Progress in Mathematics
344

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Birkhäuser

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ISSN 0743-1643 ISSN 2296-505X (electronic)
Progress in Mathematics
ISBN 978-3-031-08233-7 ISBN 978-3-031-08234-4 (eBook)
https://doi.org/10.1007/978-3-031-08234-4

Mathematics Subject Classification: 31B10, 35B65, 35C15, 35J25, 35J57, 35J67, 42B20, 42B37

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The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

We develop the theory of layer potentials in the context of δ-AR domains in $\mathbb{R}^n$ (aka δ-flat Ahlfors regular domains) where the parameter $\delta > 0$, regulating the size of the BMO semi-norm of the outward unit normal ν to Ω, is assumed to be small. This is a sub-category of the class of two-sided NTA domains with Ahlfors regular boundaries, and our results complement work carried out [61] in regular SKT domains (with SKT acronym for Semmes-Kenig-Toro). The latter brand was designed to work well when the domains in question have compact boundaries. By way of contrast, the fact that we are now demanding $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is small enough (where σ is the "surface measure" $\mathcal{H}^{n-1}\lfloor\partial\Omega$) has topological and metric implications for Ω, namely Ω is a connected unbounded open set, with a connected unbounded boundary and an unbounded connected complement. For example, in the two-dimensional setting, we show that the class of δ-AR domains with $\delta \in (0, 1)$ small agrees with the category of chord-arc domains with small constant.

Assuming $\Omega \subseteq \mathbb{R}^n$ to be a δ-AR domain with $\delta \in (0, 1)$ sufficiently small (relative to the dimension n and the Ahlfors regularity constant of $\partial\Omega$), we prove that the operator norm of Calderón-Zygmund singular integrals whose kernels exhibit a certain algebraic structure (specifically, they contain the inner product of the normal $\nu(y)$ with the "chord" $x - y$ as a factor) is $O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$. This is true in the context of Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Morrey spaces, vanishing Morrey spaces, block spaces, (weighted) Banach function spaces, as well as for the brands of Sobolev spaces naturally associated with these scales. Simply put, the problem that we solve here is that of determining when (and how) singular integral operators of double-layer type have small operator norm on domains which are relatively "flat." We also establish estimates in the opposite direction, quantifying the flatness of a "surface" by estimating the BMO semi-norm of its unit normal in terms of the operator norms of certain singular integrals canonically associated with the given surface (such as the harmonic double layer, the family of Riesz transforms, and commutators between Riesz transforms and pointwise multiplication by the components of the unit normal). Ultimately, this goes to show that the two-way bridge between geometry and analysis constructed here is in the nature of best possible.

Significantly, the operator norm estimates described in the previous paragraph permit us to invert the boundary double-layer potentials associated with certain classes of second-order PDE (such as the Laplacian, any scalar homogeneous constant complex coefficient second-order operator which is weakly elliptic when $n \geq 3$ or strongly elliptic in any dimension, the Lamé system of elasticity, and, most generally, any weakly elliptic homogeneous constant complex coefficient second-order system having a certain distinguished coefficient tensor), acting on a large variety of function spaces considered on the boundary of a sufficiently flat domain (specifically, a δ-AR domain with $\delta \in (0, 1)$ suitably small relative to other geometric characteristics of said domain). In particular, this portion of our work goes in the direction of answering the question posed by C. Kenig in [71, Problem 3.2.2, p. 117] asking to invert layer potentials in appropriate spaces on certain uniformly rectifiable sets.

In turn, these invertibility results allow us to establish solvability results for boundary value problems in the class of weakly elliptic second-order systems mentioned above, in a sufficiently flat Ahlfors regular domain, with boundary data from Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Morrey spaces, vanishing Morrey spaces, block spaces, Banach function spaces, and from Sobolev spaces naturally associated with these scales.

In summary, a central theme in Geometric Measure Theory is understanding how geometric properties translate into analytical ones, and here we explore the implications of demanding that Gauss' map $\partial\Omega \ni x \mapsto \nu(x) \in S^{n-1}$ has small BMO semi-norm in the realm of singular integral operators and boundary value problems. The theory developed here complements the results of S. Hofmann, M. Mitrea, and M. Taylor obtained in [61] and extends previously known well-posedness results for elliptic PDE in the upper half-space to the considerably more inclusive realm of δ-AR domains with $\delta \in (0, 1)$.

Acknowledgments

Portions of this work were completed at Baylor University, Waco, USA, Temple University, Philadelphia, USA, and ICMAT, Madrid, Spain. The authors thank these institutions for their hospitality. The authors also acknowledge partial support from the Simons Foundation (through grants # 426669, # 958374, # 318658, # 616050, # 637481), National Science Foundation (grant # 1900938), and MCIN/AEI/ 10.13039/501100011033 (grants CEX2019-000904-S and PID2019-107914GB-I00).

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March 2022

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Chapter 1
Introduction

More than 25 years ago, in [71, Problem 3.2.2, p. 117], C. Kenig asked to "*Prove that the layer potentials are invertible in appropriate* [...] *spaces in* [*suitable subclasses of uniformly rectifiable*] *domains*." Kenig's main motivation in this regard stems from the desire of establishing solvability results for boundary value problems formulated in a rather inclusive geometric setting. In the buildup to this open question on [71, p. 116], it is remarked that there exist some rather general classes of open sets $\Omega \subseteq \mathbb{R}^n$ with the property that if $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ (where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$) then said layer potentials are bounded operators on $L^p(\partial\Omega, \sigma)$ for each exponent $p \in (1, \infty)$. Remarkably, this is the case whenever $\Omega \subseteq \mathbb{R}^n$ is an open set with a uniformly rectifiable boundary (cf. [40]).

To further elaborate on this issue, we need some notation. Fix $n \in \mathbb{N}$ with $n \geq 2$, along with $M \in \mathbb{N}$, and consider a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system in $\mathbb{R}^n$

$$L = \big(a^{\alpha\beta}_{jk} \partial_j \partial_k\big)_{1 \leq \alpha, \beta \leq M}, \tag{1.1}$$

where the summation convention over repeated indices is in effect (here and elsewhere in the manuscript). The weak ellipticity of the system L amounts to demanding that

$$\begin{array}{l} \text{the characteristic matrix } L(\xi) := \big(- a^{\alpha\beta}_{jk} \xi_j \xi_k \big)_{1 \leq \alpha, \beta \leq M} \text{ is} \\ \text{invertible for each vector } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \end{array} \tag{1.2}$$

This should be contrasted with the more stringent Legendre–Hadamard (strong) ellipticity condition which asks for the existence of some $c > 0$ such that

$$\mathrm{Re}\big\langle - L(\xi)\zeta \,, \overline{\zeta} \,\big\rangle \geq c\,|\xi|^2\,|\zeta|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad \zeta \in \mathbb{C}^M. \tag{1.3}$$

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_1

Nonetheless, the weak ellipticity assumption which we shall enforce throughout ensures that the system L has a well-behaved fundamental solution, which is an even matrix-valued function $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^M$ whose first-order derivatives are positive homogeneous of degree $1-n$, of the sort discussed at length in [102] (see Theorem 3.1 for a brief review).

The given system L does not determine uniquely the coefficient tensor

$$A := \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}} \tag{1.4}$$

since employing $\widetilde{A} := \big(\widetilde{a}_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}}$ in place of A in the right-hand side of (1.1) yields the same system whenever the difference $a_{jk}^{\alpha\beta} - \widetilde{a}_{jk}^{\alpha\beta}$ is antisymmetric in the indices j,k (for each $\alpha,\beta \in \{1,\dots,M\}$). Hence, there are a multitude of coefficient tensors A which may be used to represent the given system L as in (1.1). For each such coefficient tensor $A := \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}}$ we shall associate a double layer potential operator K_A on the boundary of a given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ (see Definition 2.6). Specifically, if $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is the "surface measure" on $\partial\Omega$ and if $\nu = (\nu_1,\dots,\nu_n)$ denotes the geometric measure theoretic outward unit normal to Ω, then for each function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{1.5}$$

we define, at σ-a.e. point $x \in \partial\Omega$,

$$K_A f(x) := \Big(-\lim_{\varepsilon\to 0^+}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \nu_k(y) a_{jk}^{\beta\alpha}\big(\partial_j E_{\gamma\beta}\big)(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}. \tag{1.6}$$

(Note that (1.5) is the most general environment in which each truncated integral in (1.6) is absolutely convergent.)

To offer a simple example, consider the case when $L = \Delta$, the Laplacian, in $\mathbb{R}^2$. Then $n = 2$ and $M = 1$. In this scalar case, we agree to drop the Greek superscripts labeling the entries of the coefficient tensor (1.4) used to express L as in (1.1). Hence, we shall consider writings $\Delta = a_{jk}\partial_j\partial_k$ corresponding to various choices of the matrix $A = (a_{jk})_{1\leq j,k\leq 2} \in \mathbb{C}^{2\times 2}$. Two such natural choices are

$$A_0 := \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \qquad A_1 := \begin{pmatrix} 1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}, \tag{1.7}$$

corresponding to which the recipe given in (1.6) yields

$$K_{A_0}f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle\nu(y), y-x\rangle}{|x-y|^2} f(y)\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x\in\partial\Omega, \tag{1.8}$$

i.e., the (two-dimensional) harmonic boundary-to-boundary double layer potential operator and, under the natural identification $\mathbb{R}^2 \equiv \mathbb{C}$,

$$K_{A_1}f(z) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega\setminus\overline{B(z,\varepsilon)}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \ \text{ for } \sigma\text{-a.e. } z\in\partial\Omega, \tag{1.9}$$

i.e., the boundary-to-boundary Cauchy integral operator, respectively.

Returning to the mainstream discussion in the general setting considered earlier, fundamental work in [40] guarantees that, if $\Omega\subseteq\mathbb{R}^n$ is a uniformly rectifiable domain, then for each coefficient tensor A as in (1.4) which may be employed to write the given system L as in (1.1), the boundary-to-boundary double layer potential K_A from (1.6) is a well-defined, linear, and bounded operator on $\big[L^p(\partial\Omega,\sigma)\big]^M$ for each $p\in(1,\infty)$. This property is particularly relevant in the treatment of the Dirichlet Problem for the system L in the uniformly rectifiable domain Ω when the boundary data are selected from the space $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p\in(1,\infty)$, i.e.,

$$(D)_p \begin{cases} u\in\big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu=0 \text{ in } \Omega,\\ \mathcal{N}_\kappa u\in L^p(\partial\Omega,\sigma),\\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = g\in\big[L^p(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{1.10}$$

where $\mathcal{N}_\kappa u$ is the nontangential maximal function, and $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ is the nontangential boundary trace, of the solution u (see the body of the manuscript for precise definitions; cf. (2.5) and (2.12)). Indeed, the essence of the boundary layer method is to consider as a candidate for the solution of the Dirichlet Problem (1.10) the $\mathbb{C}^M$-valued function u defined at each point $x\in\Omega$ by

$$u(x) := \Big(-\int_{\partial\Omega} \nu_k(y) a^{\beta\alpha}_{jk}\big(\partial_j E_{\gamma\beta}\big)(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}, \tag{1.11}$$

for some yet-to-be-determined function $f=(f_\alpha)_{1\le\alpha\le M}\in\big[L^p(\partial\Omega,\sigma)\big]^M$. In light of the special format of u (in particular, thanks to the jump-formula (3.123)), this ultimately reduces the entire aforementioned Dirichlet Problem to the issue of solving the boundary integral equation

$$\big(\tfrac{1}{2}I + K_A\big)f = g \ \text{ on } \ \partial\Omega, \tag{1.12}$$

where I is the identity operator (see Sect. 6 for the actual implementation of this approach). As such, having the operator K_A well defined, linear, and bounded on

$\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in (1,\infty)$ opens the door for bringing in functional analytic techniques for inverting $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega,\sigma)\big]^M$ and eventually expressing the solution f as $\big(\frac{1}{2}I + K_A\big)^{-1}g$.

A breakthrough in this regard has been registered by S. Hofmann, M. Mitrea, and M. Taylor in [61], where they have employed Fredholm theory in order to solve the boundary integral equation (1.12). To describe one of their main results, suppose $L = \Delta$, the Laplacian in $\mathbb{R}^n$, is written as $\Delta = a_{jk}\partial_j\partial_k$ for $A := (\delta_{jk})_{1\leq j,k\leq n}$. The blueprint provided in (1.6) then produces the classical harmonic double layer potential operator K_Δ, acting on each $f \in L^p(\partial\Omega,\sigma)$ with $p \in (1,\infty)$ according to

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{1.13}$$

where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. In regard to this operator, S. Hofmann, M. Mitrea, and M. Taylor have proved in [61, Theorem 4.36, pp. 2728-2729] that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, then for every threshold $\varepsilon > 0$ there exists some $\delta > 0$ (which depends only on said geometric characteristics of Ω, n, p, and ε) such that

$$\mathrm{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta \Longrightarrow \mathrm{dist}\big(K_\Delta, \mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) < \varepsilon. \tag{1.14}$$

The distance in the left-hand side of (1.14) is measured in the John-Nirenberg space $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ of vector-valued functions of bounded mean oscillations on $\partial\Omega$ (with respect to the surface measure σ), from the unit vector $\nu \in \big[L^\infty(\partial\Omega,\sigma)\big]^n$ to the Sarason space $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$ of vector-valued functions of vanishing mean oscillations on $\partial\Omega$ (with respect to the surface measure σ), which is a closed subspace of $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ (cf. (2.111)). The distance in the right-hand side of (1.14) is considered from $K_\Delta \in \mathrm{Bd}(L^p(\partial\Omega,\sigma))$, the Banach space of all linear and bounded operators on $L^p(\partial\Omega,\sigma)$ equipped with the operator norm, to $\mathrm{Cp}(L^p(\partial\Omega,\sigma))$ which is the closed linear subspace of $\mathrm{Bd}(L^p(\partial\Omega,\sigma))$ consisting of all compact operators on $L^p(\partial\Omega,\sigma)$. In particular, in the class of domains currently considered, K_Δ is a *compact* operator on $L^p(\partial\Omega,\sigma)$ whenever ν belongs to $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$. This is remarkable in as much that a purely geometric condition implies a functional analytic property of a singular integral operator. Most importantly, (1.14) ensures the existence of some small threshold $\delta > 0$ (which depends only on said geometric characteristics of Ω, n, and p) with the property that

$$\begin{aligned} \mathrm{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta &\Longrightarrow \mathrm{dist}\big(K_\Delta, \mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) < \tfrac{1}{2} \\ &\Longrightarrow \tfrac{1}{2}I + K_\Delta \text{ Fredholm operator with index zero on } L^p(\partial\Omega,\sigma). \end{aligned} \tag{1.15}$$

This is the main step in establishing that $\frac{1}{2}I + K_\Delta$ is actually an invertible operator on $L^p(\partial\Omega, \sigma)$ in said geometric setting, under the additional assumption that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected (see [61, Theorem 6.13, p. 2806]).

Another key result of a similar flavor to (1.14) proved in [61] pertains to the commutators $\big[M_{\nu_k}, R_j\big] := M_{\nu_k}R_j - R_jM_{\nu_k}$, where $j, k \in \{1, \dots, n\}$, between the operator M_{ν_k} of pointwise multiplication by ν_k, the k-th scalar component of the geometric measure theoretic outward unit normal ν to Ω, and the j-th Riesz transform R_j on $\partial\Omega$, acting on any given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.16}$$

Specifically, [61, Theorem 2.19, p. 2608] states that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, and if some $p \in (1, \infty)$ has been fixed, then there exists some $C \in (0, \infty)$ (depending only on the aforementioned geometric characteristics of Ω, n, and p) such that

$$\sum_{j,k=1}^{n} \operatorname{dist}\big(\big[M_{\nu_k}, R_j\big], \operatorname{Cp}(L^p(\partial\Omega, \sigma))\big) \le C \operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big). \tag{1.17}$$

Estimates of this type (with the Riesz transforms replaced by more general singular integral operators of the same nature) turned out to be a key ingredient in the proof of the fact that, if Ω is as above and $p \in (1, \infty)$, then for every threshold $\varepsilon > 0$ there exists some $\delta > 0$ (of the same nature as before) such that

$$\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big) < \delta \Longrightarrow \operatorname{dist}\big(K_\Delta, \operatorname{Cp}(L^p_1(\partial\Omega, \sigma))\big) < \varepsilon, \tag{1.18}$$

where $L^p_1(\partial\Omega, \sigma)$ is a certain brand of L^p-based Sobolev space of order one on $\partial\Omega$, introduced in [61] (and further developed in [109], [112, Chapter 11]).

These considerations have led to the development of a theory of boundary layer potentials in what was labeled in [61] as δ-regular SKT domains, a subclass of the family of bounded uniformly rectifiable domains inspired by work of S. Semmes [123, 124], and C. Kenig and T. Toro [72–74], whose trademark feature is the fact that the distance $\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big)$, measured in the John-Nirenberg space $\big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^n$, is $< \delta$. In turn, this was used in [61] to establish the well-posedness of the Dirichlet, Regularity, Neumann, and Transmission Problems for the Laplacian in the class of δ-regular SKT domains with δ sufficiently small (relative to other geometric characteristics of Ω). Quite recently, this theory has been extended in [90] to the case when the boundary data belong to Muckenhoupt weighted Lebesgue and Sobolev spaces.

In addition, the class of δ-regular SKT domains also turns out to be in the nature of best possible as far as the "close-to-compactness" results mentioned in (1.14) and

(1.17) are concerned. Indeed, [61, Theorem 4.41, p. 2743] states that, if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain with compact boundary and if some $p \in (1, \infty)$ has been fixed, then there exists some $C \in (0, \infty)$ (depending only on the uniform rectifiability character of Ω, n, and p) such that

$$\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big) \le C\Big\{\operatorname{dist}\big(K_\Delta, \mathrm{Cp}(L^p(\partial\Omega, \sigma))\big) + \sum_{j,k=1}^{n} \operatorname{dist}\big(\big[M_{\nu_k}, R_j\big], \mathrm{Cp}(L^p(\partial\Omega, \sigma))\big)\Big\}^{1/n}. \tag{1.19}$$

In particular, if K_Δ and all commutators $\big[M_{\nu_k}, R_j\big]$ are compact on $L^p(\partial\Omega, \sigma)$ then ν belongs to $\big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n$.

The stated goal of [61] was to "*find the optimal geometric measure theoretic context in which Fredholm theory can be successfully implemented, along the lines of its original development, for solving boundary value problems with L^p data via the method of layer potentials* [*in domains with compact boundaries*]." In particular, [61] may be regarded as a sharp version of the fundamental work of E. Fabes, M. Jodeit, and N. Rivière in [49], dealing with the method of boundary layer potentials in bounded $\mathscr{C}^1$ domains. As such, the theory developed in [61] goes some way toward answering Kenig's open question formulated at the beginning of this introduction.

However, the insistence on $\partial\Omega$ being a *compact* set is prevalent in this work. In particular, the classical fact that the Dirichlet Problem (1.10) is uniquely solvable in the case when $\Omega = \mathbb{R}^n_+$ (by taking the convolution of the boundary datum g with the harmonic Poisson kernel in the upper half-space; cf. [9], [52], [132], [134]) does *not* fall under the tutelage of [61]. The issue is that once the uniformly rectifiable domain Ω is allowed to have an unbounded boundary then, generally speaking, singular integral operators like the harmonic double layer (1.13) are no longer (close to being) compact on $L^p(\partial\Omega, \sigma)$, though they remain well defined, linear, and bounded on this space, as long as $1 < p < \infty$. The fact that the theory developed in [61] is not applicable in this scenario leads one to speculate whether the treatment of layer potentials may be extended to a class of unbounded domains that includes the upper half-space. In particular, it is natural to ask whether there is a parallel theory for unbounded domains $\Omega \subseteq \mathbb{R}^n$ in which we control the mean oscillations of its outward unit normal ν by suitably adapting the condition $\operatorname{dist}\big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\big) < \delta$ which is ubiquitous in [61]. This is indeed the main goal in the present monograph.

A seemingly peculiar aspect of the harmonic double layer operator (which, in hindsight turns out to be one of its salient features) is that, as visible from (1.13), if $\Omega = \mathbb{R}^n_+$ then $K_\Delta = 0$. Indeed, in such a case we have $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ and $\nu = (0, \dots, 0, -1)$, hence $\langle \nu(y), y - x\rangle = 0$ for all $x, y \in \partial\Omega$. This observation lends some credence to the conjecture loosely formulated as follows:

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a uniformly rectifiable domain and } 1 < p < \infty, \text{ then the operator norm } \|K_\Delta\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \text{ is small if } \Omega \text{ is close to being a half-space in } \mathbb{R}^n. \tag{1.20}$$

To make this precise, one needs to choose an appropriate way of quantifying the proximity of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ to a half-space in $\mathbb{R}^n$. Since a result from [111, §5.10] (based on work in [59]) gives that a uniformly rectifiable domain $\Omega \subsetneq \mathbb{R}^n$ actually is a half-space in $\mathbb{R}^n$ if and only if its geometric measure theoretic outward unit normal ν is a constant vector field, in which scenario $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 0$, it is natural to formulate the following problem (which is a precise, quantitative version of (1.20)):

find a continuous non-decreasing function $\phi : [0, 1] \to [0, \infty)$ which vanishes at the origin with the property that for any given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ and any given integrability exponent $p \in (1, \infty)$ there exists some constant $C \in (0, \infty)$ (which depends only on the uniform rectifiability character of Ω, the dimension n, and the exponent p) such that $\|K_\Delta\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \leq C\phi(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n})$. (1.21)

We may go a step further and adopt a broader perspective, by replacing the Laplacian with a more general system of the sort discussed in (1.1). Specifically, consider a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system L in $\mathbb{R}^n$ written as in (1.1) for some coefficient tensor A as in (1.4). Then one may speculate whether there exists some continuous non-decreasing function $\phi : [0, 1] \to [0, \infty)$ which vanishes at the origin with the property that for any given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ and any given exponent $p \in (1, \infty)$ there exists some constant $C \in (0, \infty)$ (which depends only on the uniform rectifiability character of Ω, the dimension n, the exponent p, and the coefficient tensor A) such that the double layer potential operator K_A associated with the set Ω and the coefficient tensor A as in (1.6) satisfies

$$\|K_A\|_{[L^p(\partial\Omega,\sigma)]^M\to[L^p(\partial\Omega,\sigma)]^M} \leq C\phi(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}). \tag{1.22}$$

It turns out that the choice of the coefficient tensor A used to write the given system L drastically affects the veracity of (1.22). Indeed, consider the case when $L := \Delta$ is the Laplacian in $\mathbb{R}^2$, and $\Omega := \mathbb{R}^2_+$. Observe that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} = 0$ in this case, since ν is constant. From (1.7)–(1.8) we see that $K_{A_0} = 0$, which is in agreement with what (1.22) predicts in this case. On the other hand, the operator K_{A_1} from (1.9) becomes (under the natural identification $\partial\Omega \equiv \mathbb{R}$)

$$K_{A_1} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{f(y)}{y-x}\, dy \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}, \tag{1.23}$$

i.e., $K_{A_1} = (\mathrm{i}/2)H$ where

$$Hf(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{f(y)}{x-y}\,\mathrm{d}y \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R} \tag{1.24}$$

is the classical Hilbert transform on the real line. In particular, since $H^2 = -I$ we have $\big(K_{A_1}\big)^2 = 4^{-1} I$ which goes to show that

$$\|K_{A_1}\|_{L^p(\mathbb{R},\mathcal{L}^1)\to L^p(\mathbb{R},\mathcal{L}^1)} \geq 2^{-1} \tag{1.25}$$

invalidating (1.22) in this case.

A higher-dimensional version of the above considerations goes as follows. Given $n \in \mathbb{N}$ with $n \geq 2$, let $\{E_j\}_{1\leq j\leq n}$ be a family of $2^n \times 2^n$ matrices satisfying, with $I_{2^n\times 2^n}$ denoting the $2^n \times 2^n$ identity matrix,

$$\begin{aligned} \big(E_j\big)^2 &= -I_{2^n\times 2^n} \quad \text{for each } \ j \in \{1,\dots,n\} \ \text{ and} \\ E_j E_k &= -E_k E_j \ \text{ for all } \ j,k \in \{1,\dots,n\} \ \text{ with } \ j \neq k. \end{aligned} \tag{1.26}$$

Specifically, consider the double-indexed family of matrices $\big\{E_j^m\big\}_{\substack{1\leq m\leq n\\ 1\leq j\leq m}}$ defined inductively by

$$E_1^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{1.27}$$

and, in general, given any $m \in \{1,\dots,n-1\}$,

$$E_j^{m+1} := \begin{pmatrix} E_j^m & 0 \\ 0 & -E_j^m \end{pmatrix} \in \mathbb{R}^{2^{m+1}\times 2^{m+1}} \quad \text{for each } \ j \in \{1,\dots,m\}, \tag{1.28}$$

and

$$E_{m+1}^{m+1} := \begin{pmatrix} 0 & -I_{2^m\times 2^m} \\ I_{2^m\times 2^m} & 0 \end{pmatrix} \in \mathbb{R}^{2^{m+1}\times 2^{m+1}}, \tag{1.29}$$

where $I_{2^m\times 2^m}$ denotes the $2^m \times 2^m$ identity matrix. Induction then shows that for each $m \in \{1,\dots,n\}$ we have

$$\begin{aligned} \big(E_j^m\big)^2 &= -I_{2^m\times 2^m} \quad \text{for each } \ j \in \{1,\dots,m\} \ \text{ and} \\ E_j^m E_k^m &= -E_k^m E_j^m \ \text{ for all } \ j,k \in \{1,\dots,m\} \ \text{ with } \ j \neq k. \end{aligned} \tag{1.30}$$

In particular, abbreviating $E_j := E_j^n$ for each $j \in \{1,\dots,n\}$ then guarantees that the conditions in (1.26) are satisfied.

To proceed, define $M := 2^n$ and denote by $I_{M\times M}$ the $M \times M$ identity matrix. Consider the $M \times M$ second-order system in $\mathbb{R}^n$ defined as

$$L := \Delta \cdot I_{M\times M}, \tag{1.31}$$

where $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian in $\mathbb{R}^n$. In particular, the fundamental solution E_L associated with the weakly elliptic system L as in Theorem 3.1 is given by

$$E_L := E_\Delta \cdot I_{M\times M}, \tag{1.32}$$

where E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, defined in (3.27).

Next, for each $j, k \in \{1, \dots, n\}$ let us denote by $(a_{jk}^{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ the entries of the $M \times M$ matrix $-E_j E_k$, i.e.,

$$-E_j E_k = \left(a_{jk}^{\alpha\beta}\right)_{1\leq\alpha,\beta\leq M} \in \mathbb{R}^{M\times M} \quad \text{for each} \quad j, k \in \{1, \dots, n\}. \tag{1.33}$$

Then, with the summation convention over repeated indices in effect, we have

$$\left(a_{jk}^{\alpha\beta}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M} = -E_j E_k \partial_j \partial_k = -(E_j)^2\partial_j^2 = \Delta \cdot I_{M\times M}, \tag{1.34}$$

thanks to (1.26). Hence,

$$L = \left(a_{jk}^{\alpha\beta}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}. \tag{1.35}$$

Consider next the boundary-to-boundary double layer potential operator K_{A_1} associated as in (1.6) with the coefficient tensor

$$A_1 := \left(a_{jk}^{\alpha\beta}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \text{ with entries as in (1.33)} \tag{1.36}$$

and the domain $\Omega := \mathbb{R}^n_+$. In view of (1.32) and the fact that the outward unit normal vector to $\mathbb{R}^n_+$ is given by $\nu = (0, \dots, 0, -1)$, the action of said double layer potential operator on each function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \left[L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\right)\right]^M$ is given at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ by

$$K_{A_1} f(x') = \left(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} a_{jn}^{\beta\alpha}\left(\partial_j E_\Delta\right)(x'-y') f_\alpha(y')\,dy'\right)_{1\leq\beta\leq M}$$

$$= \lim_{\varepsilon \to 0^+} \int\limits_{\substack{y \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} \big(\partial_j E_\Delta\big)(x'-y') E_j E_n f(y')\, \mathrm{d}y'. \tag{1.37}$$

Hence, with $(R_j)_{1\le j\le n-1}$ denoting the Riesz transforms in $\mathbb{R}^{n-1}$ (cf. (1.16)), we may recast (1.37) simply as

$$K_{A_1} = \sum_{j=1}^{n-1} \tfrac{1}{2} E_j E_n R_j \quad \text{on} \quad \Big[L^1\Big(\mathbb{R}^{n-1}, \tfrac{dx'}{1+|x'|^{n-1}}\Big)\Big]^M. \tag{1.38}$$

Fix now an arbitrary integrability exponent $p \in (1, \infty)$. Then (1.38), (1.26), together with the usual Riesz transform identities (i.e., $\sum_{j=1}^{n-1} R_j^2 = -I$ and $R_j R_k = R_k R_j$ for each $j, k \in \{1, \dots, n\}$) imply that

$$\begin{aligned}
\big(K_{A_1}\big)^2 &= \Big(\sum_{j=1}^{n-1} \tfrac{1}{2} E_j E_n R_j\Big)^2 = \frac{1}{4}\sum_{j,k=1}^{n-1} E_j E_n E_k E_n R_j R_k \\
&= \frac{1}{4}\sum_{j,k=1}^{n-1} E_j E_k R_j R_k = \frac{1}{4}\sum_{j=1}^{n-1} E_j^2 R_j^2 \\
&= \frac{1}{4}\Big(-\sum_{j=1}^{n-1} R_j^2\Big) I_{M\times M} = \tfrac{1}{4} I_{M\times M}
\end{aligned} \tag{1.39}$$

as operators on $\big[L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^M$. Much as with its two-dimensional counterpart in (1.25), this goes to show that

$$\big\| K_{A_1} \big\|_{[L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^M \to [L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^M} \ge 2^{-1} \tag{1.40}$$

once again invalidating (1.22) for the current choice of coefficient tensor. On the other hand, the choice of the coefficient tensor

$$\begin{gathered}
A_0 := \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\le\alpha,\beta\le M \\ 1\le j,k\le n}} \text{ with } a_{jk}^{\alpha\beta} := \delta_{\alpha\beta}\delta_{jk} \\
\text{for all } 1 \le \alpha, \beta \le M \text{ and } 1 \le j, k \le n
\end{gathered} \tag{1.41}$$

allows the system (1.31) to be written as in (1.35) and the boundary-to-boundary double layer potential operator K_{A_0} associated as in (1.6) with the coefficient tensor A_0 and the domain $\Omega := \mathbb{R}^n_+$ is $K_{A_0} = 0$ (cf. the first line in (1.37)).

The above considerations bring up the question of determining which of the many coefficient tensors A that may be used in the representation of the given system

L as in (1.1) actually give rise to double layer potential operators K_A (via the blueprint (1.6)) that have a chance of satisfying the estimate formulated in (1.22). This question is of an algebraic nature. To answer it, we find it convenient to adopt a more general point of view and consider the class of singular integral operators acting at σ-a.e. point $x \in \partial\Omega$ on functions f as in (1.5) according to

$$T_\Theta f(x) := \left(\lim_{\varepsilon \to 0^+} \int_{\partial\Omega \setminus \overline{B(x,\varepsilon)}} \big\langle \Theta_\gamma(x-y)\nu(y),\, f(y) \big\rangle \, d\sigma(y) \right)_{1 \le \gamma \le M}, \tag{1.42}$$

where

$$\begin{gathered} \Theta = (\Theta_\gamma)_{1\le\gamma\le M} \ \text{ with each } \Theta_\gamma \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times n} \\ \text{odd and positive homogeneous of degree } \ 1-n. \end{gathered} \tag{1.43}$$

Note that K_A fits into this class, as it corresponds to (1.42) with $\Theta = (\Theta_\gamma)_{1\le\gamma\le M}$ given by $\Theta_\gamma := \big(-a^{\beta\alpha}_{jk}\partial_j E_{\gamma\beta}\big)_{\substack{1\le\alpha\le M \\ 1\le k\le n}}$ for each index $\gamma \in \{1, \dots, M\}$.

In this notation, the question is to find what additional condition should be imposed on $\Theta = (\Theta_\gamma)_{1\le\gamma\le M}$ so that the analogue of (1.22) holds with the operator K_A replaced by T_Θ. The latter inequality implies that

$$T_\Theta \text{ must vanish whenever } \Omega \text{ is a half-space in } \mathbb{R}^n. \tag{1.44}$$

Choosing $\Omega := \{z \in \mathbb{R}^n : \langle z, \omega\rangle > 0\}$ with $\omega \in S^{n-1}$ arbitrary then leads to the conclusion that for each index $\gamma \in \{1, \dots, M\}$ we have

$$\Theta_\gamma(x-y)\omega = 0 \ \text{ for each } \omega \in S^{n-1} \text{ and each } x, y \in \langle\omega\rangle^\perp \text{ with } x \neq y. \tag{1.45}$$

Specializing this to the case when $y = 0$ and observing that $x \in \langle\omega\rangle^\perp$ is equivalent to having $\omega \in \langle x\rangle^\perp$, we arrive at

$$\Theta_\gamma(x)\omega = 0 \in \mathbb{C}^M \ \text{ whenever } x \neq 0 \text{ and } \omega \in \langle x\rangle^\perp, \tag{1.46}$$

which is the same as saying that for each vector $x \in \mathbb{R}^n \setminus \{0\}$ the rows of the matrix $\Theta_\gamma(x) \in \mathbb{C}^{M\times n}$ are scalar multiples of x. Thus, there exists a family of scalar functions $k_{\gamma,1}, \dots, k_{\gamma,M}$ defined in $\mathbb{R}^n \setminus \{0\}$ such that

$$\begin{gathered} \text{for each } x \in \mathbb{R}^n \setminus \{0\}, \ \text{the rows of } \Theta_\gamma(x) \\ \text{are } k_{\gamma,1}(x)x, \dots, k_{\gamma,M}(x)x. \end{gathered} \tag{1.47}$$

Ultimately, this implies that $k := \big(k_{\gamma,\alpha}\big)_{\substack{1\le\gamma\le M \\ 1\le\alpha\le M}}$ is a matrix-valued function belonging to $\big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ which is even, positive homogeneous of degree $-n$, and such that for each $\gamma \in \{1, \dots, M\}$ we have

$$\Theta_\gamma(x)\omega = \langle x, \omega\rangle k_{\gamma\cdot}(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \omega \in \mathbb{R}^n. \tag{1.48}$$

Consequently, T_Θ from (1.42) may be simply recast as

$$Tf(x) = \lim_{\varepsilon \to 0^+} \int_{\partial\Omega \setminus \overline{B(x,\varepsilon)}} \langle x - y, \nu(y)\rangle k(x - y) f(y)\, d\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.49}$$

In terms of the original double layer potential operator K_A, the above argument proves that

$$\text{if (1.22) holds then the integral kernel of } K_A \text{ is necessarily of the form } \langle x - y, \nu(y)\rangle k(x - y) \text{ for some matrix-valued function } k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M} \text{ which is even and positive homogeneous of degree } -n. \tag{1.50}$$

Algebraic conditions, formulated solely in terms of A, guaranteeing that the integral kernel of K_A has the distinguished structure singled out in (1.50) have been identified in [115, Chapter 1] (see Definition 3.1). Henceforth, we shall refer to such a coefficient tensor A as being "distinguished," and we shall denote by $\mathfrak{A}_L^{\rm dis}$ the collection of all distinguished coefficient tensors which may be employed in the writing of a given system L.

In (3.223) we show that *all scalar* second-order homogeneous constant complex coefficient weakly elliptic operators L in $\mathbb{R}^n$ with $n \geq 3$ possess precisely one distinguished coefficient tensor. Consequently, $\mathfrak{A}_L^{\rm dis}$ is nonempty (in fact, a singleton) whenever $L = {\rm div} A \nabla$ in $\mathbb{R}^n$ with $n \geq 3$, with the coefficient matrix $A = (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$ satisfying the weak ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \neq 0, \qquad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{1.51}$$

In particular, this is the case for the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$.

Other examples of weakly elliptic second-order homogeneous constant coefficient systems which possess distinguished coefficient tensors are obtained by considering the complex version of the Lamé system of elasticity in $\mathbb{R}^n$, with $n \geq 2$,

$$L_{\mu,\lambda} := \mu\Delta + (\lambda + \mu)\nabla{\rm div}, \tag{1.52}$$

where the Lamé moduli $\lambda, \mu \in \mathbb{C}$ are assumed to satisfy

$$\mu \neq 0, \quad 2\mu + \lambda \neq 0, \quad 3\mu + \lambda \neq 0. \tag{1.53}$$

The first two requirements in (1.53) are equivalent to having the system $L_{\mu,\lambda}$ weakly elliptic (in the sense of (1.2)), while the last requirement in (1.53) ensures the existence of a distinguished coefficient tensor for $L_{\mu,\lambda}$. It turns out that if the last condition in (1.53) is violated then $L_{\mu,\lambda}$ fails to have a distinguished coefficient tensor.

It is of interest to remark that the (strong) Legendre–Hadamard ellipticity condition (1.3) holds for the complex Lamé system $L_{\mu,\lambda}$ if and only if

$$\operatorname{Re}\mu > 0 \ \text{ and } \ \operatorname{Re}(2\mu + \lambda) > 0. \tag{1.54}$$

As such, our results apply to certain classes of weakly elliptic second-order systems which are not necessarily strongly elliptic (in the sense of Legendre–Hadamard). Also, while the Lamé system is symmetric, we stress that the main results in this monograph require no symmetry for the systems involved.

Recall that ${}^m\mathrm{e}$ denotes the m-th tetration of e (involving m copies of e, combined via exponentiation), i.e.,

$${}^m\mathrm{e} := \underbrace{\mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot^{\mathrm{e}}}}}}}_{m \text{ copies of e}}, \quad \text{the } m\text{-th fold exponentiation of e.} \tag{1.55}$$

For each $t \geq 0$ let us define

$$t^{\langle m \rangle} := \begin{cases} 0 & \text{if } \ t = 0, \\ t \cdot \underbrace{\ln\Big(\cdots \ln\big(\ln(1/t)\big)\cdots\Big)}_{m \text{ natural logarithms}} & \text{if } \ 0 < t \leq ({}^m\mathrm{e})^{-1}, \\ ({}^m\mathrm{e})^{-1} & \text{if } \ t > ({}^m\mathrm{e})^{-1}. \end{cases} \tag{1.56}$$

One of the main results in this work asserts that if L is a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, and if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, then for each $m \in \mathbb{N}$, each $A \in \mathfrak{A}_L^{\mathrm{dis}}$, and each $p \in (1, \infty)$ there exists a constant $C_m \in (0, \infty)$ (which depends only on m, n, p, A, and the uniform rectifiability character of Ω) such that estimate (1.22) actually holds for the choice of the function $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(t) := t^{\langle m \rangle}$ for each $t \in [0, \infty)$. In particular, this offers a solution to the problem formulated in (1.21).

See Theorem 4.7 for a result of a more general flavor, formulated in terms of Muckenhoupt weighted Lebesgue spaces. Specifically, if the system L, the coefficient tensor A, and the set Ω are as just described, then for each $m \in \mathbb{N}$ and Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$ there exists a constant $C_m \in (0, \infty)$ (which now also depends on $[w]_{A_p}$, defined in (2.517)) with the property that

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}. \tag{1.57}$$

In turn, Theorem 4.7 is painlessly implied by the even more general result presented in Theorem 4.2 which is one of the focal points of this monograph. The proof of Theorem 4.2 uses a combination of tools of a purely geometric nature (such as Theorem 2.6 containing a versatile version of a decomposition result originally established by S. Semmes for smooth surfaces in [123] then subsequently strengthened as to apply to rough settings in [61], and the estimate from Proposition 2.15 controlling the inner product between the integral average of the outward unit normal and the "chord" in terms of the BMO semi-norm of the outward unit normal to a domain), techniques of a purely harmonic analytic nature (like good-λ inequalities, maximal operator estimates, stopping time arguments, and Muckenhoupt weight theory), and a bootstrap argument designed to successively improve the nature of the function ϕ in (1.22).

These considerations lead us to adopt (as we do in Definition 2.15) the following basic piece of terminology. Given $\delta > 0$, an open, nonempty, proper subset Ω of $\mathbb{R}^n$ is said to be a δ-`flat Ahlfors regular domain` (or δ-AR domain, for short) if $\partial\Omega$ is an Ahlfors regular set, and if $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then the geometric measure theoretic outward unit normal ν to Ω is well defined at σ-a.e. point on $\partial\Omega$ and satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta. \tag{1.58}$$

Remarkably, demanding that δ in (1.58) is small has topological and metric implications for the underlying domain, namely Ω is two-sided NTA domain, which is a connected unbounded open set, with a connected unbounded boundary, and an unbounded connected complement (see Theorem 2.4). In the two-dimensional setting we actually show that the class of δ-AR domains with $\delta \in (0, 1)$ small agrees with the category of chord-arc domains with small constant (see Theorem 2.7 for a precise statement). Most importantly, (1.57) shows that the oscillatory behavior of the outward unit normal is a key factor in determining the size of the operator norm for the double layer potential operator K_A on $\big[L^p(\partial\Omega, w)\big]^M$.

Inspired by the format of a double layer operator (cf. (1.6)), so far we have been searching for singular integral operators fitting the general template in (1.42) for which it may be possible to control their operator norm in terms of $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$. While $\{T_\Theta : \Theta \text{ as in (1.43)}\}$ is a linear space, this is not stable under transposition (which is an isometric transformation and, hence, preserves the quality of having a small norm). This suggests that we cast a wider net and consider the class of singular integrals acting at σ-a.e. point $x \in \partial\Omega$ on functions f as in (1.5) according to

$$T_{\Theta^1,\Theta^2} f(x)$$

$$:= \left(\lim_{\varepsilon\to 0^+} \int\limits_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \Big\langle \Theta^1_\gamma(x-y)\nu(y) - \Theta^2_\gamma(x-y)\nu(x),\, f(y)\Big\rangle \, d\sigma(y) \right)_{1\le\gamma\le M} \tag{1.59}$$

where $\Theta_1 = (\Theta^1_\gamma)_{1\le\gamma\le M}$ and $\Theta_2 = (\Theta^2_\gamma)_{1\le\gamma\le M}$ are as in (1.43). The latter condition ensures that T_{Θ^1,Θ^2} is a well-defined, linear, and bounded operator on $\big[L^p(\partial\Omega, w)\big]^M$ (recall that we are assuming Ω to be a uniformly rectifiable domain). Consequently, $\{T_{\Theta^1,\Theta^2} : \Theta^1, \Theta^2$ as in (1.43)$\}$ is a linear subspace of the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$ which contains each double layer K_A as in (1.6) as well as its formal transpose $K^\#_A$, whose action on each function f as in (1.5) at σ-a.e. $x \in \partial\Omega$ is given by

$$K^\#_A f(x) := \left(\lim_{\varepsilon\to 0^+} \int\limits_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \nu_k(x) a^{\beta\alpha}_{jk} (\partial_j E_{\gamma\beta})(x-y) f_\gamma(y) \, d\sigma(y) \right)_{1\le\alpha\le M}. \tag{1.60}$$

If an estimate like (1.57) would hold for the operator (1.59), then we would have $T_{\Theta^1,\Theta^2} = 0$ whenever $\Omega \subseteq \mathbb{R}^n$ is a half-space. Taking $\Omega := \{z \in \mathbb{R}^n : \langle z, \omega\rangle > 0\}$ with $\omega \in S^{n-1}$ arbitrary then forces that for each index $\gamma \in \{1, \dots, M\}$ we have

$$\begin{aligned} \big[\Theta^1_\gamma(x-y) - \Theta^2_\gamma(x-y)\big]\omega = 0 \quad &\text{for each} \quad \omega \in S^{n-1} \\ \text{and each } x, y \in \langle\omega\rangle^\perp &\text{ with } x \neq y. \end{aligned} \tag{1.61}$$

The same type of reasoning which, starting with (1.45), has produced (1.48) then shows that there exists a matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$, which is even as well as positive homogeneous of degree $-n$, such that for each index $\gamma \in \{1, \dots, M\}$ we have

$$[\Theta^1_\gamma(z) - \Theta^2_\gamma(z)]\omega = \langle x, \omega\rangle k_{\gamma\cdot}(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \omega \in \mathbb{R}^n. \tag{1.62}$$

In turn, this implies that (1.59) may be recast as

$$\begin{aligned} T_{\Theta^1,\Theta^2} f(x) = \lim_{\varepsilon\to 0^+} &\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle x-y, \nu(y)\rangle k(x-y) f(y) \, d\sigma(y) \\ &+ \left(\lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle \Theta^2_\gamma(x-y)(\nu(y)-\nu(x)),\, f(y)\rangle \, d\sigma(y) \right)_{1\le\gamma\le M} \end{aligned} \tag{1.63}$$

for σ-a.e. $x \in \partial\Omega$. The first principal-value integral in (1.63) has been encountered earlier in (1.49), while the second one is of commutator type. Specifically, the second principal-value integral in (1.63) may be thought of as a finite linear combination of commutators between singular integral operators of convolution type with kernels which are odd and positive homogeneous of degree $1 - n$ (like the entries in any of the matrices Θ_γ^2) and operators M_{ν_j} of pointwise multiplication with the scalar components ν_j, $1 \leq j \leq n$, of the outward unit normal ν.

The ultimate conclusion is that, in addition to the family of operators described in (1.49), the class of commutators of the sort just described provides the only other viable candidates for operators whose norms become small when the ambient surface on which they are defined becomes flatter. That such an eventuality actually materializes is implied by Hofmann et al. [61, Theorem 2.16, p. 2603] which, in particular, gives (in the same setting as above)

$$\sum_{j,k=1}^{n} \big\| \big[M_{\nu_k}, R_j\big] \big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{1.64}$$

In the opposite direction, in Theorem 5.2 we prove that whenever $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, $1 < p < \infty$, and $w \in A_p(\partial\Omega, \sigma)$, there exists some $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ &\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \\ &+ \max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \Big\}. \end{aligned} \tag{1.65}$$

This is done using the Clifford algebra machinery (briefly recalled in Sect. 5.1) and exploiting the relationship between the Cauchy–Clifford operator (cf. (5.12)) and the operators K_Δ, $[M_{\nu_k}, R_j]$ with $1 \leq j, k \leq n$, intervening in (1.65). Collectively, these results point to the optimality of the class of δ-AR domains with $\delta \in (0, 1)$ small as the geometric environment in which $\|K_\Delta\|_{[L^p(\partial\Omega,w)]^M\to[L^p(\partial\Omega,w)]^M}$ and $\big\|\big[M_{\nu_k}, R_j\big]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ for $1 \leq j, k \leq n$ can possibly be small (relative to n, p, $[w]_{A_p}$, and the uniform rectifiability character of $\partial\Omega$).

We also succeed in characterizing flatness solely in terms of the behavior of the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ (defined in (1.16)). In one direction, in Theorem 5.3 we show that if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain with an unbounded boundary and $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, then there exists some $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, and the uniform rectifiability character of $\partial\Omega$ with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \Big\| I + \sum_{j=1}^{n} R_j^2 \Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \tag{1.66}$$

$$+ \max_{1 \le j,k \le n} \big\| [R_j, R_k] \big\|_{L^p(\partial\Omega, w) \to L^p(\partial\Omega, w)} \Big\}.$$

In the opposite direction, in Theorem 5.4 we prove that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set, then for each Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ and each $m \in \mathbb{N}$ there exists some constant $C_m \in (0, \infty)$ which depends only on m, n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ such that

$$\Big\| I + \sum_{j=1}^{n} R_j^2 \Big\|_{L^p(\partial\Omega, w) \to L^p(\partial\Omega, w)} \le C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m \rangle}, \tag{1.67}$$

and

$$\max_{1 \le j < k \le n} \big\| [R_j, R_k] \big\|_{L^p(\partial\Omega, w) \to L^p(\partial\Omega, w)} \le C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m \rangle}. \tag{1.68}$$

Collectively, (1.66)–(1.68) give a fully satisfactory answer to the question of quantifying flatness of a given "surface" Σ (thought of as the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$) in terms of the operator theoretic nature of the Riesz transforms on Σ. Informally, these estimates amount to saying that the flatter Σ is, the closer $\{R_j\}_{1 \le j \le n}$ are to satisfying the "usual" Riesz transform identities

$$\sum_{j=1}^{n} R_j^2 = -I \quad \text{and} \quad R_j R_k = R_k R_j \quad \text{for all} \quad j, k \in \{1, \dots, n\}, \tag{1.69}$$

when all operators are considered on Muckenhoupt weighted Lebesgue spaces on Σ, and vice versa. In the limit case when Σ is genuinely flat (manifested through the vanishing of the BMO semi-norm of its unit normal), all formulas in (1.69) hold as stated. The best known case is that when Σ is the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ in $\mathbb{R}^n$, a scenario in which (1.69) may be readily checked when $p = 2$ and $w \equiv 1$ based on the fact that each R_j is a Fourier multiplier corresponding to the symbol $\mathrm{i}\xi_j/|\xi|$.

The insistence on Muckenhoupt weights is justified by the fact that the boundedness of the Riesz transforms on a weighted Lebesgue space L^p with $p \in (1, \infty)$ actually forces the intervening weight to belong to the Muckenhoupt class A_p. See the discussion in Sect. 5.4 in this regard, where other related results may be found.

While estimate (1.57) is valid irrespective of whether $\partial\Omega$ is bounded or not, its usefulness is most apparent when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small (relative to the geometry of Ω and the weight w) since, in the context of (1.57), having $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ small implies that $\frac{1}{2}I + K_A$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$ and $(\frac{1}{2}I + K_A)^{-1}$ may be expressed as the Neumann series $2^{-1}\sum_{j=0}^{\infty}(-2K_A)^j$, which is convergent in the operator norm, (1.70)

and one can actually show that having $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < 1$ forces $\partial\Omega$ to be unbounded (see Lemma 2.8). We may therefore recast (1.70) as saying that we may invert $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega, w)\big]^M$ whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain for some $\delta \in (0, 1)$ sufficiently small (relative to the dimension n, the Ahlfors regularity constant of $\partial\Omega$, the exponent p, and the weight w), and the latter condition implies that $\partial\Omega$ is unbounded.

A precise formulation of this result goes as follows: Fix $n, M \in \mathbb{N}$ and consider a weakly elliptic homogeneous constant complex coefficient second-order $M \times M$ system L in $\mathbb{R}^n$ with $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$. Then for each constants $C_A, C_W \in (0, \infty)$, each compact interval $I \subset (1, \infty)$, and each coefficient tensor $A \in \mathfrak{A}_L^{\mathrm{dis}}$ there exists a threshold $\delta \in (0, 1)$ which depends only on n, C_A, C_W, I, and A with the following significance. Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain such that the Ahlfors regularity constant of $\partial\Omega$ is $\leq C_A$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in I$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq C_W$. Finally, consider the boundary-to-boundary double layer potential operator K_A, associated with the set Ω and the coefficient tensor A as in (1.6). Then $\frac{1}{2}I + K_A$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$ provided $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$.

Estimate (1.57) then becomes a powerful tool in the proof of similar results on other function spaces. First, in concert with the homogeneous space version of the commutator theorem of Coifman et al. [31], proved in [61, Theorem 2.16, p. 2603], this implies an analogous estimate on Muckenhoupt weighted Sobolev spaces (see (2.587)). That is, retaining the assumptions on the domain Ω and the system L made in the buildup to (1.57), whenever $A \in \mathfrak{A}_L^{\mathrm{dis}}$, $m \in \mathbb{N}$, and $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$ we have

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M \to [L_1^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{1.71}$$

for some constant $C_m \in (0, \infty)$ of the same nature as before. To elaborate on this crucial estimate, one should think of our Muckenhoupt weighted Sobolev space $L_1^p(\partial\Omega, w)$ as being naturally associated with a family $\big\{\partial_{\tau_{jk}}\big\}_{1\leq j,k\leq n}$ of first-order "tangential" differential operators along $\partial\Omega$, which may loosely be described as $\partial_{\tau_{jk}} = \nu_j\partial_k - \nu_k\partial_j$ for each $j, k \in \{1, \dots, n\}$. Specifically, $L_1^p(\partial\Omega, w)$ is the linear space consisting of functions $f \in L^p(\partial\Omega, w)$ with $\partial_{\tau_{jk}} f \in L^p(\partial\Omega, w)$ for each $j, k \in \{1, \dots, n\}$ (see the discussion in Sect. 2.8 in this regard). From this perspective it is then of paramount importance to understand the manner in which a double layer operator K_A commutes with a generic tangential differential operators $\partial_{\tau_{jk}}$. It turns out that

$$\text{each commutator } [K_A, \partial_{\tau_{jk}}] \text{ acting on a function } f \text{ belonging to a Muckenhoupt weighted Sobolev space may be expressed as a finite linear combination of commutators of the form } [M_\nu, R] \text{ acting on the components of } \nabla_{\tan} f, \text{ the tangential gradient of } f, \text{ where } M_\nu \text{ stands for the operator of pointwise multiplication by (generic components of) the unit normal } \nu, \text{ and } R \text{ is a convolution type singular integral operator on } \partial\Omega \text{ of similar nature as the Riesz transforms on } \partial\Omega \text{ (cf. (1.16)).} \tag{1.72}$$

Based on this, (1.57), and a suitable analogue of (1.64), we then conclude that the key estimate stated in (1.71) holds. In turn, (1.71) permits us to invert $\frac{1}{2}I + K_A$ on the Muckenhoupt weighted Sobolev space $\big[L^p_1(\partial\Omega, w)\big]^M$, for each $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$, via a Neumann series converging in the operator norm, whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain for some $\delta \in (0, 1)$ sufficiently small (a condition that forces $\partial\Omega$ to be unbounded) relative to the Ahlfors regularity constant of $\partial\Omega$ and the weight w.

Second, we use the operator norm estimate on Muckenhoupt weighted Lebesgue spaces from (1.57) as a gateway to establishing similar estimates via extrapolation procedures. One of the best known embodiments of this principle is Rubio de Francia's celebrated extrapolation theorem, according to which estimates on Muckenhoupt weighted Lebesgue spaces for a fixed integrability exponent and all weights imply similar estimates for all integrability exponents (prompting Antonio Córdoba to famously declare that "there are no L^p spaces, only weighted L^2 spaces"). Here we use (1.57) together with an extrapolation procedure from [112, §6.2] (recalled in Proposition 7.4) to obtain norm estimates for double layer operators on the scale of Morrey spaces on the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$, i.e.,

$$M^{p,\lambda}(\partial\Omega, \sigma) := \big\{f \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma) : \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty\big\} \tag{1.73}$$

with $p \in (1, \infty)$ and $\lambda \in (0, n-1)$, where[1]

$$\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} := \sup_{\substack{x\in\partial\Omega \text{ and} \\ 0<R<2\,\mathrm{diam}(\partial\Omega)}} \bigg\{R^{\frac{n-1-\lambda}{p}} \Big(⨍_{\partial\Omega\cap B(x,R)} |f|^p \, d\sigma\Big)^{\frac{1}{p}}\bigg\}. \tag{1.74}$$

(Note that the scale of ordinary Lebesgue spaces on $\partial\Omega$ corresponds to the end-point case $\lambda = 0$, while the end-point $\lambda = n - 1$ corresponds to the space of essentially bounded functions on $\partial\Omega$.) Retaining the same geometric context as before and assuming $A \in \mathfrak{A}^{\mathrm{dis}}_L$, the extrapolation procedure alluded to above yields, for each $m \in \mathbb{N}$,

$$\|K_A\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{1.75}$$

[1] throughout, given any nonempty set $E \subseteq \mathbb{R}^n$, we let $\mathrm{diam}(E)$ denote the diameter of E.

for some constant $C_m \in (0,\infty)$ of the same nature as before (cf. Theorem 7.8 for this, and other related results). We may take this a step further and establish a similar operator norm estimate involving the Morrey-based Sobolev space $M_1^{p,\lambda}(\partial\Omega,\sigma)$. These, in turn, allow us to invert $\frac{1}{2}I + K_A$ both on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and on the Morrey-based Sobolev space $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, under similar assumptions as before. See Theorem 7.9 where this and other invertibility results on related spaces are proved. In addition, (1.57) implies (via real interpolation) norm estimates and invertibility results for double layer potential operators on Lorentz spaces and Lorentz-based Sobolev spaces (cf. Remarks 4.11 and 4.16).

Concisely put, in this work we are able to answer Kenig's open question (formulated at the outset of the introduction) pertaining to any given weakly elliptic homogeneous constant complex coefficient second-order system L in $\mathbb{R}^n$ with $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, in the setting of δ-AR domains $\Omega \subseteq \mathbb{R}^n$ with $\delta \in (0,1)$ small (relative to n and the Ahlfors regularity constant of $\partial\Omega$), for ordinary Lebesgue spaces, Lorentz spaces, Muckenhoupt weighted Lebesgue, Morrey spaces, as well as Sobolev spaces on $\partial\Omega$ suitably defined in relation to each of the aforementioned scales (see Theorem 4.8, Remark 4.16, Theorems 4.9, 7.9, 7.10). As indicated in Remark 4.19, the smallness condition imposed on the parameter δ is actually in the nature of best possible as far as these invertibility results are concerned.

In turn, the aforementioned invertibility results open the door for solving boundary value problems of Dirichlet, Regularity, Neumann, and Transmission type in the class of δ-AR domains with $\delta \in (0,1)$ small (relative to the dimension n, the Ahlfors regularity constant of $\partial\Omega$, and the specific nature of the space of boundary data) for second-order weakly elliptic constant complex coefficient systems which (either themselves and/or their transpose) possess distinguished coefficient tensors.

For example, in such a setting, we succeed in establishing the well-posedness of the Muckenhoupt weighted Dirichlet Problem and the Muckenhoupt weighted Regularity Problem (formulated using the nontangential maximal operator introduced in (2.5), and nontangential boundary traces defined as in (2.12), for some fixed aperture parameter $\kappa > 0$):

$$
(D)_{p,w}\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^p(\partial\Omega,w)\big]^M, \end{cases}
\qquad
(R)_{p,w}\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ \mathcal{N}_\kappa (\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L_1^p(\partial\Omega,w)\big]^M, \end{cases}
\tag{1.76}
$$

for each given integrability exponent $p \in (1,\infty)$ and each given Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, under the assumption that both L and $L^\top$ have a distinguished coefficient tensor. Moreover, we provide counterexamples which show that the well-posedness result just described may fail if these assumptions on the

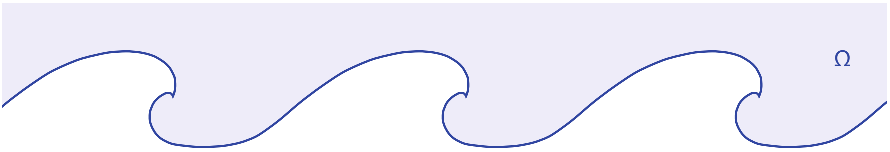

Fig. 1.1 A prototype of an unbounded δ-AR domain for which $\delta > 0$ may be made as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$ (cf. (2.325), (2.327))

existence of distinguished coefficient tensors are simply dropped. See Theorems 6.2 and 6.5 for more nuanced statements. Our results are therefore optimal in this regard. We wish to note that the present work marks the first occasion when boundary problems like (1.76) have been treated in a class of sets large enough as to contain domains with spiral points of the sort described in Fig. 1.1. This being said, even in the scalar (i.e., $M = 1$), unweighted case (i.e., $w \equiv 1$), the well-posedness of the problems in (1.76) would still be new for such basic constant complex coefficient differential operators as

$$L = \partial_1^2 + \cdots + \partial_{n-1}^2 + \mathrm{i}\partial_n^2. \tag{1.77}$$

Existence for the boundary value problems $(D)_{p,w}$, $(R)_{p,w}$ is established by looking for a solution which is expressed as in (1.11), making use of the jump-formula (3.123), and the fact that $\frac{1}{2}I + K_A$ is invertible both on the Muckenhoupt weighted Lebesgue space $\big[L^p(\partial\Omega, w)\big]^M$ as well as on the Muckenhoupt weighted Sobolev space $\big[L_1^p(\partial\Omega, w)\big]^M$. The issue of uniqueness requires a new set of techniques, and this is subtle even in the classical setting of the upper half-space $\Omega := \mathbb{R}^n_+$. In the particular case when $L = \Delta$, the Laplacian in $\mathbb{R}^n$, the Dirichlet boundary value problem $(D)_{p,w}$ in $\Omega := \mathbb{R}^n_+$ has been treated at length in a number of monographs in the unweighted case (i.e., when $w = 1$), including [9], [52], [132], [133], and [134]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel, while the uniqueness relies on either the Maximum Principle or the Schwarz reflection principle for harmonic functions. Neither of these techniques may be adapted successfully to prove uniqueness in the case of general systems treated here. Subsequently, the Dirichlet boundary value problem $(D)_{p,w}$ in $\Omega := \mathbb{R}^n_+$ for a general strongly elliptic, second-order, homogeneous, constant complex coefficient, system L, and for an arbitrary Muckenhoupt weight w has been treated in [92], where existence employs the Agmon-Douglis-Nirenberg Poisson kernel for L, while uniqueness relies on special properties of the Green function for L in the upper half-space $\mathbb{R}^n_+$.

In the present setting, when Ω is merely a δ-AR domain with $\delta \in (0, 1)$ small (relative to n, p, w, and the Ahlfors regularity constant of $\partial\Omega$), in order to deal with the issue of uniqueness for the Muckenhoupt weighted Dirichlet Problem $(D)_{p,w}$ we construct a Green function G for L in Ω by correcting the fundamental solution E of L in $\mathbb{R}^n$ (as to ensure its boundary trace on $\partial\Omega$ vanishes) using the existence part for the Regularity Problem $(R)_{p',w'}$ (formulated for the transpose system $L^\top$, the conjugate exponent p', and the dual weight w') and then employ a rather general Poisson integral representation formula recently established in [113, §4.4] (cf. Theorem 6.1 for a precise statement).

For each given integrability exponent $p \in (1, \infty)$ and each given Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ we also prove (see Theorem 6.8) that what we call the Homogeneous Regularity Problem, namely the boundary value problem

$$(HR)_{p,w} \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M, \end{cases} \tag{1.78}$$

whose formulation involves a homogeneous Muckenhoupt weighted Sobolev space, denoted by $\dot{L}^p_1(\partial\Omega, w)$ (introduced in Definition 2.18), is well posed provided both L and $L^\top$ have a distinguished coefficient tensor and the Ahlfors regular domain Ω is sufficiently flat.

In the same geometric setting, of δ-AR domains, we also discuss the solvability of the Muckenhoupt weighted Neumann Problem (in Theorem 6.11) and the Muckenhoupt weighted Transmission Problem (in Theorem 6.15), i.e.,

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial^A_\nu u = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \qquad \begin{cases} u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega, w), \\ u^+\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \sigma\text{-a.e. on } \partial\Omega, \\ \partial^A_\nu u^+ - \mu \cdot \partial^A_\nu u^- = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \tag{1.79}$$

(where ∂^A_ν is the conormal derivative operator associated with the coefficient tensor A used to represent the given system L, and $\mu \in \mathbb{C}$ is a transmission, or coupling, parameter), as well as variants of those boundary value problems involving Lorentz spaces. In all cases, we show that the boundary layer method may be successfully implemented for any second-order homogeneous constant complex coefficient weakly elliptic system L in $\mathbb{R}^n$ whose transpose possesses a distinguished coefficient tensor, assuming $A \in \mathfrak{A}^{\text{dis}}_{L^\top}$. Moreover, in the two-dimensional setting we show that the Neumann and Transmission Problems (1.79) remain solvable for a larger spectrum of choices of the coefficient tensor for the Lamé system (see the results in Sect. 4.5, as well as Remarks 6.10 and 6.16, in this regard).

In [114], a robust Calderón-Zygmund theory for singular integral operators of boundary layer type associated with weakly elliptic systems and uniformly rectifiable domains has been developed. Here we use such a platform (consisting of results recalled in Proposition 7.5, Theorems 7.1, and 7.2) to prove solvability results for a variety of boundary value problems of Dirichlet, (inhomogeneous and homogeneous) Regularity, Neumann, and Transmission type (akin those formulated in (1.76), (1.78), and (1.79)) with data in Morrey spaces, vanishing Morrey spaces, and block spaces (cf. Theorems 7.18, 7.20, 7.21, 7.22, and 7.23).

In addition, we develop a perturbation theory to the effect that, in all cases discussed so far in this narrative, solvability of a boundary value problem for a certain system L_o implies solvability for any other system L which is sufficiently close to L_o (with proximity quantified using the norm introduced in (3.12)). For results of this nature, the reader is referred to Theorems 6.4, 6.6, 6.12, 6.16, and 7.19.

Lastly, in Sect. 8 we study singular integral operators in more general functional analytic settings. The goal here is to show that these are effective tools in obtaining well-posedness results for boundary problems for second-order systems, formulated in sufficiently flat Ahlfors regular domains, and with boundary data from abstract weighted Banach function spaces. A key result in this regard is a remarkable link between this class of abstract spaces and concrete Muckenhoupt weighted Lebesgue spaces. To briefly elaborate on this topic, we need some notation. Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. Let $\mathbb{X}$ be a Banach function space over (Σ, σ), i.e., the space associated with a function norm as in (8.5) (also referred to as a Köthe function space). With $\mathbb{X}'$ denoting the Köthe dual of $\mathbb{X}$ (also known as the associated space of $\mathbb{X}$ in the terminology of [15]; cf. (8.6)), and with $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on (Σ, σ), assume that

$$\mathcal{M} \text{ is bounded both on } \mathbb{X} \text{ and on } \mathbb{X}'. \tag{1.80}$$

In this setting we then show that (see Proposition 8.1 for a more general and precise result), in a quantitative fashion, for each fixed $p_0 \in [1, \infty)$ we have

$$\mathbb{X} \subseteq \bigcup_{w \in A_{p_0}(\Sigma,\sigma)} L^{p_0}(\Sigma, w). \tag{1.81}$$

Subsequently, in Theorem 8.1, we show that for each pair of σ-measurable functions f, g on Σ, having an inequality of the form

$$\|f\|_{L^{p_0}(\Sigma,w)} \leq C_w \, \|g\|_{L^{p_0}(\Sigma,w)} \tag{1.82}$$

valid for some fixed integrability exponent $p_0 \in [1, \infty)$ and arbitrary Muckenhoupt weights $w \in A_{p_0}(\Sigma, \sigma)$ (where the constant C_w depends in a non-decreasing fashion on $[w]_{A_{p_0}}$) implies

$$\|f\|_{\mathbb{X}} \leq C\,\|g\|_{\mathbb{X}}, \tag{1.83}$$

where $C \in (0, \infty)$ depends only on p_0 and the operator norms of $\mathcal{M}$ on $\mathbb{X}$ and $\mathbb{X}'$. This result, which is in the spirit of Rubio de Francia's celebrated extrapolation theorem, then opens the door for transferring our earlier results in ordinary Muckenhoupt weighted Lebesgue spaces to the setting of abstract weighted Banach function spaces. We methodically explore this venue, and the theory we develop ultimately shows the effectiveness of the boundary layer approach in the treatment of boundary problems for second-order systems, formulated in sufficiently flat Ahlfors regular domains, and with boundary data in abstract Banach function spaces. See Sect. 8 for details; here we only mention that in the last part of this chapter we provide a multitude of relevant examples, including variable exponent Lebesgue spaces, generic rearrangement invariant Banach function spaces (RIBFS for short), Orlicz spaces, Zygmund space, Lorentz spaces, and their weighted versions.

To close, we wish to emphasize that it is natural to consider boundary value problems with boundary data from a large library of function spaces (as done here: Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Morrey spaces, block spaces, abstract weighted Banach function spaces, as well as various Sobolev spaces naturally adapted to these scales, among others). To elaborate on this aspect, assume $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta \in (0, 1)$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an arbitrary aperture parameter $\kappa > 0$ along with some power $a \in (0, n-1)$, and pick some point $x_o \in \partial\Omega$. In this setting, consider the Dirichlet Problem for the Laplacian in Ω, corresponding to the boundary datum

$$f(x) := |x - x_o|^{-a} \quad \text{for each} \quad x \in \partial\Omega \setminus \{x_o\}, \tag{1.84}$$

assumed in a nontangential fashion, i.e.,

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{cases} \tag{1.85}$$

The question which naturally arises is: what size/regularity conditions is the solution u expected to satisfy? The answer very much depends on the actual qualities of the boundary datum f and on the specific frameworks in which we know the Dirichlet Problem to be well-posed. For example, f from (1.84) does not belong to any Lebesgue space $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, so one does not expect $\mathcal{N}_\kappa u$ to belong to any ordinary Lebesgue space $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. This being said, for each fixed point $x_* \in \partial\Omega$ and each exponent $b \in (0, n-1)$ the function

$$w(x) := |x - x_*|^{-b}, \quad \forall\, x \in \partial\Omega \setminus \{x_*\} \tag{1.86}$$

is a Muckenhoupt weight, in the class $A_p(\partial\Omega, \sigma)$, and the function f from (1.84) belongs to the Muckenhoupt weighted Lebesgue space $L^p(\partial\Omega, w)$ if $x_* \neq x_o$ and

$$\max\left\{1, \frac{n-1-b}{a}\right\} < p < \frac{n-1}{a}. \tag{1.87}$$

Assuming $\delta \in (0, 1)$ is sufficiently small, the theory developed here then guarantees that there exists a unique function u solving (1.85) with the additional property that

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, w). \tag{1.88}$$

Since the boundary datum f also belongs to the Lorentz space $L^{(n-1)/a,\infty}(\partial\Omega, \sigma)$ which turns out to be an environment in which we are able to establish the well-posedness of the Dirichlet Problem with appropriate nontangential maximal function control, we then conclude that for the unique function u satisfying (1.85) and (1.88) we also have (assuming $\delta \in (0, 1)$ is sufficiently small)

$$\mathcal{N}_\kappa u \in L^{(n-1)/a,\infty}(\partial\Omega, \sigma). \tag{1.89}$$

Likewise, the fact that the boundary datum f from (1.84) also belongs to the Morrey space $M^{(n-1-\lambda)/a,\lambda}(\partial\Omega, \sigma)$ whenever $\lambda \in (0, n-1-a)$ further entails (again, assuming $\delta \in (0, 1)$ is sufficiently small)

$$\mathcal{N}_\kappa u \in M^{(n-1-\lambda)/a,\lambda}(\partial\Omega, \sigma) \text{ for each } \lambda \in (0, n-1-a). \tag{1.90}$$

The tangential derivatives of the boundary datum f also enjoy integrability properties which translate well in terms of regularity properties for the solution u of (1.85)–(1.88). For example, if

$$\begin{gathered} a \in (0, n-2), \quad \lambda \in (0, n-2-a), \\ \text{and} \quad \max\left\{1, \tfrac{n-1-b}{a+1}\right\} < q < \tfrac{n-1}{a+1}, \end{gathered} \tag{1.91}$$

then for each $j, k \in \{1, \dots, n\}$ we have

$$\partial_{\tau_{jk}} f \in L^q(\partial\Omega, w) \cap L^{(n-1)/(a+1),\infty}(\partial\Omega, \sigma) \cap M^{(n-1-\lambda)/(a+1),\lambda}(\partial\Omega, \sigma), \tag{1.92}$$

which, granted that $\delta \in (0, 1)$ is sufficiently small, ultimately imply

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega, w) \cap L^{(n-1)/(a+1),\infty}(\partial\Omega, \sigma) \cap M^{(n-1-\lambda)/(a+1),\lambda}(\partial\Omega, \sigma). \tag{1.93}$$

It is also interesting to ponder on the nature of the nontangential maximal function for solutions of (1.85) in the case when the boundary datum is a characteristic function, say, $f = \mathbf{1}_E$ for some bounded σ-measurable set $E \subseteq \partial\Omega$. If one regards the latter merely as a function in $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, then the best one can say is that $\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma)$, assuming $\delta \in (0, 1)$ is sufficiently small. However, through the consideration of weights, one may find solutions of said boundary problem for which the nontangential maximal function has better decay properties at infinity. Specifically, fix a point $x_* \in \partial\Omega$, an integrability exponent $p \in (1, \infty)$, a power $b \in \big(-(p-1)(n-1), n-1\big)$, and define the weight w as in (1.86). In particular, we have $f = \mathbf{1}_E \in L^p(\partial\Omega, w)$, and since $w \in A_p(\partial\Omega, \sigma)$ the well-posedness of the Dirichlet problem with data in Muckenhoupt weighted Lebesgue spaces implies we may find a solution u of the boundary value problem (1.85) satisfying

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(x)^p}{|x - x_*|^b}\, d\sigma(x) < +\infty, \tag{1.94}$$

once more, assuming $\delta \in (0, 1)$ is sufficiently small (relative to n, p, b, and the Ahlfors regularity constant of $\partial\Omega$).

Lastly, we wish to note that there is a wealth of sources for boundary value problems in non-smooth domains with boundary data and solutions in Besov and Triebel-Lizorkin spaces, including [11], [50], [68], [100], [98], [103], [104], [106], [107], [115], and the references therein.

Chapter 2
Geometric Measure Theory

We begin with a quick review of notational conventions used in the monograph. Throughout, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ with $n \geq 2$, and $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure in $\mathbb{R}^n$. Also, we shall denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. It is a well-known fact (cf. [47, Theorem 1, p. 61]) that the $(n-1)$-dimensional Hausdorff outer-measure is a Borel-regular outer-measure in $\mathbb{R}^n$. Since the measure induced by an arbitrary outer-measure (as in Carathéodory's theorem) is automatically complete, it follows that

$$\mathcal{H}^{n-1} \text{ is a complete Borel-regular measure in } \mathbb{R}^n. \tag{2.1}$$

Next, for each set $E \subseteq \mathbb{R}^n$, we let $\mathbf{1}_E$ denote the characteristic function of E (defined as $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \in \mathbb{R}^n \setminus E$). Also, δ_{jk} is the Kronecker symbol (i.e., $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$). By $\{\mathbf{e}_j\}_{1 \leq j \leq n}$ we shall denote the standard orthonormal basis in $\mathbb{R}^n$, i.e., $\mathbf{e}_j := (\delta_{jk})_{1 \leq k \leq n}$ for each $j \in \{1, \dots, n\}$. For each $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ set $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The dot product of two vectors $u, v \in \mathbb{R}^n$ is denoted by $u \cdot v = \langle u, v \rangle$, and for each vector $v \in \mathbb{R}^n$ we set $\langle v \rangle^{\perp} := \{u \in \mathbb{R}^n : u \cdot v = 0\}$. Next, $\mathbb{R}^n_{\pm} := \{x \in \mathbb{R}^n : \pm\langle x, \mathbf{e}_n \rangle > 0\}$ denote, respectively, the upper half-space and the lower half-space in $\mathbb{R}^n$.

Given an arbitrary set $\Omega \subseteq \mathbb{R}^n$, we shall denote by $\mathscr{C}^0(\Omega)$ the space of continuous scalar-valued functions defined on Ω. Assuming now that $\Omega \subseteq \mathbb{R}^n$ is actually open, for each $k \in \mathbb{N} \cup \{0\}$ we shall denote by $\mathscr{C}^k(\Omega)$ the space of scalar-valued functions which have continuous partial derivatives of order $\leq k$ in Ω. Also, $\mathscr{C}^\infty_0(\Omega)$ stands for the space of compactly supported functions from $\mathscr{C}^\infty(\Omega)$. We shall let $\mathcal{D}'(\Omega)$ stand for the space of distributions in the set Ω and, for each integrability exponent $p \in [1, \infty]$ and integer $k \in \mathbb{N}$, we shall define the local L^p-based Sobolev space of order k in Ω as $W^{k,p}_{\mathrm{loc}}(\Omega) := \big\{u \in \mathcal{D}'(\Omega) : \partial^\alpha u \in L^p_{\mathrm{loc}}(\Omega, \mathcal{L}^n),\ |\alpha| \leq k\big\}$. The Jacobian matrix of a differentiable $\mathbb{C}^M$-valued function $u = (u_\alpha)_{1 \leq \alpha \leq M}$ defined in an open subset of $\mathbb{R}^n$ is the $\mathbb{C}^{M \cdot n}$-valued function

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_2

$$\nabla u := \big(\partial_j u_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq j\leq n}} = \begin{bmatrix} \partial_1 u_1 & \cdots & \partial_n u_1 \\ \vdots & \vdots & \vdots \\ \partial_1 u_M & \cdots & \partial_n u_M \end{bmatrix}. \tag{2.2}$$

We shall retain the same symbol ∇u when the components of u are actually distributions. Next, we agree to denote by $S^{n-1} := \partial B(0,1)$ the unit sphere in $\mathbb{R}^n$, and use $\omega_{n-1} := \mathcal{H}^{n-1}(S^{n-1})$ for the surface area of S^{n-1}. In addition, we shall let υ_{n-1} denote the volume of the unit ball in $\mathbb{R}^{n-1}$. Given any $x, y \in \mathbb{R}^n$, by $[x, y]$ we shall denote the line segment with endpoints x, y. We shall also need $\operatorname{dist}(x, E) := \inf\{|x-y| : y \in E\}$, the distance from a given point $x \in \mathbb{R}^n$ to a nonempty set $E \subseteq \mathbb{R}^n$. If (X, μ) is a given measure space, for each $p \in (0, \infty]$ we shall denote by $L^p(X, \mu)$ the Lebesgue space of μ-measurable functions which are p-th power integrable on X with respect to μ. Also, by $L^{p,q}(X, \mu)$ with $p, q \in (0, \infty]$ we shall denote the scale of Lorentz spaces on X with respect to the measure μ. In the same setting, for each μ-measurable set $E \subseteq X$ with $0 < \mu(E) < \infty$ and each function f which is absolutely integrable on E we set $⨍_E f\,\mathrm{d}\mu := \mu(E)^{-1}\int_E f\,\mathrm{d}\mu$. For two operators T and S, the symbol $[T, S] := T \circ S - S \circ T$ denotes the commutator of T and S. For a measurable function b, we let M_b be the pointwise multiplication by b, that is, $M_b(f)(x) := b(x)\cdot f(x)$. Given $N, M \in \mathbb{N}$, for any $a = (a_1, \dots, a_N) \in \mathbb{C}^N$ and $b = (b_1, \dots, b_M) \in \mathbb{C}^M$, we agree to define $a \otimes b$ to be the $N \times M$ matrix

$$a \otimes b := \big(a_j b_k\big)_{\substack{1\leq j\leq N\\ 1\leq k\leq M}} \in \mathbb{C}^{N\times M}. \tag{2.3}$$

Finally, we adopt the common convention of writing $A \approx B$ if there exists a constant $C \in (1, \infty)$ with the property that $A/C \leq B \leq CA$ for all values of the relevant parameters entering the definitions of A, B (something that is self-evident in each context we employ this notation).

2.1 Classes of Euclidean Sets of Locally Finite Perimeter

Given an open set $\Omega \subseteq \mathbb{R}^n$ and an aperture parameter $\kappa \in (0, \infty)$, define the nontangential approach regions

$$\Gamma_\kappa(x) := \big\{y \in \Omega : |y - x| < (1+\kappa)\operatorname{dist}(y, \partial\Omega)\big\} \text{ for each } x \in \partial\Omega. \tag{2.4}$$

In turn, these are used to define the nontangential maximal operator $\mathcal{N}_\kappa$, acting on each $\mathcal{L}^n$-measurable function u defined in Ω according to

$$\big(\mathcal{N}_\kappa u\big)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x), \mathcal{L}^n)} \text{ for each } x \in \partial\Omega, \tag{2.5}$$

with the convention that $\big(\mathcal{N}_\kappa u\big)(x) := 0$ whenever $x \in \partial\Omega$ is such that $\Gamma_\kappa(x) = \varnothing$. Note that, if we work (as one usually does) with equivalence classes, obtained by identifying functions which coincide $\mathcal{L}^n$-a.e., the nontangential maximal operator is independent of the specific choice of a representative in a given equivalence class. It turns out that (see [111, §8.2] for a proof)

$$\mathcal{N}_\kappa u : \partial\Omega \to [0, +\infty] \text{ is a lower-semicontinuous function.} \tag{2.6}$$

Also, it is apparent from definitions that

$$\begin{gathered}\text{whenever } u \in \mathscr{C}^0(\Omega) \text{ one actually has} \\ \big(\mathcal{N}_\kappa u\big)(x) = \sup_{y\in\Gamma_\kappa(x)} |u(y)| \ \text{ for all } \ x \in \partial\Omega.\end{gathered} \tag{2.7}$$

More generally, if $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function and $E \subseteq \Omega$ is a $\mathcal{L}^n$-measurable set, we denote by $\mathcal{N}_\kappa^E u$ the nontangential maximal function of u restricted to E, i.e.,

$$\begin{gathered}\mathcal{N}_\kappa^E u : \partial\Omega \longrightarrow [0, +\infty] \ \text{ defined as} \\ (\mathcal{N}_\kappa^E u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x)\cap E, \mathcal{L}^n)} \ \text{ for each } \ x \in \partial\Omega.\end{gathered} \tag{2.8}$$

Hence, $\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa(u \cdot \mathbf{1}_E)$. Throughout, we agree to use the simpler notation $\mathcal{N}_\kappa^\delta$ in the case when $E = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $\delta \in (0, \infty)$, i.e.,

$$\mathcal{N}_\kappa^\delta u := \mathcal{N}_\kappa\big(u\mathbf{1}_{O_\delta}\big) \ \text{ where } \ O_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}. \tag{2.9}$$

It turns out that, when the background measure is doubling, membership of the nontangential maximal function to Lorentz spaces is not contingent on the size of the aperture parameter. This is made precise in the proposition below (see [111, §8.4] for a proof).

Proposition 2.1 *Assume that Ω is an open nonempty proper subset of $\mathbb{R}^n$ and consider a doubling Borel measure σ on $\partial\Omega$. Fix two integrability exponents $p, q \in (0, \infty]$. Then for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$ and any two aperture parameters $\kappa_1, \kappa_2 \in (0, \infty)$ one has, in a quantitative sense,*

$$\mathcal{N}_{\kappa_1} u \in L^{p,q}(\partial\Omega, \sigma) \ \text{ if and only if } \ \mathcal{N}_{\kappa_2} u \in L^{p,q}(\partial\Omega, \sigma), \tag{2.10}$$

and, for each truncation parameter $\delta \in (0, \infty)$,

$$\mathcal{N}_{\kappa_1}^\delta u \in L^p_{loc}(\partial\Omega, \sigma) \ \text{ if and only if } \ \mathcal{N}_{\kappa_2}^\delta u \in L^p_{loc}(\partial\Omega, \sigma). \tag{2.11}$$

Continue to assume that Ω is an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$ and suppose u is some vector-valued $\mathcal{L}^n$-measurable function defined in Ω. Also, fix an aperture parameter $\kappa > 0$ and consider a point $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$ (i.e., x is an accumulation point for the nontangential approach region $\Gamma_\kappa(x)$). In this context, we shall say that the nontangential limit of u at x from within $\Gamma_\kappa(x)$ exists, and its value is the vector $a \in \mathbb{C}^M$, provided

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ there exists } r > 0 \text{ with the property}\\ &|u(y) - a| < \varepsilon \text{ for } \mathcal{L}^n\text{-a.e. point } y \in \Gamma_\kappa(x) \cap B(x,r).\end{aligned} \tag{2.12}$$

Whenever the nontangential limit of u at x from within $\Gamma_\kappa(x)$ exists, we agree to denote its value by the symbol $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$. It is then clear from definitions that whenever the latter exists we have

$$\Big|\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big| \leq \big(\mathcal{N}^\delta_\kappa u\big)(x) \leq \big(\mathcal{N}_\kappa u\big)(x), \quad \text{for all } \delta > 0. \tag{2.13}$$

Moving on, recall that an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has `locally finite perimeter` if its `measure theoretic boundary`, i.e.,

$$\partial_*\Omega := \Big\{x \in \partial\Omega :\ \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap\Omega)}{r^n} > 0,\ \ \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus\Omega)}{r^n} > 0\Big\}, \tag{2.14}$$

satisfies

$$\mathcal{H}^{n-1}\big(\partial_*\Omega \cap K\big) < +\infty \quad \text{for each compact } \ K \subseteq \mathbb{R}^n \tag{2.15}$$

(cf. [47, Sections 5.7 and 5.11]). Alternatively, an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter if, with the gradient taken in the sense of distributions in $\mathbb{R}^n$,

$$\mu_\Omega := -\nabla \mathbf{1}_\Omega \tag{2.16}$$

is an $\mathbb{R}^n$-valued Borel measure in $\mathbb{R}^n$ of locally finite total variation. Occasionally, μ_Ω is referred to as the `Gauss-Green measure` of Ω (see, e.g., [89, Remark 12.2, p. 122]). Fundamental work of De Giorgi-Federer (cf., e.g., [47], [89] for modern accounts) then gives the following Polar Decomposition of the Radon measure μ_Ω:

$$\mu_\Omega = -\nabla\mathbf{1}_\Omega = \nu\,|\nabla\mathbf{1}_\Omega|, \tag{2.17}$$

where $|\nabla\mathbf{1}_\Omega|$, the total variation measure of the measure $\nabla\mathbf{1}_\Omega$, is given by

$$|\nabla\mathbf{1}_\Omega| = \mathcal{H}^{n-1}\lfloor\partial_*\Omega, \tag{2.18}$$

and where

$$\begin{aligned}&\nu \in \big[L^\infty(\partial_*\Omega, \mathcal{H}^{n-1})\big]^n \text{ is an } \mathbb{R}^n\text{-valued function}\\ &\text{satisfying } |\nu(x)| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial_*\Omega.\end{aligned} \tag{2.19}$$

We shall refer to ν above as the geometric measure theoretic outward unit normal to Ω. Note here that by simply eliminating the distribution theory jargon implicit in the interpretation of (2.17) (and using a straightforward limiting argument involving a mollifier) one already arrives at the formula

$$\int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\big|_{\partial\Omega}\big)\,d\mathcal{H}^{n-1} \quad \text{for each vector field } \vec{F}\in\big[\mathscr{C}_0^1(\mathbb{R}^n)\big]^n. \tag{2.20}$$

For a set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter, we let $\partial^*\Omega$ denote the reduced boundary of Ω, that is,

$$\partial^*\Omega \text{ consists of all points } x\in\partial\Omega \text{ satisfying the following two properties: } 0<\mathcal{H}^{n-1}\big(B(x,r)\cap\partial_*\Omega\big)<+\infty \text{ for each radius } r\in(0,\infty), \text{ and } \lim_{r\to 0^+} ⨍_{B(x,r)\cap\partial_*\Omega}\nu\,d\mathcal{H}^{n-1}=\nu(x)\in S^{n-1}. \tag{2.21}$$

From [47, Lemma 2, p. 222] we know that

$$\text{any } \mathcal{L}^n\text{-measurable set } \Omega\subseteq\mathbb{R}^n \text{ has the property that } \partial_*\Omega \text{ is a Borel set in } \mathbb{R}^n \text{ (in particular, } \partial_*\Omega \text{ is } \mathcal{H}^{n-1}\text{-measurable).} \tag{2.22}$$

In addition, given any set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter, from the structure theorem for sets of locally finite perimeter (cf. [47, Theorem 2, p. 205]) it follows that

$$\partial^*\Omega \text{ is countably rectifiable, of dimension } n-1 \quad \text{(hence, the set } \partial^*\Omega \text{ is also } \mathcal{H}^{n-1}\text{-measurable).} \tag{2.23}$$

Moreover, for any set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter we have (cf. [47, p. 208])

$$\partial^*\Omega\subseteq\partial_*\Omega\subseteq\partial\Omega \quad\text{and}\quad \mathcal{H}^{n-1}\big(\partial_*\Omega\setminus\partial^*\Omega\big)=0. \tag{2.24}$$

It is also useful to note that, as remarked in [111, §5.6],

if $\Omega\subseteq\mathbb{R}^n$ is a set of locally finite perimeter and $m\in\mathbb{N}$, then $\widetilde{\Omega}:=\mathbb{R}^m\times\Omega\subseteq\mathbb{R}^{m+n}$ is a set of locally finite perimeter, satisfying $\partial_*\widetilde{\Omega}=\mathbb{R}^m\times\partial_*\Omega$, and whose geometric measure theoretic outward unit normal $\widetilde{\nu}$ is $\widetilde{\nu}(x,y)=(0,\nu(y))$ for $(\mathcal{L}^m\otimes\mathcal{H}^{n-1})$-a.e. point $(x,y)\in\partial_*\widetilde{\Omega}=\mathbb{R}^m\times\partial_*\Omega$, where $0\in\mathbb{R}^m$ and ν is the geometric measure theoretic outward unit normal to the set Ω. (2.25)

The following result, comparing the geometric measure theoretic outward unit normals of two sets of locally finite perimeter (on the intersection of their reduced boundaries), is going to be relevant for us later on, in Theorem 2.6 (and, by extension, in the proof of Theorem 4.2).

Proposition 2.2 *Let E, F be two sets of locally finite perimeter in $\mathbb{R}^n$. If ν_E and ν_F denote the geometric measure theoretic outward unit normal vectors to E and F, respectively, then at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$ one has either $\nu_E(x) = \nu_F(x)$ or $\nu_E(x) = -\nu_F(x)$.*

Proof This is a consequence of [89, Proposition 10.5, p. 101] according to which

any two locally $\mathcal{H}^{n-1}$-rectifiable sets $M_1, M_2 \subseteq \mathbb{R}^n$ have identical approximate tangent planes at $\mathcal{H}^{n-1}$-a.e. point in $M_1 \cap M_2$, (2.26)

and [129, Theorem 14.3, (1), pp. 72-73] where it has been shown that

given any set of locally finite perimeter $\Omega \subseteq \mathbb{R}^n$, its approximate tangent plane exists at each point $x \in \partial^*\Omega$ and is equal to $\langle \nu(x)\rangle^{\perp}$ (where ν denotes the geometric measure theoretic outward unit normal vector to Ω). (2.27)

Indeed, (2.15) and (2.24) tell us that $\partial^* E, \partial^* F$ are locally $\mathcal{H}^{n-1}$-rectifiable sets (cf. [89, p. 96]), so (2.26) (used with $M_1 := \partial^* E$ and $M_2 := \partial^* F$) together with (2.27) imply that $\langle \nu_E(x)\rangle^{\perp} = \langle \nu_F(x)\rangle^{\perp}$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$, from which the desired conclusion follows. □

Given a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, another piece of notation commonly used (cf., e.g., [47, p. 169]) is

$$\|\partial\Omega\| := \mathcal{H}^{n-1} \lfloor \partial^*\Omega. \tag{2.28}$$

From (2.28), (2.24), and (2.18) (cf. also [89, (15.10), p. 170]) we then see that

$$\begin{gathered}\|\partial\Omega\| \text{ agrees with the total variation of } \mu_\Omega, \\ \text{the Gauss-Green measure of } \Omega, \end{gathered} \tag{2.29}$$

and we also claim that[1]

$$\operatorname{supp} \|\partial\Omega\| = \overline{\partial^*\Omega}. \tag{2.30}$$

Indeed, from (2.28), (2.24), (2.21) we see that $\partial^*\Omega \subseteq \operatorname{supp} \|\partial\Omega\|$ and, as a consequence, $\overline{\partial^*\Omega} \subseteq \operatorname{supp} \|\partial\Omega\|$ since the latter set is closed. This proves the right-

[1] Given a topological space X along with some (non-negative) Borel measure μ on X, the support of μ is denoted by $\operatorname{supp} \mu$ and is defined as the set of all points $x \in X$ with the property that $\mu(O) > 0$ for each open set $O \subseteq X$ containing x.

to-left inclusion in (2.30). As for the opposite inclusion, if $x \in \mathbb{R}^n \setminus \overline{\partial^*\Omega}$, then there exists $r > 0$ with the property that $B(x,r) \cap \partial^*\Omega = \varnothing$. In concert with (2.28), this implies $\|\partial\Omega\|\big(B(x,r)\big) = 0$, hence $x \notin \operatorname{supp}\|\partial\Omega\|$. The proof of (2.30) is therefore complete. As a consequence of this, (2.29), and definitions[2] we therefore have

$$\operatorname{supp}\mu_\Omega = \operatorname{supp}\|\partial\Omega\| = \overline{\partial^*\Omega}. \tag{2.31}$$

See also [89, p. 168] in this regard.

Definition 2.1 A closed set $\Sigma \subseteq \mathbb{R}^n$ is called an `Ahlfors regular set` (or an Ahlfors-David regular set) if there exists a constant $C \in [1,\infty)$ such that

$$r^{n-1}/C \leq \mathcal{H}^{n-1}\big(B(x,r)\cap\Sigma\big) \leq Cr^{n-1}, \quad \forall\, r \in \big(0, 2\operatorname{diam}(\Sigma)\big), \quad \forall\, x \in \Sigma. \tag{2.32}$$

Also, given a closed set $\Sigma \subseteq \mathbb{R}^n$ and some $R \in (0,\infty]$, say that Σ is `Ahlfors regular up to scale` R, `with constant` $C \in [1,\infty)$, provided the double inequality in (2.32) is valid for each $r \in (0,R)$.

Finally, the labels `lower Ahlfors regular` and `upper Ahlfors regular` are employed when only the lower, respectively, upper, inequality in (2.32) is required to hold.

For a given closed set $\Sigma \subseteq \mathbb{R}^n$, the quality of being Ahlfors regular is not a regularity condition in a traditional analytic sense, but rather a property guaranteeing that, at all locations, Σ behaves (in a quantitative, scale-invariant fashion) like an $(n-1)$-dimensional "surface," with respect to the Hausdorff measure $\mathcal{H}^{n-1}$. For example, the classical four-corner Cantor set in the plane is an Ahlfors regular set (cf., e.g., [108, Proposition 4.79, p. 238]). Let us also observe that

> if $\Omega \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set whose boundary is upper Ahlfors regular up to some scale $R \in (0,\infty]$ with some constant $C \in [1,\infty)$ then necessarily Ω is of locally finite perimeter. (2.33)

Indeed, this follows from (2.15) (bearing in mind that $\partial_*\Omega \subseteq \partial\Omega$; cf. (2.14)) and Definition 2.1.

Lemma 2.1 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed set which is lower Ahlfors regular with some constant $C \in [1,\infty)$ up to some scale $R \in (0,\infty]$. Then any set $E \subseteq \Sigma$ satisfying $\mathcal{H}^{n-1}(\Sigma \setminus E) = 0$ is necessarily dense in Σ, i.e., $\overline{E} = \Sigma$.*

Proof Seeking a contradiction, assume E is not dense in Σ. Then $\Sigma \setminus \overline{E} \neq \varnothing$. This means that there exist $x \in \Sigma$ and $r > 0$ such that $B(x,r) \cap E = \varnothing$. Without loss of generality we may assume that $r \in (0,R)$. We may then use the lower Ahlfors regularity property of Σ and the fact that $B(x,r) \cap \Sigma \subseteq \Sigma \setminus E$ to write

[2] Recall that the support of a vector measure μ is defined as the support of its total variation, i.e., $\operatorname{supp}\mu := \operatorname{supp}|\mu|$.

$$r^{n-1}/C \leq \mathcal{H}^{n-1}\big(B(x,r)\cap\Sigma\big) \leq \mathcal{H}^{n-1}(\Sigma\setminus E) = 0, \tag{2.34}$$

a contradiction. □

In analogy with Definition 2.1 we introduce the notion of Ahlfors regularity for measures:

Definition 2.2 A (non-negative) Borel measure μ in $\mathbb{R}^n$ is said to be `Ahlfors regular up to scale` $R \in (0,\infty]$, `with constant` $C \in [1,\infty)$, provided

$$r^{n-1}/C \leq \mu\big(B(x,r)\big) \leq Cr^{n-1}, \quad \forall\, r\in(0,R), \quad \forall\, x \in \operatorname{supp}\mu. \tag{2.35}$$

Also, say that μ is `lower Ahlfors regular`, or `upper Ahlfors regular`, when only the lower, respectively, upper, inequality in (2.35) is required to hold.

One may check straight from definitions that if $\Sigma \subseteq \mathbb{R}^n$ is a closed set, then Σ is an Ahlfors regular set up to scale $R \in (0,\infty]$ with constant $C \in [1,\infty)$ if and only if $\mu := \mathcal{H}^{n-1}\lfloor\Sigma$ is an Ahlfors regular measure up to scale $R \in (0,\infty]$ with constant $C \in [1,\infty)$. Moreover, similar considerations apply to lower/upper Ahlfors regularity.

Next, we recall the notion of Radon measure:

Definition 2.3 Let (X,τ) be a topological space, and let $\mathfrak{M}$ be a sigma-algebra of subsets of X containing all Borel sets in X. Call a measure $\mu : \mathfrak{M} \to [0,+\infty]$ `Radon` provided μ is locally finite (i.e., $\mu(K) < +\infty$ for every compact $K \subseteq X$), every open set is inner-regular, i.e.,

$$\mu(O) = \sup_{\substack{K \text{ compact}\\ K\subseteq O}} \mu(K) \quad \text{for each open set} \quad O \subseteq X, \tag{2.36}$$

and every Borel set is outer-regular, i.e.,

$$\mu(E) = \inf_{\substack{O \text{ open}\\ E\subseteq O}} \mu(O) \quad \text{for all Borel sets} \quad E \subseteq X. \tag{2.37}$$

We have the following well-known regularity result (cf., e.g., [51, Theorem 7.8, p. 217]).

Proposition 2.3 *Let (X,τ) be a locally compact Hausdorff topological space in which every open set is sigma-compact (recall that the latter condition automatically holds if (X,τ) is second countable hence, in particular, if (X,τ) is metrizable and separable). Then every locally finite Borel measure μ on X is a Radon measure.*

Let μ be a locally finite Borel measure in $\mathbb{R}^n$. In particular, Proposition 2.3 guarantees that μ is a Radon measure. If μ is also assumed to be lower Ahlfors regular up to scale $R \in (0,\infty]$ with constant $C \in [1,\infty)$, we may invoke [95, Theorem 6.9(2), p. 95] to conclude that

$$\mathcal{H}^{n-1}(A) \leq 2^{n-1}C\mu(A) \quad \text{for each } \mu\text{-measurable set } \; A \subseteq \operatorname{supp}\mu. \tag{2.38}$$

In particular,

$$\begin{aligned} &\mathcal{H}^{n-1}(A) = 0 \;\; \text{whenever} \;\; A \subseteq \operatorname{supp}\mu \\ &\text{is a } \mu\text{-measurable set with} \;\; \mu(A) = 0. \end{aligned} \tag{2.39}$$

Given a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, we are interested when the measure $\|\partial\Omega\|$ is Ahlfors regular.

Proposition 2.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a set of locally finite perimeter, and fix some scale $R \in (0, \infty]$ along with a constant $C \in [1, \infty)$. Then the measure $\|\partial\Omega\|$ is lower Ahlfors regular with constant C up to scale R if and only if*

$$\mathcal{H}^{n-1}\big(\overline{\partial^*\Omega} \setminus \partial^*\Omega\big) = 0 \tag{2.40}$$

and the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R.

Furthermore, the measure $\|\partial\Omega\|$ is actually Ahlfors regular with constant C up to scale R if and only if (2.40) holds and the set $\overline{\partial^*\Omega}$ is Ahlfors regular with constant C up to scale R.

Proof Since $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, it follows that $\mu := \|\partial\Omega\|$ is a locally finite Borel measure in $\mathbb{R}^n$ (cf. (2.15), (2.24), and (2.28)). In addition, $A := \overline{\partial^*\Omega} \setminus \partial^*\Omega$ is a μ-measurable set contained in $\operatorname{supp}\mu$ with $\mu(A) = 0$ (cf. (2.22), (2.23), (2.30), (2.28)). Let us also note that, as apparent from (2.28), we have

$$\begin{aligned} &\|\partial\Omega\|\big(B(x,r)\big) = \mathcal{H}^{n-1}\big(\partial^*\Omega \cap B(x,r)\big) \\ &\text{for each } \;\; x \in \mathbb{R}^n \;\; \text{and each } \;\; r \in (0,\infty). \end{aligned} \tag{2.41}$$

In one direction, assume the measure $\|\partial\Omega\|$ is lower Ahlfors regular with constant C up to scale R. Then (2.39) (used with μ and A as above) implies (2.40). Also, from Definition 2.2, (2.30), (2.41), and (2.40) we see that the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R. In the opposite direction, if (2.40) holds and the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R, we conclude from (2.41), Definition 2.2, and (2.30) that the measure $\|\partial\Omega\|$ is lower Ahlfors regular with constant C up to scale R. This finishes the proof of the first equivalence claimed in the statement of the proposition.

As regards the equivalence in the last part of the statement, assume the measure $\|\partial\Omega\|$ is in fact Ahlfors regular with constant C up to scale R. Then, from what we have proved already, the set $\overline{\partial^*\Omega}$ is lower Ahlfors regular with constant C up to scale R and (2.40) holds. Since now the measure $\|\partial\Omega\|$ is additionally assumed to be upper Ahlfors regular with constant C up to scale R, we deduce from Definition 2.2, (2.30), (2.40), and (2.41) that the set $\overline{\partial^*\Omega}$ is also upper Ahlfors regular with constant

C up to scale R. This establishes one implication. Finally, the opposite implication is seen from Definition 2.2, (2.30), (2.41), and (2.40). □

For future use, let us record here the following off-diagonal Carleson measure estimate of reverse Hölder type, proved in [111, §8.6].

Proposition 2.5 *Let Ω be an open subset of $\mathbb{R}^n$ with an unbounded Ahlfors regular boundary and define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix $\kappa \in (0,\infty)$, and pick $\theta \in (0,1)$ along with $p \in (0,\infty)$, all arbitrary. Then there exists $C \in (0,\infty)$ with the property that for every $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{R}$, every point $x \in \partial\Omega$, and every radius $r \in (0,\infty)$ one has*

$$\Big(\int_{\Omega\cap B(x,r)} |u|^{\frac{np}{n-1}}\, d\mathcal{L}^n\Big)^{\frac{n-1}{np}} \leq C\Big(\int_{\partial\Omega\cap B(x,Cr)} \big(\mathcal{N}_\kappa^{Cr} u\big)^p\, d\sigma\Big)^{\frac{1}{p}}, \tag{2.42}$$

where $\mathcal{N}_\kappa^{Cr}$ is the truncated nontangential maximal operator (defined as in (2.9) with $\delta := Cr$).

Following [61] we now introduce the class of Ahlfors regular domains.

Definition 2.4 An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is called an `Ahlfors regular domain` provided $\partial\Omega$ is an Ahlfors regular set and $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big) = 0$.

If $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, then the upper Ahlfors regularity condition satisfied by $\partial\Omega$ (i.e., the second inequality in (2.32) with $\Sigma := \partial\Omega$) guarantees that (2.15) holds, hence Ω is a set of locally finite perimeter. Also, the fact that the measure theoretic boundary $\partial_*\Omega$ is presently assumed to have full measure (with respect to $\mathcal{H}^{n-1}$) in the topological boundary $\partial\Omega$ ensures that the geometric measure theoretic outward unit normal ν to Ω (cf. (2.19)) is actually well defined at $\mathcal{H}^{n-1}$-a.e. point on $\partial\Omega$. Ultimately,

$$\begin{gathered} \text{if } \Omega \subseteq \mathbb{R}^n \text{ is an Ahlfors regular domain then} \\ \nu \in \big[L^\infty(\partial\Omega, \mathcal{H}^{n-1})\big]^n \text{ is an } \mathbb{R}^n\text{-valued function} \\ \text{satisfying } |\nu(x)| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial\Omega. \end{gathered} \tag{2.43}$$

From [61, Proposition 2.9, p. 2588] we also know that

> if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, and if $\kappa \in (0,\infty)$ is an arbitrary aperture parameter, then $x \in \overline{\Gamma_\kappa(x)}$ (that is, x is an accumulation point for the nontangential approach region $\Gamma_\kappa(x)$) for $\mathcal{H}^{n-1}$-a.e. point x in the topological boundary $\partial\Omega$. (2.44)

In particular, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and u is an $\mathcal{L}^n$-measurable function defined in Ω, then for any fixed aperture parameter $\kappa > 0$ it is meaningful to attempt to define the nontangential boundary trace $\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial\Omega$.

It turns out that the class of Ahlfors regular domains is bi-Lipschitz invariant.

Lemma 2.2 *Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, and $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$. Then for any given bi-Lipschitz mapping $F : \mathcal{O} \to \mathbb{R}^n$ the set $\widetilde{\Omega} := F(\Omega)$ is also an Ahlfors regular domain, with the Ahlfors regularity constant of $\partial\widetilde{\Omega}$ controlled in terms of the Ahlfors regularity constant of $\partial\Omega$ and the bi-Lipschitz constants of F.*

Proof This is a consequence of [59, Proposition 3.1, p. 610] and the proof of [59, Proposition 3.7, (3.88), p. 621]. □

We shall also need the following result, appearing in [111, §5.10].

Lemma 2.3 *If $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain (in the sense of Definition 2.4) then $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ is also an Ahlfors regular domain, whose topological boundary coincides with that of Ω, and whose geometric measure theoretic boundary agrees with that of Ω, i.e.,*

$$\partial(\Omega_-) = \partial\Omega \quad \textit{and} \quad \partial_*(\Omega_-) = \partial_*\Omega. \tag{2.45}$$

Moreover, the geometric measure theoretic outward unit normal to Ω_- is $-\nu$ at σ-a.e. point on $\partial\Omega$.

The following definition is due to G. David and S. Semmes (cf. [41]).

