σ-measurable function on Σ with $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}$ small enough (with $\ln|\cdot|$ a representative example of this local exponential integrability phenomenon). Second, (2.94) guarantees that

$$
\mathrm{BMO}(\Sigma,\sigma)\subseteq L^p_{\mathrm{loc}}(\Sigma,\sigma) \ \text{ for each } \ p\in(0,\infty).
\tag{2.96}
$$

Third, (2.94) allows for more flexibility in describing the size of the BMO semi-norm. Specifically, for each $p\in[1,\infty)$ and $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ define

$$
\|f\|_{\mathrm{BMO}_p(\Sigma,\sigma)} := \sup_{\Delta\subseteq\Sigma}\Big(\fint_\Delta |f-f_\Delta|^p\,d\sigma\Big)^{1/p},
\tag{2.97}
$$

where the supremum in (2.97) is taken over all surface balls $\Delta\subseteq\Sigma$. Then for each integrability exponent $p\in[1,\infty)$ there exists some constant $C_{\Sigma,p}\in(0,\infty)$ with the property that for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ we have

$$
\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq \|f\|_{\mathrm{BMO}_p(\Sigma,\sigma)} \leq C_{\Sigma,p}\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}.
\tag{2.98}
$$

Indeed, the first estimate in (2.98) is a direct consequence of definitions and Hölder's inequality, while the second estimate in (2.98) relies on the John-Nirenberg inequality (2.94). Parenthetically, we wish to note that when $\Sigma := \mathbb{R}$ (hence $\sigma = \mathcal{L}^1$) and $p := 2$ the value of the optimal constant in (2.98) is known. Concretely, for each $f \in L^1_{\rm loc}(\mathbb{R}, \mathcal{L}^1)$ we have

$$\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \leq \|f\|_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \leq \tfrac{1}{2}\mathrm{e}^{1+(2/\mathrm{e})}\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}. \tag{2.99}$$

The justification of the second estimate in (2.99) uses a sharp version of the one-dimensional version of the John-Nirenberg inequality (cf. [86]) according to which for each function $f \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$, each nonempty finite sub-interval $I \subset \mathbb{R}$, and each $\lambda \in (0, \infty)$ we have (with $f_I := \fint_I f\,\mathrm{d}\mathcal{L}^1$)

$$\mathcal{L}^1\Big(\big\{t \in I : |f(t) - f_I| > \lambda\big\}\Big) \leq \frac{1}{2}\mathrm{e}^{4/\mathrm{e}}\mathcal{L}^1(I)\cdot\exp\Big\{-\frac{2\lambda/\mathrm{e}}{\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}\Big\}. \tag{2.100}$$

Specifically, for each nonempty finite sub-interval $I \subset \mathbb{R}$ we may write

$$\begin{aligned}\fint_I |f(t) - f_I|^2\,\mathrm{d}t &= \frac{1}{\mathcal{L}^1(I)}\int_0^\infty 2\lambda\cdot\mathcal{L}^1\Big(\big\{t \in I : |f(t) - f_I| > \lambda\big\}\Big)\,\mathrm{d}\lambda\\ &\leq \mathrm{e}^{4/\mathrm{e}}\int_0^\infty \lambda\cdot\exp\Big\{-\tfrac{2\lambda/\mathrm{e}}{\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}\Big\}\,\mathrm{d}\lambda\\ &= \mathrm{e}^{4/\mathrm{e}}(\mathrm{e}/2)^2\,\|f\|^2_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}\int_0^\infty \lambda\cdot\,\mathrm{e}^{-\lambda}\,\mathrm{d}\lambda\\ &= \mathrm{e}^{4/\mathrm{e}}(\mathrm{e}/2)^2\,\|f\|^2_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)},\end{aligned} \tag{2.101}$$

thanks to (2.100) and some natural changes of variables, so the second estimate in (2.99) readily follows from (2.101) and (2.97).

Returning to the mainstream discussion, observe that (2.98) implies that for each integrability exponent $p \in [1, \infty)$ we have

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \approx \sup_{\Delta\subseteq\Sigma}\Big(\fint_\Delta |f - f_\Delta|^p\,\mathrm{d}\sigma\Big)^{\frac{1}{p}} \approx \sup_{\Delta\subseteq\Sigma}\inf_{c\in\mathbb{R}}\Big(\fint_\Delta |f - c|^p\,\mathrm{d}\sigma\Big)^{\frac{1}{p}}, \tag{2.102}$$

uniformly for $f \in L^1_{\rm loc}(\Sigma, \sigma)$. For further use, let us also note here that if Δ and Δ' are two concentric surface balls in Σ then for any $f \in L^1_{\rm loc}(\Sigma, \sigma)$ and any $q \in [1, \infty)$ we have

$$\Big(\fint_{\Delta}|f-f_{\Delta'}|^q\,d\sigma\Big)^{\frac{1}{q}}\leq C_{q,n}\Big[1+\Big(\frac{\sigma(\Delta\cup\Delta')}{\sigma(\Delta\cap\Delta')}\Big)^{\frac{1}{q}}\Big]\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{2.103}$$

In particular, (2.103) readily implies that there exists some constant $C\in(0,\infty)$ which depends only on n and the Ahlfors regular constant of Σ with the property that for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ and each surface ball $\Delta\subseteq\Sigma$ we have

$$\big|f_{2\Delta}-f_{\Delta}\big|\leq C\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{2.104}$$

In turn, (2.104) may be used to estimate

$$\big|f_{2^j\Delta}-f_{\Delta}\big|\leq\sum_{k=1}^{j}\big|f_{2^k\Delta}-f_{2^{k-1}\Delta}\big|\leq Cj\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}\,, \tag{2.105}$$

for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$, each surface ball $\Delta\subseteq\Sigma$, and each integer $j\in\mathbb{N}$. For future use, let us also note here that there exists some $C\in(0,\infty)$ which depends only on n and the Ahlfors regular constant of Σ with the property that for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$, each pair of points $x,y\in\Sigma$, and each radius $R>|x-y|$ we have

$$\big|f_{\Delta(x,R)}-f_{\Delta(y,R)}\big|\leq C\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{2.106}$$

More generally, suppose $\Sigma\subseteq\mathbb{R}^n$ is a closed set and assume μ is a doubling Borel measure on Σ. This means that there exists $C\in(0,\infty)$ with the property that for each surface ball $\Delta\subseteq\Sigma$ we have

$$0<\mu(2\Delta)\leq C\mu(\Delta)<+\infty. \tag{2.107}$$

In this setting, we shall denote by $\mathrm{BMO}(\Sigma,\mu)$ the space consisting of all functions $f\in L^1_{\mathrm{loc}}(\Sigma,\mu)$ with the property that

$$\|f\|_{\mathrm{BMO}(\Sigma,\mu)}:=\sup_{\Delta\subseteq\Sigma}\fint_{\Delta}\Big|f-\fint_{\Delta}f\,d\mu\Big|\,d\mu<+\infty, \tag{2.108}$$

where the supremum is once again taken over all surface balls $\Delta\subseteq\Sigma$. Much as before, since the John-Nirenberg inequality holds for generic Borel doubling measures (as noted in the discussion pertaining to (2.92)–(2.94)), for each integrability exponent $p\in[1,\infty)$ we then have

$$\|f\|_{\mathrm{BMO}(\Sigma,\mu)}\approx\|f\|_{\mathrm{BMO}_p(\Sigma,\mu)}$$

$$\approx\sup_{\Delta\subseteq\Sigma}\Big(\fint_{\Delta}\fint_{\Delta}|f(x)-f(y)|^p\,d\mu(x)d\mu(y)\Big)^{\frac{1}{p}}$$

$$\approx \sup_{\Delta\subseteq\Sigma}\inf_{c\in\mathbb{R}}\left(\fint_{\Delta}\left|f-c\right|^{p}\,\mathrm{d}\mu\right)^{\frac{1}{p}}, \tag{2.109}$$

uniformly for $f \in L^1_{\mathrm{loc}}(\Sigma,\mu)$, where

$$\|f\|_{\mathrm{BMO}_p(\Sigma,\mu)} := \sup_{\Delta\subseteq\Sigma}\left(\fint_{\Delta}\left|f-\fint_{\Delta} f\,\mathrm{d}\mu\right|^{p}\mathrm{d}\mu\right)^{1/p}, \tag{2.110}$$

with the supremum above taken over all surface balls $\Delta \subseteq \Sigma$. As before, for any given integer $N \in \mathbb{N}$, we shall denote by $\big[\mathrm{BMO}(\Sigma,\mu)\big]^N$ the space of $\mathbb{C}^N$-valued functions $f \in \big[L^1_{\mathrm{loc}}(\Sigma,\mu)\big]^N$ with the property that $\|f\|_{[\mathrm{BMO}(\Sigma,\mu)]^N} < \infty$, where the semi-norm $\|\cdot\|_{[\mathrm{BMO}(\Sigma,\mu)]^N}$ is defined much as in (2.91). Finally, given a function $f \in \big[L^1_{\mathrm{loc}}(\Sigma,\mu)\big]^N$ we agree to define $\|f\|_{[\mathrm{BMO}_p(\Sigma,\mu)]^N}$ as in (2.110), now interpreting $|\cdot|$ as the standard Euclidean norm in $\mathbb{C}^N$.

Let us also briefly discuss the space VMO which, heuristically, should be thought of as an integral version[3] of uniform continuity. Specifically, let Σ be a closed Ahlfors regular subset of $\mathbb{R}^n$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. In this setting, define the Sarason space $\mathrm{VMO}(\Sigma,\sigma)$ of functions of vanishing mean oscillations (cf. [121]) as

$$\mathrm{VMO}(\Sigma,\sigma) := \text{the closure of } \mathrm{UC}(\Sigma)\cap\mathrm{BMO}(\Sigma,\sigma) \text{ in } \mathrm{BMO}(\Sigma,\sigma), \tag{2.111}$$

where $\mathrm{UC}(\Sigma)$ stands for the space of uniformly continuous functions on Σ. Then for each given function $f \in \mathrm{BMO}(\Sigma,\sigma)$ one has the equivalence

$$f \in \mathrm{VMO}(\Sigma,\sigma) \Longleftrightarrow \lim_{R\to 0^+}\sup_{\substack{x\in\Sigma \text{ and}\\ r\in(0,R)}}\left(\fint_{\Delta(x,r)}\left|f-\fint_{\Delta(x,r)} f\,\mathrm{d}\sigma\right|^{p}\mathrm{d}\sigma\right)^{\frac{1}{p}} = 0 \tag{2.112}$$

for some (or all) $p \in [1,\infty)$. See [112, §3.1] for a proof.

Moving on, in the lemma below we collect a number of useful formulas and estimates for unimodular functions (i.e., vector-valued functions of modulus one).

Lemma 2.7 *Let (X,μ) be a measure space with the property that $0 < \mu(X) < \infty$. Also, fix an integer $N \in \mathbb{N}$ and suppose $f \in \big[L^1(X,\mu)\big]^N$. Then*

$$\fint_X\left|f-\fint_X f\,d\mu\right|^2 d\mu = \fint_X |f|^2\,d\mu - \left|\fint_X f\,d\mu\right|^2. \tag{2.113}$$

In particular,

[3] As opposed to a pointwise version.

$$\text{if } |f(x)| = 1 \text{ for } \mu\text{-a.e. } x \in X \text{ then}$$

$$\fint_X \Big| f - \fint_X f\, d\mu \Big|^2 d\mu = 1 - \Big| \fint_X f\, d\mu \Big|^2 \quad \text{and}$$

$$\Big(1 - \Big|\fint_X f\, d\mu\Big|\Big)^2 \leq \fint_X \Big| f - \fint_X f\, d\mu \Big|^2 d\mu \leq 2\Big(1 - \Big|\fint_X f\, d\mu\Big|\Big), \tag{2.114}$$

$$0 \leq 1 - \Big|\fint_X f\, d\mu\Big| \leq \fint_X \Big| f - \fint_X f\, d\mu \Big| d\mu \leq \sqrt{2}\sqrt{1 - \Big|\fint_X f\, d\mu\Big|}.$$

Proof Keeping in mind that $|Z - W|^2 = |Z|^2 - 2\text{Re}(Z \cdot \overline{W}) + |W|^2$ for each $Z, W \in \mathbb{C}^N$, we may compute

$$\begin{aligned}
\fint_X \Big| f - \fint_X f\, d\mu \Big|^2 d\mu &= \fint_X \Big(|f|^2 - 2\text{Re}\Big[f \cdot \Big(\fint_X \overline{f}\, d\mu \Big)\Big] + \Big| \fint_X f\, d\mu \Big|^2 \Big) d\mu \\
&= \fint_X |f|^2\, d\mu - 2\text{Re} \fint_X f \cdot \Big(\fint_X \overline{f}\, d\mu \Big) d\mu + \Big| \fint_X f\, d\mu \Big|^2 \\
&= \fint_X |f|^2\, d\mu - \Big| \fint_X f\, d\mu \Big|^2,
\end{aligned} \tag{2.115}$$

proving (2.113). Then (2.114) follows from this by observing that

$$1 - \Big|\fint_X f\, d\mu\Big|^2 = \Big(1 + \Big|\fint_X f\, d\mu\Big|\Big)\Big(1 - \Big|\fint_X f\, d\mu\Big|\Big) \leq 2\Big(1 - \Big|\fint_X f\, d\mu\Big|\Big) \tag{2.116}$$

and

$$\begin{aligned}
0 \leq 1 - \Big|\fint_X f\, d\mu\Big| &= \fint_X |f|\, d\mu - \Big|\fint_X f\, d\mu\Big| \\
&\leq \fint_X \Big| f - \fint_X f\, d\mu \Big| d\mu \leq \Big(\fint_X \Big| f - \fint_X f\, d\mu \Big|^2 d\mu \Big)^{1/2},
\end{aligned} \tag{2.117}$$

by the fact that $|f| = 1$, the reverse triangle inequality, and the Cauchy–Schwarz inequality. □

Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, Lemma 2.7 applies to the geometric measure theoretic outward unit normal ν to Ω, in the setting in which $X := \Delta$, an arbitrary surface ball on $\partial\Omega$, and the measure is $\mu := \mathcal{H}^{n-1}\lfloor\Delta$. As indicated below, this yields a better bound for the BMO semi-norm of ν than directly estimating $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} \leq 2\,\|\nu\|_{[L^\infty(\partial\Omega,\sigma)]^n} = 2$.

Lemma 2.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then*

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \leq \|\nu\|_{[BMO_2(\partial\Omega,\sigma)]^n} \leq 1, \tag{2.118}$$

and

$$1 - \inf_{\Delta\subseteq\partial\Omega}\left|\fint_{\Delta} \nu\, d\sigma\right| \leq \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \leq \sqrt{2}\sqrt{1 - \inf_{\Delta\subseteq\partial\Omega}\left|\fint_{\Delta} \nu\, d\sigma\right|}, \tag{2.119}$$

where the two infima are taken over all surface balls $\Delta \subseteq \partial\Omega$. In particular,

$$1 \geq \left|\fint_{\Delta} \nu\, d\sigma\right| \geq 1 - \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \text{ for each surface ball } \Delta \subseteq \partial\Omega. \tag{2.120}$$

Also,

$$\text{if } \partial\Omega \text{ is bounded then } \quad \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} = \|\nu\|_{[BMO_2(\partial\Omega,\sigma)]^n} = 1. \tag{2.121}$$

As a consequence,

$$\partial\Omega \text{ is unbounded whenever } \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < 1. \tag{2.122}$$

In relation to (2.121) we wish to note that, in the class of Ahlfors regular domains, having the BMO semi-norm of its geometric measure theoretic outward unit normal precisely 1 is not an exclusive attribute of bounded domains. For example, a straightforward computation shows that an infinite strip in $\mathbb{R}^n$ (i.e., the region in between two parallel hyperplanes in $\mathbb{R}^n$) is an unbounded Ahlfors regular domain with the property that the BMO semi-norm of its outward unit normal is equal to 1.

Proof of Lemma 2.8 Hölder's inequality and Lemma 2.7 imply that for each surface ball $\Delta \subseteq \partial\Omega$ we have

$$\left(\fint_{\Delta} |\nu - \nu_{\Delta}|\, d\sigma\right)^2 \leq \fint_{\Delta} |\nu - \nu_{\Delta}|^2\, d\sigma = 1 - \left|\fint_{\Delta} \nu\, d\sigma\right|^2 \leq 1, \tag{2.123}$$

from which (2.118) follows on account of (2.91), (2.97), and (2.98). Next, (2.119) follows from (2.114), used with $X := \Delta$, arbitrary surface ball on $\partial\Omega$, and with $\mu := \mathcal{H}^{n-1}\lfloor\Delta$.

To justify the claim made in (2.121), assume first that the set Ω is bounded. In such a case, fix some point $x_0 \in \partial\Omega$ along with some real number $r_0 > \mathrm{diam}(\partial\Omega)$ and note that the latter choice entails $\Delta_0 := B(x_0, r_0) \cap \partial\Omega = \partial\Omega$. Also, since $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ (cf. Definition 2.4) the Divergence Formula (2.20) gives

$$\nu_{\Delta_0} = \Big(\frac{1}{\sigma(\partial\Omega)}\int_{\partial\Omega}\nu\cdot\mathbf{e}_j\,d\sigma\Big)_{1\le j\le n} = \Big(\frac{1}{\sigma(\partial\Omega)}\int_{\Omega}\operatorname{div}\mathbf{e}_j\,d\mathcal{L}^n\Big)_{1\le j\le n} = 0. \tag{2.124}$$

In concert with (2.19) this implies (bearing in mind that $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big)=0$)

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \sup_{\Delta\subseteq\partial\Omega} ⨍_{\Delta}\big|\nu-\nu_\Delta\big|\,d\sigma \ge ⨍_{\partial\Omega}\big|\nu-\nu_{\Delta_0}\big|\,d\sigma = 1. \tag{2.125}$$

In light of (2.118), we then conclude that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} = 1$ in this case. When Ω is an unbounded Ahlfors regular domain with compact boundary in $\mathbb{R}^n$, having $n\ge 2$ implies that $\mathbb{R}^n\setminus\overline{\Omega}$ is a bounded Ahlfors regular domain whose topological boundary coincides with that of Ω, whose geometric measure theoretic boundary agrees with that of Ω, and whose geometric measure theoretic outward unit normal is $-\nu$ at σ-a.e. point on $\partial\Omega$ (cf. [111, §5.10] for a proof). Granted these properties, we may run the same argument as in (2.124)–(2.125) with $\mathbb{R}^n\setminus\overline{\Omega}$ in place of Ω and conclude that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 1$ in this case as well. This finishes the proof of (2.121). □

To close this section, recall for further use that $\mathrm{CMO}(\mathbb{R}^n,\mathcal{L}^n)$ is the closure of $\mathscr{C}_0^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)$. As may be seen with the help of [22, Théorème 7, p. 198], the space $\mathrm{CMO}(\mathbb{R}^n,\mathcal{L}^n)$ may be alternatively described as the linear subspace of $\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)$ consisting of functions f satisfying the following three conditions:

$$\lim_{r\to 0^+}\Big[\sup_{x\in\mathbb{R}^n}\Big(⨍_{B(x,r)}\Big|f-⨍_{B(x,r)}f\,d\mathcal{L}^n\Big|\,d\mathcal{L}^n\Big)\Big]=0, \tag{2.126}$$

$$\lim_{r\to \infty}\Big[\sup_{x\in\mathbb{R}^n}\Big(⨍_{B(x,r)}\Big|f-⨍_{B(x,r)}f\,d\mathcal{L}^n\Big|\,d\mathcal{L}^n\Big)\Big]=0, \tag{2.127}$$

and

$$\lim_{|x|\to \infty}\Big[\sup_{r\in[R_0,R_1]}\Big(⨍_{B(x,r)}\Big|f-⨍_{B(x,r)}f\,d\mathcal{L}^n\Big|\,d\mathcal{L}^n\Big)\Big]=0 \quad \text{for each } R_0,R_1\in(0,\infty) \text{ with } R_0<R_1. \tag{2.128}$$

This is going to be relevant later on, in Proposition 2.11.

2.2 Reifenberg Flat Domains

In this section we explore the notion of flatness (in the Reifenberg sense). To facilitate the subsequent discussion, the reader is reminded that the Hausdorff distance between two arbitrary nonempty sets $A, B \subset \mathbb{R}^n$ is defined as

$$\operatorname{Dist}[A, B] := \max\Big\{\sup\{\operatorname{dist}(a, B) : a \in A\}, \sup\{\operatorname{dist}(b, A) : b \in B\}\Big\}. \tag{2.129}$$

We start by recalling the following definitions from [72].

Definition 2.11 Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$ and let $\Sigma \subset \mathbb{R}^n$ be a closed set. Then Σ is said to be a (R, δ)-`Reifenberg flat set` if for each $x \in \Sigma$ and each $r \in (0, R)$ there exists an $(n-1)$-dimensional plane $\pi(x, r)$ in $\mathbb{R}^n$ which contains x and satisfies

$$\operatorname{Dist}\big[\Sigma \cap B(x, r)\,,\ \pi(x, r) \cap B(x, r)\big] \le \delta\, r. \tag{2.130}$$

For example, given $\delta > 0$, the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$ with a sufficiently small Lipschitz constant is a (∞, δ)-Reifenberg flat set.

Definition 2.12 Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$. A nonempty, proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, δ)-`separation property` if for each $x \in \partial\Omega$ and $r \in (0, R)$ there exists an $(n-1)$-dimensional plane $\widetilde{\pi}(x, r)$ in $\mathbb{R}^n$ passing through x and a choice of unit normal vector $\vec{n}_{x,r}$ to $\widetilde{\pi}(x, r)$ such that

$$\begin{aligned} &\big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \widetilde{\pi}(x, r),\ t > 2\delta r\big\} \subset \Omega \ \text{ and} \\ &\big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \widetilde{\pi}(x, r),\ t < -2\delta r\big\} \subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{2.131}$$

Definition 2.13 Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$. A nonempty, proper subset Ω of $\mathbb{R}^n$ is called an (R, δ)-`Reifenberg flat domain` (or simply a `Reifenberg flat domain` if the particular values of R, δ are not important) provided Ω satisfies the (R, δ)-separation property and $\partial\Omega$ is an (R, δ)-Reifenberg flat set.

Recall the two-sided corkscrew condition from Definition 2.7.

Proposition 2.6 *Let Ω be a nonempty proper subset of $\mathbb{R}^n$ with the property that it satisfies the (R, c)-two-sided corkscrew condition for some $R \in (0, \infty]$ and some $c \in (0, 1)$. In addition, suppose $\partial\Omega$ is an (R, δ)-Reifenberg flat set for some number $\delta \in \big(0, \frac{\sqrt{3}}{4}c\big)$. Then Ω is an (R, δ)-Reifenberg flat domain.*

Proof Pick a location $x \in \partial\Omega$ along with a scale $r \in (0, R)$. Definition 2.11 ensures the existence of an $(n-1)$-dimensional plane $\pi(x, r)$ in $\mathbb{R}^n$ passing through x which satisfies (2.130). Make a choice of a unit normal vector $\vec{n}_{x,r}$ to $\pi(x, r)$ and abbreviate

$$\begin{aligned} C^{+}(x, r) &:= \big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \pi(x, r),\ t > 2\delta r\big\}, \\ C^{-}(x, r) &:= \big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \pi(x, r),\ t < -2\delta r\big\}. \end{aligned} \tag{2.132}$$

We claim that matters may be arranged (by taking $\delta \in \left(0, \frac{\sqrt{3}}{4}c\right)$ and by making a judicious choice of the orientation of $\vec{n}_{x,r}$) so that

$$C^{+}(x,r) \subset \Omega \quad \text{and} \quad C^{-}(x,r) \subset \mathbb{R}^n \setminus \Omega. \tag{2.133}$$

To prove this claim, first observe that (2.130) guarantees that the connected sets $C^{\pm}(x,r)$ do not intersect $\partial\Omega$. As such, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ form a disjoint, open cover of $C^{\pm}(x,r)$, hence

$$\begin{aligned} &C^{+}(x,r) \text{ is entirely contained in either } \Omega_+ \text{ or } \Omega_-, \text{ and} \\ &C^{-}(x,r) \text{ is entirely contained in either } \Omega_+ \text{ or } \Omega_-. \end{aligned} \tag{2.134}$$

To proceed, denote by $x_r^{\pm} \in \Omega_{\pm}$ the two corkscrew points corresponding to the location x and scale r. In particular,

$$|x_r^{\pm} - x| < r \quad \text{and} \quad B\big(x_r^{\pm}, cr\big) \subseteq \Omega_{\pm}, \tag{2.135}$$

where the constant $c \in (0,1)$ is as in Definition 2.7. Hence, if we consider the balls $B\big(x_r^{+}, c\,r\big)$, $B\big(x_r^{-}, c\,r\big)$, their centers $x_r^{\pm}$ belong to $B(x,r)$. The fact that we are presently assuming $0 < \delta < \frac{\sqrt{3}}{4}c$ with $c \in (0,1)$ ensures that $\delta < (c/2)\sqrt{1 - c^2/4}$ which, as some elementary geometry shows, forces each of the balls $B\big(x_r^{+}, c\,r\big)$, $B\big(x_r^{-}, c\,r\big)$ to intersect one of the sets $C^{+}(x,r)$, $C^{-}(x,r)$. As such, one of the following four alternatives is true:

$$B\big(x_r^{+}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing, \tag{2.136}$$

$$B\big(x_r^{+}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing, \tag{2.137}$$

$$B\big(x_r^{+}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing, \tag{2.138}$$

$$B\big(x_r^{+}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing. \tag{2.139}$$

Observe that the alternative described in (2.136) cannot hold. Otherwise, the existence of points $z_1 \in B\big(x_r^{+}, c\,r\big) \cap C^{+}(x,r)$ and $z_2 \in B\big(x_r^{-}, c\,r\big) \cap C^{+}(x,r)$ would imply that, on the one hand, the line segment $[z_1, z_2]$ lies in the convex set $C^{+}(x,r)$, hence also either in Ω_+ or in Ω_- by (2.134). This being said, the fact that $z_1 \in B\big(x_r^{+}, c\,r\big) \subseteq \Omega_+$ and $z_2 \in B\big(x_r^{-}, c\,r\big) \subseteq \Omega_-$ prevents either one of these eventualities form materializing. This contradiction therefore excludes (2.136). Reasoning in a similar fashion we may rule out (2.137). When (2.138) holds, from (2.134) and the fact that $B\big(x_r^{\pm}, c\,r\big) \subseteq \Omega_{\pm}$ (cf. (2.135)) we conclude that the inclusions in (2.133) hold as stated. Finally, when (2.139) holds, from (2.426) and (2.135) we deduce that $C^{+}(x,r) \subseteq \Omega_-$ and $C^{-}(x,r) \subseteq \Omega_+$. In such a scenario, we may ensure that the inclusions in (2.133) are valid simply by re-denoting $\vec{n}_{x,r}$ as $-\vec{n}_{x,r}$ which amounts to reversing the roles of $C^{+}(x,r)$ and

$C^-(x,r)$. This concludes the proof of (2.133). In turn, from (2.133) and (2.132) we conclude that (2.131) holds with $\widetilde{\pi}(x,r) := \pi(x,r)$. Definition 2.12 then implies that Ω is, indeed, an (R,δ)-Reifenberg flat domain. □

It turns out that sufficiently flat Reifenberg domains are NTA domains. More specifically, from [72, Theorem 3.1, p. 524] and its proof we see that:

> there exists a purely dimensional constant $\delta_n \in (0,\infty)$ with the property that for each $\delta \in (0,\delta_n)$ and $R \in (0,\infty]$ one may find some number $N = N(\delta, R) \in \mathbb{N}$ such that any (R,δ)-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^n$ also happens to be an (R,N)-nontangentially accessible domain (in the sense of Definition 2.9). (2.140)

The result recorded in (2.140) has a number of useful consequences. For example, it allows us to conclude that any open set satisfying a two-sided corkscrew condition and whose topological boundary is a sufficiently flat Reifenberg set is actually an NTA domain.

Proposition 2.7 *Let Ω be a nonempty proper subset of $\mathbb{R}^n$ satisfying the (R,c)-two-sided corkscrew condition for some $R \in (0,\infty]$ and $c \in (0,1)$. In addition, suppose $\partial\Omega$ is a (R,δ)-Reifenberg flat set with $0 < \delta < \min\{c/2, \delta_n\}$, where $\delta_n \in (0,\infty)$ is the purely dimensional constant from (2.140). Then there exists $N = N(\delta, R) \in \mathbb{N}$ with the property that Ω is an (R,N)-nontangentially accessible domain.*

Proof The desired conclusion is a direct consequence of Proposition 2.6, (2.140), and Definition 2.9. □

Moving on, recall the Gauss-Green measure associated with sets of locally finite perimeter as in (2.16). As in [20], given $C \in [1,\infty)$ and $R \in (0,\infty]$ define

$$\mathscr{A}(C,R) := \Big\{\Omega \subseteq \mathbb{R}^n : \Omega \text{ has locally finite perimeter, } \operatorname{supp}\mu_\Omega = \partial\Omega, \text{ and } \|\partial\Omega\| \text{ is an Ahlfors regular measure with constant } C \text{ up to scale } R\Big\}. \tag{2.141}$$

Proposition 2.8 *Fix $C \in [1,\infty)$ along with $R \in (0,\infty]$, and consider an arbitrary set $\Omega \subseteq \mathbb{R}^n$. Then $\Omega \in \mathscr{A}(C,R)$ if and only if Ω is $\mathcal{L}^n$-measurable, $\partial\Omega$ is an Ahlfors regular set with constant $C \in [1,\infty)$ up to scale $R \in (0,\infty]$, and*

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0. \tag{2.142}$$

Proof The left-to-right implication is deduced from (2.141), (2.31), and Proposition 2.4, while the right-to-left implication follows from (2.141), (2.33), (2.24), (2.31), Proposition 2.4, and Lemma 2.1. □

In particular, the above result ensures that Ahlfors regular domains (and the complements of their closures) belong to the class (2.141). A formal statement to this effect is recorded below.

Proposition 2.9 *Suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain (in the sense of Definition 2.4), and denote by $C \in [1, \infty)$ the Ahlfors regularity constant of $\partial\Omega$. Also, define*

$$\Omega_+ := \Omega \;\; \textit{and} \;\; \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.143}$$

Then

$$\Omega_\pm \in \bigcap_{0<R\leq 2\,diam(\partial\Omega)} \mathscr{A}(C, R). \tag{2.144}$$

Proof This is a consequence of Definition 2.4, Proposition 2.8, and Lemma 2.3. □

To be able to continue, we shall need more notation. The cylinder $C(x_0, r, \omega)$ with center at $x_0 \in \mathbb{R}^n$, radius $r \in (0, \infty)$, and axial direction $\omega \in S^{n-1}$ is defined as

$$C(x_0, r, \omega) := \Big\{x \in \mathbb{R}^n : |\langle x - x_0, \omega\rangle| < r \text{ and } \big|x - x_0 - \langle x - x_0, \omega\rangle\omega\big| < r\Big\}. \tag{2.145}$$

As in [89, p. 290], given a set of locally finite perimeter $\Omega \subseteq \mathbb{R}^n$, the `cylindrical excess` of Ω at the point $x_0 \in \partial\Omega$, for the scale $r \in (0, \infty)$, and with respect to the direction $\omega \in S^{n-1}$ is defined as

$$\mathbf{e}(\Omega, x_0, r, \omega) := \frac{1}{r^{n-1}} \int_{C(x_0,r,\omega)\cap\partial^*\Omega} \frac{|\nu(x) - \omega|^2}{2}\, d\mathcal{H}^{n-1}(x), \tag{2.146}$$

where ν is the geometric measure theoretic outward unit normal to Ω. This notion is studied at length in [89, Chapter 22], where a number of basic properties of the excess (having to do with rescaling, change of direction, lower-semicontinuity) are established. Here, we shall need the following result.

Lemma 2.9 *Let $\Omega \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then for every point $x_0 \in \partial\Omega$, every radius $r \in (0, \infty)$, and every vector $\omega \in \mathbb{R}^n \setminus \{0\}$ there holds*

$$\mathbf{e}\Big(\Omega, x_0, r, \frac{\omega}{|\omega|}\Big) \leq \frac{2}{r^{n-1}} \int_{C(x_0,r,\omega/|\omega|)\cap\partial^*\Omega} |\nu(x) - \omega|^2\, d\mathcal{H}^{n-1}(x), \tag{2.147}$$

where ν is the geometric measure theoretic outward unit normal to Ω.

This lemma facilitates estimating the excess in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal. Specifically, suppose $\Omega \subset \mathbb{R}^n$ is actually an Ahlfors regular domain and write $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.

Having fixed a point $x_0 \in \partial\Omega$ along with a radius $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$, denote $\Delta(x_0,r) := B(x_0,r) \cap \partial\Omega$ and $\nu_{\Delta(x_0,r)} := \fint_{\Delta(x_0,r)} \nu\, d\sigma$. Then since $C(x_0, r, \omega/|\omega|) \subseteq B(x_0, \sqrt{2}r)$ for each $\omega \in \mathbb{R}^n \setminus \{0\}$, we conclude from (2.147) that whenever

$$\nu_{\Delta(x_0,r)} \neq 0 \tag{2.148}$$

we have (with the piece of notation introduced in (2.110))

$$\begin{aligned}
\mathbf{e}\Big(\Omega, x_0, r, \frac{\nu_{\Delta(x_0,r)}}{|\nu_{\Delta(x_0,r)}|}\Big) &\leq 2^{\frac{n+1}{2}} C_A \Big\{\Big(\fint_{\Delta(x_0,\sqrt{2}r)} |\nu - \nu_{\Delta(x_0,r)}|^2\, d\sigma\Big)^{1/2}\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\Big(\fint_{\Delta(x_0,\sqrt{2}r)} |\nu - \nu_{\Delta(x_0,\sqrt{2}r)}|^2\, d\sigma\Big)^{1/2} \\
&\qquad + \Big(\fint_{\Delta(x_0,\sqrt{2}r)} |\nu_{\Delta(x_0,r)} - \nu_{\Delta(x_0,\sqrt{2}r)}|^2\, d\sigma\Big)^{1/2}\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + |\nu_{\Delta(x_0,r)} - \nu_{\Delta(x_0,\sqrt{2}r)}|\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + \fint_{\Delta(x_0,r)} |\nu - \nu_{\Delta(x_0,\sqrt{2}r)}|\, d\sigma\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + \Big(\frac{\sigma\big(\Delta(x_0,\sqrt{2}r)\big)}{\sigma\big(\Delta(x_0,r)\big)}\Big)^{1/2} \times \\
&\qquad \times \Big(\fint_{\Delta(x_0,\sqrt{2}r)} |\nu - \nu_{\Delta(x_0,\sqrt{2}r)}|^2\, d\sigma\Big)^{1/2}\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + C_A \cdot (\sqrt{2})^{\frac{n-1}{2}} \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n}\Big\}^2 \\
&= 2^{\frac{n+1}{2}} C_A \big(1 + C_A \cdot 2^{\frac{n-1}{4}}\big)^2 \|\nu\|^2_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n},
\end{aligned} \tag{2.149}$$

where $C_A \in [1,\infty)$ is the Ahlfors regularity constant of $\partial\Omega$.

Here is the proof of Lemma 2.9:

Proof of Lemma 2.9 Abbreviate $\omega_0 := \omega/|\omega| \in S^{n-1}$ and observe that we have the equality $|\omega - \omega_0| = \big|1 - |\omega|\big|$. Hence,

$$|\omega - \omega_0|^2 \cdot \mathcal{H}^{n-1}\big(C(x_0, r, \omega_0) \cap \partial^*\Omega\big)$$

$$= \big|1 - |\omega|\big|^2 \cdot \mathcal{H}^{n-1}\big(C(x_0, r, \omega_0) \cap \partial^*\Omega\big)$$
$$= \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} \big||\nu(x)| - |\omega|\big|^2 \, d\mathcal{H}^{n-1}(x)$$
$$\leq \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} |\nu(x) - \omega|^2 \, d\mathcal{H}^{n-1}(x). \tag{2.150}$$

Also, $|\nu - \omega_0|^2/2 \leq |\nu - \omega|^2 + |\omega - \omega_0|^2$. Based on these observations we may then write

$$\mathbf{e}(\Omega, x_0, r, \omega_0) = \frac{1}{r^{n-1}} \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} \frac{|\nu(x) - \omega_0|^2}{2} \, d\mathcal{H}^{n-1}(x)$$
$$\leq \frac{1}{r^{n-1}} \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} |\nu(x) - \omega|^2 \, d\mathcal{H}^{n-1}(x)$$
$$+ \frac{\mathcal{H}^{n-1}\big(C(x_0, r, \omega_0) \cap \partial^*\Omega\big)}{r^{n-1}} |\omega - \omega_0|^2$$
$$\leq \frac{2}{r^{n-1}} \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} |\nu(x) - \omega|^2 \, d\mathcal{H}^{n-1}(x), \tag{2.151}$$

which is the desired estimate. □

The basic height bound, recorded in (2.153) below, has been proved in [89, Theorem 22.8, p. 294] for sets Ω in a class of perimeter minimizers (a notion discussed at length in [89, Chapter 21, pp. 278–289]). In [20] the authors have observed that this height bound continues to hold for sets Ω in $\mathscr{A}(C, R)$, the class recalled in (2.141). Specifically, the following result has been proved in [20] along the lines of the argument in [89, Section 22.2, pp. 294–302]:

Theorem 2.1 *Given any* $C_0 \in [1, \infty)$ *and* $n \in \mathbb{N}$ *with* $n \geq 2$, *there exist two constants,* $\varepsilon_1 \in (0, 1)$ *and* $C_1 \in [1, \infty)$, *depending only on n and* C_0 *such that if* $\Omega \in \mathscr{A}(C_0, R_0)$ *for some* $R_0 \in (0, \infty]$, *and* $x_0 \in \partial\Omega$, $r \in (0, R_0/2)$, $\omega \in S^{n-1}$ *are such that*

$$\mathbf{e}(\Omega, x_0, 2r, \omega) \leq \varepsilon_1, \tag{2.152}$$

then the following conditions hold (with the cylinder $C(x_0, r, \omega)$ *defined as in* (2.145)):

$C(x_0, r, \omega) \cap \partial\Omega$

$$\subseteq \Big\{x \in C(x_0, r, \omega) : |\langle x - x_0, \omega\rangle| \leq C_1 r \cdot \mathbf{e}(\Omega, x_0, 2r, \omega)^{\frac{1}{2(n-1)}}\Big\}, \tag{2.153}$$

$$\Big\{x \in C(x_0, r, \omega) \cap \Omega : \langle x - x_0, \omega\rangle > C_1 r \cdot \mathbf{e}(\Omega, x_0, 2r, \omega)^{\frac{1}{2(n-1)}}\Big\} = \varnothing, \tag{2.154}$$

$$\Big\{x \in C(x_0, r, \omega) \setminus \Omega : \langle x - x_0, \omega\rangle < -C_1 r \cdot \mathbf{e}(\Omega, x_0, 2r, \omega)^{\frac{1}{2(n-1)}}\Big\} = \varnothing. \tag{2.155}$$

Recall the class of (R, δ)-Reifenberg flat domains from Definition 2.13.

Corollary 2.1 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Then for each given $C_0 \in [1, \infty)$ there exist two constants, $\varepsilon_2 \in (0, 1)$ and $C_2 \in [1, \infty)$, depending only on n and C_0 with the following significance. Whenever $R_0 \in (0, \infty]$, $R \in (0, R_0/2)$, and $\Omega \in \mathscr{A}(C_0, R_0)$ are such that*

$$\delta := \sup_{x_0 \in \partial\Omega} \sup_{r \in (0, R)} \inf_{\omega \in S^{n-1}} \mathbf{e}(\Omega, x_0, 2r, \omega) < \varepsilon_2 \tag{2.156}$$

it follows that Ω is a $\big(R, C_2 \cdot \delta^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat domain.

Proof Let $\varepsilon_1 = \varepsilon_1(C_0, n) \in (0, 1)$ and $C_1 = C_1(C_0, n) \in (0, \infty)$ be as in Theorem 2.1. Take

$$\varepsilon_2 := \min\big\{\varepsilon_1, 2^{-1} C_1^{2(1-n)}\big\}. \tag{2.157}$$

Fix an arbitrary location $x_0 \in \partial\Omega$ along with an arbitrary scale $r \in (0, R)$. Since having $0 \leq \delta < \varepsilon_2$ (cf. (2.156)) ensures that $1 < (2\varepsilon_2)/(\varepsilon_2 + \delta) \leq 2$, it is possible to choose some $\omega_{x_0,r} \in S^{n-1}$ such that

$$\begin{aligned} \mathbf{e}\big(\Omega, x_0, 2r, \omega_{x_0,r}\big) &< \Big(\frac{2\varepsilon_2}{\varepsilon_2 + \delta}\Big) \cdot \inf_{\omega \in S^{n-1}} \mathbf{e}(\Omega, x_0, 2r, \omega) \\ &\leq 2 \cdot \inf_{\omega \in S^{n-1}} \mathbf{e}(\Omega, x_0, 2r, \omega) \leq 2\delta, \end{aligned} \tag{2.158}$$

the last inequality being a consequence of (2.156). Thanks to (2.156), the first inequality in (2.158) forces

$$\mathbf{e}\big(\Omega, x_0, 2r, \omega_{x_0,r}\big) < \Big(\frac{2\varepsilon_2}{\varepsilon_2 + \delta}\Big) \cdot \delta < \varepsilon_2 < \varepsilon_1. \tag{2.159}$$

Granted this, Theorem 2.1 guarantees that the properties (2.153)–(2.155) hold for the vector $\omega := \omega_{x_0,r} \in S^{n-1}$. In particular, from this version of (2.153) and the last inequality in (2.158) it follows that for each $x_0 \in \partial\Omega$ and $r \in (0, R)$ we have identified a vector $\omega_{x_0,r} \in S^{n-1}$ such that

$$\begin{gathered}\text{the set } C(x_0, r, \omega_{x_0,r}) \cap \partial\Omega \text{ is contained in} \\ \Big\{x \in C(x_0, r, \omega_{x_0,r}) : |\langle x - x_0, \omega_{x_0,r}\rangle| \leq C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}\Big\}.\end{gathered} \tag{2.160}$$

For each location $x_0 \in \partial\Omega$ and scale $r \in (0, R)$, the versions of (2.154)–(2.155) written for $\omega := \omega_{x_0,r} \in S^{n-1}$ also prove (once again keeping in mind the last inequality in (2.158)) that

$$\begin{gathered}\Omega^c := \mathbb{R}^n \setminus \Omega \text{ contains the set} \\ C^- := \Big\{x \in C(x_0, r, \omega_{x_0,r}) : \langle x - x_0, \omega_{x_0,r}\rangle > C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}\Big\}\end{gathered} \tag{2.161}$$

and

$$\begin{gathered}\Omega \text{ contains the set} \\ C^+ := \Big\{x \in C(x_0, r, \omega_{x_0,r}) : \langle x - x_0, \omega_{x_0,r}\rangle < -C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}\Big\}.\end{gathered} \tag{2.162}$$

Moreover, from (2.156) and (2.157) we see that

$$C_1 \cdot (2\delta)^{\frac{1}{2(n-1)}} < 1, \tag{2.163}$$

hence

$$C^{\pm} \neq \varnothing. \tag{2.164}$$

To proceed, introduce

$$\pi(x_0, r) := x_0 + \langle \omega_{x_0,r}\rangle^{\top} \tag{2.165}$$

which is an $(n-1)$-dimensional plane in $\mathbb{R}^n$ containing the point x_0. Given that $B(x_0, r) \subseteq C(x_0, r, \omega_{x_0,r})$, from (2.160) we see that

$$\sup_{x \in B(x_0,r)\cap\partial\Omega} \operatorname{dist}\big(x, \pi(x_0, r) \cap B(x_0, r)\big) \leq C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.166}$$

We also claim that

$$\sup_{x \in B(x_0,r)\cap\pi(x_0,r)} \operatorname{dist}\big(x, \partial\Omega \cap B(x_0, r)\big) \leq 2C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.167}$$

To justify (2.167), consider an arbitrary point $x \in B(x_0, r) \cap \pi(x_0, r)$. We distinguish two cases.

