Chapter 1
Introduction

More than 25 years ago, in [71, Problem 3.2.2, p. 117], C. Kenig asked to "*Prove that the layer potentials are invertible in appropriate* [...] *spaces in* [*suitable subclasses of uniformly rectifiable*] *domains*." Kenig's main motivation in this regard stems from the desire of establishing solvability results for boundary value problems formulated in a rather inclusive geometric setting. In the buildup to this open question on [71, p. 116], it is remarked that there exist some rather general classes of open sets $\Omega \subseteq \mathbb{R}^n$ with the property that if $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ (where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$) then said layer potentials are bounded operators on $L^p(\partial\Omega, \sigma)$ for each exponent $p \in (1, \infty)$. Remarkably, this is the case whenever $\Omega \subseteq \mathbb{R}^n$ is an open set with a uniformly rectifiable boundary (cf. [40]).

To further elaborate on this issue, we need some notation. Fix $n \in \mathbb{N}$ with $n \geq 2$, along with $M \in \mathbb{N}$, and consider a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system in $\mathbb{R}^n$

$$L = \big(a^{\alpha\beta}_{jk} \partial_j \partial_k\big)_{1 \leq \alpha, \beta \leq M}, \tag{1.1}$$

where the summation convention over repeated indices is in effect (here and elsewhere in the manuscript). The weak ellipticity of the system L amounts to demanding that

$$\begin{array}{l} \text{the characteristic matrix } L(\xi) := \big(-a^{\alpha\beta}_{jk} \xi_j \xi_k\big)_{1 \leq \alpha, \beta \leq M} \text{ is} \\ \text{invertible for each vector } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \end{array} \tag{1.2}$$

This should be contrasted with the more stringent Legendre–Hadamard (strong) ellipticity condition which asks for the existence of some $c > 0$ such that

$$\mathrm{Re}\big\langle -L(\xi)\zeta\,, \overline{\zeta}\,\big\rangle \geq c\,|\xi|^2\,|\zeta|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad \zeta \in \mathbb{C}^M. \tag{1.3}$$

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Nonetheless, the weak ellipticity assumption which we shall enforce throughout ensures that the system L has a well-behaved fundamental solution, which is an even matrix-valued function $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^M$ whose first-order derivatives are positive homogeneous of degree $1-n$, of the sort discussed at length in [102] (see Theorem 3.1 for a brief review).

The given system L does not determine uniquely the coefficient tensor

$$A := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}} \tag{1.4}$$

since employing $\widetilde{A} := \big(\widetilde{a}^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}}$ in place of A in the right-hand side of (1.1) yields the same system whenever the difference $a^{\alpha\beta}_{jk} - \widetilde{a}^{\alpha\beta}_{jk}$ is antisymmetric in the indices j,k (for each $\alpha,\beta \in \{1,\dots,M\}$). Hence, there are a multitude of coefficient tensors A which may be used to represent the given system L as in (1.1). For each such coefficient tensor $A := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}}$ we shall associate a double layer potential operator K_A on the boundary of a given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ (see Definition 2.6). Specifically, if $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is the "surface measure" on $\partial\Omega$ and if $\nu = (\nu_1,\dots,\nu_n)$ denotes the geometric measure theoretic outward unit normal to Ω, then for each function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{1.5}$$

we define, at σ-a.e. point $x \in \partial\Omega$,

$$K_A f(x) := \Big(-\lim_{\varepsilon\to 0^+}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \nu_k(y) a^{\beta\alpha}_{jk}\big(\partial_j E_{\gamma\beta}\big)(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}. \tag{1.6}$$

(Note that (1.5) is the most general environment in which each truncated integral in (1.6) is absolutely convergent.)

To offer a simple example, consider the case when $L = \Delta$, the Laplacian, in $\mathbb{R}^2$. Then $n = 2$ and $M = 1$. In this scalar case, we agree to drop the Greek superscripts labeling the entries of the coefficient tensor (1.4) used to express L as in (1.1). Hence, we shall consider writings $\Delta = a_{jk}\partial_j\partial_k$ corresponding to various choices of the matrix $A = (a_{jk})_{1\leq j,k\leq 2} \in \mathbb{C}^{2\times 2}$. Two such natural choices are

$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}, \tag{1.7}$$

corresponding to which the recipe given in (1.6) yields

$$K_{A_0}f(x) = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}}\frac{\langle\nu(y), y-x\rangle}{|x-y|^2}f(y)\,\mathrm{d}\sigma(y)\ \text{ for } \sigma\text{-a.e. } x\in\partial\Omega, \tag{1.8}$$

i.e., the (two-dimensional) harmonic boundary-to-boundary double layer potential operator and, under the natural identification $\mathbb{R}^2 \equiv \mathbb{C}$,

$$K_{A_1}f(z) = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega\setminus\overline{B(z,\varepsilon)}}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\ \text{ for } \sigma\text{-a.e. } z\in\partial\Omega, \tag{1.9}$$

i.e., the boundary-to-boundary Cauchy integral operator, respectively.

Returning to the mainstream discussion in the general setting considered earlier, fundamental work in [40] guarantees that, if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, then for each coefficient tensor A as in (1.4) which may be employed to write the given system L as in (1.1), the boundary-to-boundary double layer potential K_A from (1.6) is a well-defined, linear, and bounded operator on $\big[L^p(\partial\Omega,\sigma)\big]^M$ for each $p \in (1,\infty)$. This property is particularly relevant in the treatment of the Dirichlet Problem for the system L in the uniformly rectifiable domain Ω when the boundary data are selected from the space $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p\in(1,\infty)$, i.e.,

$$(D)_p \begin{cases} u\in\big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = g \in \big[L^p(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{1.10}$$

where $\mathcal{N}_\kappa u$ is the nontangential maximal function, and $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ is the nontangential boundary trace, of the solution u (see the body of the manuscript for precise definitions; cf. (2.5) and (2.12)). Indeed, the essence of the boundary layer method is to consider as a candidate for the solution of the Dirichlet Problem (1.10) the $\mathbb{C}^M$-valued function u defined at each point $x\in\Omega$ by

$$u(x) := \Big(-\int_{\partial\Omega}\nu_k(y)a_{jk}^{\beta\alpha}\,\big(\partial_j E_{\gamma\beta}\big)(x-y)f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}, \tag{1.11}$$

for some yet-to-be-determined function $f = (f_\alpha)_{1\le\alpha\le M} \in \big[L^p(\partial\Omega,\sigma)\big]^M$. In light of the special format of u (in particular, thanks to the jump-formula (3.123)), this ultimately reduces the entire aforementioned Dirichlet Problem to the issue of solving the boundary integral equation

$$\big(\tfrac{1}{2}I + K_A\big)f = g \ \text{ on } \ \partial\Omega, \tag{1.12}$$

where I is the identity operator (see Sect. 6 for the actual implementation of this approach). As such, having the operator K_A well defined, linear, and bounded on

$\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in (1,\infty)$ opens the door for bringing in functional analytic techniques for inverting $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega,\sigma)\big]^M$ and eventually expressing the solution f as $\big(\frac{1}{2}I + K_A\big)^{-1}g$.

A breakthrough in this regard has been registered by S. Hofmann, M. Mitrea, and M. Taylor in [61], where they have employed Fredholm theory in order to solve the boundary integral equation (1.12). To describe one of their main results, suppose $L = \Delta$, the Laplacian in $\mathbb{R}^n$, is written as $\Delta = a_{jk}\partial_j\partial_k$ for $A := (\delta_{jk})_{1\le j,k\le n}$. The blueprint provided in (1.6) then produces the classical harmonic double layer potential operator K_Δ, acting on each $f \in L^p(\partial\Omega,\sigma)$ with $p \in (1,\infty)$ according to

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{1.13}$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. In regard to this operator, S. Hofmann, M. Mitrea, and M. Taylor have proved in [61, Theorem 4.36, pp. 2728-2729] that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, then for every threshold $\varepsilon > 0$ there exists some $\delta > 0$ (which depends only on said geometric characteristics of Ω, n, p, and ε) such that

$$\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta \Longrightarrow \operatorname{dist}\big(K_\Delta, \mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) < \varepsilon. \tag{1.14}$$