Definition 2.5 A closed set $\Sigma \subseteq \mathbb{R}^n$ is said to be a `uniformly rectifiable` set (or simply a UR set) if Σ is an Ahlfors regular set and there exist $\varepsilon, M \in (0, \infty)$ such that for each location $x \in \Sigma$ and each scale $R \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ it is possible to find a Lipschitz map $\varphi : B_R^{n-1} \to \mathbb{R}^n$ (where B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$ and such that

$$\mathcal{H}^{n-1}\big(\Sigma \cap B(x, R) \cap \varphi(B_R^{n-1})\big) \geq \varepsilon R^{n-1}. \tag{2.46}$$

Collectively, ε, M are referred to as the UR `constants of` Σ.

The following definition appears in [61].

Definition 2.6 An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is called a `UR domain` (short for uniformly rectifiable domain) provided $\partial\Omega$ is a UR set (in the sense of Definition 2.5) and $\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0$.

By design, any UR domain is an Ahlfors regular domain. A basic subclass of UR domains has been identified by G. David and D. Jerison in [39]. To state (a version of) their result, we first recall the following definition.

Definition 2.7 Fix $R \in (0, \infty]$ and $c \in (0, 1)$. A nonempty proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, c)-`corkscrew condition` (or, simply, a `corkscrew condition` if the particular values of R, c are not important) if for each location

$x \in \partial\Omega$ and each scale $r \in (0, R)$ there exists a point $z \in \Omega$ (called a corkscrew point relative to x and r) with the property that $B(z, c\,r) \subseteq B(x, r) \cap \Omega$.

Also, a nonempty proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, c)-two-sided corkscrew condition provided both Ω and $\mathbb{R}^n \setminus \Omega$ satisfy the (R, c)-corkscrew condition (with the same convention regarding the omission of R, c).

It is then clear from definitions that we have

$$\partial_*\Omega = \partial\Omega \text{ for any } \mathcal{L}^n\text{-measurable set } \Omega \subseteq \mathbb{R}^n \text{ satisfying a two-sided corkscrew condition.} \tag{2.47}$$

Also, [39, Theorem 1, p. 840] implies that, in a quantitative fashion,

> if Ω is a nonempty proper open subset of $\mathbb{R}^n$ satisfying a two-sided corkscrew condition and whose boundary is an Ahlfors regular set, then Ω is a UR domain. (2.48)

Following [66], we define the class of nontangentially accessible domains as those open sets satisfying a two-sided corkscrew condition and the following Harnack chain condition.

Definition 2.8 Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open set $\Omega \subseteq \mathbb{R}^n$ is said to satisfy the (R, N)-Harnack chain condition (or, simply, a Harnack chain condition if the particular values of R, N are irrelevant) provided whenever $\varepsilon > 0$, $k \in \mathbb{N}$, $z \in \partial\Omega$, and $x, y \in \Omega$ with $\max\big\{|x - z|, |y - z|\big\} < R/4$ as well as $|x - y| \leq 2^k\varepsilon$ and $\min\big\{\text{dist}(x, \partial\Omega)\,,\ \text{dist}(y, \partial\Omega)\big\} \geq \varepsilon$, one may find a chain of balls $B_1, B_2, \dots, B_K$ with $K \leq Nk$, such that $x \in B_1$, $y \in B_K$, $B_i \cap B_{i+1} \neq \varnothing$ for every $i \in \{1, \dots, K - 1\}$, and

$$N^{-1} \cdot \text{diam}(B_i) \leq \text{dist}(B_i, \partial\Omega) \leq N \cdot \text{diam}(B_i), \tag{2.49}$$

$$\text{diam}(B_i) \geq N^{-1} \cdot \min\big\{\text{dist}(x, B_i)\,,\ \text{dist}(y, B_i)\big\}, \tag{2.50}$$

for every $i \in \{1, \dots, K\}$.

Note that, in the context of Definition 2.8, consecutive balls must have comparable radii. The "nontangentiality" condition (2.49) further implies that

$$\lambda B_i \subseteq \Omega \quad \text{for each} \quad \lambda \in (0, 2N^{-1} + 1] \quad \text{and} \quad i \in \{1, \dots, K\}. \tag{2.51}$$

The Harnack chain condition described in Definition 2.8 should be thought of as a quantitative local connectivity condition. In particular,

> any open set $\Omega \subseteq \mathbb{R}^n$ satisfying an (∞, N)-Harnack chain condition (for some $N \in \mathbb{N}$) is pathwise connected (hence also connected) in a quantitative fashion. (2.52)

To elaborate on the latter aspect, we find it convenient to eliminate the parameter $\varepsilon > 0$ and also relabel 2^k simply as k in Definition 2.8. Assuming $R = \infty$, this implies that for each $k \geq 2$ there exists $L_k \in \mathbb{N}$ (which is bounded by $N \cdot \log_2 k$) with the property that for each

$$x_1, x_2 \in \Omega \;\text{ with }\; |x_1 - x_2| \leq k \cdot \min\big\{\mathrm{dist}(x_1, \partial\Omega)\,,\; \mathrm{dist}(x_2, \partial\Omega)\big\} \tag{2.53}$$

one can find a sequence of balls

$$\begin{aligned}&B_j := B(y_j, r_j) \text{ with } 1 \leq j \leq \ell, \text{ where } \ell \in \mathbb{N} \text{ satisfies } \ell \leq L_k,\\ &\text{such that } B\big(y_j, (2N^{-1} + 1)r_j\big) \subseteq \Omega \text{ for every } j \in \{1, \dots, \ell\},\\ &x_1 \in B(y_1, r_1),\, x_2 \in B(y_\ell, r_\ell), \text{ and such that there exists a point}\\ &z_j \in B(y_j, r_j) \cap B(y_{j+1}, r_{j+1}) \text{ for each } j \in \{1, \dots, \ell - 1\}.\end{aligned} \tag{2.54}$$

The fact that $L_k = O(\log_2 k)$ as $k \to \infty$ quantifies the intuitive idea that the closer to the boundary the points x_1, x_2 are, and the further apart from each other they happen to be, the larger the numbers of balls in the Harnack chain joining them. To proceed, we agree to abbreviate

$$\delta_{\partial\Omega}(x) := \mathrm{dist}(x, \partial\Omega) \;\text{ for each }\; x \in \Omega. \tag{2.55}$$

Then the first property in (2.54) implies that we have

$$\delta_{\partial\Omega}(x) \geq 2N^{-1} r_j \;\text{ for all }\; j \in \{1, \dots, \ell\} \;\text{ and all }\; x \in B(y_j, r_j). \tag{2.56}$$

In concert with the second inequality in (2.49) this further permits us to estimate

$$\delta_{\partial\Omega}(a) \leq (N+1) \cdot \delta_{\partial\Omega}(b) \;\text{ for all }\; j \in \{1, \dots, \ell\} \;\text{ and }\; a, b \in B(y_j, r_j). \tag{2.57}$$

Indeed, whenever $a, b \in B(y_j, r_j)$ with $j \in \{1, \dots, \ell\}$ we may use (2.56) to write

$$\begin{aligned}\delta_{\partial\Omega}(a) &\leq |a - b| + \delta_{\partial\Omega}(b) \leq 2r_j + \delta_{\partial\Omega}(b)\\ &\leq N \cdot \delta_{\partial\Omega}(b) + \delta_{\partial\Omega}(b) = (N+1) \cdot \delta_{\partial\Omega}(b),\end{aligned} \tag{2.58}$$

proving (2.57). In particular, for each index $j \in \{1, \dots, \ell - 1\}$ we have

$$(N+1)^{-1} \cdot \delta_{\partial\Omega}(z_j) \leq \delta_{\partial\Omega}(z_{j+1}) \leq (N+1) \cdot \delta_{\partial\Omega}(z_j). \tag{2.59}$$

Joining $x_1, y_1, z_1, y_2, z_2, y_3, \dots, y_{\ell-1}, z_{\ell-1}, y_\ell, x_2$ with line segments yields a polygonal arc γ joining x_1 with x_2 in Ω, whose length may be estimated as follows:

$$\mathrm{length}(\gamma) \leq \sum_{j=1}^{\ell} 2r_j \leq N \cdot \delta_{\partial\Omega}(x_1) + N \sum_{j=1}^{\ell-1} \delta_{\partial\Omega}(z_j)$$

$$\le N \cdot \delta_{\partial\Omega}(x_1) + N \sum_{j=1}^{\ell-1} (N+1)^{j-1} \cdot \delta_{\partial\Omega}(z_1)$$

$$\le N \cdot \delta_{\partial\Omega}(x_1) + N \sum_{j=1}^{\ell-1} (N+1)^{j} \cdot \delta_{\partial\Omega}(x_1)$$

$$= N \sum_{j=0}^{\ell-1} (N+1)^{j} \cdot \delta_{\partial\Omega}(x_1) = N \frac{(N+1)^\ell - 1}{N} \delta_{\partial\Omega}(x_1)$$

$$\le (N+1)^{L_k} \cdot \delta_{\partial\Omega}(x_1), \tag{2.60}$$

thanks to (2.56) (used with x replaced by $x_1, z_1, \dots, z_{\ell-1}$), iterations of (2.59), and (2.57) (with $j := 1$, $a := z_1$, $b := x_1$), while also keeping in mind that $\ell \le L_k$. In a similar fashion, emphasizing x_2 in place of x_1 yields length$(\gamma) \le (N+1)^{L_k} \cdot \delta_{\partial\Omega}(x_2)$ hence, ultimately,

$$\text{length}(\gamma) \le (N+1)^{L_k} \cdot \min\big\{\delta_{\partial\Omega}(x_1)\,,\ \delta_{\partial\Omega}(x_2)\big\}. \tag{2.61}$$

In addition, for each $x \in \gamma$ there exists $j_x \in \{1, \dots, \ell\}$ such that $x \in B(y_{j_x}, r_{j_x})$. If $j_x \ge 2$ we write

$$\delta_{\partial\Omega}(x) \ge (N+1)^{-1} \cdot \delta_{\partial\Omega}(z_{j_x-1}) \ge (N+1)^{1-j_x} \cdot \delta_{\partial\Omega}(z_1)$$

$$\ge (N+1)^{-j_x} \cdot \delta_{\partial\Omega}(x_1) \ge (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1), \tag{2.62}$$

by (2.57) with $b := x$ and $a := z_{j_x-1}$, iterations of (2.59), and (2.57) applied with $b := z_1$ and $a := x_1$. If $j_x = 1$ we simply have

$$\delta_{\partial\Omega}(x) \ge (N+1)^{-1} \cdot \delta_{\partial\Omega}(x_1) \ge (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1). \tag{2.63}$$

Thus, in all cases we reach the conclusion that $\delta_{\partial\Omega}(x) \ge (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1)$. Analogously, $\delta_{\partial\Omega}(x) \ge (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_2)$ which goes to show that

$$\delta_{\partial\Omega}(x) \ge (N+1)^{-L_k} \cdot \max\big\{\delta_{\partial\Omega}(x_1)\,,\ \delta_{\partial\Omega}(x_2)\big\} \ \text{ for each } \ x \in \gamma. \tag{2.64}$$

The existence of such a path γ is going to be used in Lemma 2.4 and Lemma 2.5 which, in turn, play a significant role in the proof of Theorem 2.7. For now, following [66, pp. 93-94] (cf. also [75, Definition 2.1, p. 3]), we introduce the class of NTA domains.

Definition 2.9 Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is said to be an (R, N)-`nontangentially accessible domain` (or simply an NTA domain if the particular values of R, N are not important) if Ω satisfies both the (R, N^{-1})-two-sided corkscrew condition and the (R, N)-Harnack chain condition.

Call Ω a (R, N)-`two-sided nontangentially accessible domain` (or, simply, a `two-sided NTA domain` if the particular values of R, N are not relevant) provided Ω is an open, nonempty, proper subset of $\mathbb{R}^n$ with the property that both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are (R, N)-nontangentially accessible domains.

A set $\Omega \subseteq \mathbb{R}^n$ is said to be an (R, N)-`one-sided` NTA `domain` provided Ω satisfies the (R, N)-Harnack chain condition and the (R, N^{-1})-corkscrew condition (once again, with the convention that the parameters R, N are dropped if theirs values are not relevant).

Finally, it is agreed that, in all cases, one takes $R = \infty$ if and only if $\partial\Omega$ is unbounded.

For example, the complement of the classical four-corner Cantor set in the plane is a one-sided NTA domain with an Ahlfors regular boundary. Also, from the last convention in Definition 2.9 and (2.52) we see that

any NTA domain with an unbounded boundary (or, equivalently, any (∞, N)-nontangentially accessible domain for some number $N \in \mathbb{N}$) is pathwise connected, hence also connected. (2.65)

It turns out that from any point in a given one-sided NTA domain one may proceed along a path toward to the interior of said domain, which progressively distances itself from the boundary. This is made precise in the lemma below.

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^n$ be an (∞, N)-one-sided* NTA *domain for some $N \in \mathbb{N}$. Then there exists a constant $C_N \in (1, \infty)$ with the following significance. For each location $x \in \Omega$ and each scale $r \in (0, \infty)$ there exists a point $x_* \in \Omega$ and a polygonal arc γ joining x with x_* in Ω such that*

$$
\begin{aligned}
&|x - x_*| < 2r, \quad \delta_{\partial\Omega}(x_*) \geq r/N^2, \quad \textit{length}(\gamma) \leq C_N \cdot r, \\
&\textit{and} \;\; \textit{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \;\textit{for each point}\; y \in \gamma,
\end{aligned}
\tag{2.66}
$$

where $\gamma_{x,y}$ is the sub-arc of γ joining x with y.

Proof Without loss of generality assume $N \geq 2$. In the case when $\delta_{\partial\Omega}(x) \geq r/N$, we shall simply take $x_* := x$ and $\gamma := \{x\}$. If $\delta_{\partial\Omega}(x) < r/N$, there exists $m \in \mathbb{N}$ such that $r/N^{m+1} \leq \delta_{\partial\Omega}(x) < r/N^m$. Pick some $z \in \partial\Omega$ so that $\delta_{\partial\Omega}(x) = |x - z|$ and define $r_j := N^j \cdot \delta_{\partial\Omega}(x) \in (0, \infty)$ for each $j \in \{1, \dots, m\}$. The fact that Ω satisfies (∞, N^{-1})-corkscrew condition guarantees that for each $j \in \{1, \dots, m\}$ there exists a corkscrew point $x_j \in \Omega$ relative to the location z and scale r_j. Hence, for each $j \in \{1, \dots, m\}$ we have $B(x_j, r_j/N) \subseteq B(z, r_j) \cap \Omega$ which entails

$$
\begin{aligned}
&N^j \cdot \delta_{\partial\Omega}(x) = r_j > \delta_{\partial\Omega}(x_j) > r_j/N = N^{j-1} \cdot \delta_{\partial\Omega}(x) \\
&\text{and } \ |x_j - z| < r_j = N^j \cdot \delta_{\partial\Omega}(x) \ \text{ for each } \ j \in \{1, \dots, m\}.
\end{aligned}
\tag{2.67}
$$

Denote $x_0 := x$ and observe that for each $j \in \{1, \dots, m\}$ we have that the points $x_{j-1}, x_j \in B(z, r_j)$. Together with (2.67), for each $j \in \{1, \dots, m\}$ this permits us to estimate

$$
|x_{j-1} - x_j| < 2r_j = 2N^j \cdot \delta_{\partial\Omega}(x) \leq 2N^2 \cdot \min\big\{\delta_{\partial\Omega}(x_{j-1}),\ \delta_{\partial\Omega}(x_j)\big\}. \tag{2.68}
$$

Hence, we are in the scenario described in (2.53) with x_{j-1}, x_j playing the roles of x_1, x_2, and with $k := 2N^2$. From (2.61), (2.64) we then conclude that there exists $\widetilde{C}_N \in (1, \infty)$ with the property that for each $j \in \{1, \dots, m\}$ we may find a polygonal arc γ_j joining x_{j-1} with x_j in Ω such that

$$
\text{length}(\gamma_j) \leq \widetilde{C}_N \cdot \min\big\{\delta_{\partial\Omega}(x_{j-1}),\ \delta_{\partial\Omega}(x_j)\big\} \leq \widetilde{C}_N \cdot N^j \cdot \delta_{\partial\Omega}(x), \tag{2.69}
$$

and

$$
\widetilde{C}_N \cdot \delta_{\partial\Omega}(y) \geq \max\big\{\delta_{\partial\Omega}(x_{j-1}),\ \delta_{\partial\Omega}(x_j)\big\} \geq N^{j-1} \cdot \delta_{\partial\Omega}(x) \ \text{ for each } \ y \in \gamma_j. \tag{2.70}
$$

If we now define $x_* := x_m$ and take $\gamma := \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_m$ then γ is a polygonal arc joining $x = x_0$ with $x_* = x_m$ in Ω whose length satisfies

$$
\begin{aligned}
\text{length}(\gamma) &= \sum_{j=1}^{m} \text{length}(\gamma_j) \leq \sum_{j=1}^{m} \widetilde{C}_N \cdot N^j \cdot \delta_{\partial\Omega}(x) \\
&\leq \frac{N \cdot \widetilde{C}_N}{N-1} N^m \cdot \delta_{\partial\Omega}(x) \leq \Big(\frac{N \cdot \widetilde{C}_N}{N-1}\Big) r,
\end{aligned}
\tag{2.71}
$$

thanks to (2.69) and our choice of m. Also, for each point $y \in \gamma$ there exists some $j_y \in \{1, \dots, m\}$ such that $y \in \gamma_{j_y}$, hence we may use (2.70) to bound the length of the sub-arc $\gamma_{x,y}$ of γ joining x with y by

$$
\begin{aligned}
\text{length}(\gamma_{x,y}) &\leq \sum_{j=1}^{j_y} \text{length}(\gamma_j) \leq \sum_{j=1}^{j_y} \widetilde{C}_N \cdot N^j \cdot \delta_{\partial\Omega}(x) \\
&\leq \frac{N^2 \cdot \widetilde{C}_N}{N-1} N^{j_y - 1} \cdot \delta_{\partial\Omega}(x) \leq \Big(\frac{N^2 \cdot \widetilde{C}_N^2}{N-1}\Big) \delta_{\partial\Omega}(y).
\end{aligned}
\tag{2.72}
$$

Our choice of x_*, the first line in (2.67), and our choice of m also permit us to conclude that

$$\delta_{\partial\Omega}(x_*) = \delta_{\partial\Omega}(x_m) > N^{m-1} \cdot \delta_{\partial\Omega}(x) \geq r/N^2. \tag{2.73}$$

Finally, since $x, x_* \in B(z, r_m)$ it follows that $|x - x_*| < 2r_m = 2N^m \cdot \delta_{\partial\Omega}(x) < 2r$, so all properties claimed in (2.66) are verified. □

Our next lemma shows that one-sided NTA domains satisfy a quantitative connectivity property of the sort considered by O. Martio and J. Sarvas in [93], where the class of uniform domains has been introduced. See also [10, Theorem 2.15] in this regard.

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^n$ be an (∞, N)-one-sided* NTA *domain for some $N \in \mathbb{N}$. Then there exists a constant $C_N \in (1, \infty)$ with the following significance. For any two points $x, \widetilde{x} \in \Omega$ and any scale $r \in (0, \infty)$ with $r \geq |x - \widetilde{x}|$ there exists a polygonal arc Γ joining x with $\widetilde{x}$ in Ω such that*

$$\begin{aligned} &\mathit{length}(\Gamma) \leq C_N \cdot r, \quad \textit{and for each point} \quad y \in \Gamma \\ &\min\left\{\mathit{length}(\Gamma_{x,y}),\ \mathit{length}(\Gamma_{y,\widetilde{x}})\right\} \leq C_N \cdot \delta_{\partial\Omega}(y), \end{aligned} \tag{2.74}$$

where $\Gamma_{x,y}$ and $\Gamma_{y,\widetilde{x}}$ are the sub-arcs of Γ joining x with y and, respectively, y with $\widetilde{x}$.

Proof Fix two points $x, \widetilde{x} \in \Omega$ and pick a scale $r \in (0, \infty)$ with $r \geq |x - \widetilde{x}|$. If $\delta_{\partial\Omega}(x) > 2r$ then $\widetilde{x} \in \overline{B(x, r)} \subseteq B(x, 2r) \subseteq \Omega$. In such a scenario, take Γ to be the line segment with endpoints $x, \widetilde{x}$ and all desired properties follow. There remains to treat the case when

$$\delta_{\partial\Omega}(x) \leq 2r. \tag{2.75}$$

To proceed, let $x_*, \widetilde{x}_*$ be associated with the given points $x, \widetilde{x}$ as in Lemma 2.4, and denote by $\gamma, \widetilde{\gamma}$ the polygonal arcs joining x with x_* and $\widetilde{x}$ with $\widetilde{x}_*$ in Ω, having the properties described in (2.66), for the current scale r. Specifically, for this choice of the scale, (2.66) gives

$$\begin{aligned} &|x - x_*| < 2r, \quad |\widetilde{x} - \widetilde{x}_*| < 2r, \\ &\delta_{\partial\Omega}(x_*) \geq r/N^2, \quad \delta_{\partial\Omega}(\widetilde{x}_*) \geq r/N^2, \\ &\text{length}(\gamma) \leq C_N \cdot r, \quad \text{length}(\widetilde{\gamma}) \leq C_N \cdot r, \\ &\text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \ \text{ for each } \ y \in \gamma, \\ &\text{length}(\widetilde{\gamma}_{\widetilde{x},y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \ \text{ for each } \ y \in \widetilde{\gamma}. \end{aligned} \tag{2.76}$$

Note that

$$|x_* - \widetilde{x}_*| \leq |x_* - x| + |x - \widetilde{x}| + |\widetilde{x} - \widetilde{x}_*| < 2r + r + 2r = 5r. \tag{2.77}$$

From (2.77) and the second line in (2.76) we then see that

$$|x_* - \widetilde{x}_*| < 5r \leq 5N^2 \cdot \min\big\{\delta_{\partial\Omega}(x_*)\,,\ \delta_{\partial\Omega}(\widetilde{x}_*)\big\}. \tag{2.78}$$

Thus, we are in the scenario described in (2.53) with $x_1 := x_*$, $x_2 := \widetilde{x}_*$, and with $k := 5N^2$. From (2.61), (2.64) we then conclude that there exist a constant $C_N \in (1, \infty)$ along with a polygonal arc $\widehat{\gamma}$ joining x_* with $\widetilde{x}_*$ in Ω such that

$$\text{length}(\widehat{\gamma}) \leq C_N \cdot \min\big\{\delta_{\partial\Omega}(x_*)\,,\ \delta_{\partial\Omega}(\widetilde{x}_*)\big\} \leq 2C_N \cdot r, \tag{2.79}$$

where the last inequality comes from (2.75), and

$$C_N \cdot \delta_{\partial\Omega}(y) \geq \max\big\{\delta_{\partial\Omega}(x_*)\,,\ \delta_{\partial\Omega}(\widetilde{x}_*)\big\} \geq r/N^2 \ \text{ for each } \ y \in \widehat{\gamma}, \tag{2.80}$$

with the last inequality provided by the second line in (2.76).

If we now define

$$\Gamma := \gamma \cup \widehat{\gamma} \cup \widetilde{\gamma}, \tag{2.81}$$

then Γ is a polygonal arc joining x with $\widetilde{x}$ in Ω. Also, (2.76) and (2.79) allow us to estimate

$$\text{length}(\Gamma) = \text{length}(\gamma) + \text{length}(\widehat{\gamma}) + \text{length}(\widetilde{\gamma}) \leq C_N \cdot r, \tag{2.82}$$

proving the first estimate in (2.74). Fix now a point $y \in \Gamma$. If y belongs to γ, then $\Gamma_{x,y} = \gamma_{x,y}$ which further entails $\text{length}(\Gamma_{x,y}) = \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y)$ by (2.76). Thus, the last estimate in (2.74) holds in this case. Similarly, if $y \in \widetilde{\gamma}$, then $\text{length}(\Gamma_{y,\widetilde{x}}) = \text{length}(\widetilde{\gamma}_{\widetilde{x},y}) \leq C_N \cdot \delta_{\partial\Omega}(y)$ again by (2.76), so the last estimate in (2.74) holds in this case as well. Finally, in the case when $y \in \widehat{\gamma}$ we may write

$$\min\big\{\text{length}(\Gamma_{x,y})\,,\ \text{length}(\Gamma_{y,\widetilde{x}})\big\} \leq \text{length}(\Gamma) \leq C_N \cdot r \leq C_N \cdot \delta_{\partial\Omega}(y), \tag{2.83}$$

by (2.82) and (2.80). □

When its endpoints belong to a suitable neighborhood of infinity, the polygonal arc constructed in Lemma 2.5 may be chosen as to avoid any given bounded set. This property, established in the next lemma, is going to be relevant later on, in the course of the proof of Theorem 2.7.

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^n$ be an (∞, N)-one-sided* NTA *domain for some $N \in \mathbb{N}$ such that $\mathbb{R}^n \setminus \overline{\Omega} \neq \varnothing$. Fix some point $z_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and some radius $R \in (0, \infty)$. Then there exist a large constant $C = C(N) \in (0, \infty)$ together with a small number $\varepsilon = \varepsilon(N) \in (0, 1)$ with the property that for any two points $x, \widetilde{x} \in \Omega \setminus B(z_0, R)$ and any scale $r \in (0, \infty)$ with $r \geq \max\big\{|x - \widetilde{x}|, C \cdot R\big\}$ the polygonal arc Γ joining x with $\widetilde{x}$ in Ω as in Lemma 2.5 is disjoint from $B(z_0, \varepsilon R)$.*

Proof Consider $\varepsilon \in (0,1)$ and $C \in (0,\infty)$ to be specified momentarily. Recall formula (2.81). Assume there exists a point $y \in \gamma \cap B(z_0, \varepsilon R)$. Then $y \in \gamma \subseteq \Omega$ so the line segment with endpoints y and z_0 intersects $\partial\Omega$. As such, $\delta_{\partial\Omega}(y) \leq \varepsilon R$. Also, $\gamma_{x,y}$ joins the point $x \in \mathbb{R}^n \setminus B(z_0, R)$ with the point $y \in B(z_0, \varepsilon R)$, which forces $\text{length}(\gamma_{x,y}) \geq (1-\varepsilon)R$. In concert with the last line in (2.66) this permits us to write

$$(1-\varepsilon)R \leq \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \leq C_N \cdot \varepsilon R, \tag{2.84}$$

which leads to a contradiction if we choose $\varepsilon := 1/[2(C_N + 1)]$. Thus, for this choice of ε we have $\gamma \cap B(z_0, \varepsilon R) = \varnothing$. In a similar fashion, $\widetilde{\gamma} \cap B(z_0, \varepsilon R) = \varnothing$. Finally, if there exists a point $y \in \widehat{\gamma} \cap B(z_0, \varepsilon R)$ then based on (2.80) and the nature of the scale r we may estimate

$$\varepsilon R \geq \delta_{\partial\Omega}(y) \geq r/(N^2 \cdot C_N) \geq (C \cdot R)/(N^2 \cdot C_N) \tag{2.85}$$

which leads to a contradiction if $C = C(N) \in (0,\infty)$ is sufficiently large. □

The following definition of yet another brand of local path connectivity condition first appeared in [61].

Definition 2.10 An open, nonempty, proper subset Ω of $\mathbb{R}^n$ is said to satisfy a `local John condition` if there exist $\theta \in (0,1)$ and $R \in (0,\infty]$ (with the requirement that $R = \infty$ if $\partial\Omega$ is unbounded) such that for every point $x \in \partial\Omega$ and every scale $r \in (0,R)$ one may find $x_r \in B(x,r) \cap \Omega$ such that $B(x_r, \theta r) \subseteq \Omega$ and with the property that for each $y \in B(x,r) \cap \partial\Omega$ there exists a rectifiable path $\gamma_y : [0,1] \to \overline{\Omega}$ whose length is $\leq \theta^{-1} r$ and such that

$$\gamma_y(0) = y, \quad \gamma_y(1) = x_r, \quad \text{dist}\big(\gamma_y(t), \partial\Omega\big) > \theta|\gamma_y(t) - y| \text{ for all } t \in (0,1]. \tag{2.86}$$

Finally, a nonempty open set $\Omega \subseteq \mathbb{R}^n$ which is not dense in $\mathbb{R}^n$ is said to satisfy a `two-sided local John condition` if both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a local John condition.

It is clear from the definitions that, in a quantitative sense,

> any set satisfying a local John condition (respectively, a two-sided local John condition) also satisfies a corkscrew condition (respectively, a two-sided corkscrew condition). (2.87)

Moreover, given any $R \in (0,\infty]$ and $N \in \mathbb{N}$, from [61, Lemma 3.13, p. 2634] we know that

> any (R,N)-nontangentially accessible domain satisfies a local John condition, and any (R,N)-two-sided nontangentially accessible domain satisfies a two-sided local John condition. (2.88)

To be able to define the class of δ-flat Ahlfors regular domains we first need to formally introduce the John-Nirenberg space of functions of bounded mean oscillations on Ahlfors regular sets. Specifically, given a closed set $\Sigma \subseteq \mathbb{R}^n$, for each $x \in \Sigma$ and $r > 0$define the surface ball $\Delta := \Delta(x,r) := B(x,r) \cap \Sigma$. For any constant $\lambda > 0$ we also agree to define $\lambda\Delta := \Delta(x,\lambda r) := B(x,\lambda r) \cap \Sigma$. Make the assumption that Σ is Ahlfors regular and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. For each $f \in L^1_{\rm loc}(\Sigma,\sigma)$ introduce

$$f_\Delta := \fint_\Delta f \, d\sigma \quad \text{for each surface ball } \ \Delta \subseteq \Sigma, \tag{2.89}$$

then consider the semi-norm

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} := \sup_{\Delta \subseteq \Sigma} \fint_\Delta \big|f - f_\Delta\big| \, d\sigma, \tag{2.90}$$

where the supremum in the right side of (2.90) is taken over all surface balls $\Delta \subseteq \Sigma$. We shall then denote by $\mathrm{BMO}(\Sigma,\sigma)$ the space of all functions $f \in L^1_{\rm loc}(\Sigma,\sigma)$ with the property that $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} < \infty$.

The above considerations may be naturally adapted to the case of vector-valued functions. Specifically, given $N \in \mathbb{N}$, for each $f : \Sigma \to \mathbb{C}^N$ with locally integrable scalar components, we define

$$\|f\|_{[\mathrm{BMO}(\Sigma,\sigma)]^N} := \sup_{\Delta \subseteq \Sigma} \fint_\Delta \big|f - f_\Delta\big| \, d\sigma, \tag{2.91}$$

where the supremum in the right side of (2.91) is taken over all surface balls $\Delta \subseteq \Sigma$, the integral average $f_\Delta \in \mathbb{C}^N$ is taken componentwise, and $|\cdot|$ is the standard Euclidean norm in $\mathbb{C}^N$. In an analogous fashion, we then define $\big[\mathrm{BMO}(\Sigma,\sigma)\big]^N$ as the space of all $\mathbb{C}^N$-valued functions $f \in \big[L^1_{\rm loc}(\Sigma,\sigma)\big]^N$ with the property that $\|f\|_{[\mathrm{BMO}(\Sigma,\sigma)]^N} < \infty$.

A natural version of the classical John-Nirenberg inequality concerning exponential integrability of functions of bounded mean oscillations remains valid in this setting. Specifically, [88, Theorem 1.4, p. 2000] (see also [5], [30], [135, Theorem 2, p. 33]) implies that there exists a small constant $c \in (0,\infty)$ and a large constant $C \in (0,\infty)$, both of which depend only on the doubling character of σ, with the property that

$$\fint_\Delta \exp\left\{\frac{c\,|f - f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\right\} d\sigma \leq C \tag{2.92}$$

for each non-constant function $f \in \mathrm{BMO}(\Sigma,\sigma)$ and each surface ball $\Delta \subseteq \Sigma$. Note that, trivially, for each surface ball $\Delta \subseteq \Sigma$ and each $\lambda \in (0,\infty)$ we have

$$1 \le \exp\Big\{ -\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\} \cdot \exp\Big\{\tfrac{c\,|f(x)-f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\} \quad \text{for every} \quad x \in \Delta \quad \text{with} \quad |f(x) - f_\Delta| > \lambda. \tag{2.93}$$

This shows that (2.92) implies the following level set estimate with exponential decay:

$$\begin{aligned}\sigma\Big(\big\{x \in \Delta : |f(x) - f_\Delta| > \lambda\big\}\Big) &\le \exp\Big\{ -\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\} \int_\Delta \exp\Big\{\frac{c\,|f - f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\, d\sigma \\ &\le C \cdot \exp\Big\{ -\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\sigma(\Delta)\end{aligned} \tag{2.94}$$

for each non-constant function $f \in \mathrm{BMO}(\Sigma, \sigma)$, each surface ball $\Delta \subseteq \Sigma$, and each $\lambda \in (0, \infty)$. Conversely, (2.94) implies an estimate like (2.92), namely

$$⨍_\Delta \exp\Big\{\frac{c_o\,|f - f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\, d\sigma \le 1 + \frac{C}{c/c_o - 1}, \tag{2.95}$$

for each non-constant function $f \in \mathrm{BMO}(\Sigma, \sigma)$ and each surface ball $\Delta \subseteq \Sigma$, as long as $c_o \in (0, c)$. See also [18, Theorem 3.15], [44, Theorem 3.1, p. 1397], [77, Lemma 2.4, p. 409], [94], and [135, Theorem 2, p. 33] in this regard. Here we wish to emphasize that only the doubling property of the underlying measure plays a role. In turn, the John-Nirenberg level set estimate (2.94) has many notable consequences. For one thing, (2.92) implies that $e^f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$ if f is a σ-measurable function on Σ with $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}$ small enough (with $\ln|\cdot|$ a representative example of this local exponential integrability phenomenon). Second, (2.94) guarantees that

$$\mathrm{BMO}(\Sigma, \sigma) \subseteq L^p_{\mathrm{loc}}(\Sigma, \sigma) \quad \text{for each} \quad p \in (0, \infty). \tag{2.96}$$

Third, (2.94) allows for more flexibility in describing the size of the BMO seminorm. Specifically, for each $p \in [1, \infty)$ and $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$ define

$$\|f\|_{\mathrm{BMO}_p(\Sigma,\sigma)} := \sup_{\Delta \subseteq \Sigma} \Big(⨍_\Delta |f - f_\Delta|^p\, d\sigma\Big)^{1/p}, \tag{2.97}$$

where the supremum in (2.97) is taken over all surface balls $\Delta \subseteq \Sigma$. Then for each integrability exponent $p \in [1, \infty)$ there exists some constant $C_{\Sigma,p} \in (0, \infty)$ with the property that for each function $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$ we have

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \le \|f\|_{\mathrm{BMO}_p(\Sigma,\sigma)} \le C_{\Sigma,p}\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}. \tag{2.98}$$

Indeed, the first estimate in (2.98) is a direct consequence of definitions and Hölder's inequality, while the second estimate in (2.98) relies on the John-Nirenberg inequality (2.94). Parenthetically, we wish to note that when $\Sigma := \mathbb{R}$ (hence $\sigma = \mathcal{L}^1$) and $p := 2$ the value of the optimal constant in (2.98) is known. Concretely, for each $f \in L^1_{\rm loc}(\mathbb{R}, \mathcal{L}^1)$ we have

$$\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \leq \|f\|_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \leq \tfrac{1}{2}\mathrm{e}^{1+(2/\mathrm{e})}\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}. \tag{2.99}$$

The justification of the second estimate in (2.99) uses a sharp version of the one-dimensional version of the John-Nirenberg inequality (cf. [86]) according to which for each function $f \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$, each nonempty finite sub-interval $I \subset \mathbb{R}$, and each $\lambda \in (0, \infty)$ we have (with $f_I := \fint_I f \, \mathrm{d}\mathcal{L}^1$)

$$\mathcal{L}^1\Big(\big\{t \in I : |f(t) - f_I| > \lambda\big\}\Big) \leq \frac{1}{2}\mathrm{e}^{4/\mathrm{e}}\mathcal{L}^1(I) \cdot \exp\Big\{-\frac{2\lambda/\mathrm{e}}{\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}\Big\}. \tag{2.100}$$

Specifically, for each nonempty finite sub-interval $I \subset \mathbb{R}$ we may write

$$\begin{aligned}
\fint_I |f(t) - f_I|^2 \, \mathrm{d}t &= \frac{1}{\mathcal{L}^1(I)} \int_0^\infty 2\lambda \cdot \mathcal{L}^1\Big(\big\{t \in I : |f(t) - f_I| > \lambda\big\}\Big) \, \mathrm{d}\lambda \\
&\leq \mathrm{e}^{4/\mathrm{e}} \int_0^\infty \lambda \cdot \exp\Big\{-\tfrac{2\lambda/\mathrm{e}}{\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}\Big\} \, \mathrm{d}\lambda \\
&= \mathrm{e}^{4/\mathrm{e}}(\mathrm{e}/2)^2 \, \|f\|^2_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \int_0^\infty \lambda \cdot \mathrm{e}^{-\lambda} \, \mathrm{d}\lambda \\
&= \mathrm{e}^{4/\mathrm{e}}(\mathrm{e}/2)^2 \, \|f\|^2_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}, \qquad\qquad (2.101)
\end{aligned}$$

thanks to (2.100) and some natural changes of variables, so the second estimate in (2.99) readily follows from (2.101) and (2.97).

Returning to the mainstream discussion, observe that (2.98) implies that for each integrability exponent $p \in [1, \infty)$ we have

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \approx \sup_{\Delta \subseteq \Sigma} \Big(\fint_\Delta |f - f_\Delta|^p \, \mathrm{d}\sigma\Big)^{\frac{1}{p}} \approx \sup_{\Delta \subseteq \Sigma} \inf_{c \in \mathbb{R}} \Big(\fint_\Delta |f - c|^p \, \mathrm{d}\sigma\Big)^{\frac{1}{p}}, \tag{2.102}$$

uniformly for $f \in L^1_{\rm loc}(\Sigma, \sigma)$. For further use, let us also note here that if Δ and Δ' are two concentric surface balls in Σ then for any $f \in L^1_{\rm loc}(\Sigma, \sigma)$ and any $q \in [1, \infty)$ we have

$$\Big(⨍_{\Delta} |f - f_{\Delta'}|^q \,d\sigma \Big)^{\frac{1}{q}} \leq C_{q,n} \Big[1 + \Big(\frac{\sigma(\Delta \cup \Delta')}{\sigma(\Delta \cap \Delta')} \Big)^{\frac{1}{q}} \Big] \|f\|_{\mathrm{BMO}(\Sigma,\sigma)} . \tag{2.103}$$

In particular, (2.103) readily implies that there exists some constant $C \in (0, \infty)$ which depends only on n and the Ahlfors regular constant of Σ with the property that for each function $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$ and each surface ball $\Delta \subseteq \Sigma$ we have

$$\big| f_{2\Delta} - f_{\Delta} \big| \leq C \, \|f\|_{\mathrm{BMO}(\Sigma,\sigma)} . \tag{2.104}$$

In turn, (2.104) may be used to estimate

$$\big| f_{2^j\Delta} - f_{\Delta} \big| \leq \sum_{k=1}^{j} \big| f_{2^k\Delta} - f_{2^{k-1}\Delta} \big| \leq Cj \, \|f\|_{\mathrm{BMO}(\Sigma,\sigma)} , \tag{2.105}$$

for each function $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$, each surface ball $\Delta \subseteq \Sigma$, and each integer $j \in \mathbb{N}$. For future use, let us also note here that there exists some $C \in (0, \infty)$ which depends only on n and the Ahlfors regular constant of Σ with the property that for each function $f \in L^1_{\mathrm{loc}}(\Sigma, \sigma)$, each pair of points $x, y \in \Sigma$, and each radius $R > |x - y|$ we have

$$\big| f_{\Delta(x,R)} - f_{\Delta(y,R)} \big| \leq C \, \|f\|_{\mathrm{BMO}(\Sigma,\sigma)} . \tag{2.106}$$

More generally, suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set and assume μ is a doubling Borel measure on Σ. This means that there exists $C \in (0, \infty)$ with the property that for each surface ball $\Delta \subseteq \Sigma$ we have

$$0 < \mu(2\Delta) \leq C\mu(\Delta) < +\infty. \tag{2.107}$$

In this setting, we shall denote by $\mathrm{BMO}(\Sigma, \mu)$ the space consisting of all functions $f \in L^1_{\mathrm{loc}}(\Sigma, \mu)$ with the property that

$$\|f\|_{\mathrm{BMO}(\Sigma,\mu)} := \sup_{\Delta \subseteq \Sigma} ⨍_{\Delta} \Big| f - ⨍_{\Delta} f \, d\mu \Big| \, d\mu < +\infty, \tag{2.108}$$

where the supremum is once again taken over all surface balls $\Delta \subseteq \Sigma$. Much as before, since the John-Nirenberg inequality holds for generic Borel doubling measures (as noted in the discussion pertaining to (2.92)–(2.94)), for each integrability exponent $p \in [1, \infty)$ we then have

$$\begin{aligned} \|f\|_{\mathrm{BMO}(\Sigma,\mu)} &\approx \|f\|_{\mathrm{BMO}_p(\Sigma,\mu)} \\ &\approx \sup_{\Delta \subseteq \Sigma} \Big(⨍_{\Delta} ⨍_{\Delta} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \Big)^{\frac{1}{p}} \end{aligned}$$

$$\approx \sup_{\Delta\subseteq\Sigma}\inf_{c\in\mathbb{R}}\left(\fint_{\Delta}\big|f-c\big|^{p}\,\mathrm{d}\mu\right)^{\frac{1}{p}}, \tag{2.109}$$

uniformly for $f \in L^1_{\mathrm{loc}}(\Sigma,\mu)$, where

$$\|f\|_{\mathrm{BMO}_p(\Sigma,\mu)} := \sup_{\Delta\subseteq\Sigma}\left(\fint_{\Delta}\Big|f-\fint_{\Delta} f\,\mathrm{d}\mu\Big|^{p}\,\mathrm{d}\mu\right)^{1/p}, \tag{2.110}$$

with the supremum above taken over all surface balls $\Delta \subseteq \Sigma$. As before, for any given integer $N \in \mathbb{N}$, we shall denote by $\big[\mathrm{BMO}(\Sigma,\mu)\big]^N$ the space of $\mathbb{C}^N$-valued functions $f \in \big[L^1_{\mathrm{loc}}(\Sigma,\mu)\big]^N$ with the property that $\|f\|_{[\mathrm{BMO}(\Sigma,\mu)]^N} < \infty$, where the semi-norm $\|\cdot\|_{[\mathrm{BMO}(\Sigma,\mu)]^N}$ is defined much as in (2.91). Finally, given a function $f \in \big[L^1_{\mathrm{loc}}(\Sigma,\mu)\big]^N$ we agree to define $\|f\|_{[\mathrm{BMO}_p(\Sigma,\mu)]^N}$ as in (2.110), now interpreting $|\cdot|$ as the standard Euclidean norm in $\mathbb{C}^N$.

Let us also briefly discuss the space VMO which, heuristically, should be thought of as an integral version[3] of uniform continuity. Specifically, let Σ be a closed Ahlfors regular subset of $\mathbb{R}^n$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. In this setting, define the Sarason space $\mathrm{VMO}(\Sigma,\sigma)$ of functions of vanishing mean oscillations (cf. [121]) as

$$\mathrm{VMO}(\Sigma,\sigma) := \text{the closure of } \mathrm{UC}(\Sigma)\cap\mathrm{BMO}(\Sigma,\sigma) \text{ in } \mathrm{BMO}(\Sigma,\sigma), \tag{2.111}$$

where $\mathrm{UC}(\Sigma)$ stands for the space of uniformly continuous functions on Σ. Then for each given function $f \in \mathrm{BMO}(\Sigma,\sigma)$ one has the equivalence

$$f \in \mathrm{VMO}(\Sigma,\sigma) \Longleftrightarrow \lim_{R\to 0^+}\sup_{\substack{x\in\Sigma \text{ and}\\ r\in(0,R)}}\left(\fint_{\Delta(x,r)}\Big|f-\fint_{\Delta(x,r)} f\,\mathrm{d}\sigma\Big|^{p}\,\mathrm{d}\sigma\right)^{\frac{1}{p}} = 0 \tag{2.112}$$

for some (or all) $p \in [1,\infty)$. See [112, §3.1] for a proof.

Moving on, in the lemma below we collect a number of useful formulas and estimates for unimodular functions (i.e., vector-valued functions of modulus one).

Lemma 2.7 *Let (X,μ) be a measure space with the property that $0 < \mu(X) < \infty$. Also, fix an integer $N \in \mathbb{N}$ and suppose $f \in \big[L^1(X,\mu)\big]^N$. Then*

$$\fint_X\Big|f-\fint_X f\,d\mu\Big|^2\,d\mu = \fint_X |f|^2\,d\mu - \Big|\fint_X f\,d\mu\Big|^2. \tag{2.113}$$

In particular,

[3] As opposed to a pointwise version.

$$\textit{if } |f(x)| = 1 \textit{ for } \mu\textit{-a.e. } x \in X \textit{ then}$$

$$\fint_X \Big|f - \fint_X f\,d\mu\Big|^2 d\mu = 1 - \Big|\fint_X f\,d\mu\Big|^2 \quad \textit{and}$$

$$\Big(1 - \Big|\fint_X f\,d\mu\Big|\Big)^2 \leq \fint_X \Big|f - \fint_X f\,d\mu\Big|^2 d\mu \leq 2\Big(1 - \Big|\fint_X f\,d\mu\Big|\Big), \tag{2.114}$$

$$0 \leq 1 - \Big|\fint_X f\,d\mu\Big| \leq \fint_X \Big|f - \fint_X f\,d\mu\Big| d\mu \leq \sqrt{2}\sqrt{1 - \Big|\fint_X f\,d\mu\Big|}.$$

Proof Keeping in mind that $|Z - W|^2 = |Z|^2 - 2\mathrm{Re}(Z \cdot \overline{W}) + |W|^2$ for each $Z, W \in \mathbb{C}^N$, we may compute

$$\begin{aligned}\fint_X \Big|f - \fint_X f\,\mathrm{d}\mu\Big|^2 \mathrm{d}\mu &= \fint_X \Big(|f|^2 - 2\mathrm{Re}\Big[f \cdot \Big(\fint_X \overline{f}\,\mathrm{d}\mu\Big)\Big] + \Big|\fint_X f\,\mathrm{d}\mu\Big|^2\Big)\,\mathrm{d}\mu \\ &= \fint_X |f|^2\,\mathrm{d}\mu - 2\mathrm{Re}\fint_X f \cdot \Big(\fint_X \overline{f}\,\mathrm{d}\mu\Big)\,\mathrm{d}\mu + \Big|\fint_X f\,\mathrm{d}\mu\Big|^2 \\ &= \fint_X |f|^2\,\mathrm{d}\mu - \Big|\fint_X f\,\mathrm{d}\mu\Big|^2,\end{aligned} \tag{2.115}$$

proving (2.113). Then (2.114) follows from this by observing that

$$1 - \Big|\fint_X f\,\mathrm{d}\mu\Big|^2 = \Big(1 + \Big|\fint_X f\,\mathrm{d}\mu\Big|\Big)\Big(1 - \Big|\fint_X f\,\mathrm{d}\mu\Big|\Big) \leq 2\Big(1 - \Big|\fint_X f\,\mathrm{d}\mu\Big|\Big) \tag{2.116}$$

and

$$\begin{aligned}0 \leq 1 - \Big|\fint_X f\,\mathrm{d}\mu\Big| &= \fint_X |f|\,\mathrm{d}\mu - \Big|\fint_X f\,\mathrm{d}\mu\Big| \\ &\leq \fint_X \Big|f - \fint_X f\,\mathrm{d}\mu\Big|\,\mathrm{d}\mu \leq \Big(\fint_X \Big|f - \fint_X f\,\mathrm{d}\mu\Big|^2\,\mathrm{d}\mu\Big)^{1/2},\end{aligned} \tag{2.117}$$

by the fact that $|f| = 1$, the reverse triangle inequality, and the Cauchy–Schwarz inequality. □

Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, Lemma 2.7 applies to the geometric measure theoretic outward unit normal ν to Ω, in the setting in which $X := \Delta$, an arbitrary surface ball on $\partial\Omega$, and the measure is $\mu := \mathcal{H}^{n-1}\lfloor\Delta$. As indicated below, this yields a better bound for the BMO semi-norm of ν than directly estimating $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 2\,\|\nu\|_{[L^\infty(\partial\Omega,\sigma)]^n} = 2$.

Lemma 2.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then*

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \leq \|\nu\|_{[BMO_2(\partial\Omega,\sigma)]^n} \leq 1, \tag{2.118}$$

and

$$1 - \inf_{\Delta\subseteq\partial\Omega}\Big|\fint_\Delta \nu\, d\sigma\Big| \leq \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \leq \sqrt{2}\sqrt{1 - \inf_{\Delta\subseteq\partial\Omega}\Big|\fint_\Delta \nu\, d\sigma\Big|}, \tag{2.119}$$

where the two infima are taken over all surface balls $\Delta \subseteq \partial\Omega$. In particular,

$$1 \geq \Big|\fint_\Delta \nu\, d\sigma\Big| \geq 1 - \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \ \textit{ for each surface ball } \Delta \subseteq \partial\Omega. \tag{2.120}$$

Also,

$$\textit{if } \partial\Omega \textit{ is bounded then } \ \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} = \|\nu\|_{[BMO_2(\partial\Omega,\sigma)]^n} = 1. \tag{2.121}$$

As a consequence,

$$\partial\Omega \ \textit{ is unbounded whenever } \ \|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < 1. \tag{2.122}$$

In relation to (2.121) we wish to note that, in the class of Ahlfors regular domains, having the BMO semi-norm of its geometric measure theoretic outward unit normal precisely 1 is not an exclusive attribute of bounded domains. For example, a straightforward computation shows that an infinite strip in $\mathbb{R}^n$ (i.e., the region in between two parallel hyperplanes in $\mathbb{R}^n$) is an unbounded Ahlfors regular domain with the property that the BMO semi-norm of its outward unit normal is equal to 1.

Proof of Lemma 2.8 Hölder's inequality and Lemma 2.7 imply that for each surface ball $\Delta \subseteq \partial\Omega$ we have

$$\Big(\fint_\Delta |\nu - \nu_\Delta|\, \mathrm{d}\sigma\Big)^2 \leq \fint_\Delta |\nu - \nu_\Delta|^2\, \mathrm{d}\sigma = 1 - \Big|\fint_\Delta \nu\, \mathrm{d}\sigma\Big|^2 \leq 1, \tag{2.123}$$

from which (2.118) follows on account of (2.91), (2.97), and (2.98). Next, (2.119) follows from (2.114), used with $X := \Delta$, arbitrary surface ball on $\partial\Omega$, and with $\mu := \mathcal{H}^{n-1}\lfloor\Delta$.

To justify the claim made in (2.121), assume first that the set Ω is bounded. In such a case, fix some point $x_0 \in \partial\Omega$ along with some real number $r_0 > \mathrm{diam}(\partial\Omega)$ and note that the latter choice entails $\Delta_0 := B(x_0, r_0) \cap \partial\Omega = \partial\Omega$. Also, since $\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0$ (cf. Definition 2.4) the Divergence Formula (2.20) gives

$$\nu_{\Delta_0} = \Big(\frac{1}{\sigma(\partial\Omega)}\int_{\partial\Omega} \nu\cdot\mathbf{e}_j\,\mathrm{d}\sigma\Big)_{1\leq j\leq n} = \Big(\frac{1}{\sigma(\partial\Omega)}\int_{\Omega} \mathrm{div}\mathbf{e}_j\,\mathrm{d}\mathcal{L}^n\Big)_{1\leq j\leq n} = 0. \tag{2.124}$$

In concert with (2.19) this implies (bearing in mind that $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big)=0$)

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \sup_{\Delta\subseteq\partial\Omega} ⨍_{\Delta}\big|\nu-\nu_\Delta\big|\,\mathrm{d}\sigma \geq ⨍_{\partial\Omega}\big|\nu-\nu_{\Delta_0}\big|\,\mathrm{d}\sigma = 1. \tag{2.125}$$

In light of (2.118), we then conclude that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} = 1$ in this case. When Ω is an unbounded Ahlfors regular domain with compact boundary in $\mathbb{R}^n$, having $n\geq 2$ implies that $\mathbb{R}^n\setminus\overline{\Omega}$ is a bounded Ahlfors regular domain whose topological boundary coincides with that of Ω, whose geometric measure theoretic boundary agrees with that of Ω, and whose geometric measure theoretic outward unit normal is $-\nu$ at σ-a.e. point on $\partial\Omega$ (cf. [111, §5.10] for a proof). Granted these properties, we may run the same argument as in (2.124)–(2.125) with $\mathbb{R}^n\setminus\overline{\Omega}$ in place of Ω and conclude that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 1$ in this case as well. This finishes the proof of (2.121). □

To close this section, recall for further use that $\mathrm{CMO}(\mathbb{R}^n,\mathcal{L}^n)$ is the closure of $\mathscr{C}_0^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)$. As may be seen with the help of [22, Théorème 7, p. 198], the space $\mathrm{CMO}(\mathbb{R}^n,\mathcal{L}^n)$ may be alternatively described as the linear subspace of $\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)$ consisting of functions f satisfying the following three conditions:

$$\lim_{r\to 0^+}\Big[\sup_{x\in\mathbb{R}^n}\Big(⨍_{B(x,r)}\Big|f-⨍_{B(x,r)} f\,\mathrm{d}\mathcal{L}^n\Big|\,\mathrm{d}\mathcal{L}^n\Big)\Big]=0, \tag{2.126}$$

$$\lim_{r\to\infty}\Big[\sup_{x\in\mathbb{R}^n}\Big(⨍_{B(x,r)}\Big|f-⨍_{B(x,r)} f\,\mathrm{d}\mathcal{L}^n\Big|\,\mathrm{d}\mathcal{L}^n\Big)\Big]=0, \tag{2.127}$$

and

$$\lim_{|x|\to\infty}\Big[\sup_{r\in[R_0,R_1]}\Big(⨍_{B(x,r)}\Big|f-⨍_{B(x,r)} f\,\mathrm{d}\mathcal{L}^n\Big|\,\mathrm{d}\mathcal{L}^n\Big)\Big]=0 \quad \text{for each } R_0, R_1\in(0,\infty) \text{ with } R_0<R_1. \tag{2.128}$$

This is going to be relevant later on, in Proposition 2.11.

2.2 Reifenberg Flat Domains

In this section we explore the notion of flatness (in the Reifenberg sense). To facilitate the subsequent discussion, the reader is reminded that the Hausdorff distance between two arbitrary nonempty sets $A, B \subset \mathbb{R}^n$ is defined as

$$\operatorname{Dist}[A, B] := \max\Big\{\sup\{\operatorname{dist}(a, B) : a \in A\}, \sup\{\operatorname{dist}(b, A) : b \in B\}\Big\}. \tag{2.129}$$

We start by recalling the following definitions from [72].

Definition 2.11 Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$ and let $\Sigma \subset \mathbb{R}^n$ be a closed set. Then Σ is said to be a (R, δ)-`Reifenberg flat set` if for each $x \in \Sigma$ and each $r \in (0, R)$ there exists an $(n-1)$-dimensional plane $\pi(x, r)$ in $\mathbb{R}^n$ which contains x and satisfies

$$\operatorname{Dist}\big[\Sigma \cap B(x, r), \pi(x, r) \cap B(x, r)\big] \leq \delta r. \tag{2.130}$$

For example, given $\delta > 0$, the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$ with a sufficiently small Lipschitz constant is a (∞, δ)-Reifenberg flat set.

Definition 2.12 Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$. A nonempty, proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, δ)-`separation property` if for each $x \in \partial\Omega$ and $r \in (0, R)$ there exists an $(n-1)$-dimensional plane $\widetilde{\pi}(x, r)$ in $\mathbb{R}^n$ passing through x and a choice of unit normal vector $\vec{n}_{x,r}$ to $\widetilde{\pi}(x, r)$ such that

$$\begin{aligned} &\big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \widetilde{\pi}(x, r),\ t > 2\delta r\big\} \subset \Omega \ \text{ and} \\ &\big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \widetilde{\pi}(x, r),\ t < -2\delta r\big\} \subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{2.131}$$

Definition 2.13 Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$. A nonempty, proper subset Ω of $\mathbb{R}^n$ is called an (R, δ)-`Reifenberg flat domain` (or simply a `Reifenberg flat domain` if the particular values of R, δ are not important) provided Ω satisfies the (R, δ)-separation property and $\partial\Omega$ is an (R, δ)-Reifenberg flat set.

Recall the two-sided corkscrew condition from Definition 2.7.

Proposition 2.6 *Let Ω be a nonempty proper subset of $\mathbb{R}^n$ with the property that it satisfies the (R, c)-two-sided corkscrew condition for some $R \in (0, \infty]$ and some $c \in (0, 1)$. In addition, suppose $\partial\Omega$ is an (R, δ)-Reifenberg flat set for some number $\delta \in \big(0, \frac{\sqrt{3}}{4}c\big)$. Then Ω is an (R, δ)-Reifenberg flat domain.*

Proof Pick a location $x \in \partial\Omega$ along with a scale $r \in (0, R)$. Definition 2.11 ensures the existence of an $(n-1)$-dimensional plane $\pi(x, r)$ in $\mathbb{R}^n$ passing through x which satisfies (2.130). Make a choice of a unit normal vector $\vec{n}_{x,r}$ to $\pi(x, r)$ and abbreviate

$$\begin{aligned} C^+(x, r) &:= \big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \pi(x, r),\ t > 2\delta r\big\}, \\ C^-(x, r) &:= \big\{y + t\,\vec{n}_{x,r} \in B(x, r) : y \in \pi(x, r),\ t < -2\delta r\big\}. \end{aligned} \tag{2.132}$$

We claim that matters may be arranged (by taking $\delta \in \big(0, \frac{\sqrt{3}}{4}c\big)$ and by making a judicious choice of the orientation of $\vec{n}_{x,r}$) so that

$$C^{+}(x,r) \subset \Omega \quad \text{and} \quad C^{-}(x,r) \subset \mathbb{R}^n \setminus \Omega. \tag{2.133}$$

To prove this claim, first observe that (2.130) guarantees that the connected sets $C^{\pm}(x,r)$ do not intersect $\partial\Omega$. As such, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ form a disjoint, open cover of $C^{\pm}(x,r)$, hence

$$\begin{aligned} &C^{+}(x,r) \text{ is entirely contained in either } \Omega_+ \text{ or } \Omega_-, \text{ and} \\ &C^{-}(x,r) \text{ is entirely contained in either } \Omega_+ \text{ or } \Omega_-. \end{aligned} \tag{2.134}$$

To proceed, denote by $x_r^{\pm} \in \Omega_{\pm}$ the two corkscrew points corresponding to the location x and scale r. In particular,

$$|x_r^{\pm} - x| < r \quad \text{and} \quad B\big(x_r^{\pm}, cr\big) \subseteq \Omega_{\pm}, \tag{2.135}$$

where the constant $c \in (0,1)$ is as in Definition 2.7. Hence, if we consider the balls $B\big(x_r^{+}, c\,r\big)$, $B\big(x_r^{-}, c\,r\big)$, their centers $x_r^{\pm}$ belong to $B(x,r)$. The fact that we are presently assuming $0 < \delta < \frac{\sqrt{3}}{4}c$ with $c \in (0,1)$ ensures that $\delta < (c/2)\sqrt{1 - c^2/4}$ which, as some elementary geometry shows, forces each of the balls $B\big(x_r^{+}, c\,r\big)$, $B\big(x_r^{-}, c\,r\big)$ to intersect one of the sets $C^{+}(x,r)$, $C^{-}(x,r)$. As such, one of the following four alternatives is true:

$$B\big(x_r^{+}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing, \tag{2.136}$$

$$B\big(x_r^{+}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing, \tag{2.137}$$

$$B\big(x_r^{+}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing, \tag{2.138}$$

$$B\big(x_r^{+}, c\,r\big) \cap C^{-}(x,r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, c\,r\big) \cap C^{+}(x,r) \neq \varnothing. \tag{2.139}$$

Observe that the alternative described in (2.136) cannot hold. Otherwise, the existence of points $z_1 \in B\big(x_r^{+}, c\,r\big) \cap C^{+}(x,r)$ and $z_2 \in B\big(x_r^{-}, c\,r\big) \cap C^{+}(x,r)$ would imply that, on the one hand, the line segment $[z_1, z_2]$ lies in the convex set $C^{+}(x,r)$, hence also either in Ω_+ or in Ω_- by (2.134). This being said, the fact that $z_1 \in B\big(x_r^{+}, c\,r\big) \subseteq \Omega_+$ and $z_2 \in B\big(x_r^{-}, c\,r\big) \subseteq \Omega_-$ prevents either one of these eventualities form materializing. This contradiction therefore excludes (2.136). Reasoning in a similar fashion we may rule out (2.137). When (2.138) holds, from (2.134) and the fact that $B\big(x_r^{\pm}, c\,r\big) \subseteq \Omega_{\pm}$ (cf. (2.135)) we conclude that the inclusions in (2.133) hold as stated. Finally, when (2.139) holds, from (2.426) and (2.135) we deduce that $C^{+}(x,r) \subseteq \Omega_-$ and $C^{-}(x,r) \subseteq \Omega_+$. In such a scenario, we may ensure that the inclusions in (2.133) are valid simply by re-denoting $\vec{n}_{x,r}$ as $-\vec{n}_{x,r}$ which amounts to reversing the roles of $C^{+}(x,r)$ and

$C^-(x, r)$. This concludes the proof of (2.133). In turn, from (2.133) and (2.132) we conclude that (2.131) holds with $\widetilde{\pi}(x, r) := \pi(x, r)$. Definition 2.12 then implies that Ω is, indeed, an (R, δ)-Reifenberg flat domain. □

It turns out that sufficiently flat Reifenberg domains are NTA domains. More specifically, from [72, Theorem 3.1, p. 524] and its proof we see that:

there exists a purely dimensional constant $\delta_n \in (0, \infty)$ with the property that for each $\delta \in (0, \delta_n)$ and $R \in (0, \infty]$ one may find some number $N = N(\delta, R) \in \mathbb{N}$ such that any (R, δ)-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^n$ also happens to be an (R, N)-nontangentially accessible domain (in the sense of Definition 2.9). (2.140)

The result recorded in (2.140) has a number of useful consequences. For example, it allows us to conclude that any open set satisfying a two-sided corkscrew condition and whose topological boundary is a sufficiently flat Reifenberg set is actually an NTA domain.

Proposition 2.7 *Let Ω be a nonempty proper subset of $\mathbb{R}^n$ satisfying the (R, c)-two-sided corkscrew condition for some $R \in (0, \infty]$ and $c \in (0, 1)$. In addition, suppose $\partial\Omega$ is a (R, δ)-Reifenberg flat set with $0 < \delta < \min\{c/2, \delta_n\}$, where $\delta_n \in (0, \infty)$ is the purely dimensional constant from (2.140). Then there exists $N = N(\delta, R) \in \mathbb{N}$ with the property that Ω is an (R, N)-nontangentially accessible domain.*

Proof The desired conclusion is a direct consequence of Proposition 2.6, (2.140), and Definition 2.9. □

Moving on, recall the Gauss-Green measure associated with sets of locally finite perimeter as in (2.16). As in [20], given $C \in [1, \infty)$ and $R \in (0, \infty]$ define

$$\mathscr{A}(C, R) := \Big\{\Omega \subseteq \mathbb{R}^n : \Omega \text{ has locally finite perimeter}, \ \operatorname{supp}\mu_\Omega = \partial\Omega, \\ \text{and } \|\partial\Omega\| \text{ is an Ahlfors regular measure} \\ \text{with constant } C \text{ up to scale } R\Big\}. \tag{2.141}$$

Proposition 2.8 *Fix $C \in [1, \infty)$ along with $R \in (0, \infty]$, and consider an arbitrary set $\Omega \subseteq \mathbb{R}^n$. Then $\Omega \in \mathscr{A}(C, R)$ if and only if Ω is $\mathcal{L}^n$-measurable, $\partial\Omega$ is an Ahlfors regular set with constant $C \in [1, \infty)$ up to scale $R \in (0, \infty]$, and*

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0. \tag{2.142}$$

Proof The left-to-right implication is deduced from (2.141), (2.31), and Proposition 2.4, while the right-to-left implication follows from (2.141), (2.33), (2.24), (2.31), Proposition 2.4, and Lemma 2.1. □

In particular, the above result ensures that Ahlfors regular domains (and the complements of their closures) belong to the class (2.141). A formal statement to this effect is recorded below.

Proposition 2.9 *Suppose* $\Omega \subseteq \mathbb{R}^n$ *is an Ahlfors regular domain (in the sense of Definition 2.4), and denote by* $C \in [1, \infty)$ *the Ahlfors regularity constant of* $\partial\Omega$. *Also, define*

$$\Omega_+ := \Omega \quad and \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.143}$$

Then

$$\Omega_\pm \in \bigcap_{0<R\leq 2\,diam(\partial\Omega)} \mathscr{A}(C, R). \tag{2.144}$$

Proof This is a consequence of Definition 2.4, Proposition 2.8, and Lemma 2.3. □

To be able to continue, we shall need more notation. The cylinder $C(x_0, r, \omega)$ with center at $x_0 \in \mathbb{R}^n$, radius $r \in (0, \infty)$, and axial direction $\omega \in S^{n-1}$ is defined as

$$C(x_0, r, \omega) := \Big\{x \in \mathbb{R}^n : |\langle x - x_0, \omega\rangle| < r \text{ and } \big|x - x_0 - \langle x - x_0, \omega\rangle\omega\big| < r\Big\}. \tag{2.145}$$

As in [89, p. 290], given a set of locally finite perimeter $\Omega \subseteq \mathbb{R}^n$, the `cylindrical excess` of Ω at the point $x_0 \in \partial\Omega$, for the scale $r \in (0, \infty)$, and with respect to the direction $\omega \in S^{n-1}$ is defined as

$$\mathbf{e}(\Omega, x_0, r, \omega) := \frac{1}{r^{n-1}} \int_{C(x_0,r,\omega)\cap\partial^*\Omega} \frac{|\nu(x) - \omega|^2}{2}\, d\mathcal{H}^{n-1}(x), \tag{2.146}$$

where ν is the geometric measure theoretic outward unit normal to Ω. This notion is studied at length in [89, Chapter 22], where a number of basic properties of the excess (having to do with rescaling, change of direction, lower-semicontinuity) are established. Here, we shall need the following result.

Lemma 2.9 *Let* $\Omega \subset \mathbb{R}^n$ *be a set of locally finite perimeter. Then for every point* $x_0 \in \partial\Omega$, *every radius* $r \in (0, \infty)$, *and every vector* $\omega \in \mathbb{R}^n \setminus \{0\}$ *there holds*

$$\mathbf{e}\Big(\Omega, x_0, r, \frac{\omega}{|\omega|}\Big) \leq \frac{2}{r^{n-1}} \int_{C(x_0,r,\omega/|\omega|)\cap\partial^*\Omega} |\nu(x) - \omega|^2\, d\mathcal{H}^{n-1}(x), \tag{2.147}$$

where ν *is the geometric measure theoretic outward unit normal to* Ω.

This lemma facilitates estimating the excess in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal. Specifically, suppose $\Omega \subset \mathbb{R}^n$ is actually an Ahlfors regular domain and write $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.

Having fixed a point $x_0 \in \partial\Omega$ along with a radius $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$, denote $\Delta(x_0, r) := B(x_0, r) \cap \partial\Omega$ and $\nu_{\Delta(x_0,r)} := ⨍_{\Delta(x_0,r)} \nu\,\mathrm{d}\sigma$. Then since $C(x_0, r, \omega/|\omega|) \subseteq B(x_0, \sqrt{2}r)$ for each $\omega \in \mathbb{R}^n \setminus \{0\}$, we conclude from (2.147) that whenever

$$\nu_{\Delta(x_0,r)} \neq 0 \tag{2.148}$$

we have (with the piece of notation introduced in (2.110))

$$\begin{aligned}
\mathbf{e}\Big(\Omega, x_0, r, \frac{\nu_{\Delta(x_0,r)}}{|\nu_{\Delta(x_0,r)}|}\Big) &\leq 2^{\frac{n+1}{2}} C_A \Big\{\Big(⨍_{\Delta(x_0,\sqrt{2}r)} |\nu - \nu_{\Delta(x_0,r)}|^2\,\mathrm{d}\sigma\Big)^{1/2}\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\Big(⨍_{\Delta(x_0,\sqrt{2}r)} |\nu - \nu_{\Delta(x_0,\sqrt{2}r)}|^2\,\mathrm{d}\sigma\Big)^{1/2} \\
&\qquad + \Big(⨍_{\Delta(x_0,\sqrt{2}r)} |\nu_{\Delta(x_0,r)} - \nu_{\Delta(x_0,\sqrt{2}r)}|^2\,\mathrm{d}\sigma\Big)^{1/2}\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + |\nu_{\Delta(x_0,r)} - \nu_{\Delta(x_0,\sqrt{2}r)}|\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + ⨍_{\Delta(x_0,r)} |\nu - \nu_{\Delta(x_0,\sqrt{2}r)}|\,\mathrm{d}\sigma\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + \Big(\frac{\sigma\big(\Delta(x_0,\sqrt{2}r)\big)}{\sigma\big(\Delta(x_0,r)\big)}\Big)^{1/2} \times \\
&\qquad \times \Big(⨍_{\Delta(x_0,\sqrt{2}r)} |\nu - \nu_{\Delta(x_0,\sqrt{2}r)}|^2\,\mathrm{d}\sigma\Big)^{1/2}\Big\}^2 \\
&\leq 2^{\frac{n+1}{2}} C_A \Big\{\|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} + C_A \cdot (\sqrt{2})^{\frac{n-1}{2}} \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n}\Big\}^2 \\
&= 2^{\frac{n+1}{2}} C_A \big(1 + C_A \cdot 2^{\frac{n-1}{4}}\big)^2 \|\nu\|^2_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n},
\end{aligned} \tag{2.149}$$

where $C_A \in [1, \infty)$ is the Ahlfors regularity constant of $\partial\Omega$.

Here is the proof of Lemma 2.9:

Proof of Lemma 2.9 Abbreviate $\omega_0 := \omega/|\omega| \in S^{n-1}$ and observe that we have the equality $|\omega - \omega_0| = \big|1 - |\omega|\big|$. Hence,

$$|\omega - \omega_0|^2 \cdot \mathcal{H}^{n-1}\big(C(x_0, r, \omega_0) \cap \partial^*\Omega\big)$$

$$= \big|1 - |\omega|\big|^2 \cdot \mathcal{H}^{n-1}\big(C(x_0, r, \omega_0) \cap \partial^*\Omega\big)$$

$$= \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} \big||\nu(x)| - |\omega|\big|^2 \, d\mathcal{H}^{n-1}(x)$$

$$\leq \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} |\nu(x) - \omega|^2 \, d\mathcal{H}^{n-1}(x). \tag{2.150}$$

Also, $|\nu - \omega_0|^2/2 \leq |\nu - \omega|^2 + |\omega - \omega_0|^2$. Based on these observations we may then write

$$\begin{aligned}
\mathbf{e}(\Omega, x_0, r, \omega_0) &= \frac{1}{r^{n-1}} \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} \frac{|\nu(x) - \omega_0|^2}{2} \, d\mathcal{H}^{n-1}(x) \\
&\leq \frac{1}{r^{n-1}} \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} |\nu(x) - \omega|^2 \, d\mathcal{H}^{n-1}(x) \\
&\quad + \frac{\mathcal{H}^{n-1}\big(C(x_0, r, \omega_0) \cap \partial^*\Omega\big)}{r^{n-1}} |\omega - \omega_0|^2 \\
&\leq \frac{2}{r^{n-1}} \int_{C(x_0,r,\omega_0)\cap\partial^*\Omega} |\nu(x) - \omega|^2 \, d\mathcal{H}^{n-1}(x),
\end{aligned} \tag{2.151}$$

which is the desired estimate. □

The basic height bound, recorded in (2.153) below, has been proved in [89, Theorem 22.8, p. 294] for sets Ω in a class of perimeter minimizers (a notion discussed at length in [89, Chapter 21, pp. 278–289]). In [20] the authors have observed that this height bound continues to hold for sets Ω in $\mathscr{A}(C, R)$, the class recalled in (2.141). Specifically, the following result has been proved in [20] along the lines of the argument in [89, Section 22.2, pp. 294–302]:

Theorem 2.1 *Given any $C_0 \in [1, \infty)$ and $n \in \mathbb{N}$ with $n \geq 2$, there exist two constants, $\varepsilon_1 \in (0, 1)$ and $C_1 \in [1, \infty)$, depending only on n and C_0 such that if $\Omega \in \mathscr{A}(C_0, R_0)$ for some $R_0 \in (0, \infty]$, and $x_0 \in \partial\Omega$, $r \in (0, R_0/2)$, $\omega \in S^{n-1}$ are such that*

$$\mathbf{e}(\Omega, x_0, 2r, \omega) \leq \varepsilon_1, \tag{2.152}$$

then the following conditions hold (with the cylinder $C(x_0, r, \omega)$ defined as in (2.145)):

$C(x_0, r, \omega) \cap \partial\Omega$

$$\subseteq \Big\{x \in C(x_0, r, \omega) : |\langle x - x_0, \omega\rangle| \leq C_1 r \cdot \mathbf{e}(\Omega, x_0, 2r, \omega)^{\frac{1}{2(n-1)}}\Big\}, \tag{2.153}$$

$$\Big\{x \in C(x_0, r, \omega) \cap \Omega : \langle x - x_0, \omega\rangle > C_1 r \cdot \mathbf{e}(\Omega, x_0, 2r, \omega)^{\frac{1}{2(n-1)}}\Big\} = \varnothing, \tag{2.154}$$

$$\Big\{x \in C(x_0, r, \omega) \setminus \Omega : \langle x - x_0, \omega\rangle < -C_1 r \cdot \mathbf{e}(\Omega, x_0, 2r, \omega)^{\frac{1}{2(n-1)}}\Big\} = \varnothing. \tag{2.155}$$

Recall the class of (R, δ)-Reifenberg flat domains from Definition 2.13.

Corollary 2.1 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Then for each given $C_0 \in [1, \infty)$ there exist two constants, $\varepsilon_2 \in (0, 1)$ and $C_2 \in [1, \infty)$, depending only on n and C_0 with the following significance. Whenever $R_0 \in (0, \infty]$, $R \in (0, R_0/2)$, and $\Omega \in \mathscr{A}(C_0, R_0)$ are such that*

$$\delta := \sup_{x_0 \in \partial\Omega} \sup_{r \in (0,R)} \inf_{\omega \in S^{n-1}} \mathbf{e}(\Omega, x_0, 2r, \omega) < \varepsilon_2 \tag{2.156}$$

it follows that Ω is a $\big(R, C_2 \cdot \delta^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat domain.

Proof Let $\varepsilon_1 = \varepsilon_1(C_0, n) \in (0, 1)$ and $C_1 = C_1(C_0, n) \in (0, \infty)$ be as in Theorem 2.1. Take

$$\varepsilon_2 := \min\big\{\varepsilon_1, 2^{-1} C_1^{2(1-n)}\big\}. \tag{2.157}$$

Fix an arbitrary location $x_0 \in \partial\Omega$ along with an arbitrary scale $r \in (0, R)$. Since having $0 \leq \delta < \varepsilon_2$ (cf. (2.156)) ensures that $1 < (2\varepsilon_2)/(\varepsilon_2 + \delta) \leq 2$, it is possible to choose some $\omega_{x_0,r} \in S^{n-1}$ such that

$$\begin{aligned} \mathbf{e}\big(\Omega, x_0, 2r, \omega_{x_0,r}\big) &< \Big(\frac{2\varepsilon_2}{\varepsilon_2 + \delta}\Big) \cdot \inf_{\omega \in S^{n-1}} \mathbf{e}(\Omega, x_0, 2r, \omega) \\ &\leq 2 \cdot \inf_{\omega \in S^{n-1}} \mathbf{e}(\Omega, x_0, 2r, \omega) \leq 2\delta, \end{aligned} \tag{2.158}$$

the last inequality being a consequence of (2.156). Thanks to (2.156), the first inequality in (2.158) forces

$$\mathbf{e}\big(\Omega, x_0, 2r, \omega_{x_0,r}\big) < \Big(\frac{2\varepsilon_2}{\varepsilon_2 + \delta}\Big) \cdot \delta < \varepsilon_2 < \varepsilon_1. \tag{2.159}$$

Granted this, Theorem 2.1 guarantees that the properties (2.153)–(2.155) hold for the vector $\omega := \omega_{x_0,r} \in S^{n-1}$. In particular, from this version of (2.153) and the last inequality in (2.158) it follows that for each $x_0 \in \partial\Omega$ and $r \in (0, R)$ we have identified a vector $\omega_{x_0,r} \in S^{n-1}$ such that

$$\text{the set } C(x_0, r, \omega_{x_0,r}) \cap \partial\Omega \text{ is contained in} \\ \Big\{x \in C(x_0, r, \omega_{x_0,r}) : |\langle x - x_0, \omega_{x_0,r}\rangle| \le C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}\Big\}. \tag{2.160}$$

For each location $x_0 \in \partial\Omega$ and scale $r \in (0, R)$, the versions of (2.154)–(2.155) written for $\omega := \omega_{x_0,r} \in S^{n-1}$ also prove (once again keeping in mind the last inequality in (2.158)) that

$$\Omega^c := \mathbb{R}^n \setminus \Omega \text{ contains the set} \\ C^- := \Big\{x \in C(x_0, r, \omega_{x_0,r}) : \langle x - x_0, \omega_{x_0,r}\rangle > C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}\Big\} \tag{2.161}$$

and

$$\Omega \text{ contains the set} \\ C^+ := \Big\{x \in C(x_0, r, \omega_{x_0,r}) : \langle x - x_0, \omega_{x_0,r}\rangle < -C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}\Big\}. \tag{2.162}$$

Moreover, from (2.156) and (2.157) we see that

$$C_1 \cdot (2\delta)^{\frac{1}{2(n-1)}} < 1, \tag{2.163}$$

hence

$$C^\pm \neq \varnothing. \tag{2.164}$$

To proceed, introduce

$$\pi(x_0, r) := x_0 + \langle \omega_{x_0,r}\rangle^\top \tag{2.165}$$

which is an $(n-1)$-dimensional plane in $\mathbb{R}^n$ containing the point x_0. Given that $B(x_0, r) \subseteq C(x_0, r, \omega_{x_0,r})$, from (2.160) we see that

$$\sup_{x \in B(x_0,r)\cap\partial\Omega} \operatorname{dist}\big(x, \pi(x_0, r) \cap B(x_0, r)\big) \le C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.166}$$

We also claim that

$$\sup_{x \in B(x_0,r)\cap\pi(x_0,r)} \operatorname{dist}\big(x, \partial\Omega \cap B(x_0, r)\big) \le 2C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.167}$$

To justify (2.167), consider an arbitrary point $x \in B(x_0, r) \cap \pi(x_0, r)$. We distinguish two cases.

`Case 1:` *Assume first that*

$$|x - x_0| < r\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}. \tag{2.168}$$

Note that (2.163) ensures that (2.168) is a meaningful demand. In this case, denote by L the line passing through x in the direction of $\omega_{x_0,r}$. Thanks to (2.164), it is possible to pick points $x_\pm \in C^\pm \cap L$. Then since $x^+ \in C^+ \subseteq \Omega$ and $x^- \in C^- \subseteq \Omega^c$, it follows that the line segment $[x^-, x^+]$ intersects $\partial\Omega$. Thus, there exists $y \in [x^-, x^+] \cap \partial\Omega$. Given that $[x^-, x^+]$ is contained in $C(x_0, r, \omega_{x_0,r})$ and that $C(x_0, r, \omega_{x_0,r}) \cap \partial\Omega$ is contained in the set described in the second line of (2.160), we conclude that y belongs to said set, hence $|x - y| \le C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}$. We may now use the Pythagorean theorem to compute

$$\begin{aligned} |y - x_0|^2 &= |x - x_0|^2 + |x - y|^2 \\ &< r^2\big(1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}\big) + C_1^2 r^2 \cdot (2\delta)^{\frac{1}{(n-1)}} = r^2, \end{aligned} \tag{2.169}$$

which places y in $B(x_0, r)$. Ultimately, $y \in B(x_0, r) \cap \partial\Omega \subseteq C(x_0, r, \omega_{x_0,r}) \cap \partial\Omega$. Keeping in mind that the vector $x - y$ is parallel to $\omega_{x_0,r}$ and that $x - x_0$ is orthogonal to $\omega_{x_0,r}$, we may then use (2.160) to compute

$$\begin{aligned} \operatorname{dist}\big(x, \partial\Omega \cap B(x_0, r)\big) &\le |x - y| = |\langle x - y, \omega_{x_0,r}\rangle| \\ &= |\langle y - x_0, \omega_{x_0,r}\rangle| < C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \end{aligned} \tag{2.170}$$

Case 2: *Assume $x \in B(x_0, r) \cap \pi(x_0, r)$ is arbitrary.* In this scenario, define

$$\widetilde{x} := x_0 + (x - x_0)\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}. \tag{2.171}$$

Then $\widetilde{x} \in \pi(x_0, r)$ and

$$|\widetilde{x} - x_0| \le |x - x_0|\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}} < r\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}. \tag{2.172}$$

This proves two things. First, we see that $\widetilde{x} \in B(x_0, r) \cap \pi(x_0, r)$. Granted this, from (2.172) and the analysis in Case 1 (cf. (2.170)) we conclude that

$$\operatorname{dist}\big(\widetilde{x}, \partial\Omega \cap B(x_0, r)\big) < C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.173}$$

Since we also have

$$|\widetilde{x} - x| = \Big|x - x_0 - (x - x_0)\sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}\Big|$$

$$= |x - x_0|\Big|1 - \sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}\Big|$$

$$\leq r \frac{C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}{1 + \sqrt{1 - C_1^2 \cdot (2\delta)^{\frac{1}{(n-1)}}}}$$

$$\leq C_1^2 r \cdot (2\delta)^{\frac{1}{(n-1)}}, \tag{2.174}$$

we may avail ourselves of (2.173) to conclude that

$$\begin{aligned}\operatorname{dist}\big(x, \partial\Omega \cap B(x_0, r)\big) &\leq \operatorname{dist}\big(\widetilde{x}, \partial\Omega \cap B(x_0, r)\big) + |\widetilde{x} - x| \\ &\leq C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}} + C_1^2 r \cdot (2\delta)^{\frac{1}{(n-1)}} \\ &\leq 2C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}},\end{aligned} \tag{2.175}$$

where the last inequality comes from (2.163).
This finishes the proof of (2.167). In concert with (2.166) and (2.129) this establishes

$$\operatorname{Dist}\big[\partial\Omega \cap B(x_0, r)\,,\ \pi(x_0, r) \cap B(x_0, r)\big] \leq 2C_1 r \cdot (2\delta)^{\frac{1}{2(n-1)}}. \tag{2.176}$$

In view of Definition 2.11, we conclude that $\partial\Omega$ is a $\big(R, 2C_1 \cdot (2\delta)^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat set. Together with the separation property implied by (2.161)–(2.162) (cf. Definition 2.12) we conclude that Ω is a $\big(R, C_2 \cdot \delta^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat domain (see Definition 2.13) for some constant $C_2 \in [1, \infty)$ depending only on n and C_0.

□

We are now ready to state an important result, asserting that any Ahlfors regular domain whose geometric measure theoretic outward unit normal has a sufficiently small BMO semi-norm is necessarily a Reifenberg flat domain.

Theorem 2.2 *For each $n \in \mathbb{N}$ with $n \geq 2$ and each $C_0 \in [1, \infty)$ there exist some small threshold $\delta_* \in (0, 1)$ along with some large constant $C_* \in [1, \infty)$, both depending only on n and C_0, with the following significance.*

Suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain (in the sense of Definition 2.4), with the Ahlfors regularity constant of $\partial\Omega$ less than or equal to C_0, and such that the geometric measure theoretic outward unit normal to Ω satisfies (where, as usual, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$)

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} \leq \delta \ \text{ for some } \ \delta \in (0, \delta_*). \tag{2.177}$$

Then $\partial\Omega$ is unbounded and

$$\begin{gathered}\text{both } \Omega_+ := \Omega \text{ and } \Omega_- := \mathbb{R}^n \setminus \overline{\Omega} \text{ are} \\ \big(\infty, C_* \cdot \delta^{\frac{1}{2(n-1)}}\big)\text{-Reifenberg flat domains.}\end{gathered} \tag{2.178}$$

Proof The fact that the set $\partial\Omega$ is unbounded follows from (2.177) (bearing in mind that $\delta_* < 1$) and Lemma 2.8. From (2.120) we also see that

$$\nu_{\Delta(x_0,r)} := \fint_{\Delta(x_0,r)} \nu \, d\sigma \neq 0 \quad \text{for each } x_0 \in \partial\Omega \text{ and } r > 0. \tag{2.179}$$

Fix an arbitrary location $x_0 \in \partial\Omega$ and an arbitrary scale $r \in (0, \infty)$. Keeping (2.179) in mind, we then deduce from (2.148)–(2.149), Lemma 2.3, and (2.98) that whenever (2.177) holds we necessarily have

$$\mathbf{e}\Big(\Omega_\pm, x_0, r, \frac{\pm\nu_{\Delta(x_0,r)}}{|\nu_{\Delta(x_0,r)}|}\Big) \leq C\|\nu\|^2_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\delta^2, \tag{2.180}$$

where $C \in (0, \infty)$ depends only on the dimension n and C_0. Since $0 < \delta < \delta_*$, we see that (2.180) implies

$$\sup_{x_0\in\partial\Omega} \sup_{r\in(0,\infty)} \inf_{\omega\in S^{n-1}} \mathbf{e}(\Omega_\pm, x_0, 2r, \omega) \leq C\delta_*^2. \tag{2.181}$$

With $\varepsilon_2 = \varepsilon_2(C_0, n) \in (0, 1)$ as in Corollary 2.1, choose $\delta_* \in (0, 1)$ such that

$$C\delta_*^2 < \varepsilon_2. \tag{2.182}$$

Given that from Proposition 2.9 we also know that

$$\Omega_\pm \in \mathscr{A}(C_0, \infty), \tag{2.183}$$

we may invoke Corollary 2.1 to conclude that there exists $C_* \in [1, \infty)$, depending only on n and C_0, such that $\Omega_\pm$ are $\big(\infty, C_* \cdot \delta^{\frac{1}{2(n-1)}}\big)$-Reifenberg flat domains. □

Some useful consequences of Theorem 2.2 are brought to light in the result below.

Theorem 2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain (in the sense described in Definition 2.4). Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.*

Then there exists a threshold $\delta_* \in (0, 1)$ and a number $N \in \mathbb{N}$, both depending only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n, with the property that if

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < \delta_*, \tag{2.184}$$

then $\partial\Omega$ *is an unbounded set and* Ω *is an* (∞, N)*-two-sided nontangentially accessible domain* (*in the sense of Definition 2.9*). *In particular,*

Ω satisfies a two-sided local John condition with constants which depend only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$, (2.185)

also

Ω is a UR *domain, with the* UR *constants of $\partial\Omega$ controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$,* (2.186)

and, finally,

Ω is a uniform domain, in the sense that it satisfies the quantitative connectivity condition described in Lemma 2.5, with a constant controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$. (2.187)

Proof This is a consequence of Lemma 2.8, Theorem 2.2, (2.140), (2.88), (2.87), (2.48), and Lemma 2.5. □

We are now in a position to show that for an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ the demand that the BMO semi-norm of its geometric measure theoretic outward unit normal is suitably small relative to the Ahlfors regularity constant of $\partial\Omega$ has a string of remarkable topological and metric consequences for the set Ω. To set the stage, from [83, Theorem 2 in 49.VI, 57.I.9(i), 57.III.1] (cf. also [78, Lemma 4(1) and Lemma 5, p. 1702]) we first note that

if $\mathcal{O} \subseteq \mathbb{R}^n$ is some arbitrary connected open set, then any connected component of $\mathbb{R}^n \setminus \overline{\mathcal{O}}$ has a connected boundary. (2.188)

Theorem 2.4 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω.*

Then there exists a threshold $\delta_* \in (0, 1)$ depending only on the ambient dimension n and the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < \delta_*$ it follows that Ω, $\overline{\Omega}$, $\partial\Omega$, $\mathbb{R}^n \setminus \overline{\Omega}$, and $\mathbb{R}^n \setminus \Omega$ are all unbounded connected sets, $\partial(\overline{\Omega}) = \partial\Omega$, $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial\Omega$, and $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$.

As is apparent from Example 2.11, the demand that the parameter $\delta > 0$ is sufficiently small cannot be dispense with in the context of Theorem 2.4. This being said, it has been shown in [112, §11.5] that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary happens to be an unbounded Ahlfors regular set, then actually $\partial\Omega$ is connected. (2.189)

Proof of Theorem 2.4 Bring in the threshold $\delta_* \in (0,1)$ from Theorem 2.3 and assume that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_*$. From Theorem 2.3, Definition 2.9, and Definition 2.8 we then conclude that both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are pathwise connected open sets (hence, connected open sets). Having established this, from (2.188) we then see that $\partial\big(\mathbb{R}^n \setminus \overline{\Omega}\big) = \partial(\overline{\Omega})$ is connected. The fact that Ω satisfies an exterior corkscrew condition further implies $\partial(\overline{\Omega}) = \partial\Omega$. Since $\delta_* < 1$, Lemma 2.8 ensures that $\partial\Omega$ is unbounded, and this forces both Ω and $\mathbb{R}^n\setminus\overline{\Omega}$ to be unbounded (given that they have $\partial\Omega$ as their topological boundary). Also, the fact that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected implies that its closure is connected. However, $\overline{\mathbb{R}^n \setminus \overline{\Omega}} = \mathbb{R}^n \setminus \mathring{\overline{\Omega}}$ and

$$\mathring{\overline{\Omega}} = \overline{\Omega} \setminus \partial(\overline{\Omega}) = \overline{\Omega} \setminus \partial\Omega = \mathring{\Omega} = \Omega, \tag{2.190}$$

so $\mathbb{R}^n \setminus \Omega = \overline{\mathbb{R}^n \setminus \overline{\Omega}}$ is connected. □

In the two-dimensional setting, it turns out that having an outward unit normal with small BMO semi-norm implies (under certain background assumptions) that the domain in question is actually simply connected. This makes the object of Corollary 2.2, which augments Theorem 2.4.

Corollary 2.2 *Let $\Omega \subseteq \mathbb{R}^2$ be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then there exists a threshold $\delta_* \in (0,1)$, depending only on the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[BMO(\partial\Omega,\sigma)]^2} < \delta_*$ it follows that Ω is an unbounded connected set which is simply connected, $\partial\Omega$ is an unbounded connected set, $\mathbb{R}^2\setminus\overline{\Omega}$ is an unbounded connected set which is simply connected, and $\partial(\mathbb{R}^2 \setminus \overline{\Omega}) = \partial\Omega$.*

Proof All claims are consequences of Theorem 2.4 together with (2.193), (2.194), and (2.195) below. □

2.3 Chord-Arc Curves in the Plane

Shifting gears, in this section we shall work in the two-dimensional setting. We begin by recalling some known results of topological flavor. First, for bounded sets, we know from [12, Corollary 1, p. 352] that

an open bounded connected set $\Omega \subseteq \mathbb{R}^2$ is simply connected if and only if its complement $\mathbb{R}^2 \setminus \Omega$ is a connected set, (2.191)

and

$$\text{an open bounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if its topological boundary, } \partial\Omega, \text{ is a connected set.} \tag{2.192}$$

For unbounded sets, [12, Corollary 2, p. 352] gives

$$\text{an open unbounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if every connected component of } \mathbb{R}^2 \setminus \Omega \text{ is unbounded,} \tag{2.193}$$

and

$$\text{an open unbounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if every connected component } \Sigma \text{ of } \partial\Omega \text{ is unbounded.} \tag{2.194}$$

(Parenthetically, it is worth noting that the boundary of an open set $\Omega \subseteq \mathbb{R}^2$ which is both connected and simply connected is not necessarily connected: for example take $\Omega := \mathbb{R}^2 \setminus E$ where $E := [0, \infty) \times \{0, 1\}$.) Finally, according to [12, Corollary 3, p. 352],

$$\text{if } E \subseteq \mathbb{R}^2 \text{ is a closed set such that each connected component of } E \text{ is unbounded, then } \mathbb{R}^2 \setminus E \text{ is a simply connected set,} \tag{2.195}$$

and according to [120, Theorem 13.11, p. 274]

$$\text{an open connected set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ is simply connected if and only if } \widehat{\mathbb{C}} \setminus \Omega \text{ is connected, where } \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \text{ is the extended complex plane (i.e., the one-point compactification of } \mathbb{C}\text{, aka Riemann's sphere).} \tag{2.196}$$

Next, recall that a (compact) `curve` in the Euclidean plane $\mathbb{R}^2$ (canonically identified with $\mathbb{C}$) is a set of the form $\Sigma = \gamma([a, b])$, where $a, b \in \mathbb{R}$ are two numbers satisfying $a < b$, and $\gamma : [a, b] \to \mathbb{R}^2$ is a continuous function, called a parametrization of Σ. We shall call the curve Σ `simple` if Σ has a parametrization $\gamma : [a, b] \to \mathbb{R}^2$ whose restriction to $[a, b)$ is injective (hence, Σ is simple if it is non self-intersecting). We shall say that the curve Σ is `closed` if it has a parametrization $\gamma : [a, b] \to \mathbb{R}^2$ satisfying $\gamma(a) = \gamma(b)$. Also, we shall call $\Sigma \subset \mathbb{C}$ a `Jordan curve`, if Σ is a simple closed curve. Thus, a curve is Jordan if and only if it is the homeomorphic image of the unit circle S^1. The classical Jordan curve theorem asserts that

$$\text{the complement of a Jordan curve } \Sigma \subset \mathbb{C} \text{ consists precisely of two connected components, one bounded } \Omega_+\text{, and one unbounded } \Omega_-\text{, called the inner and outer domains of } \Sigma\text{, satisfying } \partial\Omega_\pm = \Sigma. \tag{2.197}$$

In light of (2.192), we also conclude that

$$\text{the inner domain } \Omega_+ \text{ of a Jordan curve } \Sigma \subset \mathbb{C} \text{ is simply connected.} \tag{2.198}$$

We are also going to be interested in Jordan curves `passing through infinity` in the plane. This class consists of sets of the form $\Sigma = \gamma(\mathbb{R})$, where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with the property that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. For this class of curves a version of the Jordan separation theorem is also valid, namely

$$\begin{array}{l}\text{if } \Sigma \text{ is a Jordan curve passing through infinity, then its comple-}\\ \text{ment in } \mathbb{C} \text{ consists precisely of two open connected components,}\\ \text{called } \Omega_\pm, \text{ which satisfy } \partial\Omega_+ = \Sigma = \partial\Omega_-.\end{array} \tag{2.199}$$

Once (2.199) has been established, we deduce from (2.194) that

$$\text{in the context of (2.199), the sets } \Omega_\pm \text{ are simply connected.} \tag{2.200}$$

To justify (2.199), let Σ be a Jordan curve passing through infinity. From definitions, it follows that Σ is a closed subset of $\mathbb{C}$. Fix an arbitrary point $z_o \in \mathbb{C}\setminus\Sigma$ and consider the homeomorphisms

$$\begin{aligned} &\Phi : \mathbb{C}\setminus\{z_o\} \longrightarrow \mathbb{C}\setminus\{0\}, \quad \Phi(z) := (z-z_o)^{-1} \text{ for all } z \in \mathbb{C}\setminus\{z_o\},\\ &\Phi^{-1} : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}\setminus\{z_o\}, \quad \Phi^{-1}(\zeta) := z_o + \zeta^{-1} \text{ for all } \zeta \in \mathbb{C}\setminus\{0\}, \end{aligned} \tag{2.201}$$

which are inverse to each other. We then claim that

$$\widetilde{\Sigma} := \Phi(\Sigma) \cup \{0\} \tag{2.202}$$

is a simple closed curve which contains the origin in $\mathbb{C}$. To see that this is indeed the case, start by expressing $\Sigma = \gamma(\mathbb{R})$ where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with the property that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. Then $\widetilde{\gamma} : \big[-\pi/2, \pi/2\big] \to \mathbb{C}$ defined for each $t \in \big[-\pi/2, \pi/2\big]$ as

$$\widetilde{\gamma}(t) := \begin{cases} \big(\gamma(\tan t) - z_o\big)^{-1} & \text{if } t \in \big(-\pi/2, \pi/2\big),\\ 0 & \text{if } t \in \big\{\pm\pi/2\big\} \end{cases} \tag{2.203}$$

is a continuous function whose restriction to $\big[-\pi/2, \pi/2\big)$ is injective, and whose image is precisely $\widetilde{\Sigma}$. Also, $0 \in \widetilde{\Sigma}$ by design. Hence, as claimed, $\widetilde{\Sigma}$ is a simple closed curve passing through $0 \in \mathbb{C}$. The classical Jordan curve theorem recalled in (2.197) then ensures that $\mathbb{C}\setminus\widetilde{\Sigma}$ consists precisely of two open connected components, one bounded $\widetilde{\Omega}_+$, and one unbounded $\widetilde{\Omega}_-$, satisfying $\partial\widetilde{\Omega}_\pm = \widetilde{\Sigma}$. In particular,

$$\mathbb{C} \setminus \{0\} = \widetilde{\Omega}_+ \sqcup \big(\widetilde{\Sigma} \setminus \{0\}\big) \sqcup \widetilde{\Omega}_- \ \text{(disjoint unions)}. \tag{2.204}$$

Then $O_\pm := \Phi^{-1}(\widetilde{\Omega}_\pm)$ are open connected subsets of $\mathbb{C} \setminus \{z_o\}$, and applying the homeomorphism Φ^{-1} to (2.204) yields

$$\mathbb{C} \setminus \{z_o\} = O_+ \sqcup \Sigma \sqcup O_- \ \text{(disjoint unions)}. \tag{2.205}$$

Let us also observe that since $\widetilde{\Omega}_-$ is unbounded, there exists a sequence $\{\zeta_j\}_{j\in\mathbb{N}}$ in $\widetilde{\Omega}_-$ with $|\zeta_j| \to \infty$ as $j \to \infty$. Consequently, the sequence $\{z_j\}_{j\in\mathbb{N}}$ defined for each $j \in \mathbb{N}$ as $z_j := \Phi^{-1}(\zeta_j) = z_o + \zeta_j^{-1}$ is contained in $\Phi^{-1}(\widetilde{\Omega}_-) = O_-$ and converges to z_o. This shows that

$$z_o \in \overline{O_-}. \tag{2.206}$$

Next, since Σ is a closed set, the fact that $z_o \in \mathbb{C} \setminus \Sigma$ guarantees the existence of some $r > 0$ with the property that $B(z_o, r) \cap \Sigma = \varnothing$. In the context of (2.205) this shows that the connected set $B(z_o, r)\setminus\{z_o\}$ is covered by the open sets $O_\pm$. As such, $B(z_o, r)\setminus\{z_o\}$ is fully contained in either O_+ or O_-. In view of (2.206) we ultimately conclude that $B(z_o, r) \setminus \{z_o\} \subseteq O_-$. Then $\Omega_+ := O_+$ and $\Omega_- := O_- \cup \{z_o\}$ are open, connected, disjoint subsets of $\mathbb{C}$, with

$$\mathbb{C} = \Omega_+ \sqcup \Sigma \sqcup \Omega_- \ \text{(disjoint unions)}, \tag{2.207}$$

and

$$\partial\Omega_\pm = \partial O_\pm \setminus \{z_o\} = \Phi^{-1}\big(\partial\widetilde{\Omega}_\pm \setminus \{0\}\big) = \Phi^{-1}\big(\widetilde{\Sigma} \setminus \{0\}\big) = \Sigma. \tag{2.208}$$

This finishes the proof of (2.199).

Moving on, the `length` $L \in [0, +\infty]$ of a given compact curve $\Sigma = \gamma([a, b])$ is defined as

$$L := \sup \sum_{j=1}^{N} |\gamma(t_j) - \gamma(t_{j-1})|, \tag{2.209}$$

the supremum being taken over all partitions $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of the interval $[a, b]$. As is well known (cf., e.g., [85, Theorem 4.38, p. 135]), the length L of any simple compact curve Σ may be expressed in terms of the Hausdorff measure by

$$L = \mathcal{H}^1(\Sigma), \tag{2.210}$$

and

$$\begin{aligned}&|z_1 - z_2| \leq \mathcal{H}^1(\Sigma) \text{ for any compact curve}\\&\Sigma \text{ in the plane with endpoints } z_1, z_2 \in \mathbb{C}.\end{aligned} \tag{2.211}$$

Call a curve Σ `rectifiable` provided its length is finite (i.e., $L < +\infty$), and call Σ `locally rectifiable` if each of its compact sub-curves is rectifiable. The latter condition is equivalent to demanding that $\gamma(I)$ is a rectifiable curve for each compact sub-interval I of the domain of definition of some (or any) parametrization on Σ. In particular, a Jordan curve Σ passing through infinity in the plane, with parametrization $\gamma : \mathbb{R} \to \Sigma$, is locally rectifiable if and only if $\gamma(I)$ is a rectifiable curve for any compact sub-interval I of $\mathbb{R}$.

Suppose Σ is a rectifiable, simple, compact curve in the plane, and denote by L its length. Then there exists a parametrization $[0, L] \ni s \mapsto z(s) \in \Sigma$ of Σ, called the `arc-length parametrization` of Σ, with the property that for each $s_1, s_2 \in [0, L]$ with $s_1 < s_2$ the length of the curve with endpoints at $z(s_1)$ and $z(s_2)$ is $s_2 - s_1$. It is well known (see, e.g., [85, Definition 4.21 and Theorem 4.22, pp. 128–129]) that the arch-length parametrization exists and satisfies

$$\begin{aligned}&z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } [0, L]\\&\quad\text{and } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in [0, L].\end{aligned} \tag{2.212}$$

Also, (2.210)–(2.211) imply

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2|, \quad \forall\, s_1, s_2 \in [0, L]. \tag{2.213}$$

Lemma 2.10 *Let Σ be a rectifiable, simple, compact curve in the plane. Denote by L its length, and let $[0, L] \ni s \mapsto z(s) \in \Sigma$ be its arc-length parametrization. Given $s_1, s_2 \in [0, L]$ with $s_1 < s_2$, abbreviate $I := [s_1, s_2]$ and set $z_I' := ⨍_I z'(s)\, ds$. Then*

$$⨍_I |z'(s) - z_I'|^2\, ds = 1 - \left|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\right|^2. \tag{2.214}$$

Proof Upon observing that

$$z_I' = ⨍_I z'(s)\, \mathrm{d}s = \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} z'(s)\, \mathrm{d}s = \frac{z(s_2) - z(s_1)}{s_2 - s_1}, \tag{2.215}$$

this is a direct consequence of the formula in the second line of (2.114). □

Remark 2.1 The arch-length parametrization of a locally rectifiable Jordan curve passing through infinity in the plane is defined similarly, with $\mathbb{R}$ now playing the role of the interval $[0, L]$, and satisfies properties analogous to (2.212), (2.213), and Lemma 2.10.

We continue by recalling an important category of curves, introduced in 1936 by Mikhail A. Lavrentiev in [84] (also known as the class of Lavrentiev curves).

Definition 2.14 Given some number $\varkappa \in [0, \infty)$, recall that a set $\Sigma \subset \mathbb{C}$ is said to be a $\varkappa$-CAC, or simply CAC (acronym for chord-arc curve) if the parameter $\varkappa$ is de-emphasized, provided Σ is a locally rectifiable Jordan curve passing through infinity with the property that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \Sigma, \tag{2.216}$$

where $\ell(z_1, z_2)$ denotes the length of the sub-arc of Σ joining z_1 and z_2.

In general, the presence of a cusp prevents a curve from being chord-arc. For example, $\Sigma := \{(x, \sqrt{|x|}) : x \in \mathbb{R}\}$ is a Jordan curve passing through infinity in $\mathbb{R}^2 \equiv \mathbb{C}$ which nonetheless fails to be chord-arc. Indeed, if for $x > 0$ we set $z_1 := x + \mathrm{i}\sqrt{x} \in \Sigma$ and $z_2 := -x + \mathrm{i}\sqrt{x} \in \Sigma$ then L'Hôspital's Rule gives

$$\lim_{x \to 0^+} \frac{\ell(z_1, z_2)}{|z_1 - z_2|} = \lim_{x \to 0^+} \frac{2\int_0^x \sqrt{1 + \frac{1}{4t}}\, \mathrm{d}t}{2x} = \lim_{x \to 0^+} \sqrt{1 + \frac{1}{4x}} = +\infty, \tag{2.217}$$

which shows that condition (2.216) is violated for each $\varkappa \in [0, \infty)$.

There are fundamental links between chord-arc curves in the plane and the John-Nirenberg space BMO on the real line. Such connections, along with other basic properties of chord-arc curves, are brought to the forefront in Proposition 2.10 below. To facilitate stating and proving it, we first wish to recall the following version for bi-Lipschitz maps of the classical Kirszbraun extension theorem proved in [79, Theorem 1.2] with a linear bound on the distortion:

any function $f : \mathbb{R} \to \mathbb{C}$ with the property that there exist C, C' in $(0, \infty)$ such that $C|t_1 - t_2| \leq |f(t_1) - f(t_2)| \leq C'|t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$ extends to a homeomorphism $F : \mathbb{C} \to \mathbb{C}$ with $(C/120)|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq (2000C')|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$. (2.218)

Results of this nature have also been proved in [138], [139], [67, Proposition 1.13, p. 227] (see also [119, Theorem 7.10, p. 166] and [36] in the case when the real line is replaced by the unit circle), though the quantitative aspect is less precise, or not explicitly mentioned, in these works.

Here is the proposition dealing with basic properties of chord-arc curves mentioned above.