`Case 1:` *Assume first that*

$$|x - x_0| < r\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}. \tag{2.168}$$

Note that (2.163) ensures that (2.168) is a meaningful demand. In this case, denote by L the line passing through x in the direction of $\omega_{x_0,r}$. Thanks to (2.164), it is possible to pick points $x_\pm \in C^\pm \cap L$. Then since $x^+ \in C^+ \subseteq \Omega$ and $x^- \in C^- \subseteq \Omega^c$, it follows that the line segment $[x^-, x^+]$ intersects $\partial\Omega$. Thus, there exists $y \in [x^-, x^+] \cap \partial\Omega$. Given that $[x^-, x^+]$ is contained in $C(x_0, r, \omega_{x_0,r})$ and that $C(x_0, r, \omega_{x_0,r}) \cap \partial\Omega$ is contained in the set described in the second line of (2.160), we conclude that y belongs to said set, hence $|x - y| \leq C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}$. We may now use the Pythagorean theorem to compute

$$\begin{aligned} |y - x_0|^2 &= |x - x_0|^2 + |x - y|^2 \\ &< r^2\big(1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}\big) + C_1^2 r^2 \cdot (2\delta)^{\frac{1}{(n-1)}} = r^2, \end{aligned} \tag{2.169}$$

which places y in $B(x_0, r)$. Ultimately, $y \in B(x_0, r) \cap \partial\Omega \subseteq C(x_0, r, \omega_{x_0,r}) \cap \partial\Omega$. Keeping in mind that the vector $x - y$ is parallel to $\omega_{x_0,r}$ and that $x - x_0$ is orthogonal to $\omega_{x_0,r}$, we may then use (2.160) to compute

$$\begin{aligned} \operatorname{dist}\big(x, \partial\Omega \cap B(x_0, r)\big) &\leq |x - y| = |\langle x - y, \omega_{x_0,r}\rangle| \\ &= |\langle y - x_0, \omega_{x_0,r}\rangle| < C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \end{aligned} \tag{2.170}$$

`Case 2:` *Assume* $x \in B(x_0, r) \cap \pi(x_0, r)$ *is arbitrary*. In this scenario, define

$$\widetilde{x} := x_0 + (x - x_0)\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}. \tag{2.171}$$

Then $\widetilde{x} \in \pi(x_0, r)$ and

$$|\widetilde{x} - x_0| \leq |x - x_0|\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}} < r\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}. \tag{2.172}$$

This proves two things. First, we see that $\widetilde{x} \in B(x_0, r) \cap \pi(x_0, r)$. Granted this, from (2.172) and the analysis in Case 1 (cf. (2.170)) we conclude that

$$\operatorname{dist}\big(\widetilde{x}, \partial\Omega \cap B(x_0, r)\big) < C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.173}$$

Since we also have

$$|\widetilde{x} - x| = \Big|x - x_0 - (x - x_0)\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}\Big|$$

$$
\begin{aligned}
&= |x - x_0|\left|1 - \sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}\right| \\
&\leq r \frac{C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}{1 + \sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}} \\
&\leq C_1^2 r \cdot (2\delta)^{\frac{1}{(n-1)}}, \qquad (2.174)
\end{aligned}
$$

we may avail ourselves of (2.173) to conclude that

$$
\begin{aligned}
\operatorname{dist}\big(x, \partial\Omega \cap B(x_0, r)\big) &\leq \operatorname{dist}\big(\widetilde{x}, \partial\Omega \cap B(x_0, r)\big) + |\widetilde{x} - x| \\
&\leq C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}} + C_1^2 r \cdot (2\delta)^{\frac{1}{(n-1)}} \\
&\leq 2C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}, \qquad (2.175)
\end{aligned}
$$

where the last inequality comes from (2.163).

This finishes the proof of (2.167). In concert with (2.166) and (2.129) this establishes

$$
\operatorname{Dist}\big[\partial\Omega \cap B(x_0, r)\,,\ \pi(x_0, r) \cap B(x_0, r)\big] \leq 2C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \qquad (2.176)
$$

In view of Definition 2.11, we conclude that $\partial\Omega$ is a $\big(R, 2C_1 \cdot (2\delta)^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat set. Together with the separation property implied by (2.161)–(2.162) (cf. Definition 2.12) we conclude that Ω is a $\big(R, C_2 \cdot \delta^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat domain (see Definition 2.13) for some constant $C_2 \in [1, \infty)$ depending only on n and C_0.

□

We are now ready to state an important result, asserting that any Ahlfors regular domain whose geometric measure theoretic outward unit normal has a sufficiently small BMO semi-norm is necessarily a Reifenberg flat domain.

Theorem 2.2 *For each $n \in \mathbb{N}$ with $n \geq 2$ and each $C_0 \in [1, \infty)$ there exist some small threshold $\delta_* \in (0, 1)$ along with some large constant $C_* \in [1, \infty)$, both depending only on n and C_0, with the following significance.*

Suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain (in the sense of Definition 2.4), with the Ahlfors regularity constant of $\partial\Omega$ less than or equal to C_0, and such that the geometric measure theoretic outward unit normal to Ω satisfies (where, as usual, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$)

$$
\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \leq \delta \ \ \textit{for some} \ \ \delta \in (0, \delta_*). \qquad (2.177)
$$

Then $\partial\Omega$ *is unbounded and*

$$\begin{aligned}&\textit{both } \Omega_+ := \Omega \textit{ and } \Omega_- := \mathbb{R}^n \setminus \overline{\Omega} \textit{ are}\\&\big(\infty, C_* \cdot \delta^{\frac{1}{2(n-1)}}\big)\textit{-Reifenberg flat domains.}\end{aligned} \tag{2.178}$$

Proof The fact that the set $\partial\Omega$ is unbounded follows from (2.177) (bearing in mind that $\delta_* < 1$) and Lemma 2.8. From (2.120) we also see that

$$\nu_{\Delta(x_0,r)} := ⨍_{\Delta(x_0,r)} \nu \, d\sigma \neq 0 \quad \text{for each } x_0 \in \partial\Omega \text{ and } r > 0. \tag{2.179}$$

Fix an arbitrary location $x_0 \in \partial\Omega$ and an arbitrary scale $r \in (0, \infty)$. Keeping (2.179) in mind, we then deduce from (2.148)–(2.149), Lemma 2.3, and (2.98) that whenever (2.177) holds we necessarily have

$$\mathbf{e}\Big(\Omega_\pm, x_0, r, \frac{\pm\nu_{\Delta(x_0,r)}}{|\nu_{\Delta(x_0,r)}|}\Big) \leq C\|\nu\|^2_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\delta^2, \tag{2.180}$$

where $C \in (0, \infty)$ depends only on the dimension n and C_0. Since $0 < \delta < \delta_*$, we see that (2.180) implies

$$\sup_{x_0\in\partial\Omega} \sup_{r\in(0,\infty)} \inf_{\omega\in S^{n-1}} \mathbf{e}(\Omega_\pm, x_0, 2r, \omega) \leq C\delta_*^2. \tag{2.181}$$

With $\varepsilon_2 = \varepsilon_2(C_0, n) \in (0, 1)$ as in Corollary 2.1, choose $\delta_* \in (0, 1)$ such that

$$C\delta_*^2 < \varepsilon_2. \tag{2.182}$$

Given that from Proposition 2.9 we also know that

$$\Omega_\pm \in \mathscr{A}(C_0, \infty), \tag{2.183}$$

we may invoke Corollary 2.1 to conclude that there exists $C_* \in [1, \infty)$, depending only on n and C_0, such that $\Omega_\pm$ are $\big(\infty, C_* \cdot \delta^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat domains. □

Some useful consequences of Theorem 2.2 are brought to light in the result below.

Theorem 2.3 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain (in the sense described in Definition 2.4). Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.

Then there exists a threshold $\delta_* \in (0, 1)$ *and a number* $N \in \mathbb{N}$*, both depending only on the Ahlfors regularity constant of* $\partial\Omega$ *and the dimension* n*, with the property that if*

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < \delta_*, \tag{2.184}$$

then $\partial\Omega$ *is an unbounded set and* Ω *is an* (∞, N)*-two-sided nontangentially accessible domain* (*in the sense of Definition* 2.9). *In particular,*

Ω *satisfies a two-sided local John condition with constants which depend only on the dimension n and the Ahlfors regularity constant of* $\partial\Omega$, (2.185)

also

Ω *is a* UR *domain, with the* UR *constants of* $\partial\Omega$ *controlled solely in terms of the dimension n and the Ahlfors regularity constant of* $\partial\Omega$, (2.186)

and, finally,

Ω *is a uniform domain, in the sense that it satisfies the quantitative connectivity condition described in Lemma* 2.5, *with a constant controlled solely in terms of the dimension n and the Ahlfors regularity constant of* $\partial\Omega$. (2.187)

Proof This is a consequence of Lemma 2.8, Theorem 2.2, (2.140), (2.88), (2.87), (2.48), and Lemma 2.5. □

We are now in a position to show that for an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ the demand that the BMO semi-norm of its geometric measure theoretic outward unit normal is suitably small relative to the Ahlfors regularity constant of $\partial\Omega$ has a string of remarkable topological and metric consequences for the set Ω. To set the stage, from [83, Theorem 2 in 49.VI, 57.I.9(i), 57.III.1] (cf. also [78, Lemma 4(1) and Lemma 5, p. 1702]) we first note that

if $\mathcal{O} \subseteq \mathbb{R}^n$ is some arbitrary connected open set, then any connected component of $\mathbb{R}^n \setminus \overline{\mathcal{O}}$ has a connected boundary. (2.188)

Theorem 2.4 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω.

Then there exists a threshold $\delta_* \in (0, 1)$ *depending only on the ambient dimension n and the Ahlfors regularity constant of* $\partial\Omega$, *such that if* $\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < \delta_*$ *it follows that* Ω, $\overline{\Omega}$, $\partial\Omega$, $\mathbb{R}^n \setminus \overline{\Omega}$, *and* $\mathbb{R}^n \setminus \Omega$ *are all unbounded connected sets,* $\partial(\overline{\Omega}) = \partial\Omega$, $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial\Omega$, *and* $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$.

As is apparent from Example 2.11, the demand that the parameter $\delta > 0$ is sufficiently small cannot be dispense with in the context of Theorem 2.4. This being said, it has been shown in [112, §11.5] that

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an open set satisfying a two-sided local John condition and whose boundary happens to be an unbounded Ahlfors regular set, then actually } \partial\Omega \text{ is connected.} \tag{2.189}$$

Proof of Theorem 2.4 Bring in the threshold $\delta_* \in (0,1)$ from Theorem 2.3 and assume that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_*$. From Theorem 2.3, Definition 2.9, and Definition 2.8 we then conclude that both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are pathwise connected open sets (hence, connected open sets). Having established this, from (2.188) we then see that $\partial\big(\mathbb{R}^n \setminus \overline{\Omega}\big) = \partial(\overline{\Omega})$ is connected. The fact that Ω satisfies an exterior corkscrew condition further implies $\partial(\overline{\Omega}) = \partial\Omega$. Since $\delta_* < 1$, Lemma 2.8 ensures that $\partial\Omega$ is unbounded, and this forces both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ to be unbounded (given that they have $\partial\Omega$ as their topological boundary). Also, the fact that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected implies that its closure is connected. However, $\overline{\mathbb{R}^n \setminus \overline{\Omega}} = \mathbb{R}^n \setminus \mathring{\overline{\Omega}}$ and

$$\mathring{\overline{\Omega}} = \overline{\Omega} \setminus \partial(\overline{\Omega}) = \overline{\Omega} \setminus \partial\Omega = \mathring{\Omega} = \Omega, \tag{2.190}$$

so $\mathbb{R}^n \setminus \Omega = \overline{\mathbb{R}^n \setminus \overline{\Omega}}$ is connected. □

In the two-dimensional setting, it turns out that having an outward unit normal with small BMO semi-norm implies (under certain background assumptions) that the domain in question is actually simply connected. This makes the object of Corollary 2.2, which augments Theorem 2.4.

Corollary 2.2 *Let $\Omega \subseteq \mathbb{R}^2$ be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then there exists a threshold $\delta_* \in (0,1)$, depending only on the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[BMO(\partial\Omega,\sigma)]^2} < \delta_*$ it follows that Ω is an unbounded connected set which is simply connected, $\partial\Omega$ is an unbounded connected set, $\mathbb{R}^2 \setminus \overline{\Omega}$ is an unbounded connected set which is simply connected, and $\partial(\mathbb{R}^2 \setminus \overline{\Omega}) = \partial\Omega$.*

Proof All claims are consequences of Theorem 2.4 together with (2.193), (2.194), and (2.195) below. □

2.3 Chord-Arc Curves in the Plane

Shifting gears, in this section we shall work in the two-dimensional setting. We begin by recalling some known results of topological flavor. First, for bounded sets, we know from [12, Corollary 1, p. 352] that

$$\text{an open bounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if its complement } \mathbb{R}^2 \setminus \Omega \text{ is a connected set,} \tag{2.191}$$

and

$$\text{an open bounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if its topological boundary, } \partial\Omega \text{, is a connected set.} \tag{2.192}$$

For unbounded sets, [12, Corollary 2, p. 352] gives

$$\text{an open unbounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if every connected component of } \mathbb{R}^2 \setminus \Omega \text{ is unbounded,} \tag{2.193}$$

and

$$\text{an open unbounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if every connected component } \Sigma \text{ of } \partial\Omega \text{ is unbounded.} \tag{2.194}$$

(Parenthetically, it is worth noting that the boundary of an open set $\Omega \subseteq \mathbb{R}^2$ which is both connected and simply connected is not necessarily connected: for example take $\Omega := \mathbb{R}^2 \setminus E$ where $E := [0, \infty) \times \{0, 1\}$.) Finally, according to [12, Corollary 3, p. 352],

$$\text{if } E \subseteq \mathbb{R}^2 \text{ is a closed set such that each connected component of } E \text{ is unbounded, then } \mathbb{R}^2 \setminus E \text{ is a simply connected set,} \tag{2.195}$$

and according to [120, Theorem 13.11, p. 274]

$$\text{an open connected set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ is simply connected if and only if } \widehat{\mathbb{C}} \setminus \Omega \text{ is connected, where } \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \text{ is the extended complex plane (i.e., the one-point compactification of } \mathbb{C} \text{, aka Riemann's sphere).} \tag{2.196}$$

Next, recall that a (compact) `curve` in the Euclidean plane $\mathbb{R}^2$ (canonically identified with $\mathbb{C}$) is a set of the form $\Sigma = \gamma([a, b])$, where $a, b \in \mathbb{R}$ are two numbers satisfying $a < b$, and $\gamma : [a, b] \to \mathbb{R}^2$ is a continuous function, called a parametrization of Σ. We shall call the curve Σ `simple` if Σ has a parametrization $\gamma : [a, b] \to \mathbb{R}^2$ whose restriction to $[a, b)$ is injective (hence, Σ is simple if it is non self-intersecting). We shall say that the curve Σ is `closed` if it has a parametrization $\gamma : [a, b] \to \mathbb{R}^2$ satisfying $\gamma(a) = \gamma(b)$. Also, we shall call $\Sigma \subset \mathbb{C}$ a `Jordan curve`, if Σ is a simple closed curve. Thus, a curve is Jordan if and only if it is the homeomorphic image of the unit circle S^1. The classical Jordan curve theorem asserts that

$$\text{the complement of a Jordan curve } \Sigma \subset \mathbb{C} \text{ consists precisely of two connected components, one bounded } \Omega_+ \text{, and one unbounded } \Omega_- \text{, called the inner and outer domains of } \Sigma \text{, satisfying } \partial\Omega_\pm = \Sigma. \tag{2.197}$$

In light of (2.192), we also conclude that

$$\text{the inner domain } \Omega_+ \text{ of a Jordan curve } \Sigma \subset \mathbb{C} \text{ is simply connected.} \tag{2.198}$$

We are also going to be interested in Jordan curves `passing through infinity` in the plane. This class consists of sets of the form $\Sigma = \gamma(\mathbb{R})$, where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with the property that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. For this class of curves a version of the Jordan separation theorem is also valid, namely

if Σ is a Jordan curve passing through infinity, then its complement in $\mathbb{C}$ consists precisely of two open connected components, called $\Omega_\pm$, which satisfy $\partial\Omega_+ = \Sigma = \partial\Omega_-$. (2.199)

Once (2.199) has been established, we deduce from (2.194) that

in the context of (2.199), the sets $\Omega_\pm$ are simply connected. (2.200)

To justify (2.199), let Σ be a Jordan curve passing through infinity. From definitions, it follows that Σ is a closed subset of $\mathbb{C}$. Fix an arbitrary point $z_o \in \mathbb{C}\setminus\Sigma$ and consider the homeomorphisms

$$\begin{aligned} &\Phi : \mathbb{C} \setminus \{z_o\} \longrightarrow \mathbb{C} \setminus \{0\}, \quad \Phi(z) := (z - z_o)^{-1} \text{ for all } z \in \mathbb{C} \setminus \{z_o\}, \\ &\Phi^{-1} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{z_o\}, \quad \Phi^{-1}(\zeta) := z_o + \zeta^{-1} \text{ for all } \zeta \in \mathbb{C} \setminus \{0\}, \end{aligned} \tag{2.201}$$

which are inverse to each other. We then claim that

$$\widetilde{\Sigma} := \Phi(\Sigma) \cup \{0\} \tag{2.202}$$

is a simple closed curve which contains the origin in $\mathbb{C}$. To see that this is indeed the case, start by expressing $\Sigma = \gamma(\mathbb{R})$ where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with the property that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. Then $\widetilde{\gamma} : \big[-\pi/2, \pi/2\big] \to \mathbb{C}$ defined for each $t \in \big[-\pi/2, \pi/2\big]$ as

$$\widetilde{\gamma}(t) := \begin{cases} \big(\gamma(\tan t) - z_o\big)^{-1} & \text{if } t \in \big(-\pi/2, \pi/2\big), \\ 0 & \text{if } t \in \big\{\pm\pi/2\big\} \end{cases} \tag{2.203}$$

is a continuous function whose restriction to $\big[-\pi/2, \pi/2\big)$ is injective, and whose image is precisely $\widetilde{\Sigma}$. Also, $0 \in \widetilde{\Sigma}$ by design. Hence, as claimed, $\widetilde{\Sigma}$ is a simple closed curve passing through $0 \in \mathbb{C}$. The classical Jordan curve theorem recalled in (2.197) then ensures that $\mathbb{C} \setminus \widetilde{\Sigma}$ consists precisely of two open connected components, one bounded $\widetilde{\Omega}_+$, and one unbounded $\widetilde{\Omega}_-$, satisfying $\partial\widetilde{\Omega}_\pm = \widetilde{\Sigma}$. In particular,

$$\mathbb{C} \setminus \{0\} = \widetilde{\Omega}_+ \sqcup \big(\widetilde{\Sigma} \setminus \{0\}\big) \sqcup \widetilde{\Omega}_- \ \text{(disjoint unions)}. \tag{2.204}$$

Then $O_\pm := \Phi^{-1}(\widetilde{\Omega}_\pm)$ are open connected subsets of $\mathbb{C} \setminus \{z_o\}$, and applying the homeomorphism Φ^{-1} to (2.204) yields

$$\mathbb{C} \setminus \{z_o\} = O_+ \sqcup \Sigma \sqcup O_- \ \text{(disjoint unions)}. \tag{2.205}$$

Let us also observe that since $\widetilde{\Omega}_-$ is unbounded, there exists a sequence $\{\zeta_j\}_{j\in\mathbb{N}}$ in $\widetilde{\Omega}_-$ with $|\zeta_j| \to \infty$ as $j \to \infty$. Consequently, the sequence $\{z_j\}_{j\in\mathbb{N}}$ defined for each $j \in \mathbb{N}$ as $z_j := \Phi^{-1}(\zeta_j) = z_o + \zeta_j^{-1}$ is contained in $\Phi^{-1}(\widetilde{\Omega}_-) = O_-$ and converges to z_o. This shows that

$$z_o \in \overline{O_-}. \tag{2.206}$$

Next, since Σ is a closed set, the fact that $z_o \in \mathbb{C} \setminus \Sigma$ guarantees the existence of some $r > 0$ with the property that $B(z_o, r) \cap \Sigma = \varnothing$. In the context of (2.205) this shows that the connected set $B(z_o, r)\setminus\{z_o\}$ is covered by the open sets $O_\pm$. As such, $B(z_o, r)\setminus\{z_o\}$ is fully contained in either O_+ or O_-. In view of (2.206) we ultimately conclude that $B(z_o, r) \setminus \{z_o\} \subseteq O_-$. Then $\Omega_+ := O_+$ and $\Omega_- := O_- \cup \{z_o\}$ are open, connected, disjoint subsets of $\mathbb{C}$, with

$$\mathbb{C} = \Omega_+ \sqcup \Sigma \sqcup \Omega_- \ \text{(disjoint unions)}, \tag{2.207}$$

and

$$\partial\Omega_\pm = \partial O_\pm \setminus \{z_o\} = \Phi^{-1}\big(\partial\widetilde{\Omega}_\pm \setminus \{0\}\big) = \Phi^{-1}\big(\widetilde{\Sigma} \setminus \{0\}\big) = \Sigma. \tag{2.208}$$

This finishes the proof of (2.199).

Moving on, the `length` $L \in [0, +\infty]$ of a given compact curve $\Sigma = \gamma([a, b])$ is defined as

$$L := \sup \sum_{j=1}^{N} |\gamma(t_j) - \gamma(t_{j-1})|, \tag{2.209}$$

the supremum being taken over all partitions $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of the interval $[a, b]$. As is well known (cf., e.g., [85, Theorem 4.38, p. 135]), the length L of any simple compact curve Σ may be expressed in terms of the Hausdorff measure by

$$L = \mathcal{H}^1(\Sigma), \tag{2.210}$$

and

$$\begin{aligned}&|z_1 - z_2| \le \mathcal{H}^1(\Sigma) \text{ for any compact curve}\\&\Sigma \text{ in the plane with endpoints } z_1, z_2 \in \mathbb{C}.\end{aligned} \tag{2.211}$$

Call a curve Σ `rectifiable` provided its length is finite (i.e., $L < +\infty$), and call Σ `locally rectifiable` if each of its compact sub-curves is rectifiable. The latter condition is equivalent to demanding that $\gamma(I)$ is a rectifiable curve for each compact sub-interval I of the domain of definition of some (or any) parametrization on Σ. In particular, a Jordan curve Σ passing through infinity in the plane, with parametrization $\gamma : \mathbb{R} \to \Sigma$, is locally rectifiable if and only if $\gamma(I)$ is a rectifiable curve for any compact sub-interval I of $\mathbb{R}$.

Suppose Σ is a rectifiable, simple, compact curve in the plane, and denote by L its length. Then there exists a parametrization $[0, L] \ni s \mapsto z(s) \in \Sigma$ of Σ, called the `arc-length parametrization` of Σ, with the property that for each $s_1, s_2 \in [0, L]$ with $s_1 < s_2$ the length of the curve with endpoints at $z(s_1)$ and $z(s_2)$ is $s_2 - s_1$. It is well known (see, e.g., [85, Definition 4.21 and Theorem 4.22, pp. 128–129]) that the arch-length parametrization exists and satisfies

$$\begin{aligned}&z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } [0, L]\\&\quad\text{and } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in [0, L].\end{aligned} \tag{2.212}$$

Also, (2.210)–(2.211) imply

$$|z(s_1) - z(s_2)| \le |s_1 - s_2|, \quad \forall s_1, s_2 \in [0, L]. \tag{2.213}$$

Lemma 2.10 *Let Σ be a rectifiable, simple, compact curve in the plane. Denote by L its length, and let $[0, L] \ni s \mapsto z(s) \in \Sigma$ be its arc-length parametrization. Given $s_1, s_2 \in [0, L]$ with $s_1 < s_2$, abbreviate $I := [s_1, s_2]$ and set $z'_I := ⨍_I z'(s)\, ds$. Then*

$$⨍_I |z'(s) - z'_I|^2\, ds = 1 - \left|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\right|^2. \tag{2.214}$$

Proof Upon observing that

$$z'_I = ⨍_I z'(s)\, \mathrm{d}s = \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} z'(s)\, \mathrm{d}s = \frac{z(s_2) - z(s_1)}{s_2 - s_1}, \tag{2.215}$$

this is a direct consequence of the formula in the second line of (2.114). □

Remark 2.1 The arch-length parametrization of a locally rectifiable Jordan curve passing through infinity in the plane is defined similarly, with $\mathbb{R}$ now playing the role of the interval $[0, L]$, and satisfies properties analogous to (2.212), (2.213), and Lemma 2.10.

We continue by recalling an important category of curves, introduced in 1936 by Mikhail A. Lavrentiev in [84] (also known as the class of Lavrentiev curves).

Definition 2.14 Given some number $\varkappa \in [0, \infty)$, recall that a set $\Sigma \subset \mathbb{C}$ is said to be a $\varkappa$-CAC, or simply CAC (acronym for chord-arc curve) if the parameter $\varkappa$ is de-emphasized, provided Σ is a locally rectifiable Jordan curve passing through infinity with the property that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \Sigma, \tag{2.216}$$

where $\ell(z_1, z_2)$ denotes the length of the sub-arc of Σ joining z_1 and z_2.

In general, the presence of a cusp prevents a curve from being chord-arc. For example, $\Sigma := \{(x, \sqrt{|x|}) : x \in \mathbb{R}\}$ is a Jordan curve passing through infinity in $\mathbb{R}^2 \equiv \mathbb{C}$ which nonetheless fails to be chord-arc. Indeed, if for $x > 0$ we set $z_1 := x + \mathrm{i}\sqrt{x} \in \Sigma$ and $z_2 := -x + \mathrm{i}\sqrt{x} \in \Sigma$ then L'Hôspital's Rule gives

$$\lim_{x\to 0^+} \frac{\ell(z_1, z_2)}{|z_1 - z_2|} = \lim_{x\to 0^+} \frac{2\int_0^x \sqrt{1 + \frac{1}{4t}}\,\mathrm{d}t}{2x} = \lim_{x\to 0^+} \sqrt{1 + \frac{1}{4x}} = +\infty, \tag{2.217}$$

which shows that condition (2.216) is violated for each $\varkappa \in [0, \infty)$.

There are fundamental links between chord-arc curves in the plane and the John-Nirenberg space BMO on the real line. Such connections, along with other basic properties of chord-arc curves, are brought to the forefront in Proposition 2.10 below. To facilitate stating and proving it, we first wish to recall the following version for bi-Lipschitz maps of the classical Kirszbraun extension theorem proved in [79, Theorem 1.2] with a linear bound on the distortion:

> any function $f : \mathbb{R} \to \mathbb{C}$ with the property that there exist C, C' in $(0, \infty)$ such that $C|t_1 - t_2| \leq |f(t_1) - f(t_2)| \leq C'|t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$ extends to a homeomorphism $F : \mathbb{C} \to \mathbb{C}$ with $(C/120)|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq (2000C')|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$. (2.218)

Results of this nature have also been proved in [138], [139], [67, Proposition 1.13, p. 227] (see also [119, Theorem 7.10, p. 166] and [36] in the case when the real line is replaced by the unit circle), though the quantitative aspect is less precise, or not explicitly mentioned, in these works.

Here is the proposition dealing with basic properties of chord-arc curves mentioned above.

Proposition 2.10 *Let $\Sigma \subset \mathbb{C}$ be a $\varkappa$-CAC in the plane, for some $\varkappa \in [0, \infty)$, and consider its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$. Then the following statements are true.*

(i) For each $s_1, s_2 \in \mathbb{R}$ one has

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2| \leq (1 + \varkappa)|z(s_1) - z(s_2)|, \tag{2.219}$$

and

$$\begin{gathered} z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } \mathbb{R}, \\ \text{with } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \end{gathered} \tag{2.220}$$

(ii) *For each $z_o \in \Sigma$ and $r \in (0, \infty)$ abbreviate $\Delta(z_o, r) := B(z_o, r) \cap \Sigma$. Then for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ one has*

$$(s_o - r, s_o + r) \subseteq z^{-1}\big(\Delta(z(s_o), r)\big) \subseteq \big(s_o - (1+\varkappa)r\ ,\ s_o + (1+\varkappa)r\big). \tag{2.221}$$

(iii) *For every Lebesgue measurable set $A \subseteq \mathbb{R}$ one has*

$$\mathcal{H}^1\big(z(A)\big) = \mathcal{L}^1(A), \tag{2.222}$$

and for each $\mathcal{H}^1$-measurable set $E \subseteq \Sigma$ one has

$$\mathcal{H}^1(E) = \mathcal{L}^1\big(z^{-1}(E)\big). \tag{2.223}$$

(iv) *With the arc-length measure σ on Σ defined as*

$$\sigma := \mathcal{H}^1 \lfloor \Sigma, \tag{2.224}$$

for each σ-measurable set $E \subseteq \Sigma$ and each non-negative σ-measurable function g on E one has

$$\int_E g\, d\sigma = \int_{z^{-1}(E)} g(z(s))\, ds. \tag{2.225}$$

(v) *Denote by Ω the region of the plane that is lying to the left of the curve Σ (relative to the orientation Σ inherits from its arc-length parametrization given by $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$). Then Ω is a set of locally finite perimeter and its geometric measure theoretic outward unit normal ν is given by*

$$\nu(z(s)) = -iz'(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \tag{2.226}$$

As a consequence, for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ the line $\{z(s) + t\, z'(s) : t \in \mathbb{R}\}$ is an approximate tangent line to Σ at the point $z(s)$. Hence, Ω has an approximate tangent line at $\mathcal{H}^1$-almost every point on $\partial\Omega$.

(vi) *The set Ω introduced in item (v) is a connected, simply connected, unbounded, two-sided* NTA *domain with an Ahlfors regular boundary (hence also an Ahlfors regular domain which satisfies a two-sided local John condition and,*

in particular, a UR *domain) and whose topological boundary is precisely* Σ, *i.e.,* $\partial\Omega = \Sigma$. *In fact,*

$$\text{there exists a bi-Lipschitz homeomorphism } F : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } 120^{-1}(1+\varkappa)^{-1}|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq 2000|z_1 - z_2| \text{ for all points } z_1, z_2 \in \mathbb{C}, \text{ and with the property that } \Omega = F(\mathbb{R}^2_+), \ \mathbb{R}^2 \setminus \overline{\Omega} = F(\mathbb{R}^2_-), \text{ as well as } \partial\Omega = F(\mathbb{R} \times \{0\}). \tag{2.227}$$

(vii) With the piece of notation introduced in (2.97) *one has*

$$\frac{1}{2(1+\varkappa)}\|\nu\|_{BMO(\Sigma,\sigma)} \leq \|z'\|_{BMO(\mathbb{R},\mathcal{L}^1)} \leq \|z'\|_{BMO_2(\mathbb{R},\mathcal{L}^1)} \leq \frac{\sqrt{\varkappa(2+\varkappa)}}{1+\varkappa} < 1 \tag{2.228}$$

and

$$\frac{1}{2(1+\varkappa)}\|z'\|_{BMO(\mathbb{R},\mathcal{L}^1)} \leq \|\nu\|_{BMO(\Sigma,\sigma)} \leq 2\sqrt{\varkappa(2+\varkappa)}. \tag{2.229}$$

Moreover, Σ *is a* $\varkappa_*$-CAC *with* $\varkappa_* \in [0, \varkappa]$ *defined as*

$$\varkappa_* := \frac{1}{\sqrt{1 - \|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}} - 1 = \frac{\|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}{\sqrt{1 - \|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}\Big(1 + \sqrt{1 - \|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}\Big)}. \tag{2.230}$$

Proof The claims in item *(i)* are seen from definitions and Remark 2.1, while the claim in item *(ii)* is an elementary consequence of (2.219). Next, in view of (2.220), the area formula (cf. [47, Theorem 1, p. 96]) gives (2.222), which may be equivalently recast as in (2.223). Also, the change of variable formula (cf. [47, Theorem 2, p. 99]) gives (2.225). This takes care of items *(iii)-(iv)*.

To proceed from the version of the Jordan curve theorem recorded in (2.199) we conclude that

$$\text{the complement of the curve } \Sigma \text{ in } \mathbb{C} \text{ consists of only two open connected components, namely } \Omega_+ := \Omega \text{ and } \Omega_- := \mathbb{C} \setminus \overline{\Omega}, \text{ satisfying } \partial\Omega_+ = \Sigma = \partial\Omega_-. \tag{2.231}$$

In addition, from (2.221) and (2.223) we see that for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ we have

$$\begin{aligned}\mathcal{H}^1\big(\Delta(z(s_o),r)\big) &= \mathcal{L}^1\Big(z^{-1}\big(\Delta(z(s_o),r)\big)\Big)\\ &\leq \mathcal{L}^1\Big(\big(s_o-(1+\varkappa)r\,,\ s_o+(1+\varkappa)r\big)\Big)\\ &= 2(1+\varkappa)r. \end{aligned} \tag{2.232}$$

Based on this and the criterion for finite perimeter from [47, Theorem 1, p. 222] we then conclude that Ω is a set of locally finite perimeter. Next, if $s_o \in \mathbb{R}$ is a point of differentiability for the complex-valued function $z(\cdot)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z(s_o+s) \in B\big(z(s_o)+s\,z'(s_o)\,,\ \varepsilon|s|\big) \ \text{ for each } \ s \in (-\delta,\delta). \tag{2.233}$$

In turn, from this geometric property we deduce that for each angle $\theta \in (0,\pi)$ there exists a height $h = h(\theta) > 0$ such that if $\Gamma^{\pm}_{\theta,h}$ denote the open truncated plane sectors with common vertex at $z(s_o)$, common aperture θ, common height h, and symmetry axes along the vectors $\pm i\,z'(s_o)$, then

$$\Gamma^{+}_{\theta,h} \subseteq \Omega = \Omega_+ \ \text{ and } \ \Gamma^{-}_{\theta,h} \subseteq \mathbb{C}\setminus\overline{\Omega} = \Omega_-. \tag{2.234}$$

To proceed, observe that the measure theoretic boundary of Ω (cf. (2.14)) may be presently described as

$$\partial_*\Omega = \Big\{z \in \partial\Omega :\ \limsup_{r\to 0^+} \frac{\mathcal{L}^2\big(B(z,r)\cap\Omega_\pm\big)}{r^2} > 0\Big\}. \tag{2.235}$$

Together, (2.234) and (2.235) imply that

$$\begin{aligned} \mathcal{A} &:= \big\{z(s_o) :\ s_o \in A\big\} \subseteq \partial_*\Omega, \ \text{ where we have set}\\ A &:= \big\{s_o \in \mathbb{R} :\ s_o \text{ differentiability point for } z(\cdot)\big\}. \end{aligned} \tag{2.236}$$

Meanwhile, from (2.222) and the fact that $z(\cdot)$ is differentiable at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$ we deduce (also using $\partial\Omega = \Sigma$) that

$$\mathcal{H}^1(\partial\Omega\setminus\mathcal{A}) = \mathcal{H}^1(\Sigma\setminus\mathcal{A}) = \mathcal{H}^1\big(z(\mathbb{R}\setminus A)\big) = \mathcal{L}^1\big(\mathbb{R}\setminus A\big) = 0. \tag{2.237}$$

With this in hand, formula

$$\mathcal{H}^1(\partial\Omega\setminus\partial_*\Omega) = 0 \tag{2.238}$$

follows by combining (2.236) with (2.237). As a consequence of (2.237)–(2.238) and (2.24) we then conclude that

$$\mathcal{A} \cap \partial^*\Omega \text{ has full } \mathcal{H}^1\text{-measure in } \partial\Omega. \tag{2.239}$$

Next, pick an arbitrary point $z_o \in A$ and recall that (2.234) holds. From this and [59, Proposition 2.14, p. 606] it follows that if $\Gamma_{\pi-\theta}$ is the infinite open plane sector with vertex at z_o, aperture $\pi - \theta$, and symmetry axis along the vector $-\mathrm{i}z'(s_o)$, then the geometric measure theoretic outward unit normal to Ω satisfies

$$\nu(z(s_o)) \in \Gamma_{\pi-\theta} \tag{2.240}$$

provided $\nu(z(s_o))$ exists, i.e., if $z(s_o) \in \partial^*\Omega$. The fact that the angle $\theta \in (0, \pi)$ may be chosen arbitrarily close to π then forces $\nu(z(s_o)) = -\mathrm{i}z'(s_o)$ whenever $z(s_o) \in \partial^*\Omega$, i.e., for $s_o \in z^{-1}(\mathcal{A} \cap \partial^*\Omega)$. Given that by (2.239) and (2.222) the latter set has full one-dimensional Lebesgue measure in $\mathbb{R}$, the claim in (2.226) is established. This finishes the treatment of item *(v)*.

Turning our attention to item *(vi)*, first observe that (2.219) implies

$$(1+\varkappa)^{-1}|s_1 - s_2| \le |z(s_1) - z(s_2)| \le |s_1 - s_2| \quad \text{for all} \quad s_1, s_2 \in \mathbb{R}, \tag{2.241}$$

hence $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is a bi-Lipschitz map. When used in conjunction with (2.241), the extension result recalled in (2.218) gives that

$$\begin{aligned} &\mathbb{R} \ni s \mapsto z(s) \in \Sigma \text{ extends to a bi-Lipschitz homeomorphism} \\ &F : \mathbb{C} \to \mathbb{C} \text{ with the property that for any points } z_1, z_2 \in \mathbb{C} \text{ one} \\ &\text{has } [120(1+\varkappa)]^{-1}|z_1 - z_2| \le |F(z_1) - F(z_2)| \le 2000|z_1 - z_2|. \end{aligned} \tag{2.242}$$

As a consequence, work in [59] implies that Ω is a connected two-sided NTA domain with an Ahlfors regular boundary (hence also a connected Ahlfors regular domain which satisfies a two-sided local John condition; cf. (2.47) and (2.88)). As far as item *(vi)* is concerned, there remains to observe that $\partial\Omega = \Sigma$ has been noted earlier in (2.231).

Turning our attention to item *(vii)*, fix two numbers $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, abbreviate $I := [s_1, s_2]$ and set $z'_I := ⨍_I z'(s)\,\mathrm{d}s$. We may then use Lemma 2.10 and (2.219) to estimate

$$⨍_I |z'(s) - z'_I|^2\,\mathrm{d}s = 1 - \left|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\right|^2 \le 1 - \left(\frac{1}{1+\varkappa}\right)^2 = \frac{\varkappa(2+\varkappa)}{(1+\varkappa)^2}. \tag{2.243}$$

In view of (2.97), this readily yields the penultimate inequality in (2.228). The second inequality in (2.228) is seen directly from the first inequality in (2.99).

To prove the very first inequality in (2.228), fix an arbitrary point $z_o \in \Sigma$ along with a radius $r \in (0, \infty)$, and set $\Delta := B(z_o, r) \cap \Sigma$. Then there exists a unique number $s_o \in \mathbb{R}$ such that $z_o = z(s_o) \in \Sigma$, and in the current setting we abbreviate $\mathcal{I} := \big(s_o - (1+\varkappa)r\,,\ s_o + (1+\varkappa)r\big)$. In particular, (2.221) and (2.223) imply

$$\sigma(\Delta) = \mathcal{H}^1\big(\Delta(z(s_o), r)\big) = \mathcal{L}^1\Big(z^{-1}\big(\Delta(z(s_o), r)\big)\Big)$$

$$\geq \mathcal{L}^1\Big(\big(s_o - r, s_o + r\big)\Big) = 2r = (1+\varkappa)^{-1}\mathcal{L}^1(\mathcal{I}). \tag{2.244}$$

With $c := -\mathrm{i} ⨍_{\mathcal{I}} z'(s)\,\mathrm{d}s \in \mathbb{C}$ we may then write

$$⨍_\Delta |\nu - c|\,\mathrm{d}\sigma = \frac{1}{\sigma(\Delta)}\int_\Delta |\nu - c|\,\mathrm{d}\sigma = \frac{1}{\sigma(\Delta)}\int_{z^{-1}(\Delta)} |\nu(z(s)) - c|\,\mathrm{d}s$$

$$\leq \frac{1}{\sigma(\Delta)}\int_{\mathcal{I}} |\nu(z(s)) - c|\,\mathrm{d}s = \frac{\mathcal{L}^1(\mathcal{I})}{\sigma(\Delta)} ⨍_{\mathcal{I}} |\nu(z(s)) - c|\,\mathrm{d}s$$

$$= \frac{\mathcal{L}^1(\mathcal{I})}{\sigma(\Delta)} ⨍_{\mathcal{I}} |z'(s) - \mathrm{i}c|\,\mathrm{d}s \leq (1+\varkappa) ⨍_{\mathcal{I}} |z'(s) - \mathrm{i}c|\,\mathrm{d}s$$

$$\leq (1+\varkappa)\|z'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}, \tag{2.245}$$

making use of (2.225), (2.221), (2.226), (2.244), and the choice of c. With (2.245) in hand, the first inequality in (2.228) readily follows. The last estimate in (2.229) is implicit in (2.228). To prove the first estimate in (2.229), retain notation introduced above and, now with the choice $c := ⨍_\Delta \nu\,\mathrm{d}\sigma \in \mathbb{C}$, estimate

$$⨍_{s_o - r}^{s_o + r} |z'(s) - \mathrm{i}c|\,\mathrm{d}s = \frac{1}{2r}\int_{s_o - r}^{s_o + r} |z'(s) - \mathrm{i}c|\,\mathrm{d}s \leq \frac{1}{2r}\int_{z^{-1}(\Delta)} |z'(s) - \mathrm{i}c|\,\mathrm{d}s$$

$$= \frac{1}{2r}\int_{z^{-1}(\Delta)} |\nu(z(s)) - c|\,\mathrm{d}s = \frac{1}{2r}\int_\Delta |\nu - c|\,\mathrm{d}\sigma$$

$$= \frac{\sigma(\Delta)}{2r} ⨍_\Delta |\nu - c|\,\mathrm{d}\sigma \leq (1+\varkappa) ⨍_\Delta |\nu - c|\,\mathrm{d}\sigma$$

$$\leq (1+\varkappa)\|\nu\|_{\mathrm{BMO}(\Sigma,\sigma)}, \tag{2.246}$$

thanks to (2.221), (2.226), (2.225), and (2.232). This readily yields the first estimate in (2.229).