The distance in the left-hand side of (1.14) is measured in the John-Nirenberg space $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ of vector-valued functions of bounded mean oscillations on $\partial\Omega$ (with respect to the surface measure σ), from the unit vector $\nu \in \big[L^\infty(\partial\Omega,\sigma)\big]^n$ to the Sarason space $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$ of vector-valued functions of vanishing mean oscillations on $\partial\Omega$ (with respect to the surface measure σ), which is a closed subspace of $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ (cf. (2.111)). The distance in the right-hand side of (1.14) is considered from $K_\Delta \in \mathrm{Bd}(L^p(\partial\Omega,\sigma))$, the Banach space of all linear and bounded operators on $L^p(\partial\Omega,\sigma)$ equipped with the operator norm, to $\mathrm{Cp}(L^p(\partial\Omega,\sigma))$ which is the closed linear subspace of $\mathrm{Bd}(L^p(\partial\Omega,\sigma))$ consisting of all compact operators on $L^p(\partial\Omega,\sigma)$. In particular, in the class of domains currently considered, K_Δ is a *compact* operator on $L^p(\partial\Omega,\sigma)$ whenever ν belongs to $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$. This is remarkable in as much that a purely geometric condition implies a functional analytic property of a singular integral operator. Most importantly, (1.14) ensures the existence of some small threshold $\delta > 0$ (which depends only on said geometric characteristics of Ω, n, and p) with the property that

$$\begin{aligned} \operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta &\Longrightarrow \operatorname{dist}\big(K_\Delta, \mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) < \tfrac{1}{2} \\ &\Longrightarrow \tfrac{1}{2}I + K_\Delta \ \text{Fredholm operator with index zero on } L^p(\partial\Omega,\sigma). \end{aligned} \tag{1.15}$$

This is the main step in establishing that $\frac{1}{2}I + K_\Delta$ is actually an invertible operator on $L^p(\partial\Omega, \sigma)$ in said geometric setting, under the additional assumption that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected (see [61, Theorem 6.13, p. 2806]).

Another key result of a similar flavor to (1.14) proved in [61] pertains to the commutators $\big[M_{\nu_k}, R_j\big] := M_{\nu_k} R_j - R_j M_{\nu_k}$, where $j, k \in \{1, \dots, n\}$, between the operator M_{ν_k} of pointwise multiplication by ν_k, the k-th scalar component of the geometric measure theoretic outward unit normal ν to Ω, and the j-th Riesz transform R_j on $\partial\Omega$, acting on any given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{n-1}} \int_{\partial\Omega \setminus \overline{B(x,\varepsilon)}} \frac{x_j - y_j}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.16}$$

Specifically, [61, Theorem 2.19, p. 2608] states that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, and if some $p \in (1, \infty)$ has been fixed, then there exists some $C \in (0, \infty)$ (depending only on the aforementioned geometric characteristics of Ω, n, and p) such that

$$\sum_{j,k=1}^{n} \operatorname{dist}\big(\big[M_{\nu_k}, R_j\big], \operatorname{Cp}(L^p(\partial\Omega, \sigma))\big) \leq C \operatorname{dist}\big(\nu, \big[\operatorname{VMO}(\partial\Omega, \sigma)\big]^n\big). \tag{1.17}$$

Estimates of this type (with the Riesz transforms replaced by more general singular integral operators of the same nature) turned out to be a key ingredient in the proof of the fact that, if Ω is as above and $p \in (1, \infty)$, then for every threshold $\varepsilon > 0$ there exists some $\delta > 0$ (of the same nature as before) such that

$$\operatorname{dist}\big(\nu, \big[\operatorname{VMO}(\partial\Omega, \sigma)\big]^n\big) < \delta \implies \operatorname{dist}\big(K_\Delta, \operatorname{Cp}(L^p_1(\partial\Omega, \sigma))\big) < \varepsilon, \tag{1.18}$$

where $L^p_1(\partial\Omega, \sigma)$ is a certain brand of L^p-based Sobolev space of order one on $\partial\Omega$, introduced in [61] (and further developed in [109], [112, Chapter 11]).

These considerations have led to the development of a theory of boundary layer potentials in what was labeled in [61] as δ-regular SKT domains, a subclass of the family of bounded uniformly rectifiable domains inspired by work of S. Semmes [123, 124], and C. Kenig and T. Toro [72–74], whose trademark feature is the fact that the distance $\operatorname{dist}\big(\nu, \big[\operatorname{VMO}(\partial\Omega, \sigma)\big]^n\big)$, measured in the John-Nirenberg space $\big[\operatorname{BMO}(\partial\Omega, \sigma)\big]^n$, is $< \delta$. In turn, this was used in [61] to establish the well-posedness of the Dirichlet, Regularity, Neumann, and Transmission Problems for the Laplacian in the class of δ-regular SKT domains with δ sufficiently small (relative to other geometric characteristics of Ω). Quite recently, this theory has been extended in [90] to the case when the boundary data belong to Muckenhoupt weighted Lebesgue and Sobolev spaces.

In addition, the class of δ-regular SKT domains also turns out to be in the nature of best possible as far as the “close-to-compactness” results mentioned in (1.14) and

(1.17) are concerned. Indeed, [61, Theorem 4.41, p. 2743] states that, if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain with compact boundary and if some $p \in (1, \infty)$ has been fixed, then there exists some $C \in (0, \infty)$ (depending only on the uniform rectifiability character of Ω, n, and p) such that

$$\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big) \leq C\Big\{\operatorname{dist}\big(K_\Delta, \mathrm{Cp}(L^p(\partial\Omega, \sigma))\big) + \sum_{j,k=1}^{n} \operatorname{dist}\big(\big[M_{\nu_k}, R_j\big], \mathrm{Cp}(L^p(\partial\Omega, \sigma))\big)\Big\}^{1/n}. \tag{1.19}$$

In particular, if K_Δ and all commutators $\big[M_{\nu_k}, R_j\big]$ are compact on $L^p(\partial\Omega, \sigma)$ then ν belongs to $\big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n$.

The stated goal of [61] was to "*find the optimal geometric measure theoretic context in which Fredholm theory can be successfully implemented, along the lines of its original development, for solving boundary value problems with L^p data via the method of layer potentials* [*in domains with compact boundaries*]." In particular, [61] may be regarded as a sharp version of the fundamental work of E. Fabes, M. Jodeit, and N. Rivière in [49], dealing with the method of boundary layer potentials in bounded $\mathscr{C}^1$ domains. As such, the theory developed in [61] goes some way toward answering Kenig's open question formulated at the beginning of this introduction.

However, the insistence on $\partial\Omega$ being a *compact* set is prevalent in this work. In particular, the classical fact that the Dirichlet Problem (1.10) is uniquely solvable in the case when $\Omega = \mathbb{R}^n_+$ (by taking the convolution of the boundary datum g with the harmonic Poisson kernel in the upper half-space; cf. [9], [52], [132], [134]) does *not* fall under the tutelage of [61]. The issue is that once the uniformly rectifiable domain Ω is allowed to have an unbounded boundary then, generally speaking, singular integral operators like the harmonic double layer (1.13) are no longer (close to being) compact on $L^p(\partial\Omega, \sigma)$, though they remain well defined, linear, and bounded on this space, as long as $1 < p < \infty$. The fact that the theory developed in [61] is not applicable in this scenario leads one to speculate whether the treatment of layer potentials may be extended to a class of unbounded domains that includes the upper half-space. In particular, it is natural to ask whether there is a parallel theory for unbounded domains $\Omega \subseteq \mathbb{R}^n$ in which we control the mean oscillations of its outward unit normal ν by suitably adapting the condition $\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big) < \delta$ which is ubiquitous in [61]. This is indeed the main goal in the present monograph.

A seemingly peculiar aspect of the harmonic double layer operator (which, in hindsight turns out to be one of its salient features) is that, as visible from (1.13), if $\Omega = \mathbb{R}^n_+$ then $K_\Delta = 0$. Indeed, in such a case we have $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ and $\nu = (0, \dots, 0, -1)$, hence $\langle \nu(y), y - x\rangle = 0$ for all $x, y \in \partial\Omega$. This observation lends some credence to the conjecture loosely formulated as follows:

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a uniformly rectifiable domain and } 1 < p < \infty, \text{ then the operator norm } \|K_\Delta\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \text{ is small if } \Omega \text{ is close to being a half-space in } \mathbb{R}^n. \tag{1.20}$$

To make this precise, one needs to choose an appropriate way of quantifying the proximity of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ to a half-space in $\mathbb{R}^n$. Since a result from [111, §5.10] (based on work in [59]) gives that a uniformly rectifiable domain $\Omega \subsetneq \mathbb{R}^n$ actually is a half-space in $\mathbb{R}^n$ if and only if its geometric measure theoretic outward unit normal ν is a constant vector field, in which scenario $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 0$, it is natural to formulate the following problem (which is a precise, quantitative version of (1.20)):

find a continuous non-decreasing function $\phi : [0, 1] \to [0, \infty)$ which vanishes at the origin with the property that for any given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ and any given integrability exponent $p \in (1, \infty)$ there exists some constant $C \in (0, \infty)$ (which depends only on the uniform rectifiability character of Ω, the dimension n, and the exponent p) such that $\|K_\Delta\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \leq C\phi(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n})$. (1.21)