Proposition 2.10 *Let $\Sigma \subset \mathbb{C}$ be a $\varkappa$-CAC in the plane, for some $\varkappa \in [0, \infty)$, and consider its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$. Then the following statements are true.*

(i) For each $s_1, s_2 \in \mathbb{R}$ one has

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2| \leq (1 + \varkappa)|z(s_1) - z(s_2)|, \tag{2.219}$$

and

$$\begin{gathered} z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } \mathbb{R}, \\ \text{with } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \end{gathered} \tag{2.220}$$

(ii) *For each $z_o \in \Sigma$ and $r \in (0, \infty)$ abbreviate $\Delta(z_o, r) := B(z_o, r) \cap \Sigma$. Then for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ one has*

$$(s_o - r, s_o + r) \subseteq z^{-1}\big(\Delta(z(s_o), r)\big) \subseteq \big(s_o - (1+\varkappa)r\ ,\ s_o + (1+\varkappa)r\big). \tag{2.221}$$

(iii) *For every Lebesgue measurable set $A \subseteq \mathbb{R}$ one has*

$$\mathcal{H}^1\big(z(A)\big) = \mathcal{L}^1(A), \tag{2.222}$$

and for each $\mathcal{H}^1$-measurable set $E \subseteq \Sigma$ one has

$$\mathcal{H}^1(E) = \mathcal{L}^1\big(z^{-1}(E)\big). \tag{2.223}$$

(iv) *With the arc-length measure σ on Σ defined as*

$$\sigma := \mathcal{H}^1 \lfloor \Sigma, \tag{2.224}$$

for each σ-measurable set $E \subseteq \Sigma$ and each non-negative σ-measurable function g on E one has

$$\int_E g \, d\sigma = \int_{z^{-1}(E)} g(z(s))\, ds. \tag{2.225}$$

(v) *Denote by Ω the region of the plane that is lying to the left of the curve Σ (relative to the orientation Σ inherits from its arc-length parametrization given by $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$). Then Ω is a set of locally finite perimeter and its geometric measure theoretic outward unit normal ν is given by*

$$\nu(z(s)) = -iz'(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } \ s \in \mathbb{R}. \tag{2.226}$$

As a consequence, for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ the line $\{z(s) + t\, z'(s) \,:\, t \in \mathbb{R}\}$ is an approximate tangent line to Σ at the point $z(s)$. Hence, Ω has an approximate tangent line at $\mathcal{H}^1$-almost every point on $\partial\Omega$.

(vi) *The set Ω introduced in item (v) is a connected, simply connected, unbounded, two-sided* NTA *domain with an Ahlfors regular boundary (hence also an Ahlfors regular domain which satisfies a two-sided local John condition and,*

in particular, a UR *domain) and whose topological boundary is precisely* Σ, *i.e.,* $\partial\Omega = \Sigma$. *In fact,*

$$\begin{aligned}&\text{there exists a bi-Lipschitz homeomorphism } F : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such}\\ &\text{that } 120^{-1}(1+\varkappa)^{-1}|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq 2000|z_1 - z_2|\\ &\text{for all points } z_1, z_2 \in \mathbb{C}, \text{ and with the property that } \Omega = F(\mathbb{R}^2_+),\\ &\mathbb{R}^2 \setminus \overline{\Omega} = F(\mathbb{R}^2_-), \text{ as well as } \partial\Omega = F(\mathbb{R}\times\{0\}).\end{aligned} \tag{2.227}$$

(vii) *With the piece of notation introduced in* (2.97) *one has*

$$\begin{aligned}\frac{1}{2(1+\varkappa)}\|\nu\|_{BMO(\Sigma,\sigma)} &\leq \|z'\|_{BMO(\mathbb{R},\mathcal{L}^1)}\\ &\leq \|z'\|_{BMO_2(\mathbb{R},\mathcal{L}^1)} \leq \frac{\sqrt{\varkappa(2+\varkappa)}}{1+\varkappa} < 1\end{aligned} \tag{2.228}$$

and

$$\frac{1}{2(1+\varkappa)}\|z'\|_{BMO(\mathbb{R},\mathcal{L}^1)} \leq \|\nu\|_{BMO(\Sigma,\sigma)} \leq 2\sqrt{\varkappa(2+\varkappa)}. \tag{2.229}$$

Moreover, Σ *is a* $\varkappa_*$-CAC *with* $\varkappa_* \in [0, \varkappa]$ *defined as*

$$\begin{aligned}\varkappa_* &:= \frac{1}{\sqrt{1 - \|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}} - 1\\ &= \frac{\|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}{\sqrt{1 - \|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}\Big(1 + \sqrt{1 - \|z'\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}\Big)}.\end{aligned} \tag{2.230}$$

Proof The claims in item *(i)* are seen from definitions and Remark 2.1, while the claim in item *(ii)* is an elementary consequence of (2.219). Next, in view of (2.220), the area formula (cf. [47, Theorem 1, p. 96]) gives (2.222), which may be equivalently recast as in (2.223). Also, the change of variable formula (cf. [47, Theorem 2, p. 99]) gives (2.225). This takes care of items *(iii)-(iv)*.

To proceed from the version of the Jordan curve theorem recorded in (2.199) we conclude that

$$\begin{aligned}&\text{the complement of the curve } \Sigma \text{ in } \mathbb{C} \text{ consists of only two open}\\ &\text{connected components, namely } \Omega_+ := \Omega \text{ and } \Omega_- := \mathbb{C} \setminus \overline{\Omega},\\ &\text{satisfying } \partial\Omega_+ = \Sigma = \partial\Omega_-.\end{aligned} \tag{2.231}$$

In addition, from (2.221) and (2.223) we see that for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ we have

$$\begin{aligned}\mathcal{H}^1\big(\Delta(z(s_o),r)\big) &= \mathcal{L}^1\Big(z^{-1}\big(\Delta(z(s_o),r)\big)\Big)\\ &\leq \mathcal{L}^1\Big(\big(s_o-(1+\varkappa)r\,,\ s_o+(1+\varkappa)r\big)\Big)\\ &= 2(1+\varkappa)r. \end{aligned} \tag{2.232}$$

Based on this and the criterion for finite perimeter from [47, Theorem 1, p. 222] we then conclude that Ω is a set of locally finite perimeter. Next, if $s_o \in \mathbb{R}$ is a point of differentiability for the complex-valued function $z(\cdot)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z(s_o+s) \in B\big(z(s_o)+s\,z'(s_o)\,,\ \varepsilon|s|\big) \ \text{ for each } \ s \in (-\delta,\delta). \tag{2.233}$$

In turn, from this geometric property we deduce that for each angle $\theta \in (0,\pi)$ there exists a height $h = h(\theta) > 0$ such that if $\Gamma^{\pm}_{\theta,h}$ denote the open truncated plane sectors with common vertex at $z(s_o)$, common aperture θ, common height h, and symmetry axes along the vectors $\pm i\,z'(s_o)$, then

$$\Gamma^{+}_{\theta,h} \subseteq \Omega = \Omega_{+} \ \text{ and } \ \Gamma^{-}_{\theta,h} \subseteq \mathbb{C}\setminus\overline{\Omega} = \Omega_{-}. \tag{2.234}$$

To proceed, observe that the measure theoretic boundary of Ω (cf. (2.14)) may be presently described as

$$\partial_*\Omega = \Big\{z \in \partial\Omega :\ \limsup_{r\to 0^+} \frac{\mathcal{L}^2\big(B(z,r)\cap\Omega_{\pm}\big)}{r^2} > 0\Big\}. \tag{2.235}$$

Together, (2.234) and (2.235) imply that

$$\begin{aligned} &\mathcal{A} := \big\{z(s_o)\,:\ s_o \in A\big\} \subseteq \partial_*\Omega, \ \text{ where we have set}\\ &A := \big\{s_o \in \mathbb{R}\,:\ s_o \text{ differentiability point for } z(\cdot)\big\}. \end{aligned} \tag{2.236}$$

Meanwhile, from (2.222) and the fact that $z(\cdot)$ is differentiable at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$ we deduce (also using $\partial\Omega = \Sigma$) that

$$\mathcal{H}^1(\partial\Omega\setminus\mathcal{A}) = \mathcal{H}^1(\Sigma\setminus\mathcal{A}) = \mathcal{H}^1\big(z(\mathbb{R}\setminus A)\big) = \mathcal{L}^1\big(\mathbb{R}\setminus A\big) = 0. \tag{2.237}$$

With this in hand, formula

$$\mathcal{H}^1(\partial\Omega\setminus\partial_*\Omega) = 0 \tag{2.238}$$

follows by combining (2.236) with (2.237). As a consequence of (2.237)–(2.238) and (2.24) we then conclude that

$$\mathcal{A} \cap \partial^*\Omega \text{ has full } \mathcal{H}^1\text{-measure in } \partial\Omega. \tag{2.239}$$

Next, pick an arbitrary point $z_o \in A$ and recall that (2.234) holds. From this and [59, Proposition 2.14, p. 606] it follows that if $\Gamma_{\pi-\theta}$ is the infinite open plane sector with vertex at z_o, aperture $\pi - \theta$, and symmetry axis along the vector $-\mathrm{i}z'(s_o)$, then the geometric measure theoretic outward unit normal to Ω satisfies

$$\nu(z(s_o)) \in \Gamma_{\pi-\theta} \tag{2.240}$$

provided $\nu(z(s_o))$ exists, i.e., if $z(s_o) \in \partial^*\Omega$. The fact that the angle $\theta \in (0, \pi)$ may be chosen arbitrarily close to π then forces $\nu(z(s_o)) = -\mathrm{i}z'(s_o)$ whenever $z(s_o) \in \partial^*\Omega$, i.e., for $s_o \in z^{-1}(\mathcal{A} \cap \partial^*\Omega)$. Given that by (2.239) and (2.222) the latter set has full one-dimensional Lebesgue measure in $\mathbb{R}$, the claim in (2.226) is established. This finishes the treatment of item *(v)*.

Turning our attention to item *(vi)*, first observe that (2.219) implies

$$(1+\varkappa)^{-1}|s_1 - s_2| \leq |z(s_1) - z(s_2)| \leq |s_1 - s_2| \quad \text{for all} \quad s_1, s_2 \in \mathbb{R}, \tag{2.241}$$

hence $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is a bi-Lipschitz map. When used in conjunction with (2.241), the extension result recalled in (2.218) gives that

$$\begin{array}{l}\mathbb{R} \ni s \mapsto z(s) \in \Sigma \text{ extends to a bi-Lipschitz homeomorphism} \\ F : \mathbb{C} \to \mathbb{C} \text{ with the property that for any points } z_1, z_2 \in \mathbb{C} \text{ one} \\ \text{has } [120(1+\varkappa)]^{-1}|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq 2000|z_1 - z_2|.\end{array} \tag{2.242}$$

As a consequence, work in [59] implies that Ω is a connected two-sided NTA domain with an Ahlfors regular boundary (hence also a connected Ahlfors regular domain which satisfies a two-sided local John condition; cf. (2.47) and (2.88)). As far as item *(vi)* is concerned, there remains to observe that $\partial\Omega = \Sigma$ has been noted earlier in (2.231).

Turning our attention to item *(vii)*, fix two numbers $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, abbreviate $I := [s_1, s_2]$ and set $z'_I := \fint_I z'(s)\,\mathrm{d}s$. We may then use Lemma 2.10 and (2.219) to estimate

$$\fint_I |z'(s) - z'_I|^2\,\mathrm{d}s = 1 - \left|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\right|^2 \leq 1 - \left(\frac{1}{1+\varkappa}\right)^2 = \frac{\varkappa(2+\varkappa)}{(1+\varkappa)^2}. \tag{2.243}$$

In view of (2.97), this readily yields the penultimate inequality in (2.228). The second inequality in (2.228) is seen directly from the first inequality in (2.99).

To prove the very first inequality in (2.228), fix an arbitrary point $z_o \in \Sigma$ along with a radius $r \in (0, \infty)$, and set $\Delta := B(z_o, r) \cap \Sigma$. Then there exists a unique number $s_o \in \mathbb{R}$ such that $z_o = z(s_o) \in \Sigma$, and in the current setting we abbreviate $\mathcal{I} := \big(s_o - (1+\varkappa)r\,,\ s_o + (1+\varkappa)r\big)$. In particular, (2.221) and (2.223) imply

$$\sigma(\Delta) = \mathcal{H}^1\big(\Delta(z(s_o), r)\big) = \mathcal{L}^1\Big(z^{-1}\big(\Delta(z(s_o), r)\big)\Big)$$
$$\geq \mathcal{L}^1\Big(\big(s_o - r, s_o + r\big)\Big) = 2r = (1+\varkappa)^{-1}\mathcal{L}^1(\mathcal{I}). \tag{2.244}$$

With $c := -\mathrm{i} \fint_{\mathcal{I}} z'(s)\,\mathrm{d}s \in \mathbb{C}$ we may then write

$$\fint_\Delta |\nu - c|\,\mathrm{d}\sigma = \frac{1}{\sigma(\Delta)}\int_\Delta |\nu - c|\,\mathrm{d}\sigma = \frac{1}{\sigma(\Delta)}\int_{z^{-1}(\Delta)} |\nu(z(s)) - c|\,\mathrm{d}s$$
$$\leq \frac{1}{\sigma(\Delta)}\int_{\mathcal{I}} |\nu(z(s)) - c|\,\mathrm{d}s = \frac{\mathcal{L}^1(\mathcal{I})}{\sigma(\Delta)}\fint_{\mathcal{I}} |\nu(z(s)) - c|\,\mathrm{d}s$$
$$= \frac{\mathcal{L}^1(\mathcal{I})}{\sigma(\Delta)}\fint_{\mathcal{I}} |z'(s) - \mathrm{i}c|\,\mathrm{d}s \leq (1+\varkappa)\fint_{\mathcal{I}} |z'(s) - \mathrm{i}c|\,\mathrm{d}s$$
$$\leq (1+\varkappa)\|z'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}, \tag{2.245}$$

making use of (2.225), (2.221), (2.226), (2.244), and the choice of c. With (2.245) in hand, the first inequality in (2.228) readily follows. The last estimate in (2.229) is implicit in (2.228). To prove the first estimate in (2.229), retain notation introduced above and, now with the choice $c := \fint_\Delta \nu\,\mathrm{d}\sigma \in \mathbb{C}$, estimate

$$\fint_{s_o-r}^{s_o+r} |z'(s) - \mathrm{i}c|\,\mathrm{d}s = \frac{1}{2r}\int_{s_o-r}^{s_o+r} |z'(s) - \mathrm{i}c|\,\mathrm{d}s \leq \frac{1}{2r}\int_{z^{-1}(\Delta)} |z'(s) - \mathrm{i}c|\,\mathrm{d}s$$
$$= \frac{1}{2r}\int_{z^{-1}(\Delta)} |\nu(z(s)) - c|\,\mathrm{d}s = \frac{1}{2r}\int_\Delta |\nu - c|\,\mathrm{d}\sigma$$
$$= \frac{\sigma(\Delta)}{2r}\fint_\Delta |\nu - c|\,\mathrm{d}\sigma \leq (1+\varkappa)\fint_\Delta |\nu - c|\,\mathrm{d}\sigma$$
$$\leq (1+\varkappa)\|\nu\|_{\mathrm{BMO}(\Sigma,\sigma)}, \tag{2.246}$$

thanks to (2.221), (2.226), (2.225), and (2.232). This readily yields the first estimate in (2.229).

To deal with the very last claim in item *(vii)*, fix some $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, set $I := [s_1, s_2]$ and abbreviate $z'_I := \fint_I z'(s)\,\mathrm{d}s$. Lemma 2.10 then permits us to estimate

$$\|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \geq \fint_I |z'(s) - z'_I|^2\,\mathrm{d}s = 1 - \Big|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\Big|^2. \tag{2.247}$$

In turn, this implies

$$|s_1 - s_2| \leq \frac{|z(s_1) - z(s_2)|}{\sqrt{1 - \|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}} = (1 + \varkappa_*)|z(s_1) - z(s_2)|, \tag{2.248}$$

provided $\varkappa_*$ is defined as in (2.230). This shows that, indeed, Σ is a $\varkappa_*$-CAC. □

Having discussed a number of basic properties of chord-arc curves in Proposition 2.10, we now wish to elaborate on the manner in which concrete examples of chord-arc curves may be produced. To set the stage for the subsequent discussion observe that, when specialized to the one-dimensional setting, (2.126)–(2.128) imply that for each function $f \in \mathrm{CMO}(\mathbb{R}, \mathcal{L}^1)$ we have

$$\lim_{\substack{-\infty < s_1 < s_2 < +\infty \\ |s_1| + |s_2| \to \infty}} \left(\fint_{s_1}^{s_2} \left| f - \fint_{s_1}^{s_2} f \, \mathrm{d}\mathcal{L}^1 \right| \mathrm{d}\mathcal{L}^1 \right) = 0, \tag{2.249}$$

and

$$\lim_{\substack{-\infty < s_1 < s_2 < +\infty \\ s_2 - s_1 \to 0^+}} \left(\fint_{s_1}^{s_2} \left| f - \fint_{s_1}^{s_2} f \, \mathrm{d}\mathcal{L}^1 \right| \mathrm{d}\mathcal{L}^1 \right) = 0. \tag{2.250}$$

These properties are relevant in the context of the next proposition, describing a wealth of examples of chord-arc curves in the plane.

Proposition 2.11 *Suppose $b \in CMO(\mathbb{R}, \mathcal{L}^1)$ is a real-valued function and consider the assignment*

$$\mathbb{R} \ni s \longmapsto z(s) := \int_0^s e^{ib(t)} \, dt \in \mathbb{C}. \tag{2.251}$$

If said assignment is injective then $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is, in fact, the arc-length parametrization of a chord-arc curve (which, in particular, passes through infinity in the plane).

Proof Introduce

$$F(s_1, s_2) := \frac{z(s_1) - z(s_2)}{s_1 - s_2} \quad \text{for each } s_1, s_2 \in \mathbb{R} \text{ with } s_1 \neq s_2. \tag{2.252}$$

Then, whenever $-\infty < s_1 < s_2 < +\infty$ and with b_I abbreviating $\fint_{s_1}^{s_2} b(t)\,\mathrm{d}t$, we may write

$$F(s_1, s_2) = \fint_{s_1}^{s_2} \mathrm{e}^{\mathrm{i}b(t)} \, \mathrm{d}t = \fint_{s_1}^{s_2} \left(\mathrm{e}^{\mathrm{i}b(t)} - \mathrm{e}^{\mathrm{i}b_I} \right) \mathrm{d}t + \mathrm{e}^{\mathrm{i}b_I}. \tag{2.253}$$

Recall that

$$|e^{i\theta} - 1| = \left| \int_0^\theta ie^{it}\, dt \right| \le \left| \int_0^\theta \left| ie^{it} \right| dt \right| = |\theta| \text{ for each } \theta \in \mathbb{R}. \tag{2.254}$$

Then, since b is real-valued, we may use (2.254) to estimate

$$\left| \fint_{s_1}^{s_2} \left(e^{ib(t)} - e^{ib_I} \right) dt \right| = \left| \fint_{s_1}^{s_2} \left(e^{i(b(t)-b_I)} - 1 \right) dt \right|$$
$$\le \fint_{s_1}^{s_2} \left| e^{i(b(t)-b_I)} - 1 \right| dt \le \fint_{s_1}^{s_2} |b(t) - b_I|\, dt. \tag{2.255}$$

According to (2.249)–(2.250) (written for b in place of f), the last integral in (2.255) converges to zero as either $|s_1| + |s_2| \to \infty$ or $s_2 - s_1 \to 0^+$. Since $\left| e^{ib_I} \right| = 1$, we conclude that

$$\lim_{\substack{-\infty < s_1 \neq s_2 < +\infty \\ |s_1|+|s_2| \to \infty}} |F(s_1, s_2)| = 1 \quad \text{and} \quad \lim_{\substack{-\infty < s_1 \neq s_2 < +\infty \\ |s_1 - s_2| \to 0^+}} |F(s_1, s_2)| = 1. \tag{2.256}$$

Given that, by assumption, the assignment $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is injective, we also have

$$F(s_1, s_2) \neq 0 \quad \text{whenever} \quad -\infty < s_1 \neq s_2 < +\infty. \tag{2.257}$$

From (2.256), (2.257), and the fact that $F : \{(s_1, s_2) \in \mathbb{R}^2 : s_1 \neq s_2\} \to \mathbb{C}$ is continuous, we conclude that there exists $c \in (0, 1)$ with the property that $|F(s_1, s_2)| \ge c$ for each $s_1, s_2 \in \mathbb{R}$ with $s_1 \neq s_2$. In view of (2.252), this implies

$$|s_1 - s_2| \le c^{-1} |z(s_1) - z(s_2)| \quad \text{for each} \quad s_1, s_2 \in \mathbb{R}. \tag{2.258}$$

In particular, this entails $\lim_{s \to \pm\infty} |z(s)| = \infty$. Also, the assignment $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is continuous, and it is assumed to be injective. Given that $|z'(s)| = \left| e^{ib(s)} \right| = 1$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$, since b is real-valued, it follows that $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is the arc-length parametrization of a Jordan curve in the plane which passes through infinity. □

Here is a version of Proposition 2.11 in which the membership of b to $\mathrm{CMO}(\mathbb{R}, \mathcal{L}^1)$ is replaced by the demand that $\|b\|_{L^\infty(\mathbb{R},\mathcal{L}^1)} < \frac{\pi}{2}$. In an interesting twist, this forces the image of (2.251) to be a Lipschitz graph.

Proposition 2.12 *If $b \in L^\infty(\mathbb{R}, \mathcal{L}^1)$ is a real-valued function with the property that $\|b\|_{L^\infty(\mathbb{R},\mathcal{L}^1)} < \frac{\pi}{2}$ then the assignment* (2.251) *is actually the arc-length parametrization of a Lipschitz graph in the plane* (*hence, in particular, a chord-arc curve*).

Proof Suppose there exists $\theta \in (0, \pi/2)$ such that $b(t) \in (-\theta, \theta)$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Since for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $z'(t) = e^{ib(t)} = \cos(b(t)) + i\sin(b(t))$ given that b is real-valued, it follows that

$$\operatorname{Re} z'(t) = \cos(b(t)) \geq \cos\theta > 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}. \tag{2.259}$$

Granted this, whenever $-\infty < s_1 < s_2 < +\infty$ we may estimate

$$\begin{aligned} |z(s_2) - z(s_1)| &\geq \operatorname{Re}\big(z(s_2) - z(s_1)\big) = \operatorname{Re}\int_{s_1}^{s_2} z'(t)\,dt = \int_{s_1}^{s_2} \operatorname{Re} z'(t)\,dt \\ &\geq \int_{s_1}^{s_2} \cos\theta\,dt = (\cos\theta)(s_2 - s_1), \end{aligned} \tag{2.260}$$

which, as in the end-game of the proof of Proposition 2.11, implies that the image of $z(\cdot)$ is a chord-arc curve Σ in the plane. As such, Proposition 2.10 applies and gives that if Ω denotes the region in $\mathbb{C}$ lying to the left of the curve Σ (relative to the orientation Σ inherits from its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$), then Ω is an Ahlfors regular domain whose topological boundary is Σ, and whose geometric measure theoretic outward unit normal ν is given at $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ by $\nu(z(s)) = -iz'(s)$. Consider next the constant vector field $h := (0, -1) \equiv -i$ in $\mathbb{C}$ and regard ν as a $\mathbb{R}^2$-valued function. Then, with $\langle \cdot, \cdot \rangle$ denoting the standard inner product in $\mathbb{R}^2$, we have

$$\begin{aligned} \big\langle \nu(z(s)), h(z(s)) \big\rangle &= \operatorname{Re}\big(i\nu(z(s))\big) \\ &= \operatorname{Re} z'(s) \geq \cos\theta > 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \end{aligned} \tag{2.261}$$

This goes to show that there exists a constant vector field which is transverse to Ω and, as a consequence of work in [59], we conclude that Ω is the upper-graph of a Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$. The desired conclusion now follows. □

Another sub-category of chord-arc curves is offered by graphs of real-valued BMO_1 functions defined on the real line.

Proposition 2.13 *Let $\varphi \in W^{1,1}_{loc}(\mathbb{R})$ be such that $\varphi' \in BMO(\mathbb{R}, \mathcal{L}^1)$ and consider its graph $\Sigma := \big\{\big(x, \varphi(x)\big) : x \in \mathbb{R}\big\} \subseteq \mathbb{R}^2$. Then Σ is a $\varkappa$-*CAC *corresponding to $\varkappa = \|\varphi'\|_{BMO(\mathbb{R},\mathcal{L}^1)}$.*

Proof Throughout, identify $\mathbb{R}^2$ with $\mathbb{C}$. Since functions in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ are locally absolutely continuous (cf., e.g., [85, Corollary 7.14, p. 223]), we conclude that Σ is a curve in the plane, with parametrization $\mathbb{R} \ni x \mapsto x + i\varphi(x) \in \Sigma$. Hence, Σ is a Jordan curve that passes through infinity in the plane. From [61, Proposition 2.25, p. 2616] we know that Σ is an Ahlfors regular set which, in light of (2.210) implies that the curve Σ is also locally rectifiable. Consider two arbitrary points $z_1, z_2 \in \Sigma$, say $z_1 := \big(a, \varphi(a)\big)$ and $z_2 := \big(b, \varphi(b)\big)$ for some $a, b \in \mathbb{R}$ with $a < b$, and denote by Σ_{z_1,z_2} the sub-arc of Σ with endpoints z_1, z_2. From [61, Proposition 2.25, p. 2616] we also know that the arc-length measure $\sigma := \mathcal{H}^1\lfloor\Sigma$ on the curve Σ satisfies

$$\ell(z_1, z_2) = \sigma(\Sigma_{z_1,z_2}) = \int_a^b \sqrt{1 + |\varphi'(x)|^2}\,\mathrm{d}x. \tag{2.262}$$

Observe that the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(t) := \sqrt{1+t^2}$ for each $t \in \mathbb{R}$ is Lipschitz, with Lipschitz constant ≤ 1, since $|F'(t)| = |t|/\sqrt{1+t^2} \leq 1$ for each $t \in \mathbb{R}$. Consequently, if we set

$$\varphi'_I := \fint_a^b \varphi'\,\mathrm{d}\mathcal{L}^1 = \frac{\varphi(b) - \varphi(a)}{b - a}, \tag{2.263}$$

then

$$\begin{aligned}
\int_a^b \sqrt{1 + |\varphi'(x)|^2}\,\mathrm{d}x &= \int_a^b F(\varphi'(x))\,\mathrm{d}x \\
&\leq \int_a^b |F(\varphi'(x)) - F(\varphi'_I)|\,\mathrm{d}x + (b-a)F(\varphi'_I) \\
&\leq \int_a^b |\varphi'(x) - \varphi'_I|\,\mathrm{d}x + (b-a)\sqrt{1 + (\varphi'_I)^2} \\
&\leq (b-a)\|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} + (b-a)\sqrt{1 + \Big(\frac{\varphi(b)-\varphi(a)}{b-a}\Big)^2} \\
&\leq |z_1 - z_2|\|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} + |z_1 - z_2| \\
&= \big(1 + \|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}\big)|z_1 - z_2|.
\end{aligned} \tag{2.264}$$

From (2.262) and (2.264) we therefore conclude that (2.216) holds for the choice $\varkappa := \|\varphi'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$, and the desired conclusion follows. □

Another basic link between chord-arc curves in the plane and the John-Nirenberg space BMO on the real line has been noted by R. Coifman and Y. Meyer. Specifically, [28] contains the following result: if $\Sigma \subseteq \mathbb{C}$ is a chord-arc curve then its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$ satisfies $z'(s) = \mathrm{e}^{ib(s)}$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ for some real-valued function $b \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ and, in the converse direction, for any given real-function $b \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ whose BMO semi-norm is sufficiently small, the function $\mathbb{R} \ni s \mapsto z(s) := \int_0^s \mathrm{e}^{ib(t)}\,\mathrm{d}t \in \mathbb{C}$ is the arc-length parametrization of a chord-arc curve (cf. also [29] for related results). Below we further elaborate on this last part of Coifman-Meyer's result. In particular, the analysis contained in our next proposition (which may be thought of as a quantitative version of Proposition 2.11) is going to be instrumental in producing a large variety of examples of δ-AR domains a little later (see Example 2.7).

Proposition 2.14 *Let $b \in BMO\,(\mathbb{R}, \mathcal{L}^1)$ be a real-valued function with*

$$\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)} < 1 \tag{2.265}$$

and introduce

$$\varkappa := \frac{\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}}{1 - \|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}} \in [0, \infty). \tag{2.266}$$

Define $z : \mathbb{R} \to \mathbb{C}$ by setting

$$z(s) := \int_0^s e^{ib(t)}\,dt \quad \textit{for each} \quad s \in \mathbb{R}. \tag{2.267}$$

Finally, consider $\Sigma := z(\mathbb{R})$, the image of $\mathbb{R}$ under the mapping $z(\cdot)$. Then the following statements are true.

(i) The set Σ is a $\varkappa$-CAC which contains the origin $0 \in \mathbb{C}$, and the mapping given by $\mathbb{R} \ni s \longmapsto z(s) \in \Sigma$ is its arc-length parametrization. In addition,

$$\|z'\|_{BMO\,(\mathbb{R},\mathcal{L}^1)} \leq 2\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)}. \tag{2.268}$$

(ii) Denote by Ω the region of the plane that is lying to the left of the curve Σ (relative to the orientation Σ inherits from its arc-length parametrization given by $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$). Then the set Ω is the image of the upper half-plane under a global bi-Lipschitz homeomorphism of $\mathbb{C}$, and

the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of Ω stay bounded as $\|b\|_{BMO\,(\mathbb{R},\mathcal{L}^1)} \longrightarrow 0^+$. (2.269)

Furthermore, the geometric measure theoretic outward unit normal ν of Ω satisfies

$$\|\nu\|_{BMO(\Sigma,\sigma)} \leq 4\varkappa. \tag{2.270}$$

(iii) With the piece of notation introduced in (2.97), if in place of (2.265) one now assumes

$$\|b\|_{BMO_2(\mathbb{R},\mathcal{L}^1)} < \sqrt{2}, \tag{2.271}$$

then Σ is a $\varkappa_2$-CAC with

$$\varkappa_2 := \frac{\|b\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}}{2 - \|b\|^2_{BMO_2(\mathbb{R},\mathcal{L}^1)}} \in [0, \infty). \tag{2.272}$$

As a consequence of this and (2.229)*, in such a scenario one has*

$$\|\nu\|_{BMO(\Sigma,\sigma)} \leq 2\sqrt{\varkappa_2(2+\varkappa_2)}. \tag{2.273}$$

Proof The fact that b is real-valued entails that $\mathrm{e}^{\mathrm{i}b(\cdot)} \in L^\infty(\mathbb{R}, \mathcal{L}^1)$. In turn, this membership guarantees that $z(\cdot)$ in (2.267) is a well-defined Lipschitz function on $\mathbb{R}$, with $z(0) = 0 \in \mathbb{C}$, and such that $z'(s) = \mathrm{e}^{\mathrm{i}b(s)}$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. In particular,

$$|z'(s)| = 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \tag{2.274}$$

We claim that the inequalities in (2.219) hold. To see this, for each $s_1, s_2 \in \mathbb{R}$ we write (keeping in mind that b is real-valued)

$$\begin{aligned} |z(s_1) - z(s_2)| &= \Big| \int_0^{s_1} \mathrm{e}^{\mathrm{i}b(t)}\,\mathrm{d}t - \int_0^{s_2} \mathrm{e}^{\mathrm{i}b(t)}\,\mathrm{d}t \Big| = \Big| \int_{s_2}^{s_1} \mathrm{e}^{\mathrm{i}b(t)}\,\mathrm{d}t \Big| \\ &\leq \Big| \int_{s_2}^{s_1} \big|\mathrm{e}^{\mathrm{i}b(t)}\big|\,\mathrm{d}t \Big| = |s_1 - s_2|, \end{aligned} \tag{2.275}$$

justifying the first inequality in (2.219). To prove the second inequality in (2.219), for each finite, non-trivial, sub-interval I of $\mathbb{R}$ introduce

$$b_I := \fint_I b(t)\,\mathrm{d}t, \qquad m_I := \mathrm{e}^{\mathrm{i}b_I}, \tag{2.276}$$

and note that the fact that b is real-valued implies $|m_I| = 1$. Also, $m_I^{-1} = \mathrm{e}^{-\mathrm{i}b_I}$. Assume $-\infty < s_1 < s_2 < +\infty$ and set $I := [s_1, s_2]$. We may then estimate

$$\begin{aligned} \big|z(s_1) - z(s_2) - m_I \cdot (s_1 - s_2)\big| &= \Big| \int_{s_1}^{s_2} (z'(t) - m_I)\,\mathrm{d}t \Big| \\ &= \Big| \int_{s_1}^{s_2} \big(z'(t)m_I^{-1} - 1\big)\,\mathrm{d}t \Big| = \Big| \int_{s_1}^{s_2} \big(\mathrm{e}^{\mathrm{i}(b(t)-b_I)} - 1\big)\,\mathrm{d}t \Big| \\ &\leq \int_{s_1}^{s_2} \big|\mathrm{e}^{\mathrm{i}(b(t)-b_I)} - 1\big|\,\mathrm{d}t \leq \int_{s_1}^{s_2} |b(t) - b_I|\,\mathrm{d}t \\ &= |s_1 - s_2| \fint_I |b(t) - b_I|\,\mathrm{d}t \leq |s_1 - s_2| \|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \end{aligned}$$

$$= \Big(\frac{\varkappa}{1+\varkappa}\Big)|s_1 - s_2|, \tag{2.277}$$

where we have used the fact that Lipschitz functions are locally absolutely continuous (hence, the fundamental theorem of calculus applies), as well as the elementary inequality from (2.254). From (2.277), we obtain

$$\begin{aligned} |s_1 - s_2| &= |m_I \cdot (s_1 - s_2)| \leq |z(s_1) - z(s_2)| + |z(s_1) - z(s_2) - m_I \cdot (s_1 - s_2)| \\ &\leq |z(s_1) - z(s_2)| + \Big(\frac{\varkappa}{1+\varkappa}\Big)|s_1 - s_2|, \end{aligned} \tag{2.278}$$

which then readily yields the second estimate in (2.219). In particular, (2.219) implies that $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$ is a bi-Lipschitz bijection. The argument so far shows that Σ is a $\varkappa$-CAC passing through the origin $0 \in \mathbb{C}$, and $\mathbb{R} \ni s \longmapsto z(s) \in \Sigma$ is its arc-length parametrization. To finish the treatment of the claims in item *(i)*, there remains to justify (2.268). To this end, given any finite interval $I \subset \mathbb{R}$, set $b_I := ⨍_I b(t)\,\mathrm{d}t \in \mathbb{R}$ and $m_I := \mathrm{e}^{\mathrm{i}b_I} \in S^1$ (with the two memberships being a consequence of the fact that b is real-valued). With $z'_I := ⨍_I z'(s)\,\mathrm{d}s \in \mathbb{C}$ we may then estimate (bearing in mind that $m_I^{-1} = \mathrm{e}^{-\mathrm{i}b_I}$ and the inequality in (2.254))

$$\begin{aligned} ⨍_I \big|z'(s) - z'_I\big|\,\mathrm{d}s &\leq 2 ⨍_I \big|z'(s) - m_I\big|\,\mathrm{d}s = 2 ⨍_I \big|z'(s)m_I^{-1} - 1\big|\,\mathrm{d}s \\ &= 2 ⨍_I \big|\mathrm{e}^{\mathrm{i}(b(s)-b_I)} - 1\big|\,\mathrm{d}s \leq 2 ⨍_I |b(s) - b_I|\,\mathrm{d}s \\ &\leq 2\|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}, \end{aligned} \tag{2.279}$$

and (2.268) readily follows from this. Next, all but the last claim in item *(ii)* are consequences of (2.227). The estimate in (2.270) is obtained by combining the first inequality in (2.228) with (2.268) and (2.266).

To deal with the claims in item *(iii)*, make the assumption that (2.271) holds and define $\varkappa_2$ as in (2.272). Whenever $-\infty < s_1 < s_2 < +\infty$ and $I := [s_1, s_2]$ we may estimate

$$\begin{aligned} s_2 - s_1 &\leq \sqrt{(s_2 - s_1)^2 + \Big|\int_{s_1}^{s_2} (b(t) - b_I)\,\mathrm{d}t\Big|^2} \\ &= \Big|(s_2 - s_1) + \mathrm{i}\int_{s_1}^{s_2} (b(t) - b_I)\,\mathrm{d}t\Big| \\ &= \Big|m_I \cdot (s_2 - s_1) + m_I \cdot \int_{s_1}^{s_2} \mathrm{i}(b(t) - b_I)\,\mathrm{d}t\Big| \end{aligned}$$

$$\leq |z(s_2) - z(s_1)| + \Big| z(s_2) - z(s_1) - m_I \cdot (s_2 - s_1) - m_I \cdot \int_{s_1}^{s_2} \mathrm{i}(b(t) - b_I)\, \mathrm{d}t \Big|. \tag{2.280}$$

Note that the last term above may be written as

$$\begin{aligned} &\Big| z(s_2) - z(s_1) - m_I \cdot (s_2 - s_1) - m_I \cdot \int_{s_1}^{s_2} \mathrm{i}(b(t) - b_I)\, \mathrm{d}t \Big| \\ &\qquad = \Big| \int_{s_1}^{s_2} \big(z'(t) - m_I - m_I \cdot \mathrm{i}(b(t) - b_I)\big)\, \mathrm{d}t \Big| \\ &\qquad = \Big| \int_{s_1}^{s_2} \big(z'(t) m_I^{-1} - 1 - \mathrm{i}(b(t) - b_I)\big)\, \mathrm{d}t \Big| \\ &\qquad = \Big| \int_{s_1}^{s_2} \big(\mathrm{e}^{\mathrm{i}(b(t)-b_I)} - 1 - \mathrm{i}(b(t) - b_I)\big)\, \mathrm{d}t \Big|. \end{aligned} \tag{2.281}$$

Also, for each $\theta \in \mathbb{R}$ we may use (2.254) to write

$$\begin{aligned} |\mathrm{e}^{\mathrm{i}\theta} - 1 - \mathrm{i}\theta| &= \Big| \int_0^\theta \mathrm{i}(\mathrm{e}^{\mathrm{i}t} - 1)\, \mathrm{d}t \Big| \leq \Big| \int_0^\theta \big|\mathrm{i}(\mathrm{e}^{\mathrm{i}t} - 1)\big|\, \mathrm{d}t \Big| \\ &\leq \Big| \int_0^\theta |t|\, \mathrm{d}t \Big| = \theta^2/2. \end{aligned} \tag{2.282}$$

From (2.280), (2.281), (2.282), (2.97), and (2.272) we then conclude that

$$\begin{aligned} s_2 - s_1 &\leq |z(s_2) - z(s_1)| + \frac{1}{2} \int_{s_1}^{s_2} |b(t) - b_I|^2\, \mathrm{d}t \\ &\leq |z(s_2) - z(s_1)| + \tfrac{1}{2}(s_2 - s_1) \|b\|^2_{\mathrm{BMO}_2(\mathbb{R}, \mathcal{L}^1)} \\ &= |z(s_2) - z(s_1)| + \Big(\frac{\varkappa_2}{1 + \varkappa_2}\Big)(s_2 - s_1). \end{aligned} \tag{2.283}$$

From (2.283) we conclude that the version of (2.219) with $\varkappa$ replaced by $\varkappa_2$ holds. In particular, Σ is a $\varkappa_2$-CAC. The proof of Proposition 2.14 is therefore complete. □

2.4 The Class of Delta-Flat Ahlfors Regular Domains

We begin by making the following definition which is central for the present work. This should be compared with [61, Definitions 4.7-4.9, p. 2690] where related, yet rather distinct, variants have been considered. Specifically, the definitions in [61] contain additional geometric hypotheses and are designed to work well when dealing with domains with compact boundaries (as opposed to the present endeavors, where we shall mostly consider domains with unbounded boundaries).

Definition 2.15 Consider a parameter $\delta > 0$. Call a nonempty, proper subset Ω of $\mathbb{R}^n$ a δ-`flat Ahlfors regular domain` (or δ-`flat AR domain`, or simply δ-`AR domain`) provided Ω is an Ahlfors regular domain (in the sense of Definition 2.4) whose geometric measure theoretic outward unit normal ν satisfies (with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$)

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta. \tag{2.284}$$

In the class of Ahlfors regular domains we always have $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1$ (as noted in (2.118)), so condition (2.284) is redundant when $\delta > 1$. We will primarily be interested in the case when δ is small. In particular, when $\delta \in (0, 1)$, Lemma 2.8 ensures that $\partial\Omega$ is an unbounded set.

Let us also note here that, as is visible from the first inequality in (2.119), whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta \in (0, 1)$ then its geometric measure theoretic outward unit normal ν satisfies (with the infimum taken over all surface balls $\Delta \subseteq \partial\Omega$)

$$\inf_{\Delta\subseteq\partial\Omega}\left|\fint_{\Delta}\nu\,d\sigma\right| > 1-\delta. \tag{2.285}$$

Conversely, given any Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, it follows from the second inequality in (2.119) that Ω is a δ-AR domain whenever

$$\delta > \sqrt{2}\sqrt{1-\inf_{\Delta\subseteq\partial\Omega}\left|\fint_{\Delta}\nu\,d\sigma\right|}, \tag{2.286}$$

where the infimum is taken over all surface balls $\Delta \subseteq \partial\Omega$.

The discussion surrounding (2.285)–(2.286) shows that the condition that

$$\text{the number } \inf_{\Delta\subseteq\partial\Omega}\left|\fint_{\Delta}\nu\,d\sigma\right| \text{ is sufficiently close to } 1 \tag{2.287}$$

is, in many regards, a good substitute for the demand that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is small.

Our next theorem describes some of the basic topological and geometric measure theoretic properties of sets in the class of δ-flat Ahlfors regular domains, with parameter $\delta \in (0, 1)$ small.

Theorem 2.5 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ satisfies $n \geq 2$, be a δ-flat Ahlfors regular domain (aka δ-AR domain), in the sense of Definition 2.15. Make the assumption that $\delta \in (0, 1)$ is sufficiently small relative to the Ahlfors regularity constant of $\partial\Omega$ and the dimension n.*

Then Ω is a two-sided NTA *domain, in particular, a* UR *domain satisfying a two-sided local John condition (hence also a two-sided cork screw condition). In all cases, the intervening constants may be controlled solely in terms of Ahlfors regularity constant of $\partial\Omega$ and the dimension n.*

In addition, Ω, $\overline{\Omega}$, $\partial\Omega$, $\mathbb{R}^n \setminus \overline{\Omega}$, and $\mathbb{R}^n \setminus \Omega$ are all unbounded connected sets, $\partial(\overline{\Omega}) = \partial\Omega$, $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial\Omega$, and $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$.

Finally, in the case when $n = 2$, both Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$ are simply connected.

Proof All claims made in the statement of the theorem are consequences of Corollary 2.2, Theorem 2.4, and Corollary 2.2. □

Examples and counterexamples of δ-AR domains in $\mathbb{R}^n$ are as follows.

Example 2.1 The set $\Omega := \mathbb{R}^n_+$ is a δ-AR domain for each $\delta > 0$. Indeed, the outward unit normal $\nu = -\mathbf{e}_n = (0, \dots, 0, -1)$ to Ω is constant, hence its BMO semi-norm vanishes. More generally, any half-space in $\mathbb{R}^n$, i.e., any set of the form

$$\begin{gathered}\Omega_{x_o,\xi} := \big\{x \in \mathbb{R}^n : \langle x - x_o, \xi\rangle > 0\big\} \\ \text{with } \ x_o \in \mathbb{R}^n \ \text{ and } \ \xi \in S^{n-1},\end{gathered} \tag{2.288}$$

is a δ-AR domain for each $\delta > 0$.

Consider next a sector of aperture $\theta \in (0, 2\pi)$ in the two-dimensional space, i.e., a planar set of the form

$$\begin{gathered}\Omega_\theta := \Big\{x \in \mathbb{R}^2 \setminus \{x_o\} : \tfrac{x-x_o}{|x-x_o|} \cdot \xi > \cos(\theta/2)\Big\} \\ \text{with } \ x_o \in \mathbb{R}^2, \ \ \theta \in (0, 2\pi), \ \ \text{and } \ \xi \in S^1,\end{gathered} \tag{2.289}$$

and abbreviate $\sigma_\theta := \mathcal{H}^1 \lfloor \partial\Omega_\theta$. Then a direct computation shows that the outward unit normal vector ν to Ω_θ, regarded as a complex-valued function, satisfies

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)} = |\cos(\theta/2)|. \tag{2.290}$$

Hence,

$$\Omega_\theta \text{ is a } \delta\text{-AR domain if and only if } \ \delta > |\cos(\theta/2)|. \tag{2.291}$$

One last example in the same spirit is offered by the cone of aperture $\theta \in (0, 2\pi)$ in $\mathbb{R}^n$ with vertex at the origin and axis along $\mathbf{e}_n$, i.e.,

$$\Omega_\theta := \Big\{x \in \mathbb{R}^n \setminus \{0\} : \frac{x_n}{|x|} > \cos(\theta/2)\Big\}$$

$$= \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \phi(x')\big\}, \tag{2.292}$$

where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is given by $\phi(x') := |x'| \cot(\theta/2)$ for each $x' \in \mathbb{R}^{n-1}$. If we abbreviate $\sigma_\theta := \mathcal{H}^{n-1} \lfloor \partial\Omega_\theta$, then a direct computation (using (2.295) below) shows that the outward unit normal vector ν to Ω_θ satisfies

$$\begin{aligned} &\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta, \sigma_\theta)]^n} = |\cos(\theta/2)|, \quad \text{hence once again} \\ &\Omega_\theta \text{ is a } \delta\text{-AR domain if and only if } \ \delta > |\cos(\theta/2)|. \end{aligned} \tag{2.293}$$

Example 2.2 If $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain for some $\delta > 0$, then $\mathbb{R}^n \setminus \overline{\Omega}$ is also a δ-AR domain (having the same topological and measure theoretic boundaries as Ω, and whose geometric measure theoretic outward unit normal is the opposite of the one for Ω). Also, we note that any rigid transformation of $\mathbb{R}^n$ preserves the class of δ-AR domains. One may also check from definitions that there exists a dimensional constant $c_n \in (0, \infty)$ with the property that if Ω is a δ-AR domain in $\mathbb{R}^n$ for some $\delta > 0$ then $\Omega \times \mathbb{R}$ is a $(c_n\delta)$-AR domain in $\mathbb{R}^{n+1}$.

Example 2.3 Given $\delta > 0$, the region $\Omega := \big\{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(x')\big\}$ above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ whose Lipschitz constant is $< 2^{-3/2}\delta$ is a δ-AR domain. To see this is indeed the case, it is relevant to note that

$$\begin{aligned} &F : \mathbb{R}^n \to \mathbb{R}^n \ \text{ defined for all } x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \\ &\qquad \text{as } \ F(x', x_n) := x + \phi(x')\mathbf{e}_n = \big(x', x_n + \phi(x')\big) \end{aligned} \tag{2.294}$$

is a bijective function with inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given at each point $y = (y', y_n)$ in $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ by $F^{-1}(y', y_n) = y - \phi(y')\mathbf{e}_n = \big(y', y_n - \phi(y')\big)$, and that both F, F^{-1} are Lipschitz functions with constant $\leq 1 + \|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}}$. Hence, Ω is the image of the upper half-space $\mathbb{R}^n_+$ under the bi-Lipschitz homeomorphism F, which also maps $\mathbb{R}^n_-$ onto $\mathbb{R}^n \setminus \overline{\Omega}$ and $\mathbb{R}^{n-1} \times \{0\}$ onto $\partial\Omega$. This goes to show that Ω is an open set satisfying a two-sided cork screw condition and with an Ahlfors regular boundary, hence also an Ahlfors regular domain (cf. (2.47)). To conclude that Ω is a δ-AR domain we need to estimate the BMO semi-norm of its geometric measure theoretic outward unit normal. Since this satisfies

$$\nu\big(x', \phi(x')\big) = \frac{(\nabla\phi(x'), -1)}{\sqrt{1 + |\nabla\phi(x')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } \ x' \in \mathbb{R}^{n-1}, \tag{2.295}$$

it follows that for $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ we have

$$\nu\big(x', \phi(x')\big) + \mathbf{e}_n = \left(\frac{\nabla\phi(x')}{\sqrt{1 + |\nabla\phi(x')|^2}}\,,\ 1 - \frac{1}{\sqrt{1 + |\nabla\phi(x')|^2}}\right) \tag{2.296}$$

$$= \left(\frac{\nabla\phi(x')}{\sqrt{1+|\nabla\phi(x')|^2}}, \frac{|\nabla\phi(x')|^2}{\sqrt{1+|\nabla\phi(x')|^2}(1+\sqrt{1+|\nabla\phi(x')|^2})} \right).$$

Therefore, with $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$, we may estimate

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} &= \|\nu + \mathbf{e}_n\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 2\,\|\nu + \mathbf{e}_n\|_{[L^\infty(\partial\Omega,\sigma)]^n} \\ &= 2^{3/2} \left\| \frac{|\nabla\phi|}{(1+|\nabla\phi|^2)^{1/4}(1+\sqrt{1+|\nabla\phi|^2})^{1/2}} \right\|_{L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \\ &\leq 2^{3/2}\,\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \delta. \end{aligned} \tag{2.297}$$

All things considered, this analysis establishes that $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain, with $\delta = O\Big(\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \Big)$ as $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \longrightarrow 0^+$. In addition, since the Lipschitz constants of the functions F, F^{-1} stay bounded when the Lipschitz constant of ϕ, i.e., $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$, stays bounded, we ultimately conclude that

> by taking $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$ sufficiently small, matters may be arranged so that the above set $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta > 0$ as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.298)

Example 2.4 To illustrate the scope of Example 2.5 discussed above, work in the two-dimensional setting and consider upper-graphs of piecewise linear functions with (relatively) small slopes. Concretely, fix a parameter $\varepsilon \in (0, \infty)$ and suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a function whose graph is a concatenation of line segments with slope belonging to $[-\varepsilon, \varepsilon]$. Then $\phi \in \mathscr{C}^0(\mathbb{R})$ and its distributional derivative ϕ' is a simple function taking values in the interval $[-\varepsilon, \varepsilon]$. Then

$$\phi' \in L^\infty(\mathbb{R}, \mathcal{L}^1) \ \text{ and } \ \|\phi'\|_{L^\infty(\mathbb{R},\mathcal{L}^1)} \leq \varepsilon. \tag{2.299}$$

As such, ϕ is a Lipschitz function. In particular, $\Omega := \big\{(x, y) \in \mathbb{R}^2 : y > \phi(x)\big\}$ is an Ahlfors regular domain. If ν denotes its geometric measure theoretic outward unit normal, and $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$, then (2.297) presently implies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq 2^{3/2}\varepsilon. \tag{2.300}$$

Granted this, from (2.298) we then conclude that

> given any $\delta \in (0, 1)$, by taking $\varepsilon \in (0, 2^{-3/2}\delta)$ ensures that the above set $\Omega \subseteq \mathbb{R}^2$ is a δ-AR domain with the Ahlfors regularity constant of $\partial\Omega$ bounded independently of δ. (2.301)

Finally, we note that if the graph of $\phi : \mathbb{R} \to \mathbb{R}$ is a concatenation of line segments with slope alternating between $+\varepsilon$ and $-\varepsilon$, then (2.300) together with (2.290) imply

$$\frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \leq \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq 2^{3/2}\varepsilon. \tag{2.302}$$

Example 2.5 Given any $\delta > 0$, the region $\Omega := \big\{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(x')\big\}$ above the graph of some BMO_1 function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, (namely, a function $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ with $\nabla\phi$ belonging to $\big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}$), satisfying (for some purely dimensional constant $C_n \in (1, \infty)$)

$$\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \min\{1, \delta/C_n\} \tag{2.303}$$

is a δ-AR domain. Indeed, BMO_1 domains are contained in the class of Zygmund domains (cf. [61, Proposition 3.15, p. 2637]) which, in turn, are NTA domains (cf. [66, Proposition 3.6, p. 94]). In particular, Ω satisfies a two-sided cork screw condition, hence $\partial_*\Omega = \partial\Omega$ (cf. (2.47)). From [61, Corollary 2.26, p. 2622] we also know that $\partial\Omega$ is an Ahlfors regular set. Finally, [61, Proposition 2.27, p. 2622] guarantees the existence of a purely dimensional constant $C \in (0, \infty)$ such that

$$\begin{aligned} &\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \\ &\quad \leq C\,\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\,\mathcal{L}^{n-1})]^{n-1}} \cdot \Big(1 + \|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\,\mathcal{L}^{n-1})]^{n-1}}\Big). \end{aligned} \tag{2.304}$$

Hence $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ if (2.303) is satisfied with $C_n := 2C$, proving that a Ω is indeed a δ-AR domain. In addition,

taking $\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\,\mathcal{L}^{n-1})]^{n-1}}$ small enough ensures that the above set $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta > 0$ as small as wanted, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.305)

To offer concrete, interesting examples and counterexamples pertaining to BMO_1, work in the two-dimensional setting, i.e., when $n = 2$. For a fixed arbitrary number $\varepsilon \in (0, \infty)$ consider the continuous odd function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\phi_\varepsilon(x) := \begin{cases} \varepsilon\, x\big(\ln|x| - 1\big) & \text{if } x \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{for each } x \in \mathbb{R}. \tag{2.306}$$

Then from [102, Exercise 2.127, p. 89] we know that the distributional derivative of this function is $\phi'_\varepsilon = \varepsilon \ln|\cdot|$. Hence, for some absolute constant $C \in (0, \infty)$,

$$\big\|\phi'_\varepsilon\big\|_{\mathrm{BMO}(\mathbb{R},\,\mathcal{L}^1)} \leq C\varepsilon \tag{2.307}$$

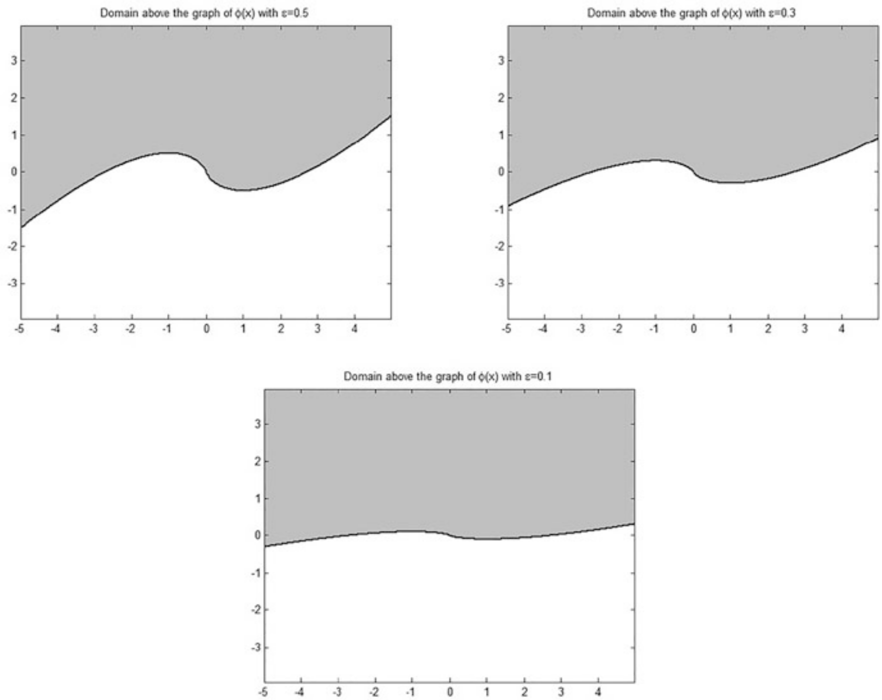

Fig. 2.1 The prototype of a non-Lipschitz δ-AR domain Ω_ε for which $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$ and such that the Ahlfors regularity constant of $\partial\Omega_\varepsilon$ and the local John constants of Ω_ε are uniformly bounded in ε

so ϕ_ε is indeed in BMO_1. This being said, ϕ_ε is not a Lipschitz function, so this example is outside the scope of Example 2.3. Consequently, the region Ω_ε lying above the graph of ϕ_ε is a non-Lipschitz δ-AR domain in the plane with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$ (as seen from (2.304) and (2.307)). See Fig. 2.1.

On the other hand, the distributional derivative of the function $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\psi_\varepsilon(x) := \begin{cases} \varepsilon\, x\big(\ln|x| - 1\big) & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0, \end{cases} \qquad \text{for each } \ x \in \mathbb{R}, \tag{2.308}$$

is $\psi_\varepsilon' = \varepsilon\big(\ln|\cdot|\big)\mathbf{1}_{(0,\infty)}$ which *fails* to be in $\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ (recall that the latter space is not stable under multiplication by cutoff functions). Hence, ψ_ε does *not* belong to BMO_1. In this vein, we wish to note that while the planar region $\widetilde{\Omega}_\varepsilon$ lying above the graph of ψ_ε continues to be an Ahlfors regular domain satisfying a two-sided local John condition for each $\varepsilon > 0$, its (complex-valued) geometric measure theoretic outward unit normal ν satisfies, due to the corner singularity at $0 \in \partial\widetilde{\Omega}_\varepsilon$ and (2.290) with $\theta = \pi/2$,

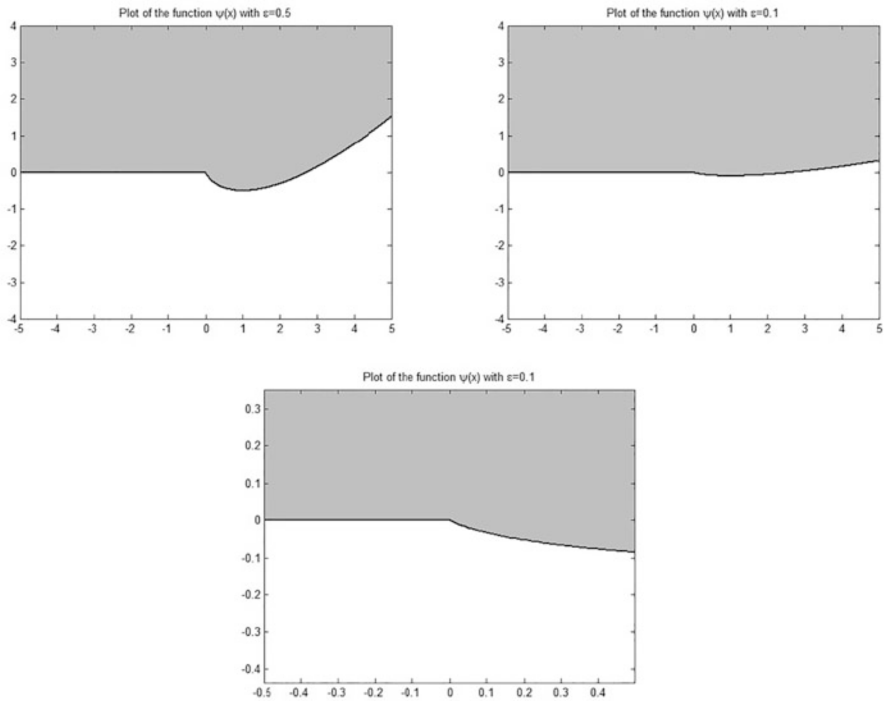

Fig. 2.2 A family $\{\widetilde{\Omega}_\varepsilon\}_{\varepsilon>0}$ of Ahlfors regular domains, with bounded Ahlfors regularity constants, which does not contain a δ-AR domain with $\delta \in (0, 1/\sqrt{2})$

$$\|\nu\|_{\mathrm{BMO}(\partial\widetilde{\Omega}_\varepsilon,\widetilde{\sigma}_\varepsilon)} \geq \frac{1}{\sqrt{2}} \quad \text{for each} \quad \varepsilon > 0, \tag{2.309}$$

where $\widetilde{\sigma}_\varepsilon := \mathcal{H}^1 \lfloor \partial\widetilde{\Omega}_\varepsilon$. Consequently, as $\varepsilon \to 0^+$, the set $\widetilde{\Omega}_\varepsilon$ never becomes a δ-AR domain if $\delta \in (0, 1/\sqrt{2})$. See Fig. 2.2.

Example 2.6 From [72, Theorem 2.1, p. 515] and [72, Remark 2.2, pp. 514-515] we know that there exist dimensional constants $\delta_n \in (0, \infty)$ and $C_n \in (0, \infty)$, with the property that if $\Omega \subseteq \mathbb{R}^n$ is a δ_o-Reifenberg flat domain, in the sense of [72, Definition 1.2, pp. 509–510] with $R = \infty$ and with $0 < \delta_o \leq \delta_n$, and if the surface measure $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ satisfies

$$\begin{gathered}\sigma\big(B(x,r) \cap \partial\Omega\big) \leq (1+\delta_o)\upsilon_{n-1} r^{n-1} \\ \text{for each } x \in \partial\Omega \text{ and } r > 0,\end{gathered} \tag{2.310}$$

(with υ_{n-1} denoting the volume of the unit ball in $\mathbb{R}^{n-1}$), then Ω is an Ahlfors regular domain whose geometric measure theoretic outward unit normal ν satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C_n\sqrt{\delta_o}. \tag{2.311}$$

See also [26, p. 11] and [123] in this regard. Consequently, given any number $\delta > 0$, any δ_o-Reifenberg flat domain with $0 < \delta_o < \min\{\delta_n, (\delta/C_n)^2\}$ which satisfies (2.310) is a δ-AR domain.

Example 2.7 Denote by Ω the region of the plane lying to one side of Σ, a $\varkappa$-CAC in $\mathbb{C}$. Then Proposition 2.10 implies that Ω is a δ-AR domain for any $\delta > 2\sqrt{\varkappa(2+\varkappa)}$.

To offer a concrete example, consider a real-valued function $b \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} < 1$ and define $z : \mathbb{R} \to \mathbb{C}$ by setting

$$z(s) := \int_0^s \mathrm{e}^{ib(t)}\,\mathrm{d}t \quad \text{for each} \quad s \in \mathbb{R}. \tag{2.312}$$

If $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ is the region of the plane to one side of the curve $\Sigma := z(\mathbb{R})$, then Proposition 2.10 and Proposition 2.14 imply that Ω is a connected Ahlfors regular domain with $\partial\Omega = \Sigma$, and whose geometric measure theoretic outward unit normal ν to Ω is given by

$$\nu(z(s)) = -i\mathrm{e}^{ib(s)} \quad \text{for } \mathcal{L}^1\text{-a.e.} \quad s \in \mathbb{R}. \tag{2.313}$$

In addition, if we set $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ then (2.270) gives

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq \frac{4\|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}{1 - \|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}. \tag{2.314}$$

As a consequence, Ω is a δ-AR domain in $\mathbb{R}^2$ for each $\delta \in (0, \infty)$ bigger than the number in the right-hand side of (2.314).

For instance, we may take b to be a small multiple of the logarithm on the real line, i.e.,

$$\begin{gathered} b(s) := \varepsilon \ln|s| \quad \text{for each} \quad s \in \mathbb{R} \setminus \{0\}, \\ \text{with} \quad 0 < \varepsilon < \|\ln|\cdot|\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \end{gathered} \tag{2.315}$$

(e.g., the computation on [55, p. 520] shows that $\|\ln|\cdot|\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \leq 3\ln(3/2)$, so taking $0 < \varepsilon < [3\ln(3/2)]^{-1} \approx 0.8221$ will do). Such a choice makes b a real-valued function with small BMO semi-norm which nonetheless maps $\mathbb{R} \setminus \{0\}$ onto $\mathbb{R}$. In view of the formula given in (2.313), this goes to show that Gauss' map $\Sigma \ni z \mapsto \nu(z) \in S^1$ is surjective, which may be interpreted as saying that the unit normal rotates arbitrarily much along the boundary. In particular, the chord-arc curve Σ produced in this fashion, which is actually the topological boundary of a δ-AR domain $\Omega \subseteq \mathbb{R}^2$ (with $\delta > 0$ which can be made as small as one pleases by taking $\varepsilon > 0$ appropriately small), fails to be a rotation of the graph of a function

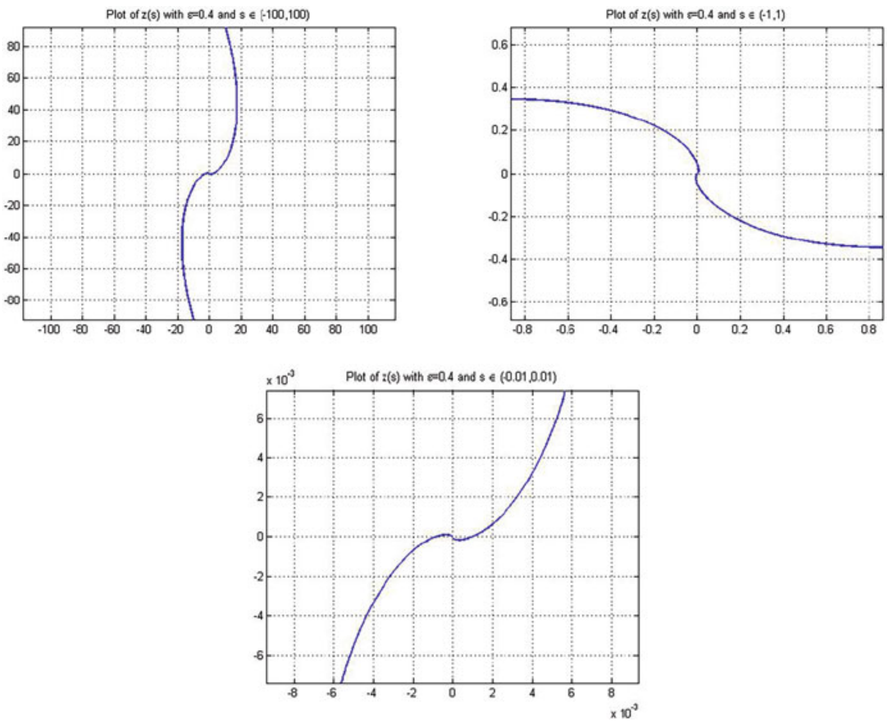

Fig. 2.3 Zooming in the curve $s \mapsto z(s)$ at the point $0 \in \mathbb{C}$

(even locally, near the origin). This being said, from Proposition 2.10 we know that

$$\text{the set } \Omega \subseteq \mathbb{R}^2 \text{ is actually bi-Lipschitz homeomorphic to the upper half-plane.} \tag{2.316}$$

Figure 2.3 depicts an unbounded δ-AR domain $\Omega \subseteq \mathbb{R}^2$ which is not the upper-graph of a function (in any system of coordinates isometric to the standard one in the plane). The set Ω is the region lying to one side of the curve $\Sigma = z(\mathbb{R})$ with $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ defined by the formula given in (2.312) for the real-valued function b as in (2.315) with $0 < \varepsilon < \|\ln|\cdot|\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$. As visible from (2.314), we have $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$.

In the above pictures we have taken $\varepsilon = 0.4 < \frac{1}{2}\|\ln|\cdot|\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$ and progressively zoomed in at the point $0 \in \partial\Omega$. The boundary of the set Ω is the plot of the curve $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ with

$$z(s) = \int_0^s e^{i\varepsilon \ln|t|}\,dt = \begin{cases} (i\varepsilon + 1)^{-1} s\, e^{i\varepsilon \ln|s|} & \text{if } s \in \mathbb{R} \setminus \{0\}, \\ 0 \in \mathbb{C} & \text{if } s = 0. \end{cases} \tag{2.317}$$

Here, $(\mathrm{i}\varepsilon+1)^{-1}$ is merely a complex constant, s is the scaling factor that determines how far $z(s)$ is from the origin (specifically, $|z(s)| = |s|/\sqrt{\varepsilon^2+1}$), and $\mathrm{e}^{\mathrm{i}\varepsilon \ln|s|}$ is the factor that determines how the two spirals (making up $\partial\Omega \setminus \{0\}$, namely $z((-\infty,0))$ and $z((0,+\infty))$) spin about the point $0 \in \mathbb{C}$. Note that $|z(s)|$ growths linearly (with respect to s) which is very fast compared to the spinning rate (which is logarithmic) and this is why we have chosen to zoom in at the point $0 \in \mathbb{C}$ in several distinct frames to get a better understanding of how $\partial\Omega$ looks near 0. The fact that $\partial\Omega$ is symmetric with respect to the origin is a direct consequence of $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ being odd. If $z(s) = r\mathrm{e}^{\mathrm{i}\theta}$ is the polar representation of (2.317) for $s \in (0,\infty)$ then, by taking $\omega := 2\pi - \arccos\big(\frac{1}{\sqrt{\varepsilon^2+1}}\big)$, it follows that $\theta = \omega + \varepsilon \ln|s|$ and that $r = |z(s)| = (\varepsilon^2+1)^{-1/2}|s| = (\varepsilon^2+1)^{-1/2}\mathrm{e}^{(\theta-\omega)/\varepsilon}$.

In polar coordinates, the curve $\Sigma_+ := z\big((0,+\infty)\big)$ has the equation $r = \alpha \mathrm{e}^{\beta\theta}$ with $\alpha := (\varepsilon^2+1)^{-1/2}\mathrm{e}^{-\omega/\varepsilon} \in (0,\infty)$ and $\beta := \varepsilon^{-1} \in (0,\infty)$ which identifies it precisely as a logarithmic spiral. In a similar fashion, the polar equation of the curve $\Sigma_- := z\big((-\infty,0)\big)$ is $r = \alpha\mathrm{e}^{\beta\theta}$ with $\alpha := (\varepsilon^2+1)^{-1/2}\mathrm{e}^{-(\omega+\pi)/\varepsilon} \in (0,\infty)$ and $\beta := \varepsilon^{-1} \in (0,\infty)$ which once again identifies it as a logarithmic spiral.

The MATLAB code that generated these pictures reads as follows:

```
s = [−100 : 0.001 : 100];
p = 0.4;
z=(1/(i*p+1.0))*s.*exp(i*p*log(abs(s)));
plot(real(z), imag(z),'LineWidth', 2), grid on, axis equal
```

Finally, we wish to elaborate on (2.316) and, in the process, get independent confirmation of (2.227) and (2.269). First, we observe that the δ-AR domain $\Omega \subseteq \mathbb{C}$ described above is the image of the upper half-plane $\mathbb{R}^2_+$ under map $F : \mathbb{C} \to \mathbb{C}$ defined for each $z \in \mathbb{C}$ by

$$F(z) := \begin{cases} (\mathrm{i}\varepsilon+1)^{-1} z\, \mathrm{e}^{\mathrm{i}\varepsilon \ln|z|} & \text{if } z \in \mathbb{C}\setminus\{0\}, \\ 0 \in \mathbb{C} & \text{if } z = 0. \end{cases} \tag{2.318}$$

Note that F is a bijective, odd function, with inverse $F^{-1} : \mathbb{C} \to \mathbb{C}$ given at each $\zeta \in \mathbb{C}$ by

$$F^{-1}(\zeta) = \begin{cases} (\mathrm{i}\varepsilon+1)\zeta\, \mathrm{e}^{-\mathrm{i}\varepsilon \ln(|\zeta|\sqrt{\varepsilon^2+1})} & \text{if } \zeta \in \mathbb{C}\setminus\{0\}, \\ 0 \in \mathbb{C} & \text{if } \zeta = 0. \end{cases} \tag{2.319}$$

Also, whenever $z_1, z_2 \in \mathbb{C}$ are such that $|z_1| \geq |z_2| > 0$ we may estimate

$$|F(z_1) - F(z_2)| \leq \frac{1}{\sqrt{\varepsilon^2+1}}\Big\{|z_1 - z_2| + |z_2|\big|\mathrm{e}^{\mathrm{i}\varepsilon\ln|z_1|} - \mathrm{e}^{\mathrm{i}\varepsilon\ln|z_2|}\big|\Big\} \tag{2.320}$$

and

$$\begin{aligned}\big|e^{i\varepsilon\ln|z_1|}-e^{i\varepsilon\ln|z_2|}\big| &= \big|e^{i\varepsilon(\ln|z_1|-\ln|z_2|)}-1\big| \leq \varepsilon\big|\ln|z_1|-\ln|z_2|\big| \\ &= \varepsilon\ln\big(\tfrac{|z_1|}{|z_2|}\big) \leq \varepsilon\big(\tfrac{|z_1|}{|z_2|}-1\big) = \varepsilon\big(\tfrac{|z_1|-|z_2|}{|z_2|}\big) \\ &\leq \varepsilon\tfrac{|z_1-z_2|}{|z_2|},\end{aligned} \tag{2.321}$$

using the fact that $|e^{i\theta}-1| \leq |\theta|$ for each $\theta \in \mathbb{R}$ (cf. (2.254)) and $0 \leq \ln x \leq x-1$ for each $x \in [1,\infty)$. From this we then eventually deduce that

$$|F(z_1)-F(z_2)| \leq \frac{\varepsilon+1}{\sqrt{\varepsilon^2+1}}|z_1-z_2| \quad \text{for all} \quad z_1, z_2 \in \mathbb{C}, \tag{2.322}$$

hence F is Lipschitz. The same type of argument also shows that F^{-1} is also Lipschitz, namely

$$|F^{-1}(\zeta_1)-F^{-1}(\zeta_2)| \leq (\varepsilon+1)\sqrt{\varepsilon^2+1}\,|\zeta_1-\zeta_2| \quad \text{for all} \quad \zeta_1, \zeta_2 \in \mathbb{C}, \tag{2.323}$$

so we ultimately conclude that $F : \mathbb{C} \to \mathbb{C}$ is an odd bi-Lipschitz homeomorphism of the complex plane. In summary,

> the δ-AR domain $\Omega \subseteq \mathbb{C}$ defined as the region of the complex plane lying to the left of the curve $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ defined in (2.317) is in fact the image of the upper half-plane $\mathbb{R}^2_+$ under the odd bi-Lipschitz homeomorphism $F : \mathbb{C} \to \mathbb{C}$ from (2.318). (2.324)

Note that F also maps the lower half-plane $\mathbb{R}^2_-$ onto $\mathbb{R}^2 \setminus \overline{\Omega}$, and $\mathbb{R} \times \{0\}$ onto $\partial\Omega$. This is in agreement with (2.227). Moreover, since the Lipschitz constants of F, F^{-1} stay bounded uniformly in $\varepsilon \in (0,1)$ (as is clear from (2.322), (2.323)) while, as noted earlier, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$, we see that (as predicted in (2.269))

> by taking $\varepsilon \in (0,1)$ sufficiently small, matters may be arranged so the above set $\Omega \subseteq \mathbb{R}^2$ is a δ-AR domain with $\delta > 0$ as small as one wishes, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.325)

Example 2.8 We may also construct examples of δ-AR domains exhibiting *multiple* spiral points. Specifically, suppose $-\infty < t_1 < t_2 < \cdots < t_{N-1} < t_N < +\infty$, for some $N \in \mathbb{N}$, and consider

$$b(t) := \varepsilon\sum_{j=1}^{N}\ln|t-t_j| \quad \text{for each} \quad t \in \mathbb{R} \setminus \{t_1, \dots, t_N\}, \tag{2.326}$$

for some sufficiently small $\varepsilon > 0$. Next, define $z : \mathbb{R} \to \mathbb{C}$ as in (2.312) for this choice of the function b. Then Proposition 2.14 and Proposition 2.10 imply that the region Ω in $\mathbb{R}^2$ lying to one side of the curve $\Sigma := z(\mathbb{R})$ is indeed a δ-AR domain and, in fact, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$. Moreover, from (2.313) and (2.326) we see that $\partial\Omega = \Sigma$ looks like a spiral at each of the points $z(t_1), \dots, z(t_N)$ (cf. Fig. 1.1). Yet, once again, there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Omega = F(\mathbb{R}^2_+)$, $\mathbb{R}^2 \setminus \overline{\Omega} = F(\mathbb{R}^2_-)$, and $\partial\Omega = F(\mathbb{R} \times \{0\})$ (cf. (2.227)). Also, (2.269) presently entails

by choosing $\varepsilon \in (0, 1)$ appropriately small, we may ensure that Ω is a δ-AR domain in $\mathbb{R}^2$ with $\delta > 0$ as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$. (2.327)

Example 2.9 We wish to note that the construction in Example 2.8 may be modified as to allow *infinitely many* spiral points. Specifically, assume $\{t_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}$ is a given sequence of real numbers and consider

$$0 < \lambda_j < 2^{-j} \min\Big\{1, \big\|\ln|\cdot - t_j|\big\|^{-1}_{L^1([-j,j],\mathcal{L}^1)}\Big\} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.328}$$

Also, suppose $0 < \varepsilon < \|\ln|\cdot|\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$ and define

$$b(t) := \varepsilon \sum_{j=1}^{\infty} \lambda_j \ln|t - t_j| \quad \text{for each} \quad t \in \mathbb{R} \setminus \{t_j\}_{j\in\mathbb{N}}. \tag{2.329}$$

The choice in (2.328) ensures that the above series converges absolutely in $L^1(K, \mathcal{L}^1)$ for any compact subset K of $\mathbb{R}$. This has two notable consequences. First, the series in (2.329) converges absolutely in a pointwise sense $\mathcal{L}^1$-a.e. in $\mathbb{R}$; in particular, b is well defined at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$ and takes real values. Second,

$$\begin{aligned}\|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} &\leq \varepsilon \sum_{j=1}^{\infty} \lambda_j \big\|\ln|\cdot - t_j|\big\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \\ &= \varepsilon \|\ln|\cdot|\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \sum_{j=1}^{\infty} \lambda_j < 1.\end{aligned} \tag{2.330}$$

Granted this, if we now define $z : \mathbb{R} \to \mathbb{C}$ as in (2.312) for this choice of the function b then Proposition 2.14 and Proposition 2.10 imply that the region Ω in $\mathbb{R}^2$ lying to one side of the curve $\Sigma := z(\mathbb{R})$ is a δ-AR domain with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$. In fact, there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ as in (2.227), and (2.269) holds. We claim that matters may be arranged so that $\partial\Omega = \Sigma$ develops a spiral at each of the points $\{z(t_j)\}_{j\in\mathbb{N}}$. To this end, start by making the assumption that the sequence $\{t_j\}_{j\in\mathbb{N}}$ does not have any finite accumulation points. Inductively, we may then select a sequence of small positive numbers $\{r_j\}_{j\in\mathbb{N}} \subseteq (0,1)$ with the property that the family of intervals $I_j := (t_j - r_j, t_j + r_j)$, $j \in \mathbb{N}$, are mutually disjoint. For each $j \in \mathbb{N}$ consider the nonempty compact set $K_j := [-j, j] \setminus I_j$ and, in addition to (2.328), impose the condition that

$$0 < \lambda_j < 2^{-j} \left\| \ln |\cdot - t_j| \right\|^{-1}_{L^\infty(K_j, \mathcal{L}^1)} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.331}$$

Pick now $j_o \in \mathbb{N}$ arbitrary. Then for each $t \in I_{j_o}$ decompose $b(t) = f(t) + g(t)$ where

$$f(t) := \varepsilon \lambda_{j_o} \ln |t - t_{j_o}| \quad \text{and} \quad g(t) := \varepsilon \sum_{j \in \mathbb{N}\setminus\{j_o\}} \lambda_j \ln |t - t_j|. \tag{2.332}$$

In view of (2.331), the series defining g converges uniformly on I_{j_o}, hence g is a continuous and bounded function on I_{j_o}. Since f is continuous and unbounded from below on $(t_{j_o}, t_{j_o} + r_{j_o})$, it follows that the restriction of b to $(t_{j_o}, t_{j_o} + r_{j_o})$ is continuous and unbounded from below. This implies that $b\big((t_{j_o}, t_{j_o} + r_{j_o})\big)$ contains an interval of the form $(-\infty, a_{j_o})$, for some $a_{j_o} \in \mathbb{R}$. Similarly, $b\big((t_{j_o} - r_{j_o}, t_{j_o})\big)$ contains an interval of the form $(-\infty, c_{j_o})$, for some $c_{j_o} \in \mathbb{R}$. Based on this and (2.313) we then conclude that the normal $\nu(z(t))$ completes infinitely many rotations on the unit circle as t approaches t_{j_o} either from the left or from the right. Hence, $\partial\Omega = \Sigma$ develops a spiral at the point $z(t_{j_o})$.

Example 2.10 Here we discuss a higher-dimensional analogue of (2.324). To set the stage, fix an integer $n \in \mathbb{N}$ with $n \geq 3$. With $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ denoting the region of the plane lying to the left of the curve $\mathbb{R} \ni s \longmapsto z(s) \in \mathbb{C}$ defined in (2.317), consider

$$\widetilde{\Omega} := \mathbb{R}^{n-2} \times \Omega \subseteq \mathbb{R}^n. \tag{2.333}$$

Bring back the odd bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{C} \equiv \mathbb{R}^2$ from (2.318), and consider

$$\begin{aligned} &\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^n \quad \text{defined as} \quad \widetilde{F}(x) := \big(x'', F(x_{n-1}, x_n)\big) \\ &\text{for each point} \quad x = (x'', x_{n-1}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{2.334}$$

Then one may check without difficulty that

$\widetilde{F}$ is an odd bi-Lipschitz homeomorphism of $\mathbb{R}^n$, and the set $\widetilde{\Omega}$ defined in (2.333) is, in fact, the image of the upper half-space $\mathbb{R}^n_+$ under the mapping $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^n$. (2.335)

From this and Lemma 2.2 we may then conclude that

$$\widetilde{\Omega} \text{ is an Ahlfors regular domain, with the Ahlfors regularity constant of } \partial\widetilde{\Omega} \text{ controlled solely in terms of the dimension } n. \tag{2.336}$$

Since, as noted earlier, F also maps the lower half-plane $\mathbb{R}^2_-$ onto $\mathbb{R}^2 \setminus \overline{\Omega}$, and $\mathbb{R} \times \{0\}$ onto $\partial\Omega$, it follows from (2.334) and (2.333) that

$$\widetilde{F}(\mathbb{R}^n_-) = \mathbb{R}^n \setminus \overline{\widetilde{\Omega}} \ \text{ and } \ \widetilde{F}(\mathbb{R}^{n-1} \times \{0\}) = \partial\widetilde{\Omega}. \tag{2.337}$$

From (2.25) and (2.336) we also know that

the geometric measure theoretic outward unit normal $\widetilde{\nu}$ to the set $\widetilde{\Omega} := \mathbb{R}^{n-2} \times \Omega \subseteq \mathbb{R}^n$ is given by $\widetilde{\nu}(x) = (0'', \nu(x_{n-1}, x_n))$ for $(\mathcal{L}^{n-2} \otimes \mathcal{H}^1)$-a.e. point $x = (x'', x_{n-1}, x_n) \in \partial\widetilde{\Omega} = \mathbb{R}^{n-2} \times \partial\Omega$, where $0'' \in \mathbb{R}^{n-2}$ and ν is the geometric measure theoretic outward unit normal to the set Ω. (2.338)

From this it readily follows that there exists some purely dimensional constant C_n in $(0, \infty)$ such that

$$\|\widetilde{\nu}\|_{[\mathrm{BMO}(\partial\widetilde{\Omega},\widetilde{\sigma})]^n} \leq C_n \|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{2.339}$$

By combining (2.339) with (2.314) we arrive at the conclusion that, for some purely dimensional constant $C_n \in (0, \infty)$,

$$\|\widetilde{\nu}\|_{[\mathrm{BMO}(\partial\widetilde{\Omega},\widetilde{\sigma})]^n} \leq C_n \frac{4\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}{1 - \|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}, \tag{2.340}$$

where $\widetilde{\sigma} := \mathcal{H}^{n-1}\lfloor\partial\widetilde{\Omega}$. As a consequence, $\widetilde{\Omega}$ is a δ-AR domain in $\mathbb{R}^n$ for each $\delta \in (0, \infty)$ bigger than the number in the right-hand side of (2.340). In particular, choosing the function b as in (2.315) allows us to conclude that $\widetilde{\Omega}$ is a δ-AR domain in $\mathbb{R}^n$ with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$.

In addition, since the Lipschitz constants of $\widetilde{F}$, $\widetilde{F}^{-1}$ stay bounded uniformly in the parameter $\varepsilon \in (0, 1)$ (as is clear from (2.334), (2.322), (2.323)) while, as just noted, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$, we see that

by taking $\varepsilon \in (0, 1)$ sufficiently small, matters may be arranged so that the set $\widetilde{\Omega} \subseteq \mathbb{R}^n$ defined in (2.333) is a δ-AR domain with $\delta > 0$ as small as one wishes, relative to the Ahlfors regularity constant of $\partial\widetilde{\Omega}$. (2.341)

Example 2.11 All sets considered so far have been connected. In the class of disconnected sets in the complex plane consider a double sector of arbitrary aperture $\theta \in (0, \pi)$, i.e., a set of the form

$$\Omega := \Big\{x \in \mathbb{R}^2 \setminus \{x_0\} : \Big|\tfrac{x-x_0}{|x-x_0|} \cdot \xi\Big| > \cos(\theta/2)\Big\}$$
$$\text{with } \; x_0 \in \mathbb{R}^2, \;\; \theta \in (0, \pi), \;\; \text{and} \;\; \xi \in S^1, \tag{2.342}$$

and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. Then simple symmetry considerations show that for each $r \in (0, \infty)$ the geometric measure theoretic outward unit normal ν to Ω satisfies $⨍_{B(x_o,r)\cap\partial\Omega} \nu \, d\sigma = 0$, hence

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \geq ⨍_{B(x_o,r)\cap\partial\Omega} \Big|\nu - ⨍_{B(x_o,r)\cap\partial\Omega} \nu \, d\sigma\Big| \, d\sigma$$
$$= ⨍_{B(x_o,r)\cap\partial\Omega} |\nu| \, d\sigma = 1. \tag{2.343}$$

As a consequence,

> the double sector Ω from (2.342) is a disconnected Ahlfors regular domain which satisfies a two-sided local John condition but fails to be a δ-AR domain for each $\delta \in (0, 1]$. (2.344)

We may even arrange matters so that the set in question has a disconnected boundary. Specifically, given any two distinct points $x_0, x_1 \in \mathbb{R}^2$, along with an angle $\theta \in (0, \pi)$, and a direction vector $\xi \in S^1$, such that

$$\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \xi < \cos(\theta/2), \tag{2.345}$$

consider

$$\Omega := \Big\{x \in \mathbb{R}^2 \setminus \{x_0\} : \frac{x - x_0}{|x - x_0|} \cdot \xi > \cos(\theta/2)\Big\} \tag{2.346}$$
$$\bigcup \Big\{x \in \mathbb{R}^2 \setminus \{x_1\} : \frac{x - x_1}{|x - x_1|} \cdot (-\xi) > \cos(\theta/2)\Big\}.$$

This is the union of two planar sectors with vertices at x_0 and x_1, axes along ξ and $-\xi$, and common aperture θ. The condition in (2.345) ensures that said sectors are disjoint, hence Ω is disconnected, with disconnected boundary. Note that if we set $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and ν stands for the geometric measure theoretic outward unit normal to Ω then

$$\lim_{r\to\infty} ⨍_{B(x_o,r)\cap\partial\Omega} \nu \, d\sigma = 0 \tag{2.347}$$

which, much as in (2.343), once again implies that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \geq 1$. Consequently,

$$\text{the set } \Omega \text{ from (2.346) is an Ahlfors regular domain satisfying a two-sided local John condition which is disconnected and has a disconnected boundary, and which fails to be a } \delta\text{-AR domain for each } \delta \in (0,1]. \tag{2.348}$$

Similar considerations apply virtually verbatim in $\mathbb{R}^n$ with $n \geq 2$ (working with cones in place of sectors).

These examples are particularly relevant in the context of Theorem 2.4.

2.5 The Decomposition Theorem

Our first result in this section, which slightly refines work in [61], identifies general geometric conditions on a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter so that the inner product between the integral average ν_Δ of outward unit normal ν to Ω in any given surface ball $\Delta \subseteq \partial\Omega$ and the "chord" $x - y$ with $x, y \in \Delta$ may be controlled in terms of the radius of said ball and the BMO semi-norm of the outward unit normal ν.

Proposition 2.15 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then there exists $C_* \in (0,\infty)$ depending only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$ such that for each dilation parameter $\lambda \in [1,\infty)$ one has*

$$\sup_{z\in\partial\Omega}\ \sup_{R>0}\ \sup_{x,y\in\Delta(z,\lambda R)} R^{-1}|\langle x-y, \nu_{\Delta(z,R)}\rangle| \leq C_*\lambda\big(1+\log_2\lambda\big)\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n}, \tag{2.349}$$

where $\nu_{\Delta(z,R)} := ⨍_{\Delta(z,R)} \nu\, d\sigma$ for each $z \in \partial\Omega$ and $R > 0$.

Proof Let $\delta_* \in (0,1)$ be the threshold associated with the set Ω as in Theorem 2.3. In particular, δ_* depends only on n and the Ahlfors regularity constant of $\partial\Omega$.

`Case I.` *Assume* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \delta_*$. For each location $z \in \partial\Omega$, each radius $R \in (0,\infty)$, each dilation parameter $\lambda \in [1,\infty)$, and any points $x, y \in \Delta(z,\lambda R)$ we then have

$$\begin{aligned} R^{-1}|\langle x-y, \nu_{\Delta(z,R)}\rangle| &\leq R^{-1}|x-y||\nu_{\Delta(z,R)}| \leq R^{-1}(2\lambda R) \\ &\leq C_*\lambda\big(1+\log_2\lambda\big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \end{aligned} \tag{2.350}$$

provided $C_* := 2\delta_*^{-1}$. This establishes (2.349) in this case.

`Case II.` *Assume* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_*$. In this scenario, (2.185) ensures that

$$\Omega \text{ satisfies a two-sided local John condition with constants which depend only on the dimension } n \text{ and the Ahlfors regularity constant of } \partial\Omega. \tag{2.351}$$

Granted this, [61, Corollary 4.15, pp. 2697–2698] applies and guarantees the existence of some constant $C \in (0, \infty)$ depending only on n and the Ahlfors regularity constant of $\partial\Omega$ such that

$$\sup_{x\in\partial\Omega}\sup_{R>0}\sup_{y\in\Delta(x,2R)} R^{-1}|\langle x-y, \nu_{\Delta(x,R)}\rangle| \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.352}$$

Fix a number $\lambda \in [1, \infty)$ along with an arbitrary point $z \in \partial\Omega$, $R > 0$, and $x, y \in \Delta(z, \lambda R)$. Then $|x - y| \leq 2\lambda R$, hence $y \in \Delta(x, 2\lambda R)$, so

$$\begin{aligned}
|\langle x-y, \nu_{\Delta(z,R)}\rangle| &\leq |\langle x-y, \nu_{\Delta(x,2\lambda R)}\rangle| + |x-y||\nu_{\Delta(x,2\lambda R)} - \nu_{\Delta(z,R)}| \\
&\leq C\lambda R\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + 2\lambda R|\nu_{\Delta(x,2\lambda R)} - \nu_{\Delta(z,3\lambda R)}| \\
&\qquad + 2\lambda R|\nu_{\Delta(z,3\lambda R)} - \nu_{\Delta(z,R)}| \\
&\leq CR\lambda\big(1+\log_2\lambda\big)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,,
\end{aligned} \tag{2.353}$$

by (2.352) and elementary estimates involving integral averages (cf. (2.103), (2.105)). After dividing the most extreme sides by R, then taking the supremum over all $z \in \partial\Omega$, $R > 0$, and $x, y \in \Delta(z, \lambda R)$, we arrive at (2.349). □

Remark 2.2 It is natural to attempt to quantify the global "tilt" of a given Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, envisioned as the maximal deviation of a chord $x - y$ with $x, y \in \Delta$ where Δ is an arbitrary surface ball on $\partial\Omega$ from being perpendicular to ν_Δ, the integral average in Δ of the geometric measure theoretic outward unit normal ν to Ω.

More specifically, we shall define the global tilt of Ω with amplitude $\lambda \in [1, \infty)$ to be

$$\mathbf{t}_\lambda(\Omega) := \sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(z,\lambda R)}\Big|\Big\langle \nu_{\Delta(z,R)}\,,\,\frac{x-y}{\lambda R}\Big\rangle\Big|, \tag{2.354}$$

where for each $z \in \partial\Omega$ and $R > 0$ we have set $\nu_{\Delta(z,R)} := ⨍_{\Delta(z,R)} \nu\,d\sigma$, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ playing the role of surface measure on $\partial\Omega$.

As an example, consider the cone of aperture $\theta \in (0, 2\pi)$ in $\mathbb{R}^n$ with vertex at the origin and axis along $\mathbf{e}_n$, i.e.,

$$\Omega_\theta := \Big\{x \in \mathbb{R}^n \setminus \{0\} : \frac{x_n}{|x|} > \cos(\theta/2)\Big\}. \tag{2.355}$$

Denote by ν the geometric measure theoretic outward unit normal to Ω_θ and abbreviate $\sigma_\theta := \mathcal{H}^{n-1}\lfloor\partial\Omega_\theta$. It may then be checked directly from the definition given in (2.354) that, on the one hand,

$$\mathbf{t}_\lambda(\Omega_\theta) = |\cos(\theta/2)| \quad \text{for each} \quad \lambda \in [1, \infty). \tag{2.356}$$

On the other hand, as noted in (2.293), the outward unit normal vector ν to Ω_θ satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^n} = |\cos(\theta/2)|. \tag{2.357}$$

In particular, in this special case we simply have

$$\mathbf{t}_\lambda(\Omega_\theta) = \|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^n} \quad \text{for each} \quad \lambda \in [1, \infty). \tag{2.358}$$

For a general Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, the best we can hope for is merely to control the global tilt $\mathbf{t}_\lambda(\Omega)$, for each fixed amplitude parameter $\lambda \in [1, \infty)$, in terms of the BMO-seminorm of the geometric measure theoretic outward unit normal ν to Ω.

Remarkably, this is possible, as (2.349) asserts that there exists some constant $C_* \in (0, \infty)$ depending only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$ such that for each amplitude parameter $\lambda \in [1, \infty)$ we have

$$\mathbf{t}_\lambda(\Omega) \leq C_*\big(1 + \log_2 \lambda\big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.359}$$

We continue by discussing a basic decomposition theorem. The general idea originated in [123, Proposition 5.1, p. 212] where such a decomposition result has been stated for surfaces of class $\mathscr{C}^2$, via a proof which makes essential use of smoothness, though the main quantitative aspects only depend on the rough character of said surface. A formulation in which the $\mathscr{C}^2$ smoothness assumption is replaced by Reifenberg flatness is stated in [73, Theorem 4.1, p. 398] (see also the comments on [26, p. 66]). A yet more potent version of such a decomposition result has been proved in [61, Theorem 4.16, p. 2701], starting with a different set of hypotheses which, a priori, do not specifically require the domain in question to be Reifenberg flat. The formulation of said result does require that the set in question satisfies a two-sided local John condition.

Below we present the most general variant of this result, valid in the class of Ahlfors regular domains $\Omega \subseteq \mathbb{R}^n$ for which the BMO semi-norm of its geometric measure theoretic outward unit normal is suitably small relative to the Ahlfors regularity constant of $\partial\Omega$. Stated as such, this result is well suited to the applications we have in mind.

Theorem 2.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then* *there exist* $C_0, C_1, C_2, C_3, C_4 \in (0, \infty)$*, depending only on the dimension n and the Ahlfors regularity constant of* $\partial\Omega$*, with the following significance.*

For each choice of a function

$$\phi : (0,1) \longrightarrow (0,\infty) \tag{2.360}$$

with

$$\lim_{t\to 0^+} \phi(t) = 0 \quad and \quad \lim_{t\to 0^+} \frac{\phi(t)}{t} \in (1,\infty], \tag{2.361}$$

there exists a threshold $\delta_* \in \big(0, \min\{1, 1/C_0\}\big)$*, depending only on the dimension n, the Ahlfors regularity constant of* $\partial\Omega$*, and the function* ϕ*, such that whenever*

$$\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n} < \delta < \delta_* \tag{2.362}$$

one has the following property:

For every location $x_0 \in \partial\Omega$ *and every scale* $r > 0$ *there exists a unit vector* $\vec{n}_{x_0,r} \in S^{n-1}$ *along with a Lipschitz function*

$$h : H(x_0,r) := \langle \vec{n}_{x_0,r}\rangle^{\perp} \to \mathbb{R} \ \ with \ \ \sup_{\substack{y_1,y_2\in H(x_0,r)\\ y_1\neq y_2}} \frac{|h(y_1)-h(y_2)|}{|y_1-y_2|} \leq C_0\phi(\delta), \tag{2.363}$$

whose graph

$$\mathcal{G} := \big\{x = x_0 + x' + t\vec{n}_{x_0,r} : x' \in H(x_0,r),\ t = h(x')\big\} \tag{2.364}$$

(*in the coordinate system* $x = (x', t) \Leftrightarrow x = x_0 + x' + t\vec{n}_{x_0,r}$, $x' \in H(x_0,r)$, $t \in \mathbb{R}$) *is a good approximation of* $\partial\Omega$ *inside the cylinder*

$$C(x_0,r) := \big\{x_0 + x' + t\vec{n}_{x_0,r} : x' \in H(x_0,r),\ |x'| < r,\ |t| < r\big\} \tag{2.365}$$

in the precise sense described below:

First, with $\triangle$ *denoting the symmetric set-theoretic difference and with* υ_{n-1} *denoting the volume of the unit ball in* $\mathbb{R}^{n-1}$*,*

$$\mathcal{H}^{n-1}\big(C(x_0,r)\cap\big(\partial\Omega\,\triangle\,\mathcal{G}\big)\big) \leq C_1\upsilon_{n-1}r^{n-1}e^{-C_2\phi(\delta)/\delta}. \tag{2.366}$$

Second, there exist two disjoint σ*-measurable subsets of* $\partial\Omega$*, call them* $G(x_0,r)$ *and* $E(x_0,r)$*, such that*

$$C(x_0,r)\cap\partial\Omega = G(x_0,r)\cup E(x_0,r), \tag{2.367}$$

$$G(x_0, r) \subseteq \mathcal{G}, \quad \sigma\big(E(x_0, r)\big) \leq C_1 \upsilon_{n-1} r^{n-1} e^{-C_2\phi(\delta)/\delta}. \tag{2.368}$$

Third, if $\Pi : \mathbb{R}^n \to H(x_0, r)$ *is defined by* $\Pi(x) := x'$ *for* $x = x_0 + x' + t\vec{n}_{x_0,r} \in \mathbb{R}^n$ *with* $x' \in H(x_0, r)$ *and* $t \in \mathbb{R}$*, then*

$$\begin{gathered}\Big|x - \big(x_0 + \Pi(x) + h(\Pi(x))\vec{n}_{x_0,r}\big)\Big| \leq 2C_0\phi(\delta) \cdot dist\big(\Pi(x), \Pi(G(x_0, r))\big) \\ \text{for each point } \; x \in E(x_0, r),\end{gathered} \tag{2.369}$$

and

$$C(x_0, r) \cap \partial\Omega \subseteq \Big\{x_0 + x' + t\vec{n}_{x_0,r} : |t| \leq C_0\delta r, \; x' \in H(x_0, r)\Big\}, \tag{2.370}$$

$$\Pi\big(C(x_0, r) \cap \partial\Omega\big) = \Big\{x' \in H(x_0, r) : |x'| < r\Big\}. \tag{2.371}$$

Fourth, if

$$\begin{aligned}C^+(x_0, r) &:= \Big\{x_0 + x' + t\,\vec{n}_{x_0,r} : x' \in H(x_0, r), \; |x'| < r, \; -r < t < -C_0\delta\, r\Big\}, \\ C^-(x_0, r) &:= \Big\{x_0 + x' + t\,\vec{n}_{x_0,r} : x' \in H(x_0, r), \; |x'| < r, \; C_0\delta\, r < t < r\Big\},\end{aligned} \tag{2.372}$$

(*having* $0 < \delta < \delta_* < 1/C_0$ *ensures that* $C^{\pm} \neq \varnothing$) *then*

$$C^+(x_0, r) \subseteq \Omega \;\; \text{and} \;\; C^-(x_0, r) \subseteq \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.373}$$

Fifth,

any line in the direction of $\vec{n}_{x_0,r}$ *passing through a point on* $G(x_0, r)$ *intersects* $\partial\Omega \cap C(x_0, r)$ *only at said point.* (2.374)

Sixth, with $\Delta(x_0, r) := B(x_0, r) \cap \partial\Omega$ *one has*

$$\begin{aligned}&\Big(1 - C_3\delta - C_1 exp\big(-C_2\phi(\delta)/\delta\big)\Big)\upsilon_{n-1} r^{n-1} \\ &\qquad \leq \sigma\big(\Delta(x_0, r)\big) \leq \Big(1 + C_3\phi(\delta) + C_1 exp\big(-C_2\phi(\delta)/\delta\big)\Big)\upsilon_{n-1} r^{n-1}.\end{aligned} \tag{2.375}$$

Finally, if $\widetilde{\nu}$ *is the unit normal vector to the Lipschitz graph* $\mathcal{G}$*, pointing toward the upper-graph of the function h then*

at $\mathcal{H}^{n-1}$*-a.e. point* $x \in \partial\Omega \cap \mathcal{G}$ *one has either* $\nu(x) = \widetilde{\nu}(x)$ *or* $\nu(x) = -\widetilde{\nu}(x)$*,* (2.376)

$$\nu(x) = \widetilde{\nu}(x) \;\; \text{at } \mathcal{H}^{n-1}\text{-a.e. point } \; x \in G(x_0, r), \tag{2.377}$$

$$\sigma\Big(\big\{x\in\mathcal{G}\cap\Delta(x_0,4r):\ \nu(x)=-\widetilde{\nu}(x)\big\}\Big)\leq C_4\cdot\phi(\delta)r^{n-1},\tag{2.378}$$

and

$$⨍_{\Delta(x_0,4r)}\Big(\sup_{y\in\mathcal{G}}|\nu(x)-\widetilde{\nu}(y)|\Big)\,d\sigma(x)\leq C_4\cdot\phi(\delta).\tag{2.379}$$

Before proving Theorem 2.6 we make a remark and record one of its immediate consequences in Corollary 2.3 below.

Remark 2.3 It is well known (cf., e.g., [47, Theorem 1, p. 251]) that there exists some $C_n\in(0,\infty)$ with the property that for each real-valued Lipschitz function $h:H\to\mathbb{R}$, where H is a hyperplane in $\mathbb{R}^n$ and each given $\varepsilon>0$ there exists $\widetilde{h}\in\mathscr{C}^1(H)$ with Lipschitz constant no larger than C_n times the Lipschitz constant of h such that

$$\mathcal{H}^{n-1}\Big(\big\{x\in H:\ h(x)\neq\widetilde{h}(x)\ \text{ or }\ (\nabla h)(x)\neq(\nabla\widetilde{h})(x)\big\}\Big)<\varepsilon.\tag{2.380}$$

Based on this, Theorem 2.6 is readily seen to self-improve to a version of itself in which the function in (2.363) is, additionally, of class $\mathscr{C}^1$.

Here is the corollary of Theorem 2.6 alluded to earlier.

Corollary 2.3 *Let $\Omega\subseteq\mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then there exists some $C\in(0,\infty)$ which depends only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\sup_{x\in\partial\Omega,\,r>0}\left|\frac{\sigma\big(\Delta(x,r)\big)}{\upsilon_{n-1}r^{n-1}}-1\right|\leq C\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n}\Big(1-\ln\|\nu\|_{[BMO(\partial\Omega,\sigma)]^n}\Big)\tag{2.381}$$

where υ_{n-1} stands for the volume of the unit ball in $\mathbb{R}^{n-1}$.

Proof In the context of (2.375) choose

$$\begin{aligned}\phi:(0,1)\to(0,\infty)\ \text{ given for each }\ t\in(0,1)\\ \text{by }\ \phi(t):=C_2^{-1}t\ln(1/t).\end{aligned}\tag{2.382}$$

This proves that there exists a threshold $\delta_*\in(0,1)$, depending only on the dimension n and the Ahlfors regularity constant of $\partial\Omega$, such that whenever (2.362) holds it follows that for each $x\in\partial\Omega$ and each $r>0$ we have

$$\begin{aligned}\big(1-(C_1+C_3)\delta\big)\upsilon_{n-1}r^{n-1}&\leq\sigma\big(\Delta(x,r)\big)\\ &\leq\big(1+(C_3/C_2)\delta\ln(1/\delta)+C_1\delta\big)\upsilon_{n-1}r^{n-1}.\end{aligned}\tag{2.383}$$

After sending $\delta \searrow \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$, this readily implies the estimate claimed in (2.381) (with $C := \max\{C_3/C_2, C_1 + C_3\}$) in this case. Finally, (2.381) is a simple consequence of the upper Ahlfors regularity of $\partial\Omega$ when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \delta_*$. □

We shall establish Theorem 2.6 by reasoning along the lines of the argument in [61, pp. 2703–2709], with (2.362) replacing the small local BMO assumption (which, in particular, frees us from having to restrict x_0 to a compact subset of $\partial\Omega$). A key observation is that, in the present context, the parameter R_* from [61, Theorem 4.16, p. 2701] (which limits the size of the scale r) may be taken to be $+\infty$.

Proof of Theorem 2.6 Throughout, for each given point $x \in \partial\Omega$ and each given radius $R > 0$ we agree to abbreviate $\Delta(x, R) := B(x, R) \cap \partial\Omega$ and also use the notation $\nu_{\Delta(x,R)} := ⨍_{\Delta(x,R)} \nu \, d\sigma$.