To deal with the very last claim in item *(vii)*, fix some $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, set $I := [s_1, s_2]$ and abbreviate $z'_I := ⨍_I z'(s)\,\mathrm{d}s$. Lemma 2.10 then permits us to estimate

$$\|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \geq ⨍_I |z'(s) - z'_I|^2\,\mathrm{d}s = 1 - \left|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\right|^2. \tag{2.247}$$

In turn, this implies

$$|s_1 - s_2| \le \frac{|z(s_1) - z(s_2)|}{\sqrt{1 - \|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}} = (1 + \varkappa_*)|z(s_1) - z(s_2)|, \tag{2.248}$$

provided $\varkappa_*$ is defined as in (2.230). This shows that, indeed, Σ is a $\varkappa_*$-CAC. □

Having discussed a number of basic properties of chord-arc curves in Proposition 2.10, we now wish to elaborate on the manner in which concrete examples of chord-arc curves may be produced. To set the stage for the subsequent discussion observe that, when specialized to the one-dimensional setting, (2.126)–(2.128) imply that for each function $f \in \mathrm{CMO}(\mathbb{R}, \mathcal{L}^1)$ we have

$$\lim_{\substack{-\infty < s_1 < s_2 < +\infty \\ |s_1| + |s_2| \to \infty}} \left(⨍_{s_1}^{s_2} \left| f - ⨍_{s_1}^{s_2} f \, \mathrm{d}\mathcal{L}^1 \right| \mathrm{d}\mathcal{L}^1 \right) = 0, \tag{2.249}$$

and

$$\lim_{\substack{-\infty < s_1 < s_2 < +\infty \\ s_2 - s_1 \to 0^+}} \left(⨍_{s_1}^{s_2} \left| f - ⨍_{s_1}^{s_2} f \, \mathrm{d}\mathcal{L}^1 \right| \mathrm{d}\mathcal{L}^1 \right) = 0. \tag{2.250}$$

These properties are relevant in the context of the next proposition, describing a wealth of examples of chord-arc curves in the plane.

Proposition 2.11 *Suppose $b \in CMO(\mathbb{R}, \mathcal{L}^1)$ is a real-valued function and consider the assignment*

$$\mathbb{R} \ni s \longmapsto z(s) := \int_0^s e^{ib(t)} \, dt \in \mathbb{C}. \tag{2.251}$$

If said assignment is injective then $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is, in fact, the arc-length parametrization of a chord-arc curve (which, in particular, passes through infinity in the plane).

Proof Introduce

$$F(s_1, s_2) := \frac{z(s_1) - z(s_2)}{s_1 - s_2} \quad \text{for each } s_1, s_2 \in \mathbb{R} \text{ with } s_1 \neq s_2. \tag{2.252}$$

Then, whenever $-\infty < s_1 < s_2 < +\infty$ and with b_I abbreviating $⨍_{s_1}^{s_2} b(t) \, \mathrm{d}t$, we may write

$$F(s_1, s_2) = ⨍_{s_1}^{s_2} \mathrm{e}^{\mathrm{i}b(t)} \, \mathrm{d}t = ⨍_{s_1}^{s_2} \left(\mathrm{e}^{\mathrm{i}b(t)} - \mathrm{e}^{\mathrm{i}b_I} \right) \mathrm{d}t + \mathrm{e}^{\mathrm{i}b_I}. \tag{2.253}$$

Recall that

$$|e^{i\theta} - 1| = \left| \int_0^\theta ie^{it}\, dt \right| \le \left| \int_0^\theta \left|ie^{it}\right| dt \right| = |\theta| \ \text{ for each } \ \theta \in \mathbb{R}. \tag{2.254}$$

Then, since b is real-valued, we may use (2.254) to estimate

$$\begin{aligned} \left| \fint_{s_1}^{s_2} \left(e^{ib(t)} - e^{ib_I}\right) dt \right| &= \left| \fint_{s_1}^{s_2} \left(e^{i(b(t)-b_I)} - 1\right) dt \right| \\ &\le \fint_{s_1}^{s_2} \left|e^{i(b(t)-b_I)} - 1\right| dt \le \fint_{s_1}^{s_2} |b(t) - b_I|\, dt. \end{aligned} \tag{2.255}$$

According to (2.249)–(2.250) (written for b in place of f), the last integral in (2.255) converges to zero as either $|s_1| + |s_2| \to \infty$ or $s_2 - s_1 \to 0^+$. Since $\left|e^{ib_I}\right| = 1$, we conclude that

$$\lim_{\substack{-\infty<s_1\neq s_2<+\infty \\ |s_1|+|s_2|\to\infty}} |F(s_1, s_2)| = 1 \ \text{ and } \ \lim_{\substack{-\infty<s_1\neq s_2<+\infty \\ |s_1-s_2|\to 0^+}} |F(s_1, s_2)| = 1. \tag{2.256}$$

Given that, by assumption, the assignment $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is injective, we also have

$$F(s_1, s_2) \neq 0 \ \text{ whenever } \ -\infty < s_1 \neq s_2 < +\infty. \tag{2.257}$$

From (2.256), (2.257), and the fact that $F : \{(s_1, s_2) \in \mathbb{R}^2 : s_1 \neq s_2\} \to \mathbb{C}$ is continuous, we conclude that there exists $c \in (0, 1)$ with the property that $|F(s_1, s_2)| \ge c$ for each $s_1, s_2 \in \mathbb{R}$ with $s_1 \neq s_2$. In view of (2.252), this implies

$$|s_1 - s_2| \le c^{-1} |z(s_1) - z(s_2)| \ \text{ for each } \ s_1, s_2 \in \mathbb{R}. \tag{2.258}$$

In particular, this entails $\lim_{s\to\pm\infty} |z(s)| = \infty$. Also, the assignment $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is continuous, and it is assumed to be injective. Given that $|z'(s)| = \left|e^{ib(s)}\right| = 1$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$, since b is real-valued, it follows that $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is the arc-length parametrization of a Jordan curve in the plane which passes through infinity. □

Here is a version of Proposition 2.11 in which the membership of b to $\mathrm{CMO}(\mathbb{R}, \mathcal{L}^1)$ is replaced by the demand that $\|b\|_{L^\infty(\mathbb{R},\mathcal{L}^1)} < \frac{\pi}{2}$. In an interesting twist, this forces the image of (2.251) to be a Lipschitz graph.

Proposition 2.12 *If $b \in L^\infty(\mathbb{R}, \mathcal{L}^1)$ is a real-valued function with the property that $\|b\|_{L^\infty(\mathbb{R},\mathcal{L}^1)} < \frac{\pi}{2}$ then the assignment* (2.251) *is actually the arc-length parametrization of a Lipschitz graph in the plane* (*hence, in particular, a chord-arc curve*).

Proof Suppose there exists $\theta \in (0, \pi/2)$ such that $b(t) \in (-\theta, \theta)$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Since for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $z'(t) = \mathrm{e}^{\mathrm{i}b(t)} = \cos(b(t)) + \mathrm{i}\sin(b(t))$ given that b is real-valued, it follows that

$$\operatorname{Re} z'(t) = \cos(b(t)) \geq \cos\theta > 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}. \tag{2.259}$$

Granted this, whenever $-\infty < s_1 < s_2 < +\infty$ we may estimate

$$\begin{aligned} |z(s_2) - z(s_1)| &\geq \operatorname{Re}\big(z(s_2) - z(s_1)\big) = \operatorname{Re}\int_{s_1}^{s_2} z'(t)\,\mathrm{d}t = \int_{s_1}^{s_2} \operatorname{Re} z'(t)\,\mathrm{d}t \\ &\geq \int_{s_1}^{s_2} \cos\theta\,\mathrm{d}t = (\cos\theta)(s_2 - s_1), \end{aligned} \tag{2.260}$$

which, as in the end-game of the proof of Proposition 2.11, implies that the image of $z(\cdot)$ is a chord-arc curve Σ in the plane. As such, Proposition 2.10 applies and gives that if Ω denotes the region in $\mathbb{C}$ lying to the left of the curve Σ (relative to the orientation Σ inherits from its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$), then Ω is an Ahlfors regular domain whose topological boundary is Σ, and whose geometric measure theoretic outward unit normal ν is given at $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ by $\nu(z(s)) = -\mathrm{i}z'(s)$. Consider next the constant vector field $h := (0, -1) \equiv -\mathrm{i}$ in $\mathbb{C}$ and regard ν as a $\mathbb{R}^2$-valued function. Then, with $\langle \cdot, \cdot \rangle$ denoting the standard inner product in $\mathbb{R}^2$, we have

$$\begin{aligned} \big\langle \nu(z(s)), h(z(s)) \big\rangle &= \operatorname{Re}\big(\mathrm{i}\nu(z(s))\big) \\ &= \operatorname{Re} z'(s) \geq \cos\theta > 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \end{aligned} \tag{2.261}$$

This goes to show that there exists a constant vector field which is transverse to Ω and, as a consequence of work in [59], we conclude that Ω is the upper-graph of a Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$. The desired conclusion now follows. □

Another sub-category of chord-arc curves is offered by graphs of real-valued BMO_1 functions defined on the real line.

Proposition 2.13 *Let $\varphi \in W^{1,1}_{loc}(\mathbb{R})$ be such that $\varphi' \in BMO\,(\mathbb{R}, \mathcal{L}^1)$ and consider its graph $\Sigma := \big\{\big(x, \varphi(x)\big) : x \in \mathbb{R}\big\} \subseteq \mathbb{R}^2$. Then Σ is a $\varkappa$-*CAC* corresponding to $\varkappa = \|\varphi'\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}$.*

Proof Throughout, identify $\mathbb{R}^2$ with $\mathbb{C}$. Since functions in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ are locally absolutely continuous (cf., e.g., [85, Corollary 7.14, p. 223]), we conclude that Σ is a curve in the plane, with parametrization $\mathbb{R} \ni x \mapsto x + \mathrm{i}\varphi(x) \in \Sigma$. Hence, Σ is a Jordan curve that passes through infinity in the plane. From [61, Proposition 2.25, p. 2616] we know that Σ is an Ahlfors regular set which, in light of (2.210) implies that the curve Σ is also locally rectifiable. Consider two arbitrary points $z_1, z_2 \in \Sigma$, say $z_1 := \big(a, \varphi(a)\big)$ and $z_2 := \big(b, \varphi(b)\big)$ for some $a, b \in \mathbb{R}$ with $a < b$, and denote by Σ_{z_1,z_2} the sub-arc of Σ with endpoints z_1, z_2. From [61, Proposition 2.25, p. 2616] we also know that the arc-length measure $\sigma := \mathcal{H}^1\lfloor\Sigma$ on the curve Σ satisfies

$$\ell(z_1, z_2) = \sigma(\Sigma_{z_1,z_2}) = \int_a^b \sqrt{1 + |\varphi'(x)|^2}\,\mathrm{d}x. \tag{2.262}$$

Observe that the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(t) := \sqrt{1+t^2}$ for each $t \in \mathbb{R}$ is Lipschitz, with Lipschitz constant ≤ 1, since $|F'(t)| = |t|/\sqrt{1+t^2} \leq 1$ for each $t \in \mathbb{R}$. Consequently, if we set

$$\varphi'_I := \fint_a^b \varphi' \,\mathrm{d}\mathcal{L}^1 = \frac{\varphi(b) - \varphi(a)}{b-a}, \tag{2.263}$$

then

$$\begin{aligned}
\int_a^b \sqrt{1 + |\varphi'(x)|^2}\,\mathrm{d}x &= \int_a^b F(\varphi'(x))\,\mathrm{d}x \\
&\leq \int_a^b |F(\varphi'(x)) - F(\varphi'_I)|\,\mathrm{d}x + (b-a)F(\varphi'_I) \\
&\leq \int_a^b |\varphi'(x) - \varphi'_I|\,\mathrm{d}x + (b-a)\sqrt{1 + (\varphi'_I)^2} \\
&\leq (b-a)\|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} + (b-a)\sqrt{1 + \Big(\frac{\varphi(b)-\varphi(a)}{b-a}\Big)^2} \\
&\leq |z_1 - z_2|\|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} + |z_1 - z_2| \\
&= \big(1 + \|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}\big)|z_1 - z_2|.
\end{aligned} \tag{2.264}$$

From (2.262) and (2.264) we therefore conclude that (2.216) holds for the choice $\varkappa := \|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$, and the desired conclusion follows. □

Another basic link between chord-arc curves in the plane and the John-Nirenberg space BMO on the real line has been noted by R. Coifman and Y. Meyer. Specifically, [28] contains the following result: if $\Sigma \subseteq \mathbb{C}$ is a chord-arc curve then its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$ satisfies $z'(s) = \mathrm{e}^{\mathrm{i}b(s)}$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ for some real-valued function $b \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ and, in the converse direction, for any given real-function $b \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ whose BMO semi-norm is sufficiently small, the function $\mathbb{R} \ni s \mapsto z(s) := \int_0^s \mathrm{e}^{\mathrm{i}b(t)}\,\mathrm{d}t \in \mathbb{C}$ is the arc-length parametrization of a chord-arc curve (cf. also [29] for related results). Below we further elaborate on this last part of Coifman-Meyer's result. In particular, the analysis contained in our next proposition (which may be thought of as a quantitative version of Proposition 2.11) is going to be instrumental in producing a large variety of examples of δ-AR domains a little later (see Example 2.7).

Proposition 2.14 *Let $b \in BMO\,(\mathbb{R}, \mathcal{L}^1)$ be a real-valued function with*

$$\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)} < 1 \tag{2.265}$$

and introduce

$$\varkappa := \frac{\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}}{1 - \|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}} \in [0, \infty). \tag{2.266}$$

Define $z : \mathbb{R} \to \mathbb{C}$ by setting

$$z(s) := \int_0^s e^{ib(t)}\, dt \quad \text{for each} \quad s \in \mathbb{R}. \tag{2.267}$$

Finally, consider $\Sigma := z(\mathbb{R})$, the image of $\mathbb{R}$ under the mapping $z(\cdot)$. Then the following statements are true.

(i) The set Σ is a $\varkappa$-CAC which contains the origin $0 \in \mathbb{C}$, and the mapping given by $\mathbb{R} \ni s \longmapsto z(s) \in \Sigma$ is its arc-length parametrization. In addition,

$$\|z'\|_{BMO\,(\mathbb{R},\mathcal{L}^1)} \leq 2\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}. \tag{2.268}$$

(ii) Denote by Ω the region of the plane that is lying to the left of the curve Σ (relative to the orientation Σ inherits from its arc-length parametrization given by $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$). Then the set Ω is the image of the upper half-plane under a global bi-Lipschitz homeomorphism of $\mathbb{C}$, and

the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of Ω stay bounded as $\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)} \longrightarrow 0^+$. (2.269)

Furthermore, the geometric measure theoretic outward unit normal ν of Ω satisfies

$$\|\nu\|_{BMO(\Sigma,\sigma)} \leq 4\varkappa. \tag{2.270}$$

(iii) With the piece of notation introduced in (2.97), if in place of (2.265) one now assumes

$$\|b\|_{BMO_2(\mathbb{R},\mathcal{L}^1)} < \sqrt{2}, \tag{2.271}$$

then Σ is a $\varkappa_2$-CAC with

$$\varkappa_2 := \frac{\|b\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}{2 - \|b\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}} \in [0,\infty). \tag{2.272}$$

As a consequence of this and (2.229)*, in such a scenario one has*

$$\|\nu\|_{BMO(\Sigma,\sigma)} \leq 2\sqrt{\varkappa_2(2+\varkappa_2)}. \tag{2.273}$$

Proof The fact that b is real-valued entails that $e^{ib(\cdot)} \in L^\infty(\mathbb{R},\mathcal{L}^1)$. In turn, this membership guarantees that $z(\cdot)$ in (2.267) is a well-defined Lipschitz function on $\mathbb{R}$, with $z(0) = 0 \in \mathbb{C}$, and such that $z'(s) = e^{ib(s)}$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. In particular,

$$|z'(s)| = 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \tag{2.274}$$

We claim that the inequalities in (2.219) hold. To see this, for each $s_1, s_2 \in \mathbb{R}$ we write (keeping in mind that b is real-valued)

$$\begin{aligned} |z(s_1) - z(s_2)| &= \Big| \int_0^{s_1} e^{ib(t)}\,dt - \int_0^{s_2} e^{ib(t)}\,dt \Big| = \Big| \int_{s_2}^{s_1} e^{ib(t)}\,dt \Big| \\ &\leq \Big| \int_{s_2}^{s_1} \big|e^{ib(t)}\big|\,dt \Big| = |s_1 - s_2|, \end{aligned} \tag{2.275}$$

justifying the first inequality in (2.219). To prove the second inequality in (2.219), for each finite, non-trivial, sub-interval I of $\mathbb{R}$ introduce

$$b_I := \fint_I b(t)\,dt, \qquad m_I := e^{ib_I}, \tag{2.276}$$

and note that the fact that b is real-valued implies $|m_I| = 1$. Also, $m_I^{-1} = e^{-ib_I}$. Assume $-\infty < s_1 < s_2 < +\infty$ and set $I := [s_1, s_2]$. We may then estimate

$$\begin{aligned} \big|z(s_1) - z(s_2) - m_I \cdot (s_1 - s_2)\big| &= \Big| \int_{s_1}^{s_2} (z'(t) - m_I)\,dt \Big| \\ &= \Big| \int_{s_1}^{s_2} \big(z'(t)m_I^{-1} - 1\big)\,dt \Big| = \Big| \int_{s_1}^{s_2} \big(e^{i(b(t)-b_I)} - 1\big)\,dt \Big| \\ &\leq \int_{s_1}^{s_2} \big|e^{i(b(t)-b_I)} - 1\big|\,dt \leq \int_{s_1}^{s_2} |b(t) - b_I|\,dt \\ &= |s_1 - s_2| \fint_I |b(t) - b_I|\,dt \leq |s_1 - s_2| \|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \end{aligned}$$

$$= \left(\frac{\varkappa}{1+\varkappa}\right)|s_1 - s_2|, \tag{2.277}$$

where we have used the fact that Lipschitz functions are locally absolutely continuous (hence, the fundamental theorem of calculus applies), as well as the elementary inequality from (2.254). From (2.277), we obtain

$$\begin{aligned} |s_1 - s_2| &= |m_I \cdot (s_1 - s_2)| \le |z(s_1) - z(s_2)| + |z(s_1) - z(s_2) - m_I \cdot (s_1 - s_2)| \\ &\le |z(s_1) - z(s_2)| + \left(\frac{\varkappa}{1+\varkappa}\right)|s_1 - s_2|, \end{aligned} \tag{2.278}$$

which then readily yields the second estimate in (2.219). In particular, (2.219) implies that $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$ is a bi-Lipschitz bijection. The argument so far shows that Σ is a $\varkappa$-CAC passing through the origin $0 \in \mathbb{C}$, and $\mathbb{R} \ni s \longmapsto z(s) \in \Sigma$ is its arc-length parametrization. To finish the treatment of the claims in item *(i)*, there remains to justify (2.268). To this end, given any finite interval $I \subset \mathbb{R}$, set $b_I := ⨍_I b(t)\,\mathrm{d}t \in \mathbb{R}$ and $m_I := \mathrm{e}^{\mathrm{i}b_I} \in S^1$ (with the two memberships being a consequence of the fact that b is real-valued). With $z'_I := ⨍_I z'(s)\,\mathrm{d}s \in \mathbb{C}$ we may then estimate (bearing in mind that $m_I^{-1} = \mathrm{e}^{-\mathrm{i}b_I}$ and the inequality in (2.254))

$$\begin{aligned} ⨍_I \big|z'(s) - z'_I\big|\,\mathrm{d}s &\le 2 ⨍_I \big|z'(s) - m_I\big|\,\mathrm{d}s = 2 ⨍_I \big|z'(s)m_I^{-1} - 1\big|\,\mathrm{d}s \\ &= 2 ⨍_I \big|\mathrm{e}^{\mathrm{i}(b(s)-b_I)} - 1\big|\,\mathrm{d}s \le 2 ⨍_I |b(s) - b_I|\,\mathrm{d}s \\ &\le 2\|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}, \end{aligned} \tag{2.279}$$

and (2.268) readily follows from this. Next, all but the last claim in item *(ii)* are consequences of (2.227). The estimate in (2.270) is obtained by combining the first inequality in (2.228) with (2.268) and (2.266).

To deal with the claims in item *(iii)*, make the assumption that (2.271) holds and define $\varkappa_2$ as in (2.272). Whenever $-\infty < s_1 < s_2 < +\infty$ and $I := [s_1, s_2]$ we may estimate

$$\begin{aligned} s_2 - s_1 &\le \sqrt{(s_2 - s_1)^2 + \left|\int_{s_1}^{s_2} (b(t) - b_I)\,\mathrm{d}t\right|^2} \\ &= \left|(s_2 - s_1) + \mathrm{i}\int_{s_1}^{s_2} (b(t) - b_I)\,\mathrm{d}t\right| \\ &= \left|m_I \cdot (s_2 - s_1) + m_I \cdot \int_{s_1}^{s_2} \mathrm{i}(b(t) - b_I)\,\mathrm{d}t\right| \end{aligned}$$

$$\leq |z(s_2) - z(s_1)| + \left| z(s_2) - z(s_1) - m_I \cdot (s_2 - s_1) - m_I \cdot \int_{s_1}^{s_2} \mathrm{i}(b(t) - b_I)\,\mathrm{d}t \right|. \tag{2.280}$$

Note that the last term above may be written as

$$\begin{aligned} &\left| z(s_2) - z(s_1) - m_I \cdot (s_2 - s_1) - m_I \cdot \int_{s_1}^{s_2} \mathrm{i}(b(t) - b_I)\,\mathrm{d}t \right| \\ &= \left| \int_{s_1}^{s_2} \big(z'(t) - m_I - m_I \cdot \mathrm{i}(b(t) - b_I)\big)\,\mathrm{d}t \right| \\ &= \left| \int_{s_1}^{s_2} \big(z'(t) m_I^{-1} - 1 - \mathrm{i}(b(t) - b_I)\big)\,\mathrm{d}t \right| \\ &= \left| \int_{s_1}^{s_2} \big(\mathrm{e}^{\mathrm{i}(b(t) - b_I)} - 1 - \mathrm{i}(b(t) - b_I)\big)\,\mathrm{d}t \right|. \end{aligned} \tag{2.281}$$

Also, for each $\theta \in \mathbb{R}$ we may use (2.254) to write

$$\begin{aligned} |\mathrm{e}^{\mathrm{i}\theta} - 1 - \mathrm{i}\theta| &= \left| \int_0^\theta \mathrm{i}(\mathrm{e}^{\mathrm{i}t} - 1)\,\mathrm{d}t \right| \leq \left| \int_0^\theta \big|\mathrm{i}(\mathrm{e}^{\mathrm{i}t} - 1)\big|\,\mathrm{d}t \right| \\ &\leq \left| \int_0^\theta |t|\,\mathrm{d}t \right| = \theta^2/2. \end{aligned} \tag{2.282}$$

From (2.280), (2.281), (2.282), (2.97), and (2.272) we then conclude that

$$\begin{aligned} s_2 - s_1 &\leq |z(s_2) - z(s_1)| + \frac{1}{2} \int_{s_1}^{s_2} |b(t) - b_I|^2\,\mathrm{d}t \\ &\leq |z(s_2) - z(s_1)| + \tfrac{1}{2}(s_2 - s_1)\|b\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \\ &= |z(s_2) - z(s_1)| + \Big(\frac{\varkappa_2}{1 + \varkappa_2}\Big)(s_2 - s_1). \end{aligned} \tag{2.283}$$

From (2.283) we conclude that the version of (2.219) with $\varkappa$ replaced by $\varkappa_2$ holds. In particular, Σ is a $\varkappa_2$-CAC. The proof of Proposition 2.14 is therefore complete. □

2.4 The Class of Delta-Flat Ahlfors Regular Domains

We begin by making the following definition which is central for the present work. This should be compared with [61, Definitions 4.7-4.9, p. 2690] where related, yet rather distinct, variants have been considered. Specifically, the definitions in [61] contain additional geometric hypotheses and are designed to work well when dealing with domains with compact boundaries (as opposed to the present endeavors, where we shall mostly consider domains with unbounded boundaries).

Definition 2.15 Consider a parameter $\delta > 0$. Call a nonempty, proper subset Ω of $\mathbb{R}^n$ a δ-`flat Ahlfors regular domain` (or δ-`flat AR domain`, or simply δ-`AR domain`) provided Ω is an Ahlfors regular domain (in the sense of Definition 2.4) whose geometric measure theoretic outward unit normal ν satisfies (with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$)

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta. \tag{2.284}$$

In the class of Ahlfors regular domains we always have $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1$ (as noted in (2.118)), so condition (2.284) is redundant when $\delta > 1$. We will primarily be interested in the case when δ is small. In particular, when $\delta \in (0, 1)$, Lemma 2.8 ensures that $\partial\Omega$ is an unbounded set.

Let us also note here that, as is visible from the first inequality in (2.119), whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta \in (0, 1)$ then its geometric measure theoretic outward unit normal ν satisfies (with the infimum taken over all surface balls $\Delta \subseteq \partial\Omega$)

$$\inf_{\Delta\subseteq\partial\Omega} \left| ⨍_{\Delta} \nu \, d\sigma \right| > 1 - \delta. \tag{2.285}$$

Conversely, given any Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, it follows from the second inequality in (2.119) that Ω is a δ-AR domain whenever

$$\delta > \sqrt{2}\sqrt{1 - \inf_{\Delta\subseteq\partial\Omega} \left| ⨍_{\Delta} \nu \, d\sigma \right|}, \tag{2.286}$$

where the infimum is taken over all surface balls $\Delta \subseteq \partial\Omega$.

The discussion surrounding (2.285)–(2.286) shows that the condition that

$$\text{the number } \inf_{\Delta\subseteq\partial\Omega} \left| ⨍_{\Delta} \nu \, d\sigma \right| \text{ is sufficiently close to } 1 \tag{2.287}$$

is, in many regards, a good substitute for the demand that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is small.

Our next theorem describes some of the basic topological and geometric measure theoretic properties of sets in the class of δ-flat Ahlfors regular domains, with parameter $\delta \in (0, 1)$ small.

Theorem 2.5 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ satisfies $n \geq 2$, be a δ-flat Ahlfors regular domain (aka δ-AR domain), in the sense of Definition 2.15. Make the assumption that $\delta \in (0, 1)$ is sufficiently small relative to the Ahlfors regularity constant of $\partial\Omega$ and the dimension n.*

Then Ω is a two-sided NTA domain, in particular, a UR domain satisfying a two-sided local John condition (hence also a two-sided cork screw condition). In all cases, the intervening constants may be controlled solely in terms of Ahlfors regularity constant of $\partial\Omega$ and the dimension n.

In addition, Ω, $\overline{\Omega}$, $\partial\Omega$, $\mathbb{R}^n \setminus \overline{\Omega}$, and $\mathbb{R}^n \setminus \Omega$ are all unbounded connected sets, $\partial(\overline{\Omega}) = \partial\Omega$, $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial\Omega$, and $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$.

Finally, in the case when $n = 2$, both Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$ are simply connected.

Proof All claims made in the statement of the theorem are consequences of Corollary 2.2, Theorem 2.4, and Corollary 2.2. □

Examples and counterexamples of δ-AR domains in $\mathbb{R}^n$ are as follows.

Example 2.1 The set $\Omega := \mathbb{R}^n_+$ is a δ-AR domain for each $\delta > 0$. Indeed, the outward unit normal $\nu = -\mathbf{e}_n = (0, \dots, 0, -1)$ to Ω is constant, hence its BMO semi-norm vanishes. More generally, any half-space in $\mathbb{R}^n$, i.e., any set of the form

$$\begin{gathered}\Omega_{x_o,\xi} := \big\{x \in \mathbb{R}^n : \langle x - x_o, \xi\rangle > 0\big\} \\ \text{with } \ x_o \in \mathbb{R}^n \ \text{ and } \ \xi \in S^{n-1},\end{gathered} \tag{2.288}$$

is a δ-AR domain for each $\delta > 0$.

Consider next a sector of aperture $\theta \in (0, 2\pi)$ in the two-dimensional space, i.e., a planar set of the form

$$\begin{gathered}\Omega_\theta := \Big\{x \in \mathbb{R}^2 \setminus \{x_o\} : \tfrac{x-x_o}{|x-x_o|} \cdot \xi > \cos(\theta/2)\Big\} \\ \text{with } \ x_o \in \mathbb{R}^2, \ \theta \in (0, 2\pi), \ \text{ and } \ \xi \in S^1,\end{gathered} \tag{2.289}$$

and abbreviate $\sigma_\theta := \mathcal{H}^1 \lfloor \partial\Omega_\theta$. Then a direct computation shows that the outward unit normal vector ν to Ω_θ, regarded as a complex-valued function, satisfies

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)} = |\cos(\theta/2)|. \tag{2.290}$$

Hence,

$$\Omega_\theta \text{ is a } \delta\text{-AR domain if and only if } \ \delta > |\cos(\theta/2)|. \tag{2.291}$$

One last example in the same spirit is offered by the cone of aperture $\theta \in (0, 2\pi)$ in $\mathbb{R}^n$ with vertex at the origin and axis along $\mathbf{e}_n$, i.e.,

$$\Omega_\theta := \Big\{x \in \mathbb{R}^n \setminus \{0\} : \frac{x_n}{|x|} > \cos(\theta/2)\Big\}$$

$$= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \phi(x')\}, \tag{2.292}$$

where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is given by $\phi(x') := |x'| \cot(\theta/2)$ for each $x' \in \mathbb{R}^{n-1}$. If we abbreviate $\sigma_\theta := \mathcal{H}^{n-1}\lfloor\partial\Omega_\theta$, then a direct computation (using (2.295) below) shows that the outward unit normal vector ν to Ω_θ satisfies

$$\begin{aligned} &\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^n} = |\cos(\theta/2)|, \quad \text{hence once again} \\ &\Omega_\theta \text{ is a } \delta\text{-AR domain if and only if } \delta > |\cos(\theta/2)|. \end{aligned} \tag{2.293}$$

Example 2.2 If $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain for some $\delta > 0$, then $\mathbb{R}^n \setminus \overline{\Omega}$ is also a δ-AR domain (having the same topological and measure theoretic boundaries as Ω, and whose geometric measure theoretic outward unit normal is the opposite of the one for Ω). Also, we note that any rigid transformation of $\mathbb{R}^n$ preserves the class of δ-AR domains. One may also check from definitions that there exists a dimensional constant $c_n \in (0, \infty)$ with the property that if Ω is a δ-AR domain in $\mathbb{R}^n$ for some $\delta > 0$ then $\Omega \times \mathbb{R}$ is a $(c_n\delta)$-AR domain in $\mathbb{R}^{n+1}$.

Example 2.3 Given $\delta > 0$, the region $\Omega := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(x')\}$ above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ whose Lipschitz constant is $< 2^{-3/2}\delta$ is a δ-AR domain. To see this is indeed the case, it is relevant to note that

$$\begin{aligned} &F : \mathbb{R}^n \to \mathbb{R}^n \quad \text{defined for all } x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \\ &\quad \text{as} \quad F(x', x_n) := x + \phi(x')\mathbf{e}_n = (x', x_n + \phi(x')) \end{aligned} \tag{2.294}$$

is a bijective function with inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given at each point $y = (y', y_n)$ in $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ by $F^{-1}(y', y_n) = y - \phi(y')\mathbf{e}_n = (y', y_n - \phi(y'))$, and that both F, F^{-1} are Lipschitz functions with constant $\leq 1 + \|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$. Hence, Ω is the image of the upper half-space $\mathbb{R}^n_+$ under the bi-Lipschitz homeomorphism F, which also maps $\mathbb{R}^n_-$ onto $\mathbb{R}^n \setminus \overline{\Omega}$ and $\mathbb{R}^{n-1} \times \{0\}$ onto $\partial\Omega$. This goes to show that Ω is an open set satisfying a two-sided cork screw condition and with an Ahlfors regular boundary, hence also an Ahlfors regular domain (cf. (2.47)). To conclude that Ω is a δ-AR domain we need to estimate the BMO semi-norm of its geometric measure theoretic outward unit normal. Since this satisfies

$$\nu(x', \phi(x')) = \frac{(\nabla\phi(x'), -1)}{\sqrt{1 + |\nabla\phi(x')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1}, \tag{2.295}$$

it follows that for $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ we have

$$\nu(x', \phi(x')) + \mathbf{e}_n = \left(\frac{\nabla\phi(x')}{\sqrt{1 + |\nabla\phi(x')|^2}}, 1 - \frac{1}{\sqrt{1 + |\nabla\phi(x')|^2}}\right) \tag{2.296}$$

$$= \left(\frac{\nabla\phi(x')}{\sqrt{1+|\nabla\phi(x')|^2}}, \frac{|\nabla\phi(x')|^2}{\sqrt{1+|\nabla\phi(x')|^2}(1+\sqrt{1+|\nabla\phi(x')|^2})} \right).$$

Therefore, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, we may estimate

$$\begin{aligned}
\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} &= \|\nu+\mathbf{e}_n\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 2\,\|\nu+\mathbf{e}_n\|_{[L^\infty(\partial\Omega,\sigma)]^n} \\
&= 2^{3/2}\left\| \frac{|\nabla\phi|}{(1+|\nabla\phi|^2)^{1/4}(1+\sqrt{1+|\nabla\phi|^2})^{1/2}} \right\|_{L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \\
&\leq 2^{3/2}\,\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \delta. \qquad (2.297)
\end{aligned}$$

All things considered, this analysis establishes that $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain, with $\delta = O\Big(\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\Big)$ as $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \longrightarrow 0^+$. In addition, since the Lipschitz constants of the functions F, F^{-1} stay bounded when the Lipschitz constant of ϕ, i.e., $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$, stays bounded, we ultimately conclude that

> by taking $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$ sufficiently small, matters may be arranged so that the above set $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta > 0$ as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.298)

Example 2.4 To illustrate the scope of Example 2.5 discussed above, work in the two-dimensional setting and consider upper-graphs of piecewise linear functions with (relatively) small slopes. Concretely, fix a parameter $\varepsilon \in (0,\infty)$ and suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a function whose graph is a concatenation of line segments with slope belonging to $[-\varepsilon, \varepsilon]$. Then $\phi \in \mathscr{C}^0(\mathbb{R})$ and its distributional derivative ϕ' is a simple function taking values in the interval $[-\varepsilon, \varepsilon]$. Then

$$\phi' \in L^\infty(\mathbb{R},\mathcal{L}^1) \ \text{ and } \ \|\phi'\|_{L^\infty(\mathbb{R},\mathcal{L}^1)} \leq \varepsilon. \qquad (2.299)$$

As such, ϕ is a Lipschitz function. In particular, $\Omega := \big\{(x,y)\in\mathbb{R}^2 : y > \phi(x)\big\}$ is an Ahlfors regular domain. If ν denotes its geometric measure theoretic outward unit normal, and $\sigma := \mathcal{H}^1\lfloor\partial\Omega$, then (2.297) presently implies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq 2^{3/2}\varepsilon. \qquad (2.300)$$

Granted this, from (2.298) we then conclude that

> given any $\delta \in (0,1)$, by taking $\varepsilon \in (0, 2^{-3/2}\delta)$ ensures that the above set $\Omega \subseteq \mathbb{R}^2$ is a δ-AR domain with the Ahlfors regularity constant of $\partial\Omega$ bounded independently of δ. (2.301)

Finally, we note that if the graph of $\phi : \mathbb{R} \to \mathbb{R}$ is a concatenation of line segments with slope alternating between $+\varepsilon$ and $-\varepsilon$, then (2.300) together with (2.290) imply

$$\frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \le \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \le 2^{3/2}\varepsilon. \tag{2.302}$$

Example 2.5 Given any $\delta > 0$, the region $\Omega := \big\{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(x')\big\}$ above the graph of some BMO_1 function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, (namely, a function $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ with $\nabla\phi$ belonging to $\big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}$), satisfying (for some purely dimensional constant $C_n \in (1, \infty)$)

$$\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \min\{1, \delta/C_n\} \tag{2.303}$$

is a δ-AR domain. Indeed, BMO_1 domains are contained in the class of Zygmund domains (cf. [61, Proposition 3.15, p. 2637]) which, in turn, are NTA domains (cf. [66, Proposition 3.6, p. 94]). In particular, Ω satisfies a two-sided cork screw condition, hence $\partial_*\Omega = \partial\Omega$ (cf. (2.47)). From [61, Corollary 2.26, p. 2622] we also know that $\partial\Omega$ is an Ahlfors regular set. Finally, [61, Proposition 2.27, p. 2622] guarantees the existence of a purely dimensional constant $C \in (0, \infty)$ such that

$$\begin{aligned} &\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \\ &\quad \le C\, \|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\, \mathcal{L}^{n-1})]^{n-1}} \cdot \Big(1 + \|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\, \mathcal{L}^{n-1})]^{n-1}}\Big). \end{aligned} \tag{2.304}$$

Hence $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ if (2.303) is satisfied with $C_n := 2C$, proving that a Ω is indeed a δ-AR domain. In addition,

> taking $\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\, \mathcal{L}^{n-1})]^{n-1}}$ small enough ensures that the above set $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta > 0$ as small as wanted, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.305)

To offer concrete, interesting examples and counterexamples pertaining to BMO_1, work in the two-dimensional setting, i.e., when $n = 2$. For a fixed arbitrary number $\varepsilon \in (0, \infty)$ consider the continuous odd function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\phi_\varepsilon(x) := \begin{cases} \varepsilon\, x\big(\ln|x| - 1\big) & \text{if } x \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{for each } x \in \mathbb{R}. \tag{2.306}$$

Then from [102, Exercise 2.127, p. 89] we know that the distributional derivative of this function is $\phi'_\varepsilon = \varepsilon \ln|\cdot|$. Hence, for some absolute constant $C \in (0, \infty)$,

$$\|\phi'_\varepsilon\|_{\mathrm{BMO}(\mathbb{R},\, \mathcal{L}^1)} \le C\varepsilon \tag{2.307}$$

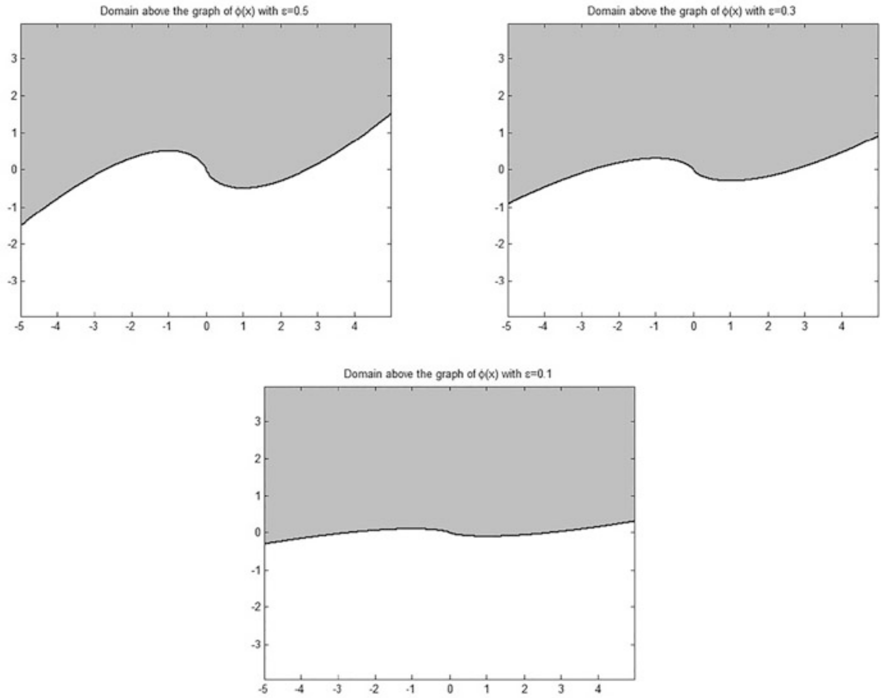

Fig. 2.1 The prototype of a non-Lipschitz δ-AR domain Ω_ε for which $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$ and such that the Ahlfors regularity constant of $\partial\Omega_\varepsilon$ and the local John constants of Ω_ε are uniformly bounded in ε

so ϕ_ε is indeed in BMO_1. This being said, ϕ_ε is not a Lipschitz function, so this example is outside the scope of Example 2.3. Consequently, the region Ω_ε lying above the graph of ϕ_ε is a non-Lipschitz δ-AR domain in the plane with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$ (as seen from (2.304) and (2.307)). See Fig. 2.1.

On the other hand, the distributional derivative of the function $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\psi_\varepsilon(x) := \begin{cases} \varepsilon\, x\big(\ln|x| - 1\big) & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0, \end{cases} \qquad \text{for each } \ x \in \mathbb{R}, \tag{2.308}$$

is $\psi_\varepsilon' = \varepsilon\big(\ln|\cdot|\big)\mathbf{1}_{(0,\infty)}$ which *fails* to be in $\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ (recall that the latter space is not stable under multiplication by cutoff functions). Hence, ψ_ε does *not* belong to BMO_1. In this vein, we wish to note that while the planar region $\widetilde{\Omega}_\varepsilon$ lying above the graph of ψ_ε continues to be an Ahlfors regular domain satisfying a two-sided local John condition for each $\varepsilon > 0$, its (complex-valued) geometric measure theoretic outward unit normal ν satisfies, due to the corner singularity at $0 \in \partial\widetilde{\Omega}_\varepsilon$ and (2.290) with $\theta = \pi/2$,

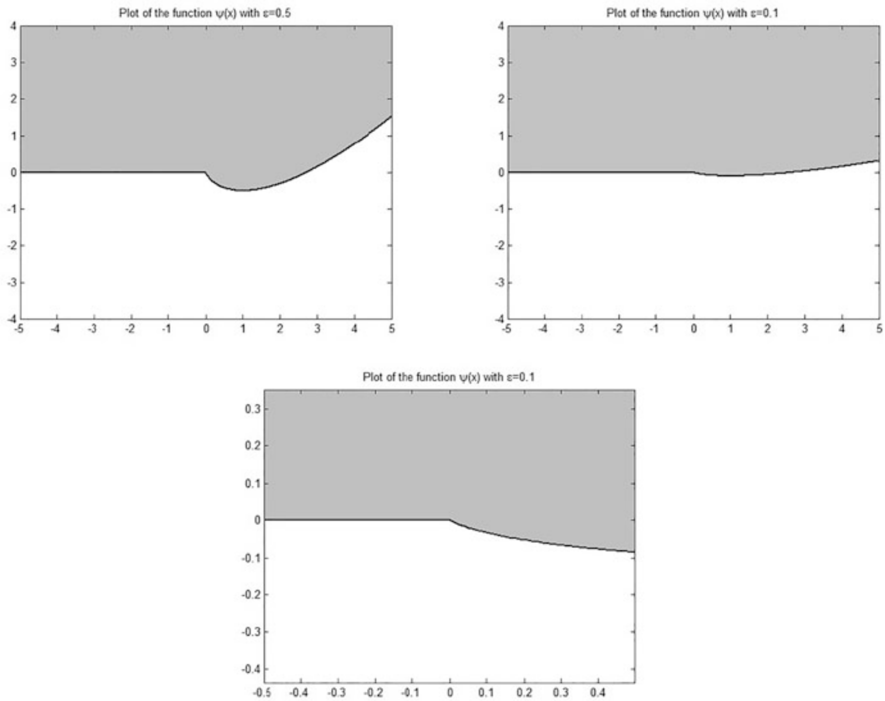

Fig. 2.2 A family $\{\widetilde{\Omega}_\varepsilon\}_{\varepsilon>0}$ of Ahlfors regular domains, with bounded Ahlfors regularity constants, which does not contain a δ-AR domain with $\delta \in (0, 1/\sqrt{2})$

$$\|\nu\|_{\mathrm{BMO}(\partial\widetilde{\Omega}_\varepsilon,\widetilde{\sigma}_\varepsilon)} \geq \frac{1}{\sqrt{2}} \quad \text{for each} \quad \varepsilon > 0, \tag{2.309}$$

where $\widetilde{\sigma}_\varepsilon := \mathcal{H}^1 \lfloor \partial\widetilde{\Omega}_\varepsilon$. Consequently, as $\varepsilon \to 0^+$, the set $\widetilde{\Omega}_\varepsilon$ never becomes a δ-AR domain if $\delta \in (0, 1/\sqrt{2})$. See Fig. 2.2.