We may go a step further and adopt a broader perspective, by replacing the Laplacian with a more general system of the sort discussed in (1.1). Specifically, consider a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system L in $\mathbb{R}^n$ written as in (1.1) for some coefficient tensor A as in (1.4). Then one may speculate whether there exists some continuous non-decreasing function $\phi : [0, 1] \to [0, \infty)$ which vanishes at the origin with the property that for any given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ and any given exponent $p \in (1, \infty)$ there exists some constant $C \in (0, \infty)$ (which depends only on the uniform rectifiability character of Ω, the dimension n, the exponent p, and the coefficient tensor A) such that the double layer potential operator K_A associated with the set Ω and the coefficient tensor A as in (1.6) satisfies

$$\|K_A\|_{[L^p(\partial\Omega,\sigma)]^M\to[L^p(\partial\Omega,\sigma)]^M} \leq C\phi(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}). \tag{1.22}$$

It turns out that the choice of the coefficient tensor A used to write the given system L drastically affects the veracity of (1.22). Indeed, consider the case when $L := \Delta$ is the Laplacian in $\mathbb{R}^2$, and $\Omega := \mathbb{R}^2_+$. Observe that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} = 0$ in this case, since ν is constant. From (1.7)–(1.8) we see that $K_{A_0} = 0$, which is in agreement with what (1.22) predicts in this case. On the other hand, the operator K_{A_1} from (1.9) becomes (under the natural identification $\partial\Omega \equiv \mathbb{R}$)

$$K_{A_1} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{f(y)}{y-x}\,\mathrm{d}y \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}, \tag{1.23}$$

i.e., $K_{A_1} = (\mathrm{i}/2)H$ where

$$Hf(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\substack{y \in \mathbb{R} \\ |x-y|>\varepsilon}} \frac{f(y)}{x-y}\, dy \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R} \tag{1.24}$$

is the classical Hilbert transform on the real line. In particular, since $H^2 = -I$ we have $\left(K_{A_1}\right)^2 = 4^{-1} I$ which goes to show that

$$\left\| K_{A_1} \right\|_{L^p(\mathbb{R},\mathcal{L}^1) \to L^p(\mathbb{R},\mathcal{L}^1)} \geq 2^{-1} \tag{1.25}$$

invalidating (1.22) in this case.

A higher-dimensional version of the above considerations goes as follows. Given $n \in \mathbb{N}$ with $n \geq 2$, let $\{E_j\}_{1 \leq j \leq n}$ be a family of $2^n \times 2^n$ matrices satisfying, with $I_{2^n \times 2^n}$ denoting the $2^n \times 2^n$ identity matrix,

$$\begin{aligned} \left(E_j\right)^2 &= -I_{2^n \times 2^n} \quad \text{for each } \ j \in \{1, \dots, n\} \ \text{ and} \\ E_j E_k &= -E_k E_j \ \text{ for all } \ j, k \in \{1, \dots, n\} \ \text{ with } \ j \neq k. \end{aligned} \tag{1.26}$$

Specifically, consider the double-indexed family of matrices $\left\{E_j^m\right\}_{\substack{1 \leq m \leq n \\ 1 \leq j \leq m}}$ defined inductively by

$$E_1^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \tag{1.27}$$

and, in general, given any $m \in \{1, \dots, n-1\}$,

$$E_j^{m+1} := \begin{pmatrix} E_j^m & 0 \\ 0 & -E_j^m \end{pmatrix} \in \mathbb{R}^{2^{m+1} \times 2^{m+1}} \quad \text{for each } \ j \in \{1, \dots, m\}, \tag{1.28}$$

and

$$E_{m+1}^{m+1} := \begin{pmatrix} 0 & -I_{2^m \times 2^m} \\ I_{2^m \times 2^m} & 0 \end{pmatrix} \in \mathbb{R}^{2^{m+1} \times 2^{m+1}}, \tag{1.29}$$

where $I_{2^m \times 2^m}$ denotes the $2^m \times 2^m$ identity matrix. Induction then shows that for each $m \in \{1, \dots, n\}$ we have

$$\begin{aligned} \left(E_j^m\right)^2 &= -I_{2^m \times 2^m} \quad \text{for each } \ j \in \{1, \dots, m\} \ \text{ and} \\ E_j^m E_k^m &= -E_k^m E_j^m \ \text{ for all } \ j, k \in \{1, \dots, m\} \ \text{ with } \ j \neq k. \end{aligned} \tag{1.30}$$

In particular, abbreviating $E_j := E_j^n$ for each $j \in \{1, \dots, n\}$ then guarantees that the conditions in (1.26) are satisfied.

To proceed, define $M := 2^n$ and denote by $I_{M\times M}$ the $M \times M$ identity matrix. Consider the $M \times M$ second-order system in $\mathbb{R}^n$ defined as

$$L := \Delta \cdot I_{M\times M}, \tag{1.31}$$

where $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian in $\mathbb{R}^n$. In particular, the fundamental solution E_L associated with the weakly elliptic system L as in Theorem 3.1 is given by

$$E_L := E_\Delta \cdot I_{M\times M}, \tag{1.32}$$

where E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, defined in (3.27).

Next, for each $j, k \in \{1, \dots, n\}$ let us denote by $(a_{jk}^{\alpha\beta})_{1\le\alpha,\beta\le M}$ the entries of the $M \times M$ matrix $-E_j E_k$, i.e.,

$$-E_j E_k = \Big(a_{jk}^{\alpha\beta}\Big)_{1\le\alpha,\beta\le M} \in \mathbb{R}^{M\times M} \quad \text{for each} \quad j, k \in \{1, \dots, n\}. \tag{1.33}$$

Then, with the summation convention over repeated indices in effect, we have

$$\big(a_{jk}^{\alpha\beta}\partial_j\partial_k\big)_{1\le\alpha,\beta\le M} = -E_j E_k \partial_j \partial_k = -(E_j)^2\partial_j^2 = \Delta \cdot I_{M\times M}, \tag{1.34}$$

thanks to (1.26). Hence,

$$L = \big(a_{jk}^{\alpha\beta}\partial_j\partial_k\big)_{1\le\alpha,\beta\le M}. \tag{1.35}$$

Consider next the boundary-to-boundary double layer potential operator K_{A_1} associated as in (1.6) with the coefficient tensor

$$A_1 := \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \text{ with entries as in (1.33)} \tag{1.36}$$

and the domain $\Omega := \mathbb{R}^n_+$. In view of (1.32) and the fact that the outward unit normal vector to $\mathbb{R}^n_+$ is given by $\nu = (0, \dots, 0, -1)$, the action of said double layer potential operator on each function $f = (f_\alpha)_{1\le\alpha\le M} \in \big[L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\big)\big]^M$ is given at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ by

$$K_{A_1} f(x') = \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} a_{jn}^{\beta\alpha}\big(\partial_j E_\Delta\big)(x'-y') f_\alpha(y')\, dy'\Bigg)_{1\le\beta\le M}$$

$$= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} \big(\partial_j E_\Delta\big)(x'-y') E_j E_n f(y')\,\mathrm{d}y'. \tag{1.37}$$

Hence, with $(R_j)_{1\le j\le n-1}$ denoting the Riesz transforms in $\mathbb{R}^{n-1}$ (cf. (1.16)), we may recast (1.37) simply as

$$K_{A_1} = \sum_{j=1}^{n-1} \tfrac{1}{2} E_j E_n R_j \quad \text{on} \quad \big[L^1\big(\mathbb{R}^{n-1}, \tfrac{dx'}{1+|x'|^{n-1}}\big)\big]^M. \tag{1.38}$$

Fix now an arbitrary integrability exponent $p \in (1,\infty)$. Then (1.38), (1.26), together with the usual Riesz transform identities (i.e., $\sum_{j=1}^{n-1} R_j^2 = -I$ and $R_j R_k = R_k R_j$ for each $j,k \in \{1,\dots,n\}$) imply that

$$\begin{aligned}
\big(K_{A_1}\big)^2 &= \Big(\sum_{j=1}^{n-1} \tfrac{1}{2} E_j E_n R_j\Big)^2 = \frac{1}{4}\sum_{j,k=1}^{n-1} E_j E_n E_k E_n R_j R_k \\
&= \frac{1}{4}\sum_{j,k=1}^{n-1} E_j E_k R_j R_k = \frac{1}{4}\sum_{j=1}^{n-1} E_j^2 R_j^2 \\
&= \frac{1}{4}\Big(-\sum_{j=1}^{n-1} R_j^2\Big) I_{M\times M} = \tfrac{1}{4} I_{M\times M}
\end{aligned} \tag{1.39}$$

as operators on $\big[L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})\big]^M$. Much as with its two-dimensional counterpart in (1.25), this goes to show that

$$\big\| K_{A_1}\big\|_{[L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^M\to[L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^M} \ge 2^{-1} \tag{1.40}$$

once again invalidating (1.22) for the current choice of coefficient tensor. On the other hand, the choice of the coefficient tensor

$$\begin{gathered}
A_0 := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \text{ with } a^{\alpha\beta}_{jk} := \delta_{\alpha\beta}\delta_{jk} \\
\text{for all } 1\le\alpha,\beta\le M \text{ and } 1\le j,k\le n
\end{gathered} \tag{1.41}$$

allows the system (1.31) to be written as in (1.35) and the boundary-to-boundary double layer potential operator K_{A_0} associated as in (1.6) with the coefficient tensor A_0 and the domain $\Omega := \mathbb{R}^n_+$ is $K_{A_0} = 0$ (cf. the first line in (1.37)).