Assume (2.362) holds for some $\delta \in (0, \delta_*)$ with $\delta_* \in (0, 1/10)$, a threshold on which we are going to impose a number of other smallness conditions, to be specified later. For now, we note that Lemma 2.8 guarantees that $\partial\Omega$ is an unbounded set, and that

$$1 \geq \left| ⨍_{\Delta} \nu \, d\sigma \right| \geq \frac{9}{10} \quad \text{for each surface ball } \Delta \subseteq \partial\Omega. \tag{2.384}$$

Recall that the constant $C_* \in (0, \infty)$ appearing in the statement of Proposition 2.15 is controlled solely in terms of the Ahlfors regularity constant of $\partial\Omega$ and the dimension n. Keeping this in mind, from (2.349) used with $\lambda = 4$ we see that

$$\sup_{R>0} \sup_{x\in\partial\Omega} \sup_{y\in\Delta(x,4R)} R^{-1} |\langle x - y, \nu_{\Delta(x,R)} \rangle| \leq 12 C_* \delta \tag{2.385}$$

with $C_* \in (0, \infty)$ depending only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n. Choose

$$C_0 := \max \left\{ 14 C_* + 4, 60 C_* \right\} \tag{2.386}$$

and, for the remainder of the proof, make the assumption that

$$\delta_* \in \big(0, \min\{1/10, 1/C_0\}\big) \tag{2.387}$$

and that δ_* is also small enough, depending on ϕ, so that

$$\delta \leq \phi(\delta) \leq (14 C_* + 4)^{-1} \quad \text{for all } \delta \in (0, \delta_*). \tag{2.388}$$

That (2.388) may be accommodated is ensured by (2.361). (The choice made in (2.386) as well as the nature of the right-most expression in (2.388) are dictated by future considerations; see (2.418).)

To proceed, for each $x \in \partial\Omega$ and $R > 0$ set

$$\nu^*_{x,R}(y) := \sup_{0<\rho<R} \fint_{\Delta(y,\rho)} |\nu(z) - \nu_{\Delta(x,2R)}| \, d\sigma(z), \qquad \forall\, y \in \partial\Omega. \tag{2.389}$$

Then (2.389) implies that for each $x \in \partial\Omega$ and $R > 0$ we have

$$\nu^*_{x,R}(y) \leq \mathcal{M}\Big(|\nu - \nu_{\Delta(x,2R)}| \cdot \mathbf{1}_{\Delta(x,2R)}\Big)(y), \qquad \forall\, y \in \Delta(x, R), \tag{2.390}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator on $\partial\Omega$. For further reference let us also note that Lebesgue's Differentiation Theorem and (2.389) imply that

$$\begin{gathered} \text{for each fixed } x \in \partial\Omega \text{ and } R > 0 \text{ we have} \\ |\nu(y) - \nu_{\Delta(x,2R)}| \leq \nu^*_{x,R}(y) \ \text{ for } \sigma\text{-a.e. } \ y \in \partial\Omega. \end{gathered} \tag{2.391}$$

Henceforth, fix a location $x_0 \in \partial\Omega$ along with a scale $r > 0$. From (2.384) we know that

$$\tfrac{9}{10} \leq |\nu_{\Delta(x_0,2r)}| \leq 1. \tag{2.392}$$

We may also conclude from (2.384) that

$$\vec{n}_{x_0,r} := \frac{\nu_{\Delta(x_0,4r)}}{|\nu_{\Delta(x_0,4r)}|} \tag{2.393}$$

is a well-defined unit vector in $\mathbb{R}^n$. Consider

$$H(x_0, r) := \big\{x \in \mathbb{R}^n : \langle x, \vec{n}_{x_0,r}\rangle = 0\big\} \tag{2.394}$$

and introduce a new system of coordinates in $\mathbb{R}^n$ by setting

$$x = (\zeta, t) \Longleftrightarrow x = x_0 + t\,\vec{n}_{x_0,r} + \zeta, \quad t \in \mathbb{R},\ \zeta \in H(x_0, r). \tag{2.395}$$

We agree to write $\zeta(x), t(x)$ in place of ζ, t whenever necessary to stress the dependence of the new coordinates on the point $x \in \mathbb{R}^n$. Let us also define the projection

$$\Pi : \mathbb{R}^n \to H(x_0, r) \ \text{ with } \ \Pi(x) := \zeta \ \text{ for each } \ x = (\zeta, t) \in \mathbb{R}^n. \tag{2.396}$$

Finally, consider the cylinder $C(x_0, r)$ defined as in (2.365) and, with the function ϕ as in (2.360)–(2.361), introduce

$$G(x_0,r) := \big\{x \in C(x_0,r)\cap\partial\Omega :\ \nu^*_{x_0,2r}(x) \le \phi(\delta)\big\},$$
$$E(x_0,r) := \big(C(x_0,r)\cap\partial\Omega\big)\setminus G(x_0,r). \tag{2.397}$$

Since $C(x_0,r) \subseteq B(x_0,\sqrt{2}r)$ (as seen from its definition), it follows from (2.397) that $G(x_0,r)$, $E(x_0,r)$ are disjoint σ-measurable subsets of $\Delta(x_0,\sqrt{2}r)$, satisfying $G(x_0,r)\cup E(x_0,r) = C(x_0,r)\cap\partial\Omega$. In particular, (2.367) holds.

Next, we claim that there exist $c, C \in (0,\infty)$, which depend only on n and the Ahlfors regularity constant of $\partial\Omega$, with the property that

$$⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,2r}\big)\,d\sigma \le C. \tag{2.398}$$

Granted this, we may then conclude that

$$\exp\big(c\,\phi(\delta)/\delta\big)\,\frac{\sigma\big(E(x_0,r)\big)}{\sigma\big(\Delta(x_0,2r)\big)} \le \frac{1}{\sigma\big(\Delta(x_0,2r)\big)}\int_{E(x_0,r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,2r}\big)\,d\sigma \le C. \tag{2.399}$$

This implies

$$\sigma\big(E(x_0,r)\big) \le C\exp\big(-c\,\phi(\delta)/\delta\big)\sigma\big(\Delta(x_0,2r)\big)$$
$$\le 2^{n-1}C_A C\,r^{n-1}\cdot\exp\big(-c\,\phi(\delta)/\delta\big), \tag{2.400}$$

where C_A is the Ahlfors regularity constant of $\partial\Omega$. In particular, the estimate claimed in (2.368) follows as long as

$$C_2 := c \ \text{ and } \ C_1 \ge 2^{n-1}C_A C/\upsilon_{n-1}. \tag{2.401}$$

To justify the claim made in (2.398), let us abbreviate

$$f := \mathcal{M}\big(|\nu - \nu_{\Delta(x_0,4r)}|\cdot\mathbf{1}_{\Delta(x_0,4r)}\big) \tag{2.402}$$

and note that, thanks to (2.390) with $R := 2r$, this entails

$$\nu^*_{x_0,2r}(x) \le f(x) \ \text{ whenever } \ x \in \Delta(x_0,2r). \tag{2.403}$$

We also make the sub-claim that there exist $A_1, A_2 \in (0,\infty)$, depending only on n and the Ahlfors regularity constant of $\partial\Omega$, such that for each $p \in [1,\infty)$ we have

$$⨍_{\Delta(x_0,4r)} |\nu(x) - \nu_{\Delta(x_0,4r)}|^p\,d\sigma(x) \le A_1\Gamma(p+1)\Big(A_2\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^p, \tag{2.404}$$

where $\Gamma(t) := \int_0^\infty \lambda^{t-1}e^{-\lambda}\,d\lambda$ for all $t \in (0,\infty)$ is the classical Gamma function. Taking this inequality for granted for the time being, we now proceed to show that

(2.398) holds for the choice

$$c := 2^{-1} A_2^{-1} \in (0, \infty) \tag{2.405}$$

and with $C \in (0, \infty)$ to be determined momentarily (see (2.409)). To implement this plan, use (2.403) plus a change of variables, then expand the exponential function into an infinite power series to write

$$\begin{aligned}
&⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,r}\big)\,\mathrm{d}\sigma \leq ⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,f\big)\,\mathrm{d}\sigma \\
&\quad = \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_0^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) : \exp\big(c\,\delta^{-1}\,f(x)\big) > \lambda\big\}\Big)\,\mathrm{d}\lambda \\
&\quad \leq 1 + \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_1^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) : \exp\big(c\,\delta^{-1}\,f(x)\big) > \lambda\big\}\Big)\,\mathrm{d}\lambda \\
&\quad = 1 + \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_0^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) : c\,\delta^{-1}\,f(x) > s\big\}\Big)\mathrm{e}^s\,\mathrm{d}s \\
&\quad \leq e + \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \sum_{k=0}^\infty \frac{1}{k!} \int_1^\infty \sigma\Big(\big\{x \in \Delta(x_0,2r) : f(x) > s\,\delta/c\big\}\Big)s^k\,\mathrm{d}s.
\end{aligned} \tag{2.406}$$

To continue, fix an arbitrary integrability exponent $p \in [2, \infty)$ along with an arbitrary number $s \in (0, \infty)$. Chebysheff's inequality, the L^p-boundedness of the Hardy–Littlewood maximal operator (with bounds independent of p, as seen by interpolation), and (2.402) then allow us to estimate

$$\begin{aligned}
&\frac{\sigma\Big(\{x \in \Delta(x_0,2r) : f(x) > s\,\delta/c\}\Big)}{\sigma\big(\Delta(x_0,2r)\big)} \\
&\qquad \leq \Big(\frac{c}{s\,\delta}\Big)^p ⨍_{\Delta(x_0,2r)} f(x)^p\,\mathrm{d}\sigma(x) \\
&\qquad \leq \Big(\frac{c}{s\,\delta}\Big)^p \frac{1}{\sigma\big(\Delta(x_0,2r)\big)} \int_{\partial\Omega} \mathcal{M}\big(|\nu - \nu_{\Delta(x_0,4r)}| \cdot \mathbf{1}_{\Delta(x_0,4r)}\big)(x)^p\,\mathrm{d}\sigma(x) \\
&\qquad \leq \Big(\frac{c}{s\,\delta}\Big)^p \frac{C'}{\sigma\big(\Delta(x_0,2r)\big)} \int_{\partial\Omega} \big(|\nu(x) - \nu_{\Delta(x_0,4r)}| \cdot \mathbf{1}_{\Delta(x_0,4r)}(x)\big)^p\,\mathrm{d}\sigma(x) \\
&\qquad \leq C''\Big(\frac{c}{s\,\delta}\Big)^p ⨍_{\Delta(x_0,4r)} |\nu(x) - \nu_{\Delta(x_0,4r)}|^p\,\mathrm{d}\sigma(x),
\end{aligned} \tag{2.407}$$

where $C', C'' \in (0, \infty)$ depend only on n and the Ahlfors regularity constant of $\partial\Omega$. Combine (2.404), (2.407), (2.405) and recall that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ to obtain

$$\frac{\sigma\Big(\{x \in \Delta(x_0, 4r) : f(x) > s\,\delta/c\}\Big)}{\sigma\big(\Delta(x_0, 4r)\big)} \leq C''\, A_1 \Gamma(p+1)\Big(\frac{c\, A_2}{s}\Big)^p$$
$$= C''\, A_1 \Gamma(p+1)\Big(\frac{1}{2s}\Big)^p, \tag{2.408}$$

for each $p \in [2, \infty)$ and each $s \in (0, \infty)$. Utilizing (2.408), in which we take $p := k + 2$ with $k = 0, 1, \dots,$ back into (2.406) then yields (upon noting that $\Gamma(k + 3) = (k + 2)!$)

$$⨍_{\Delta(x_0,2r)} \exp\big(c\,\delta^{-1}\,\nu^*_{x_0,4r}\big)\,\mathrm{d}\sigma \tag{2.409}$$
$$\leq e + C''\, A_1 \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2^{k+2}}\Big(\int_1^{\infty}\frac{\mathrm{d}s}{s^2}\Big) =: C < \infty.$$

This finishes the proof of (2.398), modulo that of (2.404). As regards the latter, we use following the John-Nirenberg level set estimate with exponential bound from (2.94). This ensures that there exist some large constant $A \in (0, \infty)$ and some small constant $a \in (0, \infty)$, both depending only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n, such that

$$\frac{\sigma\Big(\{x \in \Delta(x_0, 4r) : |\nu(x) - \nu_{\Delta(x_0,4r)}| > \lambda\}\Big)}{\sigma\big(\Delta(x_0, 4r)\big)} \leq A \cdot \exp\Big(\frac{-a\lambda}{\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}}\Big) \tag{2.410}$$

for every $\lambda > 0$. In turn, (2.410) and a natural change of variables permit us to write

$$⨍_{\Delta(x_0,4r)} |\nu(x) - \nu_{\Delta(x_0,4r)}|^p\,\mathrm{d}\sigma(x)$$
$$= p\int_0^{\infty} \lambda^{p-1}\frac{\sigma\Big(\big\{x \in \Delta(x_0, 4r) : |\nu(x) - \nu_{\Delta(x_0,4r)}| > \lambda\big\}\Big)}{\sigma\big(\Delta(x_0, 4r)\big)}\,\mathrm{d}\lambda$$
$$\leq Ap\int_0^{\infty} \lambda^{p-1}\exp\Big(\frac{-a\lambda}{\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}}\Big)\,\mathrm{d}\lambda$$
$$= Ap\Big(a^{-1}\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^p \int_0^{\infty} t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

$$= ApΓ(p)\Big(a^{-1}\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^p. \tag{2.411}$$

Since $p\,\Gamma(p) = \Gamma(p+1)$, this justifies (2.404) with $A_1 := A$ and $A_2 := a^{-1}$, and concludes the proof of (2.398).

We now turn to the task of constructing the Lipschitz function h. As a preliminary matter, we remark that

$$\begin{aligned} &|\langle x - y, \nu_{\Delta(x_0,4r)}\rangle| \leq \big(6C_*\delta + \nu^*_{x_0,2r}(x)\big)|x - y| \\ &\qquad \text{for each } x \in \partial\Omega \text{ and } y \in \Delta(x, 4r). \end{aligned} \tag{2.412}$$

To justify this, observe that (2.412) is trivially true when $x = y$, so it suffices to consider the case when $x \in \partial\Omega$ and $y \in \Delta(x, 4r)$ satisfy $x \neq y$. Assuming this is the case, based on (2.385) (used with $R := |x - y|/2 > 0$) and (2.389) we may write

$$\begin{aligned} |\langle x - y, \nu_{\Delta(x_0,4r)}\rangle| &\leq |\langle x - y, \nu_{\Delta(x,|x-y|/2)}\rangle| + |x - y||\nu_{\Delta(x,|x-y|/2)} - \nu_{\Delta(x_0,4r)}| \\ &\leq 6C_*\delta\,|x - y| + |x - y| ⨍_{\Delta(x,|x-y|/2)} |\nu - \nu_{\Delta(x_0,4r)}|\,d\sigma \\ &\leq \big(6C_*\delta + \nu^*_{x_0,2r}(x)\big)|x - y|, \end{aligned} \tag{2.413}$$

as desired. Moving on, observe from (2.395) that

$$t(x) = \langle x - x_0, \vec{n}_{x_0,r}\rangle \quad \text{for each} \quad x \in \mathbb{R}^n. \tag{2.414}$$

In concert, (2.414), (2.392)–(2.393), (2.412), (2.397), and (2.388) then allow us to control

$$\begin{aligned} |t(x) - t(y)| = |\langle x - y, \vec{n}_{x_0,r}\rangle| &\leq \tfrac{10}{9}|\langle x - y, \nu_{\Delta(x_0,4r)}\rangle| \\ &\leq \tfrac{10}{9}\big(6C_*\delta + \phi(\delta)\big)|x - y| \\ &\leq \tfrac{10}{9}(6C_* + 1)\phi(\delta)|x - y| \\ &\leq (7C_* + 2)\phi(\delta)|x - y| \\ &\qquad \text{whenever } x \in G(x_0, r) \text{ and } y \in \Delta(x, 4r). \end{aligned} \tag{2.415}$$

In turn, since for each $x, y \in \mathbb{R}^n$ we have (see (2.395))

$$\zeta(x) - \zeta(y) = x - y - \big(t(x) - t(y)\big)\vec{n}_{x_0,r}, \tag{2.416}$$

this permits us to estimate

$$|\zeta(x)-\zeta(y)| \geq |x-y| - |t(x)-t(y)| \geq \big(1-(7C_*+2)\phi(\delta)\big)|x-y|,$$
$$\text{for each } x \in G(x_0,r) \text{ and each } y \in \Delta(x,4r). \tag{2.417}$$

Combining (2.415) and (2.417) (while keeping (2.388) and (2.386) in mind) then proves that

$$\begin{aligned} |t(x)-t(y)| &\leq \frac{(7C_*+2)\phi(\delta)}{1-(7C_*+2)\phi(\delta)}|\zeta(x)-\zeta(y)| \\ &\leq (14C_*+4)\phi(\delta)|\zeta(x)-\zeta(y)| \\ &\leq C_0\phi(\delta)|\zeta(x)-\zeta(y)|, \end{aligned} \tag{2.418}$$
$$\text{for each } \ x \in G(x_0,r) \ \text{ and } \ y \in \Delta(x,4r).$$

We now claim that

$$\begin{gathered} \text{if } x \in C(x_0,r) \cap \partial\Omega \text{ and } \Pi(x) \in \Pi\big(G(x_0,r)\big) \\ \text{then } x \in G(x_0,r). \end{gathered} \tag{2.419}$$

Indeed, assume $x \in C(x_0,r) \cap \partial\Omega$ and $y \in G(x_0,r)$ are such that $\Pi(x) = \Pi(y)$. In view of (2.396), the latter condition means $\zeta(x) = \zeta(y)$. Since $x, y \in C(x_0,r)$, it follows that $|y-x| \leq \operatorname{diam}\big(C(x_0,r)\big) = 2\sqrt{2}r < 4r$, hence $x \in \Delta(y,4r)$. As such, we may invoke (2.418) (with the roles of x and y reversed) to conclude that $t(x) = t(y)$. Thus, $x = \big(\zeta(x), t(x)\big) = \big(\zeta(y), t(y)\big) = y \in G(x_0,r)$, ultimately proving (2.419).

As a consequence of the proof of (2.419) we also see that

$$\text{the projection } \Pi \text{ is one-to-one on } G(x_0,r). \tag{2.420}$$

In turn, (2.420) guarantees that the mapping

$$\begin{gathered} h : \Pi\big(G(x_0,r)\big) \longrightarrow \mathbb{R} \ \text{ given by} \\ h\big(\zeta(x)\big) := t(x) \ \text{ for each } \ x \in G(x_0,r) \end{gathered} \tag{2.421}$$

is well defined. By (2.418), this mapping satisfies a Lipschitz condition with constant $\leq C_0\phi(\delta)$ on the set $\Pi\big(G(x_0,r)\big)$. Indeed, given any $x, y \in G(x_0,r)$, the fact that $G(x_0,r) \subseteq \Delta(x_0,\sqrt{2}r)$ implies $|x-y| < 2\sqrt{2}r < 4r$, hence $y \in \Delta(x,4r)$. As such, (2.418) applies and, in view of (2.421), proves that

$$|h(x')-h(y')| \leq C_0\phi(\delta)|x'-y'| \ \text{ for each } \ x', y' \in \Pi\big(G(x_0,r)\big).$$

We may therefore extend h (using Kirszbraun's theorem; see, e.g., the discussion in [108]) as a Lipschitz function, which we continue to denote by h, to the entire hyperplane $H(x_0, r)$, with Lipschitz constant $\leq C_0\phi(\delta)$. Note that its graph $\mathcal{G}$, considered in the (ζ, t)-system of coordinates introduced in (2.395), contains the set

$$\{(\zeta(x), t(x)) : x \in G(x_0, r)\} = G(x_0, r). \tag{2.422}$$

This proves the inclusion in (2.368). Together, (2.368) and (2.419) also prove that

$$\begin{gathered}\text{if } x \in C(x_0, r) \cap \partial\Omega \text{ and } \Pi(x) \in \Pi\big(G(x_0, r)\big)\\ \text{then } x \in \mathcal{G}.\end{gathered} \tag{2.423}$$

In turn, the above property implies the claim made in (2.374). Specifically, assume $x \in G(x_0, r)$ and $y \in C(x_0, r) \cap \partial\Omega$ are such that $\Pi(y) = \Pi(x)$. Then $\Pi(y)$ belongs to $\Pi\big(G(x_0, r)\big)$ which, by virtue of (2.419), places y in $G(x_0, r)$. In particular, $x, y \in \mathcal{G}$ (cf. (2.368)) have the same projection. Thus, necessarily, $x = y$ since otherwise the Vertical Line Test would be violated for the graph $\mathcal{G}$.

To prove the inclusion claimed in (2.370), start by considering some arbitrary point $x \in C(x_0, r) \cap \partial\Omega$. Then x belongs to $B(x_0, \sqrt{2}r) \cap \partial\Omega = \Delta(x_0, \sqrt{2}r)$. Also, the convention made in (2.395) allows us to express $x = x_0 + t(x)\vec{n}_{x_0,r} + \zeta(x)$, with $\zeta(x) \in H(x_0, r)$ satisfying $|\zeta(x)| < r$ (given that $x \in C(x_0, r)$) and with

$$t(x) = \langle x - x_0, \vec{n}_{x_0,r}\rangle = \frac{\langle x - x_0, \nu_{\Delta(x_0,4r)}\rangle}{|\nu_{\Delta(x_0,4r)}|}, \tag{2.424}$$

thanks to (2.414) and (2.393). In turn, (2.424), (2.392), and (2.385) (presently used with $R := 4r$, $x := x_0$, $y := x$) permit us to estimate

$$|t(x)| \leq \frac{\big|\langle x - x_0, \nu_{\Delta(x_0,4r)}\rangle\big|}{|\nu_{\Delta(x_0,4r)}|} \leq \tfrac{10}{9}(4r)12C_*\delta \leq C_0\delta\, r, \tag{2.425}$$

since (2.386) guarantees that $C_0 \geq 60C_*$. The proof of (2.370) is therefore complete.

From (2.370) it follows that the connected sets $C^{\pm}(x_0, r)$ introduced in (2.372) do not intersect $\partial\Omega$. As such, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ form a disjoint, open cover of $C^{\pm}(x_0, r)$, hence

$$\begin{gathered}C^+(x_0, r) \text{ is fully contained in either } \Omega_+ \text{ or } \Omega_-,\\ \text{and also } C^-(x_0, r) \text{ is fully contained in either } \Omega_+\\ \text{or } \Omega_-.\end{gathered} \tag{2.426}$$

By further decreasing $\delta_* \in (0, 1)$ we may ensure (see Theorem 2.3) that

$$\Omega \text{ satisfies a two-sided local John condition with constants which depend only on the Ahlfors regularity constant of } \partial\Omega \text{ and the dimension } n. \tag{2.427}$$

In view of (2.427) and (2.87), it follows that Ω satisfies a two-sided cork screw condition (cf. Definition 2.10) for some parameter $\theta \in (0, 1)$ which depends only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n. Denote by $x_r^{\pm} \in \Omega_{\pm}$ the two corkscrew points corresponding to the location x_0 and scale r. In particular,

$$|x_r^{\pm} - x_0| < r \quad \text{and} \quad B\big(x_r^{\pm}, \theta r\big) \subseteq \Omega_{\pm}. \tag{2.428}$$

Assume $0 < \delta_* < \theta/C_0$ to begin with. Given that we are taking $\delta \in (0, \delta_*)$, this condition makes it impossible to contain either of the balls $B\big(x_r^{+}, \theta r\big)$, $B\big(x_r^{-}, \theta r\big)$ in the strip $\big\{x_0 + x' + t\vec{n}_{x_0,r} : |t| \leq C_0\delta r,\ x' \in H(x_0, r)\big\}$. Since, as seen from (2.428), their centers $x_r^{\pm}$ belong to $B(x_0, r) \subset C(x_0, r)$, in turn this forces one of the following four alternatives to be true:

$$B\big(x_r^{+}, \theta r\big) \cap C^{+}(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, \theta r\big) \cap C^{+}(x_0, r) \neq \varnothing, \tag{2.429}$$

$$B\big(x_r^{+}, \theta r\big) \cap C^{-}(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, \theta r\big) \cap C^{-}(x_0, r) \neq \varnothing, \tag{2.430}$$

$$B\big(x_r^{+}, \theta r\big) \cap C^{+}(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, \theta r\big) \cap C^{-}(x_0, r) \neq \varnothing, \tag{2.431}$$

$$B\big(x_r^{+}, \theta r\big) \cap C^{-}(x_0, r) \neq \varnothing \quad \text{and} \quad B\big(x_r^{-}, \theta r\big) \cap C^{+}(x_0, r) \neq \varnothing. \tag{2.432}$$

Note that the alternative described in (2.429) cannot possibly hold. Indeed, the existence of two points $z_1 \in B\big(x_r^{+}, \theta r\big) \cap C^{+}(x_0, r)$ and $z_2 \in B\big(x_r^{-}, \theta r\big) \cap C^{+}(x_0, r)$ would imply that, on the one hand, the line segment $[z_1, z_2]$ lies in the convex set $C^{+}(x_0, r)$, hence also either in Ω_{+} or in Ω_{-} by (2.426). Nonetheless, the fact that we have $z_1 \in B\big(x_r^{+}, \theta r\big) \subseteq \Omega_{+}$ and $z_2 \in B\big(x_r^{-}, \theta r\big) \subseteq \Omega_{-}$ prevents either one of these eventualities form materializing. This contradiction therefore excludes (2.429). Reasoning in a similar fashion we may rule out (2.430). When (2.431) holds, from the fact that $B\big(x_r^{\pm}, \theta r\big) \subseteq \Omega_{\pm}$ (cf. (2.428)) we conclude that

$$\varnothing \neq C^{+}(x_0, r) \cap B\big(x_r^{+}, \theta r\big) \subseteq B\big(x_r^{+}, \theta r\big) \subseteq \Omega_{+} \tag{2.433}$$

hence $C^{+}(x_0, r) \cap \Omega_{+} \neq \varnothing$ which, in light of (2.426), forces $C^{+}(x_0, r) \subseteq \Omega_{+}$. Similarly, $C^{-}(x_0, r) \subseteq \Omega_{-}$ so the inclusions in (2.373) hold as stated. Finally, when (2.432) holds, from (2.426) and (2.428) we deduce that $C^{+}(x_0, r) \subseteq \Omega_{-}$ and $C^{-}(x_0, r) \subseteq \Omega_{+}$. In such a scenario, we may ensure that the inclusions in (2.373) are valid simply by re-denoting $\vec{n}_{x_0,r}$ as $-\vec{n}_{x_0,r}$ (and considering the function $-h$ in place of the original h), which amounts to reversing the roles of $C^{+}(x_0, r)$ and $C^{-}(x_0, r)$ (without affecting the other properties). This concludes the proof of (2.373).

Next, observe that

$$\Pi\big(C(x_0,r)\cap\partial\Omega\big)\subseteq\Pi\big(C(x_0,r)\big)=\{\zeta\in H(x_0,r):\,|\zeta|<r\}. \tag{2.434}$$

The opposite inclusion fails only when there exists a line segment parallel to $\vec{n}_{x_0,r}$ whose two endpoints belong to $C^+(x_0,r)$ and to $C^-(x_0,r)$, respectively, and which does not intersect $\partial\Omega$ (here we implicitly use the fact that $C^\pm(x_0,r)\neq\varnothing$, itself a result of having imposed the condition that $0<\delta<\delta_*<1/C_0$; cf. (2.387)). However, (2.373) and simple connectivity arguments rule out this scenario, hence (2.371) is proved.

Going further, we note that (2.371) implies

$$\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\subseteq\Pi\big(E(x_0,r)\big). \tag{2.435}$$

The fact that $\Pi:\mathbb{R}^n\to H(x_0,r)$ is a Lipschitz function, with Lipschitz constant 1, implies (cf., e.g., [47, Theorem 1, p. 75]) that

$$\mathcal{H}^{n-1}\big(\Pi(S)\big)\leq\mathcal{H}^{n-1}(S)\quad\text{for each Borel set }\ S\subseteq\mathbb{R}^n. \tag{2.436}$$

Based on (2.435), (2.436), the definition $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, (2.367), and (2.400) we then conclude that

$$\begin{aligned}\mathcal{H}^{n-1}&\Big(\{\zeta\in H(x_0,r):|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\Big)\\&\leq\mathcal{H}^{n-1}\Big(\Pi\big(E(x_0,r)\big)\Big)\leq\mathcal{H}^{n-1}\big(E(x_0,r)\big)\\&\leq 2^{n-1}C_A C\,r^{\,n-1}\cdot\exp\big(-C_2\phi(\delta)/\delta\big).\end{aligned} \tag{2.437}$$

In addition, (2.419) gives

$$C(x_0,r)\cap\big(\mathcal{G}\setminus\partial\Omega\big)\subseteq\mathcal{G}\cap\Pi^{-1}\Big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\Big). \tag{2.438}$$

Keeping also in mind that

$$\begin{gathered}\mathcal{H}^{n-1}(S)\leq\sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\big(\Pi(S)\big),\\ \text{for each Borel set } S\subseteq\mathcal{G},\end{gathered} \tag{2.439}$$

(since $\mathcal{G}$ is the graph of a Lipschitz function), we deduce that

$$\begin{aligned}&\mathcal{H}^{n-1}\big(C(x_0,r)\cap(\mathcal{G}\setminus\partial\Omega)\big)\\&\leq\mathcal{H}^{n-1}\Big(\mathcal{G}\cap\Pi^{-1}\big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\big)\Big)\\&\leq\sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\Big(\Pi\Big(\mathcal{G}\cap\Pi^{-1}\big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\big)\Big)\Big)\end{aligned}$$

$$\leq \sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\Big(\{\zeta\in H(x_0,r):\,|\zeta|<r\}\setminus\Pi\big(G(x_0,r)\big)\Big)$$

$$\leq \sqrt{1+(C_0\phi(\delta))^2}\,2^{n-1}C_A C\,r^{n-1}\cdot\exp\big(-C_2\phi(\delta)/\delta\big)$$

$$\leq \sqrt{1+C_0^2(14C_*+4)^{-2}}\,2^{n-1}C_A C\,r^{n-1}\cdot\exp\big(-C_2\phi(\delta)/\delta\big), \tag{2.440}$$

by (2.437) and (2.388). Upon observing that $C(x_0,r)\cap\big(\partial\Omega\setminus\mathcal{G}\big)$ is contained in $E(x_0,r)$, the estimate claimed in (2.366) now follows from (2.440) and (2.400) if we choose (recall that $C_2:=c$; cf. (2.401))

$$C_1:=\sqrt{1+C_0^2(14C_*+4)^{-2}}\,2^{n-1}C_A C/\upsilon_{n-1} \tag{2.441}$$

(a choice in line with the demand formulated in (2.401)), where C is as in (2.409), and where C_A is the Ahlfors regularity constant of $\partial\Omega$.

Let us now justify the proximity condition formulated in (2.369). To this end, fix $x\in E(x_0,r)=\big(C(x_0,r)\cap\partial\Omega\big)\setminus G(x_0,r)$ and pick an arbitrary $x^*\in G(x_0,r)$. In particular, $x,x^*\in C(x_0,r)$ hence $|x-x^*|<\operatorname{diam}\big(C(x_0,r)\big)=2\sqrt{2}r$. Given that we have $x^*\in G(x_0,r)$ and $x\in\Delta(x^*,4r)$, estimate (2.418) applies and presently gives

$$\big|t(x)-h\big(\Pi(x^*)\big)\big|=|t(x)-t(x^*)|\leq C_0\phi(\delta)|\Pi(x)-\Pi(x^*)|. \tag{2.442}$$

Consequently, since $x=\big(\Pi(x),t(x)\big)$, we may write

$$\begin{aligned}\big|x-\big(\Pi(x),h(\Pi(x))\big)\big|&=\big|t(x)-h\big(\Pi(x)\big)\big|\\&\leq\big|t(x)-h\big(\Pi(x^*)\big)\big|+\big|h\big(\Pi(x^*)\big)-h\big(\Pi(x)\big)\big|\\&\leq 2C_0\phi(\delta)|\Pi(x)-\Pi(x^*)|,\end{aligned} \tag{2.443}$$

by (2.442) and the Lipschitz condition on h (cf. (2.363)). Taking the infimum over $x^*\in G(x_0,r)$ now yields (2.369).

Let us now deal with (2.375). Recall that υ_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$. Using (2.366) and (2.439) we may estimate

$$\sigma\big(\Delta(x_0,r)\big)=\mathcal{H}^{n-1}\big(B(x_0,r)\cap\partial\Omega\big)\leq\mathcal{H}^{n-1}\big(C(x_0,r)\cap\partial\Omega\big)$$

$$\leq\mathcal{H}^{n-1}\big(C(x_0,r)\cap\mathcal{G}\big)+\mathcal{H}^{n-1}\Big(C(x_0,r)\cap(\partial\Omega\setminus\mathcal{G})\Big)$$

$$\leq\sqrt{1+(C_0\phi(\delta))^2}\,\mathcal{H}^{n-1}\big(\Pi(C(x_0,r)\cap\mathcal{G})\big)+\mathcal{H}^{n-1}\Big(C(x_0,r)\cap(\partial\Omega\triangle\mathcal{G})\Big)$$

$$\leq\big(1+C_0\phi(\delta)\big)\,\mathcal{H}^{n-1}\big(\Pi(C(x_0,r))\big)+C_1\upsilon_{n-1}\,r^{n-1}\exp\big(-C_2\phi(\delta)/\delta\big)$$

$$= \Big(1 + C_0\phi(\delta) + C_1 \exp\big(-C_2\phi(\delta)/\delta\big)\Big)\upsilon_{n-1}\, r^{n-1}. \tag{2.444}$$

Also, by employing (2.371), (2.436), (2.366), (2.439), (2.373), and (2.388) we may write

$$\upsilon_{n-1}\, r^{n-1} = \mathcal{H}^{n-1}\big(\{\zeta \in H(x_0, r) : |\zeta| < r\}\big) \le \mathcal{H}^{n-1}\big(\mathcal{C}(x_0, r) \cap \partial\Omega\big) \tag{2.445}$$

$$= \mathcal{H}^{n-1}\big(B(x_0, r) \cap \partial\Omega\big) + \mathcal{H}^{n-1}\Big(\big(\mathcal{C}(x_0, r) \cap \partial\Omega\big) \setminus B(x_0, r)\Big)$$

$$\le \sigma\big(\Delta(x_0, r)\big) + \mathcal{H}^{n-1}\Big(\mathcal{C}(x_0, r) \cap \big(\partial\Omega \setminus \mathcal{G}\big)\Big)$$

$$+ \mathcal{H}^{n-1}\Big(\big(\mathcal{C}(x_0, r) \cap \mathcal{G}\big) \setminus \big(B(x_0, r) \cup \mathcal{C}^+(x_0, r) \cup \mathcal{C}^-(x_0, r)\big)\Big)$$

$$\le \sigma\big(\Delta(x_0, r)\big) + \mathcal{H}^{n-1}\Big(\mathcal{C}(x_0, r) \cap \big(\partial\Omega \triangle \mathcal{G}\big)\Big)$$

$$+ \sqrt{1 + (C_0\phi(\delta))^2}\, \upsilon_{n-1}\, r^{n-1}\Big(1 - \big(\sqrt{1 - C_0^2\delta^2}\big)^{n-1}\Big)$$

$$\le \sigma\big(\Delta(x_0, r)\big) + C_1 \upsilon_{n-1}\, r^{n-1} \exp\big(-C_2\phi(\delta)/\delta\big) + C_3\delta\, \upsilon_{n-1}\, r^{n-1},$$

where $C_3 := C_n\, C_0\sqrt{1 + C_0^2(14C_* + 4)^{-2}}$ with $C_n \in [1, \infty)$ depending only on the dimension n. This further implies

$$\Big(1 - C_3\delta - C_1 \exp\big(-C_2\phi(\delta)/\delta\big)\Big)\, \upsilon_{n-1}\, r^{n-1} \le \sigma\big(\Delta(x_0, r)\big). \tag{2.446}$$

Now, (2.375) follows from (2.444), (2.446), and (2.388).

Next, (2.376) is a direct consequence of Proposition 2.2 applied to Ω and the upper-graph of the function h (both of which are Ahlfors regular domains). There remains to prove the claims made in (2.377) and (2.378). To get started, we make two observations. First, (2.362) implies

$$⨍_{\Delta(x_0, 4r)} |\nu - \nu_{\Delta(x_0, 4r)}|\, d\sigma \le \delta. \tag{2.447}$$

Second, at σ-a.e. point on $\partial\Omega$ we may estimate

$$|\nu - \vec{n}_{x_0, r}| \le |\nu - \nu_{\Delta(x_0, 4r)}| + |\nu_{\Delta(x_0, 4r)} - \vec{n}_{x_0, r}| \tag{2.448}$$

and, thanks to (2.393), the fact that $|\nu| = 1$ at σ-a.e. point on $\partial\Omega$, and the reverse triangle inequality, we have

$$
\begin{aligned}
|\nu_{\Delta(x_0,4r)} - \vec{n}_{x_0,r}| &= \left|\nu_{\Delta(x_0,4r)} - \frac{\nu_{\Delta(x_0,4r)}}{|\nu_{\Delta(x_0,4r)}|}\right| = \left|\Big(1 - \frac{1}{|\nu_{\Delta(x_0,4r)}|}\Big)\nu_{\Delta(x_0,4r)}\right| \\
&= \left|1 - \frac{1}{|\nu_{\Delta(x_0,4r)}|}\right| \big|\nu_{\Delta(x_0,4r)}\big| = \big|1 - |\nu_{\Delta(x_0,4r)}|\big| \\
&= \big||\nu| - |\nu_{\Delta(x_0,4r)}|\big| \leq \big|\nu - \nu_{\Delta(x_0,4r)}\big|. \qquad (2.449)
\end{aligned}
$$

By combining (2.448) with (2.449) we arrive at the conclusion that

$$
|\nu - \vec{n}_{x_0,r}| \leq 2|\nu - \nu_{\Delta(x_0,4r)}| \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \qquad (2.450)
$$

Recall that $\widetilde{\nu}$ denotes the unit normal vector to the Lipschitz graph $\mathcal{G}$, pointing toward the upper-graph of the function h. This is well-defined at $\mathcal{H}^{n-1}$-a.e. point on $\mathcal{G}$, and we claim that

$$
|\widetilde{\nu} - \vec{n}_{x_0,r}| \leq C_0\phi(\delta) \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \mathcal{G}. \qquad (2.451)
$$

To justify this, after performing a rotation, there is no loss of generality in assuming that

$$
\vec{n}_{x_0,r} = \mathbf{e}_n = (0, \dots, 0, 1) \in \mathbb{R}^n. \qquad (2.452)
$$

Then the hyperplane

$$
H(x_0, r) = \langle \vec{n}_{x_0,r} \rangle^{\perp} = \langle \mathbf{e}_n \rangle^{\perp} = \mathbb{R}^{n-1} \times \{0\} \qquad (2.453)
$$

may be canonically identified with $\mathbb{R}^{n-1}$, a scenario in which

$$
\widetilde{\nu}\big(x', h(x')\big) = \frac{\big(-(\nabla' h)(x'), 1\big)}{\sqrt{1 + |(\nabla' h)(x')|^2}} \text{ for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1}, \qquad (2.454)
$$

where ∇' denotes the gradient operator in $\mathbb{R}^{n-1}$. From (2.452) and (2.454) we then see that at $\mathcal{H}^{n-1}$-a.e. point $x \in \mathcal{G}$, say $x = \big(x', h(x')\big)$ with $x' \in \mathbb{R}^{n-1}$, we have

$$
\begin{aligned}
\big|\widetilde{\nu}(x) - \vec{n}_{x_0,r}\big|^2 &= 2 - 2\langle \widetilde{\nu}(x), \vec{n}_{x_0,r} \rangle = 2\bigg(1 - \frac{1}{\sqrt{1 + |(\nabla' h)(x')|^2}}\bigg) \\
&= \frac{2|(\nabla' h)(x')|^2}{1 + |(\nabla' h)(x')|^2 + \sqrt{1 + |(\nabla' h)(x')|^2}} \\
&\leq |(\nabla' h)(x')|^2 \leq \big(C_0\phi(\delta)\big)^2, \qquad (2.455)
\end{aligned}
$$

where the last inequality comes from (2.363). Ultimately, this establishes (2.451).

Collectively, (2.450) and (2.451) prove that

$$|\nu - \widetilde{\nu}| \leq 2|\nu - \nu_{\Delta(x_0,4r)}| + C_0\phi(\delta) \ \text{ at } \sigma\text{-a.e. point on } \mathcal{G} \cap \partial\Omega. \tag{2.456}$$

From (2.391) and (2.397) we also see that

$$|\nu(x) - \nu_{\Delta(x_0,4r)}| \leq \nu^*_{x_0,2r}(x) \leq \phi(\delta) \ \text{ for } \sigma\text{-a.e. } x \in G(x_0, r). \tag{2.457}$$

Combining (2.456) with (2.457) and keeping in mind that $G(x_0, r) \subseteq \mathcal{G} \cap \partial\Omega$ leads to the conclusion that

$$|\nu - \widetilde{\nu}| \leq (2 + C_0)\phi(\delta) \ \text{ at } \sigma\text{-a.e. point on } G(x_0, r). \tag{2.458}$$

If $\delta_* > 0$ is taken small enough so that $\phi(t) < 2(2 + C_0)^{-1}$ for all $t \in (0, \delta_*)$ (something that may always be arranged, thanks to (2.361)), we conclude from (2.458) and (2.376) (again, mindful of the fact that $G(x_0, r) \subseteq \mathcal{G} \cap \partial\Omega$) that

$$\nu(x) = \widetilde{\nu}(x) \ \text{ at } \sigma\text{-a.e. point } x \in G(x_0, r). \tag{2.459}$$

This proves (2.377).

Let us now deal with (2.378). Together, (2.376), (2.456), (2.447), and the first inequality in (2.388) yield

$$\begin{aligned}\sigma\Big(\big\{x \in \mathcal{G} \cap \Delta(x_0, 4r) : \nu(x) = -\widetilde{\nu}(x)\big\}\Big) &= \frac{1}{2}\int_{\mathcal{G}\cap\Delta(x_0,4r)} |\nu - \widetilde{\nu}| \, d\sigma \\ &\leq \big(\delta + 2^{-1}C_0 \cdot \phi(\delta)\big) \cdot \sigma\big(\Delta(x_0, 4r)\big) \\ &\leq C_4 \cdot \phi(\delta) r^{n-1}\end{aligned} \tag{2.460}$$

provided $C_4 := 4^{n-1}(1 + 2^{-1}C_0)C_A$, where C_A is the Ahlfors regularity constant of Ω. Hence, (2.378) is established.

There remains to prove (2.379). To this end, combine (2.450) and (2.451) to obtain

$$\begin{aligned}\sup_{y\in\mathcal{G}} |\nu(x) - \widetilde{\nu}(y)| \leq 2|\nu(x) - \nu_{\Delta(x_0,4r)}| + C_0\phi(\delta) \\ \text{at } \sigma\text{-a.e. point } x \in \partial\Omega.\end{aligned} \tag{2.461}$$

Based on (2.461), (2.447), and (2.388) we then conclude that

$$⨍_{\Delta(x_0,4r)} \Big(\sup_{y\in\mathcal{G}} |\nu(x) - \widetilde{\nu}(y)|\Big) \, d\sigma(x) \leq 2\delta + C_0\phi(\delta) \leq C_4 \cdot \phi(\delta), \tag{2.462}$$

since our earlier choice of C_4 ensures that $C_4 \geq 2 + C_0$. This justifies (2.379), so the proof of Theorem 2.6 is now complete. □

2.6 Chord-Arc Domains in the Plane

In the two-dimensional setting, an important category of sets is the class of chord-arc domains, discussed next.

Definition 2.16 Given a nonempty, proper, open subset Ω of $\mathbb{R}^2$ and $\varkappa \in [0, \infty)$, one calls Ω a $\varkappa$-`CAD` (or simply `chord-arc domain`, if the value of $\varkappa$ is not important) provided $\partial\Omega$ is a locally rectifiable simple curve, which is either a closed curve or a Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$, with the property that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \partial\Omega, \tag{2.463}$$

where $\ell(z_1, z_2)$ denotes the length of the shortest arc of $\partial\Omega$ joining z_1 and z_2.

For example, a planar sector Ω_θ of full aperture $\theta \in (0, 2\pi)$ (cf. (2.289)) is a $\varkappa$-CAD with constant $\varkappa := [\sin(\theta/2)]^{-1} - 1$. While Proposition 2.13 shows that the upper-graph of any real-valued BMO_1 function defined on the real line is a chord-arc domain (hence, in particular, any Lipschitz domain in the plane is a chord-arc domain), from our earlier discussion (see, e.g., Example 2.7) we know that the boundaries of chord-arc domains may actually contain spiral points. As such, chord-arc domains may fail to be of "upper-graph type." There are also subtle connections between the quality of being a chord-arc domain and the behavior of the conformal mapping (see, e.g., [26] and the references therein).

Our next major goal is to establish, in the two-dimensional setting, the coincidence of the class of $\varkappa$-CAD domains with $\varkappa \geq 0$ small constant with that of δ-AR domains with $\delta > 0$ small. This is accomplished in Theorem 2.7. For now recall the concept of UR domain from Definition 2.6.

Proposition 2.16 *Assume $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a chord-arc domain. Then Ω is a connected* UR *domain, satisfying a two-sided local John condition. Moreover, $\partial\Omega = \partial(\overline{\Omega})$ and if either $\partial\Omega$ is unbounded or Ω is bounded, then Ω is also simply connected.*

Proof If $\partial\Omega$ is a Jordan curve passing through infinity in $\mathbb{C}$ then the desired conclusions follow from item *(vi)* of Proposition 2.10 and (2.194). If $\partial\Omega$ is bounded, then there exists a bi-Lipschitz homeomorphism F of the complex plane onto itself such that $F\big(\partial B(0, 1)\big) = \partial\Omega$ (cf. [119, Theorem 7.9, p. 165]). This implies that each of the connected sets $F\big(B(0, 1)\big)$, $F\big(\mathbb{C} \setminus \overline{B(0, 1)}\big)$ is contained in the disjoint union of Ω with $\mathbb{C} \setminus \overline{\Omega}$. Since F is surjective, this forces that either

$$F\big(B(0, 1)\big) = \Omega \quad \text{and} \quad F\big(\mathbb{C} \setminus \overline{B(0, 1)}\big) = \mathbb{C} \setminus \overline{\Omega} \tag{2.464}$$

or

$$F\big(B(0,1)\big) = \mathbb{C} \setminus \overline{\Omega} \quad \text{and} \quad F\big(\mathbb{C} \setminus \overline{B(0,1)}\big) = \Omega. \tag{2.465}$$

All desired conclusions readily follow from this and the transformational properties under bi-Lipschitz maps established in [59]. □

A chord-arc domain with a sufficiently small constant is necessarily unbounded (and, in fact, has an unbounded boundary).

Proposition 2.17 *If $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in \big[0, \sqrt{2}-1\big)$ then $\partial\Omega$ is unbounded.*

Proof Seeking a contradiction, assume $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in \big[0, \sqrt{2}-1\big)$ and such that $\partial\Omega$ is a bounded set. In particular, $\partial\Omega$ is a rectifiable closed curve. Abbreviate $L := \mathcal{H}^1(\partial\Omega) \in (0,\infty)$ and let $[0, L] \ni s \mapsto z(s) \in \partial\Omega$ be the arc-length parametrization of $\partial\Omega$. Define $z_0 := z(0)$, $z_{1/4} := z(L/4)$, $z_{1/2} := z(L/2)$, $z_{3/4} := z(3L/4)$. Since

$$\begin{aligned} |z_0 - z_{1/4}| \leq \ell(z_0, z_{1/4}) = L/4, \quad |z_{3/4} - z_0| \leq \ell(z_{3/4}, z_0) = L/4, \\ |z_{1/2} - z_{3/4}| \leq \ell(z_{1/2}, z_{3/4}) = L/4, \quad |z_{1/4} - z_{1/2}| \leq \ell(z_{1/4}, z_{1/2}) = L/4, \end{aligned} \tag{2.466}$$

it follows that

$$z_{1/4}, z_{3/4} \in D := \overline{B(z_0, L/4)} \cap \overline{B(z_{1/2}, L/4)}, \tag{2.467}$$

hence

$$|z_{1/4} - z_{3/4}| \leq \operatorname{diam}(D). \tag{2.468}$$

On the one hand, with $R := |z_0 - z_{1/2}|$, elementary geometry gives that

$$\operatorname{diam}(D) = 2\sqrt{(L/4)^2 - (R/2)^2} = \sqrt{L^2/4 - R^2}. \tag{2.469}$$

On the other hand, $L/2 = \ell(z_0, z_{1/2}) \leq (1+\varkappa)|z_0 - z_{1/2}| = (1+\varkappa)R$ so

$$\operatorname{diam}(D) \leq \sqrt{L^2/4 - (L/(2+2\varkappa))^2} = \frac{L}{2}\sqrt{1 - \Big(\frac{1}{1+\varkappa}\Big)^2}. \tag{2.470}$$

Based on the chord-arc property, (2.468), and (2.470) we then conclude that

$$\begin{aligned} \frac{L}{2} &= \ell(z_{1/4}, z_{3/4}) \leq (1+\varkappa)|z_{1/4} - z_{3/4}| \\ &\leq (1+\varkappa)\operatorname{diam}(D) \leq \frac{L}{2}\sqrt{(1+\varkappa)^2 - 1}, \end{aligned} \tag{2.471}$$

which further implies that $\varkappa \geq \sqrt{2} - 1$, a contradiction. □

By design, the boundary of any chord-arc domain is a simple curve, and this brings into focus the question: when is the boundary of an open, connected, simply connected planar set a Jordan curve? According to the classical Carathéodory theorem, this is the case if and only if some (or any) conformal mapping $\varphi : \mathbb{D} \to \Omega$ (where $\mathbb{D}$ is the unit disk in $\mathbb{C}$) extends to a homeomorphism $\varphi : \overline{\mathbb{D}} \to \overline{\Omega}$ (see, e.g., [53, Theorem 3.1, p. 13]). A characterization of bounded planar Jordan regions in terms of properties having no reference to their boundaries has been given by R.L. Moore in 1918. According to [116],

> given an open, bounded, connected, simply connected subset Ω of $\mathbb{R}^2$, in order for $\partial\Omega$ to be a simple closed curve it is necessary and sufficient that Ω is *uniformly connected im kleinen* (i.e., if for every $\varepsilon_o > 0$ there exists $\delta_o > 0$ such that any two points $P, \widetilde{P} \in \Omega$ with $|P - \widetilde{P}| < \delta_o$ lie in a connected subset Γ of Ω satisfying $|P - Q| < \varepsilon_o$ for each point $Q \in \Gamma$). (2.472)

A moment's reflection shows that the uniform connectivity condition (im kleinen) formulated above is equivalent to the demand that for every $\varepsilon_o > 0$ there exists $\delta_o > 0$ such that any two points $P, \widetilde{P} \in \Omega$ with $|P - \widetilde{P}| < \delta_o$ lie in a connected subset Γ of Ω with $\mathrm{diam}(\Gamma) < \varepsilon_o$. This condition is meant to prevent the boundary of Ω to "branch out" (like in the case of a partially slit disk).

We are now in a position to establish the coincidence of the class of $\varkappa$-CAD domains with $\varkappa \geq 0$ small constant with that of δ-AR domains with $\delta > 0$ small, in the two-dimensional Euclidean setting.

Theorem 2.7 *If $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-*CAD *with $\varkappa \in \big[0, \sqrt{2} - 1\big)$ then Ω satisfies a two-sided local John condition and is a δ-*AR *domain for any $\delta > 2\sqrt{\varkappa(2+\varkappa)}$. In particular, Ω is a δ-*AR *domain for, say, $\delta := 2\sqrt{\sqrt{2}+1}\sqrt{\varkappa}$, a choice which satisfies $\delta = O(\sqrt{\varkappa}\,)$ as $\varkappa \to 0^+$.*

*Conversely, given any $M \in (0, \infty)$ there exists $\delta_* \in (0, 1)$ with the property that whenever $\delta \in (0, \delta_*)$ it follows that any δ-*AR *domain $\Omega \subseteq \mathbb{R}^2$ with Ahlfors regularity constant $\leq M$ is a $\varkappa$-*CAD *with $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$.*

Proof Suppose $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in \big[0, \sqrt{2} - 1\big)$. Proposition 2.17 then ensures that $\partial\Omega$ is an unbounded set. Keeping this in mind, from Definition 2.16 we then conclude that $\partial\Omega$ is a Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$. Granted (2.463), it follows that $\partial\Omega$ is a $\varkappa$-CAC. From Proposition 2.10 and (2.199) we then see that Ω satisfies a two-sided local John condition and has an Ahlfors regular boundary. Moreover, if $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and ν is the geometric measure theoretic outward unit normal to Ω, from (2.228) we deduce that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq 2\sqrt{\varkappa(2+\varkappa)}. \tag{2.473}$$

It follows from Definition 2.15 and (2.473) that Ω is a δ-AR domain whenever $\delta > 2\sqrt{\varkappa(2+\varkappa)}$. This completes the proof of the claim made in the first part of the statement of the theorem.

In the converse direction, let $\Omega \subseteq \mathbb{R}^2$ be a δ-AR domain with $\delta \in (0,1)$ sufficiently small relative to the Ahlfors regularity constant of $\partial\Omega$. Then Theorem 2.3 implies that Ω is an (∞, N)-two-sided nontangentially accessible domain (in the sense of Definition 2.9), for some $N \in \mathbb{N}$. From Corollary 2.2 we also know that Ω is an unbounded connected set which is simply connected, and whose topological boundary is an unbounded connected set.

The first order of business is to show that actually $\partial\Omega$ is a simple curve. To establish this, we intend to make use of Moore's criterion recalled in (2.472). Since this pertains to bounded sets, as a preliminary step we fix a point $z_0 \in \mathbb{C} \setminus \overline{\Omega}$ and consider

$$\widetilde{\Omega} := \Phi(\Omega) \subseteq \mathbb{C}, \tag{2.474}$$

where

$$\begin{aligned} &\Phi : \mathbb{C} \setminus \{z_0\} \longrightarrow \mathbb{C} \setminus \{0\} \\ &\Phi(z) := (z - z_0)^{-1} \quad \text{for each} \quad z \in \mathbb{C} \setminus \{z_0\}. \end{aligned} \tag{2.475}$$

Note that, when restricted to Ω, the function Φ satisfies a Lipschitz condition. Specifically, if $r_0 := \operatorname{dist}(z_0, \partial\Omega)$ then $r_0 \in (0, \infty)$ and we may estimate

$$\big|\Phi(z_1) - \Phi(z_2)\big| = \frac{|z_1 - z_2|}{|z_1 - z_0||z_2 - z_0|} \leq r_0^{-2}|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \Omega. \tag{2.476}$$

Also, since Φ is a homeomorphism and $\Omega \subseteq \mathbb{C}\setminus\{z_0\}$ it follows that $\widetilde{\Omega} = \Phi(\Omega)$ is an open, connected, simply connected subset of $\mathbb{C} \setminus \{0\}$. Moreover, $\Omega \subseteq \mathbb{C} \setminus \overline{B(z_0, r_0)}$ and since Φ maps $\mathbb{C} \setminus \overline{B(z_0, r_0)}$ into $B(0, 1/r_0)$ it follows that $\widetilde{\Omega} \subseteq B(0, 1/r_0)$, hence $\widetilde{\Omega}$ is also bounded. The idea is then to check Moore's criterion (cf. (2.472)) for $\widetilde{\Omega}$, conclude that $\partial\widetilde{\Omega}$ is a simple curve, then use Φ^{-1} to reach a similar conclusion for $\partial\Omega$. Since Φ^{-1} is singular at $0 \in \partial\widetilde{\Omega}$, special care is required when checking the uniform connectivity condition (im kleinen) near the origin. This requires some preparations.

To proceed, fix some large number $R \in (0, \infty)$, to be specified later in the proof. Pick two points $P, \widetilde{P} \in \widetilde{\Omega} \cap B(0, 1/R)$ then define $x := \Phi^{-1}(P)$ and $\widetilde{x} := \Phi^{-1}(\widetilde{P})$. It follows that $x, \widetilde{x} \in \Omega \setminus \overline{B(z_0, R)}$. Bring in the polygonal arc Γ joining x with $\widetilde{x}$ in Ω as in Lemma 2.5. As noted in Lemma 2.6, there exists $\varepsilon = \varepsilon(N) \in (0,1)$ with the property that this curve is disjoint from $B(z_0, \varepsilon R)$. Next, abbreviate $L := \operatorname{length}(\Gamma) \in (0, \infty)$ and let $[0, L] \ni s \mapsto \Gamma(s) \in \Gamma$ be the arc-length parametrization of Γ. In particular, $\big|\Gamma'(s)\big| = 1$ for $\mathcal{L}^1$-a.e. $s \in (0, L)$. If we define

$$\widetilde{\Gamma}(s) := \Phi(\Gamma(s)) = \frac{1}{\Gamma(s) - z_0} \quad \text{for each} \quad s \in [0, L], \tag{2.477}$$

then the image of $\widetilde{\Gamma}$ is a rectifiable curve joining P with $\widetilde{P}$ in $\widetilde{\Omega}$. In particular, this curve is a connected subset of $\widetilde{\Omega}$ containing P, $\widetilde{P}$ and, with (2.472) in mind, the immediate goal is to estimate the length of this curve. Retaining the symbol $\widetilde{\Gamma}$ for said curve, we have

$$\begin{aligned} \text{length}(\widetilde{\Gamma}) &= \int_0^L \big|\widetilde{\Gamma}'(s)\big|\, ds = \int_0^L \big|\Phi'(\Gamma(s))\big| \cdot \big|\Gamma'(s)\big|\, ds \\ &= \int_0^L \frac{ds}{|\Gamma(s) - z_0|^2}. \end{aligned} \tag{2.478}$$

For each $s \in [0, L]$ we have $\Gamma(s) \in \Omega$. Given that $z_0 \notin \overline{\Omega}$, the line segment joining $\Gamma(s)$ with z_0 intersects $\partial\Omega$, hence $|\Gamma(s) - z_0| \geq \delta_{\partial\Omega}(\Gamma(s))$. On the other hand, for each $s \in [0, L]$ the last line in (2.74) implies that $C_N \cdot \delta_{\partial\Omega}(\Gamma(s)) \geq \min\{s, L - s\}$. Altogether, $C_N \cdot |\Gamma(s) - z_0| \geq \min\{s, L - s\}$ for each $s \in [0, L]$. Upon recalling that the polygonal arc Γ is disjoint from $B(z_0, \varepsilon R)$, we also have $|\Gamma(s) - z_0| \geq \varepsilon R$ for each $s \in [0, L]$. Ultimately, this proves that there exists some $c_N \in (0, \infty)$ with the property that

$$|\Gamma(s) - z_0| \geq c_N \cdot \big(R + \min\{s, L - s\}\big) \quad \text{for each} \quad s \in [0, L]. \tag{2.479}$$

Combining (2.478) with (2.479) then gives

$$\begin{aligned} \text{length}(\widetilde{\Gamma}) &= \int_0^L \frac{ds}{|\Gamma(s) - z_0|^2} \leq C_N \int_0^L \frac{ds}{\big(R + \min\{s, L - s\}\big)^2} \\ &= C_N \int_0^{L/2} \frac{ds}{\big(R + \min\{s, L - s\}\big)^2} + C_N \int_{L/2}^L \frac{ds}{\big(R + \min\{s, L - s\}\big)^2} \\ &= 2C_N \int_0^{L/2} \frac{ds}{(R + s)^2} \leq 2C_N \int_0^\infty \frac{ds}{(R + s)^2} = \frac{2C_N}{R}. \end{aligned} \tag{2.480}$$

Armed with (2.480), we now proceed to check that the set $\widetilde{\Omega}$ is uniformly connected im kleinen (in the sense made precise in (2.472)). To get started, suppose some threshold $\varepsilon_o > 0$ has been given. Make the assumption that

$$R > \max\Big\{r_0, \frac{2C_N}{\varepsilon_o}\Big\} \quad \text{and pick} \quad \delta_o \in \big(0, 1/(2R)\big), \tag{2.481}$$

reserving the right to make further specifications regarding the size of δ_o. Consider two points $P, \widetilde{P} \in \widetilde{\Omega}$ with $|P - \widetilde{P}| < \delta_o$. The goal is to find a connected subset of $\widetilde{\Omega}$ whose every point is at distance $\leq \varepsilon_o$ from P. To this end, we distinguish two cases.

`Case I:` *Assume* $P, \widetilde{P} \in \widetilde{\Omega} \cap B(0, 1/R)$. Then $\widetilde{\Gamma}$, the curve introduced in (2.477), is a connected subset of $\widetilde{\Omega}$ containing $P, \widetilde{P}$, and (2.480) implies (in view of (2.481)) that length$(\widetilde{\Gamma}) < \varepsilon_o$. In particular, for any point $Q \in \widetilde{\Gamma}$ we have $|P - Q| \leq \text{length}(\widetilde{\Gamma}) < \varepsilon_o$, as wanted.

`Case II:` *Assume either* $P \notin \widetilde{\Omega} \cap B(0, 1/R)$ *or* $\widetilde{P} \notin \widetilde{\Omega} \cap B(0, 1/R)$. Since $|P - \widetilde{P}| < \delta_o < 1/(2R)$ to begin with, this forces $P, \widetilde{P} \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}$. To proceed, observe that the restriction of $\Phi : \Omega \to \widetilde{\Omega}$ to $\Omega \cap B(z_0, 2R)$, i.e., the function

$$\begin{aligned} &\widetilde{\Phi} : \Omega \cap B(z_0, 2R) \longrightarrow \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}, \\ &\widetilde{\Phi}(z) := (z - z_0)^{-1} \quad \text{for each} \quad z \in \Omega \cap B(z_0, 2R), \end{aligned} \tag{2.482}$$

is a bijection, whose inverse

$$\begin{aligned} &\widetilde{\Phi}^{-1} : \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))} \longrightarrow \Omega \cap B(z_0, 2R), \\ &\widetilde{\Phi}^{-1}(\zeta) := \zeta^{-1} + z_0 \quad \text{for each} \quad \zeta \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}, \end{aligned} \tag{2.483}$$

is Lipschitz since for each $\zeta_1, \zeta_2 \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}$ we may estimate

$$\left|\widetilde{\Phi}^{-1}(\zeta_1) - \widetilde{\Phi}^{-1}(\zeta_2)\right| = \frac{|\zeta_1 - \zeta_2|}{|\zeta_1||\zeta_2|} \leq (2R)^2 |\zeta_1 - \zeta_2|. \tag{2.484}$$

In particular, if we set $x := \widetilde{\Phi}^{-1}(P) \in \Omega$ and $\widetilde{x} := \widetilde{\Phi}^{-1}(\widetilde{P}) \in \Omega$, it follows that

$$|x - \widetilde{x}| = \left|\widetilde{\Phi}^{-1}(P) - \widetilde{\Phi}^{-1}(\widetilde{P})\right| \leq (2R)^2 |P - \widetilde{P}| \leq (2R)^2 \delta_o. \tag{2.485}$$

Let Γ be the polygonal arc joining x with $\widetilde{x}$ in Ω as in Lemma 2.5 with the scale $r := |x - \widetilde{x}|$. The first inequality in (2.74) tells us that length$(\Gamma) \leq C_N \cdot |x - \widetilde{x}|$, so $L := \text{length}(\Gamma) \leq C_N \cdot (2R)^2 \delta_o$ by (2.485). Let $[a, b] \ni t \mapsto \gamma(t) \in \Gamma$ be a parametrization of the curve Γ and define $\widetilde{\Gamma} := \Phi \circ \gamma$. Then the image of $\widetilde{\Gamma}$ is a rectifiable curve joining P with $\widetilde{P}$ in $\widetilde{\Omega}$. Indeed, $\Phi(\Gamma) \subseteq \Phi(\Omega) = \widetilde{\Omega}$ and

$$\begin{aligned} &\Phi(\gamma(a)) = \Phi(x) = \Phi\big(\widetilde{\Phi}^{-1}(P)\big) = \widetilde{\Phi}\big(\widetilde{\Phi}^{-1}(P)\big) = P, \\ &\Phi(\gamma(b)) = \Phi(\widetilde{x}) = \Phi\big(\widetilde{\Phi}^{-1}(\widetilde{P})\big) = \widetilde{\Phi}\big(\widetilde{\Phi}^{-1}(\widetilde{P})\big) = \widetilde{P}, \end{aligned} \tag{2.486}$$

given that $\widetilde{\Phi}^{-1}(P), \widetilde{\Phi}^{-1}(\widetilde{P})$ belong to $\Omega \cap B(z_0, 2R)$ where Φ agrees with $\widetilde{\Phi}$. Retaining the symbol $\widetilde{\Gamma}$ for said curve, we may estimate

$$\text{length}(\widetilde{\Gamma}) \leq r_0^{-2} \cdot \text{length}(\Gamma) = L/r_0^2 \leq C_N \cdot (2R)^2 \delta_o / r_0^2, \tag{2.487}$$

where the first inequality follows from (2.209) and the fact that $\Phi : \Omega \to \widetilde{\Omega}$ is a Lipschitz function with constant $\leq r_0^{-2}$ (cf. (2.476)). Choosing $\delta_o > 0$ sufficiently small, to begin with, so that $C_N \cdot (2R)^2 \delta_o / r_0^2 < \varepsilon_o$, we ultimately conclude that $\text{length}(\widetilde{\Gamma}) < \varepsilon_o$. Hence, once again, $\widetilde{\Gamma}$ is a connected subset of $\widetilde{\Omega}$ containing P, $\widetilde{P}$, and with the property that $|P - Q| \leq \text{length}(\widetilde{\Gamma}) < \varepsilon_o$ for each point $Q \in \widetilde{\Gamma}$.

Let us summarize our progress. In view of (2.472), the proof so far gives that

$$\partial\widetilde{\Omega} \text{ a simple closed curve in the plane.} \tag{2.488}$$

Moreover, since $\Phi(\partial\Omega) \subseteq \partial\widetilde{\Omega}$, the origin $0 \in \mathbb{C}$ is an accumulation point for $\Phi(\partial\Omega)$ (as is visible from (2.475), keeping in mind that $\partial\Omega$ is unbounded), and $\partial\widetilde{\Omega}$ is a closed set, we conclude that $0 \in \partial\widetilde{\Omega}$. In turn, this implies that $\partial\widetilde{\Omega} \setminus \{0\}$ is a simple curve, and that the function given in (2.475) induces a homeomorphism $\Phi : \partial\Omega \to \partial\widetilde{\Omega} \setminus \{0\}$. As a consequence, $\partial\Omega = \Phi^{-1}(\partial\widetilde{\Omega} \setminus \{0\})$ is a simple curve in the plane. In addition, the (upper) Ahlfors regularity property of $\partial\Omega$ ensures that the curve $\partial\Omega$ is locally rectifiable, hence

$$\partial\Omega = \Phi^{-1}(\partial\widetilde{\Omega} \setminus \{0\}) \text{ is a locally rectifiable simple curve in the plane.} \tag{2.489}$$

Next, if $\widetilde{\gamma} : \big[-\frac{\pi}{2}, \frac{\pi}{2}\big] \to \partial\widetilde{\Omega}$ is a parametrization of $\partial\widetilde{\Omega}$ with $\widetilde{\gamma}(\pm\pi/2) = 0$, then

$$\gamma : \mathbb{R} \to \partial\Omega, \quad \gamma(t) := \Phi^{-1}\big(\widetilde{\gamma}(\arctan t)\big) \text{ for each } t \in \mathbb{R}, \tag{2.490}$$

becomes a parametrization of the curve $\partial\Omega$. Given that $\lim\limits_{t\to\pm\infty} |\gamma(t)| = 0$, we ultimately conclude that

$$\partial\Omega \text{ is a Jordan curve passing through infinity in the plane.} \tag{2.491}$$

At this stage, there remains to prove that $\partial\Omega$ satisfies the chord-arc condition (2.463) with a constant $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$. In this regard, we note that (2.381) with $n = 2$ gives that there exists a finite geometrical constant $C_o > 1$, independent of δ, with the property that

$$\left| \frac{\mathcal{H}^1\big(B(z,r) \cap \partial\Omega\big)}{2r} - 1 \right| \leq C_o \, \delta \ln(1/\delta), \qquad \forall z \in \partial\Omega, \quad \forall r \in (0, \infty). \tag{2.492}$$

Without loss of generality, for the remainder of the proof assume $\delta \in (0, 1)$ is small enough so that $0 < \delta \ln(1/\delta) < 1/(4C_o)$. Consider now two points $z_1, z_2 \in \partial\Omega$. Abbreviate $r := \ell(z_1, z_2)$ and denote by z_3 the first exit point of the curve $\partial\Omega$ out of $B(z_1, r)$. Hence, $|z_1 - z_3| = r$ and the ordering z_1, z_2, z_3 conforms with the positive orientation of $\partial\Omega$. Moreover,

$$\text{the portion of } \partial\Omega \text{ between } z_1 \text{ and } z_3 \text{ is contained inside } B(z_1, r). \tag{2.493}$$

To proceed, introduce $\Delta := B(z_1, r) \cap \partial\Omega$ and decompose $\Delta = \Delta^+ \cup \Delta^-$ (disjoint union), where $\Delta^\pm$ denote the sets of points in Δ lying, respectively, to the left and to the right of z_1. Also, denote by $\ell(\Delta^\pm)$ the arc-lengths of $\Delta^\pm$. Then

$$\mathcal{H}^1\big(B(z_1, r) \cap \partial\Omega\big) = \ell(\Delta^-) + \ell(\Delta^+) \quad \text{and} \quad \ell(\Delta^\pm) \geq r. \tag{2.494}$$

Making use of (2.492) and (2.494) we may therefore estimate

$$\begin{aligned} C_o\,\delta \ln(1/\delta) &\geq \left| \frac{\mathcal{H}^1\big(B(z, r) \cap \partial\Omega\big)}{2r} - 1 \right| = \left| \frac{\ell(\Delta^-) - r}{2r} + \frac{\ell(\Delta^+) - r}{2r} \right| \\ &= \frac{\ell(\Delta^-) - r}{2r} + \frac{\ell(\Delta^+) - r}{2r} \geq \frac{\ell(\Delta^+) - r}{2r}. \end{aligned} \tag{2.495}$$

Hence, by (2.493) and (2.495),

$$|z_2 - z_3| \leq \ell(\Delta^+) - r \leq 2rC_o\,\delta \ln(1/\delta) \tag{2.496}$$

which further implies

$$\begin{aligned} |z_1 - z_2| &\geq |z_1 - z_3| - |z_2 - z_3| \geq r - 2rC_o\,\delta \ln(1/\delta) \\ &= \big(1 - 2C_o\,\delta \ln(1/\delta)\big)\ell(z_1, z_2). \end{aligned} \tag{2.497}$$

This proves that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \ \text{ with } \ \varkappa := \frac{2C_o\,\delta \ln(1/\delta)}{1 - 2C_o\,\delta \ln(1/\delta)}, \tag{2.498}$$

which goes to show that $\partial\Omega$ is a chord-arc curve. Moreover, the fact that we have assumed $0 < \delta \ln(1/\delta) < 1/(4C_o)$ implies $0 < \varkappa < 4C_o\delta \ln(1/\delta)$. In particular, we have $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$. Hence, Ω is a $\varkappa$-CAD with $\varkappa = O\big(\delta \ln(1/\delta)\big)$ as $\delta \to 0^+$, finishing the proof of Theorem 2.7.

□

In closing, we briefly elaborate on a distinguished sub-class of the category of planar chord-arc domains, described in the next definition.

Definition 2.17 Say that $\Omega \subseteq \mathbb{R}^2$ is a `chord-arc domain with vanishing constant` (CAD with vanishing constant, for short) provided Ω is a chord-arc domain in the sense of Definition 2.16 and

$$\lim_{R\to 0^+}\left\{\sup_{\substack{z_1,z_2\in\partial\Omega\\ |z_1-z_2|<R}}\left(\frac{\ell(z_1,z_2)}{|z_1-z_2|}-1\right)\right\}=0, \tag{2.499}$$

where $\ell(\cdot,\cdot)$ denotes the shortest arc-length between points on $\partial\Omega$.

The proposition below shows that, in the two-dimensional setting, VMO_1 domains (of upper-graph type) are chord-arc domains with vanishing constant. Before stating it, the reader is reminded that the Sarason space of functions of vanishing mean oscillations has been introduced in (2.111).

Proposition 2.18 *Let $\varphi\in W^{1,1}_{loc}(\mathbb{R})$ be such that $\varphi'\in VMO(\mathbb{R},\mathcal{L}^1)$ and consider its upper-graph $\Omega:=\big\{(x,y):\ x\in\mathbb{R},\ \ y>\varphi(x)\big\}\subseteq\mathbb{R}^2$. Then Ω is a chord-arc domain with vanishing constant.*

Proof That Ω is a chord-arc domain follows from Definition 2.16 and Proposition 2.13. Finally, the vanishing property (2.499) is seen from Definition 2.17 and an inspection of the proof of Proposition 2.13, bearing in mind (2.112). □

2.7 Dyadic Grids and Muckenhoupt Weights on Ahlfors Regular Sets

The following result, pertaining to the existence of a dyadic grid structure on a given Ahlfors regular set, is essentially due to M. Christ [27] (cf. also [40], [41]), with some refinements worked out in [63, Proposition 2.11, pp. 19-20].

Proposition 2.19 *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed, unbounded, Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Then there are finite constants $a_1\geq a_0>0$ such that for each $m\in\mathbb{Z}$ there exists a collection*

$$\mathbb{D}_m(\Sigma):=\{Q^m_\alpha\}_{\alpha\in I_m} \tag{2.500}$$

of subsets of Σ indexed by a nonempty, at most countable set of indices I_m, as well as a family $\{x^m_\alpha\}_{\alpha\in I_m}$ of points in Σ, for which the collection of all dyadic cubes in Σ, i.e.,

$$\mathbb{D}(\Sigma):=\bigcup_{m\in\mathbb{Z}}\mathbb{D}_m(\Sigma), \tag{2.501}$$

has the following properties:

(1) [All dyadic cubes are open] *For each $m\in\mathbb{Z}$ and each $\alpha\in I_m$ the set Q^m_α is relatively open in Σ.*

(2) [Dyadic cubes are mutually disjoint within the same generation] *For each integer* $m \in \mathbb{Z}$ *and each* $\alpha, \beta \in I_m$ *with* $\alpha \neq \beta$ *there holds* $Q_\alpha^m \cap Q_\beta^m = \varnothing$.

(3) [No partial overlap across generations] *For each* $m, \ell \in \mathbb{Z}$ *with* $\ell > m$ *and each* $\alpha \in I_m$, $\beta \in I_\ell$, *either* $Q_\beta^\ell \subseteq Q_\alpha^m$ *or* $Q_\alpha^m \cap Q_\beta^\ell = \varnothing$.

(4) [Any dyadic cube has a unique ancestor in any earlier generation] *For each integers* $m, \ell \in \mathbb{Z}$ *with* $m > \ell$ *and each* $\alpha \in I_m$ *there is a unique* $\beta \in I_\ell$ *such that* $Q_\alpha^m \subseteq Q_\beta^\ell$. *In particular, for each* $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *there exists a unique* $\beta \in I_{m-1}$ *such that* $Q_\alpha^m \subseteq Q_\beta^{m-1}$ *(a scenario in which* Q_β^{m-1} *is referred to as the parent of* Q_α^m*).*

(5) [The size is dyadically related to the generation] *For each* $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *one has*

$$\Delta(x_\alpha^m, a_0 2^{-m}) \subseteq Q_\alpha^m \subseteq \Delta_{Q_\alpha^m} := \Delta(x_\alpha^m, a_1 2^{-m}). \tag{2.502}$$

(6) [Control of the number of children] *There exists an integer* $M \in \mathbb{N}$ *with the property that for each* $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *one has*

$$\#\big\{\beta \in I_{m+1} : \; Q_\beta^{m+1} \subseteq Q_\alpha^m\big\} \leq M. \tag{2.503}$$

Also, this integer may be chosen such that for each $m \in \mathbb{Z}$, *each* $x \in \Sigma$, *and each* $r \in (0, 2^{-m})$ *the number of* Q*'s in* $\mathbb{D}_m(\Sigma)$ *that intersect* $\Delta(x, r)$ *is at most* M.

(7) [Each generation covers the space σ-a.e.] *For each* $m \in \mathbb{Z}$ *one has*

$$\sigma\Big(\Sigma \setminus \bigcup_{\alpha \in I_m} Q_\alpha^m\Big) = 0. \tag{2.504}$$

In particular,

$$N := \bigcup_{m \in \mathbb{Z}} \Big(\Sigma \setminus \bigcup_{\alpha \in I_m} Q_\alpha^m\Big) \Longrightarrow \sigma(N) = 0, \tag{2.505}$$

and for each $m \in \mathbb{Z}$ *and each* $\alpha \in I_m$ *one has*

$$\sigma\Big(Q_\alpha^m \setminus \bigcup_{\beta \in I_{m+1},\, Q_\beta^{m+1} \subseteq Q_\alpha^m} Q_\beta^{m+1}\Big) = 0. \tag{2.506}$$

(8) [Dyadic cubes have thin boundaries] *There exist some small* $\vartheta \in (0, 1)$ *along with some large* $C \in (0, \infty)$, *such that for each* $m \in \mathbb{Z}$, *each* $\alpha \in I_m$, *and each* $t > 0$ *one has*

$$\sigma\left(\big\{x \in Q_\alpha^m : \; dist(x, \Sigma \setminus Q_\alpha^m) \leq t \cdot 2^{-m}\big\}\right) \leq C t^\vartheta \cdot \sigma(Q_\alpha^m). \tag{2.507}$$

Moving on, assume $\Sigma \subseteq \mathbb{R}^n$ is a closed set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. It has been noted in [111, §3.6] that

$$\begin{aligned}&\text{if } \mathcal{H}^{n-1}(K\cap\Sigma) < +\infty \text{ for each compact subset } K \text{ of } \mathbb{R}^n \text{ then } \sigma\\ &\text{is a complete, locally finite (hence also sigma-finite), separable,}\\ &\text{Borel-regular measure on } \Sigma, \text{ where the latter set is endowed with}\\ &\text{the topology canonically inherited from the ambient space.}\end{aligned} \tag{2.508}$$

Let w be a `weight` on Σ, i.e., a σ-measurable function satisfying $0 < w(x) < \infty$ for σ-a.e. point $x \in \Sigma$. We agree to also use the symbol w for the weighted measure $w\,\sigma$, i.e., define

$$w(E) := \int_E w\,\mathrm{d}\sigma \quad \text{for each } \sigma\text{-measurable set} \quad E \subseteq \Sigma. \tag{2.509}$$

Then the measures w and σ have the same sigma-algebra of measurable sets, and are mutually absolutely continuous with each other. Recall that, for a generic measure space (X, μ), the measure μ is said to be `semi-finite` if for each μ-measurable set $E \subseteq X$ with $\mu(E) = \infty$ there exists some μ-measurable set $F \subseteq E$ such that $0 < \mu(F) < \infty$ (cf., e.g., [51, p. 25]).

Lemma 2.11 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Let w be an arbitrary weight on Σ and pick an arbitrary σ-measurable set $\Delta \subseteq \Sigma$ with $\sigma(\Delta) < \infty$. Then the measure $w\lfloor\Delta$ is semi-finite and, whenever $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, it follows that*

$$\|w^{-1}\|_{L^{p'}(\Delta,w)} = \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\,d\sigma. \tag{2.510}$$

Proof Consider a w-measurable set $E \subseteq \Delta$ with $w(E) = \infty$. In particular, the set E is σ-measurable. If for each $N \in \mathbb{N}$ we define $E_N := \{x \in E : w(x) < N\}$ then E_N is a σ-measurable subset of Δ and the inclusion $E_N \subseteq E_{N+1}$ holds. In addition, $\bigcup_{N\in\mathbb{N}} E_N = \{x \in E : w(x) < \infty\}$ hence $\sigma\big(E \setminus \bigcup_{N\in\mathbb{N}} E_N\big) = 0$. Consequently,

$$\lim_{N\to\infty} w(E_N) = \lim_{N\to\infty} \int_{E_N} w\,\mathrm{d}\sigma = \int_E w\,\mathrm{d}\sigma = w(E) = \infty, \tag{2.511}$$

by Lebesgue's Monotone Convergence Theorem. In turn, (2.511) implies that there exists $N_o \in \mathbb{N}$ such that $w(E_{N_o}) > 0$. Since we also have

$$w(E_{N_o}) = \int_{E_{N_o}} w\,\mathrm{d}\sigma \le N_o \cdot \sigma(E_{N_o}) \le N_o \cdot \sigma(\Delta) < \infty, \tag{2.512}$$

we conclude that E_{N_o} is a w-measurable subset of E with $0 < w(E_{N_o}) < \infty$. This implies that $w\lfloor\Delta$ is indeed a semi-finite measure.

With an eye on the claim made in (2.510), define $S_{\rm fin}(\Delta, w)$ to be the vector space of all complex-valued functions defined on Δ which may be expressed in the form $f = \sum_{j=1}^{N} \lambda_j \mathbf{1}_{E_j}$ where $N \in \mathbb{N}$, each λ_j is a complex number, the family of sets $\{E_j\}_{1\leq j\leq N}$ consists of w-measurable mutually disjoint subsets of Δ which also satisfies $w\big(\bigcup_{j=1}^{N} E_j\big) < +\infty$. Note that each such function f happens to be σ-measurable and, for each $q \in (0, \infty)$, satisfies $\int_\Delta |f|^q \leq \sum_{j=1}^{N} |\lambda_j|^q \cdot \sigma(\Delta) < \infty$. Hence,

$$S_{\rm fin}(\Delta, w) \subseteq \bigcap_{0<q<\infty} L^q(\Delta, \sigma) \tag{2.513}$$

and, in particular,

$$f w^{-1} \in L^1(\Delta, w) \ \text{ for each } \ f \in S_{\rm fin}(\Delta, w). \tag{2.514}$$

Having picked $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$, we may then write

$$\begin{aligned} \|w^{-1}\|_{L^{p'}(\Delta,w)} &= \sup_{\substack{f\in S_{\rm fin}(\Delta,w) \\ \|f\|_{L^p(\Delta,w)}=1}} \Big| \int_\Delta f w^{-1}\,{\rm d}w \Big| = \sup_{\substack{f\in S_{\rm fin}(\Delta,w) \\ \|f\|_{L^p(\Delta,w)}=1}} \Big| \int_\Delta f\,{\rm d}\sigma \Big| \\ &\leq \sup_{\substack{f\in L^p(\Delta,w) \\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\,{\rm d}\sigma. \end{aligned} \tag{2.515}$$

The first equality above is a consequence of [51, Theorem 6.14, p. 189], whose applicability in the present setting is ensured by (2.514) and the fact that the measure $w\lfloor\Delta$ is semi-finite. The second equality in (2.515) is justified upon recalling that ${\rm d}w = w{\rm d}\sigma$, and the inequality in (2.515) is trivial. There remains to observe that for each $f \in L^p(\Delta, w)$ with $\|f\|_{L^p(\Delta,w)} = 1$ Hölder's inequality gives

$$\int_\Delta |f|\,{\rm d}\sigma = \int_\Delta |f| w^{-1}\,{\rm d}w \leq \|w^{-1}\|_{L^{p'}(\Delta,w)}. \tag{2.516}$$

At this stage, (2.510) becomes a consequence of (2.515) and (2.516). □

Next, assume that $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Given $p \in (1, \infty)$, we say that a weight w on Σ belongs to the Muckenhoupt class $A_p(\Sigma, \sigma)$ if

$$[w]_{A_p} := \sup_{\Delta\subseteq\Sigma} \Big(⨍_\Delta w(x)\,{\rm d}\sigma(x) \Big) \Big(⨍_\Delta w(x)^{1-p'}\,{\rm d}\sigma(x) \Big)^{p-1} < \infty, \tag{2.517}$$

where p' is the conjugate exponent of p (i.e., $p' \in (1, \infty)$ satisfies $1/p + 1/p' = 1$) and the supremum runs over all surface balls Δ in Σ. The expression in (2.517) arises naturally since for each weight function and each surface ball $\Delta \subseteq \Sigma$ Hölder's inequality gives

$$\begin{aligned} 1 &= \fint_\Delta 1 \, \mathrm{d}\sigma = \fint_\Delta w^{1/p} w^{-1/p} \, \mathrm{d}\sigma \\ &\leq \left(\fint_\Delta w \, \mathrm{d}\sigma \right)^{1/p} \left(\fint_\Delta w^{1-p'} \, \mathrm{d}\sigma \right)^{1/p'}, \end{aligned} \tag{2.518}$$

hence

$$1 \leq \inf_{\Delta \subseteq \Sigma} \left(\fint_\Delta w \, \mathrm{d}\sigma \right) \left(\fint_\Delta w^{1-p'} \, \mathrm{d}\sigma \right)^{p-1} \leq [w]_{A_p} \leq \infty. \tag{2.519}$$

For further use it is useful to note that (2.517) entails that, given any $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$, for each surface ball $\Delta \subseteq \Sigma$ we have

$$\left(\int_\Delta w^{-p'/p} \, \mathrm{d}\sigma \right)^{1/p'} \leq [w]_{A_p}^{1/p} \frac{\sigma(\Delta)}{w(\Delta)^{1/p}}. \tag{2.520}$$

Corresponding to $p = 1$, we say that $w \in A_1(\Sigma, \sigma)$ if

$$[w]_{A_1} := \sup_{\Delta \subseteq \Sigma} \left(\operatorname*{ess\,inf}_{x \in \Delta} w(x) \right)^{-1} \left(\fint_\Delta w \, \mathrm{d}\sigma \right) < \infty. \tag{2.521}$$

It is clear from the above definition that $[w]_{A_1} \geq 1$ for each weight w on Σ. Recall that the (non-centered) Hardy–Littlewood maximal operator $\mathcal{M}$ on Σ acts on each given σ-measurable function f on Σ according to

$$\mathcal{M}f(x) := \sup_{\Delta \ni x} \fint_\Delta |f| \, \mathrm{d}\sigma, \qquad \forall \, x \in \Sigma, \tag{2.522}$$

where the supremum is taken over all surface balls Δ in Σ which contain the point x. In particular,

> a weight w on Σ belongs to $A_1(\Sigma, \sigma)$ if and only if there exists a constant $C \in (0, \infty)$ with the property that $\mathcal{M}w(x) \leq C w(x)$ at σ-a.e. point $x \in \Sigma$, and the best constant is actually $[w]_{A_1}$. (2.523)

Corresponding to the end-point $p = \infty$,

> the class $A_\infty(\Sigma, \sigma)$ is defined as the union of all $A_p(\Sigma, \sigma)$ with $p \in [1, \infty)$. (2.524)

Lemma 2.12 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then for each $p \in (1,\infty)$, each Muckenhoupt weight $w \in A_p(\Sigma,\sigma)$, and each σ-measurable function f on Σ one has*

$$\fint_\Delta |f|\,d\sigma \leq [w]_{A_p}^{1/p}\Big(\fint_\Delta |f|^p\,dw\Big)^{1/p}, \quad \text{for each surface ball } \Delta \subseteq \Sigma. \tag{2.525}$$

Conversely, if $p \in (1,\infty)$ and w is a weight on Σ with the property that there exists a constant $C \in (0,\infty)$ such that

$$\fint_\Delta |f|\,d\sigma \leq C\Big(\fint_\Delta |f|^p\,dw\Big)^{1/p} \text{ for each function } f \in L^p_{loc}(\Sigma,w) \text{ and surface ball } \Delta \subseteq \Sigma, \tag{2.526}$$

then actually $w \in A_p(\Sigma,\sigma)$ and $[w]_{A_p} \leq C^p$.

Proof Let $p' \in (1,\infty)$ denote the Hölder conjugate exponent of p and fix an arbitrary σ-measurable function f on Σ. Then for each surface ball $\Delta \subseteq \Sigma$ we may estimate

$$\begin{aligned}
\fint_\Delta |f|\,\mathrm{d}\sigma &= \frac{1}{\sigma(\Delta)}\int_\Delta |f| w^{1/p} w^{-1/p}\,\mathrm{d}\sigma \\
&\leq \frac{1}{\sigma(\Delta)}\Big(\int_\Delta |f|^p\,w\,\mathrm{d}\sigma\Big)^{1/p}\Big(\int_\Delta w^{-p'/p}\,\mathrm{d}\sigma\Big)^{1/p'} \\
&= \Big(\fint_\Delta w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}\Big(\fint_\Delta w\,\mathrm{d}\sigma\Big)^{1/p}\Big(\fint_\Delta |f|^p\,\mathrm{d}w\Big)^{1/p} \\
&\leq [w]_{A_p}^{1/p}\Big(\fint_\Delta |f|^p\,\mathrm{d}w\Big)^{1/p},
\end{aligned} \tag{2.527}$$

by Hölder's inequality and (2.517). This proves (2.525).

As for the converse, fix $p \in (1,\infty)$ and suppose w is a generic weight function on Σ for which there exists a constant $C \in (0,\infty)$ such that (2.526) holds. Once again, denote $p' \in (1,\infty)$ the Hölder conjugate exponent of p and fix an arbitrary surface ball $\Delta \subseteq \Sigma$. Then, with tilde denoting the extension by zero of a function originally defined on Δ to the entire set Σ, we may write

$$\|w^{-1}\|_{L^{p'}(\Delta,w)} = \sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}}\int_\Delta |f|\,\mathrm{d}\sigma = \sigma(\Delta)\cdot\sup_{\substack{f\in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}}\fint_\Delta |\widetilde{f}|\,\mathrm{d}\sigma$$

$$\leq C\sigma(\Delta) \cdot \sup_{\substack{f \in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \Big(⨍_\Delta |\widetilde{f}|^p \, dw \Big)^{1/p}$$

$$\leq C \frac{\sigma(\Delta)}{w(\Delta)^{1/p}}, \tag{2.528}$$

where the first equality comes from Lemma 2.11, and the first inequality is implied by (2.526). This proves that $\|w^{-1}\|_{L^{p'}(\Delta,w)} \leq C \cdot \sigma(\Delta)/w(\Delta)^{1/p}$ which, after unraveling notation, yields

$$\Big(⨍_\Delta w \, d\sigma \Big) \Big(⨍_\Delta w^{1-p'} \, d\sigma \Big)^{p-1} \leq C^p. \tag{2.529}$$

Ultimately, in view of the arbitrariness of the surface ball $\Delta \subseteq \Sigma$, this implies that $w \in A_p(\Sigma, \sigma)$ and $[w]_{A_p} \leq C^p$. □

In this work we are particularly interested in the scale of weighted Lebesgue space $L^p(\Sigma, w) := L^p(\Sigma, w\sigma)$ with $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$. As in the Euclidean setting,

given a weight w on Σ and an integrability exponent $p \in (1, \infty)$, the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the space $L^p(\Sigma, w)$ if and only if $w \in A_p(\Sigma, \sigma)$, (2.530)

in which case there exists some constant $C = C(\Sigma, n, p) \in (0, \infty)$ (which depends on Σ only through its Ahlfors regularity constant) with the property that

$$\|\mathcal{M}f\|_{L^p(\Sigma,w)} \leq C[w]_{A_p}^{1/(p-1)} \|f\|_{L^p(\Sigma,w)} \quad \text{for all } \; f \in L^p(\Sigma, w) \tag{2.531}$$

(see, e.g., [64, Proposition 7.13]). Also, corresponding to $p = 1$, the operator $\mathcal{M}$ satisfies the weak-(1, 1) inequality

$$\begin{gathered} \sup_{0<\lambda<\infty} \lambda \cdot w\big(\{x \in \Sigma : \mathcal{M}f(x) > \lambda\}\big) \leq C\|f\|_{L^1(\Sigma,w)} \\ \text{for all } f \in L^1(\Sigma, w), \text{ with } C \in (0, \infty) \text{ independent of } f, \end{gathered} \tag{2.532}$$

if and only if $w \in A_1(\Sigma, \sigma)$. For the reader's convenience, other useful properties of Muckenhoupt weights are summarized in the proposition below (for a more extensive discussion pertaining to the theory of weights in the general context of spaces of homogeneous type the reader is referred to [6, 54, 65, 76, 135]).

Proposition 2.20 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. Then the following properties hold.*

(1) [*Openness/Self-Improving*] *If $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ then there exist some $\tau \in (1, \infty)$ and some $\varepsilon \in (0, p-1)$ (both of which depend only on p,*

$[w]_{A_p}$, *n, and the Ahlfors regularity constant of* Σ) *such that*

$$w^\tau \in A_p(\Sigma, \sigma) \;\; and \;\; w \in A_{p-\varepsilon}(\Sigma, \sigma). \tag{2.533}$$

In addition, both $\left[w^\tau\right]_{A_p}$ *and* $[w]_{A_{p-\varepsilon}}$ *are controlled in terms of p,* $[w]_{A_p}$*, n, and the Ahlfors regularity constant of* Σ*. In fact, matters may be arranged so that, in a quantitative fashion,*

$$w^\theta \in A_q(\Sigma, \sigma) \;\; for\ each \;\; \theta \in (\tau^{-1}, \tau) \;\; and \;\; q \in (p-\varepsilon, \infty). \tag{2.534}$$

(2) [*Monotonicity*] *If* $1 \leq p \leq q \leq \infty$ *then* $A_p(\Sigma, \sigma) \subseteq A_q(\Sigma, \sigma)$ *and if* $q < \infty$ *then* $[w]_{A_q} \leq [w]_{A_p}$ *for each* $w \in A_p(\Sigma, \sigma)$.

(3) [*Dual Weights*] *Given any* $w \in A_p(\Sigma, \sigma)$ *with* $p \in (1, \infty)$*, it follows that* $w^{1-p'}$ *belongs to* $A_{p'}(\Sigma, \sigma)$ *and* $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$*, where* $p' \in (1, \infty)$ *is the Hölder conjugate exponent of p.*

(4) [*Products/Factorization*] *If* $w_1, w_2 \in A_1(\Sigma, \sigma)$ *then for every* $p \in (1, \infty)$ *one has* $w_1 \cdot w_2^{1-p} \in A_p(\Sigma, \sigma)$ *and* $[w_1 \cdot w_2^{1-p}]_{A_p} \leq [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}$*. Also, given* $w_1, w_2 \in A_p(\Sigma, \sigma)$ *with* $p \in (1, \infty)$ *along with some* $\alpha \in [0, 1]$*, it follows that* $w_1^\alpha \cdot w_2^{1-\alpha} \in A_p(\Sigma, \sigma)$ *and* $[w_1^\alpha \cdot w_2^{1-\alpha}]_{A_p} \leq [w_1]_{A_p}^\alpha \cdot [w_2]_{A_p}^{1-\alpha}$.

(5) [*Doubling*] *If* $w \in A_p(\Sigma, \sigma)$ *with* $p \in (1, \infty)$ *then for every surface ball* Δ *in* Σ *and every* σ*-measurable set* $E \subseteq \Delta$ *one has*

$$\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^p \leq [w]_{A_p} \cdot \frac{w(E)}{w(\Delta)}. \tag{2.535}$$

In particular, the measure w is doubling, that is, there exists some $C \in (0, \infty)$ *which depends only on p, n, and the Ahlfors regularity constant of* Σ*, such that* $w(2\Delta) \leq C[w]_{A_p} \cdot w(\Delta)$ *for every surface ball* $\Delta \subseteq \Sigma$*. More generally, with the constant* $C \in (0, \infty)$ *of the same nature as above, one has the inequality* $w(\lambda\Delta) \leq C[w]_{A_p} \cdot \lambda^{p(n-1)} \cdot w(\Delta)$ *for each* $\lambda \in (1, \infty)$ *and each surface ball* $\Delta \subseteq \Sigma$ (*where* $\lambda\Delta$ *denotes the concentric dilate of* Δ *by a factor of* λ).

(6) [*Reverse Hölder Inequalities*] *For every* $w \in A_\infty(\Sigma, \sigma)$ *there exist* $q \in (1, \infty)$ *and some* $C \in (0, \infty)$ (*which both depend only on p,* $[w]_{A_p}$*, n, and the Ahlfors regularity constant of* Σ*, for some* $p \in (1, \infty)$ *for which* $w \in A_p(\Sigma, \sigma)$) *such that*

$$\left(⨍_\Delta w^q \, d\sigma\right)^{1/q} \leq C ⨍_\Delta w \, d\sigma, \tag{2.536}$$

for every surface ball $\Delta \subseteq \Sigma$*. This has several remarkable consequences. First, there exist some power* $\tau > 0$ *and some constant* $C \in (0, \infty)$ (*in fact, C is the same as in* (2.536) *and* $\tau = 1/q'$ *where* q' *is the Hölder conjugate of the exponent q from* (2.536)) *such that*

$$\frac{w(E)}{w(\Delta)} \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^{\tau} \tag{2.537}$$

for every surface ball $\Delta \subseteq \Sigma$ *and every* σ*-measurable set* $E \subseteq \Delta$. *Another useful consequence of the inequality in* (2.536) *and Hölder's inequality is that for each* σ*-measurable function* f *on* Σ *and each surface ball* $\Delta \subseteq \Sigma$ *one has*

$$⨍_{\Delta} |f|\, dw \leq C\left(⨍_{\Delta} |f|^{q'}\, d\sigma\right)^{1/q'}, \tag{2.538}$$

where $q' \in (1, \infty)$ *is the Hölder conjugate exponent of* q *from* (2.536)*, and the constant* $C \in (0, \infty)$ *is as in* (2.536)*. Finally, in the case when* Σ *is unbounded,* (2.537) (*used with* $\Delta = \Delta(x, r)$ *and* $E = \Delta(x, 1)$) *proves that there exists some* $c \in (0, \infty)$ *such that*

$$w\big(\Delta(x, r)\big) \geq c\, r^{(n-1)\tau} \cdot w\big(\Delta(x, 1)\big) \quad \text{for each } x \in \Sigma \text{ and } r \in (1, \infty). \tag{2.539}$$

In particular,

$$w(\Sigma) = +\infty \quad \text{if } \Sigma \text{ is unbounded.} \tag{2.540}$$

(7) [*Building* A_1 *Weights*] *There exists* $C \in (0, \infty)$ *which depends only on* n *and* Σ*, with the property that if* $f \in L^1_{loc}(\Sigma, \sigma)$ *is not identically zero and* $\mathcal{M}f < \infty$ *at* σ*-a.e. point on* Σ *then for each* $\theta \in (0, 1)$ *one has* $(\mathcal{M}f)^\theta \in A_1(\Sigma, \sigma)$ *and* $\big[(\mathcal{M}f)^\theta\big]_{A_1} \leq C(1-\theta)^{-1}$. *In addition, for each power* $\theta \in (0, 1)$ *the weight* $w := (\mathcal{M}f)^\theta$ *satisfies a reverse Hölder inequality* (*as in* (2.536)) *for each exponent* $q \in (1, \theta^{-1})$.

(8) [BMO *and Weights*] *For each* $p \in (1, \infty)$ *and* $w \in A_p(\Sigma, \sigma)$ *there exist some small* $\varepsilon = \varepsilon(\Sigma, p, [w]_{A_p}) > 0$ *and some large* $C = C(\Sigma, p, [w]_{A_p}) \in (0, \infty)$ *such that for each function* $b \in BMO(\Sigma, \sigma)$ *with* $\|b\|_{BMO(\Sigma,\sigma)} < \varepsilon$ *one has* $w \cdot e^b \in A_p(\Sigma, \sigma)$ *and* $\big[w \cdot e^b\big]_{A_p} \leq C$. *In particular, for each fixed integrability exponent* $p \in (1, \infty)$ *the set* $\mathcal{U}_p := \big\{b \in BMO(\Sigma, \sigma) : e^b \in A_p(\Sigma, \sigma)\big\}$ *is open in* $BMO(\Sigma, \sigma)$. *Also, for each weight* $w \in A_1(\Sigma, \sigma)$*, the function* $\log w$ *belongs to* $BMO(\Sigma, \sigma)$ *and* $\|\log w\|_{BMO(\Sigma,\sigma)} \leq C(\Sigma, n, [w]_{A_1})$. *Finally, for each function* $b \in BMO(\Sigma, \sigma)$ *and each exponent* $p \in (1, \infty)$*, the function* $\max\{1, |b|\}$ *belongs to* $A_p(\Sigma, \sigma)$ *and there exists* $C_{\Sigma,p} \in (0, \infty)$*, independent of* b*, such that* $[\max\{1, |b|\}]_{A_p} \leq C_{\Sigma,p}(1 + \|b\|_{BMO(\Sigma,\sigma)})$.

(9) [*Dyadic Cubes*] *If* Σ *is unbounded, then properties* (2.535)*,* (2.536)*, and* (2.537) *also hold if surface balls* Δ *are replaced by dyadic "cubes," as described in Proposition 2.19.*

Proof For the memberships in (2.533), (2.534) (including their quantitative aspects) see [65, Theorems 1.1-1.2], [21, Theorem 2.31, p. 58].

To deal with item *(2)*, suppose $1 \leq p \leq q < \infty$ and denote by p', q' the Hölder conjugate exponents of p and q, respectively. Also, fix an arbitrary weight $w \in A_p(\Sigma, \sigma)$. Then $r := (1 - p')/(1 - q')$ belongs to $[1, \infty)$, so for each surface ball Δ in Σ we may employ Hölder's inequality to write

$$
\begin{aligned}
&\Big(\fint_\Delta w \,\mathrm{d}\sigma\Big)\Big(\fint_\Delta w^{1-q'} \,\mathrm{d}\sigma\Big)^{q-1} \\
&\qquad \leq \Big(\fint_\Delta w \,\mathrm{d}\sigma\Big)\Big(\fint_\Delta w^{r(1-q')} \,\mathrm{d}\sigma\Big)^{(q-1)/r} \\
&\qquad = \Big(\fint_\Delta w \,\mathrm{d}\sigma\Big)\Big(\fint_\Delta w^{1-p'} \,\mathrm{d}\sigma\Big)^{p-1} \leq [w]_{A_p} < +\infty, \qquad (2.541)
\end{aligned}
$$

since $(q-1)/r = p-1$. In view of (2.517), this shows that $w \in A_q(\Sigma, \sigma)$ and we have $[w]_{A_q} \leq [w]_{A_p}$. Finally, the fact that the inclusion $A_p(\Sigma, \sigma) \subseteq A_q(\Sigma, \sigma)$ also holds if $q = \infty$ is clear from (2.524).

Going further, to justify the claim made in item *(3)*, fix some $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$, and denote by $p' \in (1, \infty)$ the Hölder conjugate exponent of p. Then for each surface ball Δ in Σ we may write

$$
\begin{aligned}
&\Big(\fint_\Delta w^{1-p'} \,\mathrm{d}\sigma\Big)\Big(\fint_\Delta (w^{1-p'})^{1-p} \,\mathrm{d}\sigma\Big)^{p'-1} \\
&\qquad = \Big(\fint_\Delta w^{1-p'} \,\mathrm{d}\sigma\Big)\Big(\fint_\Delta w \,\mathrm{d}\sigma\Big)^{p'-1} \\
&\qquad \leq [w]_{A_p}^{p'-1} = [w]_{A_p}^{1/(p-1)} < +\infty, \qquad (2.542)
\end{aligned}
$$

thanks to (2.517). This implies that $w^{1-p'}$ belongs to $A_{p'}(\Sigma, \sigma)$ and that we have $[w^{1-p'}]_{A_{p'}} \leq [w]_{A_p}^{1/(p-1)}$. Writing this last inequality with p replaced by p' and with w replaced by $w^{1-p'}$ yields $[(w^{1-p'})^{1-p}]_{A_p} \leq [w^{1-p'}]_{A_{p'}}^{1/(p'-1)}$. Hence, we have $[w]_{A_p}^{1/(p-1)} \leq [w^{1-p'}]_{A_{p'}}$ which ultimately proves that $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$.

To deal with the first claim made in item *(4)*, recall from (2.521) that, since $w_2 \in A_1(\Sigma, \sigma)$, for each surface ball Δ in Σ we have

$$
\fint_\Delta w_2 \,\mathrm{d}\sigma \leq [w_2]_{A_1} \cdot w_2 \quad \text{at } \sigma\text{-a.e. point in } \Delta. \qquad (2.543)
$$

Given that $1 - p < 0$, this entails

$$w_2^{1-p} \leq [w_2]_{A_1}^{p-1} \cdot \Big(\fint_\Delta w_2 \, \mathrm{d}\sigma \Big)^{1-p} \quad \text{at } \sigma\text{-a.e. point in } \ \Delta, \tag{2.544}$$

which further implies

$$\fint_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \leq [w_2]_{A_1}^{p-1} \cdot \Big(\fint_\Delta w_1 \, \mathrm{d}\sigma \Big) \Big(\fint_\Delta w_2 \, \mathrm{d}\sigma \Big)^{1-p}. \tag{2.545}$$

In a similar manner, the fact that $w_1 \in A_1(\Sigma, \sigma)$ implies

$$w_1^{-1/(p-1)} \leq [w_1]_{A_1}^{1/(p-1)} \cdot \Big(\fint_\Delta w_1 \, \mathrm{d}\sigma \Big)^{-1/(p-1)} \quad \text{at } \sigma\text{-a.e. point in } \ \Delta, \tag{2.546}$$

hence

$$\begin{aligned} \Big(\fint_\Delta (w_1 \cdot w_2^{1-p})^{-1/(p-1)} \, \mathrm{d}\sigma \Big)^{p-1} &= \Big(\fint_\Delta w_1^{-1/(p-1)} \cdot w_2 \, \mathrm{d}\sigma \Big)^{p-1} \\ &\leq [w_1]_{A_1} \cdot \Big(\fint_\Delta w_1 \, \mathrm{d}\sigma \Big)^{-1} \Big(\fint_\Delta w_2 \, \mathrm{d}\sigma \Big)^{p-1}. \end{aligned} \tag{2.547}$$

By combining (2.545) with (2.547) we therefore arrive at the conclusion that, with p' denoting the Hölder conjugate exponent of p,

$$\begin{aligned} &\Big(\fint_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \Big) \Big(\fint_\Delta (w_1 \cdot w_2^{1-p})^{1-p'} \, \mathrm{d}\sigma \Big)^{p-1} \\ &\qquad = \Big(\fint_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \Big) \Big(\fint_\Delta (w_1 \cdot w_2^{1-p})^{-1/(p-1)} \, \mathrm{d}\sigma \Big)^{p-1} \\ &\qquad \leq [w_2]_{A_1}^{p-1} \cdot \Big(\fint_\Delta w_1 \, \mathrm{d}\sigma \Big) \Big(\fint_\Delta w_2 \, \mathrm{d}\sigma \Big)^{1-p} \times \\ &\qquad\quad \times [w_1]_{A_1} \cdot \Big(\fint_\Delta w_1 \, \mathrm{d}\sigma \Big)^{-1} \Big(\fint_\Delta w_2 \, \mathrm{d}\sigma \Big)^{p-1} \\ &\qquad = [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}. \end{aligned} \tag{2.548}$$

Thus, with the supremum running over all surface balls Δ in Σ, we have (cf. (2.517))

$$[w_1 \cdot w_2^{1-p}]_{A_p} = \sup_{\Delta \subseteq \Sigma} \Big(\fint_\Delta w_1 \cdot w_2^{1-p} \, \mathrm{d}\sigma \Big) \Big(\fint_\Delta (w_1 \cdot w_2^{1-p})^{1-p'} \, \mathrm{d}\sigma \Big)^{p-1}$$

$$= [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1} < +\infty, \tag{2.549}$$

proving that the weight $w_1 \cdot w_2^{1-p}$ belongs to the Muckenhoupt class $A_p(\Sigma, \sigma)$ and that we have $[w_1 \cdot w_2^{1-p}]_{A_p} \leq [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}$.