Example 2.6 From [72, Theorem 2.1, p. 515] and [72, Remark 2.2, pp. 514-515] we know that there exist dimensional constants $\delta_n \in (0, \infty)$ and $C_n \in (0, \infty)$, with the property that if $\Omega \subseteq \mathbb{R}^n$ is a δ_o-Reifenberg flat domain, in the sense of [72, Definition 1.2, pp. 509–510] with $R = \infty$ and with $0 < \delta_o \leq \delta_n$, and if the surface measure $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ satisfies

$$\begin{gathered} \sigma\big(B(x,r) \cap \partial\Omega\big) \leq (1+\delta_o)\upsilon_{n-1} r^{n-1} \\ \text{for each } x \in \partial\Omega \text{ and } r > 0, \end{gathered} \tag{2.310}$$

(with υ_{n-1} denoting the volume of the unit ball in $\mathbb{R}^{n-1}$), then Ω is an Ahlfors regular domain whose geometric measure theoretic outward unit normal ν satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C_n\sqrt{\delta_o}. \tag{2.311}$$

See also [26, p. 11] and [123] in this regard. Consequently, given any number $\delta > 0$, any δ_o-Reifenberg flat domain with $0 < \delta_o < \min\{\delta_n, (\delta/C_n)^2\}$ which satisfies (2.310) is a δ-AR domain.

Example 2.7 Denote by Ω the region of the plane lying to one side of Σ, a $\varkappa$-CAC in $\mathbb{C}$. Then Proposition 2.10 implies that Ω is a δ-AR domain for any $\delta > 2\sqrt{\varkappa(2+\varkappa)}$.

To offer a concrete example, consider a real-valued function $b \in \mathrm{BMO}\,(\mathbb{R}, \mathcal{L}^1)$ with $\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} < 1$ and define $z : \mathbb{R} \to \mathbb{C}$ by setting

$$z(s) := \int_0^s e^{ib(t)}\,dt \quad \text{for each } \ s \in \mathbb{R}. \tag{2.312}$$

If $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ is the region of the plane to one side of the curve $\Sigma := z(\mathbb{R})$, then Proposition 2.10 and Proposition 2.14 imply that Ω is a connected Ahlfors regular domain with $\partial\Omega = \Sigma$, and whose geometric measure theoretic outward unit normal ν to Ω is given by

$$\nu(z(s)) = -ie^{ib(s)} \quad \text{for } \mathcal{L}^1\text{-a.e. } \ s \in \mathbb{R}. \tag{2.313}$$

In addition, if we set $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ then (2.270) gives

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \le \frac{4\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}{1 - \|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}. \tag{2.314}$$

As a consequence, Ω is a δ-AR domain in $\mathbb{R}^2$ for each $\delta \in (0, \infty)$ bigger than the number in the right-hand side of (2.314).

For instance, we may take b to be a small multiple of the logarithm on the real line, i.e.,

$$\begin{aligned} &b(s) := \varepsilon \ln|s| \quad \text{for each } \ s \in \mathbb{R} \setminus \{0\}, \\ &\text{with } \ 0 < \varepsilon < \|\ln|\cdot|\|^{-1}_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} \end{aligned} \tag{2.315}$$

(e.g., the computation on [55, p. 520] shows that $\|\ln|\cdot|\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} \le 3\ln(3/2)$, so taking $0 < \varepsilon < [3\ln(3/2)]^{-1} \approx 0.8221$ will do). Such a choice makes b a real-valued function with small BMO semi-norm which nonetheless maps $\mathbb{R} \setminus \{0\}$ onto $\mathbb{R}$. In view of the formula given in (2.313), this goes to show that Gauss' map $\Sigma \ni z \mapsto \nu(z) \in S^1$ is surjective, which may be interpreted as saying that the unit normal rotates arbitrarily much along the boundary. In particular, the chord-arc curve Σ produced in this fashion, which is actually the topological boundary of a δ-AR domain $\Omega \subseteq \mathbb{R}^2$ (with $\delta > 0$ which can be made as small as one pleases by taking $\varepsilon > 0$ appropriately small), fails to be a rotation of the graph of a function

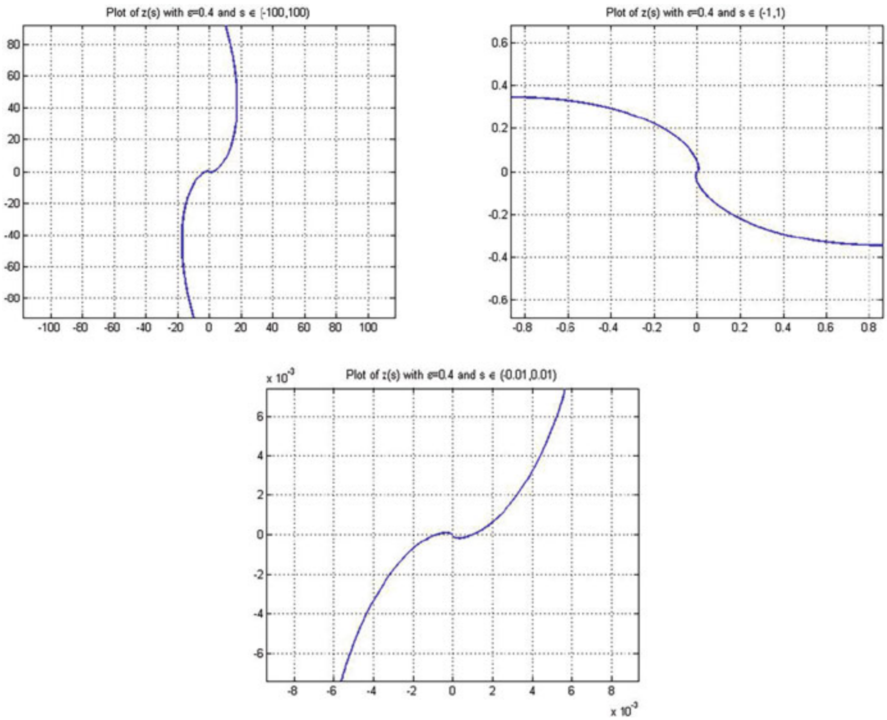

Fig. 2.3 Zooming in the curve $s \mapsto z(s)$ at the point $0 \in \mathbb{C}$

(even locally, near the origin). This being said, from Proposition 2.10 we know that

$$\text{the set } \Omega \subseteq \mathbb{R}^2 \text{ is actually bi-Lipschitz homeomorphic to the upper half-plane.} \tag{2.316}$$

Figure 2.3 depicts an unbounded δ-AR domain $\Omega \subseteq \mathbb{R}^2$ which is not the upper-graph of a function (in any system of coordinates isometric to the standard one in the plane). The set Ω is the region lying to one side of the curve $\Sigma = z(\mathbb{R})$ with $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ defined by the formula given in (2.312) for the real-valued function b as in (2.315) with $0 < \varepsilon < \|\ln|\cdot|\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$. As visible from (2.314), we have $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$.

In the above pictures we have taken $\varepsilon = 0.4 < \frac{1}{2}\|\ln|\cdot|\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$ and progressively zoomed in at the point $0 \in \partial\Omega$. The boundary of the set Ω is the plot of the curve $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ with

$$z(s) = \int_0^s e^{i\varepsilon \ln|t|}\,dt = \begin{cases} (i\varepsilon + 1)^{-1} s\, e^{i\varepsilon \ln|s|} & \text{if } s \in \mathbb{R} \setminus \{0\}, \\ 0 \in \mathbb{C} & \text{if } s = 0. \end{cases} \tag{2.317}$$

Here, $(i\varepsilon+1)^{-1}$ is merely a complex constant, s is the scaling factor that determines how far $z(s)$ is from the origin (specifically, $|z(s)| = |s|/\sqrt{\varepsilon^2+1}$), and $e^{i\varepsilon \ln|s|}$ is the factor that determines how the two spirals (making up $\partial\Omega \setminus \{0\}$, namely $z((-\infty, 0))$ and $z((0, +\infty))$) spin about the point $0 \in \mathbb{C}$. Note that $|z(s)|$ growths linearly (with respect to s) which is very fast compared to the spinning rate (which is logarithmic) and this is why we have chosen to zoom in at the point $0 \in \mathbb{C}$ in several distinct frames to get a better understanding of how $\partial\Omega$ looks near 0. The fact that $\partial\Omega$ is symmetric with respect to the origin is a direct consequence of $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ being odd. If $z(s) = re^{i\theta}$ is the polar representation of (2.317) for $s \in (0,\infty)$ then, by taking $\omega := 2\pi - \arccos\big(\frac{1}{\sqrt{\varepsilon^2+1}}\big)$, it follows that $\theta = \omega + \varepsilon \ln|s|$ and that $r = |z(s)| = (\varepsilon^2+1)^{-1/2}|s| = (\varepsilon^2+1)^{-1/2}e^{(\theta-\omega)/\varepsilon}$.

In polar coordinates, the curve $\Sigma_+ := z\big((0,+\infty)\big)$ has the equation $r = \alpha e^{\beta\theta}$ with $\alpha := (\varepsilon^2+1)^{-1/2}e^{-\omega/\varepsilon} \in (0,\infty)$ and $\beta := \varepsilon^{-1} \in (0,\infty)$ which identifies it precisely as a logarithmic spiral. In a similar fashion, the polar equation of the curve $\Sigma_- := z\big((-\infty,0)\big)$ is $r = \alpha e^{\beta\theta}$ with $\alpha := (\varepsilon^2+1)^{-1/2}e^{-(\omega+\pi)/\varepsilon} \in (0,\infty)$ and $\beta := \varepsilon^{-1} \in (0,\infty)$ which once again identifies it as a logarithmic spiral.

The MATLAB code that generated these pictures reads as follows:

```
s = [−100 : 0.001 : 100];
p = 0.4;
z=(1/(i*p+1.0))*s.*exp(i*p*log(abs(s)));
plot(real(z), imag(z),'LineWidth', 2), grid on, axis equal
```

Finally, we wish to elaborate on (2.316) and, in the process, get independent confirmation of (2.227) and (2.269). First, we observe that the δ-AR domain $\Omega \subseteq \mathbb{C}$ described above is the image of the upper half-plane $\mathbb{R}^2_+$ under map $F : \mathbb{C} \to \mathbb{C}$ defined for each $z \in \mathbb{C}$ by

$$F(z) := \begin{cases} (i\varepsilon+1)^{-1} z\, e^{i\varepsilon \ln|z|} & \text{if } z \in \mathbb{C}\setminus\{0\}, \\ 0 \in \mathbb{C} & \text{if } z = 0. \end{cases} \tag{2.318}$$

Note that F is a bijective, odd function, with inverse $F^{-1} : \mathbb{C} \to \mathbb{C}$ given at each $\zeta \in \mathbb{C}$ by

$$F^{-1}(\zeta) = \begin{cases} (i\varepsilon+1)\zeta\, e^{-i\varepsilon \ln(|\zeta|\sqrt{\varepsilon^2+1})} & \text{if } \zeta \in \mathbb{C}\setminus\{0\}, \\ 0 \in \mathbb{C} & \text{if } \zeta = 0. \end{cases} \tag{2.319}$$

Also, whenever $z_1, z_2 \in \mathbb{C}$ are such that $|z_1| \geq |z_2| > 0$ we may estimate

$$|F(z_1) - F(z_2)| \leq \frac{1}{\sqrt{\varepsilon^2+1}}\Big\{|z_1 - z_2| + |z_2|\big|e^{i\varepsilon \ln|z_1|} - e^{i\varepsilon \ln|z_2|}\big|\Big\} \tag{2.320}$$

and

$$\begin{aligned}\big|\mathrm{e}^{\mathrm{i}\varepsilon\ln|z_1|}-\mathrm{e}^{\mathrm{i}\varepsilon\ln|z_2|}\big| &= \big|\mathrm{e}^{\mathrm{i}\varepsilon(\ln|z_1|-\ln|z_2|)}-1\big| \leq \varepsilon\big|\ln|z_1|-\ln|z_2|\big| \\ &= \varepsilon\ln\big(\tfrac{|z_1|}{|z_2|}\big) \leq \varepsilon\big(\tfrac{|z_1|}{|z_2|}-1\big) = \varepsilon\big(\tfrac{|z_1|-|z_2|}{|z_2|}\big) \\ &\leq \varepsilon\tfrac{|z_1-z_2|}{|z_2|}, \end{aligned} \tag{2.321}$$

using the fact that $|\mathrm{e}^{\mathrm{i}\theta}-1|\leq|\theta|$ for each $\theta\in\mathbb{R}$ (cf. (2.254)) and $0\leq\ln x\leq x-1$ for each $x\in[1,\infty)$. From this we then eventually deduce that

$$|F(z_1)-F(z_2)|\leq\frac{\varepsilon+1}{\sqrt{\varepsilon^2+1}}|z_1-z_2| \quad \text{for all } \ z_1,z_2\in\mathbb{C}, \tag{2.322}$$

hence F is Lipschitz. The same type of argument also shows that F^{-1} is also Lipschitz, namely

$$|F^{-1}(\zeta_1)-F^{-1}(\zeta_2)|\leq(\varepsilon+1)\sqrt{\varepsilon^2+1}\,|\zeta_1-\zeta_2| \quad \text{for all } \ \zeta_1,\zeta_2\in\mathbb{C}, \tag{2.323}$$

so we ultimately conclude that $F:\mathbb{C}\to\mathbb{C}$ is an odd bi-Lipschitz homeomorphism of the complex plane. In summary,

> the δ-AR domain $\Omega\subseteq\mathbb{C}$ defined as the region of the complex plane lying to the left of the curve $\mathbb{R}\ni s\mapsto z(s)\in\mathbb{C}$ defined in (2.317) is in fact the image of the upper half-plane $\mathbb{R}^2_+$ under the odd bi-Lipschitz homeomorphism $F:\mathbb{C}\to\mathbb{C}$ from (2.318). (2.324)

Note that F also maps the lower half-plane $\mathbb{R}^2_-$ onto $\mathbb{R}^2\setminus\overline{\Omega}$, and $\mathbb{R}\times\{0\}$ onto $\partial\Omega$. This is in agreement with (2.227). Moreover, since the Lipschitz constants of F, F^{-1} stay bounded uniformly in $\varepsilon\in(0,1)$ (as is clear from (2.322), (2.323)) while, as noted earlier, $\delta=O(\varepsilon)$ as $\varepsilon\to 0^+$, we see that (as predicted in (2.269))

> by taking $\varepsilon\in(0,1)$ sufficiently small, matters may be arranged so the above set $\Omega\subseteq\mathbb{R}^2$ is a δ-AR domain with $\delta>0$ as small as one wishes, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.325)

Example 2.8 We may also construct examples of δ-AR domains exhibiting *multiple* spiral points. Specifically, suppose $-\infty<t_1<t_2<\cdots<t_{N-1}<t_N<+\infty$, for some $N\in\mathbb{N}$, and consider

$$b(t):=\varepsilon\sum_{j=1}^{N}\ln|t-t_j| \quad \text{for each } \ t\in\mathbb{R}\setminus\{t_1,\dots,t_N\}, \tag{2.326}$$

for some sufficiently small $\varepsilon > 0$. Next, define $z : \mathbb{R} \to \mathbb{C}$ as in (2.312) for this choice of the function b. Then Proposition 2.14 and Proposition 2.10 imply that the region Ω in $\mathbb{R}^2$ lying to one side of the curve $\Sigma := z(\mathbb{R})$ is indeed a δ-AR domain and, in fact, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$. Moreover, from (2.313) and (2.326) we see that $\partial\Omega = \Sigma$ looks like a spiral at each of the points $z(t_1), \dots, z(t_N)$ (cf. Fig. 1.1). Yet, once again, there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Omega = F(\mathbb{R}^2_+)$, $\mathbb{R}^2 \setminus \overline{\Omega} = F(\mathbb{R}^2_-)$, and $\partial\Omega = F(\mathbb{R} \times \{0\})$ (cf. (2.227)). Also, (2.269) presently entails

$$\text{by choosing } \varepsilon \in (0,1) \text{ appropriately small, we may ensure that } \Omega \text{ is a } \delta\text{-AR domain in } \mathbb{R}^2 \text{ with } \delta > 0 \text{ as small as desired, relative to the Ahlfors regularity constant of } \partial\Omega. \tag{2.327}$$

Example 2.9 We wish to note that the construction in Example 2.8 may be modified as to allow *infinitely many* spiral points. Specifically, assume $\{t_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}$ is a given sequence of real numbers and consider

$$0 < \lambda_j < 2^{-j} \min\Big\{1, \big\| \ln|\cdot - t_j| \big\|^{-1}_{L^1([-j,j],\mathcal{L}^1)}\Big\} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.328}$$

Also, suppose $0 < \varepsilon < \| \ln|\cdot| \|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$ and define

$$b(t) := \varepsilon \sum_{j=1}^{\infty} \lambda_j \ln|t - t_j| \quad \text{for each} \quad t \in \mathbb{R} \setminus \{t_j\}_{j\in\mathbb{N}}. \tag{2.329}$$

The choice in (2.328) ensures that the above series converges absolutely in $L^1(K, \mathcal{L}^1)$ for any compact subset K of $\mathbb{R}$. This has two notable consequences. First, the series in (2.329) converges absolutely in a pointwise sense $\mathcal{L}^1$-a.e. in $\mathbb{R}$; in particular, b is well defined at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$ and takes real values. Second,

$$\begin{aligned} \|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} &\le \varepsilon \sum_{j=1}^{\infty} \lambda_j \big\| \ln|\cdot - t_j| \big\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \\ &= \varepsilon \| \ln|\cdot| \|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \sum_{j=1}^{\infty} \lambda_j < 1. \end{aligned} \tag{2.330}$$

Granted this, if we now define $z : \mathbb{R} \to \mathbb{C}$ as in (2.312) for this choice of the function b then Proposition 2.14 and Proposition 2.10 imply that the region Ω in $\mathbb{R}^2$ lying to one side of the curve $\Sigma := z(\mathbb{R})$ is a δ-AR domain with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$. In fact, there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ as in (2.227), and (2.269) holds. We claim that matters may be arranged so that $\partial\Omega = \Sigma$ develops a spiral at each of the points $\{z(t_j)\}_{j\in\mathbb{N}}$. To this end, start by making the assumption that the sequence $\{t_j\}_{j\in\mathbb{N}}$ does not have any finite accumulation points. Inductively, we may then select a sequence of small positive numbers $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, 1)$ with the property that the family of intervals $I_j := (t_j - r_j, t_j + r_j)$, $j \in \mathbb{N}$, are mutually disjoint. For each $j \in \mathbb{N}$ consider the nonempty compact set $K_j := [-j, j] \setminus I_j$ and, in addition to (2.328), impose the condition that

$$0 < \lambda_j < 2^{-j} \left\| \ln |\cdot - t_j| \right\|^{-1}_{L^\infty(K_j,\mathcal{L}^1)} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.331}$$

Pick now $j_o \in \mathbb{N}$ arbitrary. Then for each $t \in I_{j_o}$ decompose $b(t) = f(t) + g(t)$ where

$$f(t) := \varepsilon\lambda_{j_o} \ln |t - t_{j_o}| \quad \text{and} \quad g(t) := \varepsilon \sum_{j\in\mathbb{N}\setminus\{j_o\}} \lambda_j \ln |t - t_j|. \tag{2.332}$$

In view of (2.331), the series defining g converges uniformly on I_{j_o}, hence g is a continuous and bounded function on I_{j_o}. Since f is continuous and unbounded from below on $(t_{j_o}, t_{j_o} + r_{j_o})$, it follows that the restriction of b to $(t_{j_o}, t_{j_o} + r_{j_o})$ is continuous and unbounded from below. This implies that $b\big((t_{j_o}, t_{j_o} + r_{j_o})\big)$ contains an interval of the form $(-\infty, a_{j_o})$, for some $a_{j_o} \in \mathbb{R}$. Similarly, $b\big((t_{j_o} - r_{j_o}, t_{j_o})\big)$ contains an interval of the form $(-\infty, c_{j_o})$, for some $c_{j_o} \in \mathbb{R}$. Based on this and (2.313) we then conclude that the normal $\nu(z(t))$ completes infinitely many rotations on the unit circle as t approaches t_{j_o} either from the left or from the right. Hence, $\partial\Omega = \Sigma$ develops a spiral at the point $z(t_{j_o})$.

Example 2.10 Here we discuss a higher-dimensional analogue of (2.324). To set the stage, fix an integer $n \in \mathbb{N}$ with $n \geq 3$. With $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ denoting the region of the plane lying to the left of the curve $\mathbb{R} \ni s \longmapsto z(s) \in \mathbb{C}$ defined in (2.317), consider

$$\widetilde{\Omega} := \mathbb{R}^{n-2} \times \Omega \subseteq \mathbb{R}^n. \tag{2.333}$$

Bring back the odd bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{C} \equiv \mathbb{R}^2$ from (2.318), and consider

$$\begin{aligned} &\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{defined as} \quad \widetilde{F}(x) := \big(x'', F(x_{n-1}, x_n)\big) \\ &\text{for each point} \quad x = (x'', x_{n-1}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{2.334}$$

Then one may check without difficulty that

$$\begin{aligned} &\widetilde{F} \text{ is an odd bi-Lipschitz homeomorphism of } \mathbb{R}^n \text{, and the set } \widetilde{\Omega} \\ &\text{defined in (2.333) is, in fact, the image of the upper half-space} \\ &\mathbb{R}^n_+ \text{ under the mapping } \widetilde{F} : \mathbb{R}^n \to \mathbb{R}^n. \end{aligned} \tag{2.335}$$

From this and Lemma 2.2 we may then conclude that

$$\widetilde{\Omega} \text{ is an Ahlfors regular domain, with the Ahlfors regularity constant of } \partial\widetilde{\Omega} \text{ controlled solely in terms of the dimension } n. \tag{2.336}$$

Since, as noted earlier, F also maps the lower half-plane $\mathbb{R}^2_-$ onto $\mathbb{R}^2 \setminus \overline{\Omega}$, and $\mathbb{R} \times \{0\}$ onto $\partial\Omega$, it follows from (2.334) and (2.333) that

$$\widetilde{F}(\mathbb{R}^n_-) = \mathbb{R}^n \setminus \overline{\widetilde{\Omega}} \quad \text{and} \quad \widetilde{F}(\mathbb{R}^{n-1} \times \{0\}) = \partial\widetilde{\Omega}. \tag{2.337}$$

From (2.25) and (2.336) we also know that

the geometric measure theoretic outward unit normal $\widetilde{\nu}$ to the set $\widetilde{\Omega} := \mathbb{R}^{n-2} \times \Omega \subseteq \mathbb{R}^n$ is given by $\widetilde{\nu}(x) = (0'', \nu(x_{n-1}, x_n))$ for $(\mathcal{L}^{n-2} \otimes \mathcal{H}^1)$-a.e. point $x = (x'', x_{n-1}, x_n) \in \partial\widetilde{\Omega} = \mathbb{R}^{n-2} \times \partial\Omega$, where $0'' \in \mathbb{R}^{n-2}$ and ν is the geometric measure theoretic outward unit normal to the set Ω. (2.338)

From this it readily follows that there exists some purely dimensional constant C_n in $(0, \infty)$ such that

$$\|\widetilde{\nu}\|_{[\mathrm{BMO}(\partial\widetilde{\Omega},\widetilde{\sigma})]^n} \leq C_n \|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{2.339}$$

By combining (2.339) with (2.314) we arrive at the conclusion that, for some purely dimensional constant $C_n \in (0, \infty)$,

$$\|\widetilde{\nu}\|_{[\mathrm{BMO}(\partial\widetilde{\Omega},\widetilde{\sigma})]^n} \leq C_n \frac{4\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}{1 - \|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}, \tag{2.340}$$

where $\widetilde{\sigma} := \mathcal{H}^{n-1} \lfloor \partial\widetilde{\Omega}$. As a consequence, $\widetilde{\Omega}$ is a δ-AR domain in $\mathbb{R}^n$ for each $\delta \in (0, \infty)$ bigger than the number in the right-hand side of (2.340). In particular, choosing the function b as in (2.315) allows us to conclude that $\widetilde{\Omega}$ is a δ-AR domain in $\mathbb{R}^n$ with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$.

In addition, since the Lipschitz constants of $\widetilde{F}$, $\widetilde{F}^{-1}$ stay bounded uniformly in the parameter $\varepsilon \in (0, 1)$ (as is clear from (2.334), (2.322), (2.323)) while, as just noted, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$, we see that

by taking $\varepsilon \in (0, 1)$ sufficiently small, matters may be arranged so that the set $\widetilde{\Omega} \subseteq \mathbb{R}^n$ defined in (2.333) is a δ-AR domain with $\delta > 0$ as small as one wishes, relative to the Ahlfors regularity constant of $\partial\widetilde{\Omega}$. (2.341)

Example 2.11 All sets considered so far have been connected. In the class of disconnected sets in the complex plane consider a double sector of arbitrary aperture $\theta \in (0, \pi)$, i.e., a set of the form

$$\Omega := \Big\{x \in \mathbb{R}^2 \setminus \{x_0\} : \Big|\tfrac{x-x_0}{|x-x_0|} \cdot \xi\Big| > \cos(\theta/2)\Big\}$$
$$\text{with } \; x_0 \in \mathbb{R}^2, \;\; \theta \in (0,\pi), \;\; \text{and } \;\; \xi \in S^1, \tag{2.342}$$

and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. Then simple symmetry considerations show that for each $r \in (0,\infty)$ the geometric measure theoretic outward unit normal ν to Ω satisfies $⨍_{B(x_o,r)\cap\partial\Omega} \nu \, d\sigma = 0$, hence

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \geq ⨍_{B(x_o,r)\cap\partial\Omega} \Big|\nu - ⨍_{B(x_o,r)\cap\partial\Omega} \nu \, d\sigma\Big| \, d\sigma$$
$$= ⨍_{B(x_o,r)\cap\partial\Omega} |\nu| \, d\sigma = 1. \tag{2.343}$$

As a consequence,

> the double sector Ω from (2.342) is a disconnected Ahlfors regular domain which satisfies a two-sided local John condition but fails to be a δ-AR domain for each $\delta \in (0,1]$. (2.344)

We may even arrange matters so that the set in question has a disconnected boundary. Specifically, given any two distinct points $x_0, x_1 \in \mathbb{R}^2$, along with an angle $\theta \in (0,\pi)$, and a direction vector $\xi \in S^1$, such that

$$\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \xi < \cos(\theta/2), \tag{2.345}$$

consider

$$\Omega := \Big\{x \in \mathbb{R}^2 \setminus \{x_0\} : \frac{x - x_0}{|x - x_0|} \cdot \xi > \cos(\theta/2)\Big\} \tag{2.346}$$
$$\bigcup \Big\{x \in \mathbb{R}^2 \setminus \{x_1\} : \frac{x - x_1}{|x - x_1|} \cdot (-\xi) > \cos(\theta/2)\Big\}.$$

This is the union of two planar sectors with vertices at x_0 and x_1, axes along ξ and $-\xi$, and common aperture θ. The condition in (2.345) ensures that said sectors are disjoint, hence Ω is disconnected, with disconnected boundary. Note that if we set $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and ν stands for the geometric measure theoretic outward unit normal to Ω then

$$\lim_{r\to\infty} ⨍_{B(x_o,r)\cap\partial\Omega} \nu \, d\sigma = 0 \tag{2.347}$$

which, much as in (2.343), once again implies that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \geq 1$. Consequently, the set Ω from (2.346) is an Ahlfors regular domain satisfying a two-sided local John condition which is disconnected and has a disconnected boundary, and which fails to be a δ-AR domain for each $\delta \in (0, 1]$. (2.348)

Similar considerations apply virtually verbatim in $\mathbb{R}^n$ with $n \geq 2$ (working with cones in place of sectors).

These examples are particularly relevant in the context of Theorem 2.4.

2.5 The Decomposition Theorem

Our first result in this section, which slightly refines work in [61], identifies general geometric conditions on a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter so that the inner product between the integral average ν_Δ of outward unit normal ν to Ω in any given surface ball $\Delta \subseteq \partial\Omega$ and the "chord" $x - y$ with $x, y \in \Delta$ may be controlled in terms of the radius of said ball and the BMO semi-norm of the outward unit normal ν.

Proposition 2.15 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then there exists $C_* \in (0, \infty)$ depending only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$ such that for each dilation parameter $\lambda \in [1, \infty)$ one has*

$$\sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(z,\lambda R)} R^{-1}|\langle x-y, \nu_{\Delta(z,R)}\rangle| \leq C_*\lambda\big(1+\log_2\lambda\big)\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n}, \tag{2.349}$$

where $\nu_{\Delta(z,R)} := ⨍_{\Delta(z,R)} \nu\, d\sigma$ for each $z \in \partial\Omega$ and $R > 0$.

Proof Let $\delta_* \in (0, 1)$ be the threshold associated with the set Ω as in Theorem 2.3. In particular, δ_* depends only on n and the Ahlfors regularity constant of $\partial\Omega$.

`Case I.` *Assume* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \delta_*$. For each location $z \in \partial\Omega$, each radius $R \in (0, \infty)$, each dilation parameter $\lambda \in [1, \infty)$, and any points $x, y \in \Delta(z, \lambda R)$ we then have

$$\begin{aligned} R^{-1}|\langle x-y, \nu_{\Delta(z,R)}\rangle| &\leq R^{-1}|x-y||\nu_{\Delta(z,R)}| \leq R^{-1}(2\lambda R) \\ &\leq C_*\lambda\big(1+\log_2\lambda\big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \end{aligned} \tag{2.350}$$

provided $C_* := 2\delta_*^{-1}$. This establishes (2.349) in this case.

`Case II.` *Assume* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_*$. In this scenario, (2.185) ensures that

$$\Omega \text{ satisfies a two-sided local John condition with constants which depend only on the dimension } n \text{ and the Ahlfors regularity constant of } \partial\Omega. \tag{2.351}$$

Granted this, [61, Corollary 4.15, pp. 2697–2698] applies and guarantees the existence of some constant $C \in (0, \infty)$ depending only on n and the Ahlfors regularity constant of $\partial\Omega$ such that

$$\sup_{x\in\partial\Omega}\sup_{R>0}\sup_{y\in\Delta(x,2R)} R^{-1}|\langle x-y, \nu_{\Delta(x,R)}\rangle| \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.352}$$

Fix a number $\lambda \in [1, \infty)$ along with an arbitrary point $z \in \partial\Omega$, $R > 0$, and $x, y \in \Delta(z, \lambda R)$. Then $|x-y| \leq 2\lambda R$, hence $y \in \Delta(x, 2\lambda R)$, so

$$\begin{aligned}
|\langle x-y, \nu_{\Delta(z,R)}\rangle| &\leq |\langle x-y, \nu_{\Delta(x,2\lambda R)}\rangle| + |x-y||\nu_{\Delta(x,2\lambda R)} - \nu_{\Delta(z,R)}| \\
&\leq C\lambda R\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + 2\lambda R|\nu_{\Delta(x,2\lambda R)} - \nu_{\Delta(z,3\lambda R)}| \\
&\qquad + 2\lambda R|\nu_{\Delta(z,3\lambda R)} - \nu_{\Delta(z,R)}| \\
&\leq CR\lambda\big(1+\log_2\lambda)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,,
\end{aligned} \tag{2.353}$$

by (2.352) and elementary estimates involving integral averages (cf. (2.103), (2.105)). After dividing the most extreme sides by R, then taking the supremum over all $z \in \partial\Omega$, $R > 0$, and $x, y \in \Delta(z, \lambda R)$, we arrive at (2.349). □

Remark 2.2 It is natural to attempt to quantify the global "tilt" of a given Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, envisioned as the maximal deviation of a chord $x-y$ with $x, y \in \Delta$ where Δ is an arbitrary surface ball on $\partial\Omega$ from being perpendicular to ν_Δ, the integral average in Δ of the geometric measure theoretic outward unit normal ν to Ω.

More specifically, we shall define the global tilt of Ω with amplitude $\lambda \in [1, \infty)$ to be

$$\mathbf{t}_\lambda(\Omega) := \sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(z,\lambda R)}\left|\left\langle \nu_{\Delta(z,R)}\,,\,\frac{x-y}{\lambda R}\right\rangle\right|, \tag{2.354}$$

where for each $z \in \partial\Omega$ and $R > 0$ we have set $\nu_{\Delta(z,R)} := ⨍_{\Delta(z,R)} \nu\,d\sigma$, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ playing the role of surface measure on $\partial\Omega$.

As an example, consider the cone of aperture $\theta \in (0, 2\pi)$ in $\mathbb{R}^n$ with vertex at the origin and axis along $\mathbf{e}_n$, i.e.,

$$\Omega_\theta := \left\{x \in \mathbb{R}^n \setminus \{0\} : \frac{x_n}{|x|} > \cos(\theta/2)\right\}. \tag{2.355}$$

Denote by ν the geometric measure theoretic outward unit normal to Ω_θ and abbreviate $\sigma_\theta := \mathcal{H}^{n-1}\lfloor\partial\Omega_\theta$. It may then be checked directly from the definition given in (2.354) that, on the one hand,

$$\mathbf{t}_\lambda(\Omega_\theta) = |\cos(\theta/2)| \quad \text{for each} \quad \lambda \in [1, \infty). \tag{2.356}$$

On the other hand, as noted in (2.293), the outward unit normal vector ν to Ω_θ satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^n} = |\cos(\theta/2)|. \tag{2.357}$$

In particular, in this special case we simply have

$$\mathbf{t}_\lambda(\Omega_\theta) = \|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^n} \quad \text{for each} \quad \lambda \in [1, \infty). \tag{2.358}$$

For a general Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, the best we can hope for is merely to control the global tilt $\mathbf{t}_\lambda(\Omega)$, for each fixed amplitude parameter $\lambda \in [1, \infty)$, in terms of the BMO-seminorm of the geometric measure theoretic outward unit normal ν to Ω.

Remarkably, this is possible, as (2.349) asserts that there exists some constant $C_* \in (0, \infty)$ depending only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$ such that for each amplitude parameter $\lambda \in [1, \infty)$ we have

$$\mathbf{t}_\lambda(\Omega) \leq C_*\big(1 + \log_2 \lambda\big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.359}$$

We continue by discussing a basic decomposition theorem. The general idea originated in [123, Proposition 5.1, p. 212] where such a decomposition result has been stated for surfaces of class $\mathscr{C}^2$, via a proof which makes essential use of smoothness, though the main quantitative aspects only depend on the rough character of said surface. A formulation in which the $\mathscr{C}^2$ smoothness assumption is replaced by Reifenberg flatness is stated in [73, Theorem 4.1, p. 398] (see also the comments on [26, p. 66]). A yet more potent version of such a decomposition result has been proved in [61, Theorem 4.16, p. 2701], starting with a different set of hypotheses which, a priori, do not specifically require the domain in question to be Reifenberg flat. The formulation of said result does require that the set in question satisfies a two-sided local John condition.

Below we present the most general variant of this result, valid in the class of Ahlfors regular domains $\Omega \subseteq \mathbb{R}^n$ for which the BMO semi-norm of its geometric measure theoretic outward unit normal is suitably small relative to the Ahlfors regularity constant of $\partial\Omega$. Stated as such, this result is well suited to the applications we have in mind.

Theorem 2.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then* *there exist* $C_0, C_1, C_2, C_3, C_4 \in (0, \infty)$, *depending only on the dimension n and the Ahlfors regularity constant of* $\partial\Omega$, *with the following significance.*

For each choice of a function

$$\phi : (0, 1) \longrightarrow (0, \infty) \tag{2.360}$$

with

$$\lim_{t\to 0^+} \phi(t) = 0 \;\; \textit{and} \;\; \lim_{t\to 0^+} \frac{\phi(t)}{t} \in (1, \infty], \tag{2.361}$$

there exists a threshold $\delta_* \in \big(0, \min\{1, 1/C_0\}\big)$, *depending only on the dimension n, the Ahlfors regularity constant of* $\partial\Omega$, *and the function* ϕ, *such that whenever*

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < \delta < \delta_* \tag{2.362}$$

one has the following property:

For every location $x_0 \in \partial\Omega$ *and every scale* $r > 0$ *there exists a unit vector* $\vec{n}_{x_0,r} \in S^{n-1}$ *along with a Lipschitz function*

$$h : H(x_0, r) := \langle \vec{n}_{x_0,r} \rangle^{\perp} \to \mathbb{R} \;\; \textit{with} \;\; \sup_{\substack{y_1,y_2\in H(x_0,r)\\ y_1\neq y_2}} \frac{|h(y_1) - h(y_2)|}{|y_1 - y_2|} \leq C_0\phi(\delta), \tag{2.363}$$

whose graph

$$\mathcal{G} := \big\{x = x_0 + x' + t\vec{n}_{x_0,r} : x' \in H(x_0, r),\; t = h(x')\big\} \tag{2.364}$$

(*in the coordinate system* $x = (x', t) \Leftrightarrow x = x_0 + x' + t\vec{n}_{x_0,r}$, $x' \in H(x_0, r)$, $t \in \mathbb{R}$) *is a good approximation of* $\partial\Omega$ *inside the cylinder*

$$\mathcal{C}(x_0, r) := \big\{x_0 + x' + t\vec{n}_{x_0,r} : x' \in H(x_0, r),\; |x'| < r,\; |t| < r\big\} \tag{2.365}$$

in the precise sense described below:

First, with $\triangle$ *denoting the symmetric set-theoretic difference and with* υ_{n-1} *denoting the volume of the unit ball in* $\mathbb{R}^{n-1}$,

$$\mathcal{H}^{n-1}\big(\mathcal{C}(x_0, r) \cap \big(\partial\Omega \triangle \mathcal{G}\big)\big) \leq C_1 \upsilon_{n-1} r^{n-1} e^{-C_2\phi(\delta)/\delta}. \tag{2.366}$$

Second, there exist two disjoint σ*-measurable subsets of* $\partial\Omega$, *call them* $G(x_0, r)$ *and* $E(x_0, r)$, *such that*

$$\mathcal{C}(x_0, r) \cap \partial\Omega = G(x_0, r) \cup E(x_0, r), \tag{2.367}$$

$$G(x_0, r) \subseteq \mathcal{G}, \quad \sigma\big(E(x_0, r)\big) \le C_1 \upsilon_{n-1} r^{n-1} e^{-C_2\phi(\delta)/\delta}. \tag{2.368}$$

Third, if $\Pi : \mathbb{R}^n \to H(x_0, r)$ *is defined by* $\Pi(x) := x'$ *for* $x = x_0 + x' + t\vec{n}_{x_0,r} \in \mathbb{R}^n$ *with* $x' \in H(x_0, r)$ *and* $t \in \mathbb{R}$*, then*

$$\begin{gathered}\big|x - \big(x_0 + \Pi(x) + h(\Pi(x))\vec{n}_{x_0,r}\big)\big| \le 2C_0\phi(\delta) \cdot dist\big(\Pi(x), \Pi(G(x_0, r))\big)\\ \text{for each point } \ x \in E(x_0, r),\end{gathered} \tag{2.369}$$

and

$$C(x_0, r) \cap \partial\Omega \subseteq \big\{x_0 + x' + t\vec{n}_{x_0,r} : |t| \le C_0\delta r, \ x' \in H(x_0, r)\big\}, \tag{2.370}$$

$$\Pi\big(C(x_0, r) \cap \partial\Omega\big) = \big\{x' \in H(x_0, r) : |x'| < r\big\}. \tag{2.371}$$

Fourth, if

$$\begin{aligned}C^+(x_0, r) &:= \big\{x_0 + x' + t\,\vec{n}_{x_0,r} : x' \in H(x_0, r), \ |x'| < r, \ -r < t < -C_0\delta\, r\big\},\\ C^-(x_0, r) &:= \big\{x_0 + x' + t\,\vec{n}_{x_0,r} : x' \in H(x_0, r), \ |x'| < r, \ C_0\delta\, r < t < r\big\},\end{aligned} \tag{2.372}$$

(*having* $0 < \delta < \delta_* < 1/C_0$ *ensures that* $C^\pm \neq \varnothing$) *then*

$$C^+(x_0, r) \subseteq \Omega \ \text{ and } \ C^-(x_0, r) \subseteq \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.373}$$

Fifth,

any line in the direction of $\vec{n}_{x_0,r}$ *passing through a point on* $G(x_0, r)$ *intersects* $\partial\Omega \cap C(x_0, r)$ *only at said point.* (2.374)

Sixth, with $\Delta(x_0, r) := B(x_0, r) \cap \partial\Omega$ *one has*

$$\begin{aligned}&\Big(1 - C_3\delta - C_1 exp\big(-C_2\phi(\delta)/\delta\big)\Big)\upsilon_{n-1} r^{n-1}\\ &\qquad \le \sigma\big(\Delta(x_0, r)\big) \le \Big(1 + C_3\phi(\delta) + C_1 exp\big(-C_2\phi(\delta)/\delta\big)\Big)\upsilon_{n-1} r^{n-1}.\end{aligned} \tag{2.375}$$

Finally, if $\widetilde{\nu}$ *is the unit normal vector to the Lipschitz graph* $\mathcal{G}$*, pointing toward the upper-graph of the function h then*

at $\mathcal{H}^{n-1}$*-a.e. point* $x \in \partial\Omega \cap \mathcal{G}$ *one has either* $\nu(x) = \widetilde{\nu}(x)$ *or* $\nu(x) = -\widetilde{\nu}(x)$*,* (2.376)

$$\nu(x) = \widetilde{\nu}(x) \ \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } \ x \in G(x_0, r), \tag{2.377}$$

$$\sigma\Big(\big\{x \in \mathcal{G} \cap \Delta(x_0, 4r) : \nu(x) = -\widetilde{\nu}(x)\big\}\Big) \leq C_4 \cdot \phi(\delta) r^{n-1}, \tag{2.378}$$

and

$$\fint_{\Delta(x_0,4r)} \Big(\sup_{y\in\mathcal{G}} |\nu(x) - \widetilde{\nu}(y)|\Big)\, d\sigma(x) \leq C_4 \cdot \phi(\delta). \tag{2.379}$$

Before proving Theorem 2.6 we make a remark and record one of its immediate consequences in Corollary 2.3 below.

Remark 2.3 It is well known (cf., e.g., [47, Theorem 1, p. 251]) that there exists some $C_n \in (0, \infty)$ with the property that for each real-valued Lipschitz function $h : H \to \mathbb{R}$, where H is a hyperplane in $\mathbb{R}^n$ and each given $\varepsilon > 0$ there exists $\widetilde{h} \in \mathscr{C}^1(H)$ with Lipschitz constant no larger than C_n times the Lipschitz constant of h such that

$$\mathcal{H}^{n-1}\Big(\big\{x \in H : h(x) \neq \widetilde{h}(x) \ \text{ or } \ (\nabla h)(x) \neq (\nabla\widetilde{h})(x)\big\}\Big) < \varepsilon. \tag{2.380}$$

Based on this, Theorem 2.6 is readily seen to self-improve to a version of itself in which the function in (2.363) is, additionally, of class $\mathscr{C}^1$.

Here is the corollary of Theorem 2.6 alluded to earlier.