The above considerations bring up the question of determining which of the many coefficient tensors A that may be used in the representation of the given system

L as in (1.1) actually give rise to double layer potential operators K_A (via the blueprint (1.6)) that have a chance of satisfying the estimate formulated in (1.22). This question is of an algebraic nature. To answer it, we find it convenient to adopt a more general point of view and consider the class of singular integral operators acting at σ-a.e. point $x \in \partial\Omega$ on functions f as in (1.5) according to

$$T_\Theta f(x) := \left(\lim_{\varepsilon \to 0^+} \int_{\partial\Omega \setminus \overline{B(x,\varepsilon)}} \big\langle \Theta_\gamma(x-y)\nu(y), f(y) \big\rangle \, d\sigma(y) \right)_{1 \leq \gamma \leq M}, \tag{1.42}$$

where

$$\begin{gathered} \Theta = (\Theta_\gamma)_{1 \leq \gamma \leq M} \text{ with each } \Theta_\gamma \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M \times n} \\ \text{odd and positive homogeneous of degree } 1-n. \end{gathered} \tag{1.43}$$

Note that K_A fits into this class, as it corresponds to (1.42) with $\Theta = (\Theta_\gamma)_{1 \leq \gamma \leq M}$ given by $\Theta_\gamma := \big(-a^{\beta\alpha}_{jk} \partial_j E_{\gamma\beta} \big)_{\substack{1 \leq \alpha \leq M \\ 1 \leq k \leq n}}$ for each index $\gamma \in \{1, \dots, M\}$.

In this notation, the question is to find what additional condition should be imposed on $\Theta = (\Theta_\gamma)_{1 \leq \gamma \leq M}$ so that the analogue of (1.22) holds with the operator K_A replaced by T_Θ. The latter inequality implies that

$$T_\Theta \text{ must vanish whenever } \Omega \text{ is a half-space in } \mathbb{R}^n. \tag{1.44}$$

Choosing $\Omega := \{z \in \mathbb{R}^n : \langle z, \omega \rangle > 0\}$ with $\omega \in S^{n-1}$ arbitrary then leads to the conclusion that for each index $\gamma \in \{1, \dots, M\}$ we have

$$\Theta_\gamma(x-y)\omega = 0 \text{ for each } \omega \in S^{n-1} \text{ and each } x, y \in \langle \omega \rangle^\perp \text{ with } x \neq y. \tag{1.45}$$

Specializing this to the case when $y = 0$ and observing that $x \in \langle \omega \rangle^\perp$ is equivalent to having $\omega \in \langle x \rangle^\perp$, we arrive at

$$\Theta_\gamma(x)\omega = 0 \in \mathbb{C}^M \text{ whenever } x \neq 0 \text{ and } \omega \in \langle x \rangle^\perp, \tag{1.46}$$

which is the same as saying that for each vector $x \in \mathbb{R}^n \setminus \{0\}$ the rows of the matrix $\Theta_\gamma(x) \in \mathbb{C}^{M \times n}$ are scalar multiples of x. Thus, there exists a family of scalar functions $k_{\gamma,1}, \dots, k_{\gamma,M}$ defined in $\mathbb{R}^n \setminus \{0\}$ such that

$$\begin{gathered} \text{for each } x \in \mathbb{R}^n \setminus \{0\}, \text{ the rows of } \Theta_\gamma(x) \\ \text{are } k_{\gamma,1}(x)x, \dots, k_{\gamma,M}(x)x. \end{gathered} \tag{1.47}$$

Ultimately, this implies that $k := \big(k_{\gamma,\alpha}\big)_{\substack{1 \leq \gamma \leq M \\ 1 \leq \alpha \leq M}}$ is a matrix-valued function belonging to $\big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M \times M}$ which is even, positive homogeneous of degree $-n$, and such that for each $\gamma \in \{1, \dots, M\}$ we have

$$\Theta_\gamma(x)\omega = \langle x, \omega\rangle k_{\gamma\cdot}(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \omega \in \mathbb{R}^n. \tag{1.48}$$

Consequently, T_Θ from (1.42) may be simply recast as

$$Tf(x) = \lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.49}$$

In terms of the original double layer potential operator K_A, the above argument proves that

> if (1.22) holds then the integral kernel of K_A is necessarily of the form $\langle x-y, \nu(y)\rangle k(x-y)$ for some matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ which is even and positive homogeneous of degree $-n$. (1.50)

Algebraic conditions, formulated solely in terms of A, guaranteeing that the integral kernel of K_A has the distinguished structure singled out in (1.50) have been identified in [115, Chapter 1] (see Definition 3.1). Henceforth, we shall refer to such a coefficient tensor A as being "distinguished," and we shall denote by $\mathfrak{A}_L^{\rm dis}$ the collection of all distinguished coefficient tensors which may be employed in the writing of a given system L.

In (3.223) we show that *all scalar* second-order homogeneous constant complex coefficient weakly elliptic operators L in $\mathbb{R}^n$ with $n \geq 3$ possess precisely one distinguished coefficient tensor. Consequently, $\mathfrak{A}_L^{\rm dis}$ is nonempty (in fact, a singleton) whenever $L = \operatorname{div} A \nabla$ in $\mathbb{R}^n$ with $n \geq 3$, with the coefficient matrix $A = (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$ satisfying the weak ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \neq 0, \qquad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{1.51}$$

In particular, this is the case for the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$.

Other examples of weakly elliptic second-order homogeneous constant coefficient systems which possess distinguished coefficient tensors are obtained by considering the complex version of the Lamé system of elasticity in $\mathbb{R}^n$, with $n \geq 2$,

$$L_{\mu,\lambda} := \mu\Delta + (\lambda + \mu)\nabla\operatorname{div}, \tag{1.52}$$

where the Lamé moduli $\lambda, \mu \in \mathbb{C}$ are assumed to satisfy

$$\mu \neq 0, \quad 2\mu + \lambda \neq 0, \quad 3\mu + \lambda \neq 0. \tag{1.53}$$

The first two requirements in (1.53) are equivalent to having the system $L_{\mu,\lambda}$ weakly elliptic (in the sense of (1.2)), while the last requirement in (1.53) ensures the existence of a distinguished coefficient tensor for $L_{\mu,\lambda}$. It turns out that if the last condition in (1.53) is violated then $L_{\mu,\lambda}$ fails to have a distinguished coefficient tensor.

It is of interest to remark that the (strong) Legendre–Hadamard ellipticity condition (1.3) holds for the complex Lamé system $L_{\mu,\lambda}$ if and only if

$$\operatorname{Re}\mu > 0 \text{ and } \operatorname{Re}(2\mu + \lambda) > 0. \tag{1.54}$$

As such, our results apply to certain classes of weakly elliptic second-order systems which are not necessarily strongly elliptic (in the sense of Legendre–Hadamard). Also, while the Lamé system is symmetric, we stress that the main results in this monograph require no symmetry for the systems involved.

Recall that ${}^m\mathrm{e}$ denotes the m-th tetration of e (involving m copies of e, combined via exponentiation), i.e.,

$${}^m\mathrm{e} := \underbrace{\mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot^{\mathrm{e}}}}}}}_{m \text{ copies of e}}, \quad \text{the } m\text{-th fold exponentiation of e.} \tag{1.55}$$

For each $t \geq 0$ let us define

$$t^{\langle m\rangle} := \begin{cases} 0 & \text{if } t = 0, \\ t \cdot \underbrace{\ln\Big(\cdots \ln\big(\ln(1/t)\big)\cdots\Big)}_{m \text{ natural logarithms}} & \text{if } 0 < t \leq ({}^m\mathrm{e})^{-1}, \\ ({}^m\mathrm{e})^{-1} & \text{if } t > ({}^m\mathrm{e})^{-1}. \end{cases} \tag{1.56}$$

One of the main results in this work asserts that if L is a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, and if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, then for each $m \in \mathbb{N}$, each $A \in \mathfrak{A}_L^{\mathrm{dis}}$, and each $p \in (1, \infty)$ there exists a constant $C_m \in (0, \infty)$ (which depends only on m, n, p, A, and the uniform rectifiability character of Ω) such that estimate (1.22) actually holds for the choice of the function $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(t) := t^{\langle m\rangle}$ for each $t \in [0, \infty)$. In particular, this offers a solution to the problem formulated in (1.21).