As regards the second claim in item *(4)*, pick two weights $w_1, w_2 \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ fix some $\alpha \in [0, 1]$. If $\alpha = 0$ or $\alpha = 1$ there is nothing to prove, so assume $\alpha \in (0, 1)$. With "prime" indicating a conjugate exponent, for each surface ball Δ in Σ Hölder's inequality gives

$$\begin{aligned} ⨍_\Delta w_1^\alpha \cdot w_2^{1-\alpha}\, d\sigma &\leq \Big(⨍_\Delta (w_1^\alpha)^{1/\alpha}\, d\sigma \Big)^\alpha \Big(⨍_\Delta (w_2^{1-\alpha})^{(1/\alpha)'}\, d\sigma \Big)^{1/(1/\alpha)'} \\ &= \Big(⨍_\Delta w_1\, d\sigma \Big)^\alpha \Big(⨍_\Delta w_2\, d\sigma \Big)^{1-\alpha}, \end{aligned} \tag{2.550}$$

since $(1/\alpha)' = (1-\alpha)^{-1}$. Similarly,

$$\begin{aligned} &\Big(⨍_\Delta (w_1^\alpha \cdot w_2^{1-\alpha})^{1-p'}\, d\sigma \Big)^{p-1} \\ &\qquad \leq \Big(⨍_\Delta w_1^{1-p'}\, d\sigma \Big)^{\alpha(p-1)} \Big(⨍_\Delta w_2^{1-p'}\, d\sigma \Big)^{(1-\alpha)(p-1)}. \end{aligned} \tag{2.551}$$

Together, (2.550) and (2.551) show that

$$\begin{aligned} &\Big(⨍_\Delta w_1^\alpha \cdot w_2^{1-\alpha}\, d\sigma \Big) \Big(⨍_\Delta (w_1^\alpha \cdot w_2^{1-\alpha})^{1-p'}\, d\sigma \Big)^{p-1} \\ &\leq \Big[\Big(⨍_\Delta w_1\, d\sigma \Big) \Big(⨍_\Delta w_1^{1-p'}\, d\sigma \Big)^{p-1} \Big]^\alpha \Big[\Big(⨍_\Delta w_2\, d\sigma \Big) \Big(⨍_\Delta w_2^{1-p'}\, d\sigma \Big)^{p-1} \Big]^{1-\alpha} \\ &\leq [w_1]_{A_p}^\alpha \cdot [w_2]_{A_p}^{1-\alpha} < +\infty. \end{aligned} \tag{2.552}$$

After taking the supremum over all surface balls $\Delta \subseteq \Sigma$, we then conclude from (2.552) that $w_1^\alpha \cdot w_2^{1-\alpha} \in A_p(\Sigma, \sigma)$ and $[w_1^\alpha \cdot w_2^{1-\alpha}]_{A_p} \leq [w_1]_{A_p}^\alpha \cdot [w_2]_{A_p}^{1-\alpha}$.

Moving on, the estimate in (2.535) may be seen from Lemma 2.12, used here with $f := \mathbf{1}_E$. In concert with the Ahlfors regularity of Σ, this implies all subsequent claims in item *(5)*.

The reverse Hölder inequality claimed in (2.536) is contained in [65, Theorem 2.3], [135, Theorem 15, p. 9]. Moreover, if q' is the Hölder conjugate of the exponent q from (2.536) then for every surface ball $\Delta \subseteq \Sigma$ and every σ-measurable set $E \subseteq \Delta$ we may estimate

$$\frac{w(E)}{w(\Delta)} = \fint_{\Delta} \mathbf{1}_E \, dw = \frac{\sigma(\Delta)}{w(\Delta)} \fint_{\Delta} \mathbf{1}_E \, w \, d\sigma$$

$$\leq \frac{\sigma(\Delta)}{w(\Delta)} \Big(\fint_{\Delta} \mathbf{1}_E \, d\sigma \Big)^{1/q'} \Big(\fint_{\Delta} w^q \, d\sigma \Big)^{1/q}$$

$$\leq C \frac{\sigma(\Delta)}{w(\Delta)} \Big(\fint_{\Delta} \mathbf{1}_E \, d\sigma \Big)^{1/q'} \Big(\fint_{\Delta} w \, d\sigma \Big) = C \left(\frac{\sigma(E)}{\sigma(\Delta)} \right)^{1/q'}, \tag{2.553}$$

thanks to Hölder's inequality and (2.536). This proves (2.537) with $\tau := 1/q' > 0$ and $C \in (0, \infty)$ the same constant as in (2.536).

Consider next the first claim made in item *(7)*. Suppose $f \in L^1_{\text{loc}}(\Sigma, \sigma)$ is not identically zero and has the property that $\mathcal{M} f < \infty$ at σ-a.e. point on Σ. Fix an arbitrary surface ball $\Delta \subseteq \Sigma$ and decompose $f = f_1 + f_2$ with $f_1 := f \mathbf{1}_{2\Delta}$ and $f_2 := f \mathbf{1}_{\Sigma \setminus 2\Delta}$. Having $\mathcal{M} f < \infty$ at σ-a.e. point on Σ entails $f_1 \in L^1(\Sigma, \sigma)$. Since $0 < \theta < 1$ and $0 \leq \mathcal{M} f \leq \mathcal{M} f_1 + \mathcal{M} f_2$, we conclude that

$$(\mathcal{M} f)^\theta \leq (\mathcal{M} f_1)^\theta + (\mathcal{M} f_2)^\theta \quad \text{on} \quad \Sigma. \tag{2.554}$$

Based on Kolmogorov's inequality, the fact that $\mathcal{M}$ satisfies the weak-$(1,1)$ inequality, the membership of f_1 to $L^1(\Sigma, \sigma)$, and the fact that the measure σ is doubling we may estimate

$$\Big(\fint_{\Delta} |\mathcal{M} f_1|^\theta \, d\sigma \Big)^{1/\theta} \leq \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \sigma(\Delta)^{-1} \|\mathcal{M} f_1\|_{L^{1,\infty}(\Sigma,\sigma)}$$

$$\leq C \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \sigma(\Delta)^{-1} \|f_1\|_{L^1(\Sigma,\sigma)}$$

$$\leq C \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \fint_{2\Delta} |f| \, d\sigma$$

$$\leq C \Big(\frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \inf_{x \in 2\Delta} (\mathcal{M} f)(x). \tag{2.555}$$

Hence, on the one hand,

$$\fint_{\Delta} |\mathcal{M} f_1|^\theta \, d\sigma \leq \frac{C}{1-\theta} \Big(\inf_{x \in 2\Delta} (\mathcal{M} f)(x) \Big)^\theta. \tag{2.556}$$

On the other hand, the fact that

$$\begin{array}{l} \text{for each surface ball } \Delta' \subseteq \Sigma \text{ so that } \Delta' \cap \Delta \neq \varnothing \\ \text{and } \Delta' \cap (\Sigma \setminus 2\Delta) \neq \varnothing \text{ it follows that } \Delta \subseteq 6\Delta' \end{array} \tag{2.557}$$

readily implies that there exists a geometric constant $C \in (0, \infty)$ with the property that

$$(\mathcal{M}f_2)(y) \le C(\mathcal{M}f_2)(x) \quad \text{for each} \quad x, y \in \Delta. \tag{2.558}$$

In turn, this forces

$$⨍_{\Delta} |\mathcal{M}f_2|^{\theta}\,\mathrm{d}\sigma \le C\Big(\inf_{x\in\Delta}(\mathcal{M}f_2)(x)\Big)^{\theta} \le C\Big(\inf_{x\in\Delta}(\mathcal{M}f)(x)\Big)^{\theta} \tag{2.559}$$

which, in concert with (2.556) and (2.554) proves that

$$⨍_{\Delta} |\mathcal{M}f|^{\theta}\,\mathrm{d}\sigma \le \frac{C}{1-\theta}\cdot \inf_{x\in\Delta}[(\mathcal{M}f)(x)]^{\theta}. \tag{2.560}$$

Since $0 < [(\mathcal{M}f)(x)]^{\theta} < \infty$ for σ-a.e. point $x \in \Sigma$, ultimately (2.560) implies that $(\mathcal{M}f)^{\theta} \in A_1(\Sigma, \sigma)$ and $\big[(\mathcal{M}f)^{\theta}\big]_{A_1} \le C(1-\theta)^{-1}$.

To show that for each $\theta \in (0, 1)$ and $q \in (1, \theta^{-1})$ the weight $w := (\mathcal{M}f)^{\theta}$ satisfies (2.536), observe that $\widetilde{\theta} := \theta q \in (0, 1)$ so we may invoke (2.560) (for $\widetilde{\theta}$) to write, for every surface ball $\Delta \subseteq \Sigma$,

$$\begin{aligned}\Big(⨍_{\Delta} w^q\,\mathrm{d}\sigma\Big)^{1/q} &= \Big(⨍_{\Delta} |\mathcal{M}f|^{\widetilde{\theta}}\,\mathrm{d}\sigma\Big)^{1/q} \\ &\le \Big(\frac{C}{1-\widetilde{\theta}}\Big)^{1/q}\cdot\Big(\inf_{\Delta}(\mathcal{M}f)^{\widetilde{\theta}}\Big)^{1/q} = \Big(\frac{C}{1-\theta q}\Big)^{1/q}\cdot\Big(\inf_{\Delta}(\mathcal{M}f)^{\theta}\Big) \\ &\le \Big(\frac{C}{1-\theta q}\Big)^{1/q} ⨍_{\Delta} |\mathcal{M}f|^{\theta} = \Big(\frac{C}{1-\theta q}\Big)^{1/q} ⨍_{\Delta} w\,\mathrm{d}\sigma,\end{aligned} \tag{2.561}$$

as wanted. This completes the treatment of item *(7)*.

For the first two claims in item *(8)* see [69, p. 33 and p. 60] for a proof in the Euclidean ambient which readily adapts to the present setting, given the availability of a John-Nirenberg inequality for doubling measures (see the discussion pertaining to (2.92)–(2.94)) and the results in the current items *(1)-(6)*. For the third claim in item *(8)* see [52, Theorem 3.3, p. 157] for a proof in the Euclidean space which goes through in the present setting as well. We may justify the very last claim in item *(8)* by arguing along the lines of the proof of [58, Lemma 1.12, p. 471]. Specifically, given $b \in \mathrm{BMO}(\Sigma, \sigma)$ set $w := \max\{1, |b|\}$ and fix some $p \in (1, \infty)$. Then for an arbitrary surface ball Δ in Σ we may write

$$\Big(⨍_{\Delta} w\,\mathrm{d}\sigma\Big)\Big(⨍_{\Delta} w^{-\frac{1}{p-1}}\,\mathrm{d}\sigma\Big)^{p-1}$$

$$
\begin{aligned}
&\leq \Big(⨍_{\Delta} \big[1 + |b - b_{\Delta}|\big]\,\mathrm{d}\sigma \Big) \Big(⨍_{\Delta} \Big(\frac{1}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1} \\
&\quad + |b_{\Delta}| \Big(⨍_{\Delta} \Big(\frac{1}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1} \\
&\leq 1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)} + \Big(⨍_{\Delta} \Big(\frac{|b_{\Delta}|}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1}. \qquad (2.562)
\end{aligned}
$$

Also, if $E_0 := \{x \in \Delta : |b(x)| > |b_{\Delta}|/2\}$ and $E_1 := \{x \in \Delta : |b(x)| \leq |b_{\Delta}|/2\}$, then for each point $x \in E_0$ we have $|b_{\Delta}|/|b(x)| \leq 2$ while for each point $x \in E_1$ we have $|b_{\Delta}| \leq 2|b(x) - b_{\Delta}|$. Consequently,

$$
\begin{aligned}
\Big(⨍_{\Delta} \Big(\frac{|b_{\Delta}|}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1} &\leq \max\big\{1, 2^{p-2}\big\} \cdot \Big(\frac{1}{\sigma(\Delta)} \int_{E_0} \Big(\frac{|b_{\Delta}|}{|b|}\Big)^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1} \\
&\quad + \max\big\{1, 2^{p-2}\big\} \cdot \Big(\frac{1}{\sigma(\Delta)} \int_{E_1} |b_{\Delta}|^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1} \\
&\leq \max\big\{2, 2^{p-1}\big\} \cdot \Big(\frac{\sigma(E_0)}{\sigma(\Delta)} \Big)^{p-1} \\
&\quad + \max\big\{2, 2^{p-1}\big\} \cdot \Big(⨍_{\Delta} |b - b_{\Delta}|^{\frac{1}{p-1}}\,\mathrm{d}\sigma \Big)^{p-1} \\
&\leq C_{\Sigma,p}\big(1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\big), \qquad (2.563)
\end{aligned}
$$

where the last step above uses the John-Nirenberg inequality. In view of the arbitrariness of the surface ball Δ, from the estimates in (2.562)–(2.563) we may conclude that $w \in A_p(\Sigma,\sigma)$ and $[w]_{A_p} \leq C_{\Sigma,p}(1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)})$ for some constant $C_{\Sigma,p} \in (0,\infty)$ which is independent of b. This takes care of the very last claim in item *(8)*. Finally, the claim in item *(9)* is a consequence of (2.502) and the doubling properties of σ and w (for the latter see item *(5)* above). □

Given that the class of Muckenhoupt weights is going to play a prominent role in this work, it is appropriate to include some relevant concrete examples of interest.

Example 2.12 Suppose $\Sigma \subseteq \mathbb{R}^n$ (where $n \geq 2$) is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some $p \in (1,\infty)$ along with an arbitrary point $x_0 \in \Sigma$ and a power $a \in \mathbb{R}$. Then the function

$$
w : \Sigma \to [0,\infty], \quad w(x) := |x - x_0|^a \ \text{ for each } \ x \in \Sigma \qquad (2.564)
$$

is a Muckenhoupt weight in $A_p(\Sigma, \sigma)$ if and only if $a \in \big(1-n, (p-1)(n-1)\big)$. Furthermore, whenever this happens, $[w]_{A_p}$ depends only on the Ahlfors regularity constant of Σ, p, and a.

See, for example, [54, Proposition 1.5.9, p. 42]. In a more general geometric setting, we have the following result, implied by work in [45].

Proposition 2.21 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $d \in [0, n-1)$ and consider a d-set $E \subseteq \Sigma$, i.e., a closed subset E of Σ with the property that there exists some Borel outer-measure μ on E satisfying*

$$\mu\big(B(x,r)\cap E\big) \approx r^d, \quad \textit{uniformly for } x \in E \textit{ and } r \in \big(0, 2\,\mathrm{diam}(E)\big). \tag{2.565}$$

Then for each $p \in (1,\infty)$ and each $a \in \big(d+1-n, (p-1)(n-1-d)\big)$ the function $w := \big[\mathrm{dist}(\cdot, E)\big]^a$ is a Muckenhoupt weight in the class $A_p(\Sigma,\sigma)$. Moreover, $[w]_{A_p}$ depends only on the Ahlfors regularity constant of Σ, the proportionality constants in (2.565), d, p, and a.

We continue to explore properties of Muckenhoupt weights in the context of Ahlfors regular sets which are relevant for this work.

Lemma 2.13 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then for each $w \in A_\infty(\Sigma,\sigma)$ one has*

$$BMO(\Sigma,\sigma) \subseteq L^1_{loc}(\Sigma, w). \tag{2.566}$$

Proof This is a direct consequence of (2.524), item *(2)* in Proposition 2.20, (2.538), and (2.96). □

If $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$, then for each weight function w on Σ we have $L^\infty(\Sigma,\sigma) = L^\infty(\Sigma, w)$, i.e., these vector spaces coincide and they have identical norms. Remarkably, whenever $w \in A_\infty(\Sigma,\sigma)$ it follows that the BMO spaces on Σ with respect to σ and w are once again identical. Here is a formal statement of this fact (compare with [117, Theorem 5, p. 236]).

Lemma 2.14 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some weight $w \in A_\infty(\Sigma,\sigma)$ (hence, there exists some $p \in (1,\infty)$ for which $w \in A_p(\Sigma,\sigma)$). Then there exists a constant $C \in [1,\infty)$ which depends only on p, $[w]_{A_p}$, n, and the Ahlfors regularity constant of Σ such that*

$$C^{-1}\,\|f\|_{BMO(\Sigma,\sigma)} \le \|f\|_{BMO(\Sigma,w)} \le C\,\|f\|_{BMO(\Sigma,\sigma)} \tag{2.567}$$

for each function $f \in L^1_{loc}(\Sigma,\sigma) \cap L^1_{loc}(\Sigma, w)$.

Moreover, for each σ-measurable function f on Σ one has the equivalence

$$f \in BMO(\Sigma, \sigma) \Longleftrightarrow f \in BMO(\Sigma, w) \tag{2.568}$$

and if either of these memberships materializes then $\|f\|_{BMO(\Sigma,\sigma)} \approx \|f\|_{BMO(\Sigma,w)}$ *where the implicit proportionality constants depend only on* p, $[w]_{A_p}$, n, *and the Ahlfors regularity constant of* Σ. *Succinctly put,*

$$\begin{gathered}\textit{the spaces } BMO(\Sigma, \sigma) \textit{ and } BMO(\Sigma, w) \textit{ coincide as sets} \\ \textit{and have equivalent semi-norms.}\end{gathered} \tag{2.569}$$

Proof Pick a function $f \in L^1_{\rm loc}(\Sigma, \sigma) \cap L^1_{\rm loc}(\Sigma, w)$. To prove the first inequality in (2.567), start by writing (2.525) with f replaced by $f - \fint_\Delta f\, dw$ for some arbitrary surface ball $\Delta \subseteq \Sigma$, then invoke (2.102) to obtain

$$\begin{aligned}\|f\|_{\rm BMO(\Sigma,\sigma)} &\leq 2 \sup_{\Delta\subseteq\Sigma} \inf_{c\in\mathbb{R}} \Big(\fint_\Delta \big|f - c\big|\, d\sigma\Big) \leq 2 \sup_{\Delta\subseteq\Sigma} \fint_\Delta \Big|f - \fint_\Delta f\, dw\Big|\, d\sigma \\ &\leq 2[w]_{A_p}^{1/p} \cdot \sup_{\Delta\subseteq\Sigma} \Big(\fint_\Delta \Big|f - \fint_\Delta f\, dw\Big|^p\, dw\Big)^{1/p} \\ &\leq C\, \|f\|_{\rm BMO(\Sigma,w)}\,,\end{aligned} \tag{2.570}$$

for some constant $C \in (0, \infty)$ as in the statement. To prove the second inequality in (2.567), observe first that w belongs to some Reverse Hölder class, say w satisfies (2.536) for some $q \in (1, \infty)$. If $q' \in (1, \infty)$ denotes the Hölder conjugate exponent of q, then (2.538) allows to estimate

$$\begin{aligned}\inf_{c\in\mathbb{R}} \Big(\fint_\Delta \big|f - c\big|\, dw\Big) &\leq \fint_\Delta \Big|f - \fint_\Delta f\, d\sigma\Big|\, dw \\ &\leq C\Big(\fint_\Delta \Big|f - \fint_\Delta f\, d\sigma\Big|^{q'}\, d\sigma\Big)^{1/q'}\,,\end{aligned} \tag{2.571}$$

for some constant $C \in (0, \infty)$ of the same nature as before. Taking the supremum over all surface balls $\Delta \subseteq \Sigma$ and then using John-Nirenberg's inequality, we ultimately obtain $\|f\|_{\rm BMO(\Sigma,w)} \leq C\, \|f\|_{\rm BMO(\Sigma,\sigma)}$, as desired.

As regards the equivalence in (2.568), assume first that $f \in {\rm BMO}(\Sigma, \sigma)$. Then (2.566) implies that $f \in L^1_{\rm loc}(\Sigma, \sigma) \cap L^1_{\rm loc}(\Sigma, w)$, so (2.567) holds. Conversely, assume the function f belongs to ${\rm BMO}(\Sigma, w)$. In particular, $f \in L^1_{\rm loc}(\Sigma, w)$ and the John-Nirenberg inequality (for the doubling measure w) guarantees that we also have $f \in L^p_{\rm loc}(\Sigma, w)$. In concert with (2.525) the latter membership implies that $f \in L^1_{\rm loc}(\Sigma, \sigma)$, hence once again (2.567) applies. □

The doubling and self-improving properties of Muckenhoupt weights yield the following result (see [111, §7.7] for a proof).

Lemma 2.15 *Suppose $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. In this setting, fix some $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$. Then*

$$\int_\Sigma \frac{w(x)}{(1 + |x|^{n-1})^p}\, d\sigma(x) < +\infty. \tag{2.572}$$

Also,

$$\begin{gathered} \text{there exists } \varepsilon \in (0,1) \text{ such that} \\ L^p(\Sigma, w) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1 + |x|^{n-1-\varepsilon}}\Big), \end{gathered} \tag{2.573}$$

and there exists an exponent $p_o \in (1, p]$ with the property that

$$\begin{gathered} L^p(\Sigma, w) \hookrightarrow L^q\Big(\Sigma, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \\ \text{continuously, for each fixed } q \in (0, p_o). \end{gathered} \tag{2.574}$$

As a consequence,

$$L^p(\Sigma, w) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \text{ continuously,} \tag{2.575}$$

and

$$L^p(\Sigma, w) \subseteq L^p_{loc}(\Sigma, w) \subseteq \bigcup_{1<q<p} L^q_{loc}(\Sigma, \sigma) \subseteq L^1_{loc}(\Sigma, \sigma). \tag{2.576}$$

2.8 Sobolev Spaces on Ahlfors Regular Sets

Consider an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, (2.508) implies that

$$\begin{gathered} \sigma \text{ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on } \partial\Omega, \\ \text{where the latter set is endowed with the topology canonically inherited from } \mathbb{R}^n. \end{gathered} \tag{2.577}$$

Among other things, this implies (cf. [111, §3.7]) that for every $f \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ we have

$$f = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega \iff \int_{\partial\Omega} f\phi \, d\sigma = 0 \text{ for every } \phi \in \mathscr{C}_0^\infty(\mathbb{R}^n). \tag{2.578}$$

In this context, define the family of first-order tangential derivative operators, $\partial_{\tau_{jk}}$ with $j, k \in \{1, \dots, n\}$, acting on functions $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ according to

$$\partial_{\tau_{jk}}\varphi := \nu_j(\partial_k\varphi)\big|_{\partial\Omega} - \nu_k(\partial_j\varphi)\big|_{\partial\Omega} \text{ for all } j, k \in \{1, \dots, n\}. \tag{2.579}$$

The starting point in the development of a brand of first-order Sobolev spaces on $\partial\Omega$ is the observation that for any two functions $\varphi, \psi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ and every pair of indices $j, k \in \{1, \dots, n\}$ one has the boundary integration by parts formula

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi)\psi \, d\sigma = -\int_{\partial\Omega} \varphi(\partial_{\tau_{jk}}\psi) \, d\sigma. \tag{2.580}$$

Indeed, identity (2.580) is a consequence of the Divergence Formula (2.20) applied to a suitable vector field, namely $\vec{F} := \partial_k(\varphi\psi)e_j - \partial_j(\varphi\psi)e_k$ (where $\{e_i\}_{1\leq i\leq n}$ is the standard orthonormal basis in $\mathbb{R}^n$), which is smooth, compactly supported, divergence-free, and satisfies $\nu \cdot \vec{F} = (\partial_{\tau_{jk}}\varphi)\psi + \varphi(\partial_{\tau_{jk}}\psi)$ at σ-a.e. point on $\partial\Omega$.

Next, given a function $f \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ along with two indices $j, k \in \{1, \dots, n\}$, we shall say that $\partial_{\tau_{jk}} f$ exists in (or, belongs to) the space $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ if there exists a function $f_{jk} \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ such that

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f \, d\sigma = -\int_{\partial\Omega} \varphi f_{jk} \, d\sigma \text{ for all } \varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n). \tag{2.581}$$

In view of (2.578), we conclude that the function f_{jk} is unambiguously defined (σ-a.e.) by the demand in (2.581). Henceforth we shall favor the notation

$$\partial_{\tau_{jk}} f := f_{jk} \tag{2.582}$$

which, in particular, allows us to recast (2.581) more in line with (2.580), namely as

$$\int_{\partial\Omega} f(\partial_{\tau_{jk}}\varphi) \, d\sigma = -\int_{\partial\Omega} (\partial_{\tau_{jk}} f)\varphi \, d\sigma \text{ for all } \varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n). \tag{2.583}$$

In analogy with the classical flat, Euclidean case, it is natural to regard $\partial_{\tau_{jk}} f$ as a weak (tangential) derivative of the function f. The developments so far allow us to define a convenient functional analytic environment within which is possible to consider such weak (tangential) derivatives of functions in $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$. Specifically, for each $p \in [1, \infty]$ we introduce the local Sobolev space $L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ as

$$L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma) := \big\{ f \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) : \partial_{\tau_{jk}} f \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma), \ 1 \leq j, k \leq n \big\}. \tag{2.584}$$

In such a context, we define the tangential gradient operator as (with the summation convention over repeated indices in effect)

$$L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma) \ni f \mapsto \nabla_{\tan} f := \big(\nu_k \partial_{\tau_{kj}} f\big)_{1\leq j\leq n}. \tag{2.585}$$

If Ω is actually a UR domain, we may recover the weak tangential derivatives from the components of the tangential gradient operator via (cf. [112, §11.4], [61, Lemma 3.40])

$$\begin{aligned} &\partial_{\tau_{jk}} f = \nu_j (\nabla_{\tan} f)_k - \nu_k (\nabla_{\tan} f)_j, \quad 1\leq j,k\leq n, \\ &\text{for every } \ f \in L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma) \ \text{ with } \ p\in(1,\infty). \end{aligned} \tag{2.586}$$

Going further, having fixed an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, define the (boundary) weighted Sobolev space

$$L^p_1(\partial\Omega,w) := \big\{ f\in L^p(\partial\Omega,w) : \partial_{\tau_{jk}} f \in L^p(\partial\Omega,w),\ 1\leq j,k\leq n \big\} \tag{2.587}$$

which is a Banach space when equipped with the norm

$$L^p_1(\partial\Omega,w) \ni f \mapsto \|f\|_{L^p_1(\partial\Omega,w)} := \|f\|_{L^p(\partial\Omega,w)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}. \tag{2.588}$$

Since there exists $q \in (1,\infty)$ such that $L^p(\partial\Omega,w) \hookrightarrow L^q_{\mathrm{loc}}(\partial\Omega,\sigma)$ (cf. Lemma 2.15), we see that $L^p_1(\partial\Omega,w) \hookrightarrow L^q_{1,\mathrm{loc}}(\partial\Omega,\sigma)$ for such an exponent q. In particular, the equality in (2.586) holds for every function $f \in L^p_1(\partial\Omega,w)$ whenever Ω is actually a UR domain.

In the same geometric setting, recall that $L^{p,q}(\partial\Omega,\sigma)$ with $p,q \in (0,\infty]$ stands for the scale of Lorentz spaces on $\partial\Omega$, with respect to the measure σ. These are quasi-Banach spaces which arise naturally as intermediate spaces for the real interpolation method used within the scale of ordinary Lebesgue spaces. In particular, this implies that

$$\begin{aligned} &L^{p,q}(\partial\Omega,\sigma) \hookrightarrow L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap \Big(\textstyle\bigcap_{1<s<p} L^s_{\mathrm{loc}}(\partial\Omega,\sigma)\Big) \\ &\qquad\qquad \text{whenever } \ p\in(1,\infty) \ \text{ and } \ q\in(0,\infty]. \end{aligned} \tag{2.589}$$

In relation to this scale of spaces, it is also of interest to consider (boundary) Lorentz-based Sobolev spaces. Specifically, following work in [112, §11.1], for each $p \in (1,\infty)$ and $q \in (0,\infty]$ we set

$$L^{p,q}_1(\partial\Omega,\sigma) := \big\{ f\in L^{p,q}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in L^{p,q}(\partial\Omega,\sigma),\ 1\leq j,k\leq n \big\} \tag{2.590}$$

which is a quasi-Banach space when equipped with the quasi-norm

$$L_1^{p,q}(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{L_1^{p,q}(\partial\Omega,\sigma)} := \|f\|_{L^{p,q}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{L^{p,q}(\partial\Omega,\sigma)}. \tag{2.591}$$

In the proposition below, which refines [61, Lemma 3.36, p. 2678], we study the manner in which weak tangential derivatives interact with pointwise nontangential traces. See [112, §11.3] for a proof.

Proposition 2.22 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in [1,\infty]$, an aperture parameter $\kappa \in (0,\infty)$, and a truncation parameter $\varepsilon > 0$. In this context, assume the function $u \in W^{1,1}_{loc}(\Omega)$ satisfies*

$$\mathcal{N}_\kappa^\varepsilon u \in L^p_{loc}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa^\varepsilon(\nabla u) \in L^p_{loc}(\partial\Omega,\sigma), \tag{2.592}$$

and the nontangential traces

$$u\big|_{\partial\Omega}^{\kappa-n.t.} \text{ and } (\partial_j u)\big|_{\partial\Omega}^{\kappa-n.t.} \text{ for } j \in \{1,\dots,n\} \\ \text{exist at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.593}$$

Then $u\big|_{\partial\Omega}^{\kappa-n.t.}$ belongs to $L^p_{1,loc}(\partial\Omega,\sigma)$, the functions $(\partial_1 u)\big|_{\partial\Omega}^{\kappa-n.t.}, \dots, (\partial_n u)\big|_{\partial\Omega}^{\kappa-n.t.}$ belong to $L^p_{loc}(\partial\Omega,\sigma)$ and, for each $j,k \in \{1,\dots,n\}$ and for σ-a.e. point on $\partial\Omega$, one has

$$\partial_{\tau_{jk}}\Big(u\big|_{\partial\Omega}^{\kappa-n.t.}\Big) = \nu_j\Big((\partial_k u)\big|_{\partial\Omega}^{\kappa-n.t.}\Big) - \nu_k\Big((\partial_j u)\big|_{\partial\Omega}^{\kappa-n.t.}\Big). \tag{2.594}$$

In particular, for each $j,k \in \{1,\dots,n\}$ one has

$$\Big|\partial_{\tau_{jk}}\Big(u\big|_{\partial\Omega}^{\kappa-n.t.}\Big)\Big| \leq 2\mathcal{N}_\kappa^\varepsilon(\nabla u) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.595}$$

The following result from [112, §11.3] may be regarded as a weighted counterpart of Proposition 2.22, in which no assumptions are made regarding the existence of the nontangential boundary traces of the derivatives of the function involved. The reader is reminded that the truncated nontangential maximal operator $\mathcal{N}_\kappa^\varepsilon$ has been defined in (2.9).

Proposition 2.23 *Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture parameter $\kappa \in (0,\infty)$ and an integrability exponent $p \in (1,\infty)$. Assume $w : \partial\Omega \to [0,+\infty]$ is a σ-measurable function with $0 < w(x) < \infty$ for σ-a.e. $x \in \partial\Omega$ and $w^{-1/p} \in L^{p'}_{loc}(\partial\Omega,\sigma)$, where $p' \in (1,\infty)$ denotes the Hölder conjugate exponent of p; in particular, $L^p(\partial\Omega, w\sigma) \hookrightarrow L^1_{loc}(\partial\Omega,\sigma)$. Finally, fix* *a truncation parameter* $\varepsilon > 0$. *In this setting, suppose that some complex-valued function* $u \in W^{1,1}_{loc}(\Omega)$ *has been given which satisfies the following conditions:*

$$\begin{aligned} & u\big|^{\kappa-n.t.}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and} \\ & \mathcal{N}^{\varepsilon}_{\kappa} u \in L^1_{loc}(\partial\Omega, \sigma), \quad \mathcal{N}^{\varepsilon}_{\kappa}(\nabla u) \in L^p(\partial\Omega, w\sigma). \end{aligned} \tag{2.596}$$

Then the nontangential trace $u\big|^{\kappa-n.t.}_{\partial\Omega}$ *belongs to* $L^1_{1,loc}(\partial\Omega, \sigma)$ *and satisfies*

$$\begin{aligned} & \partial_{\tau_{jk}}\Big(u\big|^{\kappa-n.t.}_{\partial\Omega}\Big) \in L^p(\partial\Omega, w\sigma) \text{ for each } j,k \in \{1,\dots,n\} \\ & \text{and } \sum_{j,k=1}^{n} \Big\| \partial_{\tau_{jk}}\Big(u\big|^{\kappa-n.t.}_{\partial\Omega}\Big) \Big\|_{L^p(\partial\Omega, w\sigma)} \leq C \left\| \mathcal{N}^{\varepsilon}_{\kappa}(\nabla u) \right\|_{L^p(\partial\Omega, w\sigma)} \end{aligned} \tag{2.597}$$

for some constant $C \in (0,\infty)$ *independent of* u.

For further use, let us also consider `homogeneous Muckenhoupt weighted boundary Sobolev spaces`. Specifically, we make the following definition.

Definition 2.18 Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Given some integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, define

$$\begin{aligned} \dot{L}^p_1(\partial\Omega, w) := \Big\{ & f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\text{loc}}(\partial\Omega, w) : \\ & \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \text{ for each } j,k \in \{1,\dots,n\}\Big\}, \end{aligned} \tag{2.598}$$

and equip this space with the semi-norm

$$\dot{L}^p_1(\partial\Omega, w) \ni f \longmapsto \|f\|_{\dot{L}^p_1(\partial\Omega, w)} := \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^p(\partial\Omega, w)}. \tag{2.599}$$

It is clear from definitions and (2.575) that we have a continuous embedding

$$L^p_1(\partial\Omega, w) \hookrightarrow \dot{L}^p_1(\partial\Omega, w). \tag{2.600}$$

Also, all constant functions on $\partial\Omega$ belong to $\dot{L}^p_1(\partial\Omega, w)$ and their semi-norm vanishes. As such, we will occasionally find it useful to work with $\dot{L}^p_1(\partial\Omega, w)\big/\sim$, the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, which we equip with the semi-norm

$$\dot{L}_1^p(\partial\Omega, w)\big/\sim\ \ni [f] \mapsto \|[f]\|_{\dot{L}_1^p(\partial\Omega,w)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}. \tag{2.601}$$

We shall next prove a membership criterion to a global weighted Lebesgue space, formulated in the lemma below.

Lemma 2.16 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed, unbounded set, which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Pick $p \in (1,\infty)$ along with $w \in A_p(\Sigma,\sigma)$, and fix a reference point $x_0 \in \Sigma$. Suppose $f \in L^1_{loc}(\Sigma, w)$ is such that*

$$C_* := \sup_{r>0} \frac{1}{r}\Big(\int_{\Delta_r} |f - f_{r,w}|^p \, dw\Big)^{1/p} < +\infty, \tag{2.602}$$

where, for each $r \in (0,\infty)$,

$$\Delta_r := B(x_0, r) \cap \Sigma \quad \text{and} \quad f_{r,w} := \fint_{\Delta_r} f \, dw. \tag{2.603}$$

Then there exists some constant $C = C\big(\Sigma, n, p, [w]_{A_p}\big) \in (0,\infty)$ with the property that for each $r \in (0,\infty)$ one has

$$\int_{\Sigma} \frac{|f(x) - f_{r,w}|}{(r + |x - x_0|)^n} \, d\sigma(x) \leq \frac{C \cdot C_*}{w(\Delta_r)^{1/p}}. \tag{2.604}$$

In particular, f belongs to the space $L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^n}\big)$.

Proof For starters, observe that for each $r > 0$ we have

$$\begin{aligned} |f_{2r,w} - f_{r,w}| &\leq \fint_{\Delta_r} |f - f_{2r,w}| \,\mathrm{d}w \leq C \fint_{\Delta_{2r}} |f - f_{2r,w}| \,\mathrm{d}w \\ &\leq C\Big(\fint_{\Delta_{2r}} |f - f_{2r,w}|^p \,\mathrm{d}w\Big)^{1/p} \leq \frac{C \cdot C_* \cdot r}{w(\Delta_{2r})^{1/p}}, \end{aligned} \tag{2.605}$$

thanks to the fact that w is doubling, Hölder's inequality, and (2.602). With this in hand (and keeping in mind that both σ and w are doubling), for each given $r > 0$ we may then estimate

$$\int_{\Sigma\setminus\Delta_r} \frac{|f(x) - f_{r,w}|}{|x - x_0|^n} \,\mathrm{d}\sigma(x) \leq C \sum_{j=0}^{\infty} \frac{1}{(2^j r)^n} \int_{\Delta_{2^{j+1}r}\setminus\Delta_{2^j r}} |f - f_{r,w}| w^{1/p} w^{-1/p} \,\mathrm{d}\sigma$$

$$\le C\sum_{j=0}^{\infty}\frac{1}{(2^j r)^n}\Big(\int_{\Delta_{2^{j+1}r}}|f-f_{r,w}|^p\,\mathrm{d}w\Big)^{1/p}\Big(\int_{\Delta_{2^{j+1}r}}w^{-p'/p}\,\mathrm{d}\sigma\Big)^{1/p'}$$

$$\le C\sum_{j=0}^{\infty}\frac{1}{(2^j r)^n}\,\big\|f-f_{r,w}\big\|_{L^p(\Delta_{2^{j+1}r},w)}\cdot\frac{\sigma\big(\Delta_{2^{j+1}r}\big)}{w\big(\Delta_{2^{j+1}r}\big)^{1/p}}$$

$$\le C\sum_{j=0}^{\infty}\frac{1}{2^j r\cdot w\big(\Delta_{2^j r}\big)^{1/p}}\Big\{\big\|f-f_{2^{j+1}r,w}\big\|_{L^p(\Delta_{2^{j+1}r},w)}$$

$$+\sum_{k=0}^{j}\big\|f_{2^{k+1}r,w}-f_{2^k r,w}\big\|_{L^p(\Delta_{2^{j+1}r},w)}\Big\}$$

$$\le C\sum_{j=0}^{\infty}\frac{1}{2^j r\cdot w\big(\Delta_{2^j r}\big)^{1/p}}\Big\{C_*\cdot 2^{j+1}r$$

$$+\sum_{k=0}^{j}|f_{2^{k+1}r,w}-f_{2^k r,w}|\cdot w\big(\Delta_{2^{j+1}r}\big)^{1/p}\Big\}$$

$$\le C\sum_{j=0}^{\infty}\frac{1}{2^j r\cdot w\big(\Delta_{2^j r}\big)^{1/p}}\Big\{C_*\cdot 2^{j+1}r$$

$$+\sum_{k=0}^{j}\frac{C\cdot C_*\cdot 2^k r}{w(\Delta_{2^{k+1}r})^{1/p}}\cdot w\big(\Delta_{2^{j+1}r}\big)^{1/p}\Big\}$$

$$\le C\sum_{j=0}^{\infty}\frac{1}{2^j r\cdot w\big(\Delta_{2^j r}\big)^{1/p}}\Big\{\sum_{k=0}^{j}\frac{C_*\cdot 2^k r}{w(\Delta_{2^{k+1}r})^{1/p}}\cdot w\big(\Delta_{2^{j+1}r}\big)^{1/p}\Big\}$$

$$\le C\cdot C_*\sum_{j=0}^{\infty}\frac{1}{2^j}\Big\{\sum_{k=0}^{j}\frac{2^k}{w(\Delta_{2^{k+1}r})^{1/p}}\Big\}$$

$$\le C\cdot C_*\sum_{k=0}^{\infty}\Big\{\sum_{j=k}^{\infty}\frac{1}{2^j}\Big\}\frac{2^k}{w(\Delta_{2^{k+1}r})^{1/p}}$$

$$\le C\cdot C_*\sum_{k=0}^{\infty}\frac{1}{w(\Delta_{2^{k+1}r})^{1/p}}$$

$$= C \cdot C_* \frac{1}{w(\Delta_r)^{1/p}} \sum_{k=0}^{\infty} \Big(\frac{w(\Delta_r)}{w(\Delta_{2^{k+1}r})} \Big)^{1/p}$$

$$\leq C \cdot C_* \frac{1}{w(\Delta_r)^{1/p}} \sum_{k=0}^{\infty} \Big(\frac{\sigma(\Delta_r)}{\sigma(\Delta_{2^{k+1}r})} \Big)^{\tau/p}$$

$$\leq C \cdot C_* \frac{1}{w(\Delta_r)^{1/p}} \sum_{k=0}^{\infty} \Big(\frac{1}{2^k} \Big)^{(n-1)\tau/p} = \frac{C \cdot C_*}{w(\Delta_r)^{1/p}}, \tag{2.606}$$

where $\tau > 0$ is as in (2.537). Above, the second inequality is a consequence of Hölder's inequality, the third inequality uses (2.520), the fifth and sixth inequalities are based on (2.602) and (2.605), while the penultimate inequality is implied by (2.537).

In addition, as a consequence of (2.602), (2.520), and Hölder's inequality we have

$$\int_{\Delta_r} \frac{|f(x) - f_{r,w}|}{r^n} \, d\sigma(x) = r^{-n} \int_{\Delta_r} |f - f_{r,w}| w^{1/p} w^{-1/p} \, d\sigma$$

$$= r^{-n} \Big(\int_{\Delta_r} |f - f_{r,w}|^p \, dw \Big)^{1/p} \Big(\int_{\Delta_r} w^{-p'/p} \, d\sigma \Big)^{1/p'}$$

$$\leq C_* \cdot r^{1-n} [w]_{A_p}^{1/p} \frac{\sigma(\Delta_r)}{w(\Delta_r)^{1/p}}$$

$$\leq \frac{C \cdot C_*}{w(\Delta_r)^{1/p}}. \tag{2.607}$$

Together, (2.606) and (2.607) prove (2.604). □

In the proposition below we explore consequences of the integrability of the nontangential maximal operator of the gradient of a given function.

Proposition 2.24 *Make the assumption that* $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with the property that* $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ *is a doubling measure on* $\partial\Omega$*. Pick an arbitrary aperture parameter* $\kappa > 0$ *and fix a reference point* $x_o \in \partial\Omega$*. Finally, select a function* $u \in \mathscr{C}^1(\Omega)$*.*

Then there exist $\widetilde{\kappa} > 0$ *large enough along with some threshold* $R \in (0, +\infty]$ (*which may be taken* $+\infty$ *if* $\partial\Omega$ *is unbounded*) *and some constant* $C \in (1, \infty)$*, all independent of the given function* u*, such that for each* $\delta \in (0, R)$ *one may find a compact subset* K_δ *of* Ω*, of diameter* $\approx \delta$ *and distance to the boundary* $\approx \delta$*, with the property that*

$$\big(\mathcal{N}_\kappa^\delta u\big)(x) \le C\delta \cdot \mathcal{N}_{\widetilde{\kappa}}^{C\delta}(\nabla u)(x) + \sup_{K_\delta} |u|, \qquad \forall\, x \in B(x_o, \delta) \cap \partial\Omega. \tag{2.608}$$

Moreover, there exists some sufficiently large $C > 1$ such that

$$\text{if } \mathcal{N}_\kappa^\varepsilon(\nabla u) \text{ belongs to } L^p_{loc}(\partial\Omega, \sigma) \text{ for some } p \in (0, \infty] \text{ and some } \varepsilon > 0 \text{ then } \mathcal{N}_\kappa^{\varepsilon/C} u \in L^p_{loc}(\partial\Omega, \sigma)\text{, the nontangential trace } \big(u\big|_{\partial\Omega}^{\kappa-n.t.}\big)(x) \text{ exists at } \sigma\text{-a.e. } x \in \partial\Omega\text{, and the function } u\big|_{\partial\Omega}^{\kappa-n.t.} \text{ is } \sigma\text{-measurable on } \partial\Omega. \tag{2.609}$$

In addition, if $\partial\Omega$ is unbounded then there exists $C = C(\Omega) \in (0, \infty)$ such that

$$\Big|\big(u\big|_{\partial\Omega}^{\kappa-n.t.}\big)(x) - \big(u\big|_{\partial\Omega}^{\kappa-n.t.}\big)(y)\Big| \le C|x-y| \cdot \big[\mathcal{N}_\kappa(\nabla u)(x) + \mathcal{N}_\kappa(\nabla u)(y)\big] \quad \text{for } \sigma\text{-a.e. points } x, y \in \partial\Omega. \tag{2.610}$$

Finally, if the original hypotheses are strengthened by now assuming that $\partial\Omega$ is an unbounded Ahlfors regular set and that the nontangential maximal function of the Jacobian of u satisfies $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w)$ for some integrability exponent $p \in (1, \infty)$ and some weight $w \in A_p(\partial\Omega, \sigma)$ then

$$\text{the nontangential trace } u\big|_{\partial\Omega}^{\kappa-n.t.} \text{ belongs to the Muckenhoupt weighted homogeneous boundary Sobolev space } \dot{L}^p_1(\partial\Omega, w) \text{ and one has } \Big\|u\big|_{\partial\Omega}^{\kappa-n.t.}\Big\|_{\dot{L}^p_1(\partial\Omega,w)} \le C\,\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \text{ for a constant } C \in (0,\infty) \text{ independent of the function } u. \tag{2.611}$$

Proof The claims in (2.608)–(2.610) have been established in [111, §8.4]. To justify (2.611), work under the additional assumptions that $\partial\Omega$ is an unbounded Ahlfors regular set and that $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w)$ for some $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. Observe that the latter condition implies, in light of (2.576), that $\mathcal{N}_\kappa(\nabla u) \in L^1_{\rm loc}(\partial\Omega, \sigma)$, so the current assumptions are indeed stronger. To lighten the exposition, abbreviate

$$f := u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{and} \quad g := \mathcal{N}_\kappa(\nabla u). \tag{2.612}$$

From (2.609), (2.13), (2.608), (2.11) (used with $\sigma := w$), and Proposition 2.23 (whose applicability is ensured by (2.576)) it follows that

$$f \in L^p_{\rm loc}(\partial\Omega, w), \quad \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \ \text{ for all } \ j,k \in \{1,\dots,n\}, \quad \text{and} \quad \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)} \le C\,\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)}, \tag{2.613}$$

for some constant $C \in (0, \infty)$ independent of u. Also, we may recast (2.610) as

$$|f(x) - f(y)| \leq C|x - y| \cdot \big[g(x) + g(y)\big] \text{ for } \sigma\text{-a.e. } x, y \in \partial\Omega. \tag{2.614}$$

To proceed, fix a reference point $x_0 \in \partial\Omega$ and for each given scale $r \in (0, \infty)$ define $\Delta_r := B(x_0, r) \cap \partial\Omega$ and $f_{r,w} := ⨍_{\Delta_r} f \, dw$. Then using (2.614) and Hölder's inequality for each $r \in (0, \infty)$ we may estimate

$$\begin{aligned}
\Big(\int_{\Delta_r} |f(x) - f_{r,w}|^p \, dw(x)\Big)^{1/p} &\\
&= \Big(\int_{\Delta_r} \Big|f(x) - ⨍_{\Delta_r} f(y) \, dw(y)\Big|^p \, dw(x)\Big)^{1/p} \\
&\leq \Big(\int_{\Delta_r} ⨍_{\Delta_r} |f(x) - f(y)|^p \, dw(x) \, dw(y)\Big)^{1/p} \\
&\leq C\Big(\int_{\Delta_r} ⨍_{\Delta_r} |x - y|^p \big(g(x) + g(y)\big)^p \, dw(x) \, dw(y)\Big)^{1/p} \\
&\leq Cr\Big(\int_{\Delta_r} g^p \, dw\Big)^{1/p} \leq Cr\Big(\int_{\partial\Omega} g^p \, dw\Big)^{1/p} \\
&= Cr \, \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega, w)},
\end{aligned} \tag{2.615}$$

since $x, y \in \Delta_r$ forces $|x - y| < 2r$. As a consequence,

$$\sup_{r>0} \frac{1}{r}\Big(\int_{\Delta_r} |f - f_{r,w}|^p \, dw\Big)^{1/p} \leq C \, \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega, w)} < +\infty. \tag{2.616}$$

Having established estimate (2.616), from Lemma 2.16 we conclude that the function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. In view of this, (2.598)–(2.599), and (2.613) we then deduce that all claims in (2.611) are true. □

We next discuss the equivalence between membership to a global weighted Lebesgue space and certain Poincaré-type inequalities.

Proposition 2.25 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Fix some reference point $x_0 \in \partial\Omega$, along with some integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, assume that*

$$\begin{aligned}&f \text{ is a function belonging to } L^1_{loc}(\partial\Omega, \sigma) \text{ with the property that} \\ &\partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \text{ for all } j, k \in \{1, \ldots, n\}.\end{aligned} \tag{2.617}$$

Then the following statements are equivalent:

(i) *The function* f *belongs to the space* $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$.

(ii) *There exists a constant* $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0, \infty)$ *which stays bounded when* $[w]_{A_p}$ *stays bounded and which is independent of the function* f, *with the property that if for each scale* $r \in (0, \infty)$ *one defines the surface ball* $\Delta_r := B(x_0, r) \cap \partial\Omega$ *and* $f_{r,\sigma} := ⨍_{\Delta_r} f \, d\sigma$ *then*

$$\sup_{r>0} \frac{1}{r} \left(\int_{\Delta_r} |f - f_{r,\sigma}|^p \, dw \right)^{1/p} \leq C \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^p(\partial\Omega, w)}. \tag{2.618}$$

(ii)' *The function* f *belongs to the space* $L^1_{loc}(\partial\Omega, w)$ *and there exists some constant* $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0, \infty)$ *which stays bounded when* $[w]_{A_p}$ *stays bounded and which is independent of the function* f, *with the property that if for each* $r \in (0, \infty)$ *one defines* $\Delta_r := B(x_0, r) \cap \partial\Omega$ *and* $f_{r,w} := ⨍_{\Delta_r} f \, dw$ *then*

$$\sup_{r>0} \frac{1}{r} \left(\int_{\Delta_r} |f - f_{r,w}|^p \, dw \right)^{1/p} \leq C \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^p(\partial\Omega, w)}. \tag{2.619}$$

(iii) *For each* $r \in (0, \infty)$ *there exists a constant* $C_r \in (0, \infty)$ *which depends only on* Ω, p, $[w]_{A_p}$, x_0, *and* r *such that, with* $f_{r,\sigma}$ *as before, one has*

$$\int_{\partial\Omega} \frac{|f(x) - f_{r,\sigma}|}{1 + |x|^n} \, d\sigma(x) \leq \frac{C_r}{w(\Delta_r)^{1/p}} \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^p(\partial\Omega, w)}. \tag{2.620}$$

(iii)' *The function* f *belongs to* $L^1_{loc}(\partial\Omega, w)$ *and for each* $r \in (0, \infty)$ *there exists a constant* $C_r \in (0, \infty)$ *which depends only on* Ω, p, $[w]_{A_p}$, x_0, *and* r *such that, with* $f_{r,w}$ *as before,*

$$\int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1 + |x|^n} \, d\sigma(x) \leq \frac{C_r}{w(\Delta_r)^{1/p}} \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^p(\partial\Omega, w)}. \tag{2.621}$$

(iv) *There exists a constant* $C = C(\Omega, p, w, x_0) \in (0, \infty)$ *independent of* f, *and some constant* $c_f \in \mathbb{C}$ *which is allowed to depend on* f, *such that*

$$\|f - c_f\|_{L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\right)} \leq C \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^p(\partial\Omega, w)}. \tag{2.622}$$

(v) *The function* f *belongs to the space* $\dot{L}^p_1(\partial\Omega, w)$.

Proof We start by proving the implication *(i)* $\Rightarrow$ *(ii)*. To this end, assume that in addition to (2.617) we have $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Denote by ν the geometric measure theoretic outward unit normal to Ω and set

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.623}$$

With ω_{n-1} denoting the surface area of the unit sphere in $\mathbb{R}^n$, at each point $x \in \Omega_\pm$ define

$$u_\pm(x) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \Big\{ \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} - \frac{\langle \nu(y), y\rangle}{|y|^n} \cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(y) \Big\} f(y)\, \mathrm{d}\sigma(y). \tag{2.624}$$

Then work in [114, §1.5] ensures that for an arbitrary, fixed, aperture parameter $\kappa > 0$ there exists a constant $C \in (0,\infty)$ independent of f and which stays bounded when $[w]_{A_p}$ stays bounded, such that

$$\begin{aligned} & u_\pm \in \mathscr{C}^\infty(\Omega_\pm), \quad \mathcal{N}_\kappa(\nabla u_\pm) \in L^p(\partial\Omega, w), \\ & \|\mathcal{N}_\kappa(\nabla u_\pm)\|_{L^p(\partial\Omega,w)} \le C \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}, \\ & f = u_+\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - u_-\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \end{aligned} \tag{2.625}$$

Hence,

$$\begin{aligned} & g := \mathcal{N}_\kappa(\nabla u_+) + \mathcal{N}_\kappa(\nabla u_-) \in L^p(\partial\Omega, w) \\ & \text{has } \|g\|_{L^p(\partial\Omega,w)} \le C \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{L^p(\partial\Omega,w)}, \end{aligned} \tag{2.626}$$

for some constant $C \in (0,\infty)$ independent of f and which stays bounded when $[w]_{A_p}$ stays bounded. In addition, thanks to (2.610), the function g satisfies

$$|f(x) - f(y)| \le C|x-y| \cdot \big[g(x) + g(y)\big] \quad \text{for } \sigma\text{-a.e. } x, y \in \partial\Omega. \tag{2.627}$$

Granted these properties, we may proceed as in (2.615) to conclude that

$$\begin{aligned} & \Big(\int_{\Delta_r} |f(x) - f_{r,\sigma}|^p \, \mathrm{d}w(x)\Big)^{1/p} \\ & \qquad = \Big(\int_{\Delta_r} \Big| f(x) - \fint_{\Delta_r} f(y)\, \mathrm{d}\sigma(y)\Big|^p \, \mathrm{d}w(x)\Big)^{1/p} \\ & \qquad \le \Big(\int_{\Delta_r} \Big(\fint_{\Delta_r} |f(x) - f(y)|\, \mathrm{d}\sigma(y)\Big)^p \, \mathrm{d}w(x)\Big)^{1/p} \end{aligned}$$

$$\le C\Big(\int_{\Delta_r}\Big(⨍_{\Delta_r}|x-y|\big(g(x)+g(y)\big)\,\mathrm{d}\sigma(y)\Big)^p\,\mathrm{d}w(x)\Big)^{1/p}$$
$$\le Cr\Big(\int_{\Delta_r}g^p\,\mathrm{d}w\Big)^{1/p}+w(\Delta_r)^{\frac{1}{p}}⨍_{\Delta_r}g\,\mathrm{d}\sigma(y)$$
$$\le Cr\Big(\int_{\partial\Omega}g^p\,\mathrm{d}w\Big)^{1/p}, \tag{2.628}$$

since $x,y\in\Delta_r$ forces $|x-y|<2r$ and we have used (2.525). Eventually we conclude that, for some constant $C\in(0,\infty)$ independent of f and which stays bounded when $[w]_{A_p}$ stays bounded, we have

$$\sup_{r>0}\frac{1}{r}\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}\le C\|g\|_{L^p(\partial\Omega,w)}\le C\sum_{j,k=1}^n\big\|\partial_{\tau_{jk}}f\big\|_{L^p(\partial\Omega,w)}. \tag{2.629}$$

This completes the proof of the implication *(i)* ⇒ *(ii)*.

To see that *(ii)* ⇒ *(ii)'* we first note that (2.617) and (2.618) imply that for each $r>0$ we have

$$\Big(\int_{\Delta_r}|f|^p\,\mathrm{d}w\Big)^{1/p}\le\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}+w(\Delta_r)^{1/p}|f_{r,\sigma}|$$
$$\le C\,r\sum_{j,k=1}^n\big\|\partial_{\tau_{jk}}f\big\|_{L^p(\partial\Omega,w)}+w(\Delta_r)^{1/p}⨍_{\Delta_r}|f|\,\mathrm{d}\sigma<\infty. \tag{2.630}$$

This goes to show that $f\in L^p_{\rm loc}(\partial\Omega,w)\subseteq L^1_{\rm loc}(\partial\Omega,w)$. Granted this, for each $r>0$ we may estimate

$$\Big(\int_{\Delta_r}|f-f_{r,w}|^p\,\mathrm{d}w\Big)^{1/p}\le\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}+w(\Delta_r)^{1/p}\,|f_{r,\sigma}-f_{r,w}|$$
$$\le 2\Big(\int_{\Delta_r}|f-f_{r,\sigma}|^p\,\mathrm{d}w\Big)^{1/p}. \tag{2.631}$$

With (2.631) in hand, (2.618) readily gives (2.619).

We next note that the implication *(ii)'* ⇒ *(iii)'* is seen from Lemma 2.16, the implication *(iii)'* ⇒ *(iv)* (respectively, *(iii)* ⇒ *(iv)*) follows by taking $r:=1$ and $c_f:=f_{1,w}$ (respectively, $c_f:=f_{1,\sigma}$), while the implication *(iv)* ⇒ *(i)* is a direct consequence that any constant belongs to the space $L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)$. The fact that *(iii)'* ⇒ *(iii)* may be justified writing (using the Ahlfors regularity of $\partial\Omega$; cf. (2.32))

$$\begin{aligned}\int_{\partial\Omega} \frac{|f(x) - f_{r,\sigma}|}{1+|x|^n} \, d\sigma(x) &\le \int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n} \, d\sigma(x) + C\,|f_{r,\sigma} - f_{r,w}| \\ &\le \int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n} \, d\sigma(x) + C \fint_{\Delta_r} |f - f_{r,w}| \, d\sigma \\ &\le C \int_{\partial\Omega} \frac{|f(x) - f_{r,w}|}{1+|x|^n} \, d\sigma(x), \end{aligned} \tag{2.632}$$

where the constant $C \in (0, \infty)$ depends only on Ω, x_0, and r.

Hence, the claims in items *(i)*, *(ii)*, *(ii)'*, *(iii)*, *(iii)'*, and *(iv)* are all equivalent. In view of (2.598) it follows that the implication *(v)* $\Rightarrow$ *(i)* also holds. To finish the proof of the proposition it suffices to check that, collectively, (2.617) and items *(i)*-*(ii)'* imply the claim in item *(v)*. This, however, is apparent from (2.598) and the fact that (2.619) guarantees that $f \in L^p_{\rm loc}(\partial\Omega, w)$. □

Remark 2.4 Consider a two-sided NTA domain $\Omega \subseteq \mathbb{R}^n$ such that $\partial\Omega$ is an unbounded Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then Proposition 2.25 implies that the local L^p integrability property with respect to the measure w for functions in the homogeneous Muckenhoupt weighted boundary Sobolev space $\dot{L}^p_1(\partial\Omega, w)$ may be replaced by a (seemingly weaker) local absolute integrability property with respect to the measure w, or may be even suppressed altogether. Specifically, in such a setting we have (compare with (2.598))

$$\begin{aligned}\dot{L}^p_1(\partial\Omega, w) &= \Big\{ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{\rm loc}(\partial\Omega, w) : \\ &\qquad\qquad \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \text{ for each } \ j,k \in \{1,\dots,n\}\Big\} \end{aligned} \tag{2.633}$$

$$\begin{aligned} &= \Big\{ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) : \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w) \\ &\qquad\qquad \text{for each } \ j,k \in \{1,\dots,n\}\Big\}. \end{aligned} \tag{2.634}$$

When considered on the boundaries of two-sided NTA domains, the quotient space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ turns out to be Banach. Here is a formal statement:

Proposition 2.26 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Pick some integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Recall that $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ denotes the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, equipped with the semi-norm* (2.601).

Then (2.601) *is a genuine norm on* $\dot{L}^p_1(\partial\Omega, w)\big/\sim$*, and* $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ *is a Banach space when equipped with the norm* (2.601).

Proof The fact that the semi-norm (2.601) is actually a norm on the space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ follows from (2.621).

To prove that $\dot{L}^p_1(\partial\Omega, w)\big/\sim$ is complete when equipped with the norm (2.601), let $\{f_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \dot{L}^p_1(\partial\Omega, w)$ be such that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the quotient space $\dot{L}^p_1(\partial\Omega, w)\big/\sim$. Then for each fixed $j, k \in \{1, \dots, n\}$ it follows that $\big\{\partial_{\tau_{jk}} f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\partial\Omega, w)$. Since the latter is complete, it follows that there exists $g_{jk} \in L^p(\partial\Omega, w)$ such that

$$\partial_{\tau_{jk}} f_\alpha \to g_{jk} \ \text{ in } \ L^p(\partial\Omega, w) \ \text{ as } \ \alpha \to \infty. \tag{2.635}$$

Fix a reference point $x_0 \in \partial\Omega$ and, for each $r \in (0, \infty)$, define $\Delta_r := B(x_0, r)\cap\partial\Omega$. Also, set $f_{\alpha,r,w} := ⨍_{\Delta_r} f_\alpha \, dw$ for each $r \in (0, \infty)$ and each $\alpha \in \mathbb{N}$. From (2.621) (written for $f := f_\alpha - f_\beta$) it follows that for each $r \in (0, \infty)$ there exists a constant $C_r \in (0, \infty)$ which depends on Ω, p, $[w]_{A_p}$, and r such that for each $\alpha, \beta \in \mathbb{N}$ we have

$$\begin{aligned} &\big\|\big(f_\alpha - f_{\alpha,r,w}\big) - \big(f_\beta - f_{\beta,r,w}\big)\big\|_{L^1\left(\partial\Omega,\, \frac{\sigma(x)}{1+|x|^n}\right)} \\ &\qquad\qquad \le \frac{C_r}{w(\Delta_r)^{1/p}} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta\big\|_{L^p(\partial\Omega, w)}. \end{aligned} \tag{2.636}$$

In view of (2.635), this estimate implies that for each fixed $r \in (0, \infty)$ the sequence $\big\{f_\alpha - f_{\alpha,r,w}\big\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Hence, for each fixed $r \in (0, \infty)$ there exists $h_r \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ such that

$$f_\alpha - f_{\alpha,r,w} \to h_r \ \text{ in } \ L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \ \text{ as } \ \alpha \to \infty. \tag{2.637}$$

Next, the estimate recorded in (2.619) (written for $f := f_\alpha - f_\beta$) implies that there exists some constant $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0, \infty)$ with the property that for each fixed $r \in (0, \infty)$ we have

$$\begin{aligned} &\Big(\int_{\Delta_r} \big|\big(f_\alpha - f_{\alpha,r,w}\big) - \big(f_\beta - f_{\beta,r,w}\big)\big|^p \, dw\Big)^{1/p} \\ &\qquad\qquad \le C \cdot r \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta\big\|_{L^p(\partial\Omega, w)}. \end{aligned} \tag{2.638}$$

By once again relying on (2.635), we conclude that for each fixed $r \in (0, \infty)$ the sequence $\big\{f_\alpha\big|_{\Delta_r} - f_{\alpha,r,w}\big\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $L^p(\Delta_r, w)$. As such,

$$\begin{aligned}&\text{for each } r\in(0,\infty) \text{ there exists some } k_r\in L^p(\Delta_r,w)\\&\text{such that } \ f_\alpha\big|_{\Delta_r}-f_{\alpha,r,w}\to k_r \ \text{ in } \ L^p(\Delta_r,w) \ \text{ as } \ \alpha\to\infty.\end{aligned} \tag{2.639}$$

Since convergence in Lebesgue spaces implies, after eventually passing to a subsequence, pointwise a.e. convergence, from (2.637) and (2.639) we see that, in fact,

$$h_r\big|_{\Delta_r}=k_r\in L^p(\Delta_r,w) \ \text{ for each } \ r\in(0,\infty). \tag{2.640}$$

From (2.637) we also see that for each fixed $r_1,r_2\in(0,\infty)$ we have

$$f_{\alpha,r_2,w}-f_{\alpha,r_1,w}\to h_{r_1}-h_{r_2} \ \text{ in } \ L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\big) \ \text{ as } \ \alpha\to\infty. \tag{2.641}$$

This forces $h_{r_1}-h_{r_2}$ to be a constant which, in concert with (2.640), ultimately shows that actually

$$h_r\in L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\big)\cap L^p_{\rm loc}(\partial\Omega,w) \ \text{ for each } \ r\in(0,\infty). \tag{2.642}$$

Henceforth, we agree to simply write h for h_r with $r=1$, and c_α for $f_{\alpha,r,w}$ with $r=1$. Then (2.642), (2.637) tell us that the function

$$h \ \text{ belongs to } \ L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\big)\cap L^p_{\rm loc}(\partial\Omega,w), \tag{2.643}$$

and the sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}}\subseteq\mathbb{C}$ is such that

$$f_\alpha-c_\alpha\to h \ \text{ in } \ L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\big) \ \text{ as } \ \alpha\to\infty. \tag{2.644}$$

For each $j,k\in\{1,\dots,n\}$ and each test function $\varphi\in\mathscr{C}^\infty_0(\mathbb{R}^n)$ we may then write

$$\begin{aligned}\int_{\partial\Omega}h(\partial_{\tau_{jk}}\varphi)\,d\sigma&=\lim_{\alpha\to\infty}\int_{\partial\Omega}(f_\alpha-c_\alpha)(\partial_{\tau_{jk}}\varphi)\,d\sigma\\&=-\lim_{\alpha\to\infty}\int_{\partial\Omega}\partial_{\tau_{jk}}(f_\alpha-c_\alpha)\varphi\,d\sigma=-\lim_{\alpha\to\infty}\int_{\partial\Omega}(\partial_{\tau_{jk}}f_\alpha)\varphi\,d\sigma\\&=-\int_{\partial\Omega}g_{jk}\varphi\,d\sigma,\end{aligned} \tag{2.645}$$

thanks to (2.644), (2.583), and (2.635). From this and (2.581)–(2.582) we then conclude that

$$\partial_{\tau_{jk}}h=g_{jk}\in L^p(\partial\Omega,w) \ \text{ for each } \ j,k\in\{1,\dots,n\}. \tag{2.646}$$

Collectively, (2.643) and (2.646) prove that $h \in \dot{L}_1^p(\partial\Omega, w)$. Finally, from (2.635), (2.646), and (2.601) we conclude that the sequence $\{[f_\alpha]\}_{\alpha\in\mathbb{N}}$ converges to $[h]$, the class of h, in the quotient space $\dot{L}_1^p(\partial\Omega, w)\big/\sim$. □

Chapter 3
Calderón–Zygmund Theory for Boundary Layers in UR Domains

In [25], A.P. Calderón has initiated a breakthrough by proving the L^p-boundedness of the principal-value Cauchy integral operator on Lipschitz curves with small Lipschitz constant. Subsequently, R. Coifman, A. McIntosh, and Y. Meyer have successfully extended Calderón's estimate on Cauchy integrals to general Lipschitz curves in [32] and used this to establish the boundedness of higher-dimensional singular integral operators (such as the harmonic double layer K_Δ) on Lebesgue spaces $L^p(\Sigma, \mathcal{H}^{n-1})$ with $p \in (1, \infty)$, whenever Σ is a strongly Lipschitz surface in $\mathbb{R}^n$. This gave the impetus for studying such singular integral operators on surfaces more general than the boundaries of Lipschitz domains. Works of G. David [37, 38], G. David and D. Jerison [39], G. David and S. Semmes [40, 41], and S. Semmes [122] yield such boundedness when the $\Sigma \subseteq \mathbb{R}^n$ is a UR set, i.e., Σ is a closed Ahlfors regular set which contains "big pieces" of Lipschitz images in a quantitative, uniform, scale-invariant fashion (cf. Definition 2.5).

This body of results, which interfaced tightly with geometric measure theory, has been applied to problems in PDEs for the first time by S. Hofmann, M. Mitrea, and M. Taylor in [61] (see also [109] for PDEs in the setting of Riemannian manifolds). Here we continue this line of work with two specific goals in mind. First, we consider singular integral operators (SIOs) acting on a larger variety of function spaces and, second, we seek finer bounds on the operator norm of the singular integrals of double layer type. We begin by discussing the general setup.

3.1 Boundary Layer Potentials: The Setup

Fix $n \in \mathbb{N}$ with $n \geq 2$ along with some $M \in \mathbb{N}$, and denote by $\mathfrak{L}$ the collection of all homogeneous constant complex coefficient second-order $M \times M$ systems L in $\mathbb{R}^n$. Hence, any element L in $\mathfrak{L}$ may be written as a matrix of differential operators of the

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_3

form $L = \left(a^{\alpha\beta}_{jk}\partial_j\partial_k\right)_{1\le\alpha,\beta\le M}$ for some complex numbers $a^{\alpha\beta}_{jk}$ (here and elsewhere, we shall use the usual convention of summation over repeated indices). In particular, the action of L on any given vector-valued distribution $u = (u_\beta)_{1\le\beta\le M}$ may be described as

$$Lu = \left(a^{\alpha\beta}_{jk}\partial_j\partial_k u_\beta\right)_{1\le\alpha\le M}, \tag{3.1}$$

and we denote by $L^\top := \left(a^{\beta\alpha}_{kj}\partial_j\partial_k\right)_{1\le\alpha,\beta\le M}$ the (real) transpose of L. We also define the characteristic matrix of L as

$$L(\xi) := \Big[\big(-a^{\alpha\beta}_{jk}\xi_j\xi_k\big)_{1\le\alpha,\beta\le M}\Big] \quad \text{for each} \quad \xi = (\xi_i)_{1\le i\le n} \in \mathbb{R}^n \tag{3.2}$$

and introduce

$$\mathfrak{L}_* := \big\{L \in \mathfrak{L} : \det[L(\xi)] \neq 0 \ \text{ for each } \ \xi \in \mathbb{R}^n \setminus \{0\}\big\}. \tag{3.3}$$

We shall refer to a system $L \in \mathfrak{L}$ as being `weakly elliptic` if actually $L \in \mathfrak{L}_*$. This should be contrasted with the more stringent `Legendre-Hadamard (strong) ellipticity` condition which asks for the existence of some $c > 0$ such that

$$\mathrm{Re}\big\langle -L(\xi)\zeta\,,\overline{\zeta}\,\big\rangle \ge c\,|\xi|^2\,|\zeta|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad \zeta \in \mathbb{C}^M. \tag{3.4}$$

Next, let us consider

$$\mathfrak{A} := \Big\{A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} : \text{ each } a^{\alpha\beta}_{jk} \text{ belongs to } \mathbb{C}\Big\}, \tag{3.5}$$

the collection of coefficient tensors with constant complex entries. Adopting natural operations (i.e., componentwise addition and multiplication by scalars), this becomes a finite dimensional vector space (over $\mathbb{C}$), which we endow with the norm

$$\|A\| := \sum_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \big|a^{\alpha\beta}_{jk}\big| \quad \text{for each} \quad A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}. \tag{3.6}$$

Hence, if the transpose of each given $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}$ is the coefficient tensor $A^\top := \big(a^{\beta\alpha}_{kj}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}}$, then $\mathfrak{A} \ni A \mapsto A^\top \in \mathfrak{A}$ is an isometry. With each coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}$ associate the system $L_A \in \mathfrak{L}$ according to

$$L_A := \Big(a_{jk}^{\alpha\beta}\partial_j\partial_k\Big)_{1\leq\alpha,\beta\leq M}. \tag{3.7}$$

Then the map

$$\mathfrak{A} \ni A \longmapsto L_A \in \mathfrak{L} \tag{3.8}$$

is linear and surjective, though it fails to be injective. Specifically, if we introduce

$$\mathfrak{A}^{\rm ant} := \Big\{B = \big(b_{jk}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A} : b_{jk}^{\alpha\beta} = -b_{kj}^{\alpha\beta} \text{ whenever } 1\leq j,k\leq n \text{ and } 1\leq\alpha,\beta\leq M\Big\}, \tag{3.9}$$

the collection of all coefficient tensors that are antisymmetric in the lower indices, then $\mathfrak{A}^{\rm ant}$ is a closed linear subspace of $\mathfrak{A}$, and for each $A, \widetilde{A} \in \mathfrak{A}$, we have

$$L_A = L_{\widetilde{A}} \Longleftrightarrow A - \widetilde{A} \in \mathfrak{A}^{\rm ant}. \tag{3.10}$$

If we now define

$$\mathfrak{A}_L := \big\{A \in \mathfrak{A} : L = L_A\big\} \text{ for each } L \in \mathfrak{L}, \tag{3.11}$$

and for each $L \in \mathfrak{L}$, we set (with the distance considered in the normed vector space $\mathfrak{A}$)

$$\|L\| := \operatorname{dist}\big(A, \mathfrak{A}^{\rm ant}\big) \text{ for each/some } A \in \mathfrak{A}_L, \tag{3.12}$$

then $\mathfrak{L} \ni L \mapsto \|L\|$ is an unambiguously defined norm on the vector space $\mathfrak{L}$. In the topology induced by this norm, $\mathfrak{L}_*$ from (3.3) is an open subset of $\mathfrak{L}$, the mapping (3.8) is continuous, and $\mathfrak{L} \ni L \mapsto L^\top \in \mathfrak{L}$ is an isometry.

Finally, we denote by $\mathfrak{A}_{\rm WE}$ the collection of all coefficient tensors A with the property that the $M \times M$ homogeneous second-order system L_A associated with A in $\mathbb{R}^n$ as in (3.7) is weakly elliptic, i.e.,

$$\mathfrak{A}_{\rm WE} := \big\{A \in \mathfrak{A} : L_A \in \mathfrak{L}_*\big\}. \tag{3.13}$$

Then $\mathfrak{A}_{\rm WE}$ is an open subset of $\mathfrak{A}$.

The following theorem, itself a special case of [102, Theorem 11.1, p. 393], summarizes some of the main properties of a certain type of fundamental solution canonically associated with any given homogeneous, constant complex coefficient, weakly elliptic second-order system in $\mathbb{R}^n$.

Theorem 3.1 *Let L be a homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$, which is weakly elliptic (cf. (1.2)). Then there exists an $M \times M$*

matrix-valued function $E = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$, canonically associated with the given system L, such that the following properties are true:

1. *For any two indices $\alpha, \beta \in \{1, \dots, M\}$, one has $E_{\alpha\beta} \in \mathscr{C}^{\infty}\big(\mathbb{R}^n \setminus \{0\}\big)$ as well as $E_{\alpha\beta}(x) = E_{\alpha\beta}(-x)$ for every $x \in \mathbb{R}^n \setminus \{0\}$.*
2. *For each fixed point $y \in \mathbb{R}^n$, one has $L\big[E(\cdot - y)\big] = \delta_y I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$, where $I_{M\times M}$ is the $M \times M$ identity matrix and δ_y denotes the Dirac distribution with mass at y in $\mathbb{R}^n$. That is, using the standard Kronecker delta notation,*
$$a^{\alpha\beta}_{jk}\partial_{x_j}\partial_{x_k}\big[E_{\beta\gamma}(x - y)\big] = \delta_{\alpha\gamma}\delta_y(x), \qquad x \in \mathbb{R}^n, \tag{3.14}$$
in the sense of distributions, for every $\alpha, \gamma \in \{1, \dots, M\}$.
3. *The transpose of E, i.e., $E^{\top} = (E_{\beta\alpha})_{1\leq\alpha,\beta\leq M}$, is a fundamental solution for the transpose system $L^{\top}$. In other words, for each fixed point $y \in \mathbb{R}^n$, one has $L^{\top}\big[E^{\top}(\cdot - y)\big] = \delta_y I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$, i.e.,*
$$a^{\beta\alpha}_{kj}\partial_{x_j}\partial_{x_k}\big[E_{\gamma\beta}(x - y)\big] = \delta_{\alpha\gamma}\delta_y(x), \qquad x \in \mathbb{R}^n, \tag{3.15}$$
in the sense of distributions, for every $\alpha, \gamma \in \{1, \dots, M\}$.
4. *For every multi-index $\alpha \in \mathbb{N}^n_0$ with $n + |\alpha| > 2$, the function $\partial^{\alpha}E$ is positive homogeneous of degree $2 - n - |\alpha|$ and there exists a constant $C_{\alpha} \in (0, \infty)$ with the property that*
$$\big|(\partial^{\alpha}E)(x)\big| \leq C_{\alpha}|x|^{2-n-|\alpha|} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{3.16}$$
Finally, corresponding to $n = 2$ and $\alpha = (0, \dots, 0)$, there exists $C \in (0, \infty)$ such that $|E(x)| \leq C\big(1 + \big|\ln|x|\big|\big)$ for every $x \in \mathbb{R}^2 \setminus \{0\}$.
5. *Let 'hat' denote the Fourier transform in $\mathbb{R}^n$ (originally defined on Schwartz functions and then extended to tempered distributions via duality). Then $\widehat{E}$ is a tempered distribution in $\mathbb{R}^n$ (which is positive homogeneous of degree -2 if $n \geq 3$), whose restriction to $\mathbb{R}^n \setminus \{0\}$ is a (matrix-valued) function of class $\mathscr{C}^{\infty}$. In fact,*
$$\widehat{E}(\xi) = \big[L(\xi)\big]^{-1} \text{ for every } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3.17}$$
More generally, given any $\gamma \in \mathbb{N}^n_0$, it follows that the tempered distribution $\widehat{\partial^{\gamma}E}$ is a function of class $\mathscr{C}^{\infty}$ when restricted to $\mathbb{R}^n \setminus \{0\}$, which, regarded as such, satisfies
$$\widehat{\partial^{\gamma}E}(\xi) = \mathrm{i}^{|\gamma|}\,\xi^{\gamma}\,\big[L(\xi)\big]^{-1} \text{ for every } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{3.18}$$
and

$$\textit{if } \gamma \in \mathbb{N}_0^n, \textit{ then } \widehat{\partial^\gamma E} = \mathrm{i}^{|\gamma|}\,\xi^\gamma\,\big[L(\xi)\big]^{-1} \textit{ as tempered distributions in } \mathbb{R}^n \textit{ when either } |\gamma| > 0 \textit{ or } n \geq 3. \tag{3.19}$$

6. *Writing E_L in place of E to emphasize the dependence on L, matters may be arranged so that*

$$\begin{aligned} &\big(E_L\big)^\top = E_{L^\top}, \quad \overline{\big(E_L\big)} = E_{\overline{L}}, \quad \big(E_L\big)^* = E_{L^*}, \\ &\textit{as well as } E_{\lambda L} = \lambda^{-1} E_L \textit{ for each } \lambda \in \mathbb{C} \setminus \{0\}, \end{aligned} \tag{3.20}$$

where $\top$, $\bar{\cdot}$, and $*$ denote, respectively, transposition, complex conjugation, and complex (or Hermitian) adjunction.

Moving on, assume $\Omega \subseteq \mathbb{R}^n$ is a given UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. In addition, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, and consider the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with L as in Theorem 3.1. Finally, fix a coefficient tensor $A = \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$, and pick an arbitrary function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.21}$$

In this setting, define the action of the boundary-to-domain double layer potential operator $\mathcal{D}_A$ on f as

$$\mathcal{D}_A f(x) := \left(-\int_{\partial\Omega} \nu_k(y) a_{jk}^{\beta\alpha} \big(\partial_j E_{\gamma\beta}\big)(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\gamma\leq M}, \tag{3.22}$$

at each point $x \in \Omega$. From (3.16), we see that (3.21) is the most general functional analytic setting in which the integral in (3.22) is absolutely convergent. The double layer operator $\mathcal{D}$ may be regarded as a mechanism for generating lots of null-solutions for the given system L in Ω since, as is apparent from (3.22) and Theorem 3.1,

$$\begin{aligned} &\text{for each function } f \text{ as in (3.21), we have} \\ &\mathcal{D}_A f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \text{ and } \ L(\mathcal{D}_A f) = 0 \ \text{ in } \ \Omega. \end{aligned} \tag{3.23}$$

Going further, let us define the action of the boundary-to-boundary double layer potential operator K_A on f as in (3.21) by setting

$$K_A f(x) := \Bigg(- \lim_{\varepsilon \to 0^+} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \nu_k(y) a_{jk}^{\beta\alpha} \left(\partial_j E_{\gamma\beta}\right)(x-y) f_\alpha(y) \, d\sigma(y) \Bigg)_{1 \leq \gamma \leq M}, \tag{3.24}$$

at σ-a.e. point $x \in \partial\Omega$. Another singular integral operator that is closely related to (3.24) is the so-called transpose double layer operator $K_A^{\#}$ defined by setting

$$K_A^{\#} f(x) := \Bigg(\lim_{\varepsilon \to 0^+} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \nu_k(x) a_{jk}^{\beta\alpha} (\partial_j E_{\gamma\beta})(x-y) f_\gamma(y) \, d\sigma(y) \Bigg)_{1 \leq \alpha \leq M} \tag{3.25}$$

at σ-a.e. $x \in \partial\Omega$, for each function f as in (3.21). Since we are presently assuming that Ω is a UR domain, work in [114, Chapter 1] guarantees that the above singular integral operators are well defined in a σ-a.e. pointwise fashion for each function as in (3.21). Also, it is clear from definitions and the last line in (3.20) that

$$\begin{gathered} \mathcal{D}_{\lambda A} = \mathcal{D}_A, \quad K_{\lambda A} = K_A, \quad K_{\lambda A}^{\#} = K_A^{\#} \\ \text{for each } \lambda \in \mathbb{C} \text{ with } \lambda \neq 0. \end{gathered} \tag{3.26}$$

Example 3.1 The standard fundamental solution for the Laplacian in $\mathbb{R}^n$ is defined for $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E_\Delta(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)} \dfrac{1}{|x|^{n-2}}, & \text{if } n \geq 3, \\[2ex] \dfrac{1}{2\pi} \ln |x|, & \text{if } n = 2, \end{cases} \tag{3.27}$$

where, as usual, ω_{n-1} denotes the surface area of the unit sphere in $\mathbb{R}^n$ (cf. [102, Section 7.1]). Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Set $a_{jk}^{\alpha\beta} := a_{jk} := \delta_{jk}$ in (3.1) so that $L = \Delta$, and refer to $\mathcal{D}_\Delta$, K_Δ (constructed as in (3.22) and (3.24)) for this choice of coefficient tensor, i.e., for $A := I_{n \times n}$, the identity matrix) as being the (classical) harmonic double layer potentials. Concretely, for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, we have (writing, in this case, $\mathcal{D}_\Delta$, K_Δ, $K_\Delta^{\#}$ in place of $\mathcal{D}_{I_{n\times n}}$, $K_{I_{n\times n}}$, $K_{I_{n\times n}}^{\#}$)

$$\mathcal{D}_\Delta f(x) = \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle \nu(y), y - x \rangle}{|x-y|^n} f(y) \, d\sigma(y), \qquad \forall\, x \in \Omega, \tag{3.28}$$

and, at σ-a.e. point $x \in \partial\Omega$,

$$K_\Delta f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{\langle\nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y), \tag{3.29}$$

$$K_\Delta^{\#} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{\langle\nu(x), x-y\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y). \tag{3.30}$$

Returning to the mainstream discussion, continue to assume that $\Omega \subseteq \mathbb{R}^n$ is a UR domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, as before, continue to assume that L is a homogeneous constant complex coefficient weakly elliptic second-order $M \times M$ system in $\mathbb{R}^n$. Then, for each coefficient tensor $A \in \mathfrak{A}_L$, a basic identity relating the boundary-to-domain double layer potential operator $\mathcal{D}_A$ to the boundary-to-boundary double layer potential operator K_A is the jump-formula (proved in [114, §1.5]), to the effect that if I denotes the identity operator and $\kappa > 0$ is an arbitrary aperture parameter, then

$$\begin{aligned} &\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_A\big) f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \\ &\text{for each given function } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \end{aligned} \tag{3.31}$$

Another fundamental property of the boundary-to-domain double layer potential operator is the ability of absorbing an arbitrary spacial derivative and eventually relocating it, via integration by parts on the boundary, all the way to the function on which this was applied to begin with. This is made precise in the following basic proposition, proved in [114, §1.3].

Proposition 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some $M \in \mathbb{N}$, consider a weakly elliptic, homogeneous, constant (complex) coefficient, second-order, $M \times M$ system L in $\mathbb{R}^n$, written as in (3.1) for some choice of a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$. Finally, associate with A and Ω the double layer potential operator $\mathcal{D}_A$ as in (3.22), and consider a function*

$$\begin{aligned} &f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \ \textit{with the property that} \\ &\partial_{\tau_{jk}} f_\alpha \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \ \textit{for } 1 \le j,k \le n \textit{ and } 1 \le \alpha \le M. \end{aligned} \tag{3.32}$$

Then, for each index $\ell \in \{1, \dots, n\}$ and each point $x \in \Omega$, one has

$$\partial_\ell\big(\mathcal{D}_A f\big)(x) = \left(\int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\gamma\leq M}. \tag{3.33}$$

As a consequence, if Ω is actually a UR *domain then for each aperture parameter* $\kappa > 0$*, the nontangential boundary trace*

$$\big(\nabla\mathcal{D}_A f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \textit{exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{3.34}$$

We next recall the following result from [114, §1.5], which identifies the commutator between the double layer potential operator K_A from (3.24) and the first-order tangential derivative operators $\partial_{\tau_{jk}}$ from (2.582) as being yet another commutator, of the sort considered in detail later, in Theorem 4.3 (with the function b a scalar component of the outward unit normal ν).

Proposition 3.2 *Suppose* $\Omega \subseteq \mathbb{R}^n$ *is a UR domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu = (\nu_1, \dots, \nu_n)$ *the geometric measure theoretic outward unit normal to* Ω*. Let* L *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *and consider the matrix-valued fundamental solution* $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ *associated with* L *as in Theorem 3.1. Also, pick a coefficient tensor* $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$ *and bring in* K_A *the boundary-to-boundary double layer potential operator associated with* Ω *and* A *as in* (3.24)*. In addition, for each* $j,k \in \{1,\dots,n\}$*, define the singular integral operator* U_{jk} *acting on each given matrix-valued function* $F = (F_{\alpha s})_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}$ *with entries belonging to* $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ *as* $U_{jk}F = \big((U_{jk}F)_\gamma\big)_{1\leq\gamma\leq M}$ *where, for each index* $\gamma \in \{1,\dots,M\}$*,*

$$\begin{aligned}
&(U_{jk}F)_\gamma(x)\\
&\quad := -\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(x)-\nu_k(y)]\nu_j(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha s}(y)\,\mathrm{d}\sigma(y)\\
&\qquad + \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(x)-\nu_j(y)]\nu_k(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha s}(y)\,\mathrm{d}\sigma(y)\\
&\qquad + \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(y)-\nu_k(x)]\nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha j}(y)\,\mathrm{d}\sigma(y)
\end{aligned}$$

$$-\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}[\nu_j(y)-\nu_j(x)]\nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)F_{\alpha k}(y)\,\mathrm{d}\sigma(y) \tag{3.35}$$

at σ-a.e. point $x\in\partial\Omega$. Finally, fix some integrability exponents $p,q\in(1,\infty]$ and consider a function

$$\begin{aligned} f &\in \Big[L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M \ \textit{with the property that}\\ \partial_{\tau_{jk}} f &\in \Big[L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\cap L^q_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M \ \textit{for all}\ j,k\in\{1,\dots,n\}. \end{aligned} \tag{3.36}$$

Then, for each $j,k\in\{1,\dots,n\}$, one has

$$\partial_{\tau_{jk}}(K_A f)=K_A(\partial_{\tau_{jk}}f)+U_{jk}(\nabla_{\tan}f)\ \textit{ at } \sigma\textit{-a.e. point on }\ \partial\Omega, \tag{3.37}$$

where $\nabla_{\tan}f$ is regarded as the $M\times n$ matrix-valued function $F=(F_{\alpha s})_{\substack{1\le\alpha\le M\\1\le s\le n}}$ whose entry $F_{\alpha s}$ is the s-th component of $\nabla_{\tan}f_\alpha$.

Once again, assume $\Omega\subseteq\mathbb{R}^n$ is a UR domain and set $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, as before, continue to assume that L is a homogeneous constant complex coefficient weakly elliptic second-order $M\times M$ system in $\mathbb{R}^n$. In general, different choices of the coefficient tensor $A\in\mathfrak{A}_L$ yield different double layer potential operators, so it makes sense to use the subscript A to highlight the dependence on the choice of the coefficient tensor A. One integral operator of layer potential variety which is intrinsically associated with the given system L is the so-called single layer potential operator $\mathscr{S}$, whose integral kernel is the matrix-valued function $E(x-y)$, for all points $x,y\in\partial\Omega$. In order to make sense of the action of such an operator on any function as in (3.21), it is necessary to alter said integral kernel and consider the following modified single layer potential operator

$$\begin{aligned} &\mathscr{S}_{\mathrm{mod}}f(x):=\int_{\partial\Omega}\big\{E(x-y)-E_*(-y)\big\}f(y)\,\mathrm{d}\sigma(y)\ \text{ for each }\ x\in\Omega,\\ &\quad\text{for each }\ f\in\big[L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M,\quad\text{where}\quad E_*:=E\cdot\mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}. \end{aligned} \tag{3.38}$$

In this regard, it is worth noting that for each $f\in\Big[L^1\big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ the function $\mathscr{S}_{\mathrm{mod}}f$ is well defined, belongs to the space $\big[\mathscr{C}^\infty(\Omega)\big]^M$, and for each multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\ge 1$, one has

$$\partial^\alpha(\mathscr{S}_{\mathrm{mod}}f)(x)=\int_{\partial\Omega}(\partial^\alpha E)(x-y)f(y)\,\mathrm{d}\sigma(y)\ \text{ for each }\ x\in\Omega. \tag{3.39}$$

In particular,

$$L\big(\mathscr{S}_{\rm mod} f\big) = 0 \text{ in } \Omega \text{ for each } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.40}$$

As noted in [114, §1.5], if $n \geq 3$ then for each aperture parameter $\kappa > 0$ and each truncation parameter $\varepsilon > 0$ we have

$$\mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\rm mod} f) \in \bigcap_{0<p<\frac{n-1}{n-2}} L^p_{\rm loc}(\partial\Omega, \sigma) \ \text{ for each } \ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.41}$$

Analogously to (3.38), let us now define the following modified version of the boundary-to-boundary single layer operator

$$\begin{gathered} S_{\rm mod} f(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\, {\rm d}\sigma(y) \ \text{ at } \sigma\text{-a.e. } \ x \in \partial\Omega, \\ \text{for each } \ f \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M, \quad \text{where } \ E_* := E \cdot \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}. \end{gathered} \tag{3.42}$$

Then this operator is meaningfully defined, via an absolutely convergent integral, at σ-a.e. point in $\partial\Omega$, and it has been shown in [114, §1.5] that for each $\varepsilon > 0$ the operator

$$S_{\rm mod} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\big)\Big]^M \longrightarrow \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n}}\big)\Big]^M \tag{3.43}$$

is well defined, linear, and bounded. In particular, from (3.43) and the embedding in (2.573) we see that for each weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ the following mapping is well defined, linear, and bounded:

$$S_{\rm mod} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M. \tag{3.44}$$

In addition, it has been shown in [114, §1.5] that

$$\begin{gathered} S_{\rm mod} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\rm loc}(\partial\Omega, w)\Big]^M \longrightarrow \big[L^p_{\rm loc}(\partial\Omega, w)\big]^M \\ \text{is a well-defined, linear, and continuous mapping} \\ \text{for each weight } \ w \in A_p(\partial\Omega, \sigma) \ \text{ with } \ p \in (1, \infty), \end{gathered} \tag{3.45}$$

and (with $\mathrm{Lip}(\partial\Omega)$ denoting the space of scalar-valued Lipschitz functions on $\partial\Omega$)

given an arbitrary Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, it follows that for each sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[L^p(\partial\Omega, w)\big]^M$ which is weak-$*$ convergent to some function $f \in \big[L^p(\partial\Omega, w)\big]^M$, one has that the limit $\lim\limits_{j\to\infty} \int_{\partial\Omega} \langle S_{\rm mod} f_j, \phi\rangle \,{\rm d}\sigma = \int_{\partial\Omega} \langle S_{\rm mod} f, \phi\rangle \,{\rm d}\sigma$ holds for each test function $\phi \in \big[{\rm Lip}(\partial\Omega)\big]^M$ with compact support. (3.46)

Also, with the modified boundary-to-domain single layer operator $\mathscr{S}_{\rm mod}$ as in (3.38), for each aperture parameter $\kappa > 0$ and each $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$, one has

$$\Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x) = (S_{\rm mod} f)(x) \quad \text{at } \sigma\text{-a.e. point } x \in \partial\Omega. \tag{3.47}$$

See [114, §1.5] for proofs of all these claims, and for a more in-depth discussion on this topic.

Theorem 3.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a UR domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Recall the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with L as in Theorem 3.1 and define*

$$\begin{aligned} &k^{(r\gamma\beta)}_{\varepsilon} := (\partial_r E_{\gamma\beta}) \cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}} \quad \textit{for each } \varepsilon > 0, \\ &\textit{each } \gamma, \beta \in \{1, \dots, M\} \textit{ and } r \in \{1, \dots, n\}. \end{aligned} \tag{3.48}$$

In this setting, consider the following modified version of the double layer operator (3.22)

$$\begin{aligned} &\big(\mathcal{D}_{A,mod} f\big)(x) \\ &\quad := \Big(-\int_{\partial\Omega} \nu_s(y) a^{\beta\alpha}_{rs} \big\{(\partial_r E_{\gamma\beta})(x-y) - k^{(r\gamma\beta)}_1(-y)\big\} f_\alpha(y)\,{\rm d}\sigma(y)\Big)_{1\leq\gamma\leq M} \\ &\textit{for each } f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \textit{ and } x \in \Omega, \end{aligned} \tag{3.49}$$

and consider the following modified boundary-to-boundary double layer potential operator (3.24)

$$K_{A,mod}f(x)$$

$$:= \Bigg(-\lim_{\varepsilon\to 0^+}\int\limits_{\partial\Omega} \nu_s(y)a^{\beta\alpha}_{rs}\big\{k^{(r\gamma\beta)}_{\varepsilon}(x-y) - k^{(r\gamma\beta)}_{1}(-y)\big\} f_\alpha(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq M}$$

$$\textit{for each } f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \textit{ and } \sigma\textit{-a.e. } x\in\partial\Omega, \tag{3.50}$$

Then the following properties hold.

(1) The operator $\mathcal{D}_{A,mod}$ *is meaningfully defined, and satisfies*

$$\begin{gathered} \mathcal{D}_{A,mod}f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \;\; \textit{and} \;\; L(\mathcal{D}_{A,mod}f) = 0 \;\; \textit{in} \;\; \Omega, \\ \textit{for each} \;\; f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M. \end{gathered} \tag{3.51}$$

In addition, the operator $\mathcal{D}_{A,mod}$ *is compatible with* $\mathcal{D}_A$ *from* (3.22)*, in the sense that for each function* f *belonging to the smaller space* $\big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *the difference*

$$C_f := \mathcal{D}_{A,mod}f - \mathcal{D}f \quad \textit{is a constant (belonging to } \mathbb{C}^M\textit{) in } \Omega. \tag{3.52}$$

As a consequence,

$$\nabla\mathcal{D}_{A,mod}f = \nabla\mathcal{D}f \textit{ in } \Omega \textit{ for each } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{3.53}$$

Moreover,

$\mathcal{D}_{A,mod}$ *maps constant ($\mathbb{C}^M$-valued) functions on $\partial\Omega$ into constant ($\mathbb{C}^M$-valued) functions in Ω.* (3.54)

In addition, at each point $x\in\Omega$ one may express

$$\begin{gathered} \partial_j\big(\mathcal{D}_{A,mod}f\big)(x) = \Big(-\int_{\partial_*\Omega} \nu_s(y)a^{\beta\alpha}_{rs}(\partial_j\partial_r E_{\gamma\beta})(x-y)f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M} \\ \textit{for each } j\in\{1,\dots,n\} \textit{ and } f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^M. \end{gathered} \tag{3.55}$$

Finally, given any function

$$f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \quad \textit{with the property that}$$
$$\partial_{\tau_{jk}} f_\alpha \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \ \textit{for all } j,k\in\{1,\dots,n\} \textit{ and } \alpha\in\{1,\dots,M\}, \tag{3.56}$$

it follows that for each index $\ell \in \{1,\dots,n\}$ and each point $x \in \Omega$, one has

$$\partial_\ell\big(\mathcal{D}_{A,mod} f\big)(x) = \bigg(\int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\bigg)_{1\le\gamma\le M}. \tag{3.57}$$

(2) *Fix an aperture parameter $\kappa \in (0,\infty)$, a truncation parameter $\varepsilon > 0$, and an integrability exponent $p \in (1,\infty)$. Then the nontangential boundary trace*

$$\begin{gathered}\big(\partial_\ell \mathcal{D}_{A,mod} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{exists (in } \mathbb{C}^M\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega, \\ \textit{for each function } f \textit{ as in } (3.56) \textit{ and each index } \ell\in\{1,\dots,n\}.\end{gathered} \tag{3.58}$$

Also, one has

$$\begin{gathered}\mathcal{N}^\varepsilon_\kappa\big(\nabla(\mathcal{D}_{A,mod} f)\big) \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \ \textit{for each function} \\ f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M \ \textit{such that} \\ \partial_{\tau_{jk}} f_\alpha \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \\ \textit{for all } j,k\in\{1,\dots,n\} \textit{ and all } \alpha\in\{1,\dots,M\}.\end{gathered} \tag{3.59}$$

In addition,

$$\begin{gathered}\mathcal{N}^\varepsilon_\kappa(\mathcal{D}_{A,mod} f) \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \ \textit{for each function} \\ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M.\end{gathered} \tag{3.60}$$

Furthermore, the following jump-formula holds:

$$\begin{gathered}\big(\mathcal{D}_{A,mod} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{A,mod}\big)f \ \textit{ at } \sigma\textit{-a.e. point on } \partial\Omega, \\ \textit{for each given function } f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M,\end{gathered} \tag{3.61}$$

where, as usual, I is the identity operator. As a consequence of (3.61) *and* (3.54), *the operator* $K_{A,mod}$ *maps constant* ($\mathbb{C}^M$*-valued) functions on* $\partial\Omega$ *into constant* ($\mathbb{C}^M$*-valued) functions on* $\partial\Omega$.

(3.62)

Finally, the operator $K_{A,mod}$ *(from* (3.50)*) is compatible with* K *(acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (3.24)*) in the sense that*

$$\begin{aligned}&\textit{for each function } f \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M \textit{ the difference}\\ &c_f := K_{mod} f - Kf \textit{ is a constant (belonging to } \mathbb{C}^M\textit{) on } \partial\Omega.\end{aligned} \tag{3.63}$$

Moving on, in view of (3.63) and the fact that tangential derivatives annihilate locally constant functions, the following result from [114, §1.8] may be regarded as a generalization of Proposition 3.2.

Proposition 3.3 *Assume* $\Omega \subseteq \mathbb{R}^n$ *is a UR domain and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Consider an* $M \times M$ *homogeneous, second-order, constant complex coefficient, weakly elliptic system* L *in* $\mathbb{R}^n$*, and pick some coefficient tensor* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le r,s\le n}}$ *for which* $L_A = L$. *Let* K_A *be the boundary-to-boundary double layer potential operator associated with* Ω *and* A *as in* (3.24)*, and bring in its modified version* $K_{A,mod}$ *from* (3.50)*. Finally, recall the family of singular integral operators* U_{jk} *with* $j, k \in \{1, \dots, n\}$ *defined in* (3.35) *and fix some integrability exponent* $p \in (1, \infty)$. *Then for each function*

$$\begin{aligned}f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma)\Big]^M \textit{ such that}\\ \partial_{\tau_{jk}} f_\alpha \textit{ belongs to } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma)\\ \textit{for all } j, k \in \{1, \dots, n\} \textit{ and } \alpha \in \{1, \dots, M\},\end{aligned} \tag{3.64}$$

and each pair of indices $j, k, \in \{1, \dots, n\}$*, one has*

$$\partial_{\tau_{jk}}\big(K_{A,mod} f\big) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \tag{3.65}$$

where, as in the case of (3.37)*,* $\nabla_{\tan} f$ *is regarded as the* $M \times n$ *matrix-valued function whose* (α, s) *entry is the* s*-th component of the tangential gradient* $\nabla_{\tan} f_\alpha$.

We next introduce (and briefly elaborate on) the notion of conormal derivative operator associated with a given domain and a given coefficient tensor. Specifically, suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ is defined σ-a.e. on $\partial\Omega$. Also, fix a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ along with some aperture parameter $\kappa > 0$. In such a setting, for any function $u = (u_\beta)_{1\le\beta\le M} \in \big[W^{1,1}_{\rm loc}(\Omega)\big]^M$ with the property that the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists (in $\mathbb{C}^{M\times n}$) at σ-a.e. point on $\partial\Omega$ define the `conormal derivative` $\partial_\nu^A u$ as the $\mathbb{C}^M$-valued function

$$\partial_\nu^A u := \Big(\nu_r a_{rs}^{\alpha\beta}\big(\partial_s u_\beta\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)_{1\leq\alpha\leq M} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.66}$$

In relation to this, it has been proved in [114, §1.5] that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ then for each function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ the conormal derivative $\partial_\nu^A \mathscr{S}_{\mathrm{mod}} f$ may be meaningfully considered in the sense of (3.66), and

$$\partial_\nu^A \mathscr{S}_{\mathrm{mod}} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \quad \text{at } \sigma\text{-a.e. point in } \partial\Omega, \tag{3.67}$$

where I is the identity, and $K^{\#}_{A^\top}$ is the operator associated as in (3.25) with the UR domain Ω and the transpose coefficient tensor $A^\top$.

We shall also need the following basic integral representation formula, established in [114, §1.8], for null-solutions of weakly elliptic systems in Ahlfors regular domains, in terms of modified boundary-to-domain layer potential operators.

Theorem 3.3 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an Ahlfors regular domain which is either bounded, or has an unbounded boundary. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, consider $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,mod}$ from (3.49), and the modified version of the single layer operator $\mathscr{S}_{mod}$ from (3.38). Finally, fix an aperture parameter $\kappa \in (0,\infty)$, a truncation parameter $\varepsilon \in (0,\infty)$, and consider a function $u : \Omega \to \mathbb{C}^M$ satisfying*

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \textit{in} \ \Omega, \quad \mathcal{N}_\kappa^\varepsilon u \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \partial\Omega \ \textit{and} \ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M, \\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma\textit{-a.e. on } \partial\Omega \ \textit{and} \ \mathcal{N}_\kappa(\nabla u) \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big). \end{gathered} \tag{3.68}$$

Then there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{A,mod}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathscr{S}_{mod}\big(\partial_\nu^A u\big) + c_u \ \ \textit{in} \ \ \Omega. \tag{3.69}$$

We proceed by recalling the following Fatou-type theorem established in [113, §3.3].

Theorem 3.4 *Suppose $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is an arbitrary UR domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a homogeneous constant (complex) coefficient second-order $M \times M$ system L in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$) which is weakly elliptic, and assume $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ is a vector-valued function which, for some aperture parameter $\kappa > 0$, satisfies*

$$\begin{aligned} &\mathcal{N}_\kappa(\nabla u) \in L^p_{\rm loc}(\partial\Omega, \sigma) \;\; \textit{for some} \;\; p \in (1, \infty] \\ &\textit{and} \;\; Lu = 0 \;\; \textit{in} \;\; \Omega. \end{aligned} \tag{3.70}$$

Then the nontangential boundary trace $\Big((\nabla u)\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)(x)$ exists (in $\mathbb{C}^{M\times n}$) at σ-a.e. point $x \in \partial\Omega$,

$$\textit{the function } (\nabla u)\big|^{\kappa-\rm n.t.}_{\partial\Omega} \textit{ belongs to the space } \big[L^p_{\rm loc}(\partial\Omega, \sigma)\big]^{M\times n}, \tag{3.71}$$

and

$$\Big|(\nabla u)\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big| \leq \mathcal{N}_\kappa(\nabla u) \;\; \textit{at } \sigma\textit{-a.e. point on} \;\; \partial\Omega. \tag{3.72}$$

A combination of Theorems 3.3 and 3.4 gives the following basic result.