Corollary 2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then there exists some $C \in (0, \infty)$ which depends only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\sup_{x\in\partial\Omega,\, r>0} \left| \frac{\sigma\big(\Delta(x, r)\big)}{\upsilon_{n-1} r^{n-1}} - 1 \right| \leq C \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \Big(1 - \ln \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n}\Big) \tag{2.381}$$

where υ_{n-1} stands for the volume of the unit ball in $\mathbb{R}^{n-1}$.

Proof In the context of (2.375) choose

$$\begin{aligned} \phi : (0, 1) \to (0, \infty) \ \text{ given for each } \ t \in (0, 1) \\ \text{by } \ \phi(t) := C_2^{-1} t \ln(1/t). \end{aligned} \tag{2.382}$$

This proves that there exists a threshold $\delta_* \in (0, 1)$, depending only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$, such that whenever (2.362) holds it follows that for each $x \in \partial\Omega$ and each $r > 0$ we have

$$\begin{aligned} \big(1 - (C_1 + C_3)\delta\big)\upsilon_{n-1} r^{n-1} &\leq \sigma\big(\Delta(x, r)\big) \\ &\leq \big(1 + (C_3/C_2)\delta \ln(1/\delta) + C_1\delta\big)\upsilon_{n-1} r^{n-1}. \end{aligned} \tag{2.383}$$

After sending $\delta \searrow \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$, this readily implies the estimate claimed in (2.381) (with $C := \max\{C_3/C_2, C_1 + C_3\}$) in this case. Finally, (2.381) is a simple consequence of the upper Ahlfors regularity of $\partial\Omega$ when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \delta_*$. □

We shall establish Theorem 2.6 by reasoning along the lines of the argument in [61, pp. 2703–2709], with (2.362) replacing the small local BMO assumption (which, in particular, frees us from having to restrict x_0 to a compact subset of $\partial\Omega$). A key observation is that, in the present context, the parameter R_* from [61, Theorem 4.16, p. 2701] (which limits the size of the scale r) may be taken to be $+\infty$.

Proof of Theorem 2.6 Throughout, for each given point $x \in \partial\Omega$ and each given radius $R > 0$ we agree to abbreviate $\Delta(x, R) := B(x, R) \cap \partial\Omega$ and also use the notation $\nu_{\Delta(x,R)} := ⨍_{\Delta(x,R)} \nu \, d\sigma$.

Assume (2.362) holds for some $\delta \in (0, \delta_*)$ with $\delta_* \in (0, 1/10)$, a threshold on which we are going to impose a number of other smallness conditions, to be specified later. For now, we note that Lemma 2.8 guarantees that $\partial\Omega$ is an unbounded set, and that

$$1 \geq \left| ⨍_{\Delta} \nu \, d\sigma \right| \geq \frac{9}{10} \quad \text{for each surface ball } \Delta \subseteq \partial\Omega. \tag{2.384}$$

Recall that the constant $C_* \in (0, \infty)$ appearing in the statement of Proposition 2.15 is controlled solely in terms of the Ahlfors regularity constant of $\partial\Omega$ and the dimension n. Keeping this in mind, from (2.349) used with $\lambda = 4$ we see that

$$\sup_{R>0} \sup_{x\in\partial\Omega} \sup_{y\in\Delta(x,4R)} R^{-1} |\langle x - y, \nu_{\Delta(x,R)} \rangle| \leq 12 C_* \delta \tag{2.385}$$

with $C_* \in (0, \infty)$ depending only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n. Choose

$$C_0 := \max\left\{14C_* + 4, 60C_*\right\} \tag{2.386}$$

and, for the remainder of the proof, make the assumption that

$$\delta_* \in \big(0, \min\{1/10, 1/C_0\}\big) \tag{2.387}$$

and that δ_* is also small enough, depending on ϕ, so that

$$\delta \leq \phi(\delta) \leq (14C_* + 4)^{-1} \quad \text{for all} \quad \delta \in (0, \delta_*). \tag{2.388}$$

That (2.388) may be accommodated is ensured by (2.361). (The choice made in (2.386) as well as the nature of the right-most expression in (2.388) are dictated by future considerations; see (2.418).)

To proceed, for each $x \in \partial\Omega$ and $R > 0$ set

$$\nu^*_{x,R}(y) := \sup_{0<\rho<R} \fint_{\Delta(y,\rho)} |\nu(z) - \nu_{\Delta(x,2R)}|\, d\sigma(z), \qquad \forall\, y \in \partial\Omega. \tag{2.389}$$

Then (2.389) implies that for each $x \in \partial\Omega$ and $R > 0$ we have

$$\nu^*_{x,R}(y) \leq \mathcal{M}\Big(|\nu - \nu_{\Delta(x,2R)}| \cdot \mathbf{1}_{\Delta(x,2R)}\Big)(y), \qquad \forall\, y \in \Delta(x, R), \tag{2.390}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator on $\partial\Omega$. For further reference let us also note that Lebesgue's Differentiation Theorem and (2.389) imply that

$$\begin{gathered}\text{for each fixed } x \in \partial\Omega \text{ and } R > 0 \text{ we have}\\ |\nu(y) - \nu_{\Delta(x,2R)}| \leq \nu^*_{x,R}(y) \ \text{ for } \sigma\text{-a.e. } \ y \in \partial\Omega.\end{gathered} \tag{2.391}$$

Henceforth, fix a location $x_0 \in \partial\Omega$ along with a scale $r > 0$. From (2.384) we know that

$$\tfrac{9}{10} \leq |\nu_{\Delta(x_0,2r)}| \leq 1. \tag{2.392}$$

We may also conclude from (2.384) that

$$\vec{n}_{x_0,r} := \frac{\nu_{\Delta(x_0,4r)}}{|\nu_{\Delta(x_0,4r)}|} \tag{2.393}$$

is a well-defined unit vector in $\mathbb{R}^n$. Consider

$$H(x_0, r) := \big\{x \in \mathbb{R}^n : \langle x, \vec{n}_{x_0,r}\rangle = 0\big\} \tag{2.394}$$

and introduce a new system of coordinates in $\mathbb{R}^n$ by setting

$$x = (\zeta, t) \Longleftrightarrow x = x_0 + t\,\vec{n}_{x_0,r} + \zeta, \quad t \in \mathbb{R}, \ \zeta \in H(x_0, r). \tag{2.395}$$

We agree to write $\zeta(x), t(x)$ in place of ζ, t whenever necessary to stress the dependence of the new coordinates on the point $x \in \mathbb{R}^n$. Let us also define the projection

$$\Pi : \mathbb{R}^n \to H(x_0, r) \ \text{ with } \ \Pi(x) := \zeta \ \text{ for each } \ x = (\zeta, t) \in \mathbb{R}^n. \tag{2.396}$$

Finally, consider the cylinder $C(x_0, r)$ defined as in (2.365) and, with the function ϕ as in (2.360)–(2.361), introduce

$$G(x_0, r) := \big\{x \in C(x_0, r) \cap \partial\Omega : \, \nu^*_{x_0,2r}(x) \leq \phi(\delta)\big\},$$
$$E(x_0, r) := \big(C(x_0, r) \cap \partial\Omega\big) \setminus G(x_0, r). \tag{2.397}$$

Since $C(x_0, r) \subseteq B(x_0, \sqrt{2}r)$ (as seen from its definition), it follows from (2.397) that $G(x_0, r)$, $E(x_0, r)$ are disjoint σ-measurable subsets of $\Delta(x_0, \sqrt{2}r)$, satisfying $G(x_0, r) \cup E(x_0, r) = C(x_0, r) \cap \partial\Omega$. In particular, (2.367) holds.

Next, we claim that there exist $c, C \in (0, \infty)$, which depend only on n and the Ahlfors regularity constant of $\partial\Omega$, with the property that

$$⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,2r}\big)\,\mathrm{d}\sigma \leq C. \tag{2.398}$$

Granted this, we may then conclude that

$$\exp\big(c\,\phi(\delta)/\delta\big)\,\frac{\sigma\big(E(x_0, r)\big)}{\sigma\big(\Delta(x_0, 2r)\big)} \leq \frac{1}{\sigma\big(\Delta(x_0, 2r)\big)} \int_{E(x_0,r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,2r}\big)\,\mathrm{d}\sigma \leq C. \tag{2.399}$$

This implies

$$\sigma\big(E(x_0, r)\big) \leq C\exp\big(-c\,\phi(\delta)/\delta\big)\sigma\big(\Delta(x_0, 2r)\big)$$
$$\leq 2^{n-1} C_A C\, r^{n-1} \cdot \exp\big(-c\,\phi(\delta)/\delta\big), \tag{2.400}$$

where C_A is the Ahlfors regularity constant of $\partial\Omega$. In particular, the estimate claimed in (2.368) follows as long as

$$C_2 := c \quad \text{and} \quad C_1 \geq 2^{n-1} C_A C/\upsilon_{n-1}. \tag{2.401}$$

To justify the claim made in (2.398), let us abbreviate

$$f := \mathcal{M}\big(|\nu - \nu_{\Delta(x_0,4r)}| \cdot \mathbf{1}_{\Delta(x_0,4r)}\big) \tag{2.402}$$

and note that, thanks to (2.390) with $R := 2r$, this entails

$$\nu^*_{x_0,2r}(x) \leq f(x) \quad \text{whenever} \quad x \in \Delta(x_0, 2r). \tag{2.403}$$

We also make the sub-claim that there exist $A_1, A_2 \in (0, \infty)$, depending only on n and the Ahlfors regularity constant of $\partial\Omega$, such that for each $p \in [1, \infty)$ we have

$$⨍_{\Delta(x_0,4r)} |\nu(x) - \nu_{\Delta(x_0,4r)}|^p\,\mathrm{d}\sigma(x) \leq A_1 \Gamma(p+1)\Big(A_2 \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^p, \tag{2.404}$$

where $\Gamma(t) := \int_0^\infty \lambda^{t-1} \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda$ for all $t \in (0, \infty)$ is the classical Gamma function. Taking this inequality for granted for the time being, we now proceed to show that

(2.398) holds for the choice

$$c := 2^{-1} A_2^{-1} \in (0, \infty) \tag{2.405}$$

and with $C \in (0, \infty)$ to be determined momentarily (see (2.409)). To implement this plan, use (2.403) plus a change of variables, then expand the exponential function into an infinite power series to write

$$\begin{aligned}
&⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,r}\big)\,\mathrm{d}\sigma \leq ⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,f\big)\,\mathrm{d}\sigma \qquad (2.406)\\
&= \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_0^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) :\ \exp\big(c\,\delta^{-1}\,f(x)\big) > \lambda\big\}\Big)\,\mathrm{d}\lambda\\
&\leq 1 + \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_1^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) :\ \exp\big(c\,\delta^{-1}\,f(x)\big) > \lambda\big\}\Big)\,\mathrm{d}\lambda\\
&= 1 + \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_0^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) :\ c\,\delta^{-1}\,f(x) > s\big\}\Big)\mathrm{e}^s\,\mathrm{d}s\\
&\leq e + \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \sum_{k=0}^\infty \frac{1}{k!} \int_1^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) :\ f(x) > s\,\delta/c\big\}\Big)s^k\,\mathrm{d}s.
\end{aligned}$$

To continue, fix an arbitrary integrability exponent $p \in [2, \infty)$ along with an arbitrary number $s \in (0, \infty)$. Chebysheff's inequality, the L^p-boundedness of the Hardy–Littlewood maximal operator (with bounds independent of p, as seen by interpolation), and (2.402) then allow us to estimate

$$\begin{aligned}
&\frac{\sigma\Big(\{x \in \Delta(x_0,2r) :\ f(x) > s\,\delta/c\}\Big)}{\sigma\big(\Delta(x_0,2r)\big)}\\
&\qquad \leq \Big(\frac{c}{s\,\delta}\Big)^p ⨍_{\Delta(x_0,2r)} f(x)^p\,\mathrm{d}\sigma(x)\\
&\qquad \leq \Big(\frac{c}{s\,\delta}\Big)^p \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_{\partial\Omega} \mathcal{M}\big(|\nu - \nu_{\Delta(x_0,4r)}| \cdot \mathbf{1}_{\Delta(x_0,4r)}\big)(x)^p\,\mathrm{d}\sigma(x)\\
&\qquad \leq \Big(\frac{c}{s\,\delta}\Big)^p \frac{C'}{\sigma\big(\Delta(x_0,2r)\big)} \int_{\partial\Omega} \big(|\nu(x) - \nu_{\Delta(x_0,4r)}| \cdot \mathbf{1}_{\Delta(x_0,4r)}(x)\big)^p\,\mathrm{d}\sigma(x)\\
&\qquad \leq C''\Big(\frac{c}{s\,\delta}\Big)^p ⨍_{\Delta(x_0,4r)} |\nu(x) - \nu_{\Delta(x_0,4r)}|^p\,\mathrm{d}\sigma(x), \qquad (2.407)
\end{aligned}$$

where $C', C'' \in (0, \infty)$ depend only on n and the Ahlfors regularity constant of $\partial\Omega$. Combine (2.404), (2.407), (2.405) and recall that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ to obtain

$$\begin{aligned}\frac{\sigma\Big(\{x \in \Delta(x_0, 4r) : f(x) > s\,\delta/c\}\Big)}{\sigma\big(\Delta(x_0, 4r)\big)} &\le C'' A_1 \Gamma(p+1)\Big(\frac{c\,A_2}{s}\Big)^p \\ &= C'' A_1 \Gamma(p+1)\Big(\frac{1}{2s}\Big)^p, \end{aligned} \tag{2.408}$$

for each $p \in [2, \infty)$ and each $s \in (0, \infty)$. Utilizing (2.408), in which we take $p := k + 2$ with $k = 0, 1, \dots,$ back into (2.406) then yields (upon noting that $\Gamma(k+3) = (k+2)!$)

$$\begin{aligned}&⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,4r}\big)\,\mathrm{d}\sigma \\ &\qquad \le e + C'' A_1 \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2^{k+2}}\Big(\int_1^\infty \frac{\mathrm{d}s}{s^2}\Big) =: C < \infty. \end{aligned} \tag{2.409}$$

This finishes the proof of (2.398), modulo that of (2.404). As regards the latter, we use following the John-Nirenberg level set estimate with exponential bound from (2.94). This ensures that there exist some large constant $A \in (0, \infty)$ and some small constant $a \in (0, \infty)$, both depending only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n, such that

$$\frac{\sigma\Big(\{x \in \Delta(x_0, 4r) : |\nu(x) - \nu_{\Delta(x_0,4r)}| > \lambda\}\Big)}{\sigma\big(\Delta(x_0, 4r)\big)} \le A \cdot \exp\Big(\frac{-a\lambda}{\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}}\Big) \tag{2.410}$$

for every $\lambda > 0$. In turn, (2.410) and a natural change of variables permit us to write

$$\begin{aligned}&⨍_{\Delta(x_0,4r)} |\nu(x) - \nu_{\Delta(x_0,4r)}|^p\,\mathrm{d}\sigma(x) \\ &\qquad = p\int_0^\infty \lambda^{p-1}\frac{\sigma\Big(\big\{x \in \Delta(x_0, 4r) : |\nu(x) - \nu_{\Delta(x_0,4r)}| > \lambda\big\}\Big)}{\sigma\big(\Delta(x_0, 4r)\big)}\,\mathrm{d}\lambda \\ &\qquad \le Ap\int_0^\infty \lambda^{p-1}\exp\Big(\frac{-a\lambda}{\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}}\Big)\,\mathrm{d}\lambda \\ &\qquad = Ap\Big(a^{-1}\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^p \int_0^\infty t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t \end{aligned}$$

$$= Ap\Gamma(p)\Big(a^{-1}\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^p. \tag{2.411}$$

Since $p\,\Gamma(p) = \Gamma(p+1)$, this justifies (2.404) with $A_1 := A$ and $A_2 := a^{-1}$, and concludes the proof of (2.398).

We now turn to the task of constructing the Lipschitz function h. As a preliminary matter, we remark that

$$\begin{gathered}|\langle x-y,\nu_{\Delta(x_0,4r)}\rangle| \leq \big(6C_*\delta + \nu^*_{x_0,2r}(x)\big)|x-y| \\ \text{for each } x\in\partial\Omega \text{ and } y\in\Delta(x,4r).\end{gathered} \tag{2.412}$$

To justify this, observe that (2.412) is trivially true when $x = y$, so it suffices to consider the case when $x \in \partial\Omega$ and $y \in \Delta(x, 4r)$ satisfy $x \neq y$. Assuming this is the case, based on (2.385) (used with $R := |x-y|/2 > 0$) and (2.389) we may write

$$\begin{aligned}|\langle x-y,\nu_{\Delta(x_0,4r)}\rangle| &\leq |\langle x-y,\nu_{\Delta(x,|x-y|/2)}\rangle| + |x-y||\nu_{\Delta(x,|x-y|/2)} - \nu_{\Delta(x_0,4r)}| \\ &\leq 6C_*\delta\,|x-y| + |x-y|\fint_{\Delta(x,|x-y|/2)}|\nu-\nu_{\Delta(x_0,4r)}|\,d\sigma \\ &\leq \big(6C_*\delta + \nu^*_{x_0,2r}(x)\big)|x-y|,\end{aligned} \tag{2.413}$$

as desired. Moving on, observe from (2.395) that

$$t(x) = \langle x-x_0,\vec{n}_{x_0,r}\rangle \quad \text{for each} \quad x\in\mathbb{R}^n. \tag{2.414}$$

In concert, (2.414), (2.392)–(2.393), (2.412), (2.397), and (2.388) then allow us to control

$$\begin{aligned}|t(x)-t(y)| &= |\langle x-y,\vec{n}_{x_0,r}\rangle| \leq \tfrac{10}{9}|\langle x-y,\nu_{\Delta(x_0,4r)}\rangle| \\ &\leq \tfrac{10}{9}\big(6C_*\delta+\phi(\delta)\big)|x-y| \\ &\leq \tfrac{10}{9}(6C_*+1)\phi(\delta)|x-y| \\ &\leq (7C_*+2)\phi(\delta)|x-y| \\ &\qquad \text{whenever } x\in G(x_0,r) \text{ and } y\in\Delta(x,4r).\end{aligned} \tag{2.415}$$

In turn, since for each $x, y \in \mathbb{R}^n$ we have (see (2.395))

$$\zeta(x)-\zeta(y) = x-y-\big(t(x)-t(y)\big)\vec{n}_{x_0,r}, \tag{2.416}$$

this permits us to estimate

$$|\zeta(x)-\zeta(y)| \geq |x-y| - |t(x)-t(y)| \geq \big(1-(7C_*+2)\phi(\delta)\big)|x-y|,$$
$$\text{for each } x \in G(x_0, r) \text{ and each } y \in \Delta(x, 4r). \tag{2.417}$$

Combining (2.415) and (2.417) (while keeping (2.388) and (2.386) in mind) then proves that

$$\begin{aligned} |t(x)-t(y)| &\leq \frac{(7C_*+2)\phi(\delta)}{1-(7C_*+2)\phi(\delta)}|\zeta(x)-\zeta(y)| \\ &\leq (14C_*+4)\phi(\delta)|\zeta(x)-\zeta(y)| \\ &\leq C_0\phi(\delta)|\zeta(x)-\zeta(y)|, \end{aligned} \tag{2.418}$$
$$\text{for each } \ x \in G(x_0, r) \ \text{ and } \ y \in \Delta(x, 4r).$$

We now claim that

$$\begin{aligned} &\text{if } x \in C(x_0, r) \cap \partial\Omega \text{ and } \Pi(x) \in \Pi\big(G(x_0, r)\big) \\ &\text{then } x \in G(x_0, r). \end{aligned} \tag{2.419}$$

Indeed, assume $x \in C(x_0, r) \cap \partial\Omega$ and $y \in G(x_0, r)$ are such that $\Pi(x) = \Pi(y)$. In view of (2.396), the latter condition means $\zeta(x) = \zeta(y)$. Since $x, y \in C(x_0, r)$, it follows that $|y-x| \leq \operatorname{diam}\big(C(x_0, r)\big) = 2\sqrt{2}r < 4r$, hence $x \in \Delta(y, 4r)$. As such, we may invoke (2.418) (with the roles of x and y reversed) to conclude that $t(x) = t(y)$. Thus, $x = \big(\zeta(x), t(x)\big) = \big(\zeta(y), t(y)\big) = y \in G(x_0, r)$, ultimately proving (2.419).

As a consequence of the proof of (2.419) we also see that

$$\text{the projection } \Pi \text{ is one-to-one on } G(x_0, r). \tag{2.420}$$

In turn, (2.420) guarantees that the mapping

$$\begin{aligned} &h : \Pi\big(G(x_0, r)\big) \longrightarrow \mathbb{R} \ \text{ given by} \\ &h\big(\zeta(x)\big) := t(x) \ \text{ for each } \ x \in G(x_0, r) \end{aligned} \tag{2.421}$$

is well defined. By (2.418), this mapping satisfies a Lipschitz condition with constant $\leq C_0\phi(\delta)$ on the set $\Pi\big(G(x_0, r)\big)$. Indeed, given any $x, y \in G(x_0, r)$, the fact that $G(x_0, r) \subseteq \Delta(x_0, \sqrt{2}r)$ implies $|x-y| < 2\sqrt{2}r < 4r$, hence $y \in \Delta(x, 4r)$. As such, (2.418) applies and, in view of (2.421), proves that

$$|h(x') - h(y')| \leq C_0\phi(\delta)|x'-y'| \ \text{ for each } \ x', y' \in \Pi\big(G(x_0, r)\big).$$

We may therefore extend h (using Kirszbraun's theorem; see, e.g., the discussion in [108]) as a Lipschitz function, which we continue to denote by h, to the entire hyperplane $H(x_0, r)$, with Lipschitz constant $\le C_0\phi(\delta)$. Note that its graph $\mathcal{G}$, considered in the (ζ, t)-system of coordinates introduced in (2.395), contains the set

$$\{(\zeta(x), t(x)) : x \in G(x_0, r)\} = G(x_0, r). \tag{2.422}$$

This proves the inclusion in (2.368). Together, (2.368) and (2.419) also prove that

$$\begin{aligned} &\text{if } x \in C(x_0, r) \cap \partial\Omega \text{ and } \Pi(x) \in \Pi\big(G(x_0, r)\big) \\ &\qquad\qquad \text{then } x \in \mathcal{G}. \end{aligned} \tag{2.423}$$

In turn, the above property implies the claim made in (2.374). Specifically, assume $x \in G(x_0, r)$ and $y \in C(x_0, r) \cap \partial\Omega$ are such that $\Pi(y) = \Pi(x)$. Then $\Pi(y)$ belongs to $\Pi\big(G(x_0, r)\big)$ which, by virtue of (2.419), places y in $G(x_0, r)$. In particular, $x, y \in \mathcal{G}$ (cf. (2.368)) have the same projection. Thus, necessarily, $x = y$ since otherwise the Vertical Line Test would be violated for the graph $\mathcal{G}$.

To prove the inclusion claimed in (2.370), start by considering some arbitrary point $x \in C(x_0, r) \cap \partial\Omega$. Then x belongs to $B(x_0, \sqrt{2}r) \cap \partial\Omega = \Delta(x_0, \sqrt{2}r)$. Also, the convention made in (2.395) allows us to express $x = x_0 + t(x)\vec{n}_{x_0,r} + \zeta(x)$, with $\zeta(x) \in H(x_0, r)$ satisfying $|\zeta(x)| < r$ (given that $x \in C(x_0, r)$) and with

$$t(x) = \langle x - x_0, \vec{n}_{x_0,r}\rangle = \frac{\langle x - x_0, \nu_{\Delta(x_0,4r)}\rangle}{|\nu_{\Delta(x_0,4r)}|}, \tag{2.424}$$

thanks to (2.414) and (2.393). In turn, (2.424), (2.392), and (2.385) (presently used with $R := 4r$, $x := x_0$, $y := x$) permit us to estimate

$$|t(x)| \le \frac{\big|\langle x - x_0, \nu_{\Delta(x_0,4r)}\rangle\big|}{|\nu_{\Delta(x_0,4r)}|} \le \tfrac{10}{9}(4r)12C_*\delta \le C_0\delta\, r, \tag{2.425}$$

since (2.386) guarantees that $C_0 \ge 60C_*$. The proof of (2.370) is therefore complete.

From (2.370) it follows that the connected sets $C^{\pm}(x_0, r)$ introduced in (2.372) do not intersect $\partial\Omega$. As such, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ form a disjoint, open cover of $C^{\pm}(x_0, r)$, hence

$$\begin{aligned} &C^+(x_0, r) \text{ is fully contained in either } \Omega_+ \text{ or } \Omega_-, \\ &\text{and also } C^-(x_0, r) \text{ is fully contained in either } \Omega_+ \\ &\text{or } \Omega_-. \end{aligned} \tag{2.426}$$

By further decreasing $\delta_* \in (0, 1)$ we may ensure (see Theorem 2.3) that

$$\Omega \text{ satisfies a two-sided local John condition with constants which depend only on the Ahlfors regularity constant of } \partial\Omega \text{ and the dimension } n. \tag{2.427}$$

In view of (2.427) and (2.87), it follows that Ω satisfies a two-sided cork screw condition (cf. Definition 2.10) for some parameter $\theta \in (0, 1)$ which depends only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n. Denote by $x_r^{\pm} \in \Omega_{\pm}$ the two corkscrew points corresponding to the location x_0 and scale r. In particular,

$$|x_r^{\pm} - x_0| < r \quad \text{and} \quad B\big(x_r^{\pm}, \theta r\big) \subseteq \Omega_{\pm}. \tag{2.428}$$

Assume $0 < \delta_* < \theta/C_0$ to begin with. Given that we are taking $\delta \in (0, \delta_*)$, this condition makes it impossible to contain either of the balls $B\big(x_r^+, \theta r\big)$, $B\big(x_r^-, \theta r\big)$ in the strip $\big\{x_0 + x' + t\vec{n}_{x_0,r} \,:\, |t| \leq C_0\delta r,\ x' \in H(x_0, r)\big\}$. Since, as seen from (2.428), their centers $x_r^{\pm}$ belong to $B(x_0, r) \subset C(x_0, r)$, in turn this forces one of the following four alternatives to be true:

$$B\big(x_r^+, \theta r\big) \cap C^+(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^-, \theta r\big) \cap C^+(x_0, r) \neq \varnothing, \tag{2.429}$$

$$B\big(x_r^+, \theta r\big) \cap C^-(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^-, \theta r\big) \cap C^-(x_0, r) \neq \varnothing, \tag{2.430}$$

$$B\big(x_r^+, \theta r\big) \cap C^+(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^-, \theta r\big) \cap C^-(x_0, r) \neq \varnothing, \tag{2.431}$$

$$B\big(x_r^+, \theta r\big) \cap C^-(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^-, \theta r\big) \cap C^+(x_0, r) \neq \varnothing. \tag{2.432}$$

Note that the alternative described in (2.429) cannot possibly hold. Indeed, the existence of two points $z_1 \in B\big(x_r^+, \theta r\big) \cap C^+(x_0, r)$ and $z_2 \in B\big(x_r^-, \theta r\big) \cap C^+(x_0, r)$ would imply that, on the one hand, the line segment $[z_1, z_2]$ lies in the convex set $C^+(x_0, r)$, hence also either in Ω_+ or in Ω_- by (2.426). Nonetheless, the fact that we have $z_1 \in B\big(x_r^+, \theta r\big) \subseteq \Omega_+$ and $z_2 \in B\big(x_r^-, \theta r\big) \subseteq \Omega_-$ prevents either one of these eventualities form materializing. This contradiction therefore excludes (2.429). Reasoning in a similar fashion we may rule out (2.430). When (2.431) holds, from the fact that $B\big(x_r^{\pm}, \theta r\big) \subseteq \Omega_{\pm}$ (cf. (2.428)) we conclude that

$$\varnothing \neq C^+(x_0, r) \cap B\big(x_r^+, \theta r\big) \subseteq B\big(x_r^+, \theta r\big) \subseteq \Omega_+ \tag{2.433}$$

hence $C^+(x_0, r) \cap \Omega_+ \neq \varnothing$ which, in light of (2.426), forces $C^+(x_0, r) \subseteq \Omega_+$. Similarly, $C^-(x_0, r) \subseteq \Omega_-$ so the inclusions in (2.373) hold as stated. Finally, when (2.432) holds, from (2.426) and (2.428) we deduce that $C^+(x_0, r) \subseteq \Omega_-$ and $C^-(x_0, r) \subseteq \Omega_+$. In such a scenario, we may ensure that the inclusions in (2.373) are valid simply by re-denoting $\vec{n}_{x_0,r}$ as $-\vec{n}_{x_0,r}$ (and considering the function $-h$ in place of the original h), which amounts to reversing the roles of $C^+(x_0, r)$ and $C^-(x_0, r)$ (without affecting the other properties). This concludes the proof of (2.373).

Next, observe that

$$\Pi\big(C(x_0,r)\cap\partial\Omega\big)\subseteq\Pi\big(C(x_0,r)\big)=\{\zeta\in H(x_0,r):\,|\zeta|<r\}. \tag{2.434}$$

The opposite inclusion fails only when there exists a line segment parallel to $\vec{n}_{x_0,r}$ whose two endpoints belong to $C^+(x_0,r)$ and to $C^-(x_0,r)$, respectively, and which does not intersect $\partial\Omega$ (here we implicitly use the fact that $C^{\pm}(x_0,r)\neq\varnothing$, itself a result of having imposed the condition that $0<\delta<\delta_*<1/C_0$; cf. (2.387)). However, (2.373) and simple connectivity arguments rule out this scenario, hence (2.371) is proved.

Going further, we note that (2.371) implies

$$\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\subseteq\Pi\big(E(x_0,r)\big). \tag{2.435}$$

The fact that $\Pi:\mathbb{R}^n\to H(x_0,r)$ is a Lipschitz function, with Lipschitz constant 1, implies (cf., e.g., [47, Theorem 1, p. 75]) that

$$\mathcal{H}^{n-1}\big(\Pi(S)\big)\leq\mathcal{H}^{n-1}(S)\quad\text{for each Borel set }\ S\subseteq\mathbb{R}^n. \tag{2.436}$$

Based on (2.435), (2.436), the definition $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, (2.367), and (2.400) we then conclude that

$$\begin{aligned}\mathcal{H}^{n-1}&\Big(\{\zeta\in H(x_0,r):|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\Big)\\&\leq\mathcal{H}^{n-1}\Big(\Pi\big(E(x_0,r)\big)\Big)\leq\mathcal{H}^{n-1}\big(E(x_0,r)\big)\\&\leq 2^{n-1}C_AC\,r^{n-1}\cdot\exp\big(-C_2\phi(\delta)/\delta\big).\end{aligned} \tag{2.437}$$

In addition, (2.419) gives

$$C(x_0,r)\cap\big(\mathcal{G}\setminus\partial\Omega\big)\subseteq\mathcal{G}\cap\Pi^{-1}\Big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\Big). \tag{2.438}$$

Keeping also in mind that

$$\begin{aligned}\mathcal{H}^{n-1}(S)\leq\sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\big(\Pi(S)\big),\\ \text{for each Borel set } S\subseteq\mathcal{G},\end{aligned} \tag{2.439}$$

(since $\mathcal{G}$ is the graph of a Lipschitz function), we deduce that

$$\begin{aligned}&\mathcal{H}^{n-1}\big(C(x_0,r)\cap(\mathcal{G}\setminus\partial\Omega)\big)\\&\leq\mathcal{H}^{n-1}\Big(\mathcal{G}\cap\Pi^{-1}\big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\big)\Big)\\&\leq\sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\Big(\Pi\Big(\mathcal{G}\cap\Pi^{-1}\big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\big)\Big)\Big)\end{aligned}$$

$$\le \sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\Big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\Big)$$

$$\le \sqrt{1+(C_0\phi(\delta))^2}\,2^{n-1}C_A C\,r^{n-1}\cdot\exp\big(-C_2\phi(\delta)/\delta\big)$$

$$\le \sqrt{1+C_0^2(14C_*+4)^{-2}}\,2^{n-1}C_A C\,r^{n-1}\cdot\exp\big(-C_2\phi(\delta)/\delta\big), \tag{2.440}$$

by (2.437) and (2.388). Upon observing that $C(x_0,r)\cap\big(\partial\Omega\setminus\mathcal{G}\big)$ is contained in $E(x_0,r)$, the estimate claimed in (2.366) now follows from (2.440) and (2.400) if we choose (recall that $C_2 := c$; cf. (2.401))

$$C_1 := \sqrt{1+C_0^2(14C_*+4)^{-2}}\,2^{n-1}C_A C/\upsilon_{n-1} \tag{2.441}$$

(a choice in line with the demand formulated in (2.401)), where C is as in (2.409), and where C_A is the Ahlfors regularity constant of $\partial\Omega$.

Let us now justify the proximity condition formulated in (2.369). To this end, fix $x\in E(x_0,r)=\big(C(x_0,r)\cap\partial\Omega\big)\setminus G(x_0,r)$ and pick an arbitrary $x^*\in G(x_0,r)$. In particular, $x,x^*\in C(x_0,r)$ hence $|x-x^*|<\operatorname{diam}\big(C(x_0,r)\big)=2\sqrt{2}r$. Given that we have $x^*\in G(x_0,r)$ and $x\in\Delta(x^*,4r)$, estimate (2.418) applies and presently gives

$$\big|t(x)-h\big(\Pi(x^*)\big)\big|=|t(x)-t(x^*)|\le C_0\phi(\delta)|\Pi(x)-\Pi(x^*)|. \tag{2.442}$$

Consequently, since $x=\big(\Pi(x),t(x)\big)$, we may write

$$\begin{aligned}\big|x-\big(\Pi(x),h(\Pi(x))\big)\big| &= \big|t(x)-h\big(\Pi(x)\big)\big|\\ &\le \big|t(x)-h\big(\Pi(x^*)\big)\big|+\big|h\big(\Pi(x^*)\big)-h\big(\Pi(x)\big)\big|\\ &\le 2C_0\phi(\delta)|\Pi(x)-\Pi(x^*)|, \end{aligned}\tag{2.443}$$

by (2.442) and the Lipschitz condition on h (cf. (2.363)). Taking the infimum over $x^*\in G(x_0,r)$ now yields (2.369).

Let us now deal with (2.375). Recall that υ_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$. Using (2.366) and (2.439) we may estimate

$$\sigma\big(\Delta(x_0,r)\big)=\mathcal{H}^{n-1}\big(B(x_0,r)\cap\partial\Omega\big)\le\mathcal{H}^{n-1}\big(C(x_0,r)\cap\partial\Omega\big)$$

$$\le\mathcal{H}^{n-1}\big(C(x_0,r)\cap\mathcal{G}\big)+\mathcal{H}^{n-1}\Big(C(x_0,r)\cap(\partial\Omega\setminus\mathcal{G})\Big)$$

$$\le\sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\big(\Pi(C(x_0,r)\cap\mathcal{G})\big)+\mathcal{H}^{n-1}\Big(C(x_0,r)\cap(\partial\Omega\triangle\mathcal{G})\Big)$$

$$\le\big(1+C_0\phi(\delta)\big)\,\mathcal{H}^{n-1}\big(\Pi(C(x_0,r))\big)+C_1\upsilon_{n-1}\,r^{n-1}\exp\big(-C_2\phi(\delta)/\delta\big)$$

$$= \Big(1 + C_0\phi(\delta) + C_1\exp\big(-C_2\phi(\delta)/\delta\big)\Big)\upsilon_{n-1}\, r^{n-1}. \tag{2.444}$$

Also, by employing (2.371), (2.436), (2.366), (2.439), (2.373), and (2.388) we may write

$$\begin{aligned}
\upsilon_{n-1}\, r^{n-1} &= \mathcal{H}^{n-1}\big(\{\zeta \in H(x_0, r) : |\zeta| < r\}\big) \leq \mathcal{H}^{n-1}\big(\mathcal{C}(x_0, r) \cap \partial\Omega\big) \qquad (2.445)\\
&= \mathcal{H}^{n-1}\big(B(x_0, r) \cap \partial\Omega\big) + \mathcal{H}^{n-1}\Big(\big(\mathcal{C}(x_0, r) \cap \partial\Omega\big) \setminus B(x_0, r)\Big)\\
&\leq \sigma\big(\Delta(x_0, r)\big) + \mathcal{H}^{n-1}\Big(\mathcal{C}(x_0, r) \cap \big(\partial\Omega \setminus \mathcal{G}\big)\Big)\\
&\quad + \mathcal{H}^{n-1}\Big(\big(\mathcal{C}(x_0, r) \cap \mathcal{G}\big) \setminus \big(B(x_0, r) \cup \mathcal{C}^{+}(x_0, r) \cup \mathcal{C}^{-}(x_0, r)\big)\Big)\\
&\leq \sigma\big(\Delta(x_0, r)\big) + \mathcal{H}^{n-1}\Big(\mathcal{C}(x_0, r) \cap \big(\partial\Omega \triangle \mathcal{G}\big)\Big)\\
&\quad + \sqrt{1 + (C_0\phi(\delta))^2}\,\upsilon_{n-1}\, r^{n-1}\Big(1 - \big(\sqrt{1 - C_0^2\delta^2}\big)^{n-1}\Big)\\
&\leq \sigma\big(\Delta(x_0, r)\big) + C_1\upsilon_{n-1}\, r^{n-1}\exp\big(-C_2\phi(\delta)/\delta\big) + C_3\delta\,\upsilon_{n-1}\, r^{n-1},
\end{aligned}$$

where $C_3 := C_n\, C_0\sqrt{1 + C_0^2(14C_* + 4)^{-2}}$ with $C_n \in [1, \infty)$ depending only on the dimension n. This further implies

$$\Big(1 - C_3\delta - C_1\exp\big(-C_2\phi(\delta)/\delta\big)\Big)\,\upsilon_{n-1}\, r^{n-1} \leq \sigma\big(\Delta(x_0, r)\big). \tag{2.446}$$

Now, (2.375) follows from (2.444), (2.446), and (2.388).

Next, (2.376) is a direct consequence of Proposition 2.2 applied to Ω and the upper-graph of the function h (both of which are Ahlfors regular domains). There remains to prove the claims made in (2.377) and (2.378). To get started, we make two observations. First, (2.362) implies

$$\fint_{\Delta(x_0,4r)} |\nu - \nu_{\Delta(x_0,4r)}|\, d\sigma \leq \delta. \tag{2.447}$$

Second, at σ-a.e. point on $\partial\Omega$ we may estimate

$$|\nu - \vec{n}_{x_0,r}| \leq |\nu - \nu_{\Delta(x_0,4r)}| + |\nu_{\Delta(x_0,4r)} - \vec{n}_{x_0,r}| \tag{2.448}$$

and, thanks to (2.393), the fact that $|\nu| = 1$ at σ-a.e. point on $\partial\Omega$, and the reverse triangle inequality, we have

$$
\begin{aligned}
|\nu_{\Delta(x_0,4r)} - \vec{n}_{x_0,r}| &= \left|\nu_{\Delta(x_0,4r)} - \frac{\nu_{\Delta(x_0,4r)}}{|\nu_{\Delta(x_0,4r)}|}\right| = \left|\Big(1 - \frac{1}{|\nu_{\Delta(x_0,4r)}|}\Big)\nu_{\Delta(x_0,4r)}\right| \\
&= \left|1 - \frac{1}{|\nu_{\Delta(x_0,4r)}|}\right|\,\big|\nu_{\Delta(x_0,4r)}\big| = \big|1 - |\nu_{\Delta(x_0,4r)}|\big| \\
&= \big||\nu| - |\nu_{\Delta(x_0,4r)}|\big| \leq \big|\nu - \nu_{\Delta(x_0,4r)}\big|. \qquad (2.449)
\end{aligned}
$$

By combining (2.448) with (2.449) we arrive at the conclusion that

$$
|\nu - \vec{n}_{x_0,r}| \leq 2|\nu - \nu_{\Delta(x_0,4r)}| \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \qquad (2.450)
$$

Recall that $\widetilde{\nu}$ denotes the unit normal vector to the Lipschitz graph $\mathcal{G}$, pointing toward the upper-graph of the function h. This is well-defined at $\mathcal{H}^{n-1}$-a.e. point on $\mathcal{G}$, and we claim that

$$
|\widetilde{\nu} - \vec{n}_{x_0,r}| \leq C_0\phi(\delta) \ \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \mathcal{G}. \qquad (2.451)
$$

To justify this, after performing a rotation, there is no loss of generality in assuming that

$$
\vec{n}_{x_0,r} = \mathbf{e}_n = (0,\dots,0,1) \in \mathbb{R}^n. \qquad (2.452)
$$

Then the hyperplane

$$
H(x_0,r) = \langle \vec{n}_{x_0,r}\rangle^{\perp} = \langle \mathbf{e}_n\rangle^{\perp} = \mathbb{R}^{n-1}\times\{0\} \qquad (2.453)
$$

may be canonically identified with $\mathbb{R}^{n-1}$, a scenario in which

$$
\widetilde{\nu}\big(x', h(x')\big) = \frac{\big(-(\nabla' h)(x'), 1\big)}{\sqrt{1 + |(\nabla' h)(x')|^2}} \ \text{ for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1}, \qquad (2.454)
$$

where ∇' denotes the gradient operator in $\mathbb{R}^{n-1}$. From (2.452) and (2.454) we then see that at $\mathcal{H}^{n-1}$-a.e. point $x \in \mathcal{G}$, say $x = \big(x', h(x')\big)$ with $x' \in \mathbb{R}^{n-1}$, we have

$$
\begin{aligned}
\big|\widetilde{\nu}(x) - \vec{n}_{x_0,r}\big|^2 &= 2 - 2\langle \widetilde{\nu}(x), \vec{n}_{x_0,r}\rangle = 2\Bigg(1 - \frac{1}{\sqrt{1 + |(\nabla' h)(x')|^2}}\Bigg) \\
&= \frac{2|(\nabla' h)(x')|^2}{1 + |(\nabla' h)(x')|^2 + \sqrt{1 + |(\nabla' h)(x')|^2}} \\
&\leq |(\nabla' h)(x')|^2 \leq \big(C_0\phi(\delta)\big)^2, \qquad (2.455)
\end{aligned}
$$

where the last inequality comes from (2.363). Ultimately, this establishes (2.451).

Collectively, (2.450) and (2.451) prove that

$$|\nu - \widetilde{\nu}| \leq 2|\nu - \nu_{\Delta(x_0,4r)}| + C_0\phi(\delta) \ \text{ at } \sigma\text{-a.e. point on } \ \mathcal{G} \cap \partial\Omega. \tag{2.456}$$

From (2.391) and (2.397) we also see that

$$|\nu(x) - \nu_{\Delta(x_0,4r)}| \leq \nu^*_{x_0,2r}(x) \leq \phi(\delta) \ \text{ for } \sigma\text{-a.e. } \ x \in G(x_0, r). \tag{2.457}$$

Combining (2.456) with (2.457) and keeping in mind that $G(x_0, r) \subseteq \mathcal{G} \cap \partial\Omega$ leads to the conclusion that

$$|\nu - \widetilde{\nu}| \leq (2 + C_0)\phi(\delta) \ \text{ at } \sigma\text{-a.e. point on } \ G(x_0, r). \tag{2.458}$$

If $\delta_* > 0$ is taken small enough so that $\phi(t) < 2(2 + C_0)^{-1}$ for all $t \in (0, \delta_*)$ (something that may always be arranged, thanks to (2.361)), we conclude from (2.458) and (2.376) (again, mindful of the fact that $G(x_0, r) \subseteq \mathcal{G} \cap \partial\Omega$) that

$$\nu(x) = \widetilde{\nu}(x) \ \text{ at } \sigma\text{-a.e. point } \ x \in G(x_0, r). \tag{2.459}$$

This proves (2.377).