See Theorem 4.7 for a result of a more general flavor, formulated in terms of Muckenhoupt weighted Lebesgue spaces. Specifically, if the system L, the coefficient tensor A, and the set Ω are as just described, then for each $m \in \mathbb{N}$ and Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$ there exists a constant $C_m \in (0, \infty)$ (which now also depends on $[w]_{A_p}$, defined in (2.517)) with the property that

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{1.57}$$

In turn, Theorem 4.7 is painlessly implied by the even more general result presented in Theorem 4.2 which is one of the focal points of this monograph. The proof of Theorem 4.2 uses a combination of tools of a purely geometric nature (such as Theorem 2.6 containing a versatile version of a decomposition result originally established by S. Semmes for smooth surfaces in [123] then subsequently strengthened as to apply to rough settings in [61], and the estimate from Proposition 2.15 controlling the inner product between the integral average of the outward unit normal and the "chord" in terms of the BMO semi-norm of the outward unit normal to a domain), techniques of a purely harmonic analytic nature (like good-λ inequalities, maximal operator estimates, stopping time arguments, and Muckenhoupt weight theory), and a bootstrap argument designed to successively improve the nature of the function ϕ in (1.22).

These considerations lead us to adopt (as we do in Definition 2.15) the following basic piece of terminology. Given $\delta > 0$, an open, nonempty, proper subset Ω of $\mathbb{R}^n$ is said to be a δ-`flat Ahlfors regular domain` (or δ-AR domain, for short) if $\partial\Omega$ is an Ahlfors regular set, and if $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then the geometric measure theoretic outward unit normal ν to Ω is well defined at σ-a.e. point on $\partial\Omega$ and satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta. \tag{1.58}$$

Remarkably, demanding that δ in (1.58) is small has topological and metric implications for the underlying domain, namely Ω is two-sided NTA domain, which is a connected unbounded open set, with a connected unbounded boundary, and an unbounded connected complement (see Theorem 2.4). In the two-dimensional setting we actually show that the class of δ-AR domains with $\delta \in (0, 1)$ small agrees with the category of chord-arc domains with small constant (see Theorem 2.7 for a precise statement). Most importantly, (1.57) shows that the oscillatory behavior of the outward unit normal is a key factor in determining the size of the operator norm for the double layer potential operator K_A on $\big[L^p(\partial\Omega, w)\big]^M$.

Inspired by the format of a double layer operator (cf. (1.6)), so far we have been searching for singular integral operators fitting the general template in (1.42) for which it may be possible to control their operator norm in terms of $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$. While $\{T_\Theta : \Theta$ as in (1.43)$\}$ is a linear space, this is not stable under transposition (which is an isometric transformation and, hence, preserves the quality of having a small norm). This suggests that we cast a wider net and consider the class of singular integrals acting at σ-a.e. point $x \in \partial\Omega$ on functions f as in (1.5) according to

$$T_{\Theta^1,\Theta^2} f(x)$$

$$:= \left(\lim_{\varepsilon\to 0^+} \int\limits_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \Big\langle \Theta^1_\gamma(x-y)\nu(y) - \Theta^2_\gamma(x-y)\nu(x),\, f(y)\Big\rangle \,d\sigma(y)\right)_{1\le\gamma\le M} \tag{1.59}$$

where $\Theta_1 = (\Theta^1_\gamma)_{1\le\gamma\le M}$ and $\Theta_2 = (\Theta^2_\gamma)_{1\le\gamma\le M}$ are as in (1.43). The latter condition ensures that T_{Θ^1,Θ^2} is a well-defined, linear, and bounded operator on $\big[L^p(\partial\Omega, w)\big]^M$ (recall that we are assuming Ω to be a uniformly rectifiable domain). Consequently, $\{T_{\Theta^1,\Theta^2} : \Theta^1, \Theta^2$ as in (1.43)$\}$ is a linear subspace of the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$ which contains each double layer K_A as in (1.6) as well as its formal transpose $K^\#_A$, whose action on each function f as in (1.5) at σ-a.e. $x \in \partial\Omega$ is given by

$$K^\#_A f(x) := \left(\lim_{\varepsilon\to 0^+} \int\limits_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \nu_k(x) a^{\beta\alpha}_{jk} (\partial_j E_{\gamma\beta})(x-y) f_\gamma(y)\,d\sigma(y)\right)_{1\le\alpha\le M}. \tag{1.60}$$

If an estimate like (1.57) would hold for the operator (1.59), then we would have $T_{\Theta^1,\Theta^2} = 0$ whenever $\Omega \subseteq \mathbb{R}^n$ is a half-space. Taking $\Omega := \{z \in \mathbb{R}^n : \langle z, \omega\rangle > 0\}$ with $\omega \in S^{n-1}$ arbitrary then forces that for each index $\gamma \in \{1, \dots, M\}$ we have

$$\begin{gathered}\big[\Theta^1_\gamma(x-y) - \Theta^2_\gamma(x-y)\big]\omega = 0 \quad \text{for each} \quad \omega \in S^{n-1} \\ \text{and each } x, y \in \langle\omega\rangle^\perp \text{ with } x \neq y.\end{gathered} \tag{1.61}$$

The same type of reasoning which, starting with (1.45), has produced (1.48) then shows that there exists a matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$, which is even as well as positive homogeneous of degree $-n$, such that for each index $\gamma \in \{1, \dots, M\}$ we have

$$[\Theta^1_\gamma(z) - \Theta^2_\gamma(z)]\omega = \langle x, \omega\rangle k_{\gamma\cdot}(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \omega \in \mathbb{R}^n. \tag{1.62}$$

In turn, this implies that (1.59) may be recast as

$$\begin{aligned} T_{\Theta^1,\Theta^2} f(x) &= \lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\,d\sigma(y) \\ &\quad + \left(\lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle \Theta^2_\gamma(x-y)(\nu(y)-\nu(x)),\, f(y)\rangle\,d\sigma(y)\right)_{1\le\gamma\le M} \end{aligned} \tag{1.63}$$

for σ-a.e. $x \in \partial\Omega$. The first principal-value integral in (1.63) has been encountered earlier in (1.49), while the second one is of commutator type. Specifically, the second principal-value integral in (1.63) may be thought of as a finite linear combination of commutators between singular integral operators of convolution type with kernels which are odd and positive homogeneous of degree $1-n$ (like the entries in any of the matrices Θ^2_γ) and operators M_{ν_j} of pointwise multiplication with the scalar components ν_j, $1 \le j \le n$, of the outward unit normal ν.

The ultimate conclusion is that, in addition to the family of operators described in (1.49), the class of commutators of the sort just described provides the only other viable candidates for operators whose norms become small when the ambient surface on which they are defined becomes flatter. That such an eventuality actually materializes is implied by Hofmann et al. [61, Theorem 2.16, p. 2603] which, in particular, gives (in the same setting as above)

$$\sum_{j,k=1}^{n} \big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \le C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{1.64}$$

In the opposite direction, in Theorem 5.2 we prove that whenever $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, $1 < p < \infty$, and $w \in A_p(\partial\Omega,\sigma)$, there exists some $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{ \|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} + \max_{1\le j,k\le n} \big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \Big\}. \tag{1.65}$$

This is done using the Clifford algebra machinery (briefly recalled in Sect. 5.1) and exploiting the relationship between the Cauchy–Clifford operator (cf. (5.12)) and the operators K_Δ, $[M_{\nu_k}, R_j]$ with $1 \le j,k \le n$, intervening in (1.65). Collectively, these results point to the optimality of the class of δ-AR domains with $\delta \in (0,1)$ small as the geometric environment in which $\|K_\Delta\|_{[L^p(\partial\Omega,w)]^M\to[L^p(\partial\Omega,w)]^M}$ and $\big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ for $1 \le j,k \le n$ can possibly be small (relative to n, p, $[w]_{A_p}$, and the uniform rectifiability character of $\partial\Omega$).