Corollary 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an NTA domain with an unbounded Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For $M \in \mathbb{N}$, consider $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}}$ a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$.*

Once again, recall the modified version of the double layer operator $\mathcal{D}_{A,mod}$ from (3.49), and the modified version of the single layer operator $\mathscr{S}_{mod}$ from (3.38). Finally, fix an aperture parameter $\kappa \in (0, \infty)$ along with an integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. In this setting, consider a function $u : \Omega \to \mathbb{C}^M$ satisfying

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \;\; \textit{in} \;\; \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w). \tag{3.73}$$

Then

$$\begin{aligned} &u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \;\; \textit{exists } \sigma\textit{-a.e. on} \;\; \partial\Omega \;\; \textit{and belongs to} \;\; \big[\dot{L}^p_1(\partial\Omega, w)\big]^M, \\ &(\nabla u)\big|^{\kappa-\rm n.t.}_{\partial\Omega} \;\; \textit{exists } \sigma\textit{-a.e. on} \;\; \partial\Omega, \;\; \textit{and} \;\; \partial^A_\nu u \;\; \textit{belongs to} \;\; \big[L^p(\partial\Omega, w)\big]^M, \end{aligned} \tag{3.74}$$

and there exists some $c_u \in \mathbb{C}^M$ with the property that

$$u = \mathcal{D}_{A,mod}\big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) - \mathscr{S}_{mod}\big(\partial^A_\nu u\big) + c_u \;\; \textit{in} \;\; \Omega. \tag{3.75}$$

Proof From Proposition 2.24, we see that $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$. In concert, the membership in (3.73), (2.608), (2.11) (used with $\sigma := w$), and (2.576) also implies that $\mathcal{N}_\kappa^\varepsilon u \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ for each $\varepsilon > 0$. Next, the present hypotheses on Ω ensure (cf. (2.48)) that Ω is a UR domain. Keeping this in mind, the Fatou-type result from Theorem 3.4 guarantees that the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists (in $\mathbb{C}^{M\cdot n}$) at σ-a.e. point on $\partial\Omega$. In particular, $\partial_\nu^A u$ is well defined and belongs to the space $\big[L^p(\partial\Omega, w)\big]^M$ (cf. (3.66), (3.71)–(3.72)). Hence, all conditions in (3.68) are satisfied, and this permits us to invoke Theorem 3.3 to conclude that (3.75) holds (for some constant $c_u \in \mathbb{C}^M$, given that the hypotheses on Ω ensure that this set is connected). □

3.2 SIOs on Muckenhoupt Weighted Lebesgue and Sobolev Spaces

We begin by considering garden variety Calderón–Zygmund singular integral operators (SIOs), i.e., operators of convolution-type with odd, homogeneous, sufficiently smooth kernels, which otherwise lack any particular algebraic characteristics. The goal is to obtain estimates in Muckenhoupt weighted Lebesgue spaces on UR sets in $\mathbb{R}^n$.

Proposition 3.4 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N\big(\mathbb{R}^n \setminus \{0\}\big)$ which is odd and positive homogeneous of degree $1-n$. Also, fix an integrability exponent $p \in (1,\infty)$, along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$. In this setting, for each $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, define*

$$T_\varepsilon f(x) := \int_{\substack{y\in\Sigma\\ |x-y|>\varepsilon}} k(x-y) f(y)\,\mathrm{d}\sigma(y) \ \textit{for all}\ x \in \Sigma\ \textit{and}\ \varepsilon > 0, \tag{3.76}$$

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \ \textit{for each}\ x \in \Sigma, \tag{3.77}$$

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \ \textit{for}\ \sigma\textit{-a.e.}\ x \in \Sigma. \tag{3.78}$$

Then there exists a constant $C \in (0,\infty)$ which depends exclusively on n, p, $[w]_{A_p}$, and the UR constants of Σ (and which stays bounded as $[w]_{A_p}$ stays bounded) with the property that for each $f \in L^p(\Sigma, w)$, one has

$$\|T_* f\|_{L^p(\Sigma,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{L^p(\Sigma,w)} . \tag{3.79}$$

In particular,

the truncated integral operators $T_\varepsilon : L^p(\Sigma, w) \to L^p(\Sigma, w)$ are well defined, linear, and bounded in a uniform fashion with respect to the truncation parameter $\varepsilon > 0$. (3.80)

Moreover, for each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, the limit defining $Tf(x)$ in (3.78) exists at σ-a.e. $x \in \Sigma$ and the operator

$$T : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{3.81}$$

is well defined, linear, and bounded. Let $p' \in (1, \infty)$ denote the Hölder conjugate exponent of p, and, with $w' := w^{1-p'} \in A_{p'}(\Sigma, \sigma)$, consider the natural identification

$$\big(L^p(\Sigma, w)\big)^* = L^{p'}(\Sigma, w'). \tag{3.82}$$

Then, under the canonical integral pairing $(f, g) \mapsto \int_\Sigma fg \, \mathrm{d}\sigma$, it follows that

the (real) transpose of the operator (3.81) is the operator $-T : L^{p'}(\Sigma, w') \to L^{p'}(\Sigma, w')$. (3.83)

Finally, assume $\Omega \subseteq \mathbb{R}^n$ is an open set such that $\partial\Omega$ is a UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and pick an aperture parameter $\kappa > 0$. With the integral kernel k as before, for each $f \in L^p(\partial\Omega, w)$, define

$$\mathcal{T} f(x) := \int_{\partial\Omega} k(x - y) f(y) \, \mathrm{d}\sigma(y) \;\; \textit{for each} \;\; x \in \Omega. \tag{3.84}$$

Then there exists a constant $C \in (0, \infty)$ which depends exclusively on n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ with the property that for each $f \in L^p(\partial\Omega, w)$, one has

$$\|\mathcal{N}_\kappa(\mathcal{T} f)\|_{L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{L^p(\partial\Omega,w)} . \tag{3.85}$$

Also, for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, one has the jump-formula

$$\Big(\mathcal{T} f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) = \frac{1}{2\mathrm{i}}\widehat{k}\big(\nu(x)\big) f(x) + (Tf)(x) \;\; \textit{at } \sigma\textit{-a.e.} \;\; x \in \partial_*\Omega, \tag{3.86}$$

where $\widehat{k}$ *denotes the Fourier transform of* k. *In particular, the jump-formula* (3.86) *is valid for each function* $f \in L^p(\partial\Omega, w)$.

The above proposition points to uniform rectifiability as being intimately connected with the boundedness of a large class of Calderón–Zygmund like operators on Muckenhoupt weighted Lebesgue spaces. From the work of G. David and S. Semmes (cf. [40, 41]) and F. Nazarov, X. Tolsa, and A. Volberg in [118] (see also [96] for similar results proved earlier in the plane), we know that UR sets make up the most general context in which convolution-like singular integral operators are bounded on ordinary Lebesgue spaces. Moreover, under the background assumption of Ahlfors regularity, uniform rectifiability is implied[1] by the simultaneous L^2 boundedness of all truncated integral convolution type operators T_ε on Σ (cf. (3.76)) uniformly with respect to the truncation $\varepsilon > 0$, whose kernels are smooth, odd, and positive homogeneous of degree $1-n$ in $\mathbb{R}^n \setminus \{0\}$. In light of (3.80), the above discussion highlights the optimality of demanding that Σ is a UR set in the context of Proposition 3.4. One of the early works on the higher-dimensional theory of singular integral operators in rough geometric settings is [23]; see also the survey paper [97] for an informative account of the development of this topic.

Results like Proposition 3.4 have been recently established in [113, §2.3-§2.5]. Here we present an alternative approach that makes essential use of the Fefferman–Stein sharp maximal function, considered in the setting of spaces of homogeneous type (for the Euclidean context, see [69, p. 52], [52, Theorem 3.6, p. 161]).

Proof of Proposition 3.4 To set the stage, recall the Fefferman–Stein sharp maximal operator $M^\#$ on Σ, acting on each function $f \in L^1_{\rm loc}(\Sigma, \sigma)$ according to

$$M^\# f(x) := \sup_{\Delta \ni x} ⨍_\Delta \Big| f - ⨍_\Delta f \, {\rm d}\sigma \Big| \, {\rm d}\sigma, \qquad \forall\, x \in \Sigma, \tag{3.87}$$

where the supremum is taken over all surface balls $\Delta \subseteq \Sigma$ containing the point $x \in \Sigma$. Clearly, for each $f \in L^1_{\rm loc}(\Sigma, \sigma)$ and each $x \in \Sigma$, we have

$$\sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} ⨍_\Delta |f - a| \, {\rm d}\sigma \leq M^\# f(x) \leq 2 \sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} ⨍_\Delta |f - a| \, {\rm d}\sigma. \tag{3.88}$$

Also, given $\alpha \in (0, 1)$, for each $f \in L^1_{\rm loc}(\Sigma, \sigma)$, set

$$M^\#_\alpha f(x) := M^\#(|f|^\alpha)(x)^{1/\alpha} \quad \text{for all } \ x \in \Sigma. \tag{3.89}$$

[1] In [40], the authors have dealt with the class of truncated singular integral operators associated with kernels in $\mathbb{R}^n \setminus \{0\}$ which are smooth, odd, and satisfy $\sup_{x \in \mathbb{R}^n \setminus \{0\}} \Big[|x|^{(n-1)+|\alpha|} \big|(\partial^\alpha k)(x)\big| \Big] < +\infty$ for all $\alpha \in \mathbb{N}_0^n$. In [118], it was shown that the truncated Riesz transforms on Σ alone will do.

Since having $0 < \alpha < 1$ ensures that $|X^\alpha - Y^\alpha| \leq |X - Y|^\alpha$ for all $X, Y \in [0, \infty)$, from (3.89) and the last inequality in (3.88), one may readily check that

$$M^{\#}_{\alpha} f(x) \leq 2^{1/\alpha} \sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} \left(⨍_{\Delta} |f - a|^\alpha \, d\sigma \right)^{1/\alpha} \tag{3.90}$$

for each $f \in L^1_{\rm loc}(\Sigma, \sigma)$ and each $x \in \Sigma$. Finally, recall from (2.522) the (non-centered) Hardy–Littlewood maximal operator $\mathcal{M}$ on Σ.

From (3.76)–(3.78), it is clear that the maximal operator T_* and the principal-value singular integral operator T depend in a homogeneous fashion on the kernel function k. In view of this observation, by working with k/K (in the case when k is not identically zero) where $K := \sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k|$, there is no loss of generality in assuming that

$$\sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k| = 1. \tag{3.91}$$

The fact that for each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, the limit defining $Tf(x)$ in (3.78) exists at σ-a.e. $x \in \Sigma$ has been proved in [113, §2.3]. To proceed, denote by $L^\infty_{\rm comp}(\Sigma, \sigma)$ the subspace of $L^\infty(\Sigma, \sigma)$ consisting of functions with compact support. Also, fix a power $\alpha \in (0, 1)$. We will first show that there exists a constant $C = C(\Sigma, n, \alpha) \in (0, \infty)$ such that

$$\begin{gathered} M^{\#}_{\alpha}(Tf)(x) \leq C \cdot \mathcal{M} f(x) \\ \text{for all } f \in L^\infty_{\rm comp}(\Sigma, \sigma) \text{ and } x \in \Sigma. \end{gathered} \tag{3.92}$$

To this end, fix a function $f \in L^\infty_{\rm comp}(\Sigma, \sigma)$ along with a point $x \in \Sigma$, and consider a surface ball $\Delta = \Delta(x_0, r_0)$, with center at $x_0 \in \Sigma$ and radius $r_0 > 0$, containing the point x. Decompose $f = f_1 + f_2$, where $f_1 := f \mathbf{1}_{2\Delta}$ and $f_2 := f \mathbf{1}_{\Sigma \setminus 2\Delta}$. Then $\big|Tf_2(x_0)\big| < +\infty$ and we abbreviate $a := Tf_2(x_0) \in \mathbb{C}$. Note that

$$⨍_{\Delta} |Tf - a|^\alpha \, d\sigma \leq ⨍_{\Delta} |Tf_1|^\alpha \, d\sigma + ⨍_{\Delta} |Tf_2 - a|^\alpha \, d\sigma. \tag{3.93}$$

For the first term in the right-hand side of (3.93), using Kolmogorov's inequality, the fact that T is bounded from $L^1(\Sigma, \sigma)$ to $L^{1,\infty}(\Sigma, \sigma)$ (cf. [113, §2.3], [61, Proposition 3.19]), and the fact that Σ is an Ahlfors regular set to write

$$\begin{aligned} ⨍_{\Delta} |Tf_1|^\alpha \, d\sigma &\leq \frac{C_\alpha}{\sigma(\Delta)^\alpha} \|Tf_1\|^\alpha_{L^{1,\infty}(\Sigma,\sigma)} \leq \frac{C_\alpha}{\sigma(\Delta)^\alpha} \|f_1\|^\alpha_{L^1(\Sigma,\sigma)} \\ &\leq C_\alpha \left(⨍_{2\Delta} |f| \, d\sigma \right)^\alpha \leq C_\alpha \cdot \mathcal{M} f(x)^\alpha. \end{aligned} \tag{3.94}$$

For the second term in the right-hand side of (3.93), note that the properties of k and (3.91) entail

$$|(\nabla k)(z)| = \Big|(\nabla k)\Big(\frac{z}{|z|}|z|\Big)\Big| \leq |z|^{-n} \sup_{|\omega|=1} |(\nabla k)(\omega)| = C_n|z|^{-n}, \tag{3.95}$$

for each $z \in \mathbb{R}^n \setminus \{0\}$, where $C_n \in (0,\infty)$ is a purely dimensional constant. On account of (3.95) and the Mean Value Theorem, we see that there exists a dimensional constant $C_n \in (0,\infty)$ with the property that for each $y \in \Delta$ and $z \in \Sigma \setminus 2\Delta$ we have

$$|k(y-z) - k(x_0 - z)| \leq C_n \frac{|y - x_0|}{|x_0 - z|^n} \leq \frac{C_n r_0}{|x_0 - z|^n}. \tag{3.96}$$

Using this, for every $y \in \Delta$, we may write

$$\begin{aligned} |Tf_2(y) - a| &= |Tf_2(y) - Tf_2(x_0)| \\ &\leq \int_{\Sigma\setminus 2\Delta} |k(y-z) - k(x_0-z)||f(z)|\,\mathrm{d}\sigma(z) \\ &\leq C r_0 \sum_{j=1}^{\infty} \int_{2^j r_0 \leq |x_0 - z| < 2^{j+1} r_0} \frac{|f(z)|}{|x_0 - z|^n}\,\mathrm{d}\sigma(z) \\ &\leq C \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta} |f(z)|\,\mathrm{d}\sigma(z) \\ &\leq C \cdot \mathcal{M}f(x), \end{aligned} \tag{3.97}$$

where $C \in (0,\infty)$ depends only on dimension and the Ahlfors regularity constant of Σ. At this stage, the claim in (3.92) follows by combining (3.90), (3.93), (3.94), and (3.97).

We shall now analyze two cases, depending on whether Σ is bounded or not. Consider first the case when Σ is unbounded. In such a setting, the A_∞-weighted version of the Fefferman–Stein inequality for spaces of homogeneous type (cf., e.g., [8, Sections 3.2 and 5]) gives that for every $q \in (0,\infty)$ there exists some constant $C_w \in (0,\infty)$, which depends on the weight $w \in A_p(\Sigma,\sigma) \subseteq A_\infty(\Sigma,\sigma)$ only through its characteristic $[w]_{A_p}$ (indeed, it can be expressed as an increasing function of $[w]_{A_p}$), such that

$$\begin{gathered} \|\mathcal{M}g\|_{L^q(\Sigma,w)} \leq C_w \left\|M^{\#}g\right\|_{L^q(\Sigma,w)} \quad \text{for each} \\ g \in L^1_{\mathrm{loc}}(\Sigma,\sigma) \ \text{ such that } \ \mathcal{M}g \in L^q(\Sigma,w). \end{gathered} \tag{3.98}$$

To proceed, fix $\alpha \in (0,1)$ and $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$. Let us momentarily work under the additional assumption that the weight w belongs to $L^\infty(\Sigma,\sigma)$. This permits us to estimate

$$\begin{aligned}\big\|\mathcal{M}(|Tf|^\alpha)\big\|_{L^{p/\alpha}(\Sigma,w)} &\leq \|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)}\,\big\|\mathcal{M}(|Tf|^\alpha)\big\|_{L^{p/\alpha}(\Sigma,\sigma)}\\ &\leq C\,\|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)}\,\|Tf\|^\alpha_{L^p(\Sigma,\sigma)}\\ &\leq C\,\|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)}\,\|f\|^\alpha_{L^p(\Sigma,\sigma)} < +\infty,\end{aligned} \tag{3.99}$$

where we have used the boundedness of $\mathcal{M}$ on $L^{p/\alpha}(\Sigma,\sigma)$ and the boundedness of T on $L^p(\Sigma,\sigma)$ (cf. [61, Proposition 3.18]). This allows us to use (3.98) (with $g := |Tf|^\alpha$ and $q := p/\alpha$) to obtain, for some constant $C_w \in (0,\infty)$ (again, depending in an increasing fashion on $[w]_{A_p}$),

$$\begin{aligned}\|Tf\|_{L^p(\Sigma,w)} &\leq \Big\|\mathcal{M}(|Tf|^\alpha)^{1/\alpha}\Big\|_{L^p(\Sigma,w)} = \big\|\mathcal{M}(|Tf|^\alpha)\big\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)}\\ &\leq C_w\,\Big\|M^\#(|Tf|^\alpha)\Big\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} = C_w\,\Big\|M^\#_\alpha(Tf)\Big\|_{L^p(\Sigma,w)}\\ &\leq C_w\,\|\mathcal{M}f\|_{L^p(\Sigma,w)} \leq C_w\,\|f\|_{L^p(\Sigma,w)}\,,\end{aligned} \tag{3.100}$$

where the first inequality follows from Lebesgue's Differentiation Theorem (cf. [7]), the last equality is a consequence of (3.89), the penultimate inequality comes from (3.92), and the last inequality is implied by the boundedness of the Hardy–Littlewood operator $\mathcal{M}$ on $L^p(\Sigma,w)$.

To remove the restriction $w \in L^\infty(\Sigma,\sigma)$, we proceed as follows. For each integer $j \in \mathbb{N}$, let $w_j := \min\{w, j\} \in L^\infty(\Sigma,\sigma)$. Moreover, as in [57, Ex. 9.1.9], we have

$$[w_j]_{A_p} \leq C_p(1+[w]_{A_p}) \tag{3.101}$$

for some $C_p \in (0,\infty)$ independent of $j \in \mathbb{N}$. As such, we may invoke (3.100) written for each w_j (which now involves a constant whose dependence of w_j may be expressed in terms of a non-decreasing function acting on $[w_j]_{A_p}$) to conclude that

$$\|Tf\|_{L^p(\Sigma,w_j)} \leq C\,\|f\|_{L^p(\Sigma,w_j)} \leq C\,\|f\|_{L^p(\Sigma,w)}\,, \tag{3.102}$$

for some constant $C \in (0,\infty)$ independent of $j \in \mathbb{N}$. Upon letting $j \to \infty$ and relying on Lebesgue's Monotone Convergence Theorem, we arrive at the conclusion that $\|Tf\|_{L^p(\Sigma,w)} \leq C\,\|f\|_{L^p(\Sigma,w)}$ for every $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$. Given that $L^\infty_{\rm comp}(\Sigma,\sigma)$ is dense in $L^p(\Sigma,w)$, this ultimately establishes the boundedness of the operator T in the context of (3.81) when Σ is unbounded.

Let us now consider the case when Σ is bounded. In this case, compared to (3.98), the A_∞-weighted version of the Fefferman–Stein inequality includes an extra term; namely, it now reads (cf. [8, Sections 3.2 and 5])

$$\|\mathcal{M}g\|_{L^q(\Sigma,w)} \leq C_w \left\| M^\# g \right\|_{L^q(\Sigma,w)} + C\sigma(\Sigma)^{-1}\Big(\int_\Sigma w \,\mathrm{d}\sigma\Big)^{1/q} \|g\|_{L^1(\Sigma,\sigma)} \tag{3.103}$$

$$\text{for all } g \in L^1(\Sigma,\sigma) \text{ with } \mathcal{M}g \in L^q(\Sigma,w),$$

where $C_w \in (0,\infty)$ is as before and $C \in (0,\infty)$ is a purely geometric constant. Fix $\alpha \in (0,1)$ and $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$. Assume first that $w \in L^\infty(\Sigma,\sigma)$ and note that (3.99) holds in the same way. This permits us to invoke (3.103) (with $g := |Tf|^\alpha$ and $q := p/\alpha$), so in place of (3.100), we now get

$$\begin{aligned}
\|Tf\|_{L^p(\Sigma,w)} &\leq \big\|\mathcal{M}(|Tf|^\alpha)\big\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)}\\
&\leq C_w \big\|M^\#(|Tf|^\alpha)\big\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} + C\sigma(\Sigma)^{-1/\alpha}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/p}\big\||Tf|^\alpha\big\|^{1/\alpha}_{L^1(\Sigma,\sigma)}\\
&\leq C_w \|f\|_{L^p(\Sigma,w)} + C\sigma(\Sigma)^{-1/\alpha}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/p}\|Tf\|_{L^\alpha(\Sigma,\sigma)}\,,
\end{aligned} \tag{3.104}$$

where the first and last estimates follow as before. Here, the constant $C_w \in (0,\infty)$ depends on w only through its characteristic $[w]_{A_p}$ (again, this may be expressed as an increasing function of $[w]_{A_p}$), while $C \in (0,\infty)$ depends just on p, α, n, and the Ahlfors regularity constant of Σ.

It remains to estimate $\|Tf\|_{L^\alpha(\Sigma,\sigma)}$ in a satisfactory manner. Using Kolmogorov's inequality and the fact that T is bounded from $L^1(\Sigma,\sigma)$ into $L^{1,\infty}(\Sigma,\sigma)$ (cf. [113, §2.3], [61, Proposition 3.19]) and Hölder's inequality, we obtain

$$\begin{aligned}
\|Tf\|_{L^\alpha(\Sigma,\sigma)} &\leq (1-\alpha)^{-1/\alpha}\sigma(\Sigma)^{(1-\alpha)/\alpha}\,\|Tf\|_{L^{1,\infty}(\Sigma,\sigma)}\\
&\leq C\sigma(\Sigma)^{(1-\alpha)/\alpha}\,\|f\|_{L^1(\Sigma,\sigma)}\\
&\leq C\sigma(\Sigma)^{(1-\alpha)/\alpha}\Big(\int_\Sigma w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}\|f\|_{L^p(\Sigma,w)}\,.
\end{aligned} \tag{3.105}$$

Let us record our progress. The argument so far proves that, if Σ is bounded, then for each $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$ we have

$$\|Tf\|_{L^p(\Sigma,w)} \leq \Big(C_w + C\,\sigma(\Sigma)^{-1}\Big(\int_\Sigma w\,\mathrm{d}\sigma\Big)^{1/p}\Big(\int_\Sigma w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'}\Big)\,\|f\|_{L^p(\Sigma,w)}$$

$$\leq \big(C_w + C\,[w]_{A_p}^{1/p}\big)\,\|f\|_{L^p(\Sigma,w)}\,, \tag{3.106}$$

where $C_w \in (0,\infty)$ is as above. As before, to remove the restriction $w \in L^\infty(\Sigma,\sigma)$, we work with $w_j := \min\{w, j\}$ for $j \in \mathbb{N}$. Thanks to (3.101) the constant in the right-hand side of (3.106) may be controlled uniformly in j. After passing to limit $j \to \infty$ and once again relying on the density $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$ into $L^p(\Sigma,w)$, we eventually conclude that the operator T is bounded in the context of (3.81) in this case as well. Moreover,

$$\|T\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C, \tag{3.107}$$

where $C \in (0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the UR constants of Σ. This finishes the proof of (3.81).

Next, recall Cotlar's inequality, to the effect that there exists some $C \in (0,\infty)$ which depends only on n, and the Ahlfors regularity constant of Σ, with the property that for every function $f \in L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$, we have

$$(T_* f)(x) \leq C\cdot\mathcal{M}(Tf)(x) + C\cdot\mathcal{M}f(x) \quad \text{for each} \quad x \in \Sigma. \tag{3.108}$$

Then (3.79) follows from (3.81), (3.108), the boundedness of the Hardy–Littlewood operator $\mathcal{M}$ on $L^p(\Sigma,w)$, and a density argument. Going further, (3.83) may be justified by first establishing a similar claim for the truncated operators (3.76) using Fubini's theorem and then invoking Lebesgue's Dominated Convergence Theorem (whose applicability is guaranteed by (3.79)) to pass to limit as $\varepsilon \to 0^+$.

Consider next the claims made in the last part of the statement. It is apparent from (3.84) that the boundary-to-domain operator $\mathcal{T}$ depends in a homogeneous fashion on the kernel function k. Much as before, this permits us to work under the additional assumption that (3.91) holds. Granted this, the estimate claimed in (3.85) is a direct consequence of inequality (3.79) and the formula (cf. [61, eq. (3.2.22)])

$$\mathcal{N}_\kappa\big(\mathcal{T}f\big)(x) \leq C\cdot T_* f(x) + C\cdot\mathcal{M}f(x) \quad \text{for each} \quad x \in \Sigma, \tag{3.109}$$

where $C \in (0,\infty)$ depends only on n and the Ahlfors regularity constant of Σ and where the maximal operator T_* and the Hardy–Littlewood maximal function $\mathcal{M}$ are now associated with the UR set $\Sigma := \partial\Omega$.

That the jump-formula (3.86) holds for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ has been established in [113, §2.5]. With this in hand, the very last claim in the statement of Proposition 3.4 is implied by (2.575). □

The stage has been set for considering the action of the boundary layer potentials associated with a given weakly elliptic system L and a given UR domain Ω in $\mathbb{R}^n$ as in (3.22)–(3.25) and (3.38) on Muckenhoupt weighted Lebesgue and Sobolev spaces on $\partial\Omega$. To state our main result in this regard, given any two Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, denote

$$\mathrm{Bd}\big(X \to Y\big) := \big\{T : X \to Y : \ T \text{ linear and bounded}\big\}, \tag{3.110}$$

and equip it with the standard operator norm $\mathrm{Bd}\big(X \to Y\big) \ni T \mapsto \|T\|_{X \to Y}$ (cf. (4.1)). Finally, corresponding to the case when $Y = X$, we agree to abbreviate

$$\mathrm{Bd}(X) := \mathrm{Bd}\big(X \to X\big). \tag{3.111}$$

Proposition 3.5 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a UR domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Pick $A \in \mathfrak{A}_L$ and consider the boundary layer potential operators $\mathcal{D}_A$, K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in* (3.22)*,* (3.24)*, and* (3.25)*. Also, recall the modified single layer potential operator $\mathscr{S}_{mod}$ associated with Ω and L as in* (3.38)*. Finally, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and an aperture parameter $\kappa > 0$.*

1. *The following operators are well defined, sub-linear, and bounded:*

$$\big[L^p(\partial\Omega, w)\big]^M \ni f \longmapsto \mathcal{N}_\kappa\big(\mathcal{D}_A f\big) \in L^p(\partial\Omega, w), \tag{3.112}$$

$$\big[L^p_1(\partial\Omega, w)\big]^M \ni f \longmapsto \mathcal{N}_\kappa\,(\nabla\mathcal{D}_A f) \in L^p(\partial\Omega, w). \tag{3.113}$$

Also,

$$\begin{aligned}&\textit{for each } f \in \big[L^p_1(\partial\Omega, w)\big]^M \textit{ the nontangential trace}\\ &\big(\nabla\mathcal{D}_A f\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{ exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega.\end{aligned} \tag{3.114}$$

As a consequence of (3.114)*,* (3.33)*,* (3.66)*,* (2.586)*, and Proposition 3.4,*

$$\begin{aligned}&\textit{the map } \big[L^p_1(\partial\Omega, w)\big]^M \ni f \longmapsto \partial^A_\nu\big(\mathcal{D}_A f\big) \in \big[L^p(\partial\Omega, w)\big]^M \textit{ is}\\ &\textit{well defined, linear, and bounded, and there exists } C \in (0, \infty) \textit{ so}\\ &\textit{that } \big\|\partial^A_\nu\big(\mathcal{D}_A f\big)\big\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \textit{ for each}\\ &f \textit{ in the Muckenhoupt weighted Sobolev space } \big[L^p_1(\partial\Omega, w)\big]^M.\end{aligned} \tag{3.115}$$

2. *For every $f \in \big[L^p(\partial\Omega, w)\big]^M$, the limits in* (3.24) *and* (3.25) *exist at σ-a.e. point on $\partial\Omega$. Moreover, the operators K_A and $K_A^{\#}$ are well defined, linear, and bounded in the following contexts:*

$$K_A : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M, \tag{3.116}$$

$$K_A : \big[L^p_1(\partial\Omega, w)\big]^M \longrightarrow \big[L^p_1(\partial\Omega, w)\big]^M, \tag{3.117}$$

$$K_A^\# : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M. \tag{3.118}$$

Moreover, under the canonical integral pairing $(f, g) \mapsto \int_{\partial\Omega}\langle f, g\rangle\, d\sigma$, *it follows that*

$$\begin{array}{l}\textit{the (real) transpose of the operator } K_A \textit{ acting on the} \\ \textit{space } \big[L^p(\partial\Omega, w)\big]^M \textit{ is the operator } K_A^\# \textit{ acting on the} \\ \textit{space } \big[L^{p'}(\partial\Omega, w')\big]^M \textit{ where } p' \in (1,\infty) \textit{ is the Hölder} \\ \textit{conjugate exponent of } p \textit{ and } w' := w^{1-p'} \in A_{p'}(\Sigma,\sigma).\end{array} \tag{3.119}$$

Additionally, the operators $K_A, K_A^\#$ *in* (3.116)–(3.118) *depend continuously on the underlying coefficient tensor* A*. More specifically, with the piece of notation introduced in* (3.13)*, the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega, w)\big]^M\Big), \tag{3.120}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[L^p_1(\partial\Omega, w)\big]^M\Big), \tag{3.121}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A^\# \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega, w)\big]^M\Big). \tag{3.122}$$

Furthermore, the nontangential boundary trace of the boundary-to-domain double layer is related to the boundary-to-boundary double layer via a jump-formula, to the effect that for every $f \in \big[L^p(\partial\Omega, w)\big]^M$ *and* σ*-a.e. in* $\partial\Omega$*, one has*

$$\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_A\big)f, \tag{3.123}$$

where I *is the identity operator.*

3. *For each* $f \in \big[L^p(\partial\Omega, w)\big]^M$*, one has*

$$\mathscr{S}_{mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{mod} f\big) = 0 \ \ \textit{in} \ \ \Omega. \tag{3.124}$$

In addition, the trace

$$\big(\nabla \mathscr{S}_{mod} f\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \ \textit{exists (in } \mathbb{C}^{M\cdot n}\textit{) at } \sigma\textit{-a.e. point on } \ \partial\Omega, \tag{3.125}$$

and the conormal derivative of the modified boundary-to-domain single layer satisfies the following jump-formula:

$$\partial_\nu^A \mathscr{S}_{mod} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \quad \text{at } \sigma\text{-a.e. point in } \partial\Omega, \tag{3.126}$$

where I is the identity, and $K^\#_{A^\top}$ is the operator associated as in (3.25) with the UR domain Ω and the transpose coefficient tensor $A^\top$. Also, there exists some constant $C = C(\Omega, p, w, L, \kappa) \in (0, \infty)$ independent of f such that

$$\big\| \mathcal{N}_\kappa(\nabla \mathscr{S}_{mod} f)\big\|_{L^p(\partial\Omega, w)} \leq C\|f\|_{[L^p(\partial\Omega,w)]^M}. \tag{3.127}$$

4. *For each function $f \in \big[L^p(\partial\Omega, w)\big]^M$ and σ-a.e. point $x \in \partial\Omega$, one has*

$$\partial_{\tau_{jk}}\big(S_{mod} f\big)(x) = \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k E)(x-y) - \nu_k(x)(\partial_j E)(x-y)\big\} f(y)\, d\sigma(y) \tag{3.128}$$

for each $j, k \in \{1, \dots, n\}$, and

$$\begin{aligned} &\big(\tfrac{1}{2}I + K^\#_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f\Big)(x) \\ &= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a_{ij}^{\mu\gamma} a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x-y) \partial_{\tau_{js}}\big(S_{mod} f\big)_\alpha(y)\, d\sigma(y)\Bigg)_{1\leq\mu\leq M}. \end{aligned} \tag{3.129}$$

5. *For each $f \in \big[L^p_1(\partial\Omega, w)\big]^M$, there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$, one has*

$$\big(\tfrac{1}{2}I + K_A\big)\Big(\big(-\tfrac{1}{2}I + K_A\big) f\Big) = S_{mod}\Big(\partial_\nu^A\big(\mathcal{D}_A f\big)\Big) + c_f. \tag{3.130}$$

6. *The operator*

$$S_{mod} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \tag{3.131}$$

is well defined, linear, and bounded, when the target space is endowed with the semi-norm introduced in (2.599). As a consequence, if $\big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ denotes the M-th power of the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, equipped with the semi-norm (2.601), then the operator

$$\begin{aligned} &\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M \quad \text{defined as} \\ &\big[S_{mod}\big] f := \big[S_{mod} f\big] \in \big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M, \quad \forall\, f \in \big[L^p(\partial\Omega, w)\big]^M \end{aligned} \tag{3.132}$$

is well defined, linear, and bounded.

Proof With the exception of (3.120)–(3.122) and (3.128)–(3.131), all claims may be justified based on (3.22)–(3.40), Lemma 2.15, Proposition 3.1, Proposition 2.22, Proposition 3.4, and Theorem 3.1. The continuity properties of the operator-valued maps in (3.120)–(3.122), as well as formulas (3.128), (3.129), (3.130) have been proved in [114, §1.5]. Finally, (3.131) is a consequence of (2.598)–(2.599), (3.44)–(3.45), (3.128), and (3.81) in Proposition 3.4. □

Our next theorem contains fundamental properties of modified double layer potential operators acting on homogeneous Muckenhoupt weighted Sobolev spaces, considered on boundaries of uniformly rectifiable domains.

Theorem 3.5 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a UR domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In addition, for some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that $L := L_A$ as in (3.7) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Also, let $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ be the matrix-valued fundamental solution associated with L as in Theorem 3.1. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,mod}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (3.49). Finally, fix some aperture parameter $\kappa \in (0, \infty)$ along with an integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$.*

Then there exists some constant $C = C(\Omega, n, p, \kappa) \in (0, \infty)$ with the property that for each function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ it follows that

$$
\begin{gathered}
\mathcal{D}_{A,mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,mod} f\big) = 0 \ \text{ in } \ \Omega, \\
\big(\mathcal{D}_{A,mod} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \ \big(\nabla\mathcal{D}_{A,mod} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \textit{exist } \sigma\textit{-a.e. on } \ \partial\Omega, \\
\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,mod} f\big) \ \textit{belongs to } \ L^p(\partial\Omega, w) \ \textit{ and} \\
\big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,mod} f\big)\big\|_{L^p(\partial\Omega,w)} \leq C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}.
\end{gathered}
\tag{3.133}
$$

In fact, for each function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, one has

$$
(\mathcal{D}_{A,mod} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_{A,mod}\big) f \ \ \textit{at } \sigma\textit{-a.e. point on } \ \partial\Omega, \tag{3.134}
$$

where I is the identity operator on $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, and $K_{A,mod}$ is the modified boundary-to-boundary double layer potential operator from (3.50).

Moreover, given any function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the homogeneous boundary Sobolev space $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, at σ-a.e. point $x \in \partial\Omega$, one has

$$\big(\partial_\nu^A(\mathcal{D}_{A,mod} f)\big)(x) \tag{3.135}$$

$$= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a_{ij}^{\mu\gamma} a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x-y) \big(\partial_{\tau_{js}} f_\alpha\big)(y)\, \mathrm{d}\sigma(y) \Bigg)_{1\le\mu\le M},$$

where the conormal derivative is considered as in (3.66).

Furthermore, the operator

$$\begin{aligned} &\partial_\nu^A \mathcal{D}_{A,mod} : \big[\dot{L}_1^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M \ \textit{ defined as} \\ &\big(\partial_\nu^A \mathcal{D}_{A,mod}\big) f := \partial_\nu^A(\mathcal{D}_{A,mod} f) \ \textit{ for each } \ f \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M \end{aligned} \tag{3.136}$$

is well defined, linear, bounded (when the domain space is equipped with the semi-norm (2.599)*), and*

$$\begin{aligned} &\partial_\nu^A \mathcal{D}_{A,mod} \ \textit{ annihilates constant} \\ &(\mathbb{C}^M\textit{-valued) functions on } \ \partial\Omega. \end{aligned} \tag{3.137}$$

As a consequence of (3.136) *and* (3.137)*, the following operator is well defined and linear:*

$$\begin{aligned} &\big[\partial_\nu^A \mathcal{D}_{A,mod}\big] : \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M \ \textit{ defined as} \\ &\big[\partial_\nu^A \mathcal{D}_{A,mod}\big][f] := \partial_\nu^A(\mathcal{D}_{A,mod} f) \ \textit{ for each } \ f \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M. \end{aligned} \tag{3.138}$$

Finally, if $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator* (3.138) *is also bounded, when the quotient space is equipped with the norm* (2.601)*. Moreover, in this setting the operator* $\big[\partial_\nu^A \mathcal{D}_{A,mod}\big]$ *in* (3.138) *depends continuously on the underlying coefficient tensor A, in the sense that (with the piece of notation introduced in* (3.13)*) the following operator-valued assignment is continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto \big[\partial_\nu^A \mathcal{D}_{A,mod}\big] \in \mathrm{Bd}\Big(\big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \to \big[L^p(\partial\Omega, w)\big]^M\Big). \tag{3.139}$$

Proof For each function $f \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, the jump-formula (3.134) is seen from (3.61) (keeping in mind (2.598)). The claims in (3.133) are consequences of (2.598), (3.51), (3.61), (3.58), (3.57), (3.85), and Theorem 3.1. In particular, given an arbitrary function $f = (f_\alpha)_{1\le\alpha\le M} \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, the conormal derivative $\partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f)$ may be meaningfully defined, as in (3.66). Specifically, at σ-a.e. point $x \in \partial\Omega$, we have

$$\big(\partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f)\big)(x) = \Big(\nu_i(x)\big(a_{ij}^{\mu\gamma}\partial_j(\mathcal{D}_{A,\mathrm{mod}} f)_\gamma\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x)\Big)_{1\leq\mu\leq M} \tag{3.140}$$

$$= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x)a_{ij}^{\mu\gamma}a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}} f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\mu\leq M},$$

where the first equality comes from (3.66) and the second equality is a consequence of (3.57) and the jump-formula (3.86). Having established (3.135), the claims made in relation to (3.136) follow with the help of Proposition 3.4 and Theorem 3.1. Note that (3.137) is also a consequence of (3.135). Next, the claims pertaining to (3.138) are consequences of what we have proved so far and (3.137). Finally, the continuity of the operator-valued assignment (3.139) follows from (3.135), Theorem 3.1, and work in [114, §1.8]. □

The modified boundary-to-boundary double layer potential operator on homogeneous Muckenhoupt weighted Sobolev spaces is studied next.

Theorem 3.6 *Let $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) be an NTA domain such that $\partial\Omega$ is an Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, with $M\in\mathbb{N}$, let $A=\big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that $L:=L_A$ as in (3.7) is a weakly elliptic $M\times M$ system in $\mathbb{R}^n$. In this context, consider the modified boundary-to-boundary double layer potential operator $K_{A,mod}$ from (3.50). Finally, select an integrability exponent $p\in(1,\infty)$ along with some Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$.*

Then the operator

$$K_{A,mod}:\big[\dot{L}_1^p(\partial\Omega,w)\big]^M\longrightarrow\big[\dot{L}_1^p(\partial\Omega,w)\big]^M \tag{3.141}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (2.599).

As a consequence of (3.141) and (3.62), the following is a well-defined and linear operator:

$$\big[K_{A,mod}\big]:\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\longrightarrow\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M \quad \textit{defined as}$$
$$\big[K_{A,mod}\big][f]:=\big[K_{A,mod}f\big]\in\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M,\quad \forall\, f\in\big[\dot{L}_1^p(\partial\Omega,w)\big]^M \tag{3.142}$$

Finally, if $\Omega\subseteq\mathbb{R}^n$ is actually a two-sided NTA domain whose boundary is an unbounded Ahlfors regular set, then the operator (3.142) is also bounded when all quotient spaces are endowed with the norm introduced in (2.601). Moreover, in this setting, the operator $\big[K_{A,mod}\big]$ in (3.142) depends continuously on the underlying coefficient tensor A, in the sense that (with the piece of notation introduced in (3.13)) the following operator-valued assignment is continuous:

$$\mathfrak{A}_{\mathrm{WE}}\ni A\longmapsto\big[K_{A,mod}\big]\in\mathrm{Bd}\Big(\big[\dot{L}_1^p(\partial\Omega,w)/\sim\big]^M\Big). \tag{3.143}$$

Proof The present hypotheses guarantee (cf. (2.48)) that Ω is a UR domain. Pick an integrability exponent $p \in (1, \infty)$ and fix an aperture parameter $\kappa \in (0, \infty)$. Next, consider a function $f = (f_\alpha)_{1 \le \alpha \le M} \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ and define $u := \mathcal{D}_{A,\text{mod}} f$ in Ω. Then $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ (cf. (3.51)), and the jump-formula (3.134) gives

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{A,\text{mod}}\big) f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.144}$$

From (3.133), we also know that

$$\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w) \quad \text{and} \quad \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,w)} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \tag{3.145}$$

for some constant $C \in (0, \infty)$ independent of f. Granted these properties, we may invoke Proposition 2.24 to conclude that

$$\begin{aligned} &u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ belongs to the space } \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \\ &\text{and } \Big\|u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}. \end{aligned} \tag{3.146}$$

Collectively, (3.144) and (3.146) then prove that

$$\begin{aligned} &K_{A,\text{mod}} f \ \text{ belongs to the space } \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \\ &\text{and } \big\|K_{A,\text{mod}} f\big\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}, \end{aligned} \tag{3.147}$$

from which the claims pertaining to (3.141) follow. Next, the claims regarding the operator (3.142) are readily seen from what we have just proved and definitions. Finally, the fact that the operator-valued assignment (3.143) is continuous is seen from (2.598), (2.601), (3.65), (2.576), (3.35), (3.120), Theorem 3.1, and work in [114, §1.8]. □

We shall now use Corollary 3.1 to derive some useful operator identities, involving boundary layer potentials, of the sort described below.

Theorem 3.7 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \ge 2$) is an NTA domain whose boundary is an unbounded Ahlfors regular set. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Next, for some $M \in \mathbb{N}$, let $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ be a complex coefficient tensor with the property that $L := L_A$ as in (3.7) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Having fixed some integrability exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, recall the operators S_{mod} from (3.131), $\partial^A_\nu \mathcal{D}_{A,mod}$ from (3.136), and $K_{A,mod}$ from (3.141). Finally, let $K^{\#}_{A^\top}$ be the operator associated with the coefficient tensor $A^\top$ and the set Ω as in (3.25). Then the following statements are true.*

(1) For each $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, *there exists some* $c_f \in \mathbb{C}^M$ *with the property that at* σ*-a.e. point on* $\partial\Omega$, *one has*

$$\big(\tfrac{1}{2}I + K_{A,mod}\big)\Big(\big(-\tfrac{1}{2}I + K_{A,mod}\big)f\Big) = S_{mod}\Big(\big(\partial^A_\nu \mathcal{D}_{A,mod}\big)f\Big) + c_f. \tag{3.148}$$

In particular,

$$\begin{gathered}\big(\tfrac{1}{2}I + \big[K_{A,mod}\big]\big)\big(-\tfrac{1}{2}I + \big[K_{A,mod}\big]\big) = \big[S_{mod}\big]\big[\partial^A_\nu \mathcal{D}_{A,mod}\big]\\ \text{as operators acting from } \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M.\end{gathered} \tag{3.149}$$

(2) For each function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, *one has*

$$\big(\partial^A_\nu \mathcal{D}_{A,mod}\big)\big(K_{A,mod} f\big) = K^{\#}_{A^\top}\big(\partial^A_\nu \mathcal{D}_{A,mod}\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.150}$$

(3) For each $f \in \big[L^p(\partial\Omega, w)\big]^M$, *there exists some* $c_f \in \mathbb{C}^M$ *with the property that*

$$S_{mod}\big(K^{\#}_{A^\top} f\big) = K_{A,mod}\big(S_{mod} f\big) + c_f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.151}$$

In particular,

$$\begin{gathered}\big[S_{mod}\big]K^{\#}_{A^\top} = \big[K_{A,mod}\big]\big[S_{mod}\big]\\ \text{as operators acting from } \big[L^p(\partial\Omega, w)\big]^M.\end{gathered} \tag{3.152}$$

(4) For each $f \in \big[L^p(\partial\Omega, w)\big]^M$, *at* σ*-a.e. point on* $\partial\Omega$, *one has*

$$\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) = \big(\partial^A_\nu \mathcal{D}_{A,mod}\big)\big(S_{mod} f\big). \tag{3.153}$$

Proof The present hypotheses imply that Ω is a connected UR domain (see (2.48)). Select an aperture parameter $\kappa \in (0,\infty)$. To justify the claims made in items *(1)*–*(2)*, pick an arbitrary function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ and define $u := \mathcal{D}_{A,\text{mod}} f$ in Ω. Then, from (3.133) and (3.134), we know that

$$\begin{gathered}u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w),\\ \text{the boundary traces } u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}, \ (\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial\Omega,\\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f \text{ and } \partial^A_\nu u = \big(\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big)f.\end{gathered} \tag{3.154}$$

Then Corollary 3.1 applies and gives that $\partial_\nu^A u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$, the trace $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, and there exists some $c_f \in \mathbb{C}^M$ with the property that

$$
\begin{aligned}
u &= \mathcal{D}_{A,\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathscr{S}_{\text{mod}}\big(\partial_\nu^A u\big) + c_u \\
&= \mathcal{D}_{A,\text{mod}}\Big(\big(\tfrac{1}{2}I + K_{\text{mod}}\big)f\Big) - \mathscr{S}_{\text{mod}}\Big(\big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big)f\Big) + c_f \quad \text{in} \quad \Omega.
\end{aligned}
\tag{3.155}
$$

Going nontangentially to the boundary in (3.155) then yields, on account of (3.154), (3.134), (3.47), (3.141), and (3.136),

$$
\begin{aligned}
\big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f &= \big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)\Big(\big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f\Big) \\
&\quad - S_{\text{mod}}\Big(\big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big)f\Big) + c_f
\end{aligned}
\tag{3.156}
$$

at σ-a.e. point on $\partial\Omega$. From this, (3.148) readily follows. This takes care of the claim in item *(1)*.

To deal with the claim in item *(2)*, take the conormal derivative ∂_ν^A of the most extreme sides of (3.155) and use (3.67), (3.136), and the fact that $\partial_\nu^A c_u = 0$ (cf. (3.66)) to arrive at the conclusion that

$$
\begin{aligned}
\big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big)f &= \big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big)\Big(\big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f\Big) \\
&\quad - \big(-\tfrac{1}{2}I + K_{A^\top}^{\#}\big)\Big(\big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big)f\Big)
\end{aligned}
\tag{3.157}
$$

at σ-a.e. point on $\partial\Omega$, from which (3.150) readily follows.

Let us now turn our attention to the claims made in items *(3)-(4)*. Start with an arbitrary function $f \in \big[L^p(\partial\Omega, w)\big]^M$, and then consider $u := \mathscr{S}_{\text{mod}} f$ in Ω. From (2.575), item *(c)* of Proposition 3.5, and (3.47), we see that

$$
\begin{aligned}
& u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{in} \ \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\
& \quad \text{the traces } u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \ (\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{exist } \sigma\text{-a.e. on } \partial\Omega, \\
& \quad u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = S_{\text{mod}} f \ \text{ and } \ \partial_\nu^A u = \big(-\tfrac{1}{2}I + K_{A^\top}^{\#}\big)f.
\end{aligned}
\tag{3.158}
$$

Again, Corollary 3.1 applies and gives that $\partial_\nu^A u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$, the trace $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, and there exists some $c_f \in \mathbb{C}^M$ such that

$$
u = \mathcal{D}_{A,\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathscr{S}_{\text{mod}}\big(\partial_\nu^A u\big) + c_u
$$

$$= \mathcal{D}_{A,\mathrm{mod}}\big(S_{\mathrm{mod}} f\big) - \mathscr{S}_{\mathrm{mod}}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big) + c_f \quad \text{in } \ \Omega. \tag{3.159}$$

Taking nontangential boundary traces in (3.159) then gives, thanks to (3.158), (3.47), (3.134), (3.131), and (3.118),

$$S_{\mathrm{mod}} f = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)\big(S_{\mathrm{mod}} f\big) - S_{\mathrm{mod}}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big) + c_f \tag{3.160}$$

at σ-a.e. point on $\partial\Omega$. With this in hand, (3.151) follows after simple algebra. This justifies the claim made in item *(3)*.

As regards item *(4)*, take the conormal derivative ∂^{A}_{ν} of the most extreme sides of (3.159) and rely on (3.158), (3.136), (3.126), (3.131), (3.118), and the fact that $\partial^{A}_{\nu} c_u = 0$ (cf. (3.66)) to conclude that

$$\begin{aligned}\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f &= \big(\partial^{A}_{\nu}\mathcal{D}_{A,\mathrm{mod}}\big)\big(S_{\mathrm{mod}} f\big) \\ &\qquad - \big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big)\end{aligned} \tag{3.161}$$

at σ-a.e. point on $\partial\Omega$, from which (3.153) readily follows. □

There are direct links between the layer potential operators discussed so far in this section and boundary value problems. To elaborate on this, we introduce a piece of notation. Given two vector spaces X, Y, for linear operator $T : X \to Y$ denote by

$$\mathrm{Im}\big(T : X \to Y\big) := \big\{Tx : x \in X\big\} \tag{3.162}$$

the image of T. Moreover, corresponding to the special case when $X = Y$, we agree to abbreviate $\mathrm{Im}\big(T; X\big) := \mathrm{Im}\big(T : X \to X\big)$.

Proposition 3.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an NTA domain with the property that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick some integrability exponent $p \in (1,\infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Finally, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, and fix a coefficient tensor $A \in \mathfrak{A}_L$.*

Then for each given function f belonging to $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, the homogeneous Muckenhoupt weighted boundary Sobolev space defined in (2.598), the following statements are equivalent:

(a) The boundary value problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \ \ at\ \sigma\text{-a.e. point on } \ \partial\Omega \end{cases} \tag{3.163}$$

has a solution.

(b) The equivalence class of the function f modulo constants, denoted by $[f]$, belongs to the space

$$\begin{aligned} \operatorname{Im}\Big(\tfrac{1}{2}I + \big[K_{A,mod}\big];\ \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big) \\ + \operatorname{Im}\Big(\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big). \end{aligned} \tag{3.164}$$

(c) Again, with $[f]$ denoting the equivalence class of the function f modulo constants,

$$\begin{aligned} \Big(-\tfrac{1}{2}I + \big[K_{A,mod}\big]\Big)[f] \ \ \textit{belongs to the space} \\ \operatorname{Im}\Big(\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big). \end{aligned} \tag{3.165}$$

Proof Assume u solves (3.163). Then Corollary 3.1 guarantees that $\partial^A_\nu u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$ and that there exists some $c_u \in \mathbb{C}^M$ such that (3.75) holds. Going nontangentially to the boundary then yields, on account of (3.134), (3.47), and (2.575),

$$f = \big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f - S_{\text{mod}}\big(\partial^A_\nu u\big) + c_u \ \text{ on } \ \partial\Omega. \tag{3.166}$$

Taking equivalence classes modulo constants and keeping in mind (3.142), (3.132), we may recast (3.166) as

$$\big(-\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\big)[f] = \big[S_{\text{mod}}\big]\big(\partial^A_\nu u\big). \tag{3.167}$$

From this, we conclude that (3.165) holds, hence *(a)* ⇒ *(c)*.

Next, assume (3.165) holds. Since

$$[f] = \Big(\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\Big)[f] - \Big(-\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\Big)[f], \tag{3.168}$$

this implies that $[f]$ belongs to the space in (3.164). Thus, *(c)* ⇒ *(b)*.

Finally, if $[f]$ belongs to the space in (3.164), it follows from (3.142) and (3.132) that

$$f = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)g + S_{\mathrm{mod}}h + c \tag{3.169}$$

for some

$$g \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M, \quad h \in \big[L^p(\partial\Omega, w)\big]^M, \quad c \in \mathbb{C}^M. \tag{3.170}$$

In view of this, (3.133), (3.134), (3.124), (3.127), (3.47), and (2.575), we then see that the function

$$u := \mathcal{D}_{A,\mathrm{mod}}g + \mathscr{S}_{\mathrm{mod}}h + c \ \text{ in } \ \Omega \tag{3.171}$$

solves the boundary value problem (3.163). Hence, *(b)* ⇒ *(a)*, finishing the proof of the proposition. □

Here is a companion result to Proposition 3.6 for a Neumann type boundary value problem.

Proposition 3.7 *Let $\Omega \subseteq \mathbb{R}^n$ be an NTA domain such that its topological boundary, $\partial\Omega$, is an unbounded Ahlfors regular set. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an aperture parameter $\kappa \in (0,\infty)$, pick some integrability exponent $p \in (1,\infty)$, and consider some Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Finally, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and fix a coefficient tensor $A \in \mathfrak{A}_L$.*

Then, for each function $f \in \big[L^p(\partial\Omega, w)\big]^M$, the following statements are equivalent:

(a) The boundary value problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial^A_\nu u = f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega \end{cases} \tag{3.172}$$

has a solution.

(b) The function f belongs to the space

$$\mathrm{Im}\Big(\big[\partial^A_\nu \mathcal{D}_{A,mod}\big] : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M\Big)$$
$$+\,\mathrm{Im}\Big(-\tfrac{1}{2}I + K^{\#}_{A^\top} ; \big[L^p(\partial\Omega, w)\big]^M\Big). \tag{3.173}$$

(c) One has

$$\begin{aligned}&\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \ \text{ belongs to the space}\\ &\mathrm{Im}\Big(\big[\partial^A_\nu \mathcal{D}_{A,mod}\big] : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M\Big).\end{aligned} \tag{3.174}$$

Proof Suppose u solves (3.172). Then Corollary 3.1 gives that the nontangential boundary trace $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$ and belongs to $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ and that the integral representation formula in (3.75) holds for some $c_u \in \mathbb{C}^M$. Taking the conormal derivative of both sides then yields, in view of (3.126),

$$f = \partial^A_\nu\Big(\mathcal{D}_{A,\text{mod}}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\Big) - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \ \text{ on } \partial\Omega. \tag{3.175}$$

From this and (3.138), we then conclude that

$$\begin{aligned}&\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f = \partial^A_\nu\Big(\mathcal{D}_{A,\text{mod}}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\Big) \ \text{ belongs to the space}\\ &\mathrm{Im}\Big(\big[\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big] : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M\Big),\end{aligned} \tag{3.176}$$

hence *(a)* ⇒ *(c)*. Going further, assume (3.174) holds. Since

$$f = \big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f, \tag{3.177}$$

this implies that f belongs to the space in (3.173). As such, *(c)* ⇒ *(b)*.

Finally, suppose the function f belongs to the space in (3.173), say

$$f = \big[\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big][g] + \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)h \tag{3.178}$$

for some

$$g \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \ \text{ and } \ h \in \big[L^p(\partial\Omega, w)\big]^M. \tag{3.179}$$

Then (3.178), (3.179), (3.138), (3.133), (3.124), (3.126), and (3.127) collectively imply that the function

$$u := \mathcal{D}_{A,\text{mod}}\, g + \mathscr{S}_{\text{mod}}\, h \ \text{ in } \Omega \tag{3.180}$$

solves the boundary value problem (3.172). Thus, *(b)* ⇒ *(a)*, and the proof of the proposition is complete. □

3.3 Distinguished Coefficient Tensors

To each weakly elliptic system L, we may canonically associate a fundamental solution E as in Theorem 3.1. Having fixed a UR domain, this is then used to create a variety of double layer potential operators K_A, in relation to each choice of a coefficient tensor $A \in \mathfrak{A}_L$. While any such double layer K_A has a rich Calderón–Zygmund theory (as discussed in Proposition 3.5), seeking more specialized properties requires placing additional demands on the coefficient tensor A. We begin by recording a result proved in [115, §1.2] describing said demands phrased in several equivalent forms.

Proposition 3.8 *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and consider the matrix-valued function defined for each $\xi \in \mathbb{R}^n \setminus \{0\}$ as*

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\leq\gamma,\beta\leq M} := \big[L(\xi)\big]^{-1} \in \mathbb{C}^{M\times M} \tag{3.181}$$

(recall that the characteristic matrix $L(\xi)$ of L has been defined in (3.2)). Also, let $E = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ be the fundamental solution associated with the given system L as in Theorem 3.1.

Then, for each coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$ (cf. (3.11)), the following conditions are equivalent:

(a) For each $k, k' \in \{1,\dots,n\}$ and each $\alpha, \gamma \in \{1,\dots,M\}$, there holds

$$\big(x_{k'}a^{\beta\alpha}_{jk} - x_k a^{\beta\alpha}_{jk'}\big)(\partial_j E_{\gamma\beta})(x) = 0 \ \textit{ for all } \ x = (x_i)_{1\leq i\leq n} \in \mathbb{R}^n \setminus \{0\}. \tag{3.182}$$

(b) For each $s, s' \in \{1,\dots,n\}$ and each $\alpha, \gamma \in \{1,\dots,M\}$, in the sense of tempered distributions in $\mathbb{R}^n$, one has

$$\Big[a^{\beta\alpha}_{rs}\partial_{\xi_{s'}} - a^{\beta\alpha}_{rs'}\partial_{\xi_s}\Big]\big[\xi_r \mathcal{E}_{\gamma\beta}(\xi)\big] = 0. \tag{3.183}$$

(c) For each $k, k' \in \{1,\dots,n\}$ and each $\alpha, \gamma \in \{1,\dots,M\}$, one has

$$\big(a^{\beta\alpha}_{k'k} - a^{\beta\alpha}_{kk'} + \xi_j a^{\beta\alpha}_{jk}\partial_{\xi_{k'}} - \xi_j a^{\beta\alpha}_{jk'}\partial_{\xi_k}\big)\mathcal{E}_{\gamma\beta}(\xi) = 0 \ \textit{for all} \ \xi \in \mathbb{R}^n\setminus\{0\} \tag{3.184}$$

and also

$$\int_{S^1} \big(a^{\beta\alpha}_{jk}\xi_{k'} - a^{\beta\alpha}_{jk'}\xi_k\big)\xi_j\, \mathcal{E}_{\gamma\beta}(\xi)\, d\mathcal{H}^1(\xi) = 0 \ \textit{ if } \ n = 2. \tag{3.185}$$

(d) One has

$$\xi_r\xi_j\Big[a^{\beta\alpha}_{rs'}\big(a^{\lambda\mu}_{sj}+a^{\lambda\mu}_{js}\big)-a^{\beta\alpha}_{rs}\big(a^{\lambda\mu}_{s'j}+a^{\lambda\mu}_{js'}\big)\Big]\mathcal{E}_{\mu\beta}(\xi)+a^{\lambda\alpha}_{ss'}-a^{\lambda\alpha}_{s's}=0$$
$$\textit{for all } \xi\in S^{n-1}\textit{, all } s,s'\in\{1,\dots,n\}\textit{, and all } \alpha,\lambda\in\{1,\dots,M\}, \tag{3.186}$$

with the cancellation condition

$$\int_{S^1}\Big(a^{\beta\alpha}_{rs}\xi_{s'}-a^{\beta\alpha}_{rs'}\xi_s\Big)\xi_r\mathcal{E}_{\lambda\beta}(\xi)\,\mathrm{d}\mathcal{H}^1(\xi)=0$$
$$\textit{for all } s,s'\in\{1,\dots,n\} \textit{ and } \alpha,\lambda\in\{1,\dots,M\}, \tag{3.187}$$

additionally imposed in the case when $n=2$.

(e) *For each* $\xi\in S^{n-1}$ *and each* $\alpha,\lambda\in\{1,\dots,M\}$,

$$\textit{the expression } \big(a^{\lambda\mu}_{sj}+a^{\lambda\mu}_{js}\big)\mathcal{E}_{\mu\beta}(\xi)\xi_j\xi_r a^{\beta\alpha}_{rs'}-a^{\lambda\alpha}_{s's}$$
$$\textit{is symmetric in the indices } s,s'\in\{1,\dots,n\}, \tag{3.188}$$

with the condition that for each $\alpha,\lambda\in\{1,\dots,M\}$

$$\textit{the expression } \int_{S^1}a^{\beta\alpha}_{rs}\xi_{s'}\xi_r\mathcal{E}_{\lambda\beta}(\xi)\,\mathrm{d}\mathcal{H}^1(\xi)$$
$$\textit{is symmetric in the indices } s,s'\in\{1,2\}, \tag{3.189}$$

also imposed in the case when $n=2$.

(f) *There exists a matrix-valued function*

$$k=\big\{k_{\gamma\alpha}\big\}_{1\leq\gamma,\alpha\leq M}:\mathbb{R}^n\setminus\{0\}\longrightarrow\mathbb{C}^{M\times M} \tag{3.190}$$

with the property that for each $\gamma,\alpha\in\{1,\dots,M\}$ *and* $s\in\{1,\dots,n\}$, *one has*

$$a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x)=x_s k_{\gamma\alpha}(x)\ \ \textit{for all}\ \ x\in\mathbb{R}^n\setminus\{0\}. \tag{3.191}$$

It is worth noting that the conditions in items *(a)–(f)* above are intrinsically formulated in terms of the given weakly elliptic system L. Observe that for each $x_*\in\mathbb{R}^n\setminus\{0\}$, we may find an open neighborhood $\mathcal{O}$ of the point x_* and an index $s\in\{1,\dots,n\}$ with the property that $x_s\neq 0$ for each $x\in\mathcal{O}$. From this observation, (3.191), and Theorem 3.1, it follows that

$$\text{all entries of the matrix-valued function } k \text{ from (3.190) belong to } \mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\text{, are even, as well as positive homogeneous of degree } -n. \tag{3.192}$$

Definition 3.1 Given a second-order, weakly elliptic, homogeneous, $M \times M$ system L in $\mathbb{R}^n$, with constant complex coefficients, call

$$A = \left(a_{rs}^{\alpha\beta}\right)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}} \in \mathfrak{A}_L \tag{3.193}$$

a `distinguished coefficient tensor` for the system L provided any of the conditions *(a)-(f)* in Proposition 3.8 holds. Also, denote by $\mathfrak{A}_L^{\text{dis}}$ the family of such distinguished coefficient tensors for L, say,

$$\mathfrak{A}_L^{\text{dis}} := \Big\{ A = \left(a_{rs}^{\alpha\beta}\right)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}} \in \mathfrak{A}_L : \text{ conditions (3.184)–(3.185)} \\ \text{hold for each } \ k, k' \in \{1, \dots, n\} \ \text{ and } \ \alpha, \gamma \in \{1, \dots, M\} \Big\}. \tag{3.194}$$

Finally, introduce the class of weakly elliptic systems which posses a distinguished coefficient tensor, by setting

$$\mathfrak{L}^{\text{dis}} := \left\{ L \in \mathfrak{L}_* : \mathfrak{A}_L^{\text{dis}} \neq \varnothing \right\}. \tag{3.195}$$

For example, from Proposition 3.8 and the second line in (3.20), we see that

for any weakly elliptic, homogeneous, second-order, constant complex coefficient, $M \times M$ system L in $\mathbb{R}^n$, any coefficient tensor $A \in \mathfrak{A}_L$, and any complex number $\lambda \in \mathbb{C} \setminus \{0\}$, it follows that $A \in \mathfrak{A}_L^{\text{dis}}$ if and only if $\lambda A \in \mathfrak{A}_{\lambda L}^{\text{dis}}$. (3.196)

The relevance of the distinguished coefficient tensors is most apparent from the following result proved in [115, §1.3].

Proposition 3.9 *Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and suppose $A \in \mathfrak{A}_L$. Then the following statements are equivalent:*

(i) *The coefficient tensor A belongs to $\mathfrak{A}_L^{\text{dis}}$.*
(ii) *Whenever Ω is a half-space in $\mathbb{R}^n$, the boundary-to-boundary double layer potential K_A associated with A and Ω as in (3.24) is the zero operator.*
(iii) *Whenever Ω is a half-space in $\mathbb{R}^n$ with the property that $0 \in \partial\Omega$, the modified boundary-to-boundary double layer operator $K_{A,mod}$ associated as in (3.50) with the set Ω and the given coefficient tensor A is actually the zero operator.*
(iii') *Whenever Ω is a half-space in $\mathbb{R}^n$, the modified boundary-to-boundary double layer operator $K_{A,mod}$ associated as in (3.50) with the set Ω and the given coefficient tensor A maps each function from $\left[\mathscr{C}_c^\infty(\partial\Omega)\right]^M$ into a constant in $\mathbb{C}^M$.*
(iv) *There exists a matrix-valued function $k \in \left[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\right]^{M\times M}$ which is even, positive homogeneous of degree $-n$, and with the property that for each*

UR domain $\Omega \subseteq \mathbb{R}^n$, the (matrix-valued) integral kernel of the double layer potential operator K_A associated with A and Ω as in (3.24) has the form

$$\begin{gathered}\langle \nu(y), x-y\rangle k(x-y) \\ \text{for each } x \in \partial\Omega \text{ and } \mathcal{H}^{n-1}\text{-a.e. } y \in \partial\Omega,\end{gathered} \tag{3.197}$$

where ν is the geometric measure theoretic outward unit normal to Ω.

(v) Whenever Ω is a half-space in $\mathbb{R}^n$, the "transpose" double layer potential $K_A^{\#}$ associated with A and Ω as in (3.25) is the zero operator.

(vi) There exists a matrix-valued function $k^{\#} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ which is even, positive homogeneous of degree $-n$, and with the property that for each UR domain $\Omega \subseteq \mathbb{R}^n$, the (matrix-valued) integral kernel of the "transpose" double layer potential operator $K_A^{\#}$ associated with A and Ω as in (3.25) has the form

$$\begin{gathered}\langle \nu(x), y-x\rangle k^{\#}(x-y) \\ \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega \text{ and each } y \in \partial\Omega,\end{gathered} \tag{3.198}$$

where ν is the geometric measure theoretic outward unit normal to Ω.

Moreover, whenever either (hence all) of the above conditions materializes, the matrices $k, k^{\#}$ in items (iv), (vi) above are related to each other via $k^{\#} = k^\top$, where the superscript $\top$ indicates transposition.

In light of Proposition 3.9 and (1.50), we are particularly interested in the class of weakly elliptic homogeneous constant complex coefficient second-order systems L with $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$. The following example shows that the latter condition is always satisfied by strongly elliptic scalar operators.

Example 3.2 Assume L is a second-order, homogeneous, constant complex coefficient, scalar differential operator in $\mathbb{R}^n$ (i.e., as in (3.1) with $M = 1$), which is `strongly elliptic`. Specifically, suppose $L = a_{jk}\partial_j\partial_k$ with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1, \dots, n\}$ having the property that there exists a constant $c \in (0, \infty)$ such that

$$\text{Re}\Big[\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k\Big] \geq c|\xi|^2, \quad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n. \tag{3.199}$$

Introduce $A := (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$ and then define

$$(\widetilde{a}_{jk})_{1\leq j,k\leq n} := \text{sym}\, A := \frac{A + A^\top}{2}, \quad (b_{jk})_{1\leq j,k\leq n} := (\text{sym}\, A)^{-1}. \tag{3.200}$$

In particular, $L = L_{\mathrm{sym}\,A} := \widetilde{a}_{jk}\partial_j\partial_k$, i.e., the coefficient matrix $\mathrm{sym}\,A$ may be used to represent the given differential operator L. In this case, it turns out that the fundamental solution E canonically associated with the operator L as in Theorem 3.1 may be explicitly identified (cf. [102, Theorem 7.68, pp. 314-315]) as the function $E \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)$ given at each point $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E(x) = \begin{cases} -\dfrac{1}{(n-2)\,\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n-2}{2}} & \text{if } n \geq 3, \\[2ex] \dfrac{1}{4\pi\sqrt{\det(\mathrm{sym}\,A)}}\log(\langle(\mathrm{sym}\,A)^{-1}x, x\rangle) + c_A & \text{if } n = 2, \end{cases} \tag{3.201}$$

where log denotes the principal branch of the complex logarithm (defined for complex numbers $z \in \mathbb{C} \setminus (-\infty, 0]$ so that $z^a = \mathrm{e}^{a \log z}$ for each $a \in \mathbb{R}$), and c_A is a complex constant which depends solely on A. As both $\mathrm{sym}\,A$ and $(\mathrm{sym}\,A)^{-1}$ are symmetric matrices, for each index $j \in \{1, \dots, n\}$ and each point $x = (x_i)_{1\leq i\leq n} \in \mathbb{R}^n \setminus \{0\}$, we therefore have (in all dimensions $n \geq 2$)

$$\begin{aligned}(\partial_j E)(x) &= \frac{\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}(\delta_{rj}b_{rs}x_s + \delta_{sj}b_{rs}x_r)}{2\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}} \\ &= \frac{\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}} b_{rj}x_r}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}.\end{aligned} \tag{3.202}$$

Thus, with $C_{A,n}$ abbreviating $\big(\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}\big)^{-1} \in \mathbb{C}$, for each pair of integers $k, k' \in \{1, \dots, n\}$, we may compute

$$\begin{aligned}\big(x_{k'}\widetilde{a}_{jk} - x_k\widetilde{a}_{jk'}\big)(\partial_j E)(x) &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}\big(x_{k'}\widetilde{a}_{kj} - x_k\widetilde{a}_{k'j}\big)\big(b_{jr}x_r\big) \\ &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}\big(x_{k'}\delta_{kr} - x_k\delta_{k'r}\big)x_r \\ &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}\big(x_{k'}x_k - x_kx_{k'}\big) = 0.\end{aligned} \tag{3.203}$$

This shows that condition (3.182) is presently verified for the choice of coefficient tensor $\mathrm{sym}\,A$ in the representation of the given differential operator L. Hence, $\mathrm{sym}\,A \in \mathfrak{A}^{\mathrm{dis}}_L$, which proves that, in the case when $M = 1$, we have

$$\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing \text{ for every scalar, strongly elliptic, homogeneous, second-order, constant complex coefficient operator } L \text{ in } \mathbb{R}^n. \tag{3.204}$$

Consequently, Proposition 3.9 guarantees that for each UR domain $\Omega \subseteq \mathbb{R}^n$ the integral kernel of the double layer potential operator $K_{\mathrm{sym}\,A}$ associated with $\mathrm{sym}\,A$ and Ω as in (3.24) has the form (3.197). This being said, it is actually of interest to identify said integral kernel explicitly. Based on (3.200)–(3.202) and (3.24), we see that the kernel of if $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to Ω, then the integral kernel of the double layer potential operator $K_{\mathrm{sym}\,A}$ is

$$\begin{aligned} -\nu_k(y)\widetilde{a}_{jk}\left(\partial_j E\right)(x-y) &= -\frac{\langle(\mathrm{sym}\,A)^{-1}(x-y), x-y\rangle^{-\frac{n}{2}}\nu_k(y)b_{rj}\widetilde{a}_{jk}(x-y)_r}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}} \\ &= -\frac{\langle(\mathrm{sym}\,A)^{-1}(x-y), x-y\rangle^{-\frac{n}{2}}\langle\nu(y), x-y\rangle}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}} \qquad (3.205) \\ &\text{for each } x\in\partial\Omega \text{ and } \mathcal{H}^{n-1}\text{-a.e. } y\in\partial\Omega, \end{aligned}$$

which, as already anticipated, is of the form (3.197) with

$$k(z) := -\frac{\langle(\mathrm{sym}\,A)^{-1}z, z\rangle^{-\frac{n}{2}}}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}, \qquad \forall\, z\in\mathbb{R}^n\setminus\{0\}. \tag{3.206}$$

In the same scenario as above, we also wish to elaborate on the nature of $\mathfrak{A}_L^{\mathrm{dis}}$ (see the conclusion reached in (3.218) below). To set the stage, recall that any given matrix $A = (a_{jk})_{1\le j,k\le n} \in \mathbb{C}^{n\times n}$ may be decompose into its symmetric and antisymmetric parts, i.e.,

$$A = \mathrm{sym}\,A + \mathrm{asym}\,A \;\text{ where }\; \mathrm{asym}\,A := A - \mathrm{sym}\,A = \frac{A-A^\top}{2}. \tag{3.207}$$

Consequently, for each UR domain $\Omega \subseteq \mathbb{R}^n$ with geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$, the integral kernel of the double layer potential operator K_A is given by

$$-\nu_k(y)\widetilde{a}_{jk}\left(\partial_j E\right)(x-y) - \nu_k(y)\widehat{a}_{jk}\left(\partial_j E\right)(x-y), \tag{3.208}$$

where E is as in (3.201), the entries $(\widetilde{a}_{jk})_{1\le j,k\le n}$ are as in (3.200), and

$$(\widehat{a}_{jk})_{1\le j,k\le n} := \mathrm{asym}\,A. \tag{3.209}$$

If Ω is a half-space, then, as seen from (3.205) and (3.208), the integral kernel of the double layer potential operator K_A reduces to

$$-\nu_k(y)\widehat{a}_{jk}\left(\partial_j E\right)(x-y). \tag{3.210}$$

From this and Proposition 3.9, we then conclude that $A \in \mathfrak{A}_L^{\mathrm{dis}}$ if and only if

$$-\nu_k(y)\widehat{a}_{jk}\left(\partial_j E\right)(x-y)=0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x,y\in\partial\Omega \\ \text{whenever } \Omega \text{ is a half-space in } \mathbb{R}^n. \tag{3.211}$$

The same type of argument which, starting with (1.44), has produced (1.47) now shows that (3.211) implies the existence of a function $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ which is even, positive homogeneous of degree $-n$, and such that

$$\widehat{a}_{rs}\,(\partial_r E)\,(x) = x_s k(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and each} \quad s \in \{1,\dots,n\}. \tag{3.212}$$

Multiply this equality by $(\partial_s E)(x)$, and summing up in $s \in \{1,\dots,n\}$ yields, on account of the antisymmetry of $(\widehat{a}_{rs})_{1\le r,s\le n} = \operatorname{asym} A$,

$$x_s(\partial_s E)(x)k(x) = 0 \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{3.213}$$

On the other hand, if $n \geq 3$, it follows that $E \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ is positive homogeneous of degree $2-n$ (cf. (3.201)), so Euler's formula gives in this case

$$x_s(\partial_s E)(x) = (2-n)E(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{3.214}$$

By combining (3.213)–(3.214), we therefore arrive at the conclusion that

$$\text{if } n \geq 3 \text{ then } E(x)k(x) = 0 \text{ for each } \; x \in \mathbb{R}^n \setminus \{0\}. \tag{3.215}$$

Since as is apparent from (3.201), at each point $x \in \mathbb{R}^n \setminus \{0\}$, we have $E(x) \neq 0$, this ultimately forces $k(x) = 0$ for each $x \in \mathbb{R}^n \setminus \{0\}$. When used back in (3.212), this permits us to conclude (assuming $n \geq 3$) that

$$\widehat{a}_{rs}\,(\partial_r E)\,(x) = 0 \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and each} \quad s \in \{1,\dots,n\}. \tag{3.216}$$

Together, (3.216) and (3.202) prove (again, assuming $n \geq 3$) that for each index $s \in \{1,\dots,n\}$, we have

$$\frac{\langle(\operatorname{sym} A)^{-1}x, x\rangle^{-\frac{n}{2}}\widehat{a}_{rs}b_{kr}x_k}{\omega_{n-1}\sqrt{\det(\operatorname{sym} A)}} = 0 \quad \text{for all} \quad x = (x_k)_{1\le k\le n} \in \mathbb{R}^n \setminus \{0\}, \tag{3.217}$$

where $(b_{jk})_{1\le j,k\le n} := (\operatorname{sym} A)^{-1}$ (cf. (3.200)). Thus, assuming $n \geq 3$, we deduce from (3.217) that in fact $(\operatorname{asym} A)(\operatorname{sym} A)^{-1} = 0$. This is equivalent to having $\operatorname{asym} A = 0$, i.e., the matrix $A \in \mathfrak{A}_L^{\mathrm{dis}}$ is necessarily symmetric. In concert with (3.204) and its proof, the above argument shows that

assuming $n \geq 3$, it follows that for each given strongly elliptic, scalar, homogeneous, second-order operator $L = \operatorname{div} A\nabla$ in $\mathbb{R}^n$ with constant complex coefficients, the class $\mathfrak{A}_L^{\mathrm{dis}}$ consists precisely of one matrix, namely $\operatorname{sym} A := (A + A^\top)/2$. (3.218)

Our next example shows that, for scalar operators in dimensions $n \geq 3$, weak ellipticity itself guarantees the existence of a unique distinguished coefficient tensor.

Example 3.3 Suppose $n \geq 3$, and consider an arbitrary second-order, homogeneous, constant complex coefficient, scalar differential operator L in $\mathbb{R}^n$ (i.e., as in (3.1) with $M = 1$), which is merely *weakly elliptic*. Recall (cf. (1.2)) that this means that we may express $L = a_{jk}\partial_j\partial_k$ with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1, \dots, n\}$ having the property that

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \neq 0, \qquad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{3.219}$$

Introduce $A := (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$. It has been shown in [113, §1.4] that (here is where $n \geq 3$ is used)

> there exists an angle $\theta \in [0, 2\pi)$ such that if we set $A_\theta := e^{i\theta}A$ then the matrix $\mathrm{sym}A_\theta := (A_\theta + A_\theta^\top)/2 \in \mathbb{C}^{n\times n}$ is strongly elliptic, in the sense that there exists some $c \in (0, \infty)$ such that $\mathrm{Re}\,\langle(\mathrm{sym}\,A_\theta)\xi, \xi\rangle \geq c|\xi|^2$ for each $\xi \in \mathbb{R}^n$ (cf. (3.199)). (3.220)

From this and (3.201), we conclude that the fundamental solution $E \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)$ canonically associated as in Theorem 3.1 with the operator

$$L := e^{-i\theta}L_{A_\theta} = e^{-i\theta}L_{\mathrm{sym}A_\theta} \tag{3.221}$$

presently may be expressed at each point $x \in \mathbb{R}^n \setminus \{0\}$ as

$$E(x) = -\frac{e^{i\theta}}{(n-2)\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A_\theta)}}\big\langle(\mathrm{sym}\,A_\theta)^{-1}x, x\big\rangle^{\frac{2-n}{2}}. \tag{3.222}$$

In view of this formula and the fact that $\mathrm{sym}\,A := (A + A^\top)/2$ is related to $\mathrm{sym}\,A_\theta$ via $\mathrm{sym}\,A_\theta = e^{i\theta}\mathrm{sym}\,A$, we conclude from (3.201)–(3.203) that condition (3.182) currently holds for the choice of coefficient matrix $\mathrm{sym}\,A$ in the representation of the given differential operator L. Thus, $\mathrm{sym}\,A \in \mathfrak{A}_L^{\mathrm{dis}}$. In concert with (3.196) and (3.218), this goes to show that the following sharper version of (3.218) holds:

> if $n \geq 3$ then for each weakly elliptic, scalar, homogeneous, second-order operator $L = \mathrm{div}\,A\nabla$ in $\mathbb{R}^n$ with constant complex coefficients, the class $\mathfrak{A}_L^{\mathrm{dis}}$ consists precisely of one matrix, namely $\mathrm{sym}\,A := (A + A^\top)/2$. (3.223)

Turning our attention to genuine systems, below we pay special attention to the Lamé system of elasticity.

Example 3.4 Consider the following complexified version of the Lamé system (originally arising in the study of linear elasticity), defined for any two parameters $\mu, \lambda \in \mathbb{C}$ (referred to as Lamé moduli) as

$$L := L_{\mu,\lambda} := \mu\Delta + (\mu+\lambda)\nabla\mathrm{div}, \tag{3.224}$$

acting on vector fields $u = (u_\beta)_{1\leq\beta\leq n}$ defined in (open subsets of) $\mathbb{R}^n$, with the Laplacian applied componentwise. Hence, $L = L^\top$, and one may check (cf. [102, Proposition 10.14, p. 366]) that

$$\begin{array}{l}\text{the complex Lamé system (3.224) is weakly elliptic}\\ \text{if and only if one has } \mu \neq 0 \text{ as well as } 2\mu+\lambda \neq 0.\end{array} \tag{3.225}$$

We may express the complex Lamé system L as in (3.1) (with $M := n$) using a variety of coefficient tensors, such as those belonging to the one-parameter family

$$\begin{gathered} A(\zeta) = \big(a_{jk}^{\alpha\beta}(\zeta)\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq n}} \quad \text{defined for each } \zeta\in\mathbb{C} \text{ according to} \\ a_{jk}^{\alpha\beta}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu+\lambda-\zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \quad 1\leq j,k,\alpha,\beta\leq n. \end{gathered} \tag{3.226}$$

In other words, for each vector field $u = (u_\beta)_{1\leq\beta\leq n} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ and each parameter $\zeta \in \mathbb{C}$, the Lamé system (3.224) satisfies

$$Lu = \Big(a_{jk}^{\alpha\beta}(\zeta)\partial_j\partial_k u_\beta\Big)_{1\leq\alpha\leq n} \quad \text{in } \big[\mathcal{D}'(\mathbb{R}^n)\big]^n. \tag{3.227}$$

In relation to the coefficient tensor (3.226), it turns out that, for any $\mu, \lambda, \zeta \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu+\lambda \neq 0$, if L is as in (3.224), then we have (cf. [61] for specific details)

$$A(\zeta) \in \mathfrak{A}_L^{\mathrm{dis}} \iff 3\mu+\lambda \neq 0 \text{ and } \zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}. \tag{3.228}$$

This ultimately shows that

$$\begin{array}{l}\text{whenever the Lamé moduli } \mu,\lambda\in\mathbb{C} \text{ satisfy } \mu\neq 0,\ 2\mu+\lambda\neq 0,\\ \text{and } 3\mu+\lambda\neq 0, \text{ the Lamé operator } L \text{ defined as in (3.227) has}\\ \text{the property that } \mathfrak{A}_L^{\mathrm{dis}} = \mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing.\end{array} \tag{3.229}$$

It is of interest to concretely identify the format of the double layer potential operators associated with the complex Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$ in $\mathbb{R}^n$, associated as in (1.52) with the Lamé moduli $\mu, \lambda \in \mathbb{C}$ satisfying

$$\mu \neq 0 \text{ and } 2\mu+\lambda \neq 0 \tag{3.230}$$

(thus ensuring the weak ellipticity of $L_{\mu,\lambda}$; cf. (3.225)). For this system, the fundamental solution E of $L_{\mu,\lambda}$ from Theorem 3.1 has the explicit form $E = (E_{jk})_{1\leq j,k\leq n}$, a matrix whose (j,k) entry is defined at each point $x = (x_1,\dots,x_n) \in \mathbb{R}^n \setminus \{0\}$ according to

$$E_{jk}(x) = \begin{cases} \dfrac{-1}{2\mu(2\mu+\lambda)\omega_{n-1}}\left[\dfrac{\delta_{jk}(3\mu+\lambda)}{(n-2)|x|^{n-2}} + \dfrac{(\mu+\lambda)x_jx_k}{|x|^n}\right] & \text{if } n \geq 3, \\[2ex] \dfrac{1}{4\pi\mu(2\mu+\lambda)}\left[\delta_{jk}(3\mu+\lambda)\ln|x| - \dfrac{(\mu+\lambda)x_jx_k}{|x|^2}\right] + c_{\mu,\lambda}\delta_{jk} & \text{if } n = 2, \end{cases} \tag{3.231}$$

for every $j,k \in \{1,\dots,n\}$, where $c_{\mu,\lambda} \in \mathbb{C}$ is the constant given by

$$c_{\mu,\lambda} := \frac{(1+\ln 4)(\lambda+\mu)}{8\pi\mu(\lambda+2\mu)} - \frac{\ln 2}{2\pi\mu}. \tag{3.232}$$

Let us now fix an arbitrary UR domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. In such a setting, with each choice of $\zeta \in \mathbb{C}$, associate a double layer potential operator $K_{A(\zeta)}$ as in (3.24). A direct computation based on (3.231), (3.226), and (3.24) then shows that the integral kernel $\Theta^\zeta(x,y)$ of the principal-value double layer potential operator $K_{A(\zeta)}$ is an $n\times n$ matrix whose (j,k) entry, $1 \leq j,k \leq n$, is explicitly given by

$$\begin{aligned} \Theta^\zeta_{jk}(x,y) = {} & -C_1(\zeta)\frac{\delta_{jk}}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle}{|x-y|^n} \\ & - (1-C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle(x_j-y_j)(x_k-y_k)}{|x-y|^{n+2}} \\ & - C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x_j-y_j)\nu_k(y)-(x_k-y_k)\nu_j(y)}{|x-y|^n}, \end{aligned} \tag{3.233}$$

for σ-a.e. $x,y \in \partial\Omega$, where the constants $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ are defined as

$$C_1(\zeta) := \frac{\mu(3\mu+\lambda)-\zeta(\mu+\lambda)}{2\mu(2\mu+\lambda)}, \qquad C_2(\zeta) := \frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{2\mu(2\mu+\lambda)}. \tag{3.234}$$

Thus, with notation introduced in (2.3), for each $\zeta \in \mathbb{C}$, the integral kernel $\Theta^\zeta(x,y)$ of $K_{A(\zeta)}$ may be recast as

$$\Theta^\zeta(x,y) = -C_1(\zeta)\frac{1}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle}{|x-y|^n}I_{n\times n}$$

$$- (1 - C_1(\zeta)) \frac{n}{\omega_{n-1}} \frac{\langle x - y, \nu(y)\rangle (x - y) \otimes (x - y)}{|x - y|^{n+2}}$$

$$- C_2(\zeta) \frac{1}{\omega_{n-1}} \frac{(x - y) \otimes \nu(y) - \nu(y) \otimes (x - y)}{|x - y|^n}, \tag{3.235}$$

for σ-a.e. $x, y \in \partial\Omega$, where $I_{n\times n}$ is the $n \times n$ identity matrix. The penultimate term above suggests that for each function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$, we define

$$\begin{aligned} Qf(x) &:= \lim_{\varepsilon\to 0^+} \frac{n}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle x - y, \nu(y)\rangle (x - y) \otimes (x - y)}{|x - y|^{n+2}} f(y)\, \mathrm{d}\sigma(y) \\ &= \lim_{\varepsilon\to 0^+} \frac{n}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle x - y, \nu(y)\rangle \langle x - y, f(y)\rangle}{|x - y|^{n+2}} (x - y)\, \mathrm{d}\sigma(y), \end{aligned} \tag{3.236}$$

at σ-a.e. point $x \in \partial\Omega$. Then, if

$$3\mu + \lambda \neq 0 \text{ and } \zeta_* := \frac{\mu(\mu + \lambda)}{3\mu + \lambda}, \tag{3.237}$$

from (3.234), we see that $C_2(\zeta_*) = 0$, so the last term in (3.235) drops out and the principal-value double layer potential operator $K_{A(\zeta_*)}$ becomes

$$\begin{aligned} K_{A(\zeta_*)} &= C_1(\zeta_*) K_\Delta I_{n\times n} - (1 - C_1(\zeta_*)) Q \\ &= \frac{2\mu}{3\mu + \lambda} K_\Delta I_{n\times n} - \frac{\mu + \lambda}{3\mu + \lambda} Q, \end{aligned} \tag{3.238}$$

where K_Δ is the harmonic double layer potential operator (cf. (3.29)). In view of (3.29) and (3.236), this is in agreement with the prediction made in item *(iv)* of Proposition 3.9.

Traditionally, the singular integral operator $K_{A(\zeta_*)}$ from (3.238) has been called the (boundary-to-boundary) `pseudo-stress` double layer potential operator for the Lamé system, and the alternative notation K_Ψ has been occasionally employed.

We conclude this series of examples by discussing a case of a second-order, homogeneous, real constant coefficient, and weakly elliptic system which does *not* possess a distinguished coefficient tensor.

Example 3.5 Work in the plane $\mathbb{R}^2 \equiv \mathbb{C}$, and consider the second-order, homogeneous, real constant coefficient, 2×2 system

$$L = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}. \tag{3.239}$$

An example of a coefficient tensor in $\mathfrak{A}_L$ is given by $A = \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq 2\\ 1\leq\alpha,\beta\leq 2}}$ with

$$\begin{aligned} &a_{11}^{11} = a_{11}^{22} = \tfrac{1}{4}, \quad a_{22}^{11} = a_{22}^{22} = -\tfrac{1}{4}, \quad a_{12}^{11} = a_{21}^{11} = a_{12}^{22} = a_{21}^{22} = 0, \\ &a_{12}^{12} = a_{21}^{12} = -\tfrac{1}{4}, \quad a_{12}^{21} = a_{21}^{21} = \tfrac{1}{4}, \quad a_{11}^{21} = a_{22}^{21} = a_{22}^{12} = a_{11}^{12} = 0. \end{aligned} \tag{3.240}$$

The characteristic matrix of the system L is given by (cf. (3.2))

$$L(\xi) = \frac{-1}{4}\begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix} \text{ at each } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{3.241}$$

Hence, at each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, we have

$$\det\,[L(\xi)] = \tfrac{1}{16}\big[(\xi_1^2 - \xi_2^2)^2 + (2\xi_1\xi_2)^2\big] = \tfrac{1}{16}(\xi_1^2 + \xi_2^2)^2 = \tfrac{1}{16}|\xi|^4 \neq 0, \tag{3.242}$$

which goes to show that

$$\text{the system } L \text{ from (3.239) is weakly elliptic.} \tag{3.243}$$

In particular, L has a fundamental solution as in Theorem 3.1, which, once a UR domain in the plane has been fixed, may then be used to associate double layer potential operators K_A with any coefficient tensor $A \in \mathfrak{A}_L$ as in (3.24), and all these singular integral operators enjoy the properties discussed in Proposition 3.5.

This being said, since with $\eta := (1, 0) \in \mathbb{C}^2$, we have

$$\langle -L(\xi)\eta, \overline{\eta}\rangle = \tfrac{1}{4}(\xi_1^2 - \xi_2^2) \text{ for each } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{3.244}$$

and since the last expression above vanishes identically on the diagonal of $\mathbb{R}^2$, it follows that the system L from (3.239) *fails* to satisfy the Legendre–Hadamard strong ellipticity condition (cf. (3.4)).

To better understand this system, observe that its transpose is

$$L^\top = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & 2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}, \tag{3.245}$$

and, if $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ are the canonical coordinate projections, defined as

$$\pi_1(z_1, z_2) := z_1 \text{ and } \pi_2(z_1, z_2) = z_2 \text{ for each } (z_1, z_2) \in \mathbb{C}^2, \tag{3.246}$$

then

$$\begin{gathered} L(u_1,u_2)=\Big(\pi_1 L^\top(u_1,-u_2),-\pi_2 L^\top(u_1,-u_2)\Big) \\ \text{for any open set } \Omega\subseteq\mathbb{R}^2\equiv\mathbb{C} \text{ and any two} \\ \text{complex-valued functions } \ u_1,u_2\in\mathscr{C}^2(\Omega). \end{gathered} \tag{3.247}$$

As a consequence,

$$\begin{gathered} L(u_1,u_2)=0 \Longleftrightarrow L^\top(u_1,-u_2)=0 \\ \text{for any open set } \Omega\subseteq\mathbb{R}^2\equiv\mathbb{C} \text{ and any} \\ \text{complex-valued functions } \ u_1,u_2\in\mathscr{C}^2(\Omega). \end{gathered} \tag{3.248}$$

Pressing on, recall the Cauchy–Riemann operator and its conjugate

$$\partial_{\bar z}:=\tfrac12\big(\partial_x+\mathrm{i}\partial_y\big),\quad \partial_z:=\tfrac12\big(\partial_x-\mathrm{i}\partial_y\big),\quad \text{where } \ z=x+\mathrm{i}y, \tag{3.249}$$

then bring in Bitsadze's operator (cf. [16, 17]), which is simply the square of $\partial_{\bar z}$, i.e.,

$$\mathbb{L}:=\partial_{\bar z}^2=\tfrac14\partial_x^2+\tfrac{\mathrm{i}}{2}\partial_x\partial_y-\tfrac14\partial_y^2,\qquad z=x+\mathrm{i}y. \tag{3.250}$$

To place things into a broader perspective, recall that there are three basic prototypes of scalar, constant coefficient, second-order, elliptic operators in the plane: the Laplacian $4\partial_z\partial_{\bar z}$, plus Bitsadze's operator $\partial_{\bar z}^2$ and its complex conjugate ∂_z^2. With $\pi_1,\pi_2:\mathbb{C}^2\to\mathbb{C}$ the canonical coordinate projections from (3.246), the system L introduced in (3.239) is related to Bitsadze's operator $\mathbb{L}=\partial_{\bar z}^2$ via

$$\begin{gathered} \mathbb{L}(u_1+\mathrm{i}u_2)=\pi_1 L(u_1,u_2)+\mathrm{i}\pi_2 L(u_1,u_2) \\ \text{for any open set } \Omega\subseteq\mathbb{R}^2\equiv\mathbb{C} \text{ and any two} \\ \text{complex-valued functions } \ u_1,u_2\in\mathscr{C}^2(\Omega). \end{gathered} \tag{3.251}$$

In particular,

$$\begin{gathered} L\big(\operatorname{Re}U,\operatorname{Im}U\big)=\big(\operatorname{Re}(\mathbb{L}U),\operatorname{Im}(\mathbb{L}U)\big) \\ \text{for any open set } \Omega\subseteq\mathbb{R}^2\equiv\mathbb{C} \text{ and any} \\ \text{complex-valued function } \ U\in\mathscr{C}^2(\Omega). \end{gathered} \tag{3.252}$$

On the other hand, given any open set $\Omega\subseteq\mathbb{R}^2\equiv\mathbb{C}$ along with any complex-valued function $U\in\mathscr{C}^2(\Omega)$, we have $\partial_{\bar z}^2U=0$ if and only if $f:=-\partial_{\bar z}U$ is holomorphic in Ω, and the latter condition is further equivalent to the demand that $g(z):=U(z)+\bar z f(z)$ for each $z\in\Omega$ is a holomorphic function in Ω. As such, the general format of null-solution of $\partial_{\bar z}^2$ in an open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is

$$U(z) = g(z) - \bar z f(z) \text{ for all } z \in \Omega, \text{ where}$$
$$f \text{ and } g \text{ are holomorphic functions in } \Omega. \tag{3.253}$$

This is akin to the description of affine functions on the real line as null-solutions of the one-dimensional Laplacian $\mathrm{d}^2/\mathrm{d}x^2$, with the role of $\mathrm{d}/\mathrm{d}x$ now played by the Cauchy–Riemann operator $\partial_{\bar z}$, with $\bar z$ now playing the role of the variable x, and with holomorphic functions playing the role of constants.

Specializing the expression of U in (3.253) to the case when $g(z) := zf(z)$ for each $z \in \Omega$, we obtain the following particular family of null-solutions for Bitsadze's operator $\mathbb{L}$ in any given open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$:

$$U(z) = (z - \bar z) f(z), \text{ where } f$$
$$\text{is any holomorphic function in } \Omega. \tag{3.254}$$

From this and (3.252), we then conclude that

> given any holomorphic function f in an open set $\Omega \subseteq \mathbb{C}$, the vector-valued function $u = (u_1, u_2)$ with components given by $u_1(z) := \mathrm{Re}\big[(z - \bar z) f(z)\big]$ and $u_2(z) := \mathrm{Im}\big[(z - \bar z) f(z)\big]$ for each $z \in \Omega$ is a null-solution of the system L from (3.239). (3.255)

In particular, by further specializing this property to the case when $\Omega := \mathbb{R}^2_+ \equiv \mathbb{C}_+$ and the holomorphic function $f(z) := (z + \mathrm{i})^{-m}$ for $z \in \mathbb{C}_+$, where the integer $m \in \mathbb{N}$ is arbitrary, shows that the vector-valued function $u^{(m)} = \big(u_1^{(m)}, u_2^{(m)}\big)$ with components defined at each $z \in \mathbb{C}_+$ as

$$u_1^{(m)}(z) := \mathrm{Re}\big[(z-\bar z)(z+\mathrm{i})^{-m}\big] \text{ and } u_2^{(m)}(z) := \mathrm{Im}\big[(z-\bar z)(z+\mathrm{i})^{-m}\big] \tag{3.256}$$

is a null-solution of the system L from (3.239). Note that each function $u^{(m)}$ belongs to $\big[\mathscr{C}^\infty(\overline{\mathbb{R}^2_+})\big]^2$ and vanishes identically on $\partial\mathbb{R}^2_+ \equiv \mathbb{R}$ (since $z - \bar z = 0$ for each $z \in \mathbb{R}$), and for each multi-index $\alpha \in \mathbb{N}_0^2$, there exists some $C_\alpha \in (0, \infty)$ with the property that

$$\big|\partial^\alpha u^{(m)}(z)\big| \le C_\alpha (1 + |z|)^{1-m-|\alpha|} \text{ for all } z \in \mathbb{R}^2_+. \tag{3.257}$$

The estimate above implies that, having fixed an aperture parameter $\kappa > 0$, for each multi-index $\alpha \in \mathbb{N}_0^2$ there exists some $C_\alpha \in (0, \infty)$ such that

$$\mathcal{N}_\kappa\big(\partial^\alpha u^{(m)}\big)(x) \le C_\alpha (1 + |x|)^{1-m-|\alpha|} \text{ for all } x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{3.258}$$

As such, for any given $p \in (1, \infty)$, any Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and any multi-index $\alpha \in \mathbb{N}_0^2$, we have $\mathcal{N}_\kappa\big(\partial^\alpha u^{(m)}\big) \in L^p(\mathbb{R}, w)$ as long as $m + |\alpha| \geq 2$ (cf. (2.572)). Let us also observe that for each $m \in \mathbb{N}$, we have

$$u_2^{(2m)}(\mathrm{i}y) = 2(-1)^m y(y+1)^{-2m} \quad \text{for each} \quad y \in (0, \infty) \tag{3.259}$$

and that the functions

$$\big\{y(y+1)^{-2m}\big\}_{m\in\mathbb{N}}, \text{ for } 0 < y < \infty, \text{ are linearly independent.} \tag{3.260}$$

Indeed, suppose that for some family of positive integers $m_1 < m_2 < \cdots < m_N$ and nonzero constants $c_1, \dots, c_N$, we have $\sum_{j=1}^N c_j y(y+1)^{-2m_j} = 0$ for each $y > 0$. Divide by $y(y+1)^{-2m_1}$ to obtain $c_1 + \sum_{j=2}^N c_j(y+1)^{-2(m_j - m_1)} = 0$ for each $y \in (0, \infty)$. Sending $y \to \infty$ yields $c_1 = 0$, a contradiction that establishes (3.260). Ultimately, this proves that the linear space of all vector-valued functions u satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \quad Lu = 0 \text{ in } \mathbb{R}^2_+, \\ \mathcal{N}_\kappa\big(\partial^\alpha u\big) \in L^p(\mathbb{R}, w) \text{ for all } \alpha \in \mathbb{N}_0^2, \\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R} \end{cases} \tag{3.261}$$

is infinite dimensional, i.e.,

the null-space of the Infinite-Order Regularity Problem for the system L (from (3.239)) in $\mathbb{R}^2_+$ is infinite dimensional. (3.262)

In particular,

the space of null-solutions of the corresponding Dirichlet Problem for the system L in $\mathbb{R}^2_+$ (formulated as in (1.76) with $n = 2$, $M = 2$, L as in (3.239), and $\Omega := \mathbb{R}^2_+$) is infinite dimensional. (3.263)

Since in item *(d)* of Theorem 6.2 we shall learn that this cannot happen if $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$, we then conclude that we necessarily have $\mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing$ in this case. In other words, $L^\top$ from (3.245) is a weakly elliptic, second-order, homogeneous, real constant coefficient, 2×2 system in $\mathbb{R}^2$ which does not possess any distinguished coefficient tensor.

We may also run a variant of this argument, in which we now start with $L^\top$ instead of L. If

$$\overline{\mathbb{L}} = \partial_z^2 = \tfrac{1}{4}\partial_x^2 - \tfrac{\mathrm{i}}{2}\partial_x\partial_y - \tfrac{1}{4}\partial_y^2 \tag{3.264}$$

is the complex conjugate of Bitsadze's operator $\mathbb{L}$ from (3.250), then in place of (3.251)–(3.252) we now have

$$\overline{\mathbb{L}}(u_1 + \mathrm{i}u_2) = \pi_1 L^\top(u_1, u_2) + \mathrm{i}\pi_2 L^\top(u_1, u_2) \quad \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two complex-valued functions } u_1, u_2 \in \mathscr{C}^2(\Omega), \tag{3.265}$$

and, respectively,

$$L^\top\big(\mathrm{Re}\, U, \mathrm{Im}\, U\big) = \big(\mathrm{Re}(\overline{\mathbb{L}}U), \mathrm{Im}\,(\overline{\mathbb{L}}U)\big) \quad \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any complex-valued function } U \in \mathscr{C}^2(\Omega). \tag{3.266}$$

Keeping in mind that U is a null-solution of $\overline{\mathbb{L}}$ if and only if $\overline{U}$ is a null-solution of $\mathbb{L}$ and reasoning as before, we conclude that, for each $m \in \mathbb{N}$, the vector-valued function $v^{(m)} = \big(v_1^{(m)}, v_2^{(m)}\big)$ with components defined at each $z \in \mathbb{C}_+$ as

$$v_1^{(m)}(z) := \mathrm{Re}\big[(\bar{z}-z)(\bar{z}-\mathrm{i})^{-m}\big] \text{ and } v_2^{(m)}(z) := \mathrm{Im}\big[(\bar{z}-z)(\bar{z}-\mathrm{i})^{-m}\big] \tag{3.267}$$

is a null-solution of the system $L^\top$ from (3.245). In turn, this goes to show that the null-space of the Infinite-Order Regularity Problem for the system $L^\top$ in $\mathbb{R}^2_+$ (formulated as in (3.261) with $L^\top$ now replacing L) is infinite dimensional. Once this has been established, from item *(c)* in Theorem 6.2 we conclude that $\mathfrak{A}^{\mathrm{dis}}_L = \varnothing$. The bottom line is that

$$L \text{ in (3.239) is an example of a weakly elliptic, second-order, homogeneous, real constant coefficient, } 2 \times 2 \text{ system in } \mathbb{R}^2 \text{, with the property that } \mathfrak{A}^{\mathrm{dis}}_L = \varnothing \text{ and } \mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing. \tag{3.268}$$

In particular, this goes to show that not every weakly elliptic, second-order, homogeneous, real constant coefficient, system has a distinguished coefficient tensor.