Let us now deal with (2.378). Together, (2.376), (2.456), (2.447), and the first inequality in (2.388) yield

$$\begin{aligned} \sigma\Big(\big\{x \in \mathcal{G} \cap \Delta(x_0, 4r) : \nu(x) = -\widetilde{\nu}(x)\big\}\Big) &= \frac{1}{2}\int_{\mathcal{G}\cap\Delta(x_0,4r)} |\nu - \widetilde{\nu}|\, d\sigma \\ &\leq \big(\delta + 2^{-1}C_0 \cdot \phi(\delta)\big) \cdot \sigma\big(\Delta(x_0, 4r)\big) \\ &\leq C_4 \cdot \phi(\delta) r^{n-1} \end{aligned} \tag{2.460}$$

provided $C_4 := 4^{n-1}(1 + 2^{-1}C_0)C_A$, where C_A is the Ahlfors regularity constant of Ω. Hence, (2.378) is established.

There remains to prove (2.379). To this end, combine (2.450) and (2.451) to obtain

$$\begin{gathered} \sup_{y\in\mathcal{G}} |\nu(x) - \widetilde{\nu}(y)| \leq 2|\nu(x) - \nu_{\Delta(x_0,4r)}| + C_0\phi(\delta) \\ \text{at } \sigma\text{-a.e. point } \ x \in \partial\Omega. \end{gathered} \tag{2.461}$$

Based on (2.461), (2.447), and (2.388) we then conclude that

$$⨍_{\Delta(x_0,4r)} \Big(\sup_{y\in\mathcal{G}} |\nu(x) - \widetilde{\nu}(y)|\Big)\, d\sigma(x) \leq 2\delta + C_0\phi(\delta) \leq C_4 \cdot \phi(\delta), \tag{2.462}$$

since our earlier choice of C_4 ensures that $C_4 \geq 2 + C_0$. This justifies (2.379), so the proof of Theorem 2.6 is now complete. $\square$

2.6 Chord-Arc Domains in the Plane

In the two-dimensional setting, an important category of sets is the class of chord-arc domains, discussed next.

Definition 2.16 Given a nonempty, proper, open subset Ω of $\mathbb{R}^2$ and $\varkappa \in [0, \infty)$, one calls Ω a $\varkappa$-`CAD` (or simply `chord-arc domain`, if the value of $\varkappa$ is not important) provided $\partial\Omega$ is a locally rectifiable simple curve, which is either a closed curve or a Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$, with the property that

$$\ell(z_1, z_2) \leq (1+\varkappa)|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \partial\Omega, \tag{2.463}$$

where $\ell(z_1, z_2)$ denotes the length of the shortest arc of $\partial\Omega$ joining z_1 and z_2.

For example, a planar sector Ω_θ of full aperture $\theta \in (0, 2\pi)$ (cf. (2.289)) is a $\varkappa$-CAD with constant $\varkappa := [\sin(\theta/2)]^{-1} - 1$. While Proposition 2.13 shows that the upper-graph of any real-valued BMO_1 function defined on the real line is a chord-arc domain (hence, in particular, any Lipschitz domain in the plane is a chord-arc domain), from our earlier discussion (see, e.g., Example 2.7) we know that the boundaries of chord-arc domains may actually contain spiral points. As such, chord-arc domains may fail to be of "upper-graph type." There are also subtle connections between the quality of being a chord-arc domain and the behavior of the conformal mapping (see, e.g., [26] and the references therein).

Our next major goal is to establish, in the two-dimensional setting, the coincidence of the class of $\varkappa$-CAD domains with $\varkappa \geq 0$ small constant with that of δ-AR domains with $\delta > 0$ small. This is accomplished in Theorem 2.7. For now recall the concept of UR domain from Definition 2.6.

Proposition 2.16 *Assume $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a chord-arc domain. Then Ω is a connected* UR *domain, satisfying a two-sided local John condition. Moreover, $\partial\Omega = \partial(\overline{\Omega})$ and if either $\partial\Omega$ is unbounded or Ω is bounded, then Ω is also simply connected.*

Proof If $\partial\Omega$ is a Jordan curve passing through infinity in $\mathbb{C}$ then the desired conclusions follow from item *(vi)* of Proposition 2.10 and (2.194). If $\partial\Omega$ is bounded, then there exists a bi-Lipschitz homeomorphism F of the complex plane onto itself such that $F\big(\partial B(0,1)\big) = \partial\Omega$ (cf. [119, Theorem 7.9, p. 165]). This implies that each of the connected sets $F\big(B(0,1)\big)$, $F\big(\mathbb{C} \setminus \overline{B(0,1)}\big)$ is contained in the disjoint union of Ω with $\mathbb{C} \setminus \overline{\Omega}$. Since F is surjective, this forces that either

$$F\big(B(0,1)\big) = \Omega \quad \text{and} \quad F\big(\mathbb{C} \setminus \overline{B(0,1)}\,\big) = \mathbb{C} \setminus \overline{\Omega} \tag{2.464}$$

or

$$F\big(B(0,1)\big) = \mathbb{C} \setminus \overline{\Omega} \quad \text{and} \quad F\big(\mathbb{C} \setminus \overline{B(0,1)}\big) = \Omega. \tag{2.465}$$

All desired conclusions readily follow from this and the transformational properties under bi-Lipschitz maps established in [59]. □

A chord-arc domain with a sufficiently small constant is necessarily unbounded (and, in fact, has an unbounded boundary).

Proposition 2.17 *If $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in \big[0, \sqrt{2}-1\big)$ then $\partial\Omega$ is unbounded.*

Proof Seeking a contradiction, assume $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in \big[0, \sqrt{2}-1\big)$ and such that $\partial\Omega$ is a bounded set. In particular, $\partial\Omega$ is a rectifiable closed curve. Abbreviate $L := \mathcal{H}^1(\partial\Omega) \in (0,\infty)$ and let $[0,L] \ni s \mapsto z(s) \in \partial\Omega$ be the arc-length parametrization of $\partial\Omega$. Define $z_0 := z(0)$, $z_{1/4} := z(L/4)$, $z_{1/2} := z(L/2)$, $z_{3/4} := z(3L/4)$. Since

$$\begin{aligned} |z_0 - z_{1/4}| \le \ell(z_0, z_{1/4}) = L/4, &\quad |z_{3/4} - z_0| \le \ell(z_{3/4}, z_0) = L/4, \\ |z_{1/2} - z_{3/4}| \le \ell(z_{1/2}, z_{3/4}) = L/4, &\quad |z_{1/4} - z_{1/2}| \le \ell(z_{1/4}, z_{1/2}) = L/4, \end{aligned} \tag{2.466}$$

it follows that

$$z_{1/4}, z_{3/4} \in D := \overline{B(z_0, L/4)} \cap \overline{B(z_{1/2}, L/4)}, \tag{2.467}$$

hence

$$|z_{1/4} - z_{3/4}| \le \operatorname{diam}(D). \tag{2.468}$$

On the one hand, with $R := |z_0 - z_{1/2}|$, elementary geometry gives that

$$\operatorname{diam}(D) = 2\sqrt{(L/4)^2 - (R/2)^2} = \sqrt{L^2/4 - R^2}. \tag{2.469}$$

On the other hand, $L/2 = \ell(z_0, z_{1/2}) \le (1+\varkappa)|z_0 - z_{1/2}| = (1+\varkappa)R$ so

$$\operatorname{diam}(D) \le \sqrt{L^2/4 - (L/(2+2\varkappa))^2} = \frac{L}{2}\sqrt{1 - \Big(\frac{1}{1+\varkappa}\Big)^2}. \tag{2.470}$$

Based on the chord-arc property, (2.468), and (2.470) we then conclude that

$$\begin{aligned} \frac{L}{2} &= \ell(z_{1/4}, z_{3/4}) \le (1+\varkappa)|z_{1/4} - z_{3/4}| \\ &\le (1+\varkappa)\operatorname{diam}(D) \le \frac{L}{2}\sqrt{(1+\varkappa)^2 - 1}, \end{aligned} \tag{2.471}$$

which further implies that $\varkappa \geq \sqrt{2} - 1$, a contradiction. □

By design, the boundary of any chord-arc domain is a simple curve, and this brings into focus the question: when is the boundary of an open, connected, simply connected planar set a Jordan curve? According to the classical Carathéodory theorem, this is the case if and only if some (or any) conformal mapping $\varphi : \mathbb{D} \to \Omega$ (where $\mathbb{D}$ is the unit disk in $\mathbb{C}$) extends to a homeomorphism $\varphi : \overline{\mathbb{D}} \to \overline{\Omega}$ (see, e.g., [53, Theorem 3.1, p. 13]). A characterization of bounded planar Jordan regions in terms of properties having no reference to their boundaries has been given by R.L. Moore in 1918. According to [116],

given an open, bounded, connected, simply connected subset Ω of $\mathbb{R}^2$, in order for $\partial\Omega$ to be a simple closed curve it is necessary and sufficient that Ω is *uniformly connected im kleinen* (i.e., if for every $\varepsilon_o > 0$ there exists $\delta_o > 0$ such that any two points $P, \widetilde{P} \in \Omega$ with $|P - \widetilde{P}| < \delta_o$ lie in a connected subset Γ of Ω satisfying $|P - Q| < \varepsilon_o$ for each point $Q \in \Gamma$). (2.472)

A moment's reflection shows that the uniform connectivity condition (im kleinen) formulated above is equivalent to the demand that for every $\varepsilon_o > 0$ there exists $\delta_o > 0$ such that any two points $P, \widetilde{P} \in \Omega$ with $|P - \widetilde{P}| < \delta_o$ lie in a connected subset Γ of Ω with $\mathrm{diam}(\Gamma) < \varepsilon_o$. This condition is meant to prevent the boundary of Ω to "branch out" (like in the case of a partially slit disk).

We are now in a position to establish the coincidence of the class of $\varkappa$-CAD domains with $\varkappa \geq 0$ small constant with that of δ-AR domains with $\delta > 0$ small, in the two-dimensional Euclidean setting.

Theorem 2.7 *If $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-*CAD *with $\varkappa \in \big[0, \sqrt{2} - 1\big)$ then Ω satisfies a two-sided local John condition and is a δ-*AR *domain for any $\delta > 2\sqrt{\varkappa(2+\varkappa)}$. In particular, Ω is a δ-*AR *domain for, say, $\delta := 2\sqrt{\sqrt{2}+1}\sqrt{\varkappa}$, a choice which satisfies $\delta = O(\sqrt{\varkappa}\,)$ as $\varkappa \to 0^+$.*

*Conversely, given any $M \in (0, \infty)$ there exists $\delta_* \in (0, 1)$ with the property that whenever $\delta \in (0, \delta_*)$ it follows that any δ-*AR *domain $\Omega \subseteq \mathbb{R}^2$ with Ahlfors regularity constant $\leq M$ is a $\varkappa$-*CAD *with $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$.*

Proof Suppose $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in \big[0, \sqrt{2} - 1\big)$. Proposition 2.17 then ensures that $\partial\Omega$ is an unbounded set. Keeping this in mind, from Definition 2.16 we then conclude that $\partial\Omega$ is a Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$. Granted (2.463), it follows that $\partial\Omega$ is a $\varkappa$-CAC. From Proposition 2.10 and (2.199) we then see that Ω satisfies a two-sided local John condition and has an Ahlfors regular boundary. Moreover, if $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and ν is the geometric measure theoretic outward unit normal to Ω, from (2.228) we deduce that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq 2\sqrt{\varkappa(2+\varkappa)}. \tag{2.473}$$

It follows from Definition 2.15 and (2.473) that Ω is a δ-AR domain whenever $\delta > 2\sqrt{\varkappa(2+\varkappa)}$. This completes the proof of the claim made in the first part of the statement of the theorem.

In the converse direction, let $\Omega \subseteq \mathbb{R}^2$ be a δ-AR domain with $\delta \in (0,1)$ sufficiently small relative to the Ahlfors regularity constant of $\partial\Omega$. Then Theorem 2.3 implies that Ω is an (∞, N)-two-sided nontangentially accessible domain (in the sense of Definition 2.9), for some $N \in \mathbb{N}$. From Corollary 2.2 we also know that Ω is an unbounded connected set which is simply connected, and whose topological boundary is an unbounded connected set.

The first order of business is to show that actually $\partial\Omega$ is a simple curve. To establish this, we intend to make use of Moore's criterion recalled in (2.472). Since this pertains to bounded sets, as a preliminary step we fix a point $z_0 \in \mathbb{C} \setminus \overline{\Omega}$ and consider

$$\widetilde{\Omega} := \Phi(\Omega) \subseteq \mathbb{C}, \tag{2.474}$$

where

$$\begin{aligned} &\Phi : \mathbb{C} \setminus \{z_0\} \longrightarrow \mathbb{C} \setminus \{0\} \\ &\Phi(z) := (z - z_0)^{-1} \quad \text{for each} \quad z \in \mathbb{C} \setminus \{z_0\}. \end{aligned} \tag{2.475}$$

Note that, when restricted to Ω, the function Φ satisfies a Lipschitz condition. Specifically, if $r_0 := \operatorname{dist}(z_0, \partial\Omega)$ then $r_0 \in (0, \infty)$ and we may estimate

$$\big|\Phi(z_1) - \Phi(z_2)\big| = \frac{|z_1 - z_2|}{|z_1 - z_0||z_2 - z_0|} \leq r_0^{-2}|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \Omega. \tag{2.476}$$

Also, since Φ is a homeomorphism and $\Omega \subseteq \mathbb{C}\setminus\{z_0\}$ it follows that $\widetilde{\Omega} = \Phi(\Omega)$ is an open, connected, simply connected subset of $\mathbb{C} \setminus \{0\}$. Moreover, $\Omega \subseteq \mathbb{C} \setminus \overline{B(z_0, r_0)}$ and since Φ maps $\mathbb{C} \setminus \overline{B(z_0, r_0)}$ into $B(0, 1/r_0)$ it follows that $\widetilde{\Omega} \subseteq B(0, 1/r_0)$, hence $\widetilde{\Omega}$ is also bounded. The idea is then to check Moore's criterion (cf. (2.472)) for $\widetilde{\Omega}$, conclude that $\partial\widetilde{\Omega}$ is a simple curve, then use Φ^{-1} to reach a similar conclusion for $\partial\Omega$. Since Φ^{-1} is singular at $0 \in \partial\widetilde{\Omega}$, special care is required when checking the uniform connectivity condition (im kleinen) near the origin. This requires some preparations.

To proceed, fix some large number $R \in (0, \infty)$, to be specified later in the proof. Pick two points $P, \widetilde{P} \in \widetilde{\Omega} \cap B(0, 1/R)$ then define $x := \Phi^{-1}(P)$ and $\widetilde{x} := \Phi^{-1}(\widetilde{P})$. It follows that $x, \widetilde{x} \in \Omega \setminus \overline{B(z_0, R)}$. Bring in the polygonal arc Γ joining x with $\widetilde{x}$ in Ω as in Lemma 2.5. As noted in Lemma 2.6, there exists $\varepsilon = \varepsilon(N) \in (0,1)$ with the property that this curve is disjoint from $B(z_0, \varepsilon R)$. Next, abbreviate $L := \operatorname{length}(\Gamma) \in (0, \infty)$ and let $[0, L] \ni s \mapsto \Gamma(s) \in \Gamma$ be the arc-length parametrization of Γ. In particular, $\big|\Gamma'(s)\big| = 1$ for $\mathcal{L}^1$-a.e. $s \in (0, L)$. If we define

$$\widetilde{\Gamma}(s) := \Phi(\Gamma(s)) = \frac{1}{\Gamma(s) - z_0} \quad \text{for each} \quad s \in [0, L], \tag{2.477}$$

then the image of $\widetilde{\Gamma}$ is a rectifiable curve joining P with $\widetilde{P}$ in $\widetilde{\Omega}$. In particular, this curve is a connected subset of $\widetilde{\Omega}$ containing P, $\widetilde{P}$ and, with (2.472) in mind, the immediate goal is to estimate the length of this curve. Retaining the symbol $\widetilde{\Gamma}$ for said curve, we have

$$\begin{aligned}\text{length}(\widetilde{\Gamma}) &= \int_0^L \big|\widetilde{\Gamma}'(s)\big|\, \mathrm{d}s = \int_0^L \big|\Phi'(\Gamma(s))\big| \cdot \big|\Gamma'(s)\big|\, \mathrm{d}s \\ &= \int_0^L \frac{\mathrm{d}s}{|\Gamma(s) - z_0|^2}.\end{aligned} \tag{2.478}$$

For each $s \in [0, L]$ we have $\Gamma(s) \in \Omega$. Given that $z_0 \notin \overline{\Omega}$, the line segment joining $\Gamma(s)$ with z_0 intersects $\partial\Omega$, hence $|\Gamma(s) - z_0| \geq \delta_{\partial\Omega}(\Gamma(s))$. On the other hand, for each $s \in [0, L]$ the last line in (2.74) implies that $C_N \cdot \delta_{\partial\Omega}(\Gamma(s)) \geq \min\{s, L - s\}$. Altogether, $C_N \cdot |\Gamma(s) - z_0| \geq \min\{s, L - s\}$ for each $s \in [0, L]$. Upon recalling that the polygonal arc Γ is disjoint from $B(z_0, \varepsilon R)$, we also have $|\Gamma(s) - z_0| \geq \varepsilon R$ for each $s \in [0, L]$. Ultimately, this proves that there exists some $c_N \in (0, \infty)$ with the property that

$$|\Gamma(s) - z_0| \geq c_N \cdot \big(R + \min\{s, L - s\}\big) \quad \text{for each} \quad s \in [0, L]. \tag{2.479}$$

Combining (2.478) with (2.479) then gives

$$\begin{aligned}\text{length}(\widetilde{\Gamma}) &= \int_0^L \frac{\mathrm{d}s}{|\Gamma(s) - z_0|^2} \leq C_N \int_0^L \frac{\mathrm{d}s}{\big(R + \min\{s, L - s\}\big)^2} \\ &= C_N \int_0^{L/2} \frac{\mathrm{d}s}{\big(R + \min\{s, L - s\}\big)^2} + C_N \int_{L/2}^L \frac{\mathrm{d}s}{\big(R + \min\{s, L - s\}\big)^2} \\ &= 2C_N \int_0^{L/2} \frac{\mathrm{d}s}{(R + s)^2} \leq 2C_N \int_0^\infty \frac{\mathrm{d}s}{(R + s)^2} = \frac{2C_N}{R}.\end{aligned} \tag{2.480}$$

Armed with (2.480), we now proceed to check that the set $\widetilde{\Omega}$ is uniformly connected im kleinen (in the sense made precise in (2.472)). To get started, suppose some threshold $\varepsilon_o > 0$ has been given. Make the assumption that

$$R > \max\left\{r_0, \frac{2C_N}{\varepsilon_o}\right\} \quad \text{and pick} \quad \delta_o \in \big(0, 1/(2R)\big), \tag{2.481}$$

reserving the right to make further specifications regarding the size of δ_o. Consider two points $P, \widetilde{P} \in \widetilde{\Omega}$ with $|P - \widetilde{P}| < \delta_o$. The goal is to find a connected subset of $\widetilde{\Omega}$ whose every point is at distance $\leq \varepsilon_o$ from P. To this end, we distinguish two cases.

`Case I:` *Assume* $P, \widetilde{P} \in \widetilde{\Omega} \cap B(0, 1/R)$. Then $\widetilde{\Gamma}$, the curve introduced in (2.477), is a connected subset of $\widetilde{\Omega}$ containing $P, \widetilde{P}$, and (2.480) implies (in view of (2.481)) that $\text{length}(\widetilde{\Gamma}) < \varepsilon_o$. In particular, for any point $Q \in \widetilde{\Gamma}$ we have $|P - Q| \leq \text{length}(\widetilde{\Gamma}) < \varepsilon_o$, as wanted.

`Case II:` *Assume either* $P \notin \widetilde{\Omega} \cap B(0, 1/R)$ *or* $\widetilde{P} \notin \widetilde{\Omega} \cap B(0, 1/R)$. Since $|P - \widetilde{P}| < \delta_o < 1/(2R)$ to begin with, this forces $P, \widetilde{P} \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}$. To proceed, observe that the restriction of $\Phi : \Omega \to \widetilde{\Omega}$ to $\Omega \cap B(z_0, 2R)$, i.e., the function

$$\begin{aligned} &\widetilde{\Phi} : \Omega \cap B(z_0, 2R) \longrightarrow \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}, \\ &\widetilde{\Phi}(z) := (z - z_0)^{-1} \quad \text{for each} \quad z \in \Omega \cap B(z_0, 2R), \end{aligned} \tag{2.482}$$

is a bijection, whose inverse

$$\begin{aligned} &\widetilde{\Phi}^{-1} : \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))} \longrightarrow \Omega \cap B(z_0, 2R), \\ &\widetilde{\Phi}^{-1}(\zeta) := \zeta^{-1} + z_0 \quad \text{for each} \quad \zeta \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}, \end{aligned} \tag{2.483}$$

is Lipschitz since for each $\zeta_1, \zeta_2 \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}$ we may estimate

$$\big|\widetilde{\Phi}^{-1}(\zeta_1) - \widetilde{\Phi}^{-1}(\zeta_2)\big| = \frac{|\zeta_1 - \zeta_2|}{|\zeta_1||\zeta_2|} \leq (2R)^2 |\zeta_1 - \zeta_2|. \tag{2.484}$$

In particular, if we set $x := \widetilde{\Phi}^{-1}(P) \in \Omega$ and $\widetilde{x} := \widetilde{\Phi}^{-1}(\widetilde{P}) \in \Omega$, it follows that

$$|x - \widetilde{x}| = \big|\widetilde{\Phi}^{-1}(P) - \widetilde{\Phi}^{-1}(\widetilde{P})\big| \leq (2R)^2 |P - \widetilde{P}| \leq (2R)^2 \delta_o. \tag{2.485}$$

Let Γ be the polygonal arc joining x with $\widetilde{x}$ in Ω as in Lemma 2.5 with the scale $r := |x - \widetilde{x}|$. The first inequality in (2.74) tells us that $\text{length}(\Gamma) \leq C_N \cdot |x - \widetilde{x}|$, so $L := \text{length}(\Gamma) \leq C_N \cdot (2R)^2 \delta_o$ by (2.485). Let $[a, b] \ni t \mapsto \gamma(t) \in \Gamma$ be a parametrization of the curve Γ and define $\widetilde{\Gamma} := \Phi \circ \gamma$. Then the image of $\widetilde{\Gamma}$ is a rectifiable curve joining P with $\widetilde{P}$ in $\widetilde{\Omega}$. Indeed, $\Phi(\Gamma) \subseteq \Phi(\Omega) = \widetilde{\Omega}$ and

$$\begin{aligned} &\Phi(\gamma(a)) = \Phi(x) = \Phi\big(\widetilde{\Phi}^{-1}(P)\big) = \widetilde{\Phi}\big(\widetilde{\Phi}^{-1}(P)\big) = P, \\ &\Phi(\gamma(b)) = \Phi(\widetilde{x}) = \Phi\big(\widetilde{\Phi}^{-1}(\widetilde{P})\big) = \widetilde{\Phi}\big(\widetilde{\Phi}^{-1}(\widetilde{P})\big) = \widetilde{P}, \end{aligned} \tag{2.486}$$

given that $\widetilde{\Phi}^{-1}(P), \widetilde{\Phi}^{-1}(\widetilde{P})$ belong to $\Omega \cap B(z_0, 2R)$ where Φ agrees with $\widetilde{\Phi}$. Retaining the symbol $\widetilde{\Gamma}$ for said curve, we may estimate

$$\text{length}(\widetilde{\Gamma}) \leq r_0^{-2} \cdot \text{length}(\Gamma) = L/r_0^2 \leq C_N \cdot (2R)^2 \delta_o / r_0^2, \tag{2.487}$$

where the first inequality follows from (2.209) and the fact that $\Phi : \Omega \to \widetilde{\Omega}$ is a Lipschitz function with constant $\leq r_0^{-2}$ (cf. (2.476)). Choosing $\delta_o > 0$ sufficiently small, to begin with, so that $C_N \cdot (2R)^2 \delta_o / r_0^2 < \varepsilon_o$, we ultimately conclude that $\text{length}(\widetilde{\Gamma}) < \varepsilon_o$. Hence, once again, $\widetilde{\Gamma}$ is a connected subset of $\widetilde{\Omega}$ containing P, $\widetilde{P}$, and with the property that $|P - Q| \leq \text{length}(\widetilde{\Gamma}) < \varepsilon_o$ for each point $Q \in \widetilde{\Gamma}$.

Let us summarize our progress. In view of (2.472), the proof so far gives that

$$\partial\widetilde{\Omega} \text{ a simple closed curve in the plane.} \tag{2.488}$$

Moreover, since $\Phi(\partial\Omega) \subseteq \partial\widetilde{\Omega}$, the origin $0 \in \mathbb{C}$ is an accumulation point for $\Phi(\partial\Omega)$ (as is visible from (2.475), keeping in mind that $\partial\Omega$ is unbounded), and $\partial\widetilde{\Omega}$ is a closed set, we conclude that $0 \in \partial\widetilde{\Omega}$. In turn, this implies that $\partial\widetilde{\Omega} \setminus \{0\}$ is a simple curve, and that the function given in (2.475) induces a homeomorphism $\Phi : \partial\Omega \to \partial\widetilde{\Omega} \setminus \{0\}$. As a consequence, $\partial\Omega = \Phi^{-1}(\partial\widetilde{\Omega} \setminus \{0\})$ is a simple curve in the plane. In addition, the (upper) Ahlfors regularity property of $\partial\Omega$ ensures that the curve $\partial\Omega$ is locally rectifiable, hence

$$\partial\Omega = \Phi^{-1}(\partial\widetilde{\Omega} \setminus \{0\}) \text{ is a locally rectifiable simple curve in the plane.} \tag{2.489}$$

Next, if $\widetilde{\gamma} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \partial\widetilde{\Omega}$ is a parametrization of $\partial\widetilde{\Omega}$ with $\widetilde{\gamma}(\pm\pi/2) = 0$, then

$$\gamma : \mathbb{R} \to \partial\Omega, \quad \gamma(t) := \Phi^{-1}\big(\widetilde{\gamma}(\arctan t)\big) \text{ for each } t \in \mathbb{R}, \tag{2.490}$$

becomes a parametrization of the curve $\partial\Omega$. Given that $\lim_{t\to\pm\infty} |\gamma(t)| = 0$, we ultimately conclude that

$$\partial\Omega \text{ is a Jordan curve passing through infinity in the plane.} \tag{2.491}$$

At this stage, there remains to prove that $\partial\Omega$ satisfies the chord-arc condition (2.463) with a constant $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$. In this regard, we note that (2.381) with $n = 2$ gives that there exists a finite geometrical constant $C_o > 1$, independent of δ, with the property that

$$\left| \frac{\mathcal{H}^1\big(B(z,r) \cap \partial\Omega\big)}{2r} - 1 \right| \leq C_o\, \delta \ln(1/\delta), \qquad \forall z \in \partial\Omega, \quad \forall r \in (0,\infty). \tag{2.492}$$

Without loss of generality, for the remainder of the proof assume $\delta \in (0,1)$ is small enough so that $0 < \delta \ln(1/\delta) < 1/(4C_o)$. Consider now two points $z_1, z_2 \in \partial\Omega$. Abbreviate $r := \ell(z_1, z_2)$ and denote by z_3 the first exit point of the curve $\partial\Omega$ out of $B(z_1, r)$. Hence, $|z_1 - z_3| = r$ and the ordering z_1, z_2, z_3 conforms with the positive orientation of $\partial\Omega$. Moreover,

$$\text{the portion of } \partial\Omega \text{ between } z_1 \text{ and } z_3 \text{ is contained inside } B(z_1, r). \tag{2.493}$$

To proceed, introduce $\Delta := B(z_1, r) \cap \partial\Omega$ and decompose $\Delta = \Delta^+ \cup \Delta^-$ (disjoint union), where $\Delta^\pm$ denote the sets of points in Δ lying, respectively, to the left and to the right of z_1. Also, denote by $\ell(\Delta^\pm)$ the arc-lengths of $\Delta^\pm$. Then

$$\mathcal{H}^1\big(B(z_1, r) \cap \partial\Omega\big) = \ell(\Delta^-) + \ell(\Delta^+) \quad \text{and} \quad \ell(\Delta^\pm) \geq r. \tag{2.494}$$

Making use of (2.492) and (2.494) we may therefore estimate

$$\begin{aligned} C_o\, \delta \ln(1/\delta) &\geq \left| \frac{\mathcal{H}^1\big(B(z, r) \cap \partial\Omega\big)}{2r} - 1 \right| = \left| \frac{\ell(\Delta^-) - r}{2r} + \frac{\ell(\Delta^+) - r}{2r} \right| \\ &= \frac{\ell(\Delta^-) - r}{2r} + \frac{\ell(\Delta^+) - r}{2r} \geq \frac{\ell(\Delta^+) - r}{2r}. \end{aligned} \tag{2.495}$$

Hence, by (2.493) and (2.495),

$$|z_2 - z_3| \leq \ell(\Delta^+) - r \leq 2r C_o\, \delta \ln(1/\delta) \tag{2.496}$$

which further implies

$$\begin{aligned} |z_1 - z_2| &\geq |z_1 - z_3| - |z_2 - z_3| \geq r - 2r C_o\, \delta \ln(1/\delta) \\ &= \big(1 - 2C_o\, \delta \ln(1/\delta)\big)\ell(z_1, z_2). \end{aligned} \tag{2.497}$$

This proves that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \ \text{ with } \ \varkappa := \frac{2C_o\, \delta \ln(1/\delta)}{1 - 2C_o\, \delta \ln(1/\delta)}, \tag{2.498}$$

which goes to show that $\partial\Omega$ is a chord-arc curve. Moreover, the fact that we have assumed $0 < \delta \ln(1/\delta) < 1/(4C_o)$ implies $0 < \varkappa < 4C_o \delta \ln(1/\delta)$. In particular, we have $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$. Hence, Ω is a $\varkappa$-CAD with $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$, finishing the proof of Theorem 2.7.

□

In closing, we briefly elaborate on a distinguished sub-class of the category of planar chord-arc domains, described in the next definition.

Definition 2.17 Say that $\Omega \subseteq \mathbb{R}^2$ is a `chord-arc domain with vanishing constant` (CAD with vanishing constant, for short) provided Ω is a chord-arc domain in the sense of Definition 2.16 and

$$\lim_{R\to 0^+}\Big\{\sup_{\substack{z_1,z_2\in\partial\Omega\\ |z_1-z_2|<R}}\Big(\frac{\ell(z_1,z_2)}{|z_1-z_2|}-1\Big)\Big\}=0, \tag{2.499}$$

where $\ell(\cdot,\cdot)$ denotes the shortest arc-length between points on $\partial\Omega$.

The proposition below shows that, in the two-dimensional setting, VMO_1 domains (of upper-graph type) are chord-arc domains with vanishing constant. Before stating it, the reader is reminded that the Sarason space of functions of vanishing mean oscillations has been introduced in (2.111).

Proposition 2.18 *Let $\varphi\in W^{1,1}_{loc}(\mathbb{R})$ be such that $\varphi'\in VMO(\mathbb{R},\mathcal{L}^1)$ and consider its upper-graph $\Omega:=\big\{(x,y):\ x\in\mathbb{R},\ \ y>\varphi(x)\big\}\subseteq\mathbb{R}^2$. Then Ω is a chord-arc domain with vanishing constant.*

Proof That Ω is a chord-arc domain follows from Definition 2.16 and Proposition 2.13. Finally, the vanishing property (2.499) is seen from Definition 2.17 and an inspection of the proof of Proposition 2.13, bearing in mind (2.112). □

2.7 Dyadic Grids and Muckenhoupt Weights on Ahlfors Regular Sets

The following result, pertaining to the existence of a dyadic grid structure on a given Ahlfors regular set, is essentially due to M. Christ [27] (cf. also [40], [41]), with some refinements worked out in [63, Proposition 2.11, pp. 19-20].

Proposition 2.19 *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed, unbounded, Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Then there are finite constants $a_1\geq a_0>0$ such that for each $m\in\mathbb{Z}$ there exists a collection*

$$\mathbb{D}_m(\Sigma):=\{Q^m_\alpha\}_{\alpha\in I_m} \tag{2.500}$$

of subsets of Σ indexed by a nonempty, at most countable set of indices I_m, as well as a family $\{x^m_\alpha\}_{\alpha\in I_m}$ of points in Σ, for which the collection of all dyadic cubes in Σ, i.e.,

$$\mathbb{D}(\Sigma):=\bigcup_{m\in\mathbb{Z}}\mathbb{D}_m(\Sigma), \tag{2.501}$$

has the following properties:

(1) [All dyadic cubes are open] *For each $m\in\mathbb{Z}$ and each $\alpha\in I_m$ the set Q^m_α is relatively open in Σ.*

(2) [Dyadic cubes are mutually disjoint within the same generation] *For each integer* $m \in \mathbb{Z}$ *and each* $\alpha, \beta \in I_m$ *with* $\alpha \neq \beta$ *there holds* $Q_\alpha^m \cap Q_\beta^m = \varnothing$.

(3) [No partial overlap across generations] *For each* $m, \ell \in \mathbb{Z}$ *with* $\ell > m$ *and each* $\alpha \in I_m$, $\beta \in I_\ell$, *either* $Q_\beta^\ell \subseteq Q_\alpha^m$ *or* $Q_\alpha^m \cap Q_\beta^\ell = \varnothing$.

(4) [Any dyadic cube has a unique ancestor in any earlier generation] *For each integers* $m, \ell \in \mathbb{Z}$ *with* $m > \ell$ *and each* $\alpha \in I_m$ *there is a unique* $\beta \in I_\ell$ *such that* $Q_\alpha^m \subseteq Q_\beta^\ell$. *In particular, for each* $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *there exists a unique* $\beta \in I_{m-1}$ *such that* $Q_\alpha^m \subseteq Q_\beta^{m-1}$ *(a scenario in which* Q_β^{m-1} *is referred to as the parent of* Q_α^m*).*

(5) [The size is dyadically related to the generation] *For each* $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *one has*

$$\Delta(x_\alpha^m, a_0 2^{-m}) \subseteq Q_\alpha^m \subseteq \Delta_{Q_\alpha^m} := \Delta(x_\alpha^m, a_1 2^{-m}). \tag{2.502}$$

(6) [Control of the number of children] *There exists an integer* $M \in \mathbb{N}$ *with the property that for each* $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *one has*

$$\#\big\{\beta \in I_{m+1} : \ Q_\beta^{m+1} \subseteq Q_\alpha^m\big\} \leq M. \tag{2.503}$$

Also, this integer may be chosen such that for each $m \in \mathbb{Z}$, *each* $x \in \Sigma$, *and each* $r \in (0, 2^{-m})$ *the number of* Q*'s in* $\mathbb{D}_m(\Sigma)$ *that intersect* $\Delta(x, r)$ *is at most* M.

(7) [Each generation covers the space σ-a.e.] *For each* $m \in \mathbb{Z}$ *one has*

$$\sigma\Big(\Sigma \setminus \bigcup_{\alpha \in I_m} Q_\alpha^m\Big) = 0. \tag{2.504}$$

In particular,

$$N := \bigcup_{m \in \mathbb{Z}} \Big(\Sigma \setminus \bigcup_{\alpha \in I_m} Q_\alpha^m\Big) \Longrightarrow \sigma(N) = 0, \tag{2.505}$$

and for each $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *one has*

$$\sigma\Big(Q_\alpha^m \setminus \bigcup_{\beta \in I_{m+1},\, Q_\beta^{m+1} \subseteq Q_\alpha^m} Q_\beta^{m+1}\Big) = 0. \tag{2.506}$$

(8) [Dyadic cubes have thin boundaries] *There exist some small* $\vartheta \in (0, 1)$ *along with some large* $C \in (0, \infty)$, *such that for each* $m \in \mathbb{Z}$, *each* $\alpha \in I_m$, *and each* $t > 0$ *one has*

$$\sigma\left(\big\{x \in Q_\alpha^m : \ dist(x, \Sigma \setminus Q_\alpha^m) \leq t \cdot 2^{-m}\big\}\right) \leq C t^\vartheta \cdot \sigma(Q_\alpha^m). \tag{2.507}$$

Moving on, assume $\Sigma \subseteq \mathbb{R}^n$ is a closed set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. It has been noted in [111, §3.6] that

$$\text{if } \mathcal{H}^{n-1}(K \cap \Sigma) < +\infty \text{ for each compact subset } K \text{ of } \mathbb{R}^n \text{ then } \sigma \text{ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on } \Sigma\text{, where the latter set is endowed with the topology canonically inherited from the ambient space.} \tag{2.508}$$

Let w be a weight on Σ, i.e., a σ-measurable function satisfying $0 < w(x) < \infty$ for σ-a.e. point $x \in \Sigma$. We agree to also use the symbol w for the weighted measure $w\,\sigma$, i.e., define

$$w(E) := \int_E w\,\mathrm{d}\sigma \quad \text{for each } \sigma\text{-measurable set } \; E \subseteq \Sigma. \tag{2.509}$$

Then the measures w and σ have the same sigma-algebra of measurable sets, and are mutually absolutely continuous with each other. Recall that, for a generic measure space (X, μ), the measure μ is said to be semi-finite if for each μ-measurable set $E \subseteq X$ with $\mu(E) = \infty$ there exists some μ-measurable set $F \subseteq E$ such that $0 < \mu(F) < \infty$ (cf., e.g., [51, p. 25]).

Lemma 2.11 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Let w be an arbitrary weight on Σ and pick an arbitrary σ-measurable set $\Delta \subseteq \Sigma$ with $\sigma(\Delta) < \infty$. Then the measure $w\lfloor\Delta$ is semi-finite and, whenever $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, it follows that*

$$\|w^{-1}\|_{L^{p'}(\Delta,w)} = \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\,d\sigma. \tag{2.510}$$

Proof Consider a w-measurable set $E \subseteq \Delta$ with $w(E) = \infty$. In particular, the set E is σ-measurable. If for each $N \in \mathbb{N}$ we define $E_N := \{x \in E : \, w(x) < N\}$ then E_N is a σ-measurable subset of Δ and the inclusion $E_N \subseteq E_{N+1}$ holds. In addition, $\bigcup_{N\in\mathbb{N}} E_N = \{x \in E : \, w(x) < \infty\}$ hence $\sigma\big(E \setminus \bigcup_{N\in\mathbb{N}} E_N\big) = 0$. Consequently,

$$\lim_{N\to\infty} w(E_N) = \lim_{N\to\infty} \int_{E_N} w\,\mathrm{d}\sigma = \int_E w\,\mathrm{d}\sigma = w(E) = \infty, \tag{2.511}$$

by Lebesgue's Monotone Convergence Theorem. In turn, (2.511) implies that there exists $N_o \in \mathbb{N}$ such that $w(E_{N_o}) > 0$. Since we also have

$$w(E_{N_o}) = \int_{E_{N_o}} w\,\mathrm{d}\sigma \le N_o \cdot \sigma(E_{N_o}) \le N_o \cdot \sigma(\Delta) < \infty, \tag{2.512}$$

we conclude that E_{N_o} is a w-measurable subset of E with $0 < w(E_{N_o}) < \infty$. This implies that $w\lfloor\Delta$ is indeed a semi-finite measure.

With an eye on the claim made in (2.510), define $S_{\rm fin}(\Delta, w)$ to be the vector space of all complex-valued functions defined on Δ which may be expressed in the form $f = \sum_{j=1}^{N} \lambda_j \mathbf{1}_{E_j}$ where $N \in \mathbb{N}$, each λ_j is a complex number, the family of sets $\{E_j\}_{1\leq j\leq N}$ consists of w-measurable mutually disjoint subsets of Δ which also satisfies $w\big(\bigcup_{j=1}^{N} E_j\big) < +\infty$. Note that each such function f happens to be σ-measurable and, for each $q \in (0,\infty)$, satisfies $\int_\Delta |f|^q \leq \sum_{j=1}^{N} |\lambda_j|^q \cdot \sigma(\Delta) < \infty$. Hence,

$$S_{\rm fin}(\Delta, w) \subseteq \bigcap_{0<q<\infty} L^q(\Delta, \sigma) \tag{2.513}$$

and, in particular,

$$f w^{-1} \in L^1(\Delta, w) \ \text{ for each } \ f \in S_{\rm fin}(\Delta, w). \tag{2.514}$$

Having picked $p, p' \in (1,\infty)$ with $1/p + 1/p' = 1$, we may then write

$$\begin{aligned} \|w^{-1}\|_{L^{p'}(\Delta,w)} &= \sup_{\substack{f\in S_{\rm fin}(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \Big|\int_\Delta f w^{-1}\,\mathrm{d}w\Big| = \sup_{\substack{f\in S_{\rm fin}(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \Big|\int_\Delta f\,\mathrm{d}\sigma\Big| \\ &\leq \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\,\mathrm{d}\sigma. \end{aligned} \tag{2.515}$$

The first equality above is a consequence of [51, Theorem 6.14, p. 189], whose applicability in the present setting is ensured by (2.514) and the fact that the measure $w\lfloor\Delta$ is semi-finite. The second equality in (2.515) is justified upon recalling that $\mathrm{d}w = w\mathrm{d}\sigma$, and the inequality in (2.515) is trivial. There remains to observe that for each $f \in L^p(\Delta, w)$ with $\|f\|_{L^p(\Delta,w)} = 1$ Hölder's inequality gives

$$\int_\Delta |f|\,\mathrm{d}\sigma = \int_\Delta |f| w^{-1}\,\mathrm{d}w \leq \|w^{-1}\|_{L^{p'}(\Delta,w)}. \tag{2.516}$$

At this stage, (2.510) becomes a consequence of (2.515) and (2.516). □

Next, assume that $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Given $p \in (1,\infty)$, we say that a weight w on Σ belongs to the Muckenhoupt class $A_p(\Sigma, \sigma)$ if

$$[w]_{A_p} := \sup_{\Delta\subseteq\Sigma} \Big(⨍_\Delta w(x)\,\mathrm{d}\sigma(x)\Big)\Big(⨍_\Delta w(x)^{1-p'}\,\mathrm{d}\sigma(x)\Big)^{p-1} < \infty, \tag{2.517}$$

where p' is the conjugate exponent of p (i.e., $p' \in (1,\infty)$ satisfies $1/p + 1/p' = 1$) and the supremum runs over all surface balls Δ in Σ. The expression in (2.517) arises naturally since for each weight function and each surface ball $\Delta \subseteq \Sigma$ Hölder's inequality gives

$$\begin{aligned} 1 = \fint_{\Delta} 1\,\mathrm{d}\sigma &= \fint_{\Delta} w^{1/p} w^{-1/p}\,\mathrm{d}\sigma \\ &\leq \Big(\fint_{\Delta} w\,\mathrm{d}\sigma\Big)^{1/p} \Big(\fint_{\Delta} w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}, \end{aligned} \tag{2.518}$$

hence

$$1 \leq \inf_{\Delta\subseteq\Sigma} \Big(\fint_{\Delta} w\,\mathrm{d}\sigma\Big)\Big(\fint_{\Delta} w^{1-p'}\,\mathrm{d}\sigma\Big)^{p-1} \leq [w]_{A_p} \leq \infty. \tag{2.519}$$

For further use it is useful to note that (2.517) entails that, given any $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$, for each surface ball $\Delta \subseteq \Sigma$ we have

$$\Big(\int_{\Delta} w^{-p'/p}\,\mathrm{d}\sigma\Big)^{1/p'} \leq [w]_{A_p}^{1/p} \frac{\sigma(\Delta)}{w(\Delta)^{1/p}}. \tag{2.520}$$

Corresponding to $p = 1$, we say that $w \in A_1(\Sigma, \sigma)$ if

$$[w]_{A_1} := \sup_{\Delta\subseteq\Sigma} \Big(\operatorname*{ess\,inf}_{x\in\Delta} w(x)\Big)^{-1} \Big(\fint_{\Delta} w\,\mathrm{d}\sigma\Big) < \infty. \tag{2.521}$$

It is clear from the above definition that $[w]_{A_1} \geq 1$ for each weight w on Σ. Recall that the (non-centered) Hardy–Littlewood maximal operator $\mathcal{M}$ on Σ acts on each given σ-measurable function f on Σ according to

$$\mathcal{M}f(x) := \sup_{\Delta\ni x} \fint_{\Delta} |f|\,\mathrm{d}\sigma, \qquad \forall\, x \in \Sigma, \tag{2.522}$$

where the supremum is taken over all surface balls Δ in Σ which contain the point x. In particular,

a weight w on Σ belongs to $A_1(\Sigma, \sigma)$ if and only if there exists a constant $C \in (0, \infty)$ with the property that $\mathcal{M}w(x) \leq Cw(x)$ at σ-a.e. point $x \in \Sigma$, and the best constant is actually $[w]_{A_1}$. (2.523)

Corresponding to the end-point $p = \infty$,

the class $A_\infty(\Sigma, \sigma)$ is defined as the union of all $A_p(\Sigma, \sigma)$ with $p \in [1, \infty)$. (2.524)

Lemma 2.12 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then for each $p \in (1,\infty)$, each Muckenhoupt weight $w \in A_p(\Sigma,\sigma)$, and each σ-measurable function f on Σ one has*

$$\fint_\Delta |f|\,d\sigma \leq [w]_{A_p}^{1/p}\Big(\fint_\Delta |f|^p\,dw\Big)^{1/p}, \quad \text{for each surface ball } \Delta \subseteq \Sigma. \tag{2.525}$$

Conversely, if $p \in (1,\infty)$ and w is a weight on Σ with the property that there exists a constant $C \in (0,\infty)$ such that

$$\fint_\Delta |f|\,d\sigma \leq C\Big(\fint_\Delta |f|^p\,dw\Big)^{1/p} \text{ for each function } f \in L^p_{loc}(\Sigma,w) \text{ and surface ball } \Delta \subseteq \Sigma, \tag{2.526}$$

then actually $w \in A_p(\Sigma,\sigma)$ and $[w]_{A_p} \leq C^p$.