We also succeed in characterizing flatness solely in terms of the behavior of the Riesz transforms $\{R_j\}_{1\le j\le n}$ (defined in (1.16)). In one direction, in Theorem 5.3 we show that if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain with an unbounded boundary and $w \in A_p(\partial\Omega,\sigma)$ with $p \in (1,\infty)$, then there exists some $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the uniform rectifiability character of $\partial\Omega$ with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{ \Big\| I + \sum_{j=1}^{n} R_j^2 \Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \tag{1.66}$$

$$+ \max_{1\le j,k\le n} \big\| [R_j, R_k] \big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \Big\}.$$

In the opposite direction, in Theorem 5.4 we prove that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set, then for each Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ and each $m \in \mathbb{N}$ there exists some constant $C_m \in (0, \infty)$ which depends only on m, n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ such that

$$\Big\| I + \sum_{j=1}^{n} R_j^2 \Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \le C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{1.67}$$

and

$$\max_{1\le j<k\le n} \big\| [R_j, R_k] \big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \le C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}. \tag{1.68}$$

Collectively, (1.66)–(1.68) give a fully satisfactory answer to the question of quantifying flatness of a given “surface” Σ (thought of as the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$) in terms of the operator theoretic nature of the Riesz transforms on Σ. Informally, these estimates amount to saying that the flatter Σ is, the closer $\{R_j\}_{1\le j\le n}$ are to satisfying the “usual” Riesz transform identities

$$\sum_{j=1}^{n} R_j^2 = -I \quad \text{and} \quad R_j R_k = R_k R_j \quad \text{for all} \quad j, k \in \{1, \dots, n\}, \tag{1.69}$$

when all operators are considered on Muckenhoupt weighted Lebesgue spaces on Σ, and vice versa. In the limit case when Σ is genuinely flat (manifested through the vanishing of the BMO semi-norm of its unit normal), all formulas in (1.69) hold as stated. The best known case is that when Σ is the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ in $\mathbb{R}^n$, a scenario in which (1.69) may be readily checked when $p = 2$ and $w \equiv 1$ based on the fact that each R_j is a Fourier multiplier corresponding to the symbol $i\xi_j/|\xi|$.

The insistence on Muckenhoupt weights is justified by the fact that the boundedness of the Riesz transforms on a weighted Lebesgue space L^p with $p \in (1, \infty)$ actually forces the intervening weight to belong to the Muckenhoupt class A_p. See the discussion in Sect. 5.4 in this regard, where other related results may be found.

While estimate (1.57) is valid irrespective of whether $\partial\Omega$ is bounded or not, its usefulness is most apparent when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small (relative to the geometry of Ω and the weight w) since, in the context of (1.57),

$$\text{having } \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \text{ small implies that } \tfrac{1}{2}I + K_A \text{ is invertible on } \big[L^p(\partial\Omega, w)\big]^M \text{ and } (\tfrac{1}{2}I + K_A)^{-1} \text{ may be expressed as the Neumann series } 2^{-1}\sum_{j=0}^{\infty}(-2K_A)^j \text{, which is convergent in the operator norm,} \tag{1.70}$$

and one can actually show that having $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < 1$ forces $\partial\Omega$ to be unbounded (see Lemma 2.8). We may therefore recast (1.70) as saying that we may invert $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega, w)\big]^M$ whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain for some $\delta \in (0, 1)$ sufficiently small (relative to the dimension n, the Ahlfors regularity constant of $\partial\Omega$, the exponent p, and the weight w), and the latter condition implies that $\partial\Omega$ is unbounded.

A precise formulation of this result goes as follows: Fix $n, M \in \mathbb{N}$ and consider a weakly elliptic homogeneous constant complex coefficient second-order $M \times M$ system L in $\mathbb{R}^n$ with $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$. Then for each constants $C_A, C_W \in (0, \infty)$, each compact interval $I \subset (1, \infty)$, and each coefficient tensor $A \in \mathfrak{A}_L^{\mathrm{dis}}$ there exists a threshold $\delta \in (0, 1)$ which depends only on n, C_A, C_W, I, and A with the following significance. Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain such that the Ahlfors regularity constant of $\partial\Omega$ is $\leq C_A$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in I$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq C_W$. Finally, consider the boundary-to-boundary double layer potential operator K_A, associated with the set Ω and the coefficient tensor A as in (1.6). Then $\frac{1}{2}I + K_A$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$ provided $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$.

Estimate (1.57) then becomes a powerful tool in the proof of similar results on other function spaces. First, in concert with the homogeneous space version of the commutator theorem of Coifman et al. [31], proved in [61, Theorem 2.16, p. 2603], this implies an analogous estimate on Muckenhoupt weighted Sobolev spaces (see (2.587)). That is, retaining the assumptions on the domain Ω and the system L made in the buildup to (1.57), whenever $A \in \mathfrak{A}_L^{\mathrm{dis}}$, $m \in \mathbb{N}$, and $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$ we have

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M \to [L_1^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{1.71}$$

for some constant $C_m \in (0, \infty)$ of the same nature as before. To elaborate on this crucial estimate, one should think of our Muckenhoupt weighted Sobolev space $L_1^p(\partial\Omega, w)$ as being naturally associated with a family $\{\partial_{\tau_{jk}}\}_{1\leq j,k\leq n}$ of first-order "tangential" differential operators along $\partial\Omega$, which may loosely be described as $\partial_{\tau_{jk}} = \nu_j\partial_k - \nu_k\partial_j$ for each $j, k \in \{1, \ldots, n\}$. Specifically, $L_1^p(\partial\Omega, w)$ is the linear space consisting of functions $f \in L^p(\partial\Omega, w)$ with $\partial_{\tau_{jk}} f \in L^p(\partial\Omega, w)$ for each $j, k \in \{1, \ldots, n\}$ (see the discussion in Sect. 2.8 in this regard). From this perspective it is then of paramount importance to understand the manner in which a double layer operator K_A commutes with a generic tangential differential operators $\partial_{\tau_{jk}}$. It turns out that

$$\text{each commutator } [K_A, \partial_{\tau_{jk}}] \text{ acting on a function } f \text{ belonging to a Muckenhoupt weighted Sobolev space may be expressed as a finite linear combination of commutators of the form } [M_\nu, R] \text{ acting on the components of } \nabla_{\tan} f, \text{ the tangential gradient of } f, \text{ where } M_\nu \text{ stands for the operator of pointwise multiplication by (generic components of) the unit normal } \nu, \text{ and } R \text{ is a convolution type singular integral operator on } \partial\Omega \text{ of similar nature as the Riesz transforms on } \partial\Omega \text{ (cf. (1.16)).} \tag{1.72}$$

Based on this, (1.57), and a suitable analogue of (1.64), we then conclude that the key estimate stated in (1.71) holds. In turn, (1.71) permits us to invert $\frac{1}{2}I + K_A$ on the Muckenhoupt weighted Sobolev space $\big[L^p_1(\partial\Omega, w)\big]^M$, for each $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$, via a Neumann series converging in the operator norm, whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain for some $\delta \in (0, 1)$ sufficiently small (a condition that forces $\partial\Omega$ to be unbounded) relative to the Ahlfors regularity constant of $\partial\Omega$ and the weight w.

Second, we use the operator norm estimate on Muckenhoupt weighted Lebesgue spaces from (1.57) as a gateway to establishing similar estimates via extrapolation procedures. One of the best known embodiments of this principle is Rubio de Francia's celebrated extrapolation theorem, according to which estimates on Muckenhoupt weighted Lebesgue spaces for a fixed integrability exponent and all weights imply similar estimates for all integrability exponents (prompting Antonio Córdoba to famously declare that "there are no L^p spaces, only weighted L^2 spaces"). Here we use (1.57) together with an extrapolation procedure from [112, §6.2] (recalled in Proposition 7.4) to obtain norm estimates for double layer operators on the scale of Morrey spaces on the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$, i.e.,

$$M^{p,\lambda}(\partial\Omega, \sigma) := \big\{ f \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma) : \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty \big\} \tag{1.73}$$

with $p \in (1, \infty)$ and $\lambda \in (0, n-1)$, where[1]

$$\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} := \sup_{\substack{x\in\partial\Omega \text{ and} \\ 0<R<2\,\mathrm{diam}(\partial\Omega)}} \Big\{ R^{\frac{n-1-\lambda}{p}} \Big(\fint_{\partial\Omega\cap B(x,R)} |f|^p \, d\sigma \Big)^{\frac{1}{p}} \Big\}. \tag{1.74}$$

(Note that the scale of ordinary Lebesgue spaces on $\partial\Omega$ corresponds to the end-point case $\lambda = 0$, while the end-point $\lambda = n - 1$ corresponds to the space of essentially bounded functions on $\partial\Omega$.) Retaining the same geometric context as before and assuming $A \in \mathfrak{A}^{\mathrm{dis}}_L$, the extrapolation procedure alluded to above yields, for each $m \in \mathbb{N}$,

$$\|K_A\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \le C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{1.75}$$

[1] throughout, given any nonempty set $E \subseteq \mathbb{R}^n$, we let $\mathrm{diam}(E)$ denote the diameter of E.

for some constant $C_m \in (0, \infty)$ of the same nature as before (cf. Theorem 7.8 for this, and other related results). We may take this a step further and establish a similar operator norm estimate involving the Morrey-based Sobolev space $M_1^{p,\lambda}(\partial\Omega, \sigma)$. These, in turn, allow us to invert $\frac{1}{2}I + K_A$ both on the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ and on the Morrey-based Sobolev space $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, under similar assumptions as before. See Theorem 7.9 where this and other invertibility results on related spaces are proved. In addition, (1.57) implies (via real interpolation) norm estimates and invertibility results for double layer potential operators on Lorentz spaces and Lorentz-based Sobolev spaces (cf. Remarks 4.11 and 4.16).