Remark 3.1 There is yet another proof of (3.268) which is not based on well-posedness results, but instead uses directly the algebraic characterization of distinguished coefficient tensors in Proposition 3.8. Specifically, to conclude that $\mathfrak{A}^{\mathrm{dis}}_L = \varnothing$, from (3.185), it suffices to show that for every coefficient tensor $B = \big(b^{\alpha\beta}_{jk}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$ such that $L = L_B$ there exist indices $k, k' \in \{1, \dots, n\}$ as well as $\alpha, \gamma \in \{1, \dots, M\}$ such that

$$\int_{S^1} \big(b^{\beta\alpha}_{jk}\xi_{k'} - b^{\beta\alpha}_{jk'}\xi_k\big)\xi_j\, \mathcal{E}_{\gamma\beta}(\xi)\, \mathrm{d}\mathcal{H}^1(\xi) \neq 0. \tag{3.269}$$

To this end, we first note that, using (3.181), (3.241), and (3.242), we have

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\le\gamma,\beta\le M} = \big[L(\xi)\big]^{-1} = \left[\frac{-1}{4}\begin{pmatrix}\xi_1^2-\xi_2^2 & -2\xi_1\xi_2\\ 2\xi_1\xi_2 & \xi_1^2-\xi_2^2\end{pmatrix}\right]^{-1}$$

$$= \frac{-4}{|\xi|^4}\begin{pmatrix}\xi_1^2-\xi_2^2 & 2\xi_1\xi_2\\ -2\xi_1\xi_2 & \xi_1^2-\xi_2^2\end{pmatrix}. \tag{3.270}$$

In particular, if $\xi\in S^1$, then $\xi=(\cos\theta,\sin\theta)$ for some $\theta\in[0,2\pi)$ and hence

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\le\gamma,\beta\le M} = -4\begin{pmatrix}\cos^2(\theta)-\sin^2(\theta) & 2\cos(\theta)\sin(\theta)\\ -2\cos(\theta)\sin(\theta) & \cos^2(\theta)-\sin^2(\theta)\end{pmatrix}$$

$$= -4\begin{pmatrix}\cos(2\theta) & \sin(2\theta)\\ -\sin(2\theta) & \cos(2\theta)\end{pmatrix}. \tag{3.271}$$

In order to facilitate the presentation, for each $j,k,\gamma,\beta\in\{1,2\}$ introduce

$$\mathrm{I}_{jk}^{\gamma\beta} := \int_{S^1}\xi_j\,\xi_k\,\mathcal{E}_{\gamma\beta}\,d\mathcal{H}^1(\xi). \tag{3.272}$$

Then using elementary trigonometric formulas, we obtain

$$\mathrm{I}_{11}^{11} = -4\int_0^{2\pi}\cos^2(\theta)\cos(2\theta)\,d\theta = -\int_0^{2\pi}\big(2\cos(2\theta)+\cos(4\theta)+1\big)\,d\theta = -2\pi, \tag{3.273}$$

$$\mathrm{I}_{12}^{11} = -4\int_0^{2\pi}\sin(\theta)\cos(\theta)\cos(2\theta)\,d\theta = -\int_0^{2\pi}\sin(4\theta)\,d\theta = 0, \tag{3.274}$$

$$\mathrm{I}_{22}^{11} = -4\int_0^{2\pi}\sin^2(\theta)\cos(2\theta)\,d\theta = -4\int_0^{2\pi}\cos(2\theta)\,d\theta + \mathrm{I}_{11}^{11} = 2\pi, \tag{3.275}$$

$$\mathrm{I}_{22}^{12} = -8\int_0^{2\pi}\sin^3(\theta)\cos(\theta)\,d\theta = -2(\sin^4(2\pi)-\sin^4(0)) = 0, \tag{3.276}$$

$$\mathrm{I}_{12}^{12} = -8\int_0^{2\pi}\sin^2(\theta)\cos^2(\theta)\,d\theta = -\int_0^{2\pi}\big(1-\cos(4\theta)\big)\,d\theta = -2\pi, \tag{3.277}$$

$$\mathrm{I}_{11}^{12} = -8\int_0^{2\pi}\sin(\theta)\cos^3(\theta)\,d\theta = 2(\cos^4(2\pi)-\cos^4(0)) = 0. \tag{3.278}$$

Finally, from (3.271)–(3.272), it follows that

$$\begin{aligned}&\mathrm{I}^{22}_{11} = \mathrm{I}^{11}_{11} = -2\pi, && \mathrm{I}^{22}_{12} = \mathrm{I}^{11}_{12} = 0, && \mathrm{I}^{22}_{22} = \mathrm{I}^{11}_{22} = 2\pi,\\ &\mathrm{I}^{21}_{11} = -\mathrm{I}^{12}_{11} = 0, && \mathrm{I}^{21}_{12} = -\mathrm{I}^{12}_{12} = 2\pi, && \mathrm{I}^{21}_{22} = -\mathrm{I}^{12}_{22} = 0.\end{aligned} \tag{3.279}$$

We are now ready to compute the integral in (3.269) with $k = \alpha = 1$ and $k' = \gamma = 2$:

$$\begin{aligned}\int_{S^1} \big(b^{\beta 1}_{j1}\xi_2 - b^{\beta 1}_{j2}\xi_1\big)\xi_j\, \mathcal{E}_{2\beta}(\xi)\, \mathrm{d}\mathcal{H}^1(\xi) &= b^{11}_{11}\cdot \mathrm{I}^{21}_{12} + b^{11}_{21}\cdot \mathrm{I}^{21}_{22} + b^{21}_{11}\cdot \mathrm{I}^{22}_{12}\\ &\quad + b^{21}_{21}\cdot \mathrm{I}^{22}_{22} - b^{11}_{12}\cdot \mathrm{I}^{21}_{11} - b^{11}_{22}\cdot \mathrm{I}^{21}_{12} - b^{21}_{12}\cdot \mathrm{I}^{22}_{11} - b^{21}_{22}\cdot \mathrm{I}^{22}_{12}\\ &= 2\pi(b^{11}_{11} + b^{21}_{21} - b^{11}_{22} + b^{21}_{12}).\end{aligned} \tag{3.280}$$

Next, we use the fact that B may be expressed as $B = A + C$, where A is a fixed coefficient tensor such that $L = L_A$ and C is a coefficient tensor which is antisymmetric in the lower indices. In particular, taking A as in (3.240), we conclude from (3.280) that

$$\begin{aligned}&\int_{S^1} \big(b^{\beta 1}_{j1}\xi_2 - b^{\beta 1}_{j2}\xi_1\big)\xi_j\, \mathcal{E}_{2\beta}(\xi)\, \mathrm{d}\mathcal{H}^1(\xi)\\ &\qquad = 2\pi(a^{11}_{11} + a^{21}_{21} - a^{11}_{22} + a^{21}_{12} + c^{11}_{11} + c^{21}_{21} - c^{11}_{22} + c^{21}_{12})\\ &\qquad = 2\pi\Big(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 0 + c^{21}_{21} - 0 + c^{21}_{12}\Big)\\ &\qquad = 2\pi \neq 0.\end{aligned} \tag{3.281}$$

This justifies (3.269) and ultimately proves that $\mathfrak{A}^{\mathrm{dis}}_{L} = \varnothing$. The same argument as above works for $L^\top$, so we also conclude that $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$.

In relation to the system L from (3.239), it is of interest to identify the space of boundary traces of its null-solutions in the upper half-plane whose nontangential maximal function belongs to a Muckenhoupt weighted Lebesgue space.

Proposition 3.10 *Fix an integrability index $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and choose an aperture parameter $\kappa > 0$. Also, recall the 2×2 system L defined in the plane as in (3.239).*

Then if $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function satisfying

$$Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \tag{3.282}$$

and such that the nontangential boundary trace

$$f := u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} \ \textit{ exists (in } \mathbb{C}^2\textit{) at } \mathcal{L}^1\textit{-a.e. point on } \ \mathbb{R}, \tag{3.283}$$

it follows that the function f *belongs to* $\big[L^p(\mathbb{R}, w)\big]^2$ *and, if* $f_1, f_2 \in L^p(\mathbb{R}, w)$ *are the scalar components of* f *(i.e.,* $f = (f_1, f_2)$*), then with* H *denoting the Hilbert transform on the real line (cf. (1.24)) one has*

$$Hf_1 = f_2 \ \textit{ at } \mathcal{L}^1\textit{-a.e. point on } \ \mathbb{R}. \tag{3.284}$$

In the converse direction, for any given $f \in L^p(\mathbb{R}, w)$*, there exists a vector-valued function* $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ *satisfying*

$$\begin{aligned} &Lu = 0 \ \textit{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \quad \textit{and} \\ &u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = (f, Hf) \ \textit{ at } \mathcal{L}^1\textit{-a.e. point on } \ \mathbb{R}. \end{aligned} \tag{3.285}$$

Altogether, the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L *in the upper half-plane, i.e.,*

$$\begin{aligned} \Big\{u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} :\ u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ Lu = 0 \ \textit{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ \textit{and } \ u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} \ \textit{ exists at } \mathcal{L}^1\textit{-a.e. point on } \ \mathbb{R}\Big\}, \end{aligned} \tag{3.286}$$

is precisely

$$\big\{(f, Hf) :\ f \in L^p(\mathbb{R}, w)\big\}. \tag{3.287}$$

As a consequence of this and (3.248), one also has

$$\begin{aligned} \Big\{u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} :\ u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ L^\top u = 0 \ \textit{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ \textit{and } \ u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} \ \textit{ exists at } \mathcal{L}^1\textit{-a.e. point on } \ \mathbb{R}\Big\} \\ = \big\{(f, -Hf) :\ f \in L^p(\mathbb{R}, w)\big\}. \end{aligned} \tag{3.288}$$

Proof That the function f belongs to $\big[L^p(\mathbb{R}, w)\big]^2$ is clear from $\Big|u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}\Big| \le \mathcal{N}_\kappa u$, the fact that $u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}}$ is $\mathcal{L}^1$-measurable (cf. [111, §8.9]), and the last property in (3.282).

To proceed, fix a function $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying (3.282)–(3.283) and denote by $u_1, u_2 \in \mathscr{C}^\infty(\mathbb{R}^2_+)$ its scalar components. Hence, $u = (u_1, u_2)$ in $\mathbb{R}^2_+$. Also, pick an arbitrary $\varepsilon > 0$ and define

$$U_\varepsilon(z) := u_1(z + \varepsilon \mathrm{i}) + \mathrm{i} u_2(z + \varepsilon \mathrm{i}) \quad \text{for each} \quad z \in \big(\mathbb{R}^2_+ - \varepsilon \mathrm{i}\big). \tag{3.289}$$

Then $U_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^2_+ - \varepsilon\mathrm{i})$ and, as seen from (3.251), the fact that $Lu = 0$ in $\mathbb{R}^2_+$ translates into $\partial_{\bar z}^2 U_\varepsilon = 0$ in $\mathbb{R}^2_+ - \varepsilon\mathrm{i}$. Granted this, (3.253) then guarantees the existence of two holomorphic functions $f_\varepsilon, g_\varepsilon$ in $\mathbb{R}^2_+ - \varepsilon\mathrm{i}$ with the property that

$$U_\varepsilon(z) = g_\varepsilon(z) - \bar z f_\varepsilon(z) \quad \text{for each} \quad z \in \big(\mathbb{R}^2_+ - \varepsilon \mathrm{i}\big). \tag{3.290}$$

More specifically, the unique holomorphic functions $f_\varepsilon, g_\varepsilon$ which do the job in (3.290) are

$$f_\varepsilon(z) := -\partial_{\bar z} U_\varepsilon(z) \quad \text{and} \quad g_\varepsilon(z) := U_\varepsilon(z) + \bar z f_\varepsilon(z) \quad \text{for each} \quad z \in \big(\mathbb{R}^2_+ - \varepsilon \mathrm{i}\big). \tag{3.291}$$

Henceforth, we agree to restrict $U_\varepsilon, f_\varepsilon, g_\varepsilon$ to $\mathbb{R}^2_+$. With this interpretation, introduce

$$W_\varepsilon(z) := g_\varepsilon(z) - z f_\varepsilon(z) \quad \text{for each} \quad z \in \mathbb{R}^2_+. \tag{3.292}$$

Hence, W_ε is holomorphic in $\mathbb{R}^2_+$ and extends continuously to $\overline{\mathbb{R}^2_+}$, and

$$\begin{aligned} U_\varepsilon(z) - W_\varepsilon(z) = 2\mathrm{i} y f_\varepsilon(z) = -2\mathrm{i} y \big(\partial_{\bar z} U_\varepsilon\big)(z) \\ \text{for each} \quad z = x + \mathrm{i} y \in \mathbb{R}^2_+. \end{aligned} \tag{3.293}$$

From the fact that $\partial_{\bar z}^2 U_\varepsilon = 0$ in $\mathbb{R}^2_+$, we also conclude that $0 = \partial_z^2 \partial_{\bar z}^2 U_\varepsilon = \frac{1}{16}\Delta^2 U_\varepsilon$, i.e., the function U_ε is bi-harmonic in $\mathbb{R}^2_+$. Select $\theta \in (0,1)$ and $\widetilde{\kappa} \in (0,\kappa)$ both small so that

$$\frac{1 + \theta + \widetilde{\kappa}}{1 - \theta} < 1 + \kappa. \tag{3.294}$$

Fix an arbitrary point $x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+$ and pick some $z_o = x_o + \mathrm{i} y_o \in \Gamma_{\widetilde{\kappa}}(x)$. The inequality demanded in (3.294) ensures that

$$B(z_o, \theta y_o) \subseteq \Gamma_\kappa(x). \tag{3.295}$$

Based on interior estimates for bi-harmonic functions (cf. [102, Theorem 11.12, p. 415]), (3.293), and (3.295), we may then write

$$\begin{aligned} \big|U_\varepsilon(z_o) - W_\varepsilon(z_o)\big| &= 2y_o \big|\big(\partial_{\bar z} U_\varepsilon\big)(z_o)\big| \leq \sqrt{2} y_o |(\nabla U_\varepsilon)(z_o)| \\ &\leq C \fint_{B(z_o,\theta y_o)} |U_\varepsilon| \, \mathrm{d}\mathcal{L}^2 \leq C\big(\mathcal{N}_\kappa U_\varepsilon\big)(x), \end{aligned} \tag{3.296}$$

for some constant $C = C(\theta) \in (0, \infty)$. Taking the supremum over all $z_o \in \Gamma_{\widetilde{\kappa}}(x)$ this ultimately yields

$$\big(\mathcal{N}_{\widetilde{\kappa}}(U_\varepsilon - W_\varepsilon)\big)(x) \le C\big(\mathcal{N}_\kappa U_\varepsilon\big)(x) \quad \text{for each} \quad x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{3.297}$$

In turn, (3.297) implies

$$\begin{aligned}\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon &\le \mathcal{N}_{\widetilde{\kappa}} U_\varepsilon + \mathcal{N}_{\widetilde{\kappa}}(U_\varepsilon - W_\varepsilon) \le \mathcal{N}_\kappa U_\varepsilon + C\mathcal{N}_\kappa U_\varepsilon \\ &= (1+C)\mathcal{N}_\kappa U_\varepsilon \le (1+C)\mathcal{N}_\kappa u \quad \text{on} \quad \mathbb{R} \equiv \partial\mathbb{R}^2_+. \end{aligned}\tag{3.298}$$

Upon recalling that the nontangential maximal function $\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon$ is non-negative and lower-semicontinuous, we then conclude from (3.298), the last property in (3.282), and (2.575) that

$$\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon \in L^1\Big(\mathbb{R}, \tfrac{\mathcal{L}^1(x)}{1+|x|}\Big). \tag{3.299}$$

Let us record our progress. The argument so far shows that the function W_ε is holomorphic in $\mathbb{R}^2_+$ and extends continuously to $\overline{\mathbb{R}^2_+}$, and there exists some aperture parameter $\widetilde{\kappa} > 0$ such that $\mathcal{N}_{\widetilde{\kappa}} W_\varepsilon$ belongs to $L^1\Big(\mathbb{R}, \tfrac{\mathcal{L}^1(x)}{1+|x|}\Big)$. These properties allow us to invoke the Cauchy reproducing formula (proved in [113, §1.1] in much more general geometric settings) which asserts that

$$W_\varepsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\big(W_\varepsilon\big|_{\mathbb{R}}\big)(y)}{y - z}\, dy, \quad \text{for each} \quad z \in \mathbb{R}^2_+. \tag{3.300}$$

Since $f_\varepsilon, g_\varepsilon$ extend continuously to $\overline{\mathbb{R}^2_+}$, from (3.290), (3.292), and the fact that $z = \bar{z}$ on $\mathbb{R} \equiv \partial\mathbb{R}^2_+$, we conclude that

$$W_\varepsilon\big|_{\mathbb{R}} = U_\varepsilon\big|_{\mathbb{R}} \quad \text{on} \quad \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{3.301}$$

As such, if we abbreviate

$$h_\varepsilon := U_\varepsilon\big|_{\mathbb{R}} \quad \text{on} \quad \mathbb{R} \equiv \partial\mathbb{R}^2_+, \tag{3.302}$$

after taking the nontangential boundary traces of both sides in (3.300) and using the Plemelj jump-formula for the Cauchy operator (which continues to be valid in this setting; see [114, §1.6]), we arrive at

$$h_\varepsilon = \big(\tfrac{1}{2}I + \tfrac{\mathrm{i}}{2}H\big)h_\varepsilon \quad \text{at } \mathcal{L}^1\text{-a.e. point on} \quad \mathbb{R}, \tag{3.303}$$

where I is the identity and H is the Hilbert transform on $\mathbb{R}$. Hence, on the one hand, we may rewrite (3.303) simply as

$$Hh_\varepsilon = -\mathrm{i}h_\varepsilon \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.304}$$

On the other hand, from (3.302) and (3.289), we see that

$$h_\varepsilon(x) = u_1(x+\varepsilon\mathrm{i}) + \mathrm{i}u_2(x+\varepsilon\mathrm{i}) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}. \tag{3.305}$$

In turn, this implies

$$|h_\varepsilon(x)| \leq \sqrt{2}\big(\mathcal{N}_\kappa u\big)(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R} \tag{3.306}$$

and, when used in concert with (3.283), that

$$\begin{aligned} \lim_{\varepsilon\to 0^+} h_\varepsilon(x) &= \big(u_1\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\big)(x) + \mathrm{i}\big(u_2\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\big)(x) \\ &= f_1(x) + \mathrm{i}f_2(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}. \end{aligned} \tag{3.307}$$

Thanks to (3.306)–(3.307) and the last property in (3.282), we may invoke Lebesgue's Dominated Convergence Theorem to conclude that

$$\lim_{\varepsilon\to 0^+} h_\varepsilon = f_1 + \mathrm{i}f_2 \quad \text{in } L^p(\mathbb{R}, w). \tag{3.308}$$

Having established this, on account of (3.304) and the continuity of the Hilbert transform H on the Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R}, w)$, we obtain

$$H(f_1 + \mathrm{i}f_2) = -\mathrm{i}(f_1 + \mathrm{i}f_2) \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.309}$$

The idea is now write $u = \operatorname{Re} u + \mathrm{i}\operatorname{Im} u$ and observe that, since the coefficients of the system L are real, $\operatorname{Re} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ and $\operatorname{Im} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ enjoy the same properties as the function u in (3.282)–(3.283). Granted what we have proved already, it follows that if ϕ_1, ϕ_2 are the scalar components of $(\operatorname{Re} u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ and if ψ_1, ψ_2 are the scalar components of $(\operatorname{Im} u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ then $\phi_1, \phi_2, \psi_1, \psi_2$ are real-valued functions belonging to $L^p(\mathbb{R}, w)$, and the conclusion in (3.309) written separately for $\operatorname{Re} u$ and $\operatorname{Im} u$ gives

$$H(\phi_1 + \mathrm{i}\phi_2) = -\mathrm{i}(\phi_1 + \mathrm{i}\phi_2) \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \tag{3.310}$$

and, respectively,

$$H(\psi_1 + \mathrm{i}\psi_2) = -\mathrm{i}(\psi_1 + \mathrm{i}\psi_2) \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.311}$$

In particular, taking the real parts in (3.310)–(3.311) (keeping in mind that H maps real-valued functions into real-valued functions) leads to the conclusion that

$$H\phi_1 = \phi_2 \text{ and } H\psi_1 = \psi_2. \tag{3.312}$$

Upon observing that $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (\operatorname{Re} u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} + \mathrm{i}(\operatorname{Im} u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ implies $f_1 = \phi_1 + \mathrm{i}\psi_1$ and $f_2 = \phi_2 + \mathrm{i}\psi_2$, from (3.312), we readily obtain the formula claimed in (3.284).

In the converse direction, suppose first that the function $f \in L^p(\mathbb{R}, w)$ is real-valued. Then $Hf \in L^p(\mathbb{R}, w)$ and work in [114, §1.5-§1.6] ensures that

$$U(z) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{(f + \mathrm{i}Hf)(y)}{y - z}\, \mathrm{d}y, \quad \text{for each } z \in \mathbb{R}^2_+, \tag{3.313}$$

is a holomorphic function in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ and

$$U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \big(\tfrac{1}{2}I + \tfrac{\mathrm{i}}{2}H\big)(f + \mathrm{i}Hf) = f + \mathrm{i}Hf \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \tag{3.314}$$

since the Hilbert transform satisfies $H^2 = -I$ on $L^p(\mathbb{R}, w)$. If we now introduce $u_1 := \operatorname{Re} U$ and $u_2 := \operatorname{Im} U$, then $u := (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function with real-valued scalar components. Thanks to (3.252), we have

$$Lu = L\big(\operatorname{Re} U, \operatorname{Im} U\big) = \big(\operatorname{Re}(\partial_{\bar z}^2 U), \operatorname{Im}(\partial_{\bar z}^2 U)\big) = 0 \in \mathbb{C}^2 \text{ in } \mathbb{R}^2_+, \tag{3.315}$$

since $\partial_{\bar z} U = 0$ in $\mathbb{R}^2_+$ by the Cauchy–Riemann equations. In addition, we observe that $\mathcal{N}_\kappa u = \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ given that, by design, $|u| = |U|$. Finally, at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we have

$$u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \Big(\operatorname{Re} U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}, \operatorname{Im} U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = (f, Hf), \tag{3.316}$$

by virtue of (3.314) and the fact that f is real-valued. Thus, u satisfies all requirements in (3.285).

To deal with an arbitrary function $f \in L^p(\mathbb{R}, w)$, which is not necessarily real-valued, denote by ϕ and ψ its real and imaginary parts so that $f = \phi + \mathrm{i}\psi$. From what we have proved so far, there exist $v, \omega \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ as in (3.285) such that $v\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (\phi, H\phi)$ and $\omega\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (\psi, H\psi)$. Then it follows that the function $u := v + \mathrm{i}\omega \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is as in (3.285) and satisfies $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (f, Hf)$, as wanted. □

We continue by making four remarks in relation to Proposition 3.10 and its proof.

Remark 3.2 Suppose $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ for some exponent $p \in (1, \infty)$ and choose an aperture parameter $\kappa > 0$. Also, let L be the 2×2 system from (3.239), and assume $u : \mathbb{R}^2_+ \to \mathbb{C}^2$ is a function satisfying

$$u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \quad Lu = 0 \text{ in } \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w),$$
$$\text{and } u\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = 0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.317}$$

In particular, u satisfies (3.282)–(3.283) with $f = (f_1, f_2) = (0, 0)$. Retaining notation introduced during the proof of Proposition 3.10, from (3.300), (3.301), and (3.302), we see that

$$W_\varepsilon(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{h_\varepsilon(y)}{y - z}\, dy, \quad \text{for each } z \in \mathbb{R}^2_+. \tag{3.318}$$

Let $U := u_1 + \mathrm{i}u_2$, where u_1 and u_2 are the two scalar components of the $\mathbb{C}^2$-valued function u. On the one hand, from (3.289)–(3.292), it is clear that

$$\lim_{\varepsilon\to 0^+} W_\varepsilon(z) = U(z) + (z - \bar{z})\big(\partial_{\bar{z}} U\big)(z) \text{ for fixed each } z \in \mathbb{R}^2_+. \tag{3.319}$$

On the other hand, for each fixed $z \in \mathbb{R}^2_+$, on account of (3.308) and the fact that we currently have $f_1 + \mathrm{i} f_2 = 0$, we conclude that the limit as $\varepsilon \to 0^+$ of the integral in (3.318) is zero. Based on these observations and (3.318), we ultimately conclude that

if u is as in (3.317) then the $\mathbb{C}$-valued function $U := u_1 + \mathrm{i}u_2$ (where u_1, u_2 are the two scalar components of the $\mathbb{C}^2$-valued function u) satisfies $U(z) = (\bar{z} - z)\big(\partial_{\bar{z}} U\big)(z)$ for each $z \in \mathbb{R}^2_+$. (3.320)

The same type of argument also shows that

$$\left.\begin{array}{r} U \in \mathscr{C}^\infty(\mathbb{R}^2_+) \\ \partial_{\bar{z}}^2 U = 0 \text{ in } \mathbb{R}^2_+ \\ \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w) \\ U\Big|_{\partial\mathbb{R}^2_+}^{\kappa-\text{n.t.}} = 0 \text{ on } \mathbb{R} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} U(z) = (\bar{z} - z)\big(\partial_{\bar{z}} U\big)(z) \\ \text{for all } z \in \mathbb{R}^2_+. \end{array}\right. \tag{3.321}$$

Bearing in mind that for any U as in the left side of (3.321) the function $f := -\partial_{\bar{z}} U$ is holomorphic in $\mathbb{R}^2_+$, we may recast the conclusion in (3.321) as saying that there exists some holomorphic function f in $\mathbb{R}^2_+$ such that $U(z) = (z - \bar{z}) f(z)$ for each $z \in \mathbb{R}^2_+$. In particular, this shows that the choice $g(z) := z f(z)$ which has led to the conclusion in (3.254) is actually canonical in the case when $\Omega = \mathbb{R}^2_+$, the nontangential trace of U vanishes, and the nontangential maximal function of U belongs to a Muckenhoupt weighted Lebesgue space.

Remark 3.3 A version of (3.321) which involves the nontangential maximal operator of the gradient of the function U goes as follows:

$$\begin{aligned}&\text{given any function } U\in\mathscr{C}^\infty(\mathbb{R}^2_+) \text{ with } \partial_{\bar z}^2 U=0 \text{ in } \mathbb{R}^2_+ \text{ and}\\ &\mathcal{N}_\kappa(\nabla U)\in L^p(\mathbb{R},w) \text{ for some } p\in(1,\infty),\ w\in A_p(\mathbb{R},\mathcal{L}^1),\\ &\text{and } \kappa\in(0,\infty) \text{ then } U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}=0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R} \text{ if and}\\ &\text{only if } U(z)=(\bar z-z)\big(\partial_{\bar z}U\big)(z) \text{ for all } z\in\mathbb{R}^2_+.\end{aligned} \tag{3.322}$$

Indeed, the left-pointing implication is a consequence of the fact that (see the Fatou-type result recalled in Theorem 3.4) the nontangential boundary trace

$$(\partial_{\bar z}U)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.323}$$

To justify the right-pointing implication in (3.322), define

$$W(z):=U(z)-(\bar z-z)\big(\partial_{\bar z}U\big)(z) \quad \text{for all} \quad z\in\mathbb{R}^2_+, \tag{3.324}$$

and note that, from assumptions and (3.323), we have

$$\begin{aligned}&W\in\mathscr{C}^\infty(\mathbb{R}^2_+),\ \ \partial_{\bar z}W=0 \ \text{ in } \ \mathbb{R}^2_+, \ \text{ and}\\ &W\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}=0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}.\end{aligned} \tag{3.325}$$

In addition, based on assumptions and interior estimates, we conclude (by reasoning much as in (3.293)–(3.297)) that

$$\mathcal{N}_\kappa(\nabla W)\in L^p(\mathbb{R},w). \tag{3.326}$$

To proceed, we find it useful to bring in a modified boundary-to-domain Cauchy integral operator for the upper half-plane acting on each $f\in\dot{L}^p_1(\mathbb{R},w)$ according to

$$\mathcal{C}_{\text{mod}}f(z):=\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\Big\{\frac{1}{y-z}-\frac{\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(y)}{y}\Big\}f(y)\,\mathrm{d}y \quad \text{for all} \quad z\in\mathbb{R}^2_+. \tag{3.327}$$

Work in [114, §1.8] then shows that W may be recovered, up to an additive constant, from the action of this modified Cauchy operator on the boundary trace of W. In the present case, this guarantees the existence of some $c\in\mathbb{C}$ such that

$$W=\mathcal{C}_{\text{mod}}\Big(W\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big)+c \quad \text{in} \quad \mathbb{R}^2_+, \tag{3.328}$$

hence $W\equiv c$ in $\mathbb{R}^2_+$, thanks to the last property recorded in (3.325). In turn, this forces $c=W\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}=0$ hence, ultimately, $W=0$ in $\mathbb{R}^2_+$. In view of the definition of W, this finishes the proof of the right-pointing implication in (3.322). In closing, we wish to note that, having fixed $p \in (1, \infty)$, $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and $\kappa \in (0, \infty)$, from (3.322) and the fact that

$$\begin{aligned} &\text{for each holomorphic function } h \text{ in } \mathbb{R}^2_+ \text{ with } \mathcal{N}_\kappa h \in L^p(\mathbb{R}, w), \\ &\text{the nontangential boundary trace } h\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists } \mathcal{L}^1\text{-a.e. on } \mathbb{R} \end{aligned} \tag{3.329}$$

(e.g., this is implied by the Fatou results proved in [113, §3.1]), we conclude that

$$\begin{aligned} \Big\{ U \in \mathscr{C}^\infty(\mathbb{R}^2_+) :\ \partial_{\bar z}^2 U = 0 \ \text{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w), \ \text{ and} \\ U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R} \Big\} \\ = \Big\{ (\bar z - z) h(z) :\ h \text{ holomorphic in } \mathbb{R}^2_+ \text{ with } \ \mathcal{N}_\kappa h \in L^p(\mathbb{R}, w) \Big\}. \end{aligned} \tag{3.330}$$

This provides an explicit description of the space of null-solutions of the Homogeneous Regularity Problem for the operator $\partial_{\bar z}^2$ in the upper half-plane. In turn, this readily implies that the space of null-solutions of the Inhomogeneous Regularity Problem for the operator $\partial_{\bar z}^2$ in the upper half-plane may be described as

$$\begin{aligned} \Big\{ U \in \mathscr{C}^\infty(\mathbb{R}^2_+) : \partial_{\bar z}^2 U = 0 \ \text{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa U,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w), \ \text{ and} \\ U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R} \Big\} \\ = \Big\{ (\bar z - z) h(z) :\ h \text{ holomorphic in } \mathbb{R}^2_+ \text{ with } \ \mathcal{N}_\kappa h \in L^p(\mathbb{R}, w) \\ \text{and } \ \mathcal{N}_\kappa\big(\mathbb{R}^2_+ \ni z \mapsto (\bar z - z) h(z)\big) \in L^p(\mathbb{R}, w) \Big\}. \end{aligned} \tag{3.331}$$

Finally, we wish to mention that it is also possible to describe the space of null-solutions of the Dirichlet Problem for the operator $\partial_{\bar z}^2$ in the upper half-plane, namely

$$\begin{aligned} \Big\{ U \in \mathscr{C}^\infty(\mathbb{R}^2_+) :\ \partial_{\bar z}^2 U = 0 \ \text{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w), \ \text{ and } \ U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \Big\} \\ = \Big\{ (\bar z - z) h(z) :\ h \text{ holomorphic in } \mathbb{R}^2_+ \text{ with} \\ \big[\mathbb{R}^2_+ \ni z \mapsto (\bar z - z) h(z)\big]\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \\ \text{and } \ \mathcal{N}_\kappa\big(\mathbb{R}^2_+ \ni z \mapsto (\bar z - z) h(z)\big) \in L^p(\mathbb{R}, w) \Big\}. \end{aligned} \tag{3.332}$$

See [115, Chapter 2] for this and other similar results in the higher-dimensional setting (some of which we will review a little further).

Remark 3.4 Bring in the complexified Cauchy–Riemann equations in the upper half-plane, i.e., consider

$$\begin{gathered} A, B : \mathbb{R}^2_+ \to \mathbb{C} \ \text{ of class } \ \mathscr{C}^\infty, \ \text{ satisfying} \\ \partial_x A = \partial_y B \ \text{ and } \ \partial_y A = -\partial_x B \ \text{ in } \ \mathbb{R}^2_+. \end{gathered} \tag{3.333}$$

Write $(A, B) \in \mathrm{CR}(\mathbb{R}^2_+)$ whenever A, B are as in (3.333). Hence, $\mathrm{CR}(\mathbb{R}^2_+)$ is a complex vector space with the property that for each $(A, B) \in \mathrm{CR}(\mathbb{R}^2_+)$ we have

$$\begin{gathered} \big(\mathrm{Re}\, A, \mathrm{Re}\, B\big) \in \mathrm{CR}(\mathbb{R}^2_+), \quad \big(\mathrm{Im}\, A, \mathrm{Im}\, B\big) \in \mathrm{CR}(\mathbb{R}^2_+), \\ \text{and } A + \mathrm{i}B \text{ is a holomorphic function in } \mathbb{R}^2_+. \end{gathered} \tag{3.334}$$

Also,

$$\begin{gathered} \big(\mathrm{Re}\, U, \mathrm{Im}\, U\big) \in \mathrm{CR}(\mathbb{R}^2_+) \ \text{ for each} \\ \text{holomorphic function } U \text{ in } \mathbb{R}^2_+. \end{gathered} \tag{3.335}$$

Having fixed some $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ and an aperture parameter $\kappa > 0$, we claim that

$$\begin{aligned} &\big\{(f, Hf) : \ f \in L^p(\mathbb{R}, w)\big\} \\ &= \Big\{\Big(A\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}, B\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) : \ (A, B) \in \mathrm{CR}(\mathbb{R}^2_+), \ \mathcal{N}_\kappa A, \ \mathcal{N}_\kappa B \in L^p(\mathbb{R}, w)\Big\}. \end{aligned} \tag{3.336}$$

Formula (3.336) carries special significance in the present context. Indeed, in view of Proposition 3.10, we conclude that

> the space (described in (3.286)) of admissible boundary data for the Dirichlet Problem in the upper half-plane formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L defined in (3.239) coincides with the space of nontangential boundary traces of pairs of functions satisfying the complexified Cauchy–Riemann equations (3.333) whose nontangential maximal functions belong to said Muckenhoupt weighted Lebesgue spaces. (3.337)

Hence, in the big picture, the space of admissible boundary data for the Dirichlet Problem for the *second-order* system L from (3.239) coincides with the space of boundary traces of null-solutions of a *first-order* system, namely the complexified Cauchy–Riemann equations (3.333).

To justify (3.336), observe that since both sets involved are actually vector spaces over the field of complex numbers and since (3.334)–(3.335) hold, it suffices to show that

$$\begin{aligned}&\big\{(f, Hf) : f \in L^p(\mathbb{R}, w) \ \text{ real-valued}\big\} \\ &\quad = \Big\{\Big(\mathrm{Re}\, U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}, \mathrm{Im}\, U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) : U \ \text{ holomorphic in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)\Big\}.\end{aligned} \tag{3.338}$$

As far as the equality in (3.338) is concerned, work in [113, §3.1] implies that for any holomorphic function U in $\mathbb{R}^2_+$ with $\mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ the nontangential boundary trace $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ exists at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. Also, this trace belongs to $L^p(\mathbb{R}, w)$ and the following Cauchy reproducing formula holds:

$$U(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)(y)}{y - z}\, \mathrm{d}y, \quad \text{for each } z \in \mathbb{R}^2_+. \tag{3.339}$$

Going nontangentially to the boundary then yields

$$U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = \big(\tfrac{1}{2} I + \tfrac{\mathrm{i}}{2} H\big)\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big). \tag{3.340}$$

Hence $U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = \mathrm{i}H\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)$, from which we deduce that

$$\mathrm{Im}\, U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = H\big(\mathrm{Re}\, U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big). \tag{3.341}$$

This proves the right-to-left inclusion in (3.338). As regards the left-to-right inclusion in (3.338), given any real-valued function $f \in L^p(\mathbb{R}, w)$, it follows that

$$U(z) := \frac{1}{\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(y)}{y - z}\, \mathrm{d}y, \quad \text{for each } z \in \mathbb{R}^2_+, \tag{3.342}$$

is holomorphic in $\mathbb{R}^2_+$, has $\mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$, and satisfies $U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = (I + \mathrm{i}H)f$. In particular, $\big(\mathrm{Re}\, U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}, \mathrm{Im}\, U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big) = (f, Hf)$, finishing the proof of (3.338).

Remark 3.5 The space $\big\{(f, Hf) : f \in L^p(\mathbb{R}, w)\big\}$ appearing in (3.287) is the complexification of the space appearing in the first line of (3.338). In turn, via the identification $\mathbb{R}^2 \ni (a, b) \equiv a + \mathrm{i}b \in \mathbb{C}$, the latter space may be viewed as

$$\big\{f + \mathrm{i}Hf : f \in L^p(\mathbb{R}, w) \ \text{ real-valued}\big\}, \tag{3.343}$$

which, by virtue of (3.338), is further equal to the Muckenhoupt weighted Hardy space

$$\Big\{U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\,:\,U\ \text{ holomorphic in }\mathbb{R}^2_+,\ \ \mathcal{N}_\kappa U\in L^p(\mathbb{R},w)\Big\}. \tag{3.344}$$

From Proposition 3.10, we then conclude that the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L in the upper half-plane (cf. (3.286)) is ultimately linked to the Muckenhoupt weighted Hardy space (3.344) in the manner detailed in the above discussion.

By further building on Proposition 3.10, below we identify the space of admissible boundary data for the Muckenhoupt weighted version of the Regularity Problem for the system L from (3.239) in the upper half-plane.

Proposition 3.11 *Fix an integrability index $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\mathbb{R},\mathcal{L}^1)$, and choose an aperture parameter $\kappa>0$. Also, recall the 2×2 system L defined in the plane as in (3.239). Then the space of admissible boundary data for the Muckenhoupt weighted version of the Regularity Problem for the system L in the upper half-plane, i.e.,*

$$\Big\{u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\,:\,u\in\big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ \ Lu=0\ \ in\ \ \mathbb{R}^2_+,\ \ \mathcal{N}_\kappa u,\mathcal{N}_\kappa(\nabla u)\in L^p(\mathbb{R},w)\Big\}, \tag{3.345}$$

coincides with

$$\big\{(f,Hf)\,:\,f\in L^p_1(\mathbb{R},w)\big\}. \tag{3.346}$$

As a consequence of this and (3.248), one also has

$$\Big\{u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\,:\,u\in\big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ L^\top u=0\ \ in\ \ \mathbb{R}^2_+,\ \ \mathcal{N}_\kappa u,\mathcal{N}_\kappa(\nabla u)\in L^p(\mathbb{R},w)\Big\}$$

$$=\big\{(f,-Hf)\,:\,f\in L^p_1(\mathbb{R},w)\big\}. \tag{3.347}$$

That the nontangential boundary traces exist in the context of (3.345), (3.347) is a consequence of Proposition 2.24.

Proof of Proposition 3.11 Consider some function $u=(u_1,u_2)\in\big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying $Lu=0$ in $\mathbb{R}^2_+$, with $\mathcal{N}_\kappa u,\mathcal{N}_\kappa(\nabla u)\in L^p(\mathbb{R},w)$, and such that $u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ exists at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. Proposition 3.10 guarantees that $u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}=(f,Hf)$ for some $f\in L^p(\mathbb{R},w)$. Then actually $f=u_1\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\in L^p_1(\mathbb{R},w)$, thanks to Proposition 2.22 whose applicability with u_1 in place of u and with $\Omega:=\mathbb{R}^2_+$ is ensured by Theorem 3.4. This proves that the nontangential boundary trace $u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}}$ belongs to the space in (3.346).

Conversely, start with a function $f \in L^p_1(\mathbb{R}, w)$, which is first assumed to be real-valued. Work in [114, §1.6] (in more general settings) ensures that $Hf \in L^p_1(\mathbb{R}, w)$ and

$$U(z) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{(f + \mathrm{i}Hf)(y)}{y - z}\, \mathrm{d}y, \quad \text{for each } \ z \in \mathbb{R}^2_+, \tag{3.348}$$

is a holomorphic function in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa U,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$ and, much as in (3.314), $U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} = f + \mathrm{i}Hf$. Then $u := \big(\mathrm{Re}\, U, \mathrm{Im}\, U\big) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function with real-valued scalar components, with the property that $\mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$ and

$$u\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}} = \Big(\mathrm{Re}\, U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}}, \mathrm{Im}\, U\big|_{\partial\mathbb{R}^2_+}^{\kappa-\mathrm{n.t.}}\Big) = (f, Hf), \tag{3.349}$$

since f is real-valued. Given that, much as for (3.315), we also have $Lu = 0$ in $\mathbb{R}^2_+$, it follows that (f, Hf) belongs to the space in (3.345). Finally, the general case when $f \in L^p_1(\mathbb{R}, w)$ is not necessarily real-valued is dealt with based on what we have just proved, decomposing f into its real and imaginary parts. This eventually shows that the space from (3.346) is contained in the space from (3.345). By double inclusion, we may therefore conclude that these spaces are in fact equal. □

There is also a version of Proposition 3.11 for the *Homogeneous Regularity Problem*, involving homogeneous Muckenhoupt weighted Sobolev spaces. To state this result, we shall need the homogeneous Muckenhoupt weighted Sobolev space $\dot{L}^p_1(\mathbb{R}, w)$ defined for each integrability exponent $p \in (1, \infty)$ and for each weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ as (compare with (2.598))

$$\dot{L}^p_1(\mathbb{R}, w) := \Big\{ f \in L^1\big(\mathbb{R}, \tfrac{dx}{1+|x|^2}\big) \cap L^p_{\mathrm{loc}}(\mathbb{R}, w) :\ f' \in L^p(\mathbb{R}, w) \Big\}, \tag{3.350}$$

where the derivative is taken in the sense of distributions. We shall also need the operator H_{mod}, the modified version of the classical Hilbert transform H on the real line from (3.351), whose action on functions $f \in \dot{L}^p_1(\mathbb{R}, w)$ is given by

$$H_{\mathrm{mod}} f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \left\{ \frac{\mathbf{1}_{\mathbb{R}\setminus[x-\varepsilon, x+\varepsilon]}(y)}{x - y} - \frac{\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(y)}{-y} \right\} f(y)\, \mathrm{d}y \tag{3.351}$$

at $\mathcal{L}^1$-a.e. point $x \in \mathbb{R}$.

Proposition 3.12 *Pick an integrability index $p \in (1, \infty)$, fix a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and choose an aperture parameter $\kappa > 0$. Then the space of admissible boundary data for the Muckenhoupt weighted version of the*

Homogeneous Regularity Problem in the upper half-plane for the 2×2 *system* L *from* (3.239)*, i.e.,*

$$\Big\{ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ Lu = 0 \ \textit{in}\ \mathbb{R}^2_+,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w) \Big\} \tag{3.352}$$

is equal to

$$\Big\{ f = (f_1, f_2) \in \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2 : H(f_1') = f_2' \ \textit{at}\ \mathcal{L}^1\textit{-a.e. point on}\ \mathbb{R} \Big\} \tag{3.353}$$

$$= \Big\{ (f, H_{mod} f + c) \in \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2 : f \in \dot{L}^p_1(\mathbb{R}, w) \ \textit{and}\ c \in \mathbb{C} \Big\}.$$

The fact that the nontangential boundary traces exist in the context of (3.352) is a consequence of Proposition 2.24.

Proof of Proposition 3.12 Consider a vector-valued function

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2 \ \text{ satisfying} \\ Lu = 0 \ \text{ in }\ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w). \end{gathered} \tag{3.354}$$

From Theorem 3.4, Proposition 2.24, and (2.576), we know that

$$\begin{gathered} u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists and belongs to } \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2, \text{ the nontangential} \\ \text{boundary trace } (\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \text{ and} \\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \text{ belong to the space } L^1_{\text{loc}}(\mathbb{R}, \mathcal{L}^1). \end{gathered} \tag{3.355}$$

In particular, if we set $\widetilde{u} := \partial_x u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$, then

$$\begin{gathered} L\widetilde{u} = 0 \ \text{ in }\ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa \widetilde{u} \in L^p(\mathbb{R}, w), \ \text{ and} \\ \widetilde{u}\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \end{gathered} \tag{3.356}$$

In addition, if at $\mathcal{L}^1$-a.e. point $x \in \mathbb{R}$ we set

$$f(x) = \big(f_1(x), f_2(x)\big) := \Big(u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big)(x) \in \mathbb{C}^2, \tag{3.357}$$

then (3.355) gives $f \in \big[\dot{L}^p_1(\mathbb{R}, w)\big]^2$, and Proposition 2.22 tells us that

$$f' = \partial_x\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = \Big((\partial_x u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = \widetilde{u}\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ x \in \mathbb{R}. \tag{3.358}$$

Granted these properties, Proposition 3.10 applies to $\widetilde{u}$ and, with H denoting the Hilbert transform on the real line (cf. (1.24)), implies that we necessarily have

$$H(f_1') = f_2' \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}. \tag{3.359}$$

This proves that the set from (3.352) is included in the set described in the first line of (3.353).

To proceed, we need to recall some results from [114, Chapter 1]. First, $H_{\rm mod}$ maps $\dot{L}_1^p(\mathbb{R}, w)$ boundedly into itself, and for each $f \in \dot{L}_1^p(\mathbb{R}, w)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[H_{\rm mod} f\big] = H(f') \text{ at } \mathcal{L}^1\text{-a.e. point in } \mathbb{R}. \tag{3.360}$$

In particular,

$$H_{\rm mod} \text{ maps constants into constants.} \tag{3.361}$$

In addition, for each $f \in \dot{L}_1^p(\mathbb{R}, w)$, there exists some constant $c_f \in \mathbb{C}$ with the property that

$$H_{\rm mod}\big(H_{\rm mod} f\big) = -f + c_f. \tag{3.362}$$

Finally, recall the modified boundary-to-domain Cauchy integral operator for the upper half-plane from (3.327). Then, for each given function $f \in \dot{L}_1^p(\mathbb{R}, w)$, we have

$$\begin{array}{l} \mathcal{C}_{\rm mod} f \text{ is holomorphic in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa\big(\nabla \mathcal{C}_{\rm mod} f\big) \in L^p(\mathbb{R}, w), \text{ and} \\ \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R} \text{ we have } (\mathcal{C}_{\rm mod} f)\big|^{\kappa-\rm n.t.}_{\partial\mathbb{R}^2_+} = \big(\tfrac{1}{2}I + \tfrac{\rm i}{2}H_{\rm mod}\big)f. \end{array} \tag{3.363}$$

Next, note that for each $f_1, f_2 \in \dot{L}_1^p(\mathbb{R}, w)$ having $H(f_1') = f_2'$ at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ amounts (cf. (3.360)) to having $\frac{\mathrm{d}}{\mathrm{d}x}\big(H_{\rm mod} f_1 - f_2\big) = 0$ at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. Hence, in this case we have $f_2 = H_{\rm mod} f_1 + c$ for some constant $c \in \mathbb{C}$, proving that the set in the first line of (3.353) is contained in the set in the second line of (3.353).

At this stage, there remains to show that the set from the second line of (3.353) is contained in (3.352). To deal with this inclusion, observe that both sets are actually vector spaces over the field of complex numbers. Moreover, the vector space in the second line of (3.353) is the linear span of pairs of the form $(f, H_{\rm mod} f + c)$ with $f \in \dot{L}_1^p(\mathbb{R}, w)$ real-valued function and $c \in \mathbb{R}$. As such, it suffices to prove that for any real-valued function $f \in \dot{L}_1^p(\mathbb{R}, w)$ and any number $c \in \mathbb{R}$, there exists some vector-valued function u as in (3.354) such that

$$u\big|^{\kappa-\rm n.t.}_{\partial\mathbb{R}^2_+} = (f, H_{\rm mod} f + c). \tag{3.364}$$

To this end, define

$$U(z) := 2C_{\text{mod}} f(z) + \mathrm{i}c \quad \text{for each} \quad z \in \mathbb{R}^2_+. \tag{3.365}$$

Then (3.363) guarantees that U is a holomorphic function in $\mathbb{R}^2_+$, with the property that $\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$ and that at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we have

$$U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \big(I + \mathrm{i}H_{\text{mod}}\big) f + \mathrm{i}c. \tag{3.366}$$

If we now set $u_1 := \operatorname{Re} U$ and $u_2 := \operatorname{Im} U$, then $u := (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function, with real-valued scalar components, satisfying

$$Lu = L\big(\operatorname{Re} U, \operatorname{Im} U\big) = \big(\operatorname{Re}(\partial_{\bar{z}}^2 U), \operatorname{Im}(\partial_{\bar{z}}^2 U)\big) = 0 \in \mathbb{C}^2 \quad \text{in} \quad \mathbb{R}^2_+, \tag{3.367}$$

thanks to (3.252) and the fact that $\partial_{\bar{z}} U = 0$ in $\mathbb{R}^2_+$, by the Cauchy–Riemann equations. Also, $\mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$ given that $\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$. Finally, bearing in mind that f, $H_{\text{mod}} f$ are real-valued and that $c \in \mathbb{R}$, at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we may use (3.366) to compute

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \Big(\operatorname{Re} U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}, \operatorname{Im} U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = \big(f, H_{\text{mod}} f + c\big), \tag{3.368}$$

proving (3.364). □

A higher-dimensional version of the theory presented in connection with the planar 2×2 system L from (3.239) has been worked out in [115, Chapter 2], where analogous results to Proposition 3.10 have been established. In order to describe them, we need some notation in the n-dimensional Euclidean space, where $n \in \mathbb{N}$ with $n \geq 2$. First, recall the family of Riesz transforms $(R_j)_{1 \leq j \leq n-1}$ in the hyperplane $\mathbb{R}^{n-1} \times \{0\} \equiv \mathbb{R}^{n-1}$. Specifically, the j-th Riesz transform R_j on $\mathbb{R}^{n-1}$, with $j \in \{1, \dots, n-1\}$, is the singular integral operator acting on any given function $f \in L^1\Big(\mathbb{R}^{n-1}, \frac{\mathcal{L}^{n-1}(x')}{1+|x'|^{n-1}}\Big)$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ according to

$$R_j f(x') := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} \frac{x_j - y_j}{|x' - y'|^n} f(y')\, d\mathcal{L}^{n-1}(y'). \tag{3.369}$$

We shall also need the j-th modified Riesz transform R_j^{mod}, acting on each function $f \in L^1\Big(\mathbb{R}^{n-1}, \frac{\mathcal{L}^{n-1}(x')}{1+|x'|^n}\Big)$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ according to

$$R_j^{\text{mod}} f(x') := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\mathbb{R}^{n-1}} \Big\{ \frac{x_j - y_j}{|x' - y'|^n} \mathbf{1}_{\mathbb{R}^{n-1} \setminus \overline{B((x',0),\varepsilon)}}(y') \tag{3.370}$$

$$- \frac{-y_j}{|-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1}\setminus \overline{B(0,1)}}(y') \Big\} f(y')\, \mathrm{d}\mathcal{L}^{n-1}(y').$$

Finally, following [115, Chapter 2], we shall consider a special system, namely the homogeneous, constant real coefficient, symmetric, $n \times n$ second-order system acting on each vector-valued distribution $\vec{u} = (u_1, \dots, u_n)$ (defined in an open subset of $\mathbb{R}^n$) according to

$$L_D \vec{u} := \Delta \vec{u} - 2\nabla \mathrm{div}\, \vec{u}. \tag{3.371}$$

That is,

$$\begin{gathered} L_D = \big(a_{jk}^{\alpha\beta} \partial_j \partial_k\big)_{1 \leq \alpha,\beta \leq n} \quad \text{with} \\ a_{jk}^{\alpha\beta} = \delta_{jk}\delta_{\alpha\beta} - 2\delta_{j\alpha}\delta_{k\beta} \quad \text{for all} \quad \alpha, \beta, j, k \in \{1, \dots, n\}. \end{gathered} \tag{3.372}$$

Here is the result which amounts to a higher-dimensional version of Propositions 3.10, 3.11, and 3.12.

Proposition 3.13 *Fix $n \in \mathbb{N}$, with $n \geq 2$. Pick an integrability index $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, and choose some aperture parameter $\kappa > 0$. Also, recall the second-order, weakly elliptic, constant (real) coefficient, symmetric, $n \times n$ system L_D defined in (3.371).*

Then if $\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n$ is a vector-valued function satisfying

$$L_D \vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa \vec{u} \in L^p(\mathbb{R}^{n-1}, w), \tag{3.373}$$

and such that the nontangential boundary trace

$$\vec{f} = (f_1, \dots, f_n) := \vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ exists (in } \mathbb{C}^n\text{) at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}, \tag{3.374}$$

it follows that the vector-valued function $\vec{f}$ belongs to $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ and

$$f_n = -\sum_{j=1}^{n-1} R_j f_j \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}. \tag{3.375}$$

In the converse direction, for any given $\vec{f} = (f_1, \dots, f_n) \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ satisfying (3.375), there exists a vector-valued function $\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n$ satisfying

$$\begin{gathered} L_D \vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa \vec{u} \in L^p(\mathbb{R}^{n-1}, w), \quad \text{and} \\ \vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = \vec{f} \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}. \end{gathered} \tag{3.376}$$

Altogether, the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system L_D in the upper half-space may be described as follows:

$$\begin{aligned}\Big\{\vec{u}\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} : \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D\vec{u} = 0 \ \textit{in}\ \mathbb{R}^n_+,\ \mathcal{N}_\kappa\vec{u} \in L^p(\mathbb{R}^{n-1}, w),\\ \textit{and}\ \ u\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} \ \textit{exists at}\ \mathcal{L}^{n-1}\textit{-a.e. point on}\ \mathbb{R}^{n-1}\Big\}\\ = \Big\{(f_1, \dots, f_n) \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n : \ f_n = -\sum_{j=1}^{n-1} R_j f_j\Big\}.\end{aligned} \tag{3.377}$$

Furthermore, the space of admissible boundary data for the Inhomogeneous Regularity Dirichlet Problem with boundary data in Muckenhoupt weighted Sobolev spaces for the system L_D in the upper half-space is given by[2]

$$\begin{aligned}\Big\{\vec{u}\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} : \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D\vec{u} = 0 \ \textit{in}\ \mathbb{R}^n_+,\ \mathcal{N}_\kappa\vec{u},\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w)\Big\}\\ = \Big\{(f_1, \dots, f_n) \in \big[L^p_1(\mathbb{R}^{n-1}, w)\big]^n : \ f_n = -\sum_{j=1}^{n-1} R_j f_j\Big\}.\end{aligned} \tag{3.378}$$

Also, the space of admissible boundary data for the Homogeneous Regularity Dirichlet Problem with boundary data in homogeneous Muckenhoupt weighted Sobolev spaces for the system L_D in the upper half-space may be characterized as follows:[3]

$$\begin{aligned}\Big\{\vec{u}\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} : \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D\vec{u} = 0 \ \textit{in}\ \mathbb{R}^n_+,\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w)\Big\}\\ = \Big\{(f_1, \dots, f_n) \in \big[\dot{L}^p_1(\mathbb{R}^{n-1}, w)\big]^n : \ f_n + \sum_{j=1}^{n-1} \overset{\textit{mod}}{R_j} f_j \ \textit{is constant}\Big\}.\end{aligned} \tag{3.379}$$

Finally, similar results are valid on the scales of Morrey spaces and block spaces (cf. Sect. 7.1).

In particular, it is apparent from (3.377) that no nonzero vector-valued function from the space

[2] With the existence of the nontangential boundary traces guaranteed by Proposition 2.24.

[3] The existence of the nontangential boundary traces here being guaranteed by Proposition 2.24. Also, the homogeneous Muckenhoupt weighted Sobolev space $\dot{L}^p_1(\mathbb{R}^{n-1}, w)$ is defined as in (2.598) with $\Omega := \mathbb{R}^n_+$.

$$\Big\{(0,\dots,0,f) : f \in L^p(\mathbb{R}^{n-1},w)\Big\} \tag{3.380}$$

can possibly be an admissible boundary datum for the Dirichlet Problem for system L_D in the upper half-space. As such,

the codimension of the admissible boundary data for the Dirichlet Problem for system L_D in the upper half-space (i.e., the space in the first line of (3.377)) into the full data space $\big[L^p(\mathbb{R}^{n-1},w)\big]^n$ is $+\infty$. (3.381)

Likewise, since no nonzero vector-valued function from the space

$$\Big\{(0,\dots,0,f) : f \in L^p_1(\mathbb{R}^{n-1},w)\Big\} \tag{3.382}$$

can possibly be an admissible boundary datum for the Inhomogeneous Regularity Problem for system L_D in the upper half-space, it follows that

the codimension of the admissible boundary data for the Inhomogeneous Regularity Problem for system L_D in the upper half-space (i.e., the space in the first line of (3.378)) into the full data space $\big[L^p_1(\mathbb{R}^{n-1},w)\big]^n$ is $+\infty$. (3.383)

Finally, given that no nonzero vector-valued function from the space

$$\Big\{(0,\dots,0,f) : f \in \dot{L}^p_1(\mathbb{R}^{n-1},w)\Big\} \tag{3.384}$$

can possibly be an admissible boundary datum for the Homogeneous Regularity Problem for system L_D in the upper half-space, we see that

the codimension of the admissible boundary data for the Homogeneous Regularity Problem for system L_D in the upper half-space (i.e., the space in the first line of (3.379)) into the full data space $\big[\dot{L}^p_1(\mathbb{R}^{n-1},w)\big]^n$ is $+\infty$. (3.385)

It has also been noted in [115, §2.6] that for each scalar function

$$\omega \in \mathscr{C}^\infty(\mathbb{R}^n_+) \ \text{ with } \ \Delta\omega = 0 \ \text{ in } \ \mathbb{R}^n_+ \ \text{ and } \ \mathcal{N}_\kappa(\nabla\omega) \in L^p(\mathbb{R}^{n-1},w), \tag{3.386}$$

the vector-valued function

$$\begin{gathered} \vec{u} : \mathbb{R}^n_+ \longrightarrow \mathbb{C}^n \ \text{ given by} \\ \vec{u}(x) := x_n(\nabla\omega)(x) \ \text{ for each } \ x = (x_1,\dots,x_n) \in \mathbb{R}^n_+ \end{gathered} \tag{3.387}$$

satisfies

$$\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \quad L_D\vec{u} = 0 \text{ in } \mathbb{R}^n_+, \quad \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w),$$
$$\text{and } \vec{u}\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \mathbb{R}^{n-1}. \tag{3.388}$$

In the converse direction, each vector-valued function $\vec{u}$ as in (3.388) has the format described in the second line of (3.387) for some scalar function ω as in (3.386). Finally, it has been noted in [115, §2.6] that if in place of (3.386), one now assumes

$$\omega \in \mathscr{C}^\infty(\mathbb{R}^n_+) \text{ with } \Delta\omega = 0 \text{ in } \mathbb{R}^n_+ \text{ and}$$
$$\mathcal{N}_\kappa\omega \in L^p(\mathbb{R}^{n-1}, w), \quad \mathcal{N}_\kappa(\nabla\omega) \in L^p(\mathbb{R}^{n-1}, w), \tag{3.389}$$

then the vector-valued function $\vec{u}$ defined as in (3.387) for this choice of ω has the additional property that $\mathcal{N}_\kappa\vec{u} \in L^p(\mathbb{R}^{n-1}, w)$, i.e., satisfies

$$\vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \quad L_D\vec{u} = 0 \text{ in } \mathbb{R}^n_+, \quad \mathcal{N}_\kappa\vec{u},\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\mathbb{R}^{n-1}, w),$$
$$\text{and } \vec{u}\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \mathbb{R}^{n-1}. \tag{3.390}$$

In particular, these considerations readily imply that

the space of null-solutions for the Homogeneous Regularity Problem for the system L_D in the upper half-space (i.e., the space of functions as in (3.388)) is infinite dimensional (3.391)

and that

the space of null-solutions for the Inhomogeneous Regularity Problem for the system L_D in the upper half-space (i.e., the space of functions as in (3.390)) is infinite dimensional. (3.392)

As a corollary of (3.392), we also see that

the space of null-solutions for the Dirichlet Problem for the system L_D in the upper half-space is infinite dimensional. (3.393)

We next turn our attention to the issue of existence and uniqueness of distinguished coefficient tensors for a given weakly elliptic system and its transposed. The starting point is the following result, proved in [115, §1.5], for strongly elliptic systems.

Theorem 3.8 *Fix $M, n \in \mathbb{N}$ with $n \geq 2$. Let L be a homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$ which satisfies the strong Legendre–Hadamard ellipticity condition* (3.4)*. Then either*

$$\mathfrak{A}^{\text{dis}}_L = \varnothing \text{ and } \mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing, \tag{3.394}$$

or

$$\mathfrak{A}_L^{\mathrm{dis}} = \{A\} \textit{ and } \mathfrak{A}_{L^\top}^{\mathrm{dis}} = \{A^\top\} \textit{ for some } A \in \mathfrak{A}_L. \tag{3.395}$$

As a corollary, if $M, n \in \mathbb{N}$ *with* $n \geq 2$ *and* L *is a homogeneous, second-order, constant complex coefficient,* $M \times M$ *system in* $\mathbb{R}^n$ *satisfying the Legendre–Hadamard (strong) ellipticity condition, then*

$$\mathfrak{A}_L^{\mathrm{dis}} \textit{ is either empty or a singleton.} \tag{3.396}$$

We next state a result, augmenting Theorem 3.8, pertaining to weakly elliptic systems, also established in [115, §1.5].

Theorem 3.9 *Let* $M, n \in \mathbb{N}$ *with* $n \geq 2$ *and consider a weakly elliptic, homogeneous, second-order, constant complex coefficient,* $M \times M$ *system* L *in* $\mathbb{R}^n$ *with the property that* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ *and* $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$. *Then both* $\mathfrak{A}_L^{\mathrm{dis}}$ *and* $\mathfrak{A}_{L^\top}^{\mathrm{dis}}$ *are singletons. In fact,* $\mathfrak{A}_L^{\mathrm{dis}} = \{A\}$ *and* $\mathfrak{A}_{L^\top}^{\mathrm{dis}} = \{A^\top\}$ *for some* $A \in \mathfrak{A}_L$.

In particular, if $M, n \in \mathbb{N}$ *with* $n \geq 2$ *and* L *is a symmetric, weakly elliptic, homogeneous, second-order, constant complex coefficient,* $M \times M$ *system in* $\mathbb{R}^n$, *then* $\mathfrak{A}_L^{\mathrm{dis}}$ *is either empty or a singleton, and, in the latter case, one has* $\mathfrak{A}_L^{\mathrm{dis}} = \{A\}$ *for some* $A \in \mathfrak{A}_L$ *satisfying* $A^\top = A$.

For example, from (3.223), we know that

$$\mathfrak{A}_\Delta^{\mathrm{dis}} = \{I_{n\times n}\} \quad \text{where } \Delta \text{ is the Laplacian in } \mathbb{R}^n \text{ with } n \geq 2, \tag{3.397}$$

$$\mathfrak{A}_{\mathrm{div}\, A\nabla}^{\mathrm{dis}} = \{(A + A^\top)/2\} \quad \text{if } n \geq 3 \text{ and } A \in \mathbb{C}^{n\times n} \text{ is invertible}, \tag{3.398}$$

while Theorem 3.9 and (3.228) imply that, for the complex Lamé system $L_{\mu,\lambda}$ defined in (3.224), we have

$$\begin{aligned} \mathfrak{A}_{L_{\mu,\lambda}}^{\mathrm{dis}} = \Big\{\big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq n}}\Big\} \quad & \text{if } \mu \neq 0,\ 2\mu+\lambda \neq 0, \text{ and } 3\mu+\lambda \neq 0, \text{ where} \\ a_{jk}^{\alpha\beta} := \mu\delta_{jk}\delta_{\alpha\beta} + \tfrac{(\mu+\lambda)(2\mu+\lambda)}{3\mu+\lambda}&\delta_{j\alpha}\delta_{k\beta} + \tfrac{\mu(\mu+\lambda)}{3\mu+\lambda}\delta_{j\beta}\delta_{k\alpha}, \\ \text{for } 1 \leq j,k,\alpha,&\beta \leq n. \end{aligned} \tag{3.399}$$

Here is an equivalent characterization of the existence of a distinguished coefficient tensor proved in [115, §1.6].

Theorem 3.10 *Fix* $M, n \in \mathbb{N}$ *with* $n \geq 2$. *Let* L *be an* $M \times M$ *second-order, homogeneous, constant complex coefficient, weakly elliptic system in* $\mathbb{R}^n$. *Then the following statements are equivalent:*

(i) The system L *possesses a distinguished coefficient tensor, i.e.,* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$.

(ii) There exists a matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ *which is positive homogeneous of degree* $-n$ *and satisfies*

$$\int_{S^{n-1}} k \, \mathrm{d}\mathcal{H}^{n-1} = I_{M\times M} \tag{3.400}$$

(where $I_{M\times M}$ *is the* $M \times M$ *identity matrix), as well as*

$$L\big(x_s k(x)\big) = 0 \cdot I_{M\times M} \ \text{ in } \ \mathbb{R}^n \setminus \{0\} \ \text{ for each } \ s \in \{1, \dots, n\}. \tag{3.401}$$

Moreover, if L *has a unique distinguished coefficient tensor (i.e., if* $\#\mathfrak{A}_L^{\mathrm{dis}} = 1$*), then there is only one function* k *as in item (ii).*

It has been noted in [115, §1.6] that Theorem 3.10 has the following noteworthy consequence:

Corollary 3.2 *Fix* $M, n \in \mathbb{N}$ *with* $n \geq 2$*. Let* L *be an* $M \times M$ *second-order, homogeneous, constant complex coefficient, weakly elliptic system in* $\mathbb{R}^n$*. Assume that there exists a matrix-valued function* $k_* \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$ *which is positive homogeneous of degree* $-n$*, is not identically zero, and satisfies*

$$\int_{S^{n-1}} k_* \, \mathrm{d}\mathcal{H}^{n-1} = 0 \cdot I_{M\times M} \tag{3.402}$$

(where $I_{M\times M}$ *is the* $M \times M$ *identity matrix), as well as*

$$L\big(x_s k_*(x)\big) = 0 \cdot I_{M\times M} \ \text{ in } \ \mathbb{R}^n \setminus \{0\} \ \text{ for each } \ s \in \{1, \dots, n\}. \tag{3.403}$$

Then either $\mathfrak{A}_L^{\mathrm{dis}} = \varnothing$*, or* $\mathfrak{A}_{L^\top}^{\mathrm{dis}} = \varnothing$*.*

To proceed, we revisit the special system L_D from (3.371) which turns out not to have any distinguished coefficient tensors. Indeed, it has been noted in [115, §1.6] that if E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, defined at each point $x \in \mathbb{R}^n \setminus \{0\}$ according to

$$E_\Delta(x) := \begin{cases} \dfrac{1}{(2-n)\omega_{n-1}} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\[2ex] \dfrac{1}{2\pi} \ln|x| & \text{if } n = 2, \end{cases} \tag{3.404}$$

and if k_* is the Hessian matrix of E_Δ, defined at each point $x \in \mathbb{R}^n \setminus \{0\}$ by

$$k_*(x) := \big((\partial_i\partial_j E_\Delta)(x)\big)_{1\le i,j\le n} = \Big(\frac{\delta_{ij}}{\omega_{n-1}}\frac{1}{|x|^n} - \frac{n}{\omega_{n-1}}\frac{x_i x_j}{|x|^{n+2}}\Big)_{1\le i,j\le n}, \tag{3.405}$$

then (3.403)–(3.402) hold for $L = L_D$, the special system L_D from (3.371). In view of the fact that L_D is symmetric, Corollary 3.2 then gives

for each $n \in \mathbb{N}$ with $n \ge 2$, the $n \times n$ system L_D in $\mathbb{R}^n$ from (3.371) is weakly elliptic, second-order, homogeneous, constant real coefficient, symmetric, and $\mathfrak{A}^{\rm dis}_{L_D} = \mathfrak{A}^{\rm dis}_{L_D^\top} = \varnothing$. (3.406)

Remark 3.6 Consider the complex Lamé system $L_{\mu,\lambda}$, defined earlier in (3.224), in the regime $\mu, \lambda \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$. From (3.225), we know that this is equivalent with the weak ellipticity of $L_{\mu,\lambda}$. Hence, this is the range in which we may consider the issue of whether $L_{\mu,\lambda}$ possesses distinguished coefficient tensors. In this regard, we wish to note that from (3.229) and Theorem 3.9, it follows that $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}}$ is a singleton when $3\mu+\lambda \neq 0$. In addition, from (3.406) and (3.371), we see that $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}}$ is empty when $3\mu + \lambda = 0$. Collectively, these observations prove that

given any $\mu, \lambda \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$, then $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}} \neq \varnothing$ if and only if $3\mu + \lambda \neq 0$ if and only if $\mathfrak{A}^{\rm dis}_{L_{\mu,\lambda}}$ is a singleton (namely the coefficient tensor $A(\zeta)$ described in (3.226), corresponding to the choice $\zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}$). (3.407)

One final remark is as follows. Consider an arbitrary second-order, weakly elliptic, homogeneous, constant complex coefficient, $M \times M$ system L in $\mathbb{R}^n$, and pick a coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_L$. For each invertible matrix $C = (c_{jk})_{1\le j,k\le n} \in \mathbb{C}^{M\times M}$, define

$$AC := \big(a^{\alpha\beta}_{j\ell}c_{\ell k}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_{LC} \tag{3.408}$$

and

$$CA := \big(c_{j\ell}a^{\alpha\beta}_{\ell k}\big)_{\substack{1\le\alpha,\beta\le M\\ 1\le j,k\le n}} \in \mathfrak{A}_{CL}, \tag{3.409}$$

with the systems LC and CL naturally interpreted in the sense of multiplication of $M \times M$ matrices. With this notation, it has been noted in [115, §1.2] that for each invertible matrix $C \in \mathbb{C}^{M\times M}$ we have

$$A \in \mathfrak{A}^{\rm dis}_L \iff AC \in \mathfrak{A}^{\rm dis}_{LC}, \tag{3.410}$$

and

$$A \in \mathfrak{A}_L^{\rm dis} \iff CA \in \mathfrak{A}_{CL}^{\rm dis}. \tag{3.411}$$

A useful consequence of (3.410)–(3.411) and Corollary 3.2 is as follows. Bring back the second-order, homogeneous, real constant coefficient, 2×2 system in the plane

$$L_B = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}, \tag{3.412}$$

which is matrix representation of Bitsadze's operator $\mathbb{L}$ from (3.250). Also, recall the two-dimensional version of the special system L_D from (3.371), i.e.,

$$L_D = \begin{pmatrix} \partial_y^2 - \partial_x^2 & -2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}. \tag{3.413}$$

Hence, if we let

$$V := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.414}$$

then $V^\top = V = V^{-1}$ and

$$L_B = \tfrac{1}{4} L_D V \ \text{ and } \ L_B^\top = \tfrac{1}{4} V L_D. \tag{3.415}$$

These together with (3.410) and (3.411) imply

$$A \in \mathfrak{A}_{L_D}^{\rm dis} \iff A\,V \in \mathfrak{A}_{L_B}^{\rm dis} \tag{3.416}$$

and

$$A \in \mathfrak{A}_{L_D}^{\rm dis} \iff VA \in \mathfrak{A}_{L_B^\top}^{\rm dis}. \tag{3.417}$$

Since we have proved that $\mathfrak{A}_{L_D}^{\rm dis} = \varnothing$ (cf. (3.406)), the equivalences in (3.416)–(3.417) imply that

$$\mathfrak{A}_{L_B}^{\rm dis} = \varnothing \ \text{ and } \ \mathfrak{A}_{L_B^\top}^{\rm dis} = \varnothing. \tag{3.418}$$

Chapter 4
Boundedness and Invertibility of Layer Potential Operators

The key result in this work is Theorem 4.2 which elaborates on the nature of the operator norm of a singular integral operator T defined on the boundary of a UR domain Ω whose integral kernel has a special algebraic format, through the presence of the inner product between the outward unit normal ν to Ω and the chord, as a factor. Proving this theorem requires extensive preparations and takes quite a bit of effort, but the redeeming feature of Theorem 4.2 is that said operator norm estimate involves the BMO semi-norm of ν as a factor. This hallmark attribute (which is shared by the double layer operator K_A associated with a distinguished coefficient tensor A) entails that the flatter $\partial\Omega$ is, the smaller $\|T\|$ is. In particular, having $\partial\Omega$ sufficiently flat ultimately allows us to invert $\frac{1}{2}I + K_A$ on Muckenhoupt weighted Lebesgue spaces via a Neumann series, and this is of paramount importance later on, when dealing with boundary value problems via the method of boundary layer potentials. Subsequently, via operator identities relating the single and double layers, we also succeed in inverting the single layer potential operator in a similar geometric and algebraic setting.

4.1 Estimates for Euclidean Singular Integral Operators

We begin with a few generalities of functional analytic nature. Given two normed vector spaces $\big(X, \|\cdot\|_X\big)$ and $\big(Y, \|\cdot\|_Y\big)$, consider a positively homogeneous mapping $T : X \to Y$, i.e., a function T sending X into Y and satisfying $T(\lambda u) = \lambda T(u)$ for each $u \in X$ and each $\lambda \in (0, \infty)$ (note that taking $u := 0 \in X$ and $\lambda := 2$ implies $T(0) = 0 \in Y$). We shall denote by

$$\|T\|_{X\to Y} := \sup\Big\{\|Tu\|_Y : u \in X,\ \|u\|_X = 1\Big\} \in [0, \infty] \tag{4.1}$$

the operator norm of such a mapping T; in particular,

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_4

$$\|Tu\|_Y \leq \|T\|_{X\to Y}\|u\|_X \quad \text{for each} \quad u \in X. \tag{4.2}$$

It is then straightforward to see that a positively homogeneous mapping $T : X \to Y$ is continuous at $0 \in X$ if and only if T is bounded (i.e., it maps bounded subsets of X into bounded subsets of Y) if and only if $\|T\|_{X\to Y} < +\infty$.

Consider now the special case when X, Y are Lebesgue spaces (associated with a generic measure space) and T is a sub-linear mapping of X into Y (i.e., $T : X \to Y$ satisfies the property $T(\lambda u) = |\lambda| T(u)$ for each scalar λ and each function $u \in X$, as well as $T(u + w) \leq Tu + Tw$ at a.e. point, for each $u, w \in X$). Then, for each $u, w \in X$ we have $|Tu - Tw| \leq |T(u - w)|$ at a.e. point, which further implies that $\|Tu - Tw\|_Y \leq \|T(u - w)\|_Y \leq \|T\|_{X\to Y}\|u - w\|_X$. Consequently,

$$\begin{gathered}\text{a sub-linear map } T : X \to Y \text{ is continuous}\\ \text{if and only if } \|T\|_{X\to Y} < +\infty.\end{gathered} \tag{4.3}$$

Let us now start in earnest. To facilitate dealing with Theorem 4.1 a little later, we first isolate a useful estimate in the lemma below.

Lemma 4.1 *Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^n, \mathcal{L}^n)$. Then there exists a constant $C \in (0, \infty)$ which only depends on n, p, and $[w]_{A_p}$, with the property that for each point $x \in \mathbb{R}^n$, each radius $r \in (0, \infty)$, and real-valued function $A \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$ with*

$$\nabla A \in \big[\mathrm{BMO}(\mathbb{R}^n, \mathcal{L}^n)\big]^n \tag{4.4}$$

one has

$$\begin{aligned}\int\limits_{\substack{y\in\mathbb{R}^n\\ |x-y|>r}} & \frac{\big|A(x) - A(y) - \langle \nabla A(y), x - y\rangle\big|^p}{|x - y|^{(n+1)p}}\, dw(y)\\ & \leq C r^p\, w\big(B(x, r)\big)\big\|\nabla A\big\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}.\end{aligned} \tag{4.5}$$

Proof Fix a function A as in the statement of the lemma. From Lemma 2.13 and (4.4) we see that

$$\nabla A \in \big[L^1_{\rm loc}(\mathbb{R}^n, w)\big]^n. \tag{4.6}$$

Next, recall from (2.533) that there exists $\varepsilon \in (0, p - 1)$ which depends only on p, n, and $[w]_{A_p}$, such that

$$w \in A_{p-\varepsilon}(\mathbb{R}^n, \mathcal{L}^n). \tag{4.7}$$

Fix $x \in \mathbb{R}^n$ and $r \in (0, \infty)$. By breaking up the integral dyadically, estimating the denominator, and using the doubling property of $w \in A_{p-\varepsilon}(\mathbb{R}^n, \mathcal{L}^n)$ (cf. item *(5)* of

Proposition 2.20) we may dominate

$$\int\limits_{\substack{y\in\mathbb{R}^n\\|x-y|>r}} \frac{\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p}{|x-y|^{(n+1)p}}\,\mathrm{d}w(y)$$
$$\leq C_{n,p}\sum_{j=1}^{\infty}\frac{w(B(x,2^jr))}{2^{j(n+1)p}}\cdot \mathrm{I}_j \leq C_{n,p,w}\sum_{j=1}^{\infty}\frac{2^{jn(p-\varepsilon)}w(B(x,r))}{2^{j(n+1)p}}\cdot \mathrm{I}_j, \tag{4.8}$$

where, for each $j\in\mathbb{N}$,

$$\mathrm{I}_j := \frac{1}{w(B(x,2^jr))}\int\limits_{2^{j-1}r<|x-y|\leq 2^jr}\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p\,\mathrm{d}w(y). \tag{4.9}$$

To proceed, for each $j\in\mathbb{N}$ introduce

$$A_j(z) := A(z)-\Big(⨍_{B(x,2^jr)}\nabla A\,\mathrm{d}w\Big)\cdot z \quad \text{for each } z\in\mathbb{R}^n \tag{4.10}$$

(making use of (4.6) to ensure that this is meaningful), and observe that I_j, originally defined in (4.9), does not change if the function A is replaced by A_j. Consequently, for each $j\in\mathbb{N}$ we have

$$\mathrm{I}_j \leq C_p\cdot \mathrm{II}_j + C_p\cdot \mathrm{III}_j, \tag{4.11}$$

where

$$\mathrm{II}_j := \frac{1}{w(B(x,2^jr))}\int\limits_{2^{j-1}r<|x-y|\leq 2^jr}\big|A_j(x)-A_j(y)|^p\,\mathrm{d}w(y), \tag{4.12}$$

and

$$\mathrm{III}_j := \frac{2^{jp}r^p}{w(B(x,2^jr))}\int\limits_{2^{j-1}r<|x-y|\leq 2^jr}\big|\nabla A_j(y)\big|^p\,\mathrm{d}w(y). \tag{4.13}$$

Fix an integrability exponent $q\in(n,\infty)$ and pick $j\in\mathbb{N}$ arbitrary. Then for each $y\in\mathbb{R}^n$ such that $2^{j-1}r<|x-y|\leq 2^jr$ we may estimate

$$|A_j(x)-A_j(y)|\leq C_{q,n}|x-y|\Big(⨍_{|x-z|\leq 2|x-y|}|\nabla A_j(z)|^q\,\mathrm{d}z\Big)^{1/q}$$

$$
\begin{aligned}
&\leq C_{q,n,w}\cdot 2^j r\Big(\fint_{B(x,2|x-y|)}|\nabla A_j|^{pq}\,\mathrm{d}w\Big)^{1/(pq)}\\
&\leq C_{q,n,w}\cdot 2^j r\Big(\fint_{B(x,2|x-y|)}\Big|\nabla A-\fint_{B(x,2|x-y|)}\nabla A\,\mathrm{d}w\Big|^{pq}\,\mathrm{d}w\Big)^{1/(pq)}\\
&\qquad + C_{q,n,w}\cdot 2^j r\Big|\fint_{B(x,2^j r)}\nabla A\,\mathrm{d}w-\fint_{B(x,2|x-y|)}\nabla A\,\mathrm{d}w\Big|\\
&\leq C_{q,n,w}\cdot 2^j r\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,w)]^n}\\
&\leq C_{q,n,w}\cdot 2^j r\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}.
\end{aligned}
\tag{4.14}
$$

Above, the first estimate is provided by Mary Weiss' Lemma (cf. [24, Lemma 1.4, p. 144], or [58, Lemma 2.10, p. 477]), the second estimate uses the fact that we have $|x-y|\leq 2^j r$ and Lemma 2.12, the third estimate is implied by (4.10) which gives $\nabla A_j=\nabla A-\fint_{B(x,2^j r)}\nabla A\,\mathrm{d}w$, the penultimate estimate is a consequence of the John-Nirenberg inequality, (2.103) (written with w in place of σ), and the doubling property of w, while the final estimate in (4.14) comes from Lemma 2.14. In turn, (4.12) and (4.14) yield

$$
\mathrm{II}_j\leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}. \tag{4.15}
$$

By combining (4.13) and (4.10) we also see that

$$
\begin{aligned}
\mathrm{III}_j &\leq 2^{jp}r^p\fint_{B(x,2^j r)}\Big|\nabla A-\fint_{B(x,2^j r)}\nabla A\,\mathrm{d}w\Big|^p\,\mathrm{d}w\\
&\leq C\cdot 2^{jp}r^p\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,w)]^n}\leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n},
\end{aligned}
\tag{4.16}
$$

where the last inequality is once again provided by Lemma 2.14. From (4.15)–(4.16) and (4.11) we then conclude that

$$
\mathrm{I}_j\leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}\quad\text{for each}\quad j\in\mathbb{N}. \tag{4.17}
$$

Using this back in (4.8) now readily yields (4.5), since $\sum_{j=1}^{\infty}2^{-jn\varepsilon}<\infty$. □

The next result, dealing with boundedness for certain type of singular integral operators in the Euclidean context, refines work in [61, Theorem 4.34, p. 2725].

Theorem 4.1 *Pick an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. Denote by $p'\in(1,\infty)$ the Hölder conjugate exponent of p and by w' the dual weight $w':=w^{1-p'}\in A_{p'}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ of w. Next, fix three numbers $n,m,d\in\mathbb{N}$ with $n\geq 2$, and let $N=N(n,m)\in\mathbb{N}$ be a sufficiently large integer. Let $A\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ be a complex-valued function with* *the property that*

$$\nabla A \in \big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}. \tag{4.18}$$

Also, for each $j \in \{1, \dots, m\}$ *consider a real-valued function* $B_j \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ *with the property that*

$$\nabla B_j \in \big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}, \tag{4.19}$$

and set $B := (B_1, \dots, B_m)$*. In addition, consider a function* $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^d$ *for which there exists* $c \in (0, 1]$ *such that*

$$c|x' - y'| \leq |\Phi(x') - \Phi(y')| \leq c^{-1}|x' - y'| \;\; \textit{for all} \;\; x', y' \in \mathbb{R}^{n-1}; \tag{4.20}$$

hence, Φ *is bi-Lipschitz. Going further, suppose* $F \in \mathscr{C}^{N+2}(\mathbb{R}^m)$ *is a complex-valued function which is even, has the property that* $\partial^\alpha F$ *belongs to* $L^1(\mathbb{R}^m, \mathcal{L}^m)$ *for every multi-index* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \leq N + 2$*, and*

$$\sup_{X \in \mathbb{R}^m} \big[(1 + |X|)|F(X)|\big] < +\infty. \tag{4.21}$$

Finally, for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ *and each point* $x' \in \mathbb{R}^{n-1}$ *define*

$$T^{A,B}_{\Phi,*} g(x') := \sup_{\varepsilon > 0} \Bigg| \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |\Phi(x') - \Phi(y')| > \varepsilon}} \frac{A(x') - A(y') - \langle \nabla A(y'), x' - y' \rangle}{|x' - y'|^n} \times$$

$$\times F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y')\, \mathrm{d}y' \Bigg|. \tag{4.22}$$

Then $T^{A,B}_{\Phi,*}$ *is a well-defined, continuous, sub-linear mapping of the Muckenhoupt weighted Lebesgue space* $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ *into itself, and there exists some constant* $C(n, p, w) \in (0, \infty)$ *which depends only on* n*,* p*, and* $[w]_{A_p}$ *with the property that*

$$\begin{aligned} &\big\| T^{A,B}_{\Phi,*} \big\|_{L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \to L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})} \\ &\quad \leq C(n, p, w) \cdot c^{-3n} \Big(\sum_{|\alpha| \leq N+2} \big\| \partial^\alpha F \big\|_{L^1(\mathbb{R}^m, \mathcal{L}^m)} + \sup_{X \in \mathbb{R}^m} (1 + |X|)|F(X)| \Big) \\ &\qquad \times \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big(1 + \sum_{j=1}^m \|\nabla B_j\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big)^N. \end{aligned} \tag{4.23}$$

Theorem 4.1 is an intricate piece of machinery allowing us to estimate, in a rather detailed and specific manner, the maximal operator associated with integral kernels that exhibit a certain type of algebraic structure. We shall put this to good use in Lemma 4.2 which, in turn, is a basic ingredient in the proof of Theorem 4.2 (the main result in this section). This being said, Theorem 4.1 is useful for a variety of other purposes.

To give a significant example in this regard, work in the one-dimensional setting and recall the Hilbert transform H on the real line from (1.24). Consider a complex-valued function $A \in W^{1,1}_{\rm loc}(\mathbb{R})$ with the property that $A' \in {\rm BMO}(\mathbb{R}, \mathcal{L}^1)$. Let M_A stand for the operator of pointwise multiplication by A, and denote by D the one-dimensional derivative operator $f \mapsto {\rm d}f/{\rm d}x$ on the real line. Also, fix an integrability exponent $p \in (1,\infty)$ and a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. Then the commutator $[H, M_A D]$, originally defined on functions from $\mathscr{C}^\infty_0(\mathbb{R})$, extends to a bounded linear mapping on $L^p(\mathbb{R}, w)$ with operator norm $\leq C \left\| A' \right\|_{{\rm BMO}(\mathbb{R},\mathcal{L}^1)}$ where $C \in (0,\infty)$ is an absolute constant. Indeed, given any function $f \in \mathscr{C}^\infty_0(\mathbb{R})$, at $\mathcal{L}^1$-a.e. differentiability point $x \in \mathbb{R}$ for A (hence, at $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$) we may write (keeping in mind that, since the Hilbert transform is a multiplier, H commutes with differentiation):

$$
\begin{aligned}
[H, M_A D]f(x) &= H(Af')(x) - A(x)\frac{\rm d}{{\rm d}x}(Hf(x)) = H(Af')(x) - A(x)(Hf')(x) \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{A(y)-A(x)}{x-y} f'(y)\,{\rm d}y \\
&= -\lim_{\varepsilon\to 0^+} \Big(\frac{A(y)-A(x)}{x-y} f(y)\Big|_{y=x-\varepsilon}^{y=x+\varepsilon} \Big) \\
&\quad - \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{\rm d}{{\rm d}y}\Big(\frac{A(y)-A(x)}{x-y} \Big) f(y)\,{\rm d}y \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{A(x)-A(y)-A'(y)(x-y)}{(x-y)^2} f(y)\,{\rm d}y.
\end{aligned}
\tag{4.24}
$$

(The fact that the limit in the third line of (4.24) vanishes is ensured by the differentiability of A at x, and the continuity of f at x.) Granted this formula, Theorem 4.1 applies with $n = 2$, $m = 1$, Φ the identity, $B \equiv 0$, and taking $F \in \mathscr{C}^\infty_0(\mathbb{R})$ to be an even function with $F(0) = 1$. The desired conclusion then follows from (4.23).

To offer another example where Theorem 4.1 plays a decisive role, fix some $\varkappa \in (0, \infty)$ and suppose Σ is a $\varkappa$-CAC passing through infinity in $\mathbb{C}$. Recall the Cauchy integral operator on the chord-arc curve Σ acts on $f \in L^1\big(\Sigma, \frac{d\mathcal{H}^1(\zeta)}{1+|\zeta|}\big)$ according to

$$(C_\Sigma f)(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int_{\substack{\zeta\in\Sigma \\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \Sigma. \tag{4.25}$$

Since from Proposition 2.10 we know that Σ is the topological boundary of a UR domain, Proposition 3.4 guarantees that C_Σ is a well-defined, linear, and bounded operator on the space $L^p(\Sigma, w)$ whenever $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$, where $\sigma := \mathcal{H}^1 \lfloor \Sigma$. Let us indicate how Theorem 4.1 may be used to show that

the flatter the chord-arc curve Σ becomes, the closer the corresponding Cauchy operator becomes (with proximity measured in the operator norm on Muckenhoupt weighted Lebesgue spaces) to the (suitably normalized) Hilbert transform on the real line. (4.26)

A brief discussion on this topic may be found in [33, pp. 138-139]. In order to facilitate a direct comparison between the two singular integral operators mentioned in (4.26), it is natural to consider the pull-back of C_Σ to $\mathbb{R}$ under the arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ of Σ. After natural adjustments in notation, this corresponds to the mapping sending each $f \in L^p(\mathbb{R}, w)$ into

$$(C_{\mathbb{R}} f)(t) := \lim_{\varepsilon\to 0^+} \frac{i}{2\pi} \int_{\substack{s\in\mathbb{R} \\ |z(t)-z(s)|>\varepsilon}} \frac{z'(s)}{z(t) - z(s)} f(s)\, ds \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}, \tag{4.27}$$

where $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. Recall from (2.219) that the function $z(\cdot)$ is bi-Lipschitz, specifically,

$$(1 + \varkappa)^{-1}|t - s| \le |z(t) - z(s)| \le |t - s| \quad \text{for all } t, s \in \mathbb{R}. \tag{4.28}$$

Keeping this in mind, a suitable application[1] of [62, Proposition B.2] allows to change the truncation in (4.27) to

$$(C_{\mathbb{R}} f)(t) = \lim_{\varepsilon\to 0^+} \frac{i}{2\pi} \int_{\substack{s\in\mathbb{R} \\ |t-s|>\varepsilon}} \frac{z'(s)}{z(t) - z(s)} f(s)\, ds \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}, \tag{4.29}$$

[1] While [62, Proposition B.2] is stated for ordinary Lebesgue spaces, the same type of result holds in the class of Muckenhoupt weighted Lebesgue spaces (thanks to the fact that the phenomenon in question is local in nature, and (2.576)).

for each $f \in L^p(\mathbb{R}, w)$ with $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. We wish to compare the operator written in this form with the (suitably normalized) Hilbert transform on the real line, acting on arbitrary functions $f \in L^p(\mathbb{R}, w)$, where $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, according to

$$(Hf)(t) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\substack{s \in \mathbb{R} \\ |t-s|>\varepsilon}} \frac{f(s)}{t-s}\,\mathrm{d}s \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}. \tag{4.30}$$

Fix $p \in (1, \infty)$, $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and $f \in L^p(\mathbb{R}, w)$. Then at $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we may express

$$\begin{aligned} \big(C_{\mathbb{R}} - (\mathrm{i}/2)H\big)f(t) &= \lim_{\varepsilon \to 0^+} \frac{\mathrm{i}}{2\pi} \int_{\substack{s \in \mathbb{R} \\ |t-s|>\varepsilon}} \Big(\frac{z'(s)}{z(t)-z(s)} - \frac{1}{t-s}\Big) f(s)\,\mathrm{d}s \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{\substack{s \in \mathbb{R} \\ |t-s|>\varepsilon}} \frac{z(t)-z(s)-z'(s)(t-s)}{(z(t)-z(s))(t-s)} f(s)\,\mathrm{d}s. \end{aligned} \tag{4.31}$$

Pick an even function $\phi \in \mathscr{C}_0^\infty(\mathbb{C})$ satisfying (with $\varkappa$ as in (4.28))

$$\begin{gathered} 0 \le \phi \le 1 \ \text{ and } \ \operatorname{supp}\phi \subseteq B(0,2), \\ \phi \equiv 1 \ \text{ on } \ B(0,1) \setminus B\big(0, (1+\varkappa)^{-1}\big), \\ \phi \equiv 0 \ \text{ on } \ B\big(0, (2+2\varkappa)^{-1}\big), \end{gathered} \tag{4.32}$$

along with a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ which is even and satisfies

$$0 \le \psi \le 1, \quad \operatorname{supp}\psi \subseteq [-4,4], \quad \text{and} \quad \psi \equiv 1 \ \text{ on } \ [-2,2] \setminus \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]. \tag{4.33}$$

We may then invoke Theorem 4.1 with $n := 2$, $m := 3$, and

$$\begin{gathered} \Phi(t) := t, \quad A(t) := z(t), \quad B(t) := \big(\operatorname{Re} z(t), \operatorname{Im} z(t), t\big) \ \text{ for all } \ t \in \mathbb{R}, \\ F(a,b,c) := \frac{c}{a+\mathrm{i}b}\phi(a+\mathrm{i}b)\psi(c) \ \text{ for all } \ (a,b,c) \in \mathbb{R}^3, \end{gathered} \tag{4.34}$$

and conclude from (4.23) and (2.228) that there exist some integer $\widetilde{N} \in \mathbb{N}$ and some constant $C_{p,w} \in (0, \infty)$ such that, with $\varkappa$ as in (4.28), we have

$$\big\| C_{\mathbb{R}} - (\mathrm{i}/2)H \big\|_{L^p(\mathbb{R},w) \to L^p(\mathbb{R},w)} \le C_{p,w}(1+\varkappa)^{\widetilde{N}}\sqrt{\varkappa}. \tag{4.35}$$

This lends credence to (4.26) since it implies

$$\big\|C_{\mathbb{R}} - (\mathrm{i}/2)H\big\|_{L^p(\mathbb{R},w)\to L^p(\mathbb{R},w)} = O(\sqrt{\varkappa}\,) \quad \text{as} \quad \varkappa \to 0^+. \tag{4.36}$$

After this preamble, we are ready to present the proof of Theorem 4.1.

Proof of Theorem 4.1 Throughout, let us abbreviate

$$K(x', y') := \frac{A(x') - A(y') - \langle \nabla A(y'), x' - y'\rangle}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big), \tag{4.37}$$

for each $x' \in \mathbb{R}^{n-1}$ and $\mathcal{L}^{n-1}$-a.e. $y' \in \mathbb{R}^{n-1}$. Having $T_*^{A,B} g(x')$ in (4.56) well defined for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and each $x' \in \mathbb{R}^{n-1}$ is ensured by observing that

$$K(\cdot,\cdot) \text{ is an } \mathcal{L}^{n-1} \otimes \mathcal{L}^{n-1}\text{-measurable function on } \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \tag{4.38}$$

which is clear from (4.37), and

$$\begin{aligned} &\text{for each } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}),\ \varepsilon > 0,\ x' \in \mathbb{R}^{n-1}, \\ &\quad \text{one has} \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} |K(x', y')||g(y')|\,\mathrm{d}y' < +\infty. \end{aligned} \tag{4.39}$$

The finiteness property in (4.39) is a consequence of Hölder's inequality, (4.37), the fact that F is bounded, and Lemma 4.1 (used with n replaced by $n-1$, p' in place of p, and with w' in place of w). In concert, these give that for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, each $\varepsilon > 0$, and each $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} |K(x', y')||g(y')|\,\mathrm{d}y' &\leq C\varepsilon\big[w'\big(B(x',\varepsilon)\big)\big]^{1/p'}\Big(\sup_{X\in\mathbb{R}^m} |F(X)|\Big)\times \\ &\quad \times \|g\|_{L^p(\mathbb{R}^{n-1},w\mathcal{L}^{n-1})}\big\|\nabla A\big\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \infty. \end{aligned} \tag{4.40}$$

To proceed, for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, each truncation parameter $\varepsilon > 0$, and each point $x' \in \mathbb{R}^{n-1}$ define

$$T^{A,B}_{\Phi,\varepsilon} g(x') := \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |\Phi(x')-\Phi(y')|>\varepsilon}} K(x', y')g(y')\,\mathrm{d}y'. \tag{4.41}$$

Thanks to (4.20) and (4.38)–(4.39), the above integral is absolutely convergent, which means that $T^{A,B}_{\Phi,\varepsilon} g(x')$ is a well-defined number. If $\mathbb{Q}_+$ denotes the collection of all positive rational numbers, we next make the claim that for each arbitrary function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we have

$$\big(T^{A,B}_{\Phi,*}\, g\big)(x') = \sup_{\varepsilon\in\mathbb{Q}_+} \big|\big(T^{A,B}_{\Phi,\varepsilon}\, g\big)(x')\big| \quad \text{for every} \quad x' \in \mathbb{R}^{n-1}. \tag{4.42}$$

To justify this, pick some $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. The idea is to show that if the point $x' \in \mathbb{R}^{n-1}$ is arbitrary and fixed then for every $\varepsilon \in (0,\infty)$ and for every sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$ we have

$$\lim_{j\to\infty} \big(T^{A,B}_{\Phi,\varepsilon_j} g\big)(x') = \big(T^{A,B}_{\Phi,\varepsilon}\, g\big)(x'). \tag{4.43}$$

To justify (4.43) note that

$$\{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\} \nearrow \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\} \tag{4.44}$$

as $j \to \infty$, in the sense that

$$\begin{aligned}\{y' \in \mathbb{R}^{n-1} :& |\Phi(x') - \Phi(y')| > \varepsilon\}\\ &= \bigcup_{j\in\mathbb{N}} \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\}\end{aligned} \tag{4.45}$$

and

$$\begin{aligned}\{y' \in \mathbb{R}^{n-1} :& |\Phi(x') - \Phi(y')| > \varepsilon_j\}\\ &\subseteq \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_{j+1}\}\end{aligned} \tag{4.46}$$

for every $j \in \mathbb{N}$. Then (4.43) follows from (4.44) and Lebesgue's Dominated Convergence Theorem (whose applicability is ensured by (4.38)–(4.39)). Having established this, (4.42) readily follows on account of the density of $\mathbb{Q}_+$ in $(0,\infty)$.

Moving on, we claim that

$$\begin{gathered}\text{for each fixed threshold } \varepsilon > 0, \text{ the function}\\ \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \ni (x', y') \longmapsto \big(\mathbf{1}_{\{y'\in\mathbb{R}^{n-1},\, |\Phi(x')-\Phi(y')|>\varepsilon\}}\big)(y') \in \mathbb{R}\\ \text{is lower-semicontinuous, hence } \mathcal{L}^{n-1} \otimes \mathcal{L}^{n-1}\text{-measurable.}\end{gathered} \tag{4.47}$$

To justify this claim, observe that for every number $\lambda \in \mathbb{R}$ the set of points in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ where the given function is $> \lambda$ may be described as

$$\begin{cases} \varnothing \text{ if } \lambda \geq 1, \\ \big\{(x',y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\big\} \text{ if } \lambda \in [0,1), \\ \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \text{ if } \lambda < 0. \end{cases} \tag{4.48}$$

Thanks to the fact that Φ is a continuous function, all sets appearing in (4.48) are open in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. This proves that the function (4.47) is indeed lower-semicontinuous.

We next claim that

$$\text{given any } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}), \text{ the function } T^{A,B}_{\Phi,*}g \text{ is } \mathcal{L}^{n-1}\text{-measurable.} \tag{4.49}$$

To see that this is the case, granted (4.42) and since the supremum of some countable family of $\mathcal{L}^{n-1}$-measurable functions is itself a $\mathcal{L}^{n-1}$-measurable function, it suffices to show that

$$T^{A,B}_{\Phi,\varepsilon}g \text{ is a } \mathcal{L}^{n-1}\text{-measurable function, for each fixed } \varepsilon \in (0,\infty) \text{ and each fixed } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}). \tag{4.50}$$

With this goal in mind, fix $\varepsilon \in (0,\infty)$ along with $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, and for each $j \in \mathbb{N}$ define

$$\begin{aligned} &G_j : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \text{ given at every } (x',y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \text{ by} \\ &G_j(x',y') := \big(\mathbf{1}_{B(0',j)}\big)(x')K(x',y')g(y')\big(\mathbf{1}_{\{y' \in \mathbb{R}^{n-1},\, |\Phi(x')-\Phi(y')|>\varepsilon\}}\big)(y'). \end{aligned} \tag{4.51}$$

Then, thanks to (4.38) and (4.47), it follows that G_j is an $\mathcal{L}^{n-1} \otimes \mathcal{L}^{n-1}$-measurable function for each $j \in \mathbb{N}$. In addition, from (4.51), (4.39), and since balls have finite measure, we see that

$$\int_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} |G_j(x',y')|\, dx'dy' < +\infty. \tag{4.52}$$

Granted these properties, Fubini's Theorem (whose applicability is ensured by the fact that $(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ is a sigma-finite measure space) then guarantees that

$$\begin{aligned} &g_j : \mathbb{R}^{n-1} \to \mathbb{R}, \quad g_j(x') := \int_{\mathbb{R}^{n-1}} G_j(x',y')\, dy', \quad \forall\, x' \in \mathbb{R}^{n-1}, \\ &\text{is an } \mathcal{L}^{n-1}\text{-measurable function, for each integer } j \in \mathbb{N}. \end{aligned} \tag{4.53}$$

On the other hand, from (4.51), (4.53), and (4.41) it is apparent that for each $j \in \mathbb{N}$ we have

$$g_j = \mathbf{1}_{B(0',j)}\, T^{A,B}_{\Phi,\varepsilon} g \;\; \text{everywhere in} \;\; \mathbb{R}^{n-1}. \tag{4.54}$$

In particular, this implies

$$\lim_{j\to\infty} g_j = T^{A,B}_{\Phi,\varepsilon} g \;\; \text{pointwise everywhere in} \;\; \mathbb{R}^{n-1}. \tag{4.55}$$

At this stage, the fact that $T^{A,B}_{\Phi,\varepsilon} g$ is an $\mathcal{L}^{n-1}$-measurable function follows from (4.55) and (4.53). The claim in (4.49) is therefore established.

We next turn our attention to the main claim made in (4.23). The special case when $d := n-1$ and $\Phi(x') := x'$ for each $x' \in \mathbb{R}^{n-1}$ has been treated in [61], following basic work in [58]. Specifically, from [61, Theorem 4.34, p. 2725] we know that if for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we define

$$T^{A,B}_{*} g(x') := \sup_{\varepsilon>0} \Bigg| \int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} K(x',y') g(y')\, dy' \Bigg| \;\; \text{at each} \;\; x' \in \mathbb{R}^{n-1}, \tag{4.56}$$

then

$$\begin{array}{l} T^{A,B}_{*} \;\text{ is a well-defined sub-linear operator} \\ \text{from the space } L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \text{ into itself} \end{array} \tag{4.57}$$

and there exists a constant $C(n,p,w) \in (0,\infty)$ with the property that

$$\begin{aligned} &\Big\| T^{A,B}_{*} \Big\|_{L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \to L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})} \\ &\quad \le C(n,p,w) \Big(\sum_{|\alpha|\le N+2} \big\| \partial^\alpha F \big\|_{L^1(\mathbb{R}^m, \mathcal{L}^m)} + \sup_{X\in\mathbb{R}^m} (1+|X|)|F(X)| \Big) \\ &\qquad \times \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \Big(1 + \sum_{j=1}^m \big\| \nabla B_j \big\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \Big)^N . \end{aligned} \tag{4.58}$$

To deal with the present case, in which the truncation is performed in the more general fashion described in (4.22), for each $\varepsilon > 0$ and each $x' \in \mathbb{R}^{n-1}$ abbreviate

$$\begin{aligned} D_\varepsilon(x') :=& \big\{ y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon \text{ and } |x'-y'| \le \varepsilon \big\} \\ &\bigcup \big\{ y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| \le \varepsilon \text{ and } |x'-y'| > \varepsilon \big\}. \end{aligned} \tag{4.59}$$

Fix an arbitrary $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and define

$$Rg(x') := \sup_{\varepsilon>0} \int_{D_\varepsilon(x')} \left| \frac{A(x') - A(y') - \langle \nabla A(y'), x' - y' \rangle}{|x' - y'|^n} \times \right. \tag{4.60}$$

$$\left. \times\, F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| \mathrm{d}y'$$

at each point $x' \in \mathbb{R}^{n-1}$. The above definitions now imply that for each given function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we have

$$T^{A,B}_{\Phi,*} g(x') \leq T^{A,B}_{*} g(x') + Rg(x') \quad \text{for every} \quad x' \in \mathbb{R}^{n-1}. \tag{4.61}$$

To estimate the last term appearing in the right-hand side of (4.61), pick some

$$\gamma \in (0, p-1) \text{ such that } w \in A_{p/(1+\gamma)}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{4.62}$$

fix an arbitrary point $x' \in \mathbb{R}^{n-1}$, consider an arbitrary threshold $\varepsilon > 0$, and select a function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. Also, abbreviate

$$Q := Q_{x',\varepsilon} := \big\{ y' \in \mathbb{R}^{n-1} : |x' - y'| < \varepsilon \big\} \tag{4.63}$$

and introduce

$$A_Q(z') := A(z') - \Big(⨍_Q \nabla A \, \mathrm{d}\mathcal{L}^{n-1} \Big) \cdot z' \quad \text{for each} \quad z' \in \mathbb{R}^{n-1}. \tag{4.64}$$

Observe that the number $Rg(x')$, originally defined in (4.60), does not change if the function A is replaced by A_Q. Consequently,

$$Rg(x') \leq R_1 g(x') + R_2 g(x'), \tag{4.65}$$

where

$$R_1 g(x') := \sup_{\varepsilon>0} \int_{D_\varepsilon(x')} \left| \frac{A_Q(x') - A_Q(y')}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| \mathrm{d}y' \tag{4.66}$$

and

$$R_2 g(x') := \sup_{\varepsilon>0} \int_{D_\varepsilon(x')} \left| \frac{\langle \nabla A_Q(y'), x' - y' \rangle}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| \mathrm{d}y'. \tag{4.67}$$

Note that, thanks to (4.20) and (4.59), we have

$$c\,\varepsilon \leq |x' - y'| \leq c^{-1}\varepsilon \quad \text{for each} \quad y' \in D_\varepsilon(x'). \tag{4.68}$$

Having fixed an integrability exponent $q \in (n-1, \infty)$, for each $y' \in D_\varepsilon(x')$ we may rely on Mary Weiss' Lemma (cf. [24, Lemma 1.4, p. 144]) in concert with (2.102), (2.103), (4.63), and (4.68) to estimate

$$\begin{aligned}
\frac{|A_Q(x') - A_Q(y')|}{|x' - y'|} &\leq C_{q,n}\Big(\fint_{|x'-z'|\leq 2|x'-y'|} |\nabla A_Q(z')|^q \,\mathrm{d}z'\Big)^{1/q} \\
&\leq C_{q,n}\Big(\fint_{|x'-z'|\leq 2|x'-y'|} \Big|\nabla A(z') - \fint_{|x'-\zeta'|\leq 2|x'-y'|} \nabla A(\zeta')\,\mathrm{d}\zeta'\Big|^q \,\mathrm{d}z'\Big)^{1/q} \\
&\quad + C_{q,n}\Big|\fint_Q \nabla A\,\mathrm{d}\mathcal{L}^{n-1} - \fint_{|x'-\zeta'|\leq 2|x'-y'|} \nabla A(\zeta')\,\mathrm{d}\zeta'\Big| \\
&\leq C_{q,n} \cdot c^{-2(n-1)/q} \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}.
\end{aligned} \tag{4.69}$$

Choosing $q := 2(n-1)$ it follows that there exists a constant $C_n \in (0, \infty)$, which depends only on n, such that

$$\begin{aligned}
|A_Q(x') - A_Q(y')| &\leq (C_n/c)|x' - y'|\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \\
&\quad \text{for each point} \quad y' \in D_\varepsilon(x').
\end{aligned} \tag{4.70}$$

In concert, (4.66), (4.68), and (4.70) allow us to conclude that

$$\begin{aligned}
R_1 g(x') \leq C_n \cdot c^{1-2n}\Big(\sup_{X\in\mathbb{R}^m} |F(X)|\Big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \times \\
\times \sup_{\varepsilon>0}\Big(\fint_{|x'-y'|<c^{-1}\varepsilon} |g(y')|\,\mathrm{d}y'\Big).
\end{aligned} \tag{4.71}$$

To estimate $R_2 g(x')$, bring in a brand of the Hardy–Littlewood maximal operator which associates to each $\mathcal{L}^{n-1}$-measurable function f on $\mathbb{R}^{n-1}$ the function $\mathcal{M}_\gamma f$ defined as

$$\mathcal{M}_\gamma f(x') := \sup_{r>0}\Big(\fint_{|x'-y'|<r} |f(y')|^{1+\gamma}\,\mathrm{d}y'\Big)^{1/(1+\gamma)} \quad \text{for each} \quad x' \in \mathbb{R}^{n-1}. \tag{4.72}$$

Then, using (4.67), (4.64), Hölder's inequality, and (2.103) we may write

$$\begin{aligned}
R_2 g(x') &\leq C_n \cdot c^{2-2n}\Big(\sup_{X\in\mathbb{R}^m} |F(X)|\Big) \times \\
&\quad \times \sup_{\varepsilon>0}\Big(\fint_{|x'-y'|<c^{-1}\varepsilon} \Big|\nabla A(y') - \fint_Q \nabla A\,\mathrm{d}\mathcal{L}^{n-1}\Big||g(y')|\,\mathrm{d}y'\Big)
\end{aligned}$$

$$\le C_n \cdot c^{2-2n}\Big(\sup_{X\in\mathbb{R}^m}|F(X)|\Big)\mathcal{M}_\gamma g(x')\times$$

$$\times \sup_{\varepsilon>0}\Big(⨍_{|x'-y'|<c^{-1}\varepsilon}\Big|\nabla A(y') - ⨍_Q \nabla A\,\mathrm{d}\mathcal{L}^{n-1}\Big|^{(1+\gamma)/\gamma}\,\mathrm{d}y'\Big)^{\gamma/(1+\gamma)}$$

$$\le C_{n,\gamma}\cdot c^{3-3n}\Big(\sup_{X\in\mathbb{R}^m}|F(X)|\Big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\mathcal{M}_\gamma g(x'). \tag{4.73}$$

Collectively, (4.65), (4.71), and (4.73), and Hölder's inequality imply

$$Rg(x') \le C_{n,\gamma}\cdot c^{-3n}\Big(\sup_{X\in\mathbb{R}^m}|F(X)|\Big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\mathcal{M}_\gamma g(x'). \tag{4.74}$$

In turn, from (4.74) and (4.61) we conclude that for every $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned}0 \le\,& T^{A,B}_{\Phi,*}g(x') \\ &\le T^{A,B}_{*}g(x') + C_{n,\gamma}\cdot c^{-3n}\Big(\sup_{X\in\mathbb{R}^m}|F(X)|\Big)\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\mathcal{M}_\gamma g(x').\end{aligned} \tag{4.75}$$

Granted (4.62), the maximal operator $\mathcal{M}_\gamma$ is a well-defined sub-linear bounded mapping from $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself. Bearing this in mind, from (4.75), (4.57), (4.58), and (2.575), and the fact that the space $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ is a lattice, the estimate claimed in (4.23) now follows. As a consequence, $T^{A,B}_{\Phi,*}$ is a sub-linear mapping of finite operator norm on $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. Hence, as remarked in (4.3), the operator $T^{A,B}_{\Phi,*}$ is continuous from $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself. □

The next step is to transfer the Euclidean result from Theorem 4.1 to singular integral operators on Lipschitz graphs, a task accomplished in the following lemma.