Proof Let $p' \in (1,\infty)$ denote the Hölder conjugate exponent of p and fix an arbitrary σ-measurable function f on Σ. Then for each surface ball $\Delta \subseteq \Sigma$ we may estimate

$$\begin{aligned}
\fint_\Delta |f|\,\mathrm{d}\sigma &= \frac{1}{\sigma(\Delta)}\int_\Delta |f|w^{1/p}w^{-1/p}\,\mathrm{d}\sigma \\
&\leq \frac{1}{\sigma(\Delta)}\Big(\int_\Delta |f|^p\,w\,\mathrm{d}\sigma\Big)^{1/p}\Big(\int_\Delta w^{-p'/p}\,\mathrm{d}\sigma\Big)^{1/p'} \\
&= \Big(\fint_\Delta w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}\Big(\fint_\Delta w\,\mathrm{d}\sigma\Big)^{1/p}\Big(\fint_\Delta |f|^p\,\mathrm{d}w\Big)^{1/p} \\
&\leq [w]_{A_p}^{1/p}\Big(\fint_\Delta |f|^p\,\mathrm{d}w\Big)^{1/p},
\end{aligned} \tag{2.527}$$

by Hölder's inequality and (2.517). This proves (2.525).

As for the converse, fix $p \in (1,\infty)$ and suppose w is a generic weight function on Σ for which there exists a constant $C \in (0,\infty)$ such that (2.526) holds. Once again, denote $p' \in (1,\infty)$ the Hölder conjugate exponent of p and fix an arbitrary surface ball $\Delta \subseteq \Sigma$. Then, with tilde denoting the extension by zero of a function originally defined on Δ to the entire set Σ, we may write

$$\|w^{-1}\|_{L^{p'}(\Delta,w)} = \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\,\mathrm{d}\sigma = \sigma(\Delta)\cdot \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \fint_\Delta |\widetilde{f}|\,\mathrm{d}\sigma$$

$$\leq C\sigma(\Delta)\cdot \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \Big(\fint_\Delta |\widetilde{f}|^p\,\mathrm{d}w\Big)^{1/p}$$

$$\leq C\frac{\sigma(\Delta)}{w(\Delta)^{1/p}}, \tag{2.528}$$

where the first equality comes from Lemma 2.11, and the first inequality is implied by (2.526). This proves that $\|w^{-1}\|_{L^{p'}(\Delta,w)} \leq C\cdot\sigma(\Delta)/w(\Delta)^{1/p}$ which, after unraveling notation, yields

$$\Big(\fint_\Delta w\,\mathrm{d}\sigma\Big)\Big(\fint_\Delta w^{1-p'}\,\mathrm{d}\sigma\Big)^{p-1} \leq C^p. \tag{2.529}$$

Ultimately, in view of the arbitrariness of the surface ball $\Delta\subseteq\Sigma$, this implies that $w\in A_p(\Sigma,\sigma)$ and $[w]_{A_p}\leq C^p$. □

In this work we are particularly interested in the scale of weighted Lebesgue space $L^p(\Sigma,w) := L^p(\Sigma,\,w\sigma)$ with $p\in(1,\infty)$ and $w\in A_p(\Sigma,\sigma)$. As in the Euclidean setting,

given a weight w on Σ and an integrability exponent $p\in(1,\infty)$, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $L^p(\Sigma,w)$ if and only if $w\in A_p(\Sigma,\sigma)$, (2.530)

in which case there exists some constant $C=C(\Sigma,n,p)\in(0,\infty)$ (which depends on Σ only through its Ahlfors regularity constant) with the property that

$$\|\mathcal{M}f\|_{L^p(\Sigma,w)} \leq C[w]_{A_p}^{1/(p-1)}\|f\|_{L^p(\Sigma,w)} \quad\text{for all}\quad f\in L^p(\Sigma,w) \tag{2.531}$$

(see, e.g., [64, Proposition 7.13]). Also, corresponding to $p=1$, the operator $\mathcal{M}$ satisfies the weak-(1, 1) inequality

$$\begin{gathered}\sup_{0<\lambda<\infty}\lambda\cdot w\big(\{x\in\Sigma:\ \mathcal{M}f(x)>\lambda\}\big)\leq C\|f\|_{L^1(\Sigma,w)}\\ \text{for all } f\in L^1(\Sigma,w),\ \text{ with } C\in(0,\infty) \text{ independent of } f,\end{gathered} \tag{2.532}$$

if and only if $w\in A_1(\Sigma,\sigma)$. For the reader's convenience, other useful properties of Muckenhoupt weights are summarized in the proposition below (for a more extensive discussion pertaining to the theory of weights in the general context of spaces of homogeneous type the reader is referred to [6, 54, 65, 76, 135]).

Proposition 2.20 *Suppose* $\Sigma\subseteq\mathbb{R}^n$ *is a closed Ahlfors regular set and abbreviate* $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. *Then the following properties hold.*

(1) [*Openness/Self-Improving*] *If* $w\in A_p(\Sigma,\sigma)$ *with* $p\in(1,\infty)$ *then there exist some* $\tau\in(1,\infty)$ *and some* $\varepsilon\in(0,p-1)$ (*both of which depend only on p,*

$[w]_{A_p}$, n, and the Ahlfors regularity constant of Σ) such that

$$w^{\tau} \in A_p(\Sigma, \sigma) \ \text{ and } \ w \in A_{p-\varepsilon}(\Sigma, \sigma). \tag{2.533}$$

In addition, both $\big[w^{\tau}\big]_{A_p}$ and $[w]_{A_{p-\varepsilon}}$ are controlled in terms of p, $[w]_{A_p}$, n, and the Ahlfors regularity constant of Σ. In fact, matters may be arranged so that, in a quantitative fashion,

$$w^{\theta} \in A_q(\Sigma, \sigma) \ \text{ for each } \ \theta \in (\tau^{-1}, \tau) \ \text{ and } \ q \in (p - \varepsilon, \infty). \tag{2.534}$$

(2) [*Monotonicity*] *If $1 \leq p \leq q \leq \infty$ then $A_p(\Sigma, \sigma) \subseteq A_q(\Sigma, \sigma)$ and if $q < \infty$ then $[w]_{A_q} \leq [w]_{A_p}$ for each $w \in A_p(\Sigma, \sigma)$.*

(3) [*Dual Weights*] *Given any $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$, it follows that $w^{1-p'}$ belongs to $A_{p'}(\Sigma, \sigma)$ and $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$, where $p' \in (1, \infty)$ is the Hölder conjugate exponent of p.*

(4) [*Products/Factorization*] *If $w_1, w_2 \in A_1(\Sigma, \sigma)$ then for every $p \in (1, \infty)$ one has $w_1 \cdot w_2^{1-p} \in A_p(\Sigma, \sigma)$ and $[w_1 \cdot w_2^{1-p}]_{A_p} \leq [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}$. Also, given $w_1, w_2 \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ along with some $\alpha \in [0, 1]$, it follows that $w_1^{\alpha} \cdot w_2^{1-\alpha} \in A_p(\Sigma, \sigma)$ and $[w_1^{\alpha} \cdot w_2^{1-\alpha}]_{A_p} \leq [w_1]_{A_p}^{\alpha} \cdot [w_2]_{A_p}^{1-\alpha}$.*

(5) [*Doubling*] *If $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ then for every surface ball Δ in Σ and every σ-measurable set $E \subseteq \Delta$ one has*

$$\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^p \leq [w]_{A_p} \cdot \frac{w(E)}{w(\Delta)}. \tag{2.535}$$

In particular, the measure w is doubling, that is, there exists some $C \in (0, \infty)$ which depends only on p, n, and the Ahlfors regularity constant of Σ, such that $w(2\Delta) \leq C[w]_{A_p} \cdot w(\Delta)$ for every surface ball $\Delta \subseteq \Sigma$. More generally, with the constant $C \in (0, \infty)$ of the same nature as above, one has the inequality $w(\lambda\Delta) \leq C[w]_{A_p} \cdot \lambda^{p(n-1)} \cdot w(\Delta)$ for each $\lambda \in (1, \infty)$ and each surface ball $\Delta \subseteq \Sigma$ (where $\lambda\Delta$ denotes the concentric dilate of Δ by a factor of λ).

(6) [*Reverse Hölder Inequalities*] *For every $w \in A_{\infty}(\Sigma, \sigma)$ there exist $q \in (1, \infty)$ and some $C \in (0, \infty)$ (which both depend only on p, $[w]_{A_p}$, n, and the Ahlfors regularity constant of Σ, for some $p \in (1, \infty)$ for which $w \in A_p(\Sigma, \sigma)$) such that*

$$\left(\fint_{\Delta} w^q \, d\sigma\right)^{1/q} \leq C \fint_{\Delta} w \, d\sigma, \tag{2.536}$$

for every surface ball $\Delta \subseteq \Sigma$. This has several remarkable consequences. First, there exist some power $\tau > 0$ and some constant $C \in (0, \infty)$ (in fact, C is the same as in (2.536) and $\tau = 1/q'$ where q' is the Hölder conjugate of the exponent q from (2.536)) such that

$$\frac{w(E)}{w(\Delta)} \le C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{\tau} \tag{2.537}$$

for every surface ball $\Delta \subseteq \Sigma$ and every σ-measurable set $E \subseteq \Delta$. Another useful consequence of the inequality in (2.536) *and Hölder's inequality is that for each σ-measurable function f on Σ and each surface ball $\Delta \subseteq \Sigma$ one has*

$$⨍_{\Delta} |f|\, dw \le C\left(⨍_{\Delta} |f|^{q'}\, d\sigma\right)^{1/q'}, \tag{2.538}$$

where $q' \in (1,\infty)$ is the Hölder conjugate exponent of q from (2.536)*, and the constant $C \in (0,\infty)$ is as in* (2.536)*. Finally, in the case when Σ is unbounded,* (2.537) (*used with $\Delta = \Delta(x,r)$ and $E = \Delta(x,1)$*) *proves that there exists some $c \in (0,\infty)$ such that*

$$w\big(\Delta(x,r)\big) \ge c\, r^{(n-1)\tau} \cdot w\big(\Delta(x,1)\big) \quad \text{for each } x \in \Sigma \text{ and } r \in (1,\infty). \tag{2.539}$$

In particular,

$$w(\Sigma) = +\infty \quad \text{if } \Sigma \text{ is unbounded.} \tag{2.540}$$

(7) [*Building A_1 Weights*] *There exists $C \in (0,\infty)$ which depends only on n and Σ, with the property that if $f \in L^1_{loc}(\Sigma,\sigma)$ is not identically zero and $\mathcal{M}f < \infty$ at σ-a.e. point on Σ then for each $\theta \in (0,1)$ one has $(\mathcal{M}f)^\theta \in A_1(\Sigma,\sigma)$ and $\big[(\mathcal{M}f)^\theta\big]_{A_1} \le C(1-\theta)^{-1}$. In addition, for each power $\theta \in (0,1)$ the weight $w := (\mathcal{M}f)^\theta$ satisfies a reverse Hölder inequality* (*as in* (2.536)) *for each exponent $q \in (1,\theta^{-1})$.*

(8) [BMO *and Weights*] *For each $p \in (1,\infty)$ and $w \in A_p(\Sigma,\sigma)$ there exist some small $\varepsilon = \varepsilon(\Sigma,p,[w]_{A_p}) > 0$ and some large $C = C(\Sigma,p,[w]_{A_p}) \in (0,\infty)$ such that for each function $b \in BMO(\Sigma,\sigma)$ with $\|b\|_{BMO(\Sigma,\sigma)} < \varepsilon$ one has $w\cdot e^b \in A_p(\Sigma,\sigma)$ and $\big[w\cdot e^b\big]_{A_p} \le C$. In particular, for each fixed integrability exponent $p \in (1,\infty)$ the set $\mathcal{U}_p := \big\{b \in BMO(\Sigma,\sigma) : e^b \in A_p(\Sigma,\sigma)\big\}$ is open in $BMO(\Sigma,\sigma)$. Also, for each weight $w \in A_1(\Sigma,\sigma)$, the function* $\log w$ *belongs to $BMO(\Sigma,\sigma)$ and $\|\log w\|_{BMO(\Sigma,\sigma)} \le C(\Sigma,n,[w]_{A_1})$. Finally, for each function $b \in BMO(\Sigma,\sigma)$ and each exponent $p \in (1,\infty)$, the function* $\max\{1,|b|\}$ *belongs to $A_p(\Sigma,\sigma)$ and there exists $C_{\Sigma,p} \in (0,\infty)$, independent of b, such that* $[\max\{1,|b|\}]_{A_p} \le C_{\Sigma,p}(1+\|b\|_{BMO(\Sigma,\sigma)})$.

(9) [*Dyadic Cubes*] *If Σ is unbounded, then properties* (2.535), (2.536)*, and* (2.537) *also hold if surface balls Δ are replaced by dyadic "cubes," as described in Proposition 2.19.*

Proof For the memberships in (2.533), (2.534) (including their quantitative aspects) see [65, Theorems 1.1-1.2], [21, Theorem 2.31, p. 58].

To deal with item *(2)*, suppose $1 \leq p \leq q < \infty$ and denote by p', q' the Hölder conjugate exponents of p and q, respectively. Also, fix an arbitrary weight $w \in A_p(\Sigma, \sigma)$. Then $r := (1-p')/(1-q')$ belongs to $[1, \infty)$, so for each surface ball Δ in Σ we may employ Hölder's inequality to write

$$\begin{aligned}
&\Big(⨍_\Delta w \, d\sigma\Big)\Big(⨍_\Delta w^{1-q'} \, d\sigma\Big)^{q-1} \\
&\qquad \leq \Big(⨍_\Delta w \, d\sigma\Big)\Big(⨍_\Delta w^{r(1-q')} \, d\sigma\Big)^{(q-1)/r} \\
&\qquad = \Big(⨍_\Delta w \, d\sigma\Big)\Big(⨍_\Delta w^{1-p'} \, d\sigma\Big)^{p-1} \leq [w]_{A_p} < +\infty,
\end{aligned} \tag{2.541}$$

since $(q-1)/r = p-1$. In view of (2.517), this shows that $w \in A_q(\Sigma, \sigma)$ and we have $[w]_{A_q} \leq [w]_{A_p}$. Finally, the fact that the inclusion $A_p(\Sigma, \sigma) \subseteq A_q(\Sigma, \sigma)$ also holds if $q = \infty$ is clear from (2.524).

Going further, to justify the claim made in item *(3)*, fix some $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$, and denote by $p' \in (1, \infty)$ the Hölder conjugate exponent of p. Then for each surface ball Δ in Σ we may write

$$\begin{aligned}
&\Big(⨍_\Delta w^{1-p'} \, d\sigma\Big)\Big(⨍_\Delta (w^{1-p'})^{1-p} \, d\sigma\Big)^{p'-1} \\
&\qquad = \Big(⨍_\Delta w^{1-p'} \, d\sigma\Big)\Big(⨍_\Delta w \, d\sigma\Big)^{p'-1} \\
&\qquad \leq [w]_{A_p}^{p'-1} = [w]_{A_p}^{1/(p-1)} < +\infty,
\end{aligned} \tag{2.542}$$

thanks to (2.517). This implies that $w^{1-p'}$ belongs to $A_{p'}(\Sigma, \sigma)$ and that we have $[w^{1-p'}]_{A_{p'}} \leq [w]_{A_p}^{1/(p-1)}$. Writing this last inequality with p replaced by p' and with w replaced by $w^{1-p'}$ yields $[(w^{1-p'})^{1-p}]_{A_p} \leq [w^{1-p'}]_{A_{p'}}^{1/(p'-1)}$. Hence, we have $[w]_{A_p}^{1/(p-1)} \leq [w^{1-p'}]_{A_{p'}}$ which ultimately proves that $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$.

To deal with the first claim made in item *(4)*, recall from (2.521) that, since $w_2 \in A_1(\Sigma, \sigma)$, for each surface ball Δ in Σ we have

$$⨍_\Delta w_2 \, d\sigma \leq [w_2]_{A_1} \cdot w_2 \quad \text{at } \sigma\text{-a.e. point in } \Delta. \tag{2.543}$$

Given that $1 - p < 0$, this entails

$$w_2^{1-p} \le [w_2]_{A_1}^{p-1} \cdot \Big(⨍_\Delta w_2 \, \mathrm{d}\sigma \Big)^{1-p} \quad \text{at } \sigma\text{-a.e. point in } \Delta, \tag{2.544}$$

which further implies

$$⨍_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \le [w_2]_{A_1}^{p-1} \cdot \Big(⨍_\Delta w_1 \, \mathrm{d}\sigma \Big)\Big(⨍_\Delta w_2 \, \mathrm{d}\sigma \Big)^{1-p}. \tag{2.545}$$

In a similar manner, the fact that $w_1 \in A_1(\Sigma, \sigma)$ implies

$$w_1^{-1/(p-1)} \le [w_1]_{A_1}^{1/(p-1)} \cdot \Big(⨍_\Delta w_1 \, \mathrm{d}\sigma \Big)^{-1/(p-1)} \quad \text{at } \sigma\text{-a.e. point in } \Delta, \tag{2.546}$$

hence

$$\begin{aligned} \Big(⨍_\Delta (w_1 \cdot w_2^{1-p})^{-1/(p-1)} \, \mathrm{d}\sigma \Big)^{p-1} &= \Big(⨍_\Delta w_1^{-1/(p-1)} \cdot w_2 \, \mathrm{d}\sigma \Big)^{p-1} \\ &\le [w_1]_{A_1} \cdot \Big(⨍_\Delta w_1 \, \mathrm{d}\sigma \Big)^{-1} \Big(⨍_\Delta w_2 \, \mathrm{d}\sigma \Big)^{p-1}. \end{aligned} \tag{2.547}$$

By combining (2.545) with (2.547) we therefore arrive at the conclusion that, with p' denoting the Hölder conjugate exponent of p,

$$\begin{aligned} &\Big(⨍_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \Big) \Big(⨍_\Delta (w_1 \cdot w_2^{1-p})^{1-p'} \, \mathrm{d}\sigma \Big)^{p-1} \\ &\qquad = \Big(⨍_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \Big) \Big(⨍_\Delta (w_1 \cdot w_2^{1-p})^{-1/(p-1)} \, \mathrm{d}\sigma \Big)^{p-1} \\ &\qquad \le [w_2]_{A_1}^{p-1} \cdot \Big(⨍_\Delta w_1 \, \mathrm{d}\sigma \Big)\Big(⨍_\Delta w_2 \, \mathrm{d}\sigma \Big)^{1-p} \times \\ &\qquad\quad \times [w_1]_{A_1} \cdot \Big(⨍_\Delta w_1 \, \mathrm{d}\sigma \Big)^{-1} \Big(⨍_\Delta w_2 \, \mathrm{d}\sigma \Big)^{p-1} \\ &\qquad = [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}. \end{aligned} \tag{2.548}$$

Thus, with the supremum running over all surface balls Δ in Σ, we have (cf. (2.517))

$$[w_1 \cdot w_2^{1-p}]_{A_p} = \sup_{\Delta \subseteq \Sigma} \Big(⨍_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \Big) \Big(⨍_\Delta (w_1 \cdot w_2^{1-p})^{1-p'} \, \mathrm{d}\sigma \Big)^{p-1}$$

$$= [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1} < +\infty, \tag{2.549}$$

proving that the weight $w_1 \cdot w_2^{1-p}$ belongs to the Muckenhoupt class $A_p(\Sigma, \sigma)$ and that we have $[w_1 \cdot w_2^{1-p}]_{A_p} \leq [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}$.

As regards the second claim in item *(4)*, pick two weights $w_1, w_2 \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ fix some $\alpha \in [0, 1]$. If $\alpha = 0$ or $\alpha = 1$ there is nothing to prove, so assume $\alpha \in (0, 1)$. With "prime" indicating a conjugate exponent, for each surface ball Δ in Σ Hölder's inequality gives

$$\begin{aligned} \fint_\Delta w_1^\alpha \cdot w_2^{1-\alpha}\, d\sigma &\leq \Big(\fint_\Delta (w_1^\alpha)^{1/\alpha}\, d\sigma\Big)^\alpha \Big(\fint_\Delta (w_2^{1-\alpha})^{(1/\alpha)'}\, d\sigma\Big)^{1/(1/\alpha)'} \\ &= \Big(\fint_\Delta w_1\, d\sigma\Big)^\alpha \Big(\fint_\Delta w_2\, d\sigma\Big)^{1-\alpha}, \end{aligned} \tag{2.550}$$

since $(1/\alpha)' = (1-\alpha)^{-1}$. Similarly,

$$\begin{aligned} &\Big(\fint_\Delta (w_1^\alpha \cdot w_2^{1-\alpha})^{1-p'}\, d\sigma\Big)^{p-1} \\ &\qquad \leq \Big(\fint_\Delta w_1^{1-p'}\, d\sigma\Big)^{\alpha(p-1)} \Big(\fint_\Delta w_2^{1-p'}\, d\sigma\Big)^{(1-\alpha)(p-1)}. \end{aligned} \tag{2.551}$$

Together, (2.550) and (2.551) show that

$$\begin{aligned} &\Big(\fint_\Delta w_1^\alpha \cdot w_2^{1-\alpha}\, d\sigma\Big)\Big(\fint_\Delta (w_1^\alpha \cdot w_2^{1-\alpha})^{1-p'}\, d\sigma\Big)^{p-1} \\ &\leq \Big[\Big(\fint_\Delta w_1\, d\sigma\Big)\Big(\fint_\Delta w_1^{1-p'}\, d\sigma\Big)^{p-1}\Big]^\alpha \Big[\Big(\fint_\Delta w_2\, d\sigma\Big)\Big(\fint_\Delta w_2^{1-p'}\, d\sigma\Big)^{p-1}\Big]^{1-\alpha} \\ &\leq [w_1]_{A_p}^\alpha \cdot [w_2]_{A_p}^{1-\alpha} < +\infty. \end{aligned} \tag{2.552}$$

After taking the supremum over all surface balls $\Delta \subseteq \Sigma$, we then conclude from (2.552) that $w_1^\alpha \cdot w_2^{1-\alpha} \in A_p(\Sigma, \sigma)$ and $[w_1^\alpha \cdot w_2^{1-\alpha}]_{A_p} \leq [w_1]_{A_p}^\alpha \cdot [w_2]_{A_p}^{1-\alpha}$.

Moving on, the estimate in (2.535) may be seen from Lemma 2.12, used here with $f := \mathbf{1}_E$. In concert with the Ahlfors regularity of Σ, this implies all subsequent claims in item *(5)*.

The reverse Hölder inequality claimed in (2.536) is contained in [65, Theorem 2.3], [135, Theorem 15, p. 9]. Moreover, if q' is the Hölder conjugate of the exponent q from (2.536) then for every surface ball $\Delta \subseteq \Sigma$ and every σ-measurable set $E \subseteq \Delta$ we may estimate

$$\begin{aligned}\frac{w(E)}{w(\Delta)} &= \fint_{\Delta} \mathbf{1}_E \, dw = \frac{\sigma(\Delta)}{w(\Delta)} \fint_{\Delta} \mathbf{1}_E \, w \, d\sigma \\ &\leq \frac{\sigma(\Delta)}{w(\Delta)} \Big(\fint_{\Delta} \mathbf{1}_E \, d\sigma \Big)^{1/q'} \Big(\fint_{\Delta} w^q \, d\sigma \Big)^{1/q} \\ &\leq C \frac{\sigma(\Delta)}{w(\Delta)} \Big(\fint_{\Delta} \mathbf{1}_E \, d\sigma \Big)^{1/q'} \Big(\fint_{\Delta} w \, d\sigma \Big) = C \left(\frac{\sigma(E)}{\sigma(\Delta)} \right)^{1/q'}, \end{aligned} \tag{2.553}$$

thanks to Hölder's inequality and (2.536). This proves (2.537) with $\tau := 1/q' > 0$ and $C \in (0, \infty)$ the same constant as in (2.536).

Consider next the first claim made in item *(7)*. Suppose $f \in L^1_{\rm loc}(\Sigma, \sigma)$ is not identically zero and has the property that $\mathcal{M} f < \infty$ at σ-a.e. point on Σ. Fix an arbitrary surface ball $\Delta \subseteq \Sigma$ and decompose $f = f_1 + f_2$ with $f_1 := f \mathbf{1}_{2\Delta}$ and $f_2 := f \mathbf{1}_{\Sigma \setminus 2\Delta}$. Having $\mathcal{M} f < \infty$ at σ-a.e. point on Σ entails $f_1 \in L^1(\Sigma, \sigma)$. Since $0 < \theta < 1$ and $0 \leq \mathcal{M} f \leq \mathcal{M} f_1 + \mathcal{M} f_2$, we conclude that

$$(\mathcal{M} f)^{\theta} \leq (\mathcal{M} f_1)^{\theta} + (\mathcal{M} f_2)^{\theta} \quad \text{on} \quad \Sigma. \tag{2.554}$$

Based on Kolmogorov's inequality, the fact that $\mathcal{M}$ satisfies the weak-$(1,1)$ inequality, the membership of f_1 to $L^1(\Sigma, \sigma)$, and the fact that the measure σ is doubling we may estimate

$$\begin{aligned}\Big(\fint_{\Delta} |\mathcal{M} f_1|^{\theta} \, d\sigma \Big)^{1/\theta} &\leq \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \sigma(\Delta)^{-1} \|\mathcal{M} f_1\|_{L^{1,\infty}(\Sigma,\sigma)} \\ &\leq C \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \sigma(\Delta)^{-1} \|f_1\|_{L^1(\Sigma,\sigma)} \\ &\leq C \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \fint_{2\Delta} |f| \, d\sigma \\ &\leq C \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \inf_{x \in 2\Delta} (\mathcal{M} f)(x). \end{aligned} \tag{2.555}$$

Hence, on the one hand,

$$\fint_{\Delta} |\mathcal{M} f_1|^{\theta} \, d\sigma \leq \frac{C}{1-\theta} \Big(\inf_{x \in 2\Delta} (\mathcal{M} f)(x) \Big)^{\theta}. \tag{2.556}$$

On the other hand, the fact that

$$\begin{aligned} &\text{for each surface ball } \Delta' \subseteq \Sigma \text{ so that } \Delta' \cap \Delta \neq \varnothing \\ &\text{and } \Delta' \cap (\Sigma \setminus 2\Delta) \neq \varnothing \text{ it follows that } \Delta \subseteq 6\Delta' \end{aligned} \tag{2.557}$$

readily implies that there exists a geometric constant $C \in (0, \infty)$ with the property that

$$(\mathcal{M}f_2)(y) \leq C(\mathcal{M}f_2)(x) \quad \text{for each} \quad x, y \in \Delta. \tag{2.558}$$

In turn, this forces

$$\fint_\Delta |\mathcal{M}f_2|^\theta \, d\sigma \leq C\Big(\inf_{x\in\Delta} (\mathcal{M}f_2)(x)\Big)^\theta \leq C\Big(\inf_{x\in\Delta} (\mathcal{M}f)(x)\Big)^\theta \tag{2.559}$$

which, in concert with (2.556) and (2.554) proves that

$$\fint_\Delta |\mathcal{M}f|^\theta \, d\sigma \leq \frac{C}{1-\theta} \cdot \inf_{x\in\Delta} [(\mathcal{M}f)(x)]^\theta. \tag{2.560}$$

Since $0 < [(\mathcal{M}f)(x)]^\theta < \infty$ for σ-a.e. point $x \in \Sigma$, ultimately (2.560) implies that $(\mathcal{M}f)^\theta \in A_1(\Sigma, \sigma)$ and $\big[(\mathcal{M}f)^\theta\big]_{A_1} \leq C(1-\theta)^{-1}$.

To show that for each $\theta \in (0, 1)$ and $q \in (1, \theta^{-1})$ the weight $w := (\mathcal{M}f)^\theta$ satisfies (2.536), observe that $\widetilde{\theta} := \theta q \in (0, 1)$ so we may invoke (2.560) (for $\widetilde{\theta}$) to write, for every surface ball $\Delta \subseteq \Sigma$,

$$\begin{aligned}
\Big(\fint_\Delta w^q \, d\sigma\Big)^{1/q} &= \Big(\fint_\Delta |\mathcal{M}f|^{\widetilde{\theta}} \, d\sigma\Big)^{1/q} \\
&\leq \Big(\frac{C}{1-\widetilde{\theta}}\Big)^{1/q} \cdot \Big(\inf_\Delta (\mathcal{M}f)^{\widetilde{\theta}}\Big)^{1/q} = \Big(\frac{C}{1-\theta q}\Big)^{1/q} \cdot \Big(\inf_\Delta (\mathcal{M}f)^\theta\Big) \\
&\leq \Big(\frac{C}{1-\theta q}\Big)^{1/q} \fint_\Delta |\mathcal{M}f|^\theta = \Big(\frac{C}{1-\theta q}\Big)^{1/q} \fint_\Delta w \, d\sigma,
\end{aligned} \tag{2.561}$$

as wanted. This completes the treatment of item *(7)*.

For the first two claims in item *(8)* see [69, p. 33 and p. 60] for a proof in the Euclidean ambient which readily adapts to the present setting, given the availability of a John-Nirenberg inequality for doubling measures (see the discussion pertaining to (2.92)–(2.94)) and the results in the current items *(1)-(6)*. For the third claim in item *(8)* see [52, Theorem 3.3, p. 157] for a proof in the Euclidean space which goes through in the present setting as well. We may justify the very last claim in item *(8)* by arguing along the lines of the proof of [58, Lemma 1.12, p. 471]. Specifically, given $b \in \mathrm{BMO}(\Sigma, \sigma)$ set $w := \max\{1, |b|\}$ and fix some $p \in (1, \infty)$. Then for an arbitrary surface ball Δ in Σ we may write

$$\Big(\fint_\Delta w \, d\sigma\Big)\Big(\fint_\Delta w^{-\frac{1}{p-1}} \, d\sigma\Big)^{p-1}$$

$$\leq \Big(⨍_{\Delta} \big[1 + |b - b_\Delta|\big]\, \mathrm{d}\sigma \Big) \Big(⨍_{\Delta} \Big(\frac{1}{\max\{1, |b|\}} \Big)^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1}$$

$$+ |b_\Delta| \Big(⨍_{\Delta} \Big(\frac{1}{\max\{1, |b|\}} \Big)^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1}$$

$$\leq 1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)} + \Big(⨍_{\Delta} \Big(\frac{|b_\Delta|}{\max\{1, |b|\}} \Big)^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1}. \tag{2.562}$$

Also, if $E_0 := \{x \in \Delta : |b(x)| > |b_\Delta|/2\}$ and $E_1 := \{x \in \Delta : |b(x)| \leq |b_\Delta|/2\}$, then for each point $x \in E_0$ we have $|b_\Delta|/|b(x)| \leq 2$ while for each point $x \in E_1$ we have $|b_\Delta| \leq 2|b(x) - b_\Delta|$. Consequently,

$$\Big(⨍_{\Delta} \Big(\frac{|b_\Delta|}{\max\{1, |b|\}} \Big)^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1} \leq \max\big\{1, 2^{p-2}\big\} \cdot \Big(\frac{1}{\sigma(\Delta)} \int_{E_0} \Big(\frac{|b_\Delta|}{|b|} \Big)^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1}$$

$$+ \max\big\{1, 2^{p-2}\big\} \cdot \Big(\frac{1}{\sigma(\Delta)} \int_{E_1} |b_\Delta|^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1}$$

$$\leq \max\big\{2, 2^{p-1}\big\} \cdot \Big(\frac{\sigma(E_0)}{\sigma(\Delta)} \Big)^{p-1}$$

$$+ \max\big\{2, 2^{p-1}\big\} \cdot \Big(⨍_{\Delta} |b - b_\Delta|^{\frac{1}{p-1}} \mathrm{d}\sigma \Big)^{p-1}$$

$$\leq C_{\Sigma,p}\big(1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\big), \tag{2.563}$$

where the last step above uses the John-Nirenberg inequality. In view of the arbitrariness of the surface ball Δ, from the estimates in (2.562)–(2.563) we may conclude that $w \in A_p(\Sigma, \sigma)$ and $[w]_{A_p} \leq C_{\Sigma,p}(1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)})$ for some constant $C_{\Sigma,p} \in (0, \infty)$ which is independent of b. This takes care of the very last claim in item *(8)*. Finally, the claim in item *(9)* is a consequence of (2.502) and the doubling properties of σ and w (for the latter see item *(5)* above). □

Given that the class of Muckenhoupt weights is going to play a prominent role in this work, it is appropriate to include some relevant concrete examples of interest.

Example 2.12 Suppose $\Sigma \subseteq \mathbb{R}^n$ (where $n \geq 2$) is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. Also, fix some $p \in (1, \infty)$ along with an arbitrary point $x_0 \in \Sigma$ and a power $a \in \mathbb{R}$. Then the function

$$w : \Sigma \to [0, \infty], \quad w(x) := |x - x_0|^a \ \text{ for each } \ x \in \Sigma \tag{2.564}$$

is a Muckenhoupt weight in $A_p(\Sigma, \sigma)$ if and only if $a \in \big(1-n, (p-1)(n-1)\big)$. Furthermore, whenever this happens, $[w]_{A_p}$ depends only on the Ahlfors regularity constant of Σ, p, and a.

See, for example, [54, Proposition 1.5.9, p. 42]. In a more general geometric setting, we have the following result, implied by work in [45].

Proposition 2.21 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $d \in [0, n-1)$ and consider a d-set $E \subseteq \Sigma$, i.e., a closed subset E of Σ with the property that there exists some Borel outer-measure μ on E satisfying*

$$\mu\big(B(x,r)\cap E\big) \approx r^d, \quad \textit{uniformly for } x \in E \textit{ and } r \in \big(0, 2\,diam(E)\big). \tag{2.565}$$

Then for each $p \in (1,\infty)$ and each $a \in \big(d+1-n, (p-1)(n-1-d)\big)$ the function $w := \big[dist(\cdot, E)\big]^a$ is a Muckenhoupt weight in the class $A_p(\Sigma, \sigma)$. Moreover, $[w]_{A_p}$ depends only on the Ahlfors regularity constant of Σ, the proportionality constants in (2.565), d, p, and a.

We continue to explore properties of Muckenhoupt weights in the context of Ahlfors regular sets which are relevant for this work.

Lemma 2.13 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then for each $w \in A_\infty(\Sigma, \sigma)$ one has*

$$BMO(\Sigma, \sigma) \subseteq L^1_{loc}(\Sigma, w). \tag{2.566}$$

Proof This is a direct consequence of (2.524), item *(2)* in Proposition 2.20, (2.538), and (2.96). □

If $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$, then for each weight function w on Σ we have $L^\infty(\Sigma, \sigma) = L^\infty(\Sigma, w)$, i.e., these vector spaces coincide and they have identical norms. Remarkably, whenever $w \in A_\infty(\Sigma, \sigma)$ it follows that the BMO spaces on Σ with respect to σ and w are once again identical. Here is a formal statement of this fact (compare with [117, Theorem 5, p. 236]).

Lemma 2.14 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some weight $w \in A_\infty(\Sigma, \sigma)$ (hence, there exists some $p \in (1,\infty)$ for which $w \in A_p(\Sigma, \sigma)$). Then there exists a constant $C \in [1,\infty)$ which depends only on p, $[w]_{A_p}$, n, and the Ahlfors regularity constant of Σ such that*

$$C^{-1}\,\|f\|_{BMO(\Sigma,\sigma)} \le \|f\|_{BMO(\Sigma,w)} \le C\,\|f\|_{BMO(\Sigma,\sigma)} \tag{2.567}$$

for each function $f \in L^1_{loc}(\Sigma, \sigma) \cap L^1_{loc}(\Sigma, w)$.

Moreover, for each σ-measurable function f on Σ one has the equivalence

$$f \in BMO(\Sigma, \sigma) \Longleftrightarrow f \in BMO(\Sigma, w) \tag{2.568}$$

and if either of these memberships materializes then $\|f\|_{BMO(\Sigma,\sigma)} \approx \|f\|_{BMO(\Sigma,w)}$ *where the implicit proportionality constants depend only on* p, $[w]_{A_p}$, n, *and the Ahlfors regularity constant of* Σ. *Succinctly put,*

$$\text{the spaces } BMO(\Sigma, \sigma) \text{ and } BMO(\Sigma, w) \text{ coincide as sets and have equivalent semi-norms.} \tag{2.569}$$

Proof Pick a function $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma) \cap L^1_{\mathrm{loc}}(\Sigma, w)$. To prove the first inequality in (2.567), start by writing (2.525) with f replaced by $f - ⨍_\Delta f \, dw$ for some arbitrary surface ball $\Delta \subseteq \Sigma$, then invoke (2.102) to obtain

$$\begin{aligned} \|f\|_{\mathrm{BMO}(\Sigma,\sigma)} &\leq 2 \sup_{\Delta\subseteq\Sigma} \inf_{c\in\mathbb{R}} \Big(⨍_\Delta |f - c| \, d\sigma \Big) \leq 2 \sup_{\Delta\subseteq\Sigma} ⨍_\Delta \Big| f - ⨍_\Delta f \, dw \Big| \, d\sigma \\ &\leq 2[w]^{1/p}_{A_p} \cdot \sup_{\Delta\subseteq\Sigma} \Big(⨍_\Delta \Big| f - ⨍_\Delta f \, dw \Big|^p dw \Big)^{1/p} \\ &\leq C \, \|f\|_{\mathrm{BMO}(\Sigma,w)} , \end{aligned} \tag{2.570}$$

for some constant $C \in (0, \infty)$ as in the statement. To prove the second inequality in (2.567), observe first that w belongs to some Reverse Hölder class, say w satisfies (2.536) for some $q \in (1, \infty)$. If $q' \in (1, \infty)$ denotes the Hölder conjugate exponent of q, then (2.538) allows to estimate

$$\begin{aligned} \inf_{c\in\mathbb{R}} \Big(⨍_\Delta |f - c| \, dw \Big) &\leq ⨍_\Delta \Big| f - ⨍_\Delta f \, d\sigma \Big| \, dw \\ &\leq C \Big(⨍_\Delta \Big| f - ⨍_\Delta f \, d\sigma \Big|^{q'} d\sigma \Big)^{1/q'} , \end{aligned} \tag{2.571}$$

for some constant $C \in (0, \infty)$ of the same nature as before. Taking the supremum over all surface balls $\Delta \subseteq \Sigma$ and then using John-Nirenberg's inequality, we ultimately obtain $\|f\|_{\mathrm{BMO}(\Sigma,w)} \leq C \, \|f\|_{\mathrm{BMO}(\Sigma,\sigma)}$, as desired.

As regards the equivalence in (2.568), assume first that $f \in \mathrm{BMO}(\Sigma, \sigma)$. Then (2.566) implies that $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma) \cap L^1_{\mathrm{loc}}(\Sigma, w)$, so (2.567) holds. Conversely, assume the function f belongs to $\mathrm{BMO}(\Sigma, w)$. In particular, $f \in L^1_{\mathrm{loc}}(\Sigma, w)$ and the John-Nirenberg inequality (for the doubling measure w) guarantees that we also have $f \in L^p_{\mathrm{loc}}(\Sigma, w)$. In concert with (2.525) the latter membership implies that $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$, hence once again (2.567) applies. □

The doubling and self-improving properties of Muckenhoupt weights yield the following result (see [111, §7.7] for a proof).