Concisely put, in this work we are able to answer Kenig's open question (formulated at the outset of the introduction) pertaining to any given weakly elliptic homogeneous constant complex coefficient second-order system L in $\mathbb{R}^n$ with $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, in the setting of δ-AR domains $\Omega \subseteq \mathbb{R}^n$ with $\delta \in (0, 1)$ small (relative to n and the Ahlfors regularity constant of $\partial\Omega$), for ordinary Lebesgue spaces, Lorentz spaces, Muckenhoupt weighted Lebesgue, Morrey spaces, as well as Sobolev spaces on $\partial\Omega$ suitably defined in relation to each of the aforementioned scales (see Theorem 4.8, Remark 4.16, Theorems 4.9, 7.9, 7.10). As indicated in Remark 4.19, the smallness condition imposed on the parameter δ is actually in the nature of best possible as far as these invertibility results are concerned.

In turn, the aforementioned invertibility results open the door for solving boundary value problems of Dirichlet, Regularity, Neumann, and Transmission type in the class of δ-AR domains with $\delta \in (0, 1)$ small (relative to the dimension n, the Ahlfors regularity constant of $\partial\Omega$, and the specific nature of the space of boundary data) for second-order weakly elliptic constant complex coefficient systems which (either themselves and/or their transpose) possess distinguished coefficient tensors.

For example, in such a setting, we succeed in establishing the well-posedness of the Muckenhoupt weighted Dirichlet Problem and the Muckenhoupt weighted Regularity Problem (formulated using the nontangential maximal operator introduced in (2.5), and nontangential boundary traces defined as in (2.12), for some fixed aperture parameter $\kappa > 0$):

$$
(D)_{p,w} \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \qquad (R)_{p,w} \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L_1^p(\partial\Omega, w)\big]^M, \end{cases} \tag{1.76}
$$

for each given integrability exponent $p \in (1, \infty)$ and each given Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, under the assumption that both L and $L^\top$ have a distinguished coefficient tensor. Moreover, we provide counterexamples which show that the well-posedness result just described may fail if these assumptions on the

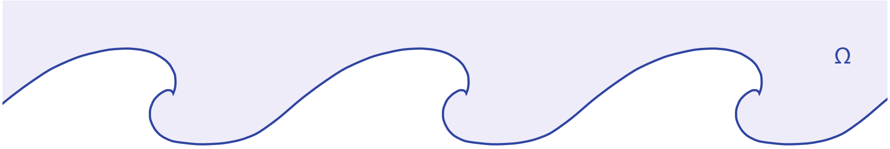

Fig. 1.1 A prototype of an unbounded δ-AR domain for which $\delta > 0$ may be made as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$ (cf. (2.325), (2.327))

existence of distinguished coefficient tensors are simply dropped. See Theorems 6.2 and 6.5 for more nuanced statements. Our results are therefore optimal in this regard. We wish to note that the present work marks the first occasion when boundary problems like (1.76) have been treated in a class of sets large enough as to contain domains with spiral points of the sort described in Fig. 1.1. This being said, even in the scalar (i.e., $M = 1$), unweighted case (i.e., $w \equiv 1$), the well-posedness of the problems in (1.76) would still be new for such basic constant complex coefficient differential operators as

$$L = \partial_1^2 + \cdots + \partial_{n-1}^2 + \mathrm{i}\partial_n^2. \tag{1.77}$$

Existence for the boundary value problems $(D)_{p,w}$, $(R)_{p,w}$ is established by looking for a solution which is expressed as in (1.11), making use of the jump-formula (3.123), and the fact that $\frac{1}{2}I + K_A$ is invertible both on the Muckenhoupt weighted Lebesgue space $\big[L^p(\partial\Omega, w)\big]^M$ as well as on the Muckenhoupt weighted Sobolev space $\big[L_1^p(\partial\Omega, w)\big]^M$. The issue of uniqueness requires a new set of techniques, and this is subtle even in the classical setting of the upper half-space $\Omega := \mathbb{R}^n_+$. In the particular case when $L = \Delta$, the Laplacian in $\mathbb{R}^n$, the Dirichlet boundary value problem $(D)_{p,w}$ in $\Omega := \mathbb{R}^n_+$ has been treated at length in a number of monographs in the unweighted case (i.e., when $w = 1$), including [9], [52], [132], [133], and [134]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel, while the uniqueness relies on either the Maximum Principle or the Schwarz reflection principle for harmonic functions. Neither of these techniques may be adapted successfully to prove uniqueness in the case of general systems treated here. Subsequently, the Dirichlet boundary value problem $(D)_{p,w}$ in $\Omega := \mathbb{R}^n_+$ for a general strongly elliptic, second-order, homogeneous, constant complex coefficient, system L, and for an arbitrary Muckenhoupt weight w has been treated in [92], where existence employs the Agmon-Douglis-Nirenberg Poisson kernel for L, while uniqueness relies on special properties of the Green function for L in the upper half-space $\mathbb{R}^n_+$.

In the present setting, when Ω is merely a δ-AR domain with $\delta \in (0, 1)$ small (relative to n, p, w, and the Ahlfors regularity constant of $\partial\Omega$), in order to deal with the issue of uniqueness for the Muckenhoupt weighted Dirichlet Problem $(D)_{p,w}$ we construct a Green function G for L in Ω by correcting the fundamental solution E of L in $\mathbb{R}^n$ (as to ensure its boundary trace on $\partial\Omega$ vanishes) using the existence part for the Regularity Problem $(R)_{p',w'}$ (formulated for the transpose system $L^\top$, the conjugate exponent p', and the dual weight w') and then employ a rather general Poisson integral representation formula recently established in [113, §4.4] (cf. Theorem 6.1 for a precise statement).

For each given integrability exponent $p \in (1, \infty)$ and each given Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ we also prove (see Theorem 6.8) that what we call the Homogeneous Regularity Problem, namely the boundary value problem

$$(HR)_{p,w} \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M, \end{cases} \tag{1.78}$$

whose formulation involves a homogeneous Muckenhoupt weighted Sobolev space, denoted by $\dot{L}_1^p(\partial\Omega, w)$ (introduced in Definition 2.18), is well posed provided both L and $L^\top$ have a distinguished coefficient tensor and the Ahlfors regular domain Ω is sufficiently flat.

In the same geometric setting, of δ-AR domains, we also discuss the solvability of the Muckenhoupt weighted Neumann Problem (in Theorem 6.11) and the Muckenhoupt weighted Transmission Problem (in Theorem 6.15), i.e.,

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial_\nu^A u = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \qquad \begin{cases} u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega, w), \\ u^+\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \sigma\text{-a.e. on } \partial\Omega, \\ \partial_\nu^A u^+ - \mu\cdot\partial_\nu^A u^- = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \tag{1.79}$$

(where ∂_ν^A is the conormal derivative operator associated with the coefficient tensor A used to represent the given system L, and $\mu \in \mathbb{C}$ is a transmission, or coupling, parameter), as well as variants of those boundary value problems involving Lorentz spaces. In all cases, we show that the boundary layer method may be successfully implemented for any second-order homogeneous constant complex coefficient weakly elliptic system L in $\mathbb{R}^n$ whose transpose possesses a distinguished coefficient tensor, assuming $A \in \mathfrak{A}_{L^\top}^{\text{dis}}$. Moreover, in the two-dimensional setting we show that the Neumann and Transmission Problems (1.79) remain solvable for a larger spectrum of choices of the coefficient tensor for the Lamé system (see the results in Sect. 4.5, as well as Remarks 6.10 and 6.16, in this regard).

In [114], a robust Calderón-Zygmund theory for singular integral operators of boundary layer type associated with weakly elliptic systems and uniformly rectifiable domains has been developed. Here we use such a platform (consisting of results recalled in Proposition 7.5, Theorems 7.1, and 7.2) to prove solvability results for a variety of boundary value problems of Dirichlet, (inhomogeneous and homogeneous) Regularity, Neumann, and Transmission type (akin those formulated in (1.76), (1.78), and (1.79)) with data in Morrey spaces, vanishing Morrey spaces, and block spaces (cf. Theorems 7.18, 7.20, 7.21, 7.22, and 7.23).

In addition, we develop a perturbation theory to the effect that, in all cases discussed so far in this narrative, solvability of a boundary value problem for a certain system L_o implies solvability for any other system L which is sufficiently close to L_o (with proximity quantified using the norm introduced in (3.12)). For results of this nature, the reader is referred to Theorems 6.4, 6.6, 6.12, 6.16, and 7.19.

Lastly, in Sect. 8 we study singular integral operators in more general functional analytic settings. The goal here is to show that these are effective tools in obtaining well-posedness results for boundary problems for second-order systems, formulated in sufficiently flat Ahlfors regular domains, and with boundary data from abstract weighted Banach function spaces. A key result in this regard is a remarkable link between this class of abstract spaces and concrete Muckenhoupt weighted Lebesgue spaces. To briefly elaborate on this topic, we need some notation. Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. Let $\mathbb{X}$ be a Banach function space over (Σ, σ), i.e., the space associated with a function norm as in (8.5) (also referred to as a Köthe function space). With $\mathbb{X}'$ denoting the Köthe dual of $\mathbb{X}$ (also known as the associated space of $\mathbb{X}$ in the terminology of [15]; cf. (8.6)), and with $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on (Σ, σ), assume that

$$\mathcal{M} \text{ is bounded both on } \mathbb{X} \text{ and on } \mathbb{X}'. \tag{1.80}$$

In this setting we then show that (see Proposition 8.1 for a more general and precise result), in a quantitative fashion, for each fixed $p_0 \in [1, \infty)$ we have

$$\mathbb{X} \subseteq \bigcup_{w \in A_{p_0}(\Sigma,\sigma)} L^{p_0}(\Sigma, w). \tag{1.81}$$

Subsequently, in Theorem 8.1, we show that for each pair of σ-measurable functions f, g on Σ, having an inequality of the form

$$\|f\|_{L^{p_0}(\Sigma,w)} \leq C_w \, \|g\|_{L^{p_0}(\Sigma,w)} \tag{1.82}$$

valid for some fixed integrability exponent $p_0 \in [1, \infty)$ and arbitrary Muckenhoupt weights $w \in A_{p_0}(\Sigma, \sigma)$ (where the constant C_w depends in a non-decreasing fashion on $[w]_{A_{p_0}}$) implies

$$\|f\|_{\mathbb{X}} \leq C \, \|g\|_{\mathbb{X}}, \tag{1.83}$$

where $C \in (0, \infty)$ depends only on p_0 and the operator norms of $\mathcal{M}$ on $\mathbb{X}$ and $\mathbb{X}'$. This result, which is in the spirit of Rubio de Francia's celebrated extrapolation theorem, then opens the door for transferring our earlier results in ordinary Muckenhoupt weighted Lebesgue spaces to the setting of abstract weighted Banach function spaces. We methodically explore this venue, and the theory we develop ultimately shows the effectiveness of the boundary layer approach in the treatment of boundary problems for second-order systems, formulated in sufficiently flat Ahlfors regular domains, and with boundary data in abstract Banach function spaces. See Sect. 8 for details; here we only mention that in the last part of this chapter we provide a multitude of relevant examples, including variable exponent Lebesgue spaces, generic rearrangement invariant Banach function spaces (RIBFS for short), Orlicz spaces, Zygmund space, Lorentz spaces, and their weighted versions.

To close, we wish to emphasize that it is natural to consider boundary value problems with boundary data from a large library of function spaces (as done here: Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Morrey spaces, block spaces, abstract weighted Banach function spaces, as well as various Sobolev spaces naturally adapted to these scales, among others). To elaborate on this aspect, assume $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta \in (0, 1)$ and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Also, fix an arbitrary aperture parameter $\kappa > 0$ along with some power $a \in (0, n-1)$, and pick some point $x_o \in \partial\Omega$. In this setting, consider the Dirichlet Problem for the Laplacian in Ω, corresponding to the boundary datum

$$f(x) := |x - x_o|^{-a} \quad \text{for each} \quad x \in \partial\Omega \setminus \{x_o\}, \tag{1.84}$$

assumed in a nontangential fashion, i.e.,

$$\begin{cases} u \in \mathscr{C}^{\infty}(\Omega), \quad \Delta u = 0 \;\text{ in }\; \Omega, \\ u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \;\text{ at } \sigma\text{-a.e. point on }\; \partial\Omega. \end{cases} \tag{1.85}$$

The question which naturally arises is: what size/regularity conditions is the solution u expected to satisfy? The answer very much depends on the actual qualities of the boundary datum f and on the specific frameworks in which we know the Dirichlet Problem to be well-posed. For example, f from (1.84) does not belong to any Lebesgue space $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, so one does not expect $\mathcal{N}_\kappa u$ to belong to any ordinary Lebesgue space $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. This being said, for each fixed point $x_* \in \partial\Omega$ and each exponent $b \in (0, n-1)$ the function

$$w(x) := |x - x_*|^{-b}, \quad \forall\, x \in \partial\Omega \setminus \{x_*\} \tag{1.86}$$

is a Muckenhoupt weight, in the class $A_p(\partial\Omega, \sigma)$, and the function f from (1.84) belongs to the Muckenhoupt weighted Lebesgue space $L^p(\partial\Omega, w)$ if $x_* \neq x_o$ and

$$\max\left\{1, \frac{n-1-b}{a}\right\} < p < \frac{n-1}{a}. \tag{1.87}$$

Assuming $\delta \in (0, 1)$ is sufficiently small, the theory developed here then guarantees that there exists a unique function u solving (1.85) with the additional property that

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, w). \tag{1.88}$$

Since the boundary datum f also belongs to the Lorentz space $L^{(n-1)/a,\infty}(\partial\Omega, \sigma)$ which turns out to be an environment in which we are able to establish the well-posedness of the Dirichlet Problem with appropriate nontangential maximal function control, we then conclude that for the unique function u satisfying (1.85) and (1.88) we also have (assuming $\delta \in (0, 1)$ is sufficiently small)

$$\mathcal{N}_\kappa u \in L^{(n-1)/a,\infty}(\partial\Omega, \sigma). \tag{1.89}$$

Likewise, the fact that the boundary datum f from (1.84) also belongs to the Morrey space $M^{(n-1-\lambda)/a,\lambda}(\partial\Omega, \sigma)$ whenever $\lambda \in (0, n-1-a)$ further entails (again, assuming $\delta \in (0, 1)$ is sufficiently small)

$$\mathcal{N}_\kappa u \in M^{(n-1-\lambda)/a,\lambda}(\partial\Omega, \sigma) \quad \text{for each} \quad \lambda \in (0, n-1-a). \tag{1.90}$$

The tangential derivatives of the boundary datum f also enjoy integrability properties which translate well in terms of regularity properties for the solution u of (1.85)–(1.88). For example, if

$$\begin{aligned} &a \in (0, n-2), \qquad \lambda \in (0, n-2-a), \\ &\text{and} \quad \max\left\{1, \tfrac{n-1-b}{a+1}\right\} < q < \tfrac{n-1}{a+1}, \end{aligned} \tag{1.91}$$

then for each $j, k \in \{1, \dots, n\}$ we have

$$\partial_{\tau_{jk}} f \in L^q(\partial\Omega, w) \cap L^{(n-1)/(a+1),\infty}(\partial\Omega, \sigma) \cap M^{(n-1-\lambda)/(a+1),\lambda}(\partial\Omega, \sigma), \tag{1.92}$$

which, granted that $\delta \in (0, 1)$ is sufficiently small, ultimately imply

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega, w) \cap L^{(n-1)/(a+1),\infty}(\partial\Omega, \sigma) \cap M^{(n-1-\lambda)/(a+1),\lambda}(\partial\Omega, \sigma). \tag{1.93}$$

It is also interesting to ponder on the nature of the nontangential maximal function for solutions of (1.85) in the case when the boundary datum is a characteristic function, say, $f = \mathbf{1}_E$ for some bounded σ-measurable set $E \subseteq \partial\Omega$. If one regards the latter merely as a function in $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, then the best one can say is that $\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma)$, assuming $\delta \in (0, 1)$ is sufficiently small. However, through the consideration of weights, one may find solutions of said boundary problem for which the nontangential maximal function has better decay properties at infinity. Specifically, fix a point $x_* \in \partial\Omega$, an integrability exponent $p \in (1, \infty)$, a power $b \in \big(-(p-1)(n-1), n-1 \big)$, and define the weight w as in (1.86). In particular, we have $f = \mathbf{1}_E \in L^p(\partial\Omega, w)$, and since $w \in A_p(\partial\Omega, \sigma)$ the well-posedness of the Dirichlet problem with data in Muckenhoupt weighted Lebesgue spaces implies we may find a solution u of the boundary value problem (1.85) satisfying

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(x)^p}{|x - x_*|^b} \, d\sigma(x) < +\infty, \tag{1.94}$$

once more, assuming $\delta \in (0, 1)$ is sufficiently small (relative to n, p, b, and the Ahlfors regularity constant of $\partial\Omega$).

Lastly, we wish to note that there is a wealth of sources for boundary value problems in non-smooth domains with boundary data and solutions in Besov and Triebel-Lizorkin spaces, including [11], [50], [68], [100], [98], [103], [104], [106], [107], [115], and the references therein.