Lemma 4.2 *Having fixed an arbitrary unit vector $\vec{n} \in S^{n-1}$, consider the hyperplane $H := \langle\vec{n}\rangle^{\perp} \subseteq \mathbb{R}^{n-1}$ and suppose $h : H \to \mathbb{R}$ is a function satisfying*

$$M := \sup_{\substack{x,y\in H\\ x\neq y}}\frac{|h(x)-h(y)|}{|x-y|} < +\infty. \tag{4.76}$$

Fix an arbitrary point $x_0 \in \mathbb{R}^n$ and let

$$\mathcal{G} := \big\{x_0 + x + h(x)\vec{n} \,:\, x \in H\big\} \subseteq \mathbb{R}^n \tag{4.77}$$

denote the graph of h in the coordinate system $X = (x,t) \Leftrightarrow X = x_0 + x + t\vec{n}$, with $x \in H$ and $t \in \mathbb{R}$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\mathcal{G}$ and denote by ν the unique unit normal to $\mathcal{G}$ satisfying $\nu\cdot\vec{n} < 0$ at σ-a.e. point on $\mathcal{G}$. Also, fix some integrability exponent

$p \in (1,\infty)$. *Given a complex-valued function* $k \in \mathscr{C}^{N+2}(\mathbb{R}^n \setminus \{0\})$, *for some sufficiently large integer* $N = N(n) \in \mathbb{N}$, *which is even and positive homogeneous of degree* $-n$, *consider the maximal singular integral operator* T *acting on each* $f \in L^p(\mathcal{G},\sigma)$ *as*

$$T_* f(x) := \sup_{\varepsilon>0} \Big| \int_{\substack{y\in\mathcal{G}\\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \Big|, \quad \forall\, x \in \mathcal{G}. \tag{4.78}$$

Then T_* *is a well-defined continuous sub-linear mapping from the space* $L^p(\mathcal{G},\sigma)$ *into itself and there exists a constant* $C(n,p) \in (0,\infty)$, *which depends only on* n, p, *with the property that*

$$\|T_*\|_{L^p(\mathcal{G},\sigma)\to L^p(\mathcal{G},\sigma)} \leq C(n,p) M (1+M)^{4n+N} \Big(\sum_{|\alpha|\leq N+2} \sup_{S^{n-1}} \big|\partial^\alpha k\big| \Big). \tag{4.79}$$

Moreover, corresponding to the end-point case $p = 1$, *the operator* T_* *induces a well-defined continuous sub-linear mapping from the space* $L^1(\mathcal{G},\sigma)$ *into the space* $L^{1,\infty}(\mathcal{G},\sigma)$ *and there exists a constant* $C_n \in (0,\infty)$ *along with some large exponent* $N_n \in \mathbb{N}$, *which depend only on n, with the property that*

$$\|T_*\|_{L^1(\mathcal{G},\sigma)\to L^{1,\infty}(\mathcal{G},\sigma)} \leq C_n (1+M)^{N_n} \Big(\sum_{|\alpha|\leq N_n} \sup_{S^{n-1}} \big|\partial^\alpha k\big| \Big). \tag{4.80}$$

Proof Recall that $\{\mathbf{e}_j\}_{1\leq j\leq n}$ stands for the standard orthonormal basis in $\mathbb{R}^n$. Let us first treat the case when $x_0 = 0 \in \mathbb{R}^n$ and $\vec{n} := \mathbf{e}_n$, a scenario in which $H = \langle \mathbf{e}_n\rangle^\perp$ may be canonically identified with $\mathbb{R}^{n-1}$. Assume this is the case, and consider an even function $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that

$$\begin{aligned} &0 \leq \psi \leq 1, \quad \psi \text{ vanishes identically in } \mathbb{R}^n \setminus B\big(0, 2\sqrt{1+M^2}\,\big), \\ &\psi \equiv 1 \text{ on } \overline{B(0,\sqrt{1+M^2})} \setminus B(0,1), \quad \psi \equiv 0 \text{ on } B(0,1/2), \\ &\text{and for each } \alpha \in \mathbb{N}_0^n \text{ there exists } C_\alpha \in (0,\infty), \text{ depending only} \\ &\text{on the given multi-index } \alpha, \text{ so that } \sup\nolimits_{x\in\mathbb{R}^n} |(\partial^\alpha \psi)(x)| \leq C_\alpha. \end{aligned} \tag{4.81}$$

Then $F := \psi k$ is an even function belonging to $\mathscr{C}^{N+2}(\mathbb{R}^n)$, and satisfying

$$\sum_{|\alpha|\leq N+2} \big\|\partial^\alpha F\big\|_{L^1(\mathbb{R}^n,\mathcal{L}^n)} + \sup_{x\in\mathbb{R}^n} (1+|x|)|F(x)| \leq C_n (1+M)^n \Big(\sum_{|\alpha|\leq N+2} \sup_{S^{n-1}} \big|\partial^\alpha k\big| \Big), \tag{4.82}$$

for some purely dimensional constant $C_n \in (0,\infty)$. Moreover, if for each point $x' \in \mathbb{R}^{n-1}$ we set $\Phi(x') := (x', h(x'))$ then $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a bi-Lipschitz function and (4.81) implies that

$$k\left(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\right) = F\left(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\right) \quad \text{for each } x', y' \in \mathbb{R}^{n-1} \text{ with } x' \neq y'. \tag{4.83}$$

To proceed, note that for each σ-measurable set $E \subseteq \mathcal{G}$ and each $g \in L^1(E,\sigma)$ we have

$$\int_E g\,\mathrm{d}\sigma = \int_{\{y'\in\mathbb{R}^{n-1}:\,(y',h(y'))\in E\}} g(y',h(y'))\sqrt{1+|(\nabla h)(y')|^2}\,\mathrm{d}y', \tag{4.84}$$

(cf., e.g., [136, Proposition 12.9, p. 164]) and

$$\nu(y',h(y')) = \frac{((\nabla h)(y'),-1)}{\sqrt{1+|(\nabla h)(y')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } y' \in \mathbb{R}^{n-1}. \tag{4.85}$$

Also, fix $f \in L^p(\mathcal{G},\sigma)$ and define $\widetilde{f}(x') := f(x', h(x'))$ for each $x' \in \mathbb{R}^{n-1}$. In particular, from (4.84) we conclude that

$$\widetilde{f} \in L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \quad \text{and} \quad \|\widetilde{f}\|_{L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \leq \|f\|_{L^p(\mathcal{G},\sigma)}. \tag{4.86}$$

Then based on (4.78), (4.84), (4.85), the homogeneity of k, and (4.83) we may write

$$\begin{aligned}
&(T_* f)(x',h(x')) \\
&= \sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{y'\in\mathbb{R}^{n-1} \text{ with} \\ \sqrt{|x'-y'|^2+(h(x')-h(y'))^2}>\varepsilon}} \big(\langle\nabla h(y'), x'-y'\rangle + h(y') - h(x')\big)\times \\
&\qquad\qquad \times k\big(x'-y', h(x')-h(y')\big)\widetilde{f}(y')\,\mathrm{d}y'\Bigg| \\
&= \sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{y'\in\mathbb{R}^{n-1} \\ |\Phi(x')-\Phi(y')|>\varepsilon}} \frac{h(x')-h(y')-\langle\nabla h(y'), x'-y'\rangle}{|x'-y'|^n}\times
\end{aligned}$$

$$\times F\Big(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\Big)\widetilde{f}(y')\,\mathrm{d}y'\Big|. \tag{4.87}$$

From (4.87), Theorem 4.1 (used with $m := n$, $d := n$, $A := h$, $B := \Phi$, and $w \equiv 1$), (4.82), and (4.84) we then conclude that (4.79) holds in this case.

To treat the case when $x_0 = 0$ but $\vec{n} \in S^{n-1}$ is arbitrary, pick an orthonormal basis $\{v_j\}_{1\leq j\leq n-1}$ in H and consider the unitary transformation in $\mathbb{R}^n$ uniquely defined by the demand that $Uv_j = \mathbf{e}_j$ for $j \in \{1,\dots,n-1\}$ and $U\vec{n} = \mathbf{e}_n$. Then $\widetilde{\mathcal{G}} := U\mathcal{G}$ becomes the graph of $\widetilde{h} := h \circ U^{-1} : \mathbb{R}^{n-1} \to \mathbb{R}$, which is a Lipschitz function with the same Lipschitz constant M as the original function h. Since the Hausdorff measure is rotation invariant, for each $g \in L^1(\mathcal{G},\sigma)$ we have

$$\int_{y\in\mathcal{G}} g(y)\,\mathrm{d}\sigma(y) = \int_{\widetilde{y}\in\widetilde{\mathcal{G}}} (g\circ U^{-1})(\widetilde{y})\,\mathrm{d}\widetilde{\sigma}(\widetilde{y}), \tag{4.88}$$

where $\widetilde{\sigma} := \mathcal{H}^{n-1}\lfloor\widetilde{\mathcal{G}}$. Moreover, the unique unit normal $\widetilde{\nu}$ to $\widetilde{\mathcal{G}}$ satisfying $\widetilde{\nu}\cdot\mathbf{e}_n < 0$ at $\mathcal{H}^{n-1}$-a.e. point on $\widetilde{\mathcal{G}}$ is $\widetilde{\nu} = U(\nu\circ U^{-1})$. Consider $\widetilde{k} := k\circ U^{-1}$ and note that this is a complex-valued function of class $\mathscr{C}^{N+2}\big(\mathbb{R}^n\setminus\{0\}\big)$, which is even and positive homogeneous of degree $-n$. Finally, fix some function $f \in L^p(\mathcal{G},\sigma)$ and abbreviate $\widetilde{f} := f\circ U^{-1}$. Bearing in mind the fact that U is a linear isometry satisfying $U^{-1} = U^\top$, from (4.78) and (4.88) we see that if $x \in \mathcal{G}$ and $\widetilde{x} := Ux$ then

$$T_* f(x) = \sup_{\varepsilon>0}\Big|\int_{\substack{\widetilde{y}\in\widetilde{\mathcal{G}}\\ |\widetilde{x}-\widetilde{y}|>\varepsilon}} \langle\widetilde{x}-\widetilde{y},\widetilde{\nu}(\widetilde{y})\rangle\widetilde{k}(\widetilde{x}-\widetilde{y})\widetilde{f}(\widetilde{y})\,\mathrm{d}\widetilde{\sigma}(\widetilde{y})\Big|. \tag{4.89}$$

Hence,

$$T_* f(x) = \widetilde{T}_*\widetilde{f}(\widetilde{x}) \quad\text{whenever}\quad x\in\mathcal{G} \quad\text{and}\quad \widetilde{x} = Ux, \tag{4.90}$$

where $\widetilde{T}_*$ is the maximal operator associated as in (4.78) with the Lipschitz graph $\widetilde{\mathcal{G}}$ and the kernel $\widetilde{k}$. In particular, given that (4.90) and (4.88) imply

$$\int_{\mathcal{G}} (T_* f)(x)^p\,\mathrm{d}\sigma(x) = \int_{\widetilde{\mathcal{G}}} (\widetilde{T}_*\widetilde{f})(\widetilde{x})^p\,\mathrm{d}\widetilde{\sigma}(\widetilde{x}), \tag{4.91}$$

the estimate claimed in (4.79) becomes a consequence of the corresponding estimate for the maximal operator $\widetilde{T}_*$ established in the first part of the current proof.

The case when both $x_0 \in \mathbb{R}^n$ and $\vec{n} \in S^{n-1}$ are arbitrary follows from what we have proved so far using the natural invariance of the maximal operator (4.78) to translations.

Finally, the estimate claimed in (4.80) becomes a consequence of (4.79) (with, say, the choice $p = 2$), and standard Calderón–Zygmund theory (based on the classical Calderón–Zygmund Lemma, and Cotlar's inequality). See, for example, [56, Theorem 8.2.1, p. 584] for more details in the standard Euclidean setting. □

4.2 Estimates for Certain Classes of Singular Integrals on UR Sets

Theorem 4.2, which is central for the present work, is the main result regarding the size of the operator norm of certain maximal integral operators acting on Muckenhoupt weighted Lebesgue spaces on the boundary of UR domains. In turn, this is going to be the key ingredient in obtaining invertibility results for the brand of boundary double layer potential operators considered in this work.

To facilitate stating Theorem 4.2 we first introduce some notation and make some remarks. Specifically, with e denoting the base of natural logarithms, for each number $m \in \mathbb{N}_0$ and $t \in [0, \infty)$ let us define

$$t^{\langle 0 \rangle} := 1 \tag{4.92}$$

and, if $m \geq 1$,

$$t^{\langle m \rangle} := \begin{cases} 0 & \text{if } \ t = 0, \\ t \cdot \underbrace{\ln\Big(\cdots \ln\big(\ln(1/t)\big)\cdots\Big)}_{m \text{ natural logarithms}} & \text{if } \ 0 < t \leq ({}^{m}\mathrm{e})^{-1}, \\ ({}^{m}\mathrm{e})^{-1} & \text{if } \ t > ({}^{m}\mathrm{e})^{-1}, \end{cases} \tag{4.93}$$

where ${}^{m}\mathrm{e}$ is the m-th tetration of e (involving m copies of e, combined by exponentiation), i.e.,

$${}^{m}\mathrm{e} := \underbrace{\mathrm{e}^{\mathrm{e}^{\cdot^{\cdot^{\cdot^{\mathrm{e}}}}}}}_{m \text{ copies of e}}, \quad \text{the } m\text{-th fold exponentiation of e.} \tag{4.94}$$

We also agree to set ${}^{0}\mathrm{e} := 1$. Hence, inductively, for each integer $m \in \mathbb{N}_0$ and each $t \in [0, \infty)$ we have

$$t^{\langle m+1\rangle} = \begin{cases} 0 & \text{if } t = 0, \\ t \cdot \ln\big(t^{\langle m\rangle}/t\big) & \text{if } 0 < t \leq ({}^{m+1}\mathrm{e})^{-1}, \\ ({}^{m+1}\mathrm{e})^{-1} & \text{if } t > ({}^{m+1}\mathrm{e})^{-1}. \end{cases} \tag{4.95}$$

For further reference, it is useful to note that elementary calculus gives that this function enjoys the following properties:

$$[0,\infty) \ni t \longmapsto t^{\langle m\rangle} \in [0,\infty) \ \text{ is continuous, non-decreasing,} \tag{4.96}$$

$$\begin{gathered} t^{\langle m\rangle} \leq t^{\langle m-1\rangle} \leq \cdots \leq t^{\langle 1\rangle} \leq \big(\mathrm{e}^{\varepsilon-1}/\varepsilon\big) \cdot t^{1-\varepsilon} \\ \text{for each } \ t \in [0,\infty), \ m \in \mathbb{N}, \ \varepsilon \in (0,1), \end{gathered} \tag{4.97}$$

$$t \leq \max\{1, ({}^{m}\mathrm{e})t\} \cdot t^{\langle m\rangle} \ \text{ for all } \ t \in [0,\infty) \ \text{ and } \ m \in \mathbb{N}_0, \tag{4.98}$$

$$(\lambda t)^{\langle m\rangle} \leq \lambda t^{\langle m\rangle} \ \text{ for all } \ t \in [0,\infty), \ m \in \mathbb{N}_0, \ \text{ and } \ \lambda \in [1,\infty), \tag{4.99}$$

$$\begin{gathered} (t^{\alpha})^{\langle m\rangle} \leq t^{\alpha} \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln(1/\min\{t, ({}^{m}\mathrm{e})^{-1}\})\big)\cdots\Big) \\ \text{for all } \ t \in [0,\infty), \ m \in \mathbb{N}, \ \text{ and } \ \alpha \in (0,1] \end{gathered} \tag{4.100}$$

(with the convention that the value at $t = 0$ for the function in the right-hand side of the inequality in (4.100) is its limit as $t \to 0^+$). In particular,

$$t^{\langle m\rangle} \leq t \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/t)\big)\cdots\Big) \ \text{ for all } \ t \in [0,1], \ m \in \mathbb{N}. \tag{4.101}$$

In fact, up to a multiplicative constant, the opposite inequality in (4.101) is true as well. Specifically,

$$({}^{m}\mathrm{e})^{-1} \cdot t \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/t)\big)\cdots\Big) \leq t^{\langle m\rangle} \ \text{ for all } \ t \in [0,1], \ m \in \mathbb{N}, \tag{4.102}$$

hence for each fixed $m \in \mathbb{N}$ we have

$$t^{\langle m\rangle} \approx t \cdot \underbrace{\ln\Big(\cdots \ln}_{m \text{ natural logarithms}}\big(\ln({}^{m}\mathrm{e}/t)\big)\cdots\Big), \ \text{ uniformly for } \ t \in [0,1]. \tag{4.103}$$

Here is the basic result mentioned earlier. Its proof is inspired by that of [61, Theorem 4.36, pp. 2728-2729].

Theorem 4.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and recall the earlier convention of using the same symbol w for the measure associated with the given weight w as in* (2.509).

Next, consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, where $N = N(n) \in \mathbb{N}$ is a sufficiently large integer. Associate with this function and the set Ω the maximal operator T_* whose action on each given function $f \in L^p(\partial\Omega, w)$ is defined as

$$T_* f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big| \quad \textit{for each} \quad x \in \partial\Omega, \tag{4.104}$$

where, for each $\varepsilon > 0$,

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x - y, \nu(y)\rangle k(x - y) f(y)\,\mathrm{d}\sigma(y) \quad \textit{for all} \quad x \in \partial\Omega. \tag{4.105}$$

Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$, which depends only on m, n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93)*, one has*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.106}$$

Moreover, when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0, \infty)$ appearing in (4.106) *to depend itself only on said entities (i.e., n, p, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$) and m.*

Before presenting the proof of this theorem, several comments are in order.

Remark 4.1 It is of interest to compare the estimate in the above theorem with the corresponding estimate from Proposition 3.4. Specifically, estimate (3.79) applied with $\Sigma := \partial\Omega$ gives that for T_* as in (4.104) we have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C(\partial\Omega, p, [w]_{A_p}) \big\|k\big|_{S^{n-1}}\big\|_{\mathscr{C}^N(S^{n-1})}, \tag{4.107}$$

where $C(\partial\Omega, p, [w]_{A_p}) \in (0, \infty)$ depends on $\partial\Omega$ solely through its UR constants. We observe that, in sharp contrast to this estimate, (4.106) features in the right-hand side $\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ as a multiplicative factor, something which the UR constants of $\partial\Omega$ cannot control.

Indeed, for (3.79) no provisions are in place to take advantage of the specific algebraic format of the present integral kernel $\langle x-y, \nu(y)\rangle k(x-y)$. For Proposition 3.4 to apply, this integral kernel needs to be dismantled into its most primordial building blocks, i.e., as $\sum_{j=1}^{n} k_j(x-y)\nu_j(y)$ with $k_j(z) := z_j k(z)$ for each point $z \in \mathbb{R}^n \setminus \{0\}$ and $j \in \{1, \dots, n\}$. Since multiplication by ν_j may be absorbed with the function f (without changing its membership, or increasing its size, in the Muckenhoupt weighted Lebesgue space $L^p(\partial\Omega, w)$), Proposition 3.4 may then finally be invoked in relation to each maximal operator associated with the kernel k_j. Estimate (3.79), the end-product of such an approach, is then rendered insensitive to the flatness of $\partial\Omega$.

As an example, consider the scenario in which Ω is a half-space in $\mathbb{R}^n$. While is apparent from (4.104)–(4.105) that in this case $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} = 0$, estimate (3.79) only gives $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} < +\infty$. By way of contrast, since in this case $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 0$ given that ν is a constant vector, (4.106) accurately predicts $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} = 0$.

Remark 4.2 In view of (2.118) and (4.103), in the estimate recorded in (4.106) we could use

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \cdot \underbrace{\ln\Big(\cdots \ln\big(\ln(}_{m \text{ natural logarithms}}{}^{m}\mathrm{e}/\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n})\big)\cdots\Big) \tag{4.108}$$

in place of $\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$. In particular, if we abbreviate

$$\|\nu\|_* := \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.109}$$

then corresponding to $m = 1$ we thus obtain

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \le C_\Omega\Big(\sum_{|\alpha|\le N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|\nu\|_* \ln\big(\mathrm{e}/\|\nu\|_*\big), \tag{4.110}$$

corresponding to $m = 2$ we have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \le C_\Omega\Big(\sum_{|\alpha|\le N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|\nu\|_* \ln\Big(\ln\big(\mathrm{e}^{\mathrm{e}}/\|\nu\|_*\big)\Big), \tag{4.111}$$

etc., where in each case $C_\Omega \in (0, \infty)$ depends only on n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$. In particular, all the aforementioned operator norms have at most linear growth in $\|\nu\|_*$, up to arbitrarily many iterated logarithms.

In the same vein, we may rely on the property recorded in (4.97) and we deduce from (4.106) that for each $\varepsilon \in (0, 1)$ we have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq \big(\mathrm{e}^{\varepsilon-1}/\varepsilon\big)\cdot C_\Omega\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{1-\varepsilon}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.112}$$

where $C_\Omega \in (0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$.

Remark 4.3 In the context of Theorem 4.2, estimate (4.106) continues to hold with a fixed constant $C_m \in (0,\infty)$ when the integrability exponent and the Muckenhoupt weight are allowed to vary with control. Specifically, an inspection of the proof of Theorem 4.2 given below shows that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a constant $C_m \in (0,\infty)$, which depends only on m, n, I, W, and the UR constants of $\partial\Omega$, with the property that (4.106) holds for each $p \in I$ and each $w \in A_p(\partial\Omega,\sigma)$ with $[w]_{A_p} \leq W$.

Remark 4.4 From Proposition 3.4 we already know that T_* is bounded on $L^p(\partial\Omega, w)$, with norm controlled in terms of n, k, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$. The crux of the matter here is the more refined version of the estimate of the operator norm of T_* given in (4.106).

Remark 4.5 We focus on establishing the estimate claimed in (4.106) in the class of operators whose integral kernel factors as the product of $\langle x-y, \nu(y)\rangle$, i.e., the inner product between the unit normal $\nu(y)$ and the "chord" $x-y$, with some matrix-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, since it has been noted in (1.50) that this is the only type of kernel (in the class of double layer-like integral operators) for which said estimate has a chance of materializing.

Remark 4.6 The class of domains to which Theorem 4.2 applies includes all NTA domains with an Ahlfors regular boundary.

Remark 4.7 In the unweighted case, i.e., for $w \equiv 1$ (or, equivalently, when the measure w coincides with σ), estimate (4.106) simply reads

$$\|T_*\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.113}$$

It turns out that whenever (4.113) is available one may produce a weighted version of such an estimate via interpolation. Specifically, recall the interpolation theorem of Stein-Weiss (cf. [14, Theorem 5.4.1, p. 115]) according to which for any two σ-measurable functions $w_0, w_1 : \partial\Omega \to [0,\infty]$ and any $\theta \in (0,1)$ we have

$$\big(L^p(\partial\Omega, w_0\sigma)\,,\, L^p(\partial\Omega, w_1\sigma)\big)_{\theta,p} = L^p(\partial\Omega, \widetilde{w}\sigma) \;\text{ where }\; \widetilde{w} := w_0^{1-\theta}\cdot w_1^{\theta}. \tag{4.114}$$

Now, given a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, from (2.533) we know that there exists some $\tau \in (1, \infty)$ (which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) such that $w^\tau \in A_p(\partial\Omega, \sigma)$. Upon specializing (4.114) to the case when $\theta := 1 - \tau^{-1} \in (0, 1)$, $w_0 := w^\tau$, and $w_1 := 1$ we therefore obtain

$$\big(L^p(\partial\Omega, w^\tau\sigma)\,,\ L^p(\partial\Omega, \sigma)\big)_{\theta,p} = L^p(\partial\Omega, w). \tag{4.115}$$

As a result, since T_* is a sub-linear operator which is bounded both on $L^p(\partial\Omega, w^\tau\sigma)$ (given that $w^\tau \in A_p(\partial\Omega, \sigma)$), and on $L^p(\partial\Omega, \sigma)$ we may write

$$\begin{aligned}
&\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}\\
&\quad\le \|T_*\|^{1-\theta}_{L^p(\partial\Omega,w^\tau\sigma)\to L^p(\partial\Omega,w^\tau\sigma)}\,\|T_*\|^{\theta}_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)}\\
&\quad\le C_{\Omega,m,n,p,k,[w]_{A_p}}\Big(\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Big)^{\theta},
\end{aligned} \tag{4.116}$$

with the last inequality provided by (4.113).

While the weighted norm inequality established in (4.116) is in the spirit of (4.106), the manner in which the BMO semi-norm of the outward unit normal vector ν is involved is less optimal, as the small exponent θ tempers the rate at which the right-hand side of (4.116) vanishes as $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \to 0^+$ (indeed, we have $\lim\limits_{t\to 0^+}(t^{\langle m\rangle})^\theta/t^{\langle m\rangle} = +\infty$ for each fixed $\theta \in (0, 1)$). Hence, a two-step approach consisting first of proving the plain estimate (4.113) and, second, deriving a weighted version based on the procedure based on interpolation described above, only yields a weaker result than the one advertised in (4.106). Given this, in the proof of (4.106) presented below we shall devise an alternative approach, which deals with the weighted case directly, incorporating the weight in all relevant intermediary steps.

We are ready to proceed to the task of providing the proof of Theorem 4.2.

Proof of Theorem 4.2 We shall write the proof of Theorem 4.2 using an approach designed to shed light on the specific manner in which the right-hand side of (4.106) depends on the BMO semi-norm of the geometric measure theoretic outward unit normal vector ν to the set Ω.

The bulk of the proof is occupied by the justification of the following result (strongly reminiscent of an induction step, that allows us to boot-strap a weaker bound on $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ to a stronger one): knowing that there exists a function

$$\psi : [0, \infty) \longrightarrow [0, \infty) \tag{4.117}$$

which is quasi-increasing near the origin, i.e.,

$$\begin{aligned}&\text{there exist } t_* > 0 \text{ and } C \in [1,\infty) \text{ such that}\\&\psi(t_0) \leq C\psi(t_1) \text{ whenever } 0 \leq t_0 < t_1 < t_*,\end{aligned} \tag{4.118}$$

such that for each exponent $p \in (1,\infty)$ and each weight $w \in A_p(\partial\Omega,\sigma)$ there exists a constant $C \in (0,\infty)$, depending only on n, p, $[w]_{A_p}$, the UR constants of $\partial\Omega$, and ψ, with the property that

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\psi\big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\big), \tag{4.119}$$

implies that for each given integrability exponent $p \in (1,\infty)$, each Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and each function

$$\phi : [0,\infty) \longrightarrow [0,\infty) \tag{4.120}$$

satisfying

$$\begin{aligned}&\inf\{\phi(t) : t \geq \widetilde{t}\} > 0 \quad \text{for each } \widetilde{t} > 0,\\&\phi(\widetilde{t}) \geq \liminf_{t\searrow\widetilde{t}}\phi(t) \quad \text{for each } \widetilde{t} > 0,\\&\phi(0) = \lim_{t\to 0^+}\phi(t) = 0, \quad \phi'(0) := \lim_{t\to 0^+}\phi(t)/t = \infty,\\&\text{and } \psi(t)\cdot\phi(t)^{-1}\cdot \mathrm{e}^{-\phi(t)/t} = O(1) \text{ as } t\to 0^+,\end{aligned} \tag{4.121}$$

there exists a constant $C \in (0,\infty)$ depending only on n, p, $[w]_{A_p}$, the UR constants of $\partial\Omega$, ψ, and ϕ, such that we also have

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\phi\big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\big). \tag{4.122}$$

Henceforth we shall summarize the above claim by simply saying that "(4.119) implies (4.122)."

In connection with (4.121) we wish to make two remarks. Our first remark pertains to the case when we assume

$$\lim_{t\to 0^+}\psi(t)/t = \infty. \tag{4.123}$$

In particular,

$$t_e := \sup\big\{t_o \in (0,\infty) : \psi(t)/t > \mathrm{e} \text{ for all } t \in (0,t_o)\big\} \in (0,\infty] \tag{4.124}$$

is well defined and

$$\psi(t)/t > \mathrm{e} \quad \text{for all} \quad t \in (0, t_e). \tag{4.125}$$

Then among all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the last property in (4.121) the smallest (up to multiplicative constants) in terms of behavior near the origin is actually the function

$$\begin{aligned} &\widehat{\psi} : [0, \infty) \longrightarrow [0, \infty) \text{ given for each } t \geq 0 \text{ by} \\ &\widehat{\psi}(0) := 0, \quad \widehat{\psi}(t) := t \ln(\psi(t)/t) \ \text{ if } \ t \in (0, t_e), \\ &\text{and} \ \ \widehat{\psi}(t) := t_e \ln(\psi(t_e)/t_e) \ \text{ for all } \ t \in [t_e, \infty). \end{aligned} \tag{4.126}$$

To justify the minimality of (4.126), observe that the property in the last line of (4.121) implies that there exist $t_b, M \in (0, \infty)$ such that

$$\psi(t) \leq M\phi(t) \cdot \mathrm{e}^{\phi(t)/t} \quad \text{for each} \quad t \in (0, t_b). \tag{4.127}$$

Elementary calculus gives

$$x\mathrm{e}^x \leq \mathrm{e}^{2x-1} \quad \text{for each} \quad x \in [0, \infty). \tag{4.128}$$

From this used with $x := \phi(t)/t$ and (4.127) we then obtain

$$\psi(t)/t \leq M\mathrm{e}^{2\phi(t)/t-1} \quad \text{for each} \quad t \in (0, t_b). \tag{4.129}$$

In turn, this forces

$$\tfrac{1}{2} t \ln\big(\mathrm{e}\,\psi(t)/Mt\big) \leq \phi(t) \quad \text{for each} \quad t \in (0, t_b), \tag{4.130}$$

and since thanks to (4.123) we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{\frac{1}{2} t \ln\big(\mathrm{e}\,\psi(t)/Mt\big)}{t \ln(\psi(t)/t)} &= \lim_{t\to 0^+} \frac{\frac{1}{2}\ln(\mathrm{e}/M) + \frac{1}{2}\ln(\psi(t)/t)}{\ln(\psi(t)/t)} \\ &= \frac{1}{2} + \frac{1}{2}\ln(\mathrm{e}/M) \lim_{t\to 0^+} \frac{1}{\ln(\psi(t)/t)} = \frac{1}{2}, \end{aligned} \tag{4.131}$$

we ultimately conclude that

given any $\phi : [0, \infty) \to [0, \infty)$ satisfying the last property in (4.121) it follows that $\phi(t)$ dominates, up to a multiplicative constant, $\widehat{\psi}(t)$ for all $t \geq 0$ sufficiently close to 0. (4.132)

This justifies the claim about the minimality of $\widehat{\psi}$ made in the previous paragraph.

The second remark we wish to make in connection with (4.121) is that if in addition to (4.118) and (4.123) we also assume that

$$\psi \text{ is continuous and } \lim_{t\to 0^+} t\ln(\psi(t)/t) = 0, \tag{4.133}$$

then

$$\begin{array}{l}\text{the function } \widehat{\psi} \text{ defined in (4.126) is continuous, quasi-increasing} \\ \text{near the origin (in the sense of (4.118)), } \lim_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) = 0, \\ \text{and the function } \phi := \widehat{\psi} \text{ satisfies all properties listed in (4.121).}\end{array} \tag{4.134}$$

That $\widehat{\psi}$ is continuous is clear from (4.126) and (4.133). In particular, $\phi := \widehat{\psi}$ satisfies the second property listed in (4.121). To check the second claim made in (4.134), observe that

$$\begin{array}{l}(0,\infty) \ni y \longmapsto x\ln(y/x) \text{ is a strictly increasing function for} \\ \text{each fixed } x \in (0,\infty)\text{, and each fixed } y \in (0,\infty) \text{ the function} \\ (0, y/\mathrm{e}) \ni x \longmapsto x\ln(y/x) \text{ is also strictly increasing.}\end{array} \tag{4.135}$$

If $t_* > 0$ and $C \in (0,\infty)$ are as in (4.118), if $t_e \in (0,\infty)$ is as in (4.125), and if $t^* > 0$ is small enough such that

$$\max\{C, \mathrm{e}/C\} \leq \psi(t)/t \quad \text{for each} \quad t \in (0, t^*), \tag{4.136}$$

(something we may always arrange, thanks to the property assumed in (4.123)) then whenever $0 \leq t_0 < t_1 < \min\{t_*, t^*, t_e\}$ we may write (using (4.126), (4.118), (4.125), (4.135), and (4.136))

$$\begin{aligned}\widehat{\psi}(t_0) &= t_0\ln(\psi(t_0)/t_0) \leq t_0\ln(C\psi(t_1)/t_0) \leq t_1\ln(C\psi(t_1)/t_1) \\ &= t_1\ln(C) + t_1\ln(\psi(t_1)/t_1) \leq 2t_1\ln(\psi(t_1)/t_1) = 2\widehat{\psi}(t_1),\end{aligned} \tag{4.137}$$

ultimately proving that $\widehat{\psi}$ is, as claimed, quasi-increasing near the origin. In fact, the same type of argument as in (4.137) (with $C := 1$) shows that

$$\begin{array}{l}\text{if the original function } \psi \text{ is genuinely non-decreasing, then the} \\ \text{function } \widehat{\psi} \text{ associated with } \psi \text{ as in (4.126) is strictly increasing} \\ \text{on } (0, t_e) \text{ and constant thereafter.}\end{array} \tag{4.138}$$

Next, (4.125) and (4.126) readily imply (bearing in mind that the function ψ is continuous) that $\inf\{\widehat{\psi}(t) : t \geq \widetilde{t}\} > 0$ for each $\widetilde{t} > 0$. The fact that $\widehat{\psi}$ is continuous at the origin is seen from (4.126) and (4.133). Furthermore, (4.123) implies

$$\lim_{t\to 0^+} \widehat{\psi}(t)/t = \lim_{t\to 0^+} \ln(\psi(t)/t) = \infty. \tag{4.139}$$

Let us also note here that (4.139), (4.126), the fact that $\ln(\ln x) \leq \ln x$ for each $x > 1$, and (4.133) allow us to write

$$0 \le \liminf_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) \le \limsup_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) = \limsup_{t\to 0^+} t\ln\big(\ln(\psi(t)/t)\big)$$
$$\le \limsup_{t\to 0^+} t\ln(\psi(t)/t) = 0, \qquad (4.140)$$

ultimately proving that, as claimed, $\lim_{t\to 0^+} t\ln(\widehat{\psi}(t)/t) = 0$. Finally, (4.126) and (4.123) give

$$\psi(t)\cdot\widehat{\psi}(t)^{-1}\cdot e^{-\widehat{\psi}(t)/t} = \psi(t)\cdot\frac{1}{t\ln(\psi(t)/t)}\cdot e^{-\ln(\psi(t)/t)}$$
$$= \frac{1}{\ln(\psi(t)/t)} = o(1) \quad \text{as} \quad t\to 0^+. \qquad (4.141)$$

This completes the proof of (4.134).

Assuming for the time being that (4.119) implies (4.122), let us explain how this inductive step may be used to establish (4.106). From Proposition 3.4 (which guarantees that the maximal operator T_* is bounded in $L^p(\partial\Omega, w)$ for each exponent $p\in(1,\infty)$ and each weight $w\in A_p(\partial\Omega,\sigma)$ with norm controlled solely in terms of n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$) we conclude that (4.119) holds for the constant function

$$\psi_0(t) := 1 \quad \text{for each} \quad t\in[0,\infty). \qquad (4.142)$$

Incidentally, we may recast this as $\psi_0(t) = t^{\langle 0\rangle}$ for each $t\in[0,\infty)$ (cf. (4.92)). This choice of function satisfies (4.118) (in fact, ψ_0 is non-decreasing), as well as (4.123) and (4.133). Granted these, we may then conclude from (4.134) and the working hypothesis, according to which (4.119) implies (4.122), that (4.122) holds with

$$\psi_1 := \widehat{\psi_0} \qquad (4.143)$$

playing the role of the function ϕ. This selection of the function ϕ is actually optimal, since $\widehat{\psi_0}$ enjoys the minimality property described in (4.132). Specifically, given any $\phi : [0,\infty) \to [0,\infty)$ satisfying the last property in (4.121) with $\psi := \psi_0$ it follows that $\phi(t)$ dominates, up to a multiplicative constant, the quantity $\psi_1(t) = \widehat{\psi_0}(t)$ for all $t\ge 0$ sufficiently close to 0.

In addition, from (4.134) and (4.138) we see that

> the function ψ_1 is continuous, strictly increasing near the origin, globally nondecreasing, and satisfies $\lim_{t\to 0^+}\psi_1(t)/t = \infty$ as well as $\lim_{t\to 0^+} t\ln(\psi_1(t)/t) = 0$. (4.144)

In fact, according to (4.124)–(4.126), we have

$$
\begin{aligned}
&\psi_1 : [0,\infty) \longrightarrow [0,\infty) \text{ is given for each } t \geq 0 \text{ by}\\
&\psi_1(0) := 0, \quad \psi_1(t) := t\ln(1/t) \ \text{ if } \ t \in (0, 1/\mathrm{e}),\\
&\qquad \text{and } \ \psi_1(t) := 1/\mathrm{e} \ \text{ for all } \ t \in [1/\mathrm{e}, \infty),
\end{aligned}
\tag{4.145}
$$

hence (cf. (4.93))

$$
\psi_1(t) = t^{\langle 1\rangle} \ \text{ for each } \ t \in [0,\infty). \tag{4.146}
$$

In view of the aforementioned properties of ψ_1 and the fact that (4.122) holds with ψ_1 playing the role of the function ϕ, the present working hypothesis (according to which (4.119) implies (4.122)) shows that (4.122) also holds with $\psi_2 := \widehat{\psi_1}$ playing the role of the function ϕ, and that ψ_2 satisfies similar properties to those listed in (4.144). Actually, (4.145) and (4.124)–(4.126) yield a concrete description of ψ_2, namely:

$$
\begin{aligned}
&\psi_2 : [0,\infty) \longrightarrow [0,\infty) \text{ is given for each } t \geq 0 \text{ by}\\
&\psi_2(0) := 0, \quad \psi_2(t) := t\ln\big(\ln(1/t)\big) \ \text{ if } \ t \in (0, 1/\mathrm{e}^{\mathrm{e}}),\\
&\qquad \text{and } \ \psi_2(t) := 1/\mathrm{e}^{\mathrm{e}} \ \text{ for all } \ t \in [1/\mathrm{e}^{\mathrm{e}}, \infty).
\end{aligned}
\tag{4.147}
$$

Equivalently (cf. (4.93)),

$$
\psi_2(t) = t^{\langle 2\rangle} \ \text{ for each } \ t \in [0,\infty). \tag{4.148}
$$

Iterating this scheme m times then proves (see (4.95)) that (4.122) holds with ϕ replaced by the function described (using notation introduced in (4.93)–(4.94)) as

$$
\begin{aligned}
&\psi_m : [0,\infty) \longrightarrow [0,\infty) \text{ given by}\\
&\psi_m(t) = t^{\langle m\rangle} \text{ for each } t \in [0,\infty).
\end{aligned}
\tag{4.149}
$$

This induction establishes (4.106), modulo the proof of the fact that (4.119) implies (4.122) (which we shall deal with momentarily). The above line of reasoning explains the format of the conclusion in (4.106), while it also makes it clear that (4.106) is the best outcome one can produce working under the assumption that (4.119) implies (4.122).

On to the proof of the fact that (4.119) implies (4.122). Our working hypothesis is that there exists some function $\psi : [0,\infty) \to [0,\infty)$ which is quasi-increasing near the origin (in the sense of (4.118)) such that for each exponent $p \in (1,\infty)$ and each weight $w \in A_p(\partial\Omega, \sigma)$ the estimate recorded in (4.119) holds for some constant $C \in (0,\infty)$ depending only on n, p, $[w]_{A_p}$, the UR constants of $\partial\Omega$, and ψ. Having fixed a function ϕ as in (4.120)–(4.121), the goal is to prove (4.122).

To get started, it is visible from (4.104)–(4.105) that the maximal operator T_* depends in a homogeneous fashion on the kernel function k. As such, by working with k/K (in the case when k is not identically zero) for the choice $K := \sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|$, matters are reduced to proving that whenever (4.118) holds for any $p \in (1, \infty)$ and, in addition, we have

$$\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \leq 1 \tag{4.150}$$

then for each integrability exponent $p \in (1, \infty)$ and each Muckenhoupt weight w in $A_p(\partial\Omega, \sigma)$ it is possible to find a constant $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the UR constants of $\partial\Omega$ such that

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\phi\big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\big). \tag{4.151}$$

Henceforth, assume (4.150).

To proceed, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Pick a parameter $\delta_* \in (0, 1)$. Along the way, we will impose further restrictions on the size of δ_*, depending only on n, p, $[w]_{A_p}$, the UR constants of $\partial\Omega$, and the functions ψ, ϕ. In the case when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \delta_*$, the estimate claimed in (4.151) follows directly (simply by adjusting constants) from the first line in (4.121) and Proposition 3.4, which ensures that the maximal operator T_* is bounded in $L^p(\partial\Omega, w)$. Therefore, there remains to consider the case when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_*$. Assume this is the case and pick some δ such that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta < \delta_*. \tag{4.152}$$

Recall that our long-term goal is to prove (4.151) for some constant $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the UR constants of $\partial\Omega$. Since we may assume that δ_* is sufficiently small relative to the Ahlfors regularity constant of $\partial\Omega$ and the dimension n, we may invoke Theorem 2.3 which guarantees that

> the set $\partial\Omega$ is unbounded and Ω satisfies a two-sided local John condition with constants which depend only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n; in particular, the UR constants of $\partial\Omega$ are also controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$. (4.153)

In addition, Proposition 2.15 ensures that there exists some constant $C_\Omega \in (0, \infty)$, which depends only on n and the Ahlfors regularity constant of $\partial\Omega$, such that for each dilation factor $\mu \in [1, \infty)$ we have

$$\sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(x,\mu R)} R^{-1}\big|\langle x - y, \nu_{\Delta(z,R)}\rangle\big| \leq C_\Omega \cdot \mu(1 + \log_2 \mu)\delta. \tag{4.154}$$

For reasons which are going to be clear momentarily, in addition to the truncated operators T_ε from (4.105) we shall need a version in which the truncation is performed using a smooth cutoff function (rather than a characteristic function). Specifically, fix a function $\zeta \in \mathscr{C}^\infty(\mathbb{R})$ satisfying $0 \leq \zeta \leq 1$ on $\mathbb{R}$ and with the property that $\zeta \equiv 0$ in $(-\infty, 1]$ and $\zeta \equiv 1$ in $[2, \infty)$. For each $\varepsilon > 0$ then define the action of the smoothly truncated operator $T_{(\varepsilon)}$ on each $f \in L^p(\partial\Omega, w)$ by setting

$$T_{(\varepsilon)} f(x) := \int_{\partial\Omega} \zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\langle x-y, \nu(y)\rangle\, k(x-y) f(y)\, d\sigma(y) \tag{4.155}$$

for each $x \in \partial\Omega$. Let us also define a smoothly truncated version of the maximal operator (4.104) by setting, for each $f \in L^p(\partial\Omega, w)$,

$$T_{(*)} f(x) := \sup_{\varepsilon>0} \big|T_{(\varepsilon)} f(x)\big| \quad \text{at every point } x \in \partial\Omega. \tag{4.156}$$

For the time being, the goal is to compare roughly truncated singular integral operators with their smoothly truncated counterparts. To accomplish this task, for each fixed $\gamma \geq 0$ bring in a brand of Hardy–Littlewood maximal operator which associates to each σ-measurable function f on $\partial\Omega$ the function $\mathcal{M}_\gamma f$ defined as

$$\mathcal{M}_\gamma f(x) := \sup_{\Delta \ni x} \Big(\fint_{\Delta} |f|^{1+\gamma}\, d\sigma\Big)^{1/(1+\gamma)} \quad \text{for each } x \in \partial\Omega, \tag{4.157}$$

where the supremum is taken over all surface balls $\Delta \subseteq \partial\Omega$ containing the point x. On to the task at hand, having fixed some $\varepsilon > 0$, for each $f \in L^p(\partial\Omega, w)$ and each $x \in \partial\Omega$ we may estimate

$$\begin{aligned}
\big|(T_\varepsilon f - T_{(\varepsilon)} f)(x)\big| &\leq \int_{\Delta(x,2\varepsilon)\setminus\overline{\Delta(x,\varepsilon)}} \big|\langle x-y, \nu(y)\rangle\big|\, |k(x-y)||f(y)|\, d\sigma(y) \\
&\leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} \big|\langle x-y, \nu(y)\rangle\big|\, |f(y)|\, d\sigma(y) \\
&\leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} \big|\big\langle x-y, \nu(y)-\nu_{\Delta(x,2\varepsilon)}\big\rangle\big||f(y)|\, d\sigma(y) \\
&\quad + C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} \big|\big\langle x-y, \nu_{\Delta(x,2\varepsilon)}\big\rangle\big||f(y)|\, d\sigma(y) \\
&\leq C\Big(\fint_{\Delta(x,2\varepsilon)} \big|\nu(y)-\nu_{\Delta(x,2\varepsilon)}\big|^{\frac{\gamma+1}{\gamma}}\, d\sigma(y)\Big)^{\frac{\gamma}{1+\gamma}} \Big(\fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\, d\sigma(y)\Big)^{\frac{1}{1+\gamma}}
\end{aligned}$$

$$+ C\Big(\sup_{y\in\Delta(x,2\varepsilon)} \varepsilon^{-1}\big|\langle x-y, \nu_{\Delta(x,2\varepsilon)}\rangle\big|\Big)\Big(⨍_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\, d\sigma(y)\Big)^{\frac{1}{1+\gamma}}$$

$$\leq C\delta \cdot \inf_{\Delta(x,2\varepsilon)} \mathcal{M}_\gamma f, \tag{4.158}$$

using Hölder's inequality, (2.102), (4.152), (4.154), and (4.157). Ultimately, the estimate recorded in (4.158) implies that there exists some $C \in (0,\infty)$, which depends only on γ, n, and the Ahlfors regularity constant of $\partial\Omega$, with the property that for each function $f \in L^p(\partial\Omega, w)$ we have

$$\big|T_* f(x) - T_{(*)} f(x)\big| \leq C\delta \cdot \mathcal{M}_\gamma f(x) \quad \text{for each } \; x \in \partial\Omega. \tag{4.159}$$

Henceforth we agree to fix $\gamma \in (0, p-1)$, which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that $w \in A_{p/(1+\gamma)}(\partial\Omega, \sigma)$, with $[w]_{A_{p/(1+\gamma)}}$ controlled in terms of n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. From (2.533) we know that such a choice is possible.

To proceed, consider a dyadic grid $\mathbb{D}(\partial\Omega)$ on the Ahlfors regular set $\partial\Omega$ (as in Proposition 2.19, presently used with $\Sigma := \partial\Omega$). Also, choose a compactly supported function $f \in L^p(\partial\Omega, w)$. Note that for each $\varepsilon > 0$ the function $T_{(\varepsilon)} f$ is continuous on $\partial\Omega$, by Lebesgue's Dominated Convergence Theorem (whose applicability in the present setting is ensured by Lemma 2.15). Since the pointwise supremum of any collection of continuous functions is lower-semicontinuous, we conclude that for each $\lambda > 0$ the set

$$\big\{x \in \partial\Omega : \; T_{(*)} f(x) > \lambda\big\} \;\text{ is relatively open in }\; \partial\Omega. \tag{4.160}$$

Next, fix a reference point $x_0 \in \partial\Omega$ and abbreviate $\Delta_0 := \Delta(x_0, 2^{-m})$ for some $m \in \mathbb{Z}$ chosen so that

$$\operatorname{supp} f \subseteq 2\Delta_0. \tag{4.161}$$

We emphasize that all subsequent constants are going to be independent of the function f, the point x_0, and the integer m. Upon recalling (2.500), define

$$\mathcal{Q}_0 := \big\{Q \in \mathbb{D}_m(\partial\Omega) : \; Q \cap 2\Delta_0 \neq \varnothing\big\} \tag{4.162}$$

then introduce

$$I_0 := \bigcup_{Q\in\mathcal{Q}_0} Q. \tag{4.163}$$

By design, I_0 is a relatively open subset of $\partial\Omega$. Recall the parameter $a_1 > 0$ appearing in (2.502) of Proposition 2.19. We claim that

$$I_0 \subseteq a\Delta_0 \text{ where } a := 2(1+a_1) > 2. \tag{4.164}$$

Indeed, if $x \in I_0$ then $x \in Q$ for some $Q \in \mathcal{Q}_0$. In particular, $Q \cap 2\Delta_0 \neq \varnothing$ so we may pick some $y \in Q \cap 2\Delta_0$. Then $x, y \in Q \subseteq \Delta(x_Q, a_1 2^{-m})$ by (2.502), where x_Q denotes the "center" of the dyadic cube Q. Consequently, $|x - y| < a_1 2^{-m+1}$ which permits us to estimate $|x - x_0| \leq |x - y| + |y - x_0| < a_1 2^{-m+1} + 2^{-m+1} = a \cdot 2^{-m}$. Thus $x \in B(x_0, a \cdot 2^{-m}) \cap \partial\Omega = a\Delta_0$, proving the inclusion in (4.164).

We also claim that

$$\begin{array}{l}\text{there exists a } \sigma\text{-measurable set } N \subseteq \partial\Omega \text{ with the property that} \\ \sigma(N) = 0 \text{ and } 2\Delta_0 \setminus N \subseteq I_0.\end{array} \tag{4.165}$$

To justify this, recall from (2.504) that

$$\begin{array}{l} N := \partial\Omega \setminus \big(\bigcup_{Q \in \mathbb{D}_m(\partial\Omega)} Q\big) \text{ is a } \sigma\text{-measurable set satisfying} \\ \sigma(N) = 0 \text{ and } \partial\Omega \setminus N = \bigcup_{Q \in \mathbb{D}_m(\partial\Omega)} Q. \end{array} \tag{4.166}$$

Intersecting both sides of the last equality in (4.166) with $2\Delta_0$ while bearing in mind (4.162)–(4.163) then yields

$$2\Delta_0 \setminus N = \bigcup_{Q \in \mathbb{D}_m(\partial\Omega)} \big(Q \cap 2\Delta_0\big) = \bigcup_{Q \in \mathcal{Q}_0} \big(Q \cap 2\Delta_0\big) \subseteq \bigcup_{Q \in \mathcal{Q}_0} Q = I_0, \tag{4.167}$$

ultimately proving (4.165).

Let us now define

$$A := \theta \cdot \phi(\delta)^{-1} \in (0, \infty) \text{ for some fixed small } \theta \in (0, 1). \tag{4.168}$$

At various stages in the proof we shall make specific demands on the size of θ, though always in relation to the background geometric parameters, the weight, and the function ϕ, namely n, p, $[w]_{A_p}$, ϕ, and the Ahlfors regularity constant of $\partial\Omega$ (the final demand of this nature is made in connection with (4.240)). We find it convenient to abbreviate

$$\begin{aligned} &\eta(\theta, \delta) \\ &\quad := C\left\{\theta^{1+\gamma} + \theta^{1+\gamma/2}\left(\frac{\psi(\delta)}{\phi(\delta)} \cdot \mathrm{e}^{-\phi(\delta)/\delta}\right)^{1+\gamma/2} + \mathrm{e}^{-(3+\gamma+2/\gamma)\phi(\delta)/\delta}\right\}, \end{aligned} \tag{4.169}$$

where $C \in (0, \infty)$ is a constant which depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the Ahlfors regularity constant of $\partial\Omega$. We agree to retain the notation $\eta(\theta, \delta)$ even when $C \in (0, \infty)$ may occasionally change in size (while retaining the same nature, however).

Since $w \in A_p(\partial\Omega, \sigma) \subseteq A_\infty(\partial\Omega, \sigma)$, there exists some small number $\tau > 0$ such that (2.537) holds. Our long-term goal is to obtain the following type of good-

λ inequality: there exists $C \in (0, \infty)$ as above (entering the makeup of the entity $\eta(\theta, \delta)$ defined in (4.169)) such that for each $\lambda > 0$ we have

$$w\Big(\big\{x \in I_0 : T_* f(x) > 4\lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \le A\lambda\big\}\Big)$$
$$\le \eta(\theta, \delta)^\tau \cdot w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}\Big). \tag{4.170}$$

Here and elsewhere, we employ our earlier convention of using the same symbol w for the measure associated with the given weight w as in (2.509). The reader is also alerted to the fact that the maximal operator appearing in the right-hand side of (4.170) employs smooth truncations (as in (4.156)).

To prove (4.170), fix an arbitrary $\lambda > 0$ and abbreviate

$$\mathcal{F}_\lambda := \big\{x \in I_0 : T_* f(x) > 4\lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \le A\lambda\big\}. \tag{4.171}$$

Proposition 3.4 implies that $T_* f$ is a σ-measurable function. Since so is $\mathcal{M}_\gamma f$ (cf. [7] or [111, §7.6] for a proof), it follows that $\mathcal{F}_\lambda$ is necessarily a σ-measurable set. From (4.160) and the fact that I_0 is a relatively open subset of $\partial\Omega$ we also conclude that $\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}$ is a relatively open subset of $\partial\Omega$ (hence, σ-measurable). As such, the good-λ inequality is meaningfully formulated in (4.170).

Clearly, it is enough to consider the case $\mathcal{F}_\lambda \neq \varnothing$ since otherwise (4.170) is trivially satisfied by any choice of $C \in (0, \infty)$. For the remainder of the proof, assume this is the case. Since $\mathcal{F}_\lambda \subseteq I_0$ and $I_0 \subseteq a\Delta_0$, we conclude that

$$\mathcal{F}_\lambda \subseteq I_0 \subseteq a\Delta_0 \ \text{ and } \ \sup_{\mathcal{F}_\lambda} \mathcal{M}_\gamma f \le A\lambda. \tag{4.172}$$

To proceed, decompose $I_0 = \mathcal{P}_\lambda \cup \mathcal{S}_\lambda$ (disjoint union) where, with the smoothly truncated maximal operator $T_{(*)}$ as in (4.156),

$$\mathcal{P}_\lambda := \big\{x \in I_0 : T_{(*)} f(x) \le \lambda\big\} \text{ and } \mathcal{S}_\lambda := \big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}. \tag{4.173}$$

As a consequence of (4.160) and the fact that I_0 is a relatively open subset of $\partial\Omega$, the set $\mathcal{S}_\lambda$ is itself a relatively open subset of $\partial\Omega$. Moreover, using (4.159) and (4.172), for each point $x \in \mathcal{F}_\lambda$ we may estimate

$$4\lambda < T_* f(x) \le T_{(*)} f(x) + C\delta \cdot \mathcal{M}_\gamma f(x) \le T_{(*)} f(x) + C\delta A\lambda$$
$$= T_{(*)} f(x) + C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda < T_{(*)} f(x) + 3\lambda, \tag{4.174}$$

by our choice of A in (4.168), the fact that $\theta \in (0, 1)$, and taking δ_* small enough to begin with (while keeping in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)). From

(4.174) we see that $T_{(*)}f(x) > \lambda$, hence $x \in \mathcal{S}_\lambda$ which ultimately goes to show that $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$. Thus,

$$\begin{aligned}&\mathcal{S}_\lambda \text{ is a nonempty relatively open subset of}\\&\partial\Omega, \text{ with the property that } \mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda \subseteq I_0.\end{aligned} \tag{4.175}$$

We first treat the case in which there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \varnothing$ or, equivalently,

$$Q_0 \subseteq \mathcal{S}_\lambda. \tag{4.176}$$

Apply Theorem 2.6 to the (center and radius of the) surface ball $a\Delta_0$. This guarantees the existence of three constants $C_0, C_1, C_2 \in (0,\infty)$ of a purely geometric nature (i.e., depending only on n and the Ahlfors regularity constant of $\partial\Omega$) with the following significance. Take

$$\widetilde{\phi} := \frac{(1+\gamma)(1+\gamma/2)}{C_2(\gamma/2)}\phi = \frac{3+\gamma+2/\gamma}{C_2}\phi \tag{4.177}$$

to play the role of the function in (2.360)–(2.361)). Assuming $\delta_* \in (0,1)$ to be sufficiently small to begin with, we then have the decomposition

$$a\Delta_0 \subseteq G \cup E, \tag{4.178}$$

where G and E are disjoint σ-measurable subsets of $\partial\Omega$ satisfying properties implied by (2.363)–(2.368) (relative to x_0 and the scale $r := a2^{-m}$) in the present setting. Also, G is contained in the graph $\mathcal{G} = \{x_0 + x + h(x)\vec{n} : x \in H\}$ of a Lipschitz function $h : H \to \mathbb{R}$ (where $\vec{n} \in S^{n-1}$ is a unit vector and $H = \langle\vec{n}\rangle^\perp$ is the hyperplane in $\mathbb{R}^n$ orthogonal to $\vec{n}$) such that

$$\sup_{\substack{x,y\in H\\ x\neq y}} \frac{|h(x)-h(y)|}{|x-y|} \leq C_0\widetilde{\phi}(\delta), \tag{4.179}$$

whereas E satisfies

$$\sigma(E) \leq C_1 \mathrm{e}^{-C_2\widetilde{\phi}(\delta)/\delta}\sigma(a\Delta_0). \tag{4.180}$$

Since $\operatorname{supp} f \subseteq 2\Delta_0$ and $a > 2$ it follows that $f = f\mathbf{1}_{a\Delta_0}$. Based on this observation and the fact that $I_0 \subseteq a\Delta_0$ (cf. (4.172)), we may then estimate

$$\sigma(\mathcal{F}_\lambda) \leq \sigma\Big(\big\{x \in a\Delta_0 : T_*\big(f\mathbf{1}_{a\Delta_0}\big)(x) > 4\lambda\big\}\Big). \tag{4.181}$$

By further decomposing $f\mathbf{1}_{a\Delta_0} = f\mathbf{1}_G + f\mathbf{1}_E$ (cf. (4.178) and the fact that we have $f = f\mathbf{1}_{a\Delta_0}$), then using the sub-linearity of T_*, as well as (4.178), (4.180), and

(4.177) we obtain

$$
\begin{aligned}
\sigma\Big(\big\{x \in a\Delta_0 : T_*\big(f\mathbf{1}_{a\Delta_0}\big)(x) > 4\lambda\big\}\Big) \\
&\leq \sigma\Big(\big\{x \in G :\ T_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) \\
&\quad + \sigma\Big(\big\{x \in G :\ T_*\big(f\mathbf{1}_E\big)(x) > 2\lambda\big\}\Big) \\
&\quad + C_1 \mathrm{e}^{-(3+\gamma+2/\gamma)\phi(\delta)/\delta}\sigma(a\Delta_0).
\end{aligned}
\tag{4.182}
$$

To bound the first term in the right-hand side of (4.182), the idea is to use the fact that G is contained in the graph $\mathcal{G}$ of the function h, then employ Lemma 4.2 while taking advantage of (4.179). Turning to specifics, denote by $\widetilde{\sigma}$ the surface measure on $\mathcal{G}$, and by $\widetilde{T}_*$ the maximal operator associated with $\mathcal{G}$ as in (4.78) (much as T_* in (4.104)–(4.105) is associated with $\partial\Omega$). That is, for each $\widetilde{f} \in L^p(\mathcal{G}, \widetilde{\sigma})$ set

$$
\widetilde{T}_*\widetilde{f}(x) := \sup_{\varepsilon>0}\big|\widetilde{T}_\varepsilon \widetilde{f}(x)\big|, \quad \forall\, x \in \mathcal{G},
\tag{4.183}
$$

where for each $\varepsilon > 0$ we have set

$$
\widetilde{T}_\varepsilon \widetilde{f}(x) := \int\limits_{\substack{y\in\mathcal{G}\\ |x-y|>\varepsilon}} \langle x-y, \widetilde{\nu}(y)\rangle k(x-y)\widetilde{f}(y)\,\mathrm{d}\widetilde{\sigma}(y), \quad \forall\, x \in \mathcal{G},
\tag{4.184}
$$

with $\widetilde{\nu}$ denoting the unit normal vector to the Lipschitz graph $\mathcal{G}$, pointing toward the upper-graph of the function h. From (2.377) we know that

$$
\nu(x) = \widetilde{\nu}(x) \ \text{ at } \sigma\text{-a.e. point } \ x \in G.
\tag{4.185}
$$

We continue by fixing a point $\widetilde{x} \in \mathcal{F}_\lambda$ (which, according to (4.172), also places $\widetilde{x}$ into $a\Delta_0$). As regards the first term in the right-hand side of (4.182), we may rely on (4.185), the fact that the measures σ and $\widetilde{\sigma}$ agree on $\partial\Omega \cap \mathcal{G}$ (as they are both manifestations of $\mathcal{H}^{n-1}$), (4.183)–(4.184), (4.104)–(4.105), Chebyshev's inequality, Lemma 4.2, (4.177), (4.161) (and the fact that $a > 2$), (4.178), (4.157), (4.172), and (4.168) to estimate

$$
\begin{aligned}
\sigma\Big(\big\{x \in G : T_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) &= \widetilde{\sigma}\Big(\big\{x \in G :\ \widetilde{T}_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) \\
&\leq \widetilde{\sigma}\Big(\big\{x \in \mathcal{G} :\ \widetilde{T}_*\big(f\mathbf{1}_G\big)(x) > 2\lambda\big\}\Big) \\
&\leq \frac{1}{(2\lambda)^{1+\gamma}} \int_{\mathcal{G}} |\widetilde{T}_*(f\mathbf{1}_G)|^{1+\gamma}\,\mathrm{d}\widetilde{\sigma} \leq C\frac{\widetilde{\phi}(\delta)^{1+\gamma}}{\lambda^{1+\gamma}} \int_{\mathcal{G}} |f\mathbf{1}_G|^{1+\gamma}\,\mathrm{d}\widetilde{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&= C\frac{\phi(\delta)^{1+\gamma}}{\lambda^{1+\gamma}}\int_G |f|^{1+\gamma}\,\mathrm{d}\sigma \leq C\phi(\delta)^{1+\gamma}\,\frac{\sigma(a\Delta_0)}{\lambda^{1+\gamma}}\fint_{a\Delta_0}|f|^{1+\gamma}\,\mathrm{d}\sigma \\
&\leq C\phi(\delta)^{1+\gamma}\,\frac{\sigma(a\Delta_0)}{\lambda^{1+\gamma}}\Big[\mathcal{M}_\gamma f(\widetilde{x})\Big]^{1+\gamma} \leq C\,(A\phi(\delta))^{1+\gamma}\,\sigma(a\Delta_0) \\
&= C\,\theta^{1+\gamma}\,\sigma(a\Delta_0),
\end{aligned}
\tag{4.186}
$$

for some constant $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the Ahlfors regularity constant of $\partial\Omega$.

As regards the second term in the right-hand side of (4.182), once again fix a point $\widetilde{x} \in \mathcal{F}_\lambda$ (which then also belongs to $a\Delta_0$). Also, assume that $\delta_* \in (0, t_*)$. We may then use Chebyshev's inequality, the hypothesis made in (4.119) (used with $p := 1 + \gamma/2$ and $w := 1$), the assumption (4.150), (4.118), (4.152), the fact that $0 < \delta_* < t_*$, Hölder's inequality, (4.180), (4.157), (4.177), (4.172), and (4.168) to obtain[2]

$$
\begin{aligned}
&\sigma\Big(\big\{x \in G : T_*\big(f\mathbf{1}_E\big)(x) > 2\lambda\big\}\Big) \\
&\qquad \leq \sigma\Big(\big\{x \in \partial\Omega :\ T_*\big(f\mathbf{1}_E\big)(x) > 2\lambda\big\}\Big) \\
&\qquad \leq \frac{1}{(2\lambda)^{1+\gamma/2}}\int_{\partial\Omega}\big(T_*\big(f\mathbf{1}_E\big)\big)^{1+\gamma/2}\,\mathrm{d}\sigma \\
&\qquad \leq \frac{\big(\,\|T_*\|_{L^{1+\gamma/2}(\partial\Omega,\sigma)\to L^{1+\gamma/2}(\partial\Omega,\sigma)}\big)^{1+\gamma/2}}{(2\lambda)^{1+\gamma/2}}\int_{\partial\Omega}\big(|f|\,\mathbf{1}_E\big)^{1+\gamma/2}\,\mathrm{d}\sigma \\
&\qquad \leq \frac{(C\psi(\delta))^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\int_{a\Delta_0}|f|^{1+\gamma/2}\,\mathbf{1}_E\,\mathrm{d}\sigma \\
&\qquad \leq \frac{(C\psi(\delta))^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\,\sigma(E)^{\frac{\gamma/2}{1+\gamma}}\left(\int_{a\Delta_0}|f|^{1+\gamma}\,\mathrm{d}\sigma\right)^{\frac{1+\gamma/2}{1+\gamma}} \\
&\qquad = \frac{(C\psi(\delta))^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\left(\frac{\sigma(E)}{\sigma(a\Delta_0)}\right)^{\frac{\gamma/2}{1+\gamma}}\left(\fint_{a\Delta_0}|f|^{1+\gamma}\,\mathrm{d}\sigma\right)^{\frac{1+\gamma/2}{1+\gamma}}\ \sigma(a\Delta_0)
\end{aligned}
$$

[2] It is from the format of (4.187) that the value of having the last property in (4.121) is most apparent. Indeed, since the left-most side of (4.187) is obviously dominated by $\sigma(G) \leq \sigma(a\Delta_0)$ (cf. (4.178)), the estimate derived in (4.187) is only useful if $\psi(\delta)\phi(\delta)^{-1}\cdot\exp\Big\{-\frac{\phi(\delta)}{\delta}\Big\}$ stays bounded for δ close to 0.

$$\le C\frac{\psi(\delta)^{1+\gamma/2}}{\lambda^{1+\gamma/2}}\exp\Big\{-\frac{C_2(\gamma/2)\widetilde{\phi}(\delta)}{(1+\gamma)\delta}\Big\}\Big[\mathcal{M}_\gamma f(\widetilde{x})\Big]^{1+\gamma/2}\sigma(a\Delta_0)$$

$$\le C\big(A\psi(\delta)\big)^{1+\gamma/2}\cdot\exp\Big\{-\frac{(1+\gamma/2)\phi(\delta)}{\delta}\Big\}\sigma(a\Delta_0)$$

$$= C\theta^{1+\gamma/2}\Big[\psi(\delta)\phi(\delta)^{-1}\cdot\exp\Big\{-\frac{\phi(\delta)}{\delta}\Big\}\Big]^{1+\gamma/2}\sigma(a\Delta_0), \tag{4.187}$$

where $C\in(0,\infty)$ depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the Ahlfors regularity constant of $\partial\Omega$. Gathering (4.182), (4.186), and (4.187) then yields

$$\sigma\Big(\big\{x\in a\Delta_0:\ T_*\big(f\mathbf{1}_{a\Delta_0}\big)(x)>4\lambda\big\}\Big)$$

$$\le C\Big\{\theta^{1+\gamma}+\theta^{1+\gamma/2}\Big(\frac{\psi(\delta)}{\phi(\delta)}\cdot\mathrm{e}^{-\phi(\delta)/\delta}\Big)^{1+\gamma/2}+\mathrm{e}^{-(3+\gamma+2/\gamma)\phi(\delta)/\delta}\Big\}\sigma(a\Delta_0)$$

$$=\eta(\theta,\delta)\sigma(a\Delta_0), \tag{4.188}$$

where $\eta(\theta,\delta)\in(0,\infty)$ is as in (4.169). Finally, from (4.188) and (4.181) we see that

$$\sigma(\mathcal{F}_\lambda)\le\eta(\theta,\delta)\sigma(a\Delta_0), \tag{4.189}$$

where $\eta(\theta,\delta)\in(0,\infty)$ is as in (4.169).

Moving on, observe that (2.502) implies that there exists a point $x_{Q_0}\in\partial\Omega$ with the property that

$$\Delta(x_{Q_0},a_0 2^{-m})\subseteq Q_0\subseteq\Delta(x_{Q_0},a_1 2^{-m}). \tag{4.190}$$

From this inclusion and (4.162) we then conclude that there exists some $c\in(0,\infty)$, which only depends on the Ahlfors regularity constant of $\partial\Omega$, with the property that $a\Delta_0\subseteq c\Delta(x_{Q_0},a_1 2^{-m})$. As a consequence of this inclusion we may write (for some $C\in(0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$)

$$w(a\Delta_0)\le w\big(c\Delta(x_{Q_0},a_1 2^{-m})\big)\le C\,w\big(\Delta(x_{Q_0},a_0 2^{-m})\big)\le Cw(Q_0), \tag{4.191}$$

where we have also used the fact that w is a doubling measure (cf. (2.535)) and (4.190). With this in hand, we may now estimate

$$w(\mathcal{F}_\lambda)\le\eta(\theta,\delta)^\tau\cdot w(a\Delta_0)\le\eta(\theta,\delta)^\tau\cdot w(Q_0)$$

$$\le\eta(\theta,\delta)^\tau\cdot w(\mathcal{S}_\lambda), \tag{4.192}$$

where the first inequality uses (2.537), the fact that $\mathcal{F}_\lambda \subseteq a\Delta_0$ (cf. (4.172)), and (4.189), the second inequality is based on (4.191), while the last inequality is a consequence of (4.176). Therefore (4.170) holds whenever there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \varnothing$.

To complete the proof of (4.170), it remains to consider the case $\mathcal{P}_\lambda \cap Q \neq \varnothing$ for each $Q \in \mathcal{Q}_0$. In this scenario, consider an arbitrary dyadic cube $Q \in \mathcal{Q}_0$. From (4.163) we know that $Q \subseteq I_0$. Subdivide Q dyadically and stop when $\mathcal{P}_\lambda \cap Q' = \varnothing$. This process produces a family of pairwise disjoint (stopping time) dyadic cubes $\{Q^j\}_{j\in J_Q} \subset \mathbb{D}(\partial\Omega)$ such that $Q^j \cap \mathcal{P}_\lambda = \varnothing$, $Q^j \subseteq Q$ but $Q^j \neq Q$ (since we have $Q^j \cap \mathcal{P}_\lambda = \varnothing$ but $Q \cap \mathcal{P}_\lambda \neq \varnothing$), and $Q' \cap \mathcal{P}_\lambda \neq \varnothing$ for all $Q' \in \mathbb{D}(\partial\Omega)$ such that $Q^j \subsetneq Q' \subseteq Q$. In particular $Q^j \subsetneq Q$ for every $j \in J_Q$ and $\widetilde{Q}^j$, the dyadic parent of Q^j, satisfies $\widetilde{Q}^j \subseteq Q$. With the σ-nullset N as in (2.505), we now claim that

$$\bigcup_{j\in J_Q} Q^j \subseteq \mathcal{S}_\lambda \cap Q \subseteq \Big(\bigcup_{j\in J_Q} Q^j\Big) \cup N. \tag{4.193}$$

To justify the first inclusion above, observe that if $j \in J_Q$ then $Q^j \subseteq \mathcal{S}_\lambda \cap Q$, since $Q^j \subseteq Q \subseteq I_0$ and $Q^j \cap \mathcal{P}_\lambda = \varnothing$ imply that $Q_j \subseteq Q \setminus \mathcal{P}_\lambda = Q \cap \mathcal{S}_\lambda$. This establishes the first inclusion in (4.193). As regards the second inclusion claimed in (4.193), consider an arbitrary point $x \in \big(\mathcal{S}_\lambda \cap Q\big) \setminus N$. Then $T_{(*)} f(x) > \lambda$ which, in view of (4.160), ensures that we may find a surface ball $\Delta_x := \Delta(x, r_x)$ such that $T_{(*)} f(y) > \lambda$ for every $y \in \Delta_x$. Thanks to (2.502) and (2.504) we may then choose a dyadic cube $Q_x \in \mathbb{D}(\partial\Omega)$ such that $x \in Q_x$ and $Q_x \subseteq \Delta_x \cap Q \subseteq I_0$. This forces $Q_x \subseteq \mathcal{S}_\lambda \cap Q$, hence $Q_x \cap \mathcal{P}_\lambda = \varnothing$. By the maximality of the family chosen above, $Q_x \subseteq Q^j$ for some $j \in J_Q$ which goes to show that $x \in Q^j$. Ultimately, this proves the second inclusion in (4.193).

Going further, the idea is to carry out the stopping time argument just described for each dyadic cube $Q \in \mathcal{Q}_0$. For ease of reference, organize the resulting collection of dyadic cubes $\big\{Q^j : Q \in \mathcal{Q}_0 \text{ and } j \in J_Q\big\}$ (which is an at most countable set) as a single-index family $\big\{Q_\ell\big\}_{\ell\in\mathcal{I}}$ of mutually disjoint dyadic cubes; in particular,

$$\bigcup_{Q\in\mathcal{Q}_0} \bigcup_{j\in J_Q} Q^j = \bigcup_{\ell\in\mathcal{I}} Q_\ell, \tag{4.194}$$

with the latter union comprised of pairwise disjoint dyadic cubes in $\partial\Omega$. Note that $\mathcal{S}_\lambda \cap Q$ might be empty for some $Q \in \mathcal{Q}_0$ and in this case $J_Q = \varnothing$ (i.e., the family of cubes $\{Q^j\}_{j\in J_Q}$ is empty, since there are no stopping time dyadic cubes produced in this case). However, (4.163) and (4.175) imply that $\mathcal{S}_\lambda \cap Q$ cannot be empty for every $Q \in \mathcal{Q}_0$ and, as a consequence, $\mathcal{I} \neq \varnothing$. Going further, using (4.163) and the fact that $\mathcal{S}_\lambda \subseteq I_0$ (cf. (4.173)) we may write

$$\bigcup_{Q\in\mathcal{Q}_0} (\mathcal{S}_\lambda \cap Q) = \mathcal{S}_\lambda \tag{4.195}$$

which further entails, on account of (4.194) and (4.193), that

$$\bigcup_{\ell\in\mathcal{I}} Q_\ell \subseteq \mathcal{S}_\lambda \subseteq \Big(\bigcup_{\ell\in\mathcal{I}} Q_\ell\Big) \cup N. \tag{4.196}$$

By construction, for each index $\ell \in \mathcal{I}$ there exists a point x_ℓ^* such that

$$x_\ell^* \in \widetilde{Q}_\ell \cap \mathcal{P}_\lambda = \widetilde{Q}_\ell \cap \big(I_0 \setminus \mathcal{S}_\lambda\big), \tag{4.197}$$

where $\widetilde{Q}_\ell$ denotes the dyadic parent of Q_ℓ (cf. item *(4)* in Proposition 2.19). For each $\ell \in \mathcal{I}$ we let $\Delta_\ell := \Delta_{Q_\ell}$ and $\widetilde{\Delta}_\ell := \Delta_{\widetilde{Q}_\ell}$ be as in (2.502). Pressing on, split the collection $\{\Delta_\ell\}_{\ell\in\mathcal{I}}$ into two sub-classes. Specifically, bring in

$$\begin{gathered}\mathcal{I}_1 := \big\{\ell \in \mathcal{I} : \text{ there exists } x_\ell^{**} \in \Delta_\ell \text{ such that } \mathcal{M}_\gamma f(x_\ell^{**}) \leq A\lambda\big\} \\ \text{and } \mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1.\end{gathered} \tag{4.198}$$

Hence, by design, $\mathcal{F}_\lambda \cap \Delta_\ell = \varnothing$ for each $\ell \in \mathcal{I}_2$. Recall now from (4.175) that $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$. From this, (4.196), and (2.502) we then obtain (bearing in mind that $\sigma(N) = 0$; cf. (2.505))

$$w(\mathcal{F}_\lambda) = \sum_{\ell\in\mathcal{I}} w(\mathcal{F}_\lambda \cap Q_\ell) \leq \sum_{\ell\in\mathcal{I}_1} w(\mathcal{F}_\lambda \cap \Delta_\ell). \tag{4.199}$$

Let us also consider

$$F_\ell := \big\{x \in \Delta_\ell : T_* f(x) > 4\lambda\big\} \text{ for each } \ell \in \mathcal{I}_1, \tag{4.200}$$

and observe that this entails

$$\mathcal{F}_\lambda \cap \Delta_\ell \subseteq F_\ell \text{ for each } \ell \in \mathcal{I}_1. \tag{4.201}$$

Our next goal is to prove that

$$\sigma(F_\ell) \leq \eta(\theta,\delta)\cdot\sigma(\Delta_\ell) \text{ for each } \ell \in \mathcal{I}_1. \tag{4.202}$$

Granted this, using (2.537) it would follow that

$$w(F_\ell) \leq \eta(\theta,\delta)^\tau \cdot w(\Delta_\ell) \text{ for each } \ell \in \mathcal{I}_1 \tag{4.203}$$

which, in concert with (4.199), (4.201), (2.502) plus the fact that w is a doubling measure, and (4.196), would then imply

$$w(\mathcal{F}_\lambda) \leq \sum_{\ell\in\mathcal{I}_1} w(\mathcal{F}_\lambda \cap \Delta_\ell) \leq \sum_{\ell\in\mathcal{I}_1} w(F_\ell) \leq \eta(\theta,\delta)^\tau \cdot \sum_{\ell\in\mathcal{I}_1} w(\Delta_\ell)$$

$$\leq \eta(\theta,\delta)^{\tau} \cdot \sum_{\ell \in \mathcal{I}_1} w(Q_\ell) \leq \eta(\theta,\delta)^{\tau} \cdot \sum_{\ell \in \mathcal{I}} w(Q_\ell)$$

$$= \eta(\theta,\delta)^{\tau} \cdot w(\mathcal{S}_\lambda), \tag{4.204}$$

finishing the justification of (4.170).

We now turn to the proof of (4.202). Fix $\ell \in \mathcal{I}_1$ and, in order to lighten notation, in the sequel we agree to suppress the dependence of Δ_ℓ, $\widetilde{\Delta}_\ell$, F_ℓ, x_ℓ^*, and x_ℓ^{**} on the index ℓ, and simply write Δ, $\widetilde{\Delta}$, F, x^*, and x^{**}, respectively. With this convention in mind, observe first that

$$\Delta \subseteq 2\widetilde{\Delta}. \tag{4.205}$$

To justify this inclusion, recall from (2.502) that we may write $\Delta = B(x_Q, r_Q) \cap \partial\Omega$ and $\widetilde{\Delta} = B(x_{\widetilde{Q}}, r_{\widetilde{Q}}) \cap \partial\Omega$; moreover, since $\widetilde{Q}$ is the parent of Q, we have $r_{\widetilde{Q}} = 2r_Q$. Then for each $x \in \Delta$ we have

$$|x - x_{\widetilde{Q}}| \leq |x - x_Q| + |x_Q - x_{\widetilde{Q}}| < r_Q + r_{\widetilde{Q}} = (3/2) r_{\widetilde{Q}} < 2r_{\widetilde{Q}} \tag{4.206}$$

which ultimately proves (4.205). Going forward, let us also denote by Δ^* the surface ball of center x^* and radius $R := \Lambda \cdot r_Q$, for a sufficiently large constant $\Lambda \in (2,\infty)$ (depending only on the implicit constants in the dyadic grid construction, which in turn depend only on the Ahlfors regularity constant of $\partial\Omega$) chosen so that

$$2\widetilde{\Delta} \subseteq \Delta^*. \tag{4.207}$$

We then decompose

$$f = f_1 + f_2 \quad \text{where} \quad f_1 := f\mathbf{1}_{\overline{2\Delta^*}} \quad \text{and} \quad f_2 := f\mathbf{1}_{\partial\Omega \setminus \overline{2\Delta^*}}. \tag{4.208}$$

By virtue of the sub-linearity of T_* and the fact that $\Delta \subseteq \Delta^* \subseteq 4\Delta^*$ (cf. (4.205)–(4.207)) this implies

$$\sigma(F) \leq \sigma\Big(\big\{x \in \Delta : T_* f_1(x) > 2\lambda\big\}\Big) + \sigma\Big(\big\{x \in \Delta : T_* f_2(x) > 2\lambda\big\}\Big)$$

$$\leq \sigma\Big(\big\{x \in 4\Delta^* : T_* f_1(x) > 2\lambda\big\}\Big) + \sigma\Big(\big\{x \in \Delta : T_* f_2(x) > 2\lambda\big\}\Big). \tag{4.209}$$

The contribution from f_1 in the last line above is handled as in (4.178)–(4.180), (4.182)–(4.188) by performing a decomposition of $4\Delta^*$ as in Theorem 2.6. Indeed, $a\Delta_0$, $\widetilde{x}$, f, and λ are replaced by $4\Delta^*$, x^{**}, f_1, and $\frac{1}{2}\lambda$, respectively, and we use the fact that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (4.198)), supp $f_1 \subseteq \overline{2\Delta^*} \subseteq 4\Delta^*$ (cf. (4.208)), and $\sigma(4\Delta^*) \leq c \cdot \sigma(\Delta)$ for some $c \in (0,\infty)$ depending only on the Ahlfors regularity constant of $\partial\Omega$ (since $\partial\Omega$ is Ahlfors regular and the surface balls $4\Delta^*$, Δ have comparable radii) to run the same proof as before. The conclusion is that

$$\sigma\Big(\big\{x \in 4\Delta^* : T_* f_1(x) > 2\lambda\big\}\Big) \leq \eta(\theta, \delta) \cdot \sigma(\Delta). \tag{4.210}$$

In view of the conclusion we seek (cf. (4.202)), this suits our purposes.

As for f_2, recall that R is the radius of the surface ball Δ^*, and for each $\varepsilon > 0$ set $\varepsilon' := \max\{\varepsilon, 2R\}$. Based on this choice of ε', the definition of the truncated singular integral operators in (4.105), the truncation in the definition of the function f_2, the estimate in (4.158) (presently used with x^* in place of x and ε' in place of ε), the fact that $x^{**} \in \Delta \subseteq \Delta^* \subseteq \Delta(x^*, 2\varepsilon')$ (cf. (4.198) and (4.205)–(4.207)), the fact that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (4.198)), the definition of $T_{(*)} f(x^*)$ (cf. (4.156)), the membership of x^* to $\mathcal{P}_\lambda$ (cf. (4.197)), and the first formula in (4.173), we may write

$$\begin{aligned} \big|T_\varepsilon f_2(x^*)\big| &= \big|T_{\varepsilon'} f(x^*)\big| \leq |T_{\varepsilon'} f(x^*) - T_{(\varepsilon')} f(x^*)\big| + \big|T_{(\varepsilon')} f(x^*)\big| \\ &\leq C\delta \cdot \mathcal{M}_\gamma f(x^{**}) + T_{(*)} f(x^*) \leq C\delta A\lambda + \lambda \\ &= C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda + \lambda \leq \tfrac{3}{2}\lambda, \end{aligned} \tag{4.211}$$

with the last line a consequence of our choice of A in (4.168), the fact that $\theta \in (0, 1)$, and the ability of taking $\delta_* \in (0, 1)$ small enough to begin with (while bearing in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)). With $\varepsilon > 0$ momentarily fixed, consider now an arbitrary point $x \in \Delta$ and bound

$$\big|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)\big| \leq \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{4.212}$$

where

$$\begin{aligned} \mathrm{I} := \int\limits_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x-y|>\varepsilon,\ |x^*-y|>\varepsilon}} & \Big|\langle x - y, \nu(y)\rangle k(x - y) \\ & - \langle x^* - y, \nu(y)\rangle k(x^* - y)\Big| |f(y)| \, d\sigma(y), \\ \mathrm{II} := \int\limits_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x-y|>\varepsilon,\ |x^*-y|\leq\varepsilon}} & |\langle x - y, \nu(y)\rangle| |k(x - y)| |f(y)| \, d\sigma(y), \end{aligned}$$

$$\mathrm{III} := \int\limits_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*}\\ |x^*-y|>\varepsilon,\ |x-y|\leq\varepsilon}} |\langle x^*-y, \nu(y)\rangle||k(x^*-y)||f(y)|\,\mathrm{d}\sigma(y). \tag{4.213}$$

In preparation for estimating the term I, we will first analyze the difference between I and a similar expression in which $\nu(y)$ has been replaced by the integral average $\nu_{\Delta^*} := \fint_{\Delta^*} \nu\,\mathrm{d}\sigma$. To set the stage, for each fixed $y \in \partial\Omega \setminus \overline{2\Delta^*}$ consider the function

$$F_y(z) := \langle z-y, \nu(y)-\nu_{\Delta^*}\rangle k(z-y) \quad \text{for each} \quad z \in B(x^*, R). \tag{4.214}$$

Then

$$|(\nabla F_y)(z)| \leq \Big(\sum_{|\alpha|\leq 1} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\frac{\big|\nu(y)-\nu_{\Delta^*}\big|}{|z-y|^n} \quad \text{for each} \quad z \in B(x^*, R). \tag{4.215}$$

Keeping in mind that $x \in \Delta \subseteq \Delta^* = B(x^*, R) \cap \partial\Omega$ (cf. (4.205)–(4.207)), we have

$$|x-x^*| < R. \tag{4.216}$$

Also, (recall that $[x, x^*]$ denotes the line segment with endpoints x, x^*),

$$|x^*-y| \leq 2|\xi-y| \quad \text{for each} \quad y \in \partial\Omega \setminus \overline{2\Delta^*} \quad \text{and each} \quad \xi \in [x, x^*]. \tag{4.217}$$

Hence, by (4.214)–(4.215), the Mean Value Theorem (bearing in mind (4.150)), (4.216)–(4.217), and Hölder's inequality it follows that

$$\begin{aligned}
&\int_{\partial\Omega\setminus\overline{2\Delta^*}} \Big|\langle x-y, \nu(y)-\nu_{\Delta^*}\rangle k(x-y) - \langle x^*-y, \nu(y)-\nu_{\Delta^*}\rangle k(x^*-y)\Big||f(y)|\,\mathrm{d}\sigma(y)\\
&= \int_{\partial\Omega\setminus\overline{2\Delta^*}} \big|F_y(x)-F_y(x^*)\big||f(y)|\,\mathrm{d}\sigma(y)\\
&\leq \int_{\partial\Omega\setminus\overline{2\Delta^*}} |x-x^*|\cdot \sup_{\xi\in[x,x^*]} \big|(\nabla F_y)(\xi)\big||f(y)|\,\mathrm{d}\sigma(y)\\
&\leq C\int_{\partial\Omega\setminus 2\Delta^*} \frac{R}{|x^*-y|^n}\big|\nu(y)-\nu_{\Delta^*}\big||f(y)|\,\mathrm{d}\sigma(y)\\
&\leq C\sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta^*\setminus 2^j\Delta^*} \big|\nu(y)-\nu_{\Delta^*}\big||f(y)|\,\mathrm{d}\sigma(y)
\end{aligned}$$

$$\leq C\sum_{j=1}^{\infty}2^{-j}\Big(⨍_{2^{j+1}\Delta^*}\big(\big|\nu(y)-\nu_{2^{j+1}\Delta^*}\big|+\big|\nu_{2^{j+1}\Delta^*}-\nu_{\Delta^*}\big|\big)^{\frac{1+\gamma}{\gamma}}\,\mathrm{d}\sigma(y)\Big)^{\frac{\gamma}{1+\gamma}}\times$$

$$\times\Big(⨍_{2^{j+1}\Delta^*}|f(y)|^{1+\gamma}\,\mathrm{d}\sigma(y)\Big)^{\frac{1}{1+\gamma}}$$

$$\leq C\Big(\sum_{j=1}^{\infty}(j+2)\,2^{-j}\Big)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,\mathcal{M}_\gamma f(x^{**})$$

$$\leq CA\,\delta\lambda, \tag{4.218}$$

for some $C\in(0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Above, the fifth inequality relies on (2.102) and the fact that

$$\big|\nu_{2^{j+1}\Delta^*}-\nu_{\Delta^*}\big|\leq C\,(j+1)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\quad\text{for each}\quad j\in\mathbb{N} \tag{4.219}$$

for some $C\in(0,\infty)$ depending only on n and the Ahlfors regular constant of $\partial\Omega$, which is a direct consequence of (2.105). The fifth inequality in (4.218) also uses the fact that $x^{**}\in\Delta\subseteq\Delta^*\subseteq 2^{j+1}\Delta^*$ for each integer $j\in\mathbb{N}$. The last inequality in (4.218) is a consequence of the fact that $\mathcal{M}_\gamma f(x^{**})\leq A\lambda$ (cf. (4.198)).

On the other hand, from the properties of the kernel k and the Mean Value Theorem we obtain

$$\int_{\partial\Omega\setminus\overline{2\Delta^*}}\Big|\langle x-y,\nu_{\Delta^*}\rangle k(x-y)-\langle x^*-y,\nu_{\Delta^*}\rangle k(x^*-y)\Big||f(y)|\,\mathrm{d}\sigma(y)$$

$$=\int_{\partial\Omega\setminus\overline{2\Delta^*}}\Big|\big(\langle x-y,\nu_{\Delta^*}\rangle-\langle x^*-y,\nu_{\Delta^*}\rangle\big)\,k(x^*-y)$$

$$+\langle x-y,\nu_{\Delta^*}\rangle\big(k(x-y)-k(x^*-y)\big)\Big||f(y)|\,\mathrm{d}\sigma(y)$$

$$\leq C_n\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\Big(\frac{|\langle x-x^*,\nu_{\Delta^*}\rangle|}{|x^*-y|^n}+R\frac{|\langle x-y,\nu_{\Delta^*}\rangle|}{|x^*-y|^{n+1}}\Big)|f(y)|\,\mathrm{d}\sigma(y)$$

$$\leq C_n\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle x-x^*,\nu_{\Delta^*}\rangle|}{|x^*-y|^n}|f(y)|\,\mathrm{d}\sigma(y)$$

$$+C_nR\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle x-y,\nu_{\Delta^*}-\nu_{2^{j+1}\Delta^*}\rangle|}{|x^*-y|^{n+1}}|f(y)|\,\mathrm{d}\sigma(y)$$

$$+ C_n R \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j \Delta^*} \frac{|\langle x - y, \nu_{2^{j+1}\Delta^*}\rangle|}{|x^* - y|^{n+1}} |f(y)| \, d\sigma(y)$$

$$=: I_1 + I_2 + I_3. \tag{4.220}$$

To estimate I_1, write

$$\begin{aligned} I_1 &\leq C_n R^{-1} |\langle x - x^*, \nu_{\Delta^*}\rangle| \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta^*} |f(y)| \, d\sigma(y) \\ &\leq C\delta \sum_{j=1}^{\infty} 2^{-j} \mathcal{M}_\gamma f(x^{**}) \leq C\delta \mathcal{M}_\gamma f(x^{**}) \\ &\leq CA\,\delta\lambda, \end{aligned} \tag{4.221}$$

where $C \in (0, \infty)$ depends only on n, and the Ahlfors regularity constant of $\partial\Omega$. The second inequality above is a consequence of (4.154) used here with $z := x^*$, $y := x^*$, $\mu := 2$ (a valid choice given that $x \in \Delta(x^*, 2R)$ since, as seen from (4.205)–(4.207), we have $x \in \Delta \subseteq \Delta^* = \Delta(x^*, R)$) and $x^{**} \in \Delta \subseteq \Delta^* \subseteq 2^{j+1}\Delta^*$ for each $j \in \mathbb{N}$. The last inequality (4.221) uses $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (4.198)).

To treat I_2, we write (for some $C \in (0, \infty)$ which depends only on n, and the Ahlfors regularity constant of $\partial\Omega$),

$$\begin{aligned} I_2 &\leq CR \sum_{j=1}^{\infty} \int_{2^{j+1}\Delta^* \setminus 2^j \Delta^*} \frac{|\nu_{\Delta^*} - \nu_{2^{j+1}\Delta^*}|}{|x^* - y|^n} |f(y)| \, d\sigma(y) \\ &\leq C \sum_{j=1}^{\infty} (j+1) \, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, 2^{-j} \fint_{2^{j+1}\Delta^*} |f(y)| \, d\sigma(y) \\ &\leq C\delta \, \mathcal{M}_\gamma f(x^{**}) \leq CA\,\delta\lambda, \end{aligned} \tag{4.222}$$

where the first inequality uses the definition of I_2 (given in (4.220)) as well as the estimate $|x - y| \leq (3/2)|x^* - y|$ valid for each $y \in \partial\Omega \setminus 2\Delta^*$, the second inequality takes into account (4.219) and the Ahlfors regularity of $\partial\Omega$, while the remaining inequalities are justified as in (4.221).

As regards I_3, write (again, with $C \in (0, \infty)$ depending only on n, and the Ahlfors regularity constant of $\partial\Omega$)

$$I_3 \leq C \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta^*} \frac{|\langle x - y, \nu_{2^{j+1}\Delta^*}\rangle|}{2^{j+1}R} |f(y)| \, d\sigma(y)$$

$$\leq C\delta \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta^*} |f(y)| \, d\sigma(y) \leq C\delta \mathcal{M}_\gamma f(x^{**})$$
$$\leq CA\,\delta\lambda. \tag{4.223}$$

The second inequality in (4.223) is based on (4.154) used with $z := x^*$ and R replaced by $2^{j+1}R$. The remaining inequalities in (4.223) are then justified much as in (4.221).

At this stage, by combining (4.218) and (4.220)–(4.223) we conclude that there exists some $C \in (0, \infty)$ which depends only on n, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{I} \leq CA\,\delta\lambda. \tag{4.224}$$

To bound II in (4.213), recall that $x, x^{**} \in \Delta$ and assume $y \in \partial\Omega \setminus \overline{2\Delta^*}$ is such that $|x^* - y| \leq \varepsilon$ and $|x - y| > \varepsilon$. Then, $2R < |x^* - y| \leq \varepsilon$ and since $x, x^{**} \in \Delta \subseteq B(x_Q, r_Q)$ (where x_Q and r_Q are, respectively, the center and radius of the surface ball Δ) and $R = \Lambda \cdot r_Q$ with $\Lambda > 2$, we have $|x - x^{**}| < 2r_Q < R < \varepsilon/2$. Hence, the point x^{**} belongs to the surface ball $\Delta(x, \varepsilon/2)$. Moreover, on account of (4.216) we may write $|x - y| \leq |x - x^*| + |x^* - y| < R + \varepsilon < (3/2)\varepsilon$ which, in particular, guarantees that $y \in \Delta(x, 2\varepsilon)$. Consequently, $\varepsilon < |x - y| < 2\varepsilon$ hence $|k(x - y)| \leq \varepsilon^{-n}$ and (for some $C \in (0, \infty)$ which depends only on depends only on n and the Ahlfors regularity constant of $\partial\Omega$),

$$\mathrm{II} \leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x - y, \nu(y)\rangle| \, |f(y)| \, d\sigma(y)$$
$$\leq C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x - y, \nu(y) - \nu_{\Delta(x,2\varepsilon)}\rangle| \, |f(y)| \, d\sigma(y)$$
$$+ C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x - y, \nu_{\Delta(x,2\varepsilon)}\rangle| \, |f(y)| \, d\sigma(y)$$
$$=: \mathrm{II}_1 + \mathrm{II}_2. \tag{4.225}$$

Using Hölder's inequality, (2.102), (4.198), and (4.152) we obtain that there exists some $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{II}_1 \leq C \left(\fint_{\Delta(x,2\varepsilon)} |\nu(y) - \nu_{\Delta(x,2\varepsilon)}|^{\frac{1+\gamma}{\gamma}} \, d\sigma(y) \right)^{\frac{\gamma}{1+\gamma}} \left(\fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma} \, d\sigma(y) \right)^{\frac{1}{1+\gamma}}$$
$$\leq C \, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, \mathcal{M}_\gamma f(x^{**}) \leq CA\,\delta\lambda, \tag{4.226}$$

since x^{**} is contained in $\Delta(x,\varepsilon/2) \subseteq \Delta(x,2\varepsilon)$ and $\mathcal{M}_\gamma f(x^{**}) \leq CA\lambda$, as already noted earlier. As for II_2, invoking (4.154), Hölder's inequality, and (4.198), it follows that (with $C \in (0,\infty)$ as above)

$$\mathrm{II}_2 \leq C\Big(\sup_{y\in\Delta(x,2\varepsilon)} \varepsilon^{-1}|\langle x-y, \nu_{\Delta(x,2\varepsilon)}\rangle|\Big) ⨍_{\Delta(x,2\varepsilon)} |f(y)|\,\mathrm{d}\sigma(y)$$

$$\leq C\delta\Big(⨍_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\,\mathrm{d}\sigma(y)\Big)^{\frac{1}{1+\gamma}}$$

$$\leq C\delta\cdot\mathcal{M}_\gamma f(x^{**}) \leq CA\,\delta\lambda. \tag{4.227}$$

From (4.225)–(4.227) we see that there exists $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{II} \leq CA\,\delta\lambda. \tag{4.228}$$

Turning our attention to III, recall that $x, x^{**} \in \Delta$ and suppose $y \in \partial\Omega \setminus \overline{2\Delta^*}$ is such that $|x^*-y| > \varepsilon$ and $|x-y| \leq \varepsilon$. Then $|x^*-y| > 2R > R + |x-x^*|$ by (4.216) which further entails $\varepsilon \geq |x-y| \geq |x^*-y| - |x-x^*| > R$. In particular, $R < \varepsilon$. If we now abbreviate $\widetilde{R} := R+\varepsilon$ then, on the one hand, we may write the estimate $|x^*-y| \leq |x^*-x| + |x-y| < R+\varepsilon = \widetilde{R}$, while on the other hand having $|x^*-y| > \varepsilon$ and $|x^*-y| > 2R$ implies $|x^*-y| > R + (\varepsilon/2) > \frac{1}{2}\widetilde{R}$. As such, $|k(x^*-y)| \leq \widetilde{R}^{-n}$ and

$$\mathrm{III} \leq C_n\widetilde{R}^{-1} ⨍_{\Delta(x^*,\widetilde{R})} |\langle x^*-y, \nu(y)\rangle|\,|f(y)|\,\mathrm{d}\sigma(y). \tag{4.229}$$

Granted this, the same type of argument which, starting with the first line in (4.225) has produced (4.228) (reasoning with $\widetilde{R}/2$ replacing ε and with x^* replacing x) will now yield (for some $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$)

$$\mathrm{III} \leq CA\,\delta\lambda, \tag{4.230}$$

as soon as we show that $x^{**} \in \Delta(x^*,\widetilde{R})$. To justify this membership, start by recalling that $|x-x^{**}| < 2r_Q < R$ and then use (4.216), the triangle inequality, and the fact that $R < \varepsilon$ to estimate $|x^*-x^{**}| \leq |x-x^*| + |x-x^{**}| < 2R < \widetilde{R}$. The proof of (4.230) is therefore complete.

Let us summarize our progress. From (4.212), (4.224), (4.228), (4.230), and our choice of A in (4.168) we conclude that there exists some $C \in (0,\infty)$, which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\left|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)\right| \leq C\,A\,\delta\lambda = C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda, \qquad \forall\, x \in \Delta, \quad \forall\, \varepsilon > 0. \tag{4.231}$$

In view of the fact that $\theta \in (0, 1)$, and taking $\delta_* \in (0, 1)$ small enough to begin with (again, keeping in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)), from (4.231) we conclude that

$$\left|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)\right| \leq \tfrac{1}{2}\lambda, \qquad \forall\, x \in \Delta, \quad \forall\, \varepsilon > 0. \tag{4.232}$$

By combining (4.211), (4.232), and (4.104) we thus obtain

$$T_* f_2(x) \leq 2\lambda \ \text{ for all } \ x \in \Delta, \tag{4.233}$$

whenever $\delta_* \in (0, 1)$ is small enough. Therefore, for this choice of δ_*, we conclude that

$$\sigma\Big(\big\{x \in \Delta : T_* f_2(x) > 2\lambda\big\}\Big) = 0 \tag{4.234}$$

which, in concert with (4.209) and (4.210), establishes (4.202). This finishes the proof of the good-λ inequality (4.170).

Once (4.170) has been established, we proceed to prove (4.151). First, using (4.159), by our definition of A, and by possibly choosing a smaller $\delta_* \in (0, 1)$ (again, bearing in mind that $\lim_{t\to 0^+} t/\phi(t) = 0$; cf. (4.121)), for each point $x \in I_0$ with $T_{(*)} f(x) > \lambda$ and $\mathcal{M}_\gamma f(x) \leq A\lambda$ we may write

$$\begin{aligned} \lambda < T_{(*)} f(x) &\leq T_* f(x) + C\delta \cdot \mathcal{M}_\gamma f(x) \\ &\leq T_* f(x) + C\delta A\lambda = T_* f(x) + C\theta\Big(\frac{\delta}{\phi(\delta)}\Big)\lambda \\ &< T_* f(x) + \tfrac{1}{2}\lambda. \end{aligned} \tag{4.235}$$

Hence, for such a choice of $\delta_* \in (0, 1)$ we have

$$\tfrac{1}{2}\lambda < T_* f(x) \text{ whenever the point } x \in I_0 \text{ is such that } T_{(*)} f(x) > \lambda \text{ and } \mathcal{M}_\gamma f(x) \leq A\lambda. \tag{4.236}$$

Consequently,

$$\begin{aligned} &\big\{x \in I_0 : T_{(*)} f(x) > \lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\big\} \\ &\qquad\qquad \subseteq \big\{x \in I_0 : T_* f(x) > \tfrac{\lambda}{2}\big\} \end{aligned} \tag{4.237}$$

which, in turn, permits us to estimate

$$\begin{aligned}
w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}\Big) &\leq w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda \text{ and } \mathcal{M}_\gamma f(x) \leq A\lambda\big\}\Big) \\
&\quad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big) \\
&\leq w\Big(\big\{x \in I_0 : T_* f(x) > \tfrac{\lambda}{2}\big\}\Big) \\
&\quad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big). \qquad (4.238)
\end{aligned}$$

From (4.169) and (4.121) it is clear that for each fixed θ we have

$$\eta(\theta, \delta) = C\big(\theta^{1+\gamma} + \theta^{1+\gamma/2} \cdot O(1) + o(1)\big) \text{ as } \delta \to 0^+. \qquad (4.239)$$

This makes it is possible to first choose the threshold $\delta_* \in (0, 1)$, then pick the coefficient $\theta \in (0, 1)$ small enough depending only on n, p, $[w]_{A_p}$, ϕ, and the Ahlfors regularity constant of $\partial\Omega$, so that

$$\eta(\theta, \delta)^\tau < (2 \cdot 8^p)^{-1}. \qquad (4.240)$$

This is the last demand imposed on δ_*, θ, and the totality of all these size specifications imply that the final choice of these parameters ultimately depends only on n, p, $[w]_{A_p}$, ϕ, and the Ahlfors regularity constant of $\partial\Omega$. Combining (4.238) with (4.170) and keeping (4.240) in mind we then get

$$\begin{aligned}
&w\Big(\big\{x \in I_0 : T_* f(x) > 4\lambda\big\}\Big) \\
&\quad \leq w\Big(\big\{x \in I_0 : T_* f(x) > 4\lambda \text{ and } \mathcal{M}_\gamma f(x) \leq A\lambda\big\}\Big) \\
&\qquad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big) \\
&\quad \leq \eta(\theta, \delta)^\tau \cdot w\Big(\big\{x \in I_0 : T_{(*)} f(x) > \lambda\big\}\Big) \\
&\qquad + w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big) \\
&\quad < (2 \cdot 8^p)^{-1}\, w\Big(\big\{x \in I_0 : T_* f(x) > \tfrac{\lambda}{2}\big\}\Big) \\
&\qquad + \big(1 + (2 \cdot 8^p)^{-1}\big)\, w\Big(\big\{x \in I_0 : \mathcal{