Lemma 2.15 *Suppose $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. In this setting, fix some $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$. Then*

$$\int_{\Sigma} \frac{w(x)}{(1 + |x|^{n-1})^p}\, d\sigma(x) < +\infty. \tag{2.572}$$

Also,

$$\begin{gathered}\text{there exists } \varepsilon \in (0,1) \text{ such that}\\ L^p(\Sigma, w) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1 + |x|^{n-1-\varepsilon}}\Big),\end{gathered} \tag{2.573}$$

and there exists an exponent $p_o \in (1, p]$ with the property that

$$\begin{gathered}L^p(\Sigma, w) \hookrightarrow L^q\Big(\Sigma, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big)\\ \text{continuously, for each fixed } q \in (0, p_o).\end{gathered} \tag{2.574}$$

As a consequence,

$$L^p(\Sigma, w) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \ \text{continuously}, \tag{2.575}$$

and

$$L^p(\Sigma, w) \subseteq L^p_{loc}(\Sigma, w) \subseteq \bigcup_{1<q<p} L^q_{loc}(\Sigma, \sigma) \subseteq L^1_{loc}(\Sigma, \sigma). \tag{2.576}$$

2.8 Sobolev Spaces on Ahlfors Regular Sets

Consider an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, (2.508) implies that

σ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on $\partial\Omega$, where the latter set is endowed with the topology canonically inherited from $\mathbb{R}^n$. (2.577)

Among other things, this implies (cf. [111, §3.7]) that for every $f \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ we have

$$f = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega \iff \int_{\partial\Omega} f\phi \, d\sigma = 0 \text{ for every } \phi \in \mathscr{C}_0^\infty(\mathbb{R}^n). \tag{2.578}$$

In this context, define the family of first-order tangential derivative operators, $\partial_{\tau_{jk}}$ with $j, k \in \{1, \dots, n\}$, acting on functions $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ according to

$$\partial_{\tau_{jk}}\varphi := \nu_j(\partial_k\varphi)\big|_{\partial\Omega} - \nu_k(\partial_j\varphi)\big|_{\partial\Omega} \quad \text{for all} \quad j, k \in \{1, \dots, n\}. \tag{2.579}$$

The starting point in the development of a brand of first-order Sobolev spaces on $\partial\Omega$ is the observation that for any two functions $\varphi, \psi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and every pair of indices $j, k \in \{1, \dots, n\}$ one has the boundary integration by parts formula

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi)\psi \, d\sigma = -\int_{\partial\Omega} \varphi(\partial_{\tau_{jk}}\psi) \, d\sigma. \tag{2.580}$$

Indeed, identity (2.580) is a consequence of the Divergence Formula (2.20) applied to a suitable vector field, namely $\vec{F} := \partial_k(\varphi\psi)e_j - \partial_j(\varphi\psi)e_k$ (where $\{e_i\}_{1 \leq i \leq n}$ is the standard orthonormal basis in $\mathbb{R}^n$), which is smooth, compactly supported, divergence-free, and satisfies $\nu \cdot \vec{F} = (\partial_{\tau_{jk}}\varphi)\psi + \varphi(\partial_{\tau_{jk}}\psi)$ at σ-a.e. point on $\partial\Omega$.

Next, given a function $f \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ along with two indices $j, k \in \{1, \dots, n\}$, we shall say that $\partial_{\tau_{jk}} f$ exists in (or, belongs to) the space $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ if there exists a function $f_{jk} \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ such that

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f \, d\sigma = -\int_{\partial\Omega} \varphi f_{jk} \, d\sigma \quad \text{for all} \quad \varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n). \tag{2.581}$$

In view of (2.578), we conclude that the function f_{jk} is unambiguously defined (σ-a.e.) by the demand in (2.581). Henceforth we shall favor the notation

$$\partial_{\tau_{jk}} f := f_{jk} \tag{2.582}$$

which, in particular, allows us to recast (2.581) more in line with (2.580), namely as

$$\int_{\partial\Omega} f(\partial_{\tau_{jk}}\varphi) \, d\sigma = -\int_{\partial\Omega} (\partial_{\tau_{jk}} f)\varphi \, d\sigma \quad \text{for all} \quad \varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n). \tag{2.583}$$

In analogy with the classical flat, Euclidean case, it is natural to regard $\partial_{\tau_{jk}} f$ as a weak (tangential) derivative of the function f. The developments so far allow us to define a convenient functional analytic environment within which is possible to consider such weak (tangential) derivatives of functions in $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$. Specifically, for each $p \in [1, \infty]$ we introduce the local Sobolev space $L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ as

$$L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma) := \big\{ f \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) : \partial_{\tau_{jk}} f \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma), \ 1 \leq j, k \leq n \big\}. \tag{2.584}$$

In such a context, we define the tangential gradient operator as (with the summation convention over repeated indices in effect)

$$L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma) \ni f \mapsto \nabla_{\tan} f := \big(\nu_k \partial_{\tau_{kj}} f\big)_{1\le j\le n}. \tag{2.585}$$

If Ω is actually a UR domain, we may recover the weak tangential derivatives from the components of the tangential gradient operator via (cf. [112, §11.4], [61, Lemma 3.40])

$$\begin{gathered}\partial_{\tau_{jk}} f = \nu_j (\nabla_{\tan} f)_k - \nu_k (\nabla_{\tan} f)_j, \quad 1 \le j,k \le n,\\ \text{for every } \ f \in L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma) \ \text{ with } \ p\in(1,\infty).\end{gathered} \tag{2.586}$$

Going further, having fixed an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, define the (boundary) weighted Sobolev space

$$L^p_1(\partial\Omega,w) := \big\{ f \in L^p(\partial\Omega,w) : \partial_{\tau_{jk}} f \in L^p(\partial\Omega,w),\ 1\le j,k\le n\big\} \tag{2.587}$$

which is a Banach space when equipped with the norm

$$L^p_1(\partial\Omega,w) \ni f \mapsto \|f\|_{L^p_1(\partial\Omega,w)} := \|f\|_{L^p(\partial\Omega,w)} + \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}. \tag{2.588}$$

Since there exists $q \in (1,\infty)$ such that $L^p(\partial\Omega,w) \hookrightarrow L^q_{\mathrm{loc}}(\partial\Omega,\sigma)$ (cf. Lemma 2.15), we see that $L^p_1(\partial\Omega,w) \hookrightarrow L^q_{1,\mathrm{loc}}(\partial\Omega,\sigma)$ for such an exponent q. In particular, the equality in (2.586) holds for every function $f \in L^p_1(\partial\Omega,w)$ whenever Ω is actually a UR domain.

In the same geometric setting, recall that $L^{p,q}(\partial\Omega,\sigma)$ with $p,q \in (0,\infty]$ stands for the scale of Lorentz spaces on $\partial\Omega$, with respect to the measure σ. These are quasi-Banach spaces which arise naturally as intermediate spaces for the real interpolation method used within the scale of ordinary Lebesgue spaces. In particular, this implies that

$$\begin{gathered}L^{p,q}(\partial\Omega,\sigma) \hookrightarrow L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap \Big(\textstyle\bigcap_{1<s<p} L^s_{\mathrm{loc}}(\partial\Omega,\sigma)\Big)\\ \text{whenever } \ p\in(1,\infty) \ \text{ and } \ q \in (0,\infty].\end{gathered} \tag{2.589}$$

In relation to this scale of spaces, it is also of interest to consider (boundary) Lorentz-based Sobolev spaces. Specifically, following work in [112, §11.1], for each $p \in (1,\infty)$ and $q \in (0,\infty]$ we set

$$L^{p,q}_1(\partial\Omega,\sigma) := \big\{ f \in L^{p,q}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in L^{p,q}(\partial\Omega,\sigma),\ 1\le j,k\le n\big\} \tag{2.590}$$

which is a quasi-Banach space when equipped with the quasi-norm

$$L_1^{p,q}(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{L_1^{p,q}(\partial\Omega,\sigma)} := \|f\|_{L^{p,q}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{L^{p,q}(\partial\Omega,\sigma)}. \tag{2.591}$$

In the proposition below, which refines [61, Lemma 3.36, p. 2678], we study the manner in which weak tangential derivatives interact with pointwise nontangential traces. See [112, §11.3] for a proof.

Proposition 2.22 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in [1,\infty]$, an aperture parameter $\kappa \in (0,\infty)$, and a truncation parameter $\varepsilon > 0$. In this context, assume the function $u \in W^{1,1}_{loc}(\Omega)$ satisfies*

$$\mathcal{N}_\kappa^\varepsilon u \in L^p_{loc}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa^\varepsilon(\nabla u) \in L^p_{loc}(\partial\Omega,\sigma), \tag{2.592}$$

and the nontangential traces

$$u\big|_{\partial\Omega}^{\kappa-n.t.} \text{ and } (\partial_j u)\big|_{\partial\Omega}^{\kappa-n.t.} \text{ for } j \in \{1,\dots,n\} \\ \text{exist at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.593}$$

Then $u\big|_{\partial\Omega}^{\kappa-n.t.}$ belongs to $L^p_{1,loc}(\partial\Omega,\sigma)$, the functions $(\partial_1 u)\big|_{\partial\Omega}^{\kappa-n.t.}, \dots, (\partial_n u)\big|_{\partial\Omega}^{\kappa-n.t.}$ belong to $L^p_{loc}(\partial\Omega,\sigma)$ and, for each $j,k \in \{1,\dots,n\}$ and for σ-a.e. point on $\partial\Omega$, one has

$$\partial_{\tau_{jk}}\Big(u\big|_{\partial\Omega}^{\kappa-n.t.}\Big) = \nu_j\Big((\partial_k u)\big|_{\partial\Omega}^{\kappa-n.t.}\Big) - \nu_k\Big((\partial_j u)\big|_{\partial\Omega}^{\kappa-n.t.}\Big). \tag{2.594}$$

In particular, for each $j,k \in \{1,\dots,n\}$ one has

$$\Big|\partial_{\tau_{jk}}\Big(u\big|_{\partial\Omega}^{\kappa-n.t.}\Big)\Big| \leq 2\mathcal{N}_\kappa^\varepsilon(\nabla u) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.595}$$

The following result from [112, §11.3] may be regarded as a weighted counterpart of Proposition 2.22, in which no assumptions are made regarding the existence of the nontangential boundary traces of the derivatives of the function involved. The reader is reminded that the truncated nontangential maximal operator $\mathcal{N}_\kappa^\varepsilon$ has been defined in (2.9).

Proposition 2.23 *Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture parameter $\kappa \in (0,\infty)$ and an integrability exponent $p \in (1,\infty)$. Assume $w : \partial\Omega \to [0,+\infty]$ is a σ-measurable function with $0 < w(x) < \infty$ for σ-a.e. $x \in \partial\Omega$ and $w^{-1/p} \in L^{p'}_{loc}(\partial\Omega,\sigma)$, where $p' \in (1,\infty)$ denotes the Hölder conjugate exponent of p; in particular, $L^p(\partial\Omega, w\sigma) \hookrightarrow L^1_{loc}(\partial\Omega,\sigma)$. Finally, fix* *a truncation parameter* $\varepsilon > 0$. *In this setting, suppose that some complex-valued function* $u \in W^{1,1}_{loc}(\Omega)$ *has been given which satisfies the following conditions:*

$$\begin{aligned} &u\big|^{\kappa-n.t.}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and} \\ &\mathcal{N}^{\varepsilon}_{\kappa} u \in L^1_{loc}(\partial\Omega, \sigma), \quad \mathcal{N}^{\varepsilon}_{\kappa}(\nabla u) \in L^p(\partial\Omega, w\sigma). \end{aligned} \tag{2.596}$$

Then the nontangential trace $u\big|^{\kappa-n.t.}_{\partial\Omega}$ *belongs to* $L^1_{1,loc}(\partial\Omega, \sigma)$ *and satisfies*

$$\begin{aligned} &\partial_{\tau_{jk}}\Big(u\big|^{\kappa-n.t.}_{\partial\Omega}\Big) \in L^p(\partial\Omega, w\sigma) \text{ for each } j,k \in \{1,\dots,n\} \\ &\text{and } \sum_{j,k=1}^{n} \Big\|\partial_{\tau_{jk}}\Big(u\big|^{\kappa-n.t.}_{\partial\Omega}\Big)\Big\|_{L^p(\partial\Omega, w\sigma)} \leq C \left\|\mathcal{N}^{\varepsilon}_{\kappa}(\nabla u)\right\|_{L^p(\partial\Omega, w\sigma)} \end{aligned} \tag{2.597}$$

for some constant $C \in (0,\infty)$ *independent of* u.

For further use, let us also consider `homogeneous Muckenhoupt weighted boundary Sobolev spaces`. Specifically, we make the following definition.

Definition 2.18 Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Given some integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, define

$$\begin{aligned} \dot{L}^p_1(\partial\Omega, w) := \Big\{ & f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\text{loc}}(\partial\Omega, w) : \\ & \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \text{ for each } j,k \in \{1,\dots,n\}\Big\}, \end{aligned} \tag{2.598}$$

and equip this space with the semi-norm

$$\dot{L}^p_1(\partial\Omega, w) \ni f \longmapsto \|f\|_{\dot{L}^p_1(\partial\Omega,w)} := \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega,w)}. \tag{2.599}$$

It is clear from definitions and (2.575) that we have a continuous embedding

$$L^p_1(\partial\Omega, w) \hookrightarrow \dot{L}^p_1(\partial\Omega, w). \tag{2.600}$$

Also, all constant functions on $\partial\Omega$ belong to $\dot{L}^p_1(\partial\Omega, w)$ and their semi-norm vanishes. As such, we will occasionally find it useful to work with $\dot{L}^p_1(\partial\Omega, w)\big/\sim$, the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, which we equip with the semi-norm

$$\dot{L}_1^p(\partial\Omega, w)\big/\sim\ni [f] \mapsto \|[f]\|_{\dot{L}_1^p(\partial\Omega,w)/\sim} := \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega,w)}. \tag{2.601}$$

We shall next prove a membership criterion to a global weighted Lebesgue space, formulated in the lemma below.

Lemma 2.16 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed, unbounded set, which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Pick $p \in (1,\infty)$ along with $w \in A_p(\Sigma,\sigma)$, and fix a reference point $x_0 \in \Sigma$. Suppose $f \in L^1_{loc}(\Sigma, w)$ is such that*

$$C_* := \sup_{r>0} \frac{1}{r}\Big(\int_{\Delta_r} |f - f_{r,w}|^p \, dw\Big)^{1/p} < +\infty, \tag{2.602}$$

where, for each $r \in (0,\infty)$,

$$\Delta_r := B(x_0, r) \cap \Sigma \quad \text{and} \quad f_{r,w} := \fint_{\Delta_r} f \, dw. \tag{2.603}$$

Then there exists some constant $C = C\big(\Sigma, n, p, [w]_{A_p}\big) \in (0,\infty)$ with the property that for each $r \in (0,\infty)$ one has

$$\int_{\Sigma} \frac{|f(x) - f_{r,w}|}{(r + |x - x_0|)^n} \, d\sigma(x) \le \frac{C \cdot C_*}{w(\Delta_r)^{1/p}}. \tag{2.604}$$

In particular, f belongs to the space $L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^n}\big)$.

Proof For starters, observe that for each $r > 0$ we have

$$\begin{aligned} |f_{2r,w} - f_{r,w}| &\le \fint_{\Delta_r} |f - f_{2r,w}| \, \mathrm{d}w \le C \fint_{\Delta_{2r}} |f - f_{2r,w}| \, \mathrm{d}w \\ &\le C\Big(\fint_{\Delta_{2r}} |f - f_{2r,w}|^p \, \mathrm{d}w\Big)^{1/p} \le \frac{C \cdot C_* \cdot r}{w(\Delta_{2r})^{1/p}}, \end{aligned} \tag{2.605}$$

thanks to the fact that w is doubling, Hölder's inequality, and (2.602). With this in hand (and keeping in mind that both σ and w are doubling), for each given $r > 0$ we may then estimate

$$\int_{\Sigma\setminus\Delta_r} \frac{|f(x) - f_{r,w}|}{|x - x_0|^n} \, \mathrm{d}\sigma(x) \le C \sum_{j=0}^{\infty} \frac{1}{(2^j r)^n} \int_{\Delta_{2^{j+1}r}\setminus\Delta_{2^j r}} |f - f_{r,w}| w^{1/p} w^{-1/p} \, \mathrm{d}\sigma$$

$$
\begin{aligned}
&\leq C \sum_{j=0}^{\infty} \frac{1}{(2^j r)^n} \Big(\int_{\Delta_{2^{j+1}r}} |f - f_{r,w}|^p \, \mathrm{d}w \Big)^{1/p} \Big(\int_{\Delta_{2^{j+1}r}} w^{-p'/p} \, \mathrm{d}\sigma \Big)^{1/p'} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{(2^j r)^n} \, \| f - f_{r,w} \|_{L^p(\Delta_{2^{j+1}r}, w)} \cdot \frac{\sigma\big(\Delta_{2^{j+1}r}\big)}{w\big(\Delta_{2^{j+1}r}\big)^{1/p}} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j r \cdot w\big(\Delta_{2^j r}\big)^{1/p}} \Big\{ \| f - f_{2^{j+1}r,w} \|_{L^p(\Delta_{2^{j+1}r}, w)} \\
&\qquad\qquad + \sum_{k=0}^{j} \| f_{2^{k+1}r,w} - f_{2^k r,w} \|_{L^p(\Delta_{2^{j+1}r}, w)} \Big\} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j r \cdot w\big(\Delta_{2^j r}\big)^{1/p}} \Big\{ C_* \cdot 2^{j+1} r \\
&\qquad\qquad + \sum_{k=0}^{j} | f_{2^{k+1}r,w} - f_{2^k r,w} | \cdot w\big(\Delta_{2^{j+1}r}\big)^{1/p} \Big\} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j r \cdot w\big(\Delta_{2^j r}\big)^{1/p}} \Big\{ C_* \cdot 2^{j+1} r \\
&\qquad\qquad + \sum_{k=0}^{j} \frac{C \cdot C_* \cdot 2^k r}{w(\Delta_{2^{k+1}r})^{1/p}} \cdot w\big(\Delta_{2^{j+1}r}\big)^{1/p} \Big\} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j r \cdot w\big(\Delta_{2^j r}\big)^{1/p}} \Big\{ \sum_{k=0}^{j} \frac{C_* \cdot 2^k r}{w(\Delta_{2^{k+1}r})^{1/p}} \cdot w\big(\Delta_{2^{j+1}r}\big)^{1/p} \Big\} \\
&\leq C \cdot C_* \sum_{j=0}^{\infty} \frac{1}{2^j} \Big\{ \sum_{k=0}^{j} \frac{2^k}{w(\Delta_{2^{k+1}r})^{1/p}} \Big\} \\
&\leq C \cdot C_* \sum_{k=0}^{\infty} \Big\{ \sum_{j=k}^{\infty} \frac{1}{2^j} \Big\} \frac{2^k}{w(\Delta_{2^{k+1}r})^{1/p}} \\
&\leq C \cdot C_* \sum_{k=0}^{\infty} \frac{1}{w(\Delta_{2^{k+1}r})^{1/p}}
\end{aligned}
$$

$$= C \cdot C_* \frac{1}{w(\Delta_r)^{1/p}} \sum_{k=0}^{\infty} \Big(\frac{w(\Delta_r)}{w(\Delta_{2^{k+1}r})} \Big)^{1/p}$$

$$\leq C \cdot C_* \frac{1}{w(\Delta_r)^{1/p}} \sum_{k=0}^{\infty} \Big(\frac{\sigma(\Delta_r)}{\sigma(\Delta_{2^{k+1}r})} \Big)^{\tau/p}$$

$$\leq C \cdot C_* \frac{1}{w(\Delta_r)^{1/p}} \sum_{k=0}^{\infty} \Big(\frac{1}{2^k} \Big)^{(n-1)\tau/p} = \frac{C \cdot C_*}{w(\Delta_r)^{1/p}}, \tag{2.606}$$

where $\tau > 0$ is as in (2.537). Above, the second inequality is a consequence of Hölder's inequality, the third inequality uses (2.520), the fifth and sixth inequalities are based on (2.602) and (2.605), while the penultimate inequality is implied by (2.537).

In addition, as a consequence of (2.602), (2.520), and Hölder's inequality we have

$$\int_{\Delta_r} \frac{|f(x) - f_{r,w}|}{r^n} \, \mathrm{d}\sigma(x) = r^{-n} \int_{\Delta_r} |f - f_{r,w}| w^{1/p} w^{-1/p} \, \mathrm{d}\sigma$$

$$= r^{-n} \Big(\int_{\Delta_r} |f - f_{r,w}|^p \, \mathrm{d}w \Big)^{1/p} \Big(\int_{\Delta_r} w^{-p'/p} \, \mathrm{d}\sigma \Big)^{1/p'}$$

$$\leq C_* \cdot r^{1-n} [w]_{A_p}^{1/p} \frac{\sigma(\Delta_r)}{w(\Delta_r)^{1/p}}$$

$$\leq \frac{C \cdot C_*}{w(\Delta_r)^{1/p}}. \tag{2.607}$$

Together, (2.606) and (2.607) prove (2.604). □

In the proposition below we explore consequences of the integrability of the nontangential maximal operator of the gradient of a given function.

Proposition 2.24 *Make the assumption that $\Omega \subseteq \mathbb{R}^n$ is an* NTA *domain with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Pick an arbitrary aperture parameter $\kappa > 0$ and fix a reference point $x_o \in \partial\Omega$. Finally, select a function $u \in \mathscr{C}^1(\Omega)$.*

Then there exist $\widetilde{\kappa} > 0$ large enough along with some threshold $R \in (0, +\infty]$ (which may be taken $+\infty$ if $\partial\Omega$ is unbounded) and some constant $C \in (1, \infty)$, all independent of the given function u, such that for each $\delta \in (0, R)$ one may find a compact subset K_δ of Ω, of diameter $\approx \delta$ and distance to the boundary $\approx \delta$, with the property that

$$\big(\mathcal{N}_\kappa^\delta u\big)(x) \le C\delta \cdot \mathcal{N}_{\widetilde{\kappa}}^{C\delta}(\nabla u)(x) + \sup_{K_\delta}|u|, \qquad \forall\, x \in B(x_o,\delta)\cap\partial\Omega. \tag{2.608}$$

Moreover, there exists some sufficiently large $C > 1$ such that

$$\text{if } \mathcal{N}_\kappa^\varepsilon(\nabla u) \text{ belongs to } L^p_{loc}(\partial\Omega,\sigma) \text{ for some } p\in(0,\infty] \text{ and some } \varepsilon>0 \text{ then } \mathcal{N}_\kappa^{\varepsilon/C}u \in L^p_{loc}(\partial\Omega,\sigma), \text{ the nontangential trace } \big(u\big|_{\partial\Omega}^{\kappa-n.t.}\big)(x) \text{ exists at } \sigma\text{-a.e. } x\in\partial\Omega, \text{ and the function } u\big|_{\partial\Omega}^{\kappa-n.t.} \text{ is } \sigma\text{-measurable on } \partial\Omega. \tag{2.609}$$

In addition, if $\partial\Omega$ is unbounded then there exists $C = C(\Omega) \in (0,\infty)$ such that

$$\Big|\big(u\big|_{\partial\Omega}^{\kappa-n.t.}\big)(x) - \big(u\big|_{\partial\Omega}^{\kappa-n.t.}\big)(y)\Big| \le C|x-y|\cdot\big[\mathcal{N}_\kappa(\nabla u)(x) + \mathcal{N}_\kappa(\nabla u)(y)\big] \quad \text{for } \sigma\text{-a.e. points } x,y\in\partial\Omega. \tag{2.610}$$

Finally, if the original hypotheses are strengthened by now assuming that $\partial\Omega$ is an unbounded Ahlfors regular set and that the nontangential maximal function of the Jacobian of u satisfies $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w)$ for some integrability exponent $p \in (1,\infty)$ and some weight $w \in A_p(\partial\Omega,\sigma)$ then

$$\text{the nontangential trace } u\big|_{\partial\Omega}^{\kappa-n.t.} \text{ belongs to the Muckenhoupt weighted homogeneous boundary Sobolev space } \dot{L}^p_1(\partial\Omega,w) \text{ and one has } \Big\|u\big|_{\partial\Omega}^{\kappa-n.t.}\Big\|_{\dot{L}^p_1(\partial\Omega,w)} \le C\,\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \text{ for a constant } C\in(0,\infty) \text{ independent of the function } u. \tag{2.611}$$

Proof The claims in (2.608)–(2.610) have been established in [111, §8.4]. To justify (2.611), work under the additional assumptions that $\partial\Omega$ is an unbounded Ahlfors regular set and that $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w)$ for some $w \in A_p(\partial\Omega,\sigma)$ with $p \in (1,\infty)$. Observe that the latter condition implies, in light of (2.576), that $\mathcal{N}_\kappa(\nabla u) \in L^1_{\rm loc}(\partial\Omega,\sigma)$, so the current assumptions are indeed stronger. To lighten the exposition, abbreviate

$$f := u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad\text{and}\quad g := \mathcal{N}_\kappa(\nabla u). \tag{2.612}$$

From (2.609), (2.13), (2.608), (2.11) (used with $\sigma := w$), and Proposition 2.23 (whose applicability is ensured by (2.576)) it follows that

$$\begin{aligned} &f \in L^p_{\rm loc}(\partial\Omega,w), \quad \partial_{\tau_{jk}} f \in L^p(\partial\Omega,w) \ \text{ for all } j,k\in\{1,\dots,n\}, \\ &\text{and} \quad \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)} \le C\,\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)}, \end{aligned} \tag{2.613}$$

for some constant $C \in (0, \infty)$ independent of u. Also, we may recast (2.610) as

$$|f(x) - f(y)| \leq C|x - y| \cdot \big[g(x) + g(y)\big] \ \text{ for } \sigma\text{-a.e. } \ x, y \in \partial\Omega. \tag{2.614}$$

To proceed, fix a reference point $x_0 \in \partial\Omega$ and for each given scale $r \in (0, \infty)$ define $\Delta_r := B(x_0, r) \cap \partial\Omega$ and $f_{r,w} := ⨍_{\Delta_r} f \, dw$. Then using (2.614) and Hölder's inequality for each $r \in (0, \infty)$ we may estimate

$$\begin{aligned}
\Big(⨍_{\Delta_r} &|f(x) - f_{r,w}|^p \, dw(x)\Big)^{1/p} \\
&= \Big(⨍_{\Delta_r} \Big|f(x) - ⨍_{\Delta_r} f(y) \, dw(y)\Big|^p \, dw(x)\Big)^{1/p} \\
&\leq \Big(⨍_{\Delta_r} ⨍_{\Delta_r} |f(x) - f(y)|^p \, dw(x) \, dw(y)\Big)^{1/p} \\
&\leq C\Big(⨍_{\Delta_r} ⨍_{\Delta_r} |x - y|^p \big(g(x) + g(y)\big)^p \, dw(x) \, dw(y)\Big)^{1/p} \\
&\leq Cr\Big(⨍_{\Delta_r} g^p \, dw\Big)^{1/p} \leq Cr\Big(⨍_{\partial\Omega} g^p \, dw\Big)^{1/p} \\
&= Cr \, \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega, w)} ,
\end{aligned} \tag{2.615}$$

since $x, y \in \Delta_r$ forces $|x - y| < 2r$. As a consequence,

$$\sup_{r>0} \frac{1}{r}\Big(⨍_{\Delta_r} |f - f_{r,w}|^p \, dw\Big)^{1/p} \leq C \, \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega, w)} < +\infty. \tag{2.616}$$

Having established estimate (2.616), from Lemma 2.16 we conclude that the function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. In view of this, (2.598)–(2.599), and (2.613) we then deduce that all claims in (2.611) are true. □

We next discuss the equivalence between membership to a global weighted Lebesgue space and certain Poincaré-type inequalities.

Proposition 2.25 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Fix some reference point $x_0 \in \partial\Omega$, along with some integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, assume that*

$$\begin{aligned}
&f \text{ is a function belonging to } L^1_{loc}(\partial\Omega, \sigma) \text{ with the property that} \\
&\partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \text{ for all } j, k \in \{1, \ldots, n\}.
\end{aligned} \tag{2.617}$$

Then the following statements are equivalent:

(i) *The function f belongs to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$.*

(ii) *There exists a constant $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0,\infty)$ which stays bounded when $[w]_{A_p}$ stays bounded and which is independent of the function f, with the property that if for each scale $r \in (0,\infty)$ one defines the surface ball $\Delta_r := B(x_0, r) \cap \partial\Omega$ and $f_{r,\sigma} := \fint_{\Delta_r} f\, d\sigma$ then*

$$\sup_{r>0} \frac{1}{r}\left(\int_{\Delta_r} |f - f_{r,\sigma}|^p\, dw\right)^{1/p} \leq C \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega, w)}. \tag{2.618}$$

(ii)' *The function f belongs to the space $L^1_{loc}(\partial\Omega, w)$ and there exists some constant $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0,\infty)$ which stays bounded when $[w]_{A_p}$ stays bounded and which is independent of the function f, with the property that if for each $r \in (0,\infty)$ one defines $\Delta_r := B(x_0, r) \cap \partial\Omega$ and $f_{r,w} := \fint_{\Delta_r} f\, dw$ then*

$$\sup_{r>0} \frac{1}{r}\left(\int_{\Delta_r} |f - f_{r,w}|^p\, dw\right)^{1/p} \leq C \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega, w)}. \tag{2.619}$$

(iii) *For each $r \in (0,\infty)$ there exists a constant $C_r \in (0,\infty)$ which depends only on Ω, p, $[w]_{A_p}$, x_0, and r such that, with $f_{r,\sigma}$ as before, one has*

$$\int_{\partial\Omega} \frac{|f(x) - f_{r,\sigma}|}{1+|x|^n}\, d\sigma(x) \leq \frac{C_r}{w(\Delta_r)^{1/p}} \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega, w)}. \tag{2.620}$$

(iii)' *The function f belongs to $L^1_{loc}(\partial\Omega, w)$ and for each $r \in (0,\infty)$ there exists a constant $C_r \in (0,\infty)$ which depends only on Ω, p, $[w]_{A_p}$, x_0, and r such that, with $f_{r,w}$ as before,*

$$\int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n}\, d\sigma(x) \leq \frac{C_r}{w(\Delta_r)^{1/p}} \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega, w)}. \tag{2.621}$$

(iv) *There exists a constant $C = C(\Omega, p, w, x_0) \in (0,\infty)$ independent of f, and some constant $c_f \in \mathbb{C}$ which is allowed to depend on f, such that*

$$\|f - c_f\|_{L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\right)} \leq C \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^p(\partial\Omega, w)}. \tag{2.622}$$

(v) *The function f belongs to the space $\dot{L}^p_1(\partial\Omega, w)$.*

Proof We start by proving the implication *(i)* ⇒ *(ii)*. To this end, assume that in addition to (2.617) we have $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Denote by ν the geometric measure theoretic outward unit normal to Ω and set

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.623}$$

With ω_{n-1} denoting the surface area of the unit sphere in $\mathbb{R}^n$, at each point $x \in \Omega_\pm$ define

$$u_\pm(x) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \Big\{ \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} - \frac{\langle \nu(y), y\rangle}{|y|^n} \cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(y) \Big\} f(y)\, \mathrm{d}\sigma(y). \tag{2.624}$$

Then work in [114, §1.5] ensures that for an arbitrary, fixed, aperture parameter $\kappa > 0$ there exists a constant $C \in (0, \infty)$ independent of f and which stays bounded when $[w]_{A_p}$ stays bounded, such that

$$\begin{aligned} & u_\pm \in \mathscr{C}^\infty(\Omega_\pm), \quad \mathcal{N}_\kappa(\nabla u_\pm) \in L^p(\partial\Omega, w), \\ & \|\mathcal{N}_\kappa(\nabla u_\pm)\|_{L^p(\partial\Omega,w)} \le C \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}, \\ & f = u_+\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - u_-\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \end{aligned} \tag{2.625}$$

Hence,

$$\begin{aligned} & g := \mathcal{N}_\kappa(\nabla u_+) + \mathcal{N}_\kappa(\nabla u_-) \in L^p(\partial\Omega, w) \\ & \text{has } \|g\|_{L^p(\partial\Omega,w)} \le C \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}, \end{aligned} \tag{2.626}$$

for some constant $C \in (0, \infty)$ independent of f and which stays bounded when $[w]_{A_p}$ stays bounded. In addition, thanks to (2.610), the function g satisfies

$$|f(x) - f(y)| \le C|x-y| \cdot \big[g(x) + g(y)\big] \text{ for } \sigma\text{-a.e. } x, y \in \partial\Omega. \tag{2.627}$$

Granted these properties, we may proceed as in (2.615) to conclude that

$$\begin{aligned} & \Big(\int_{\Delta_r} |f(x) - f_{r,\sigma}|^p \, \mathrm{d}w(x)\Big)^{1/p} \\ & \qquad = \Big(\int_{\Delta_r} \Big| f(x) - ⨍_{\Delta_r} f(y)\, \mathrm{d}\sigma(y)\Big|^p \mathrm{d}w(x)\Big)^{1/p} \\ & \qquad \le \Big(\int_{\Delta_r} \Big(⨍_{\Delta_r} |f(x) - f(y)|\, \mathrm{d}\sigma(y)\Big)^p \mathrm{d}w(x)\Big)^{1/p} \end{aligned}$$

$$\leq C\Big(\int_{\Delta_r}\Big(⨍_{\Delta_r}|x-y|\big(g(x)+g(y)\big)\,\mathrm{d}\sigma(y)\Big)^p\,\mathrm{d}w(x)\Big)^{1/p}$$

$$\leq Cr\Big(\int_{\Delta_r}g^p\,\mathrm{d}w\Big)^{1/p}+w(\Delta_r)^{\frac{1}{p}}⨍_{\Delta_r}g\,\mathrm{d}\sigma(y)$$

$$\leq Cr\Big(\int_{\partial\Omega}g^p\,\mathrm{d}w\Big)^{1/p}, \tag{2.628}$$

since $x,y\in\Delta_r$ forces $|x-y|<2r$ and we have used (2.525). Eventually we conclude that, for some constant $C\in(0,\infty)$ independent of f and which stays bounded when $[w]_{A_p}$ stays bounded, we have

$$\sup_{r>0}\frac{1}{r}\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}\leq C\|g\|_{L^p(\partial\Omega,w)}\leq C\sum_{j,k=1}^{n}\big\|\partial_{\tau_{jk}}f\big\|_{L^p(\partial\Omega,w)}. \tag{2.629}$$

This completes the proof of the implication *(i)* ⇒ *(ii)*.

To see that *(ii)* ⇒ *(ii)'* we first note that (2.617) and (2.618) imply that for each $r>0$ we have

$$\Big(\int_{\Delta_r}|f|^p\,\mathrm{d}w\Big)^{1/p}\leq\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}+w(\Delta_r)^{1/p}|f_{r,\sigma}|$$

$$\leq C\,r\sum_{j,k=1}^{n}\big\|\partial_{\tau_{jk}}f\big\|_{L^p(\partial\Omega,w)}+w(\Delta_r)^{1/p}⨍_{\Delta_r}|f|\,\mathrm{d}\sigma<\infty. \tag{2.630}$$

This goes to show that $f\in L^p_{\rm loc}(\partial\Omega,w)\subseteq L^1_{\rm loc}(\partial\Omega,w)$. Granted this, for each $r>0$ we may estimate

$$\Big(\int_{\Delta_r}|f-f_{r,w}|^p\,\mathrm{d}w\Big)^{1/p}\leq\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}+w(\Delta_r)^{1/p}\,|f_{r,\sigma}-f_{r,w}|$$

$$\leq 2\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}. \tag{2.631}$$

With (2.631) in hand, (2.618) readily gives (2.619).

We next note that the implication *(ii)'* ⇒ *(iii)'* is seen from Lemma 2.16, the implication *(iii)'* ⇒ *(iv)* (respectively, *(iii)* ⇒ *(iv)*) follows by taking $r:=1$ and $c_f:=f_{1,w}$ (respectively, $c_f:=f_{1,\sigma}$), while the implication *(iv)* ⇒ *(i)* is a direct consequence that any constant belongs to the space $L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)$. The fact that *(iii)'* ⇒ *(iii)* may be justified writing (using the Ahlfors regularity of $\partial\Omega$; cf. (2.32))

$$\begin{aligned}\int_{\partial\Omega} \frac{|f(x) - f_{r,\sigma}|}{1+|x|^n}\, d\sigma(x) &\le \int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n}\, d\sigma(x) + C\,|f_{r,\sigma} - f_{r,w}| \\ &\le \int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n}\, d\sigma(x) + C \fint_{\Delta_r} |f - f_{r,w}|\, d\sigma \\ &\le C \int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n}\, d\sigma(x), \end{aligned} \tag{2.632}$$

where the constant $C \in (0, \infty)$ depends only on Ω, x_0, and r.

Hence, the claims in items *(i)*, *(ii)*, *(ii)'*, *(iii)*, *(iii)'*, and *(iv)* are all equivalent. In view of (2.598) it follows that the implication *(v)* $\Rightarrow$ *(i)* also holds. To finish the proof of the proposition it suffices to check that, collectively, (2.617) and items *(i)*-*(ii)'* imply the claim in item *(v)*. This, however, is apparent from (2.598) and the fact that (2.619) guarantees that $f \in L^p_{\rm loc}(\partial\Omega, w)$. □

Remark 2.4 Consider a two-sided NTA domain $\Omega \subseteq \mathbb{R}^n$ such that $\partial\Omega$ is an unbounded Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then Proposition 2.25 implies that the local L^p integrability property with respect to the measure w for functions in the homogeneous Muckenhoupt weighted boundary Sobolev space $\dot{L}^p_1(\partial\Omega, w)$ may be replaced by a (seemingly weaker) local absolute integrability property with respect to the measure w, or may be even suppressed altogether. Specifically, in such a setting we have (compare with (2.598))

$$\dot{L}^p_1(\partial\Omega, w) = \Big\{ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{\rm loc}(\partial\Omega, w) : \tag{2.633}$$

$$\partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \text{ for each } \; j, k \in \{1, \dots, n\}\Big\}$$

$$= \Big\{ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) : \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \tag{2.634}$$

$$\text{for each } \; j, k \in \{1, \dots, n\}\Big\}.$$

When considered on the boundaries of two-sided NTA domains, the quotient space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ turns out to be Banach. Here is a formal statement:

Proposition 2.26 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Pick some integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Recall that $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ denotes the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, equipped with the semi-norm* (2.601).

Then (2.601) *is a genuine norm on* $\dot{L}^p_1(\partial\Omega, w)\big/\sim$, *and* $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ *is a Banach space when equipped with the norm* (2.601).

Proof The fact that the semi-norm (2.601) is actually a norm on the space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ follows from (2.621).

To prove that $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ is complete when equipped with the norm (2.601), let $\{f_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \dot{L}^p_1(\partial\Omega, w)$ be such that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the quotient space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$. Then for each fixed $j, k \in \{1, \dots, n\}$ it follows that $\big\{\partial_{\tau_{jk}} f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\partial\Omega, w)$. Since the latter is complete, it follows that there exists $g_{jk} \in L^p(\partial\Omega, w)$ such that

$$\partial_{\tau_{jk}} f_\alpha \to g_{jk} \ \text{ in } \ L^p(\partial\Omega, w) \ \text{ as } \ \alpha \to \infty. \tag{2.635}$$

Fix a reference point $x_0 \in \partial\Omega$ and, for each $r \in (0,\infty)$, define $\Delta_r := B(x_0, r)\cap\partial\Omega$. Also, set $f_{\alpha,r,w} := ⨍_{\Delta_r} f_\alpha \, dw$ for each $r \in (0,\infty)$ and each $\alpha \in \mathbb{N}$. From (2.621) (written for $f := f_\alpha - f_\beta$) it follows that for each $r \in (0,\infty)$ there exists a constant $C_r \in (0,\infty)$ which depends on Ω, p, $[w]_{A_p}$, and r such that for each $\alpha, \beta \in \mathbb{N}$ we have

$$\begin{aligned} &\big\|\big(f_\alpha - f_{\alpha,r,w}\big) - \big(f_\beta - f_{\beta,r,w}\big)\big\|_{L^1\left(\partial\Omega,\, \frac{\sigma(x)}{1+|x|^n}\right)} \\ &\qquad\qquad \le \frac{C_r}{w(\Delta_r)^{1/p}} \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta\big\|_{L^p(\partial\Omega,w)}. \end{aligned} \tag{2.636}$$

In view of (2.635), this estimate implies that for each fixed $r \in (0,\infty)$ the sequence $\big\{f_\alpha - f_{\alpha,r,w}\big\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Hence, for each fixed $r \in (0,\infty)$ there exists $h_r \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ such that

$$f_\alpha - f_{\alpha,r,w} \to h_r \ \text{ in } \ L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \ \text{ as } \ \alpha \to \infty. \tag{2.637}$$

Next, the estimate recorded in (2.619) (written for $f := f_\alpha - f_\beta$) implies that there exists some constant $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0,\infty)$ with the property that for each fixed $r \in (0,\infty)$ we have

$$\begin{aligned} &\Big(\int_{\Delta_r} \big|\big(f_\alpha - f_{\alpha,r,w}\big) - \big(f_\beta - f_{\beta,r,w}\big)\big|^p \, dw\Big)^{1/p} \\ &\qquad\qquad \le C \cdot r \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta\big\|_{L^p(\partial\Omega,w)}. \end{aligned} \tag{2.638}$$

By once again relying on (2.635), we conclude that for each fixed $r \in (0,\infty)$ the sequence $\big\{f_\alpha\big|_{\Delta_r} - f_{\alpha,r,w}\big\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $L^p(\Delta_r, w)$. As such,

$$\begin{aligned}&\text{for each } r\in(0,\infty) \text{ there exists some } k_r\in L^p(\Delta_r,w)\\ &\text{such that } \ f_\alpha\big|_{\Delta_r}-f_{\alpha,r,w}\to k_r \ \text{ in } \ L^p(\Delta_r,w) \ \text{ as } \ \alpha\to\infty.\end{aligned}\tag{2.639}$$

Since convergence in Lebesgue spaces implies, after eventually passing to a subsequence, pointwise a.e. convergence, from (2.637) and (2.639) we see that, in fact,

$$h_r\big|_{\Delta_r}=k_r\in L^p(\Delta_r,w) \ \text{ for each } \ r\in(0,\infty).\tag{2.640}$$

From (2.637) we also see that for each fixed $r_1,r_2\in(0,\infty)$ we have

$$f_{\alpha,r_2,w}-f_{\alpha,r_1,w}\to h_{r_1}-h_{r_2} \ \text{ in } \ L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big) \ \text{ as } \ \alpha\to\infty.\tag{2.641}$$

This forces $h_{r_1}-h_{r_2}$ to be a constant which, in concert with (2.640), ultimately shows that actually

$$h_r\in L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\rm loc}(\partial\Omega,w) \ \text{ for each } \ r\in(0,\infty).\tag{2.642}$$

Henceforth, we agree to simply write h for h_r with $r=1$, and c_α for $f_{\alpha,r,w}$ with $r=1$. Then (2.642), (2.637) tell us that the function

$$h \ \text{ belongs to } \ L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\rm loc}(\partial\Omega,w),\tag{2.643}$$

and the sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}}\subseteq\mathbb{C}$ is such that

$$f_\alpha-c_\alpha\to h \ \text{ in } \ L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big) \ \text{ as } \ \alpha\to\infty.\tag{2.644}$$

For each $j,k\in\{1,\dots,n\}$ and each test function $\varphi\in\mathscr{C}^\infty_0(\mathbb{R}^n)$ we may then write

$$\begin{aligned}\int_{\partial\Omega}h(\partial_{\tau_{jk}}\varphi)\,d\sigma&=\lim_{\alpha\to\infty}\int_{\partial\Omega}(f_\alpha-c_\alpha)(\partial_{\tau_{jk}}\varphi)\,d\sigma\\&=-\lim_{\alpha\to\infty}\int_{\partial\Omega}\partial_{\tau_{jk}}(f_\alpha-c_\alpha)\varphi\,d\sigma=-\lim_{\alpha\to\infty}\int_{\partial\Omega}(\partial_{\tau_{jk}}f_\alpha)\varphi\,d\sigma\\&=-\int_{\partial\Omega}g_{jk}\varphi\,d\sigma,\end{aligned}\tag{2.645}$$

thanks to (2.644), (2.583), and (2.635). From this and (2.581)–(2.582) we then conclude that

$$\partial_{\tau_{jk}}h=g_{jk}\in L^p(\partial\Omega,w) \ \text{ for each } \ j,k\in\{1,\dots,n\}.\tag{2.646}$$

Collectively, (2.643) and (2.646) prove that $h \in \dot{L}^p_1(\partial\Omega, w)$. Finally, from (2.635), (2.646), and (2.601) we conclude that the sequence $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ converges to $[h]$, the class of h, in the quotient space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$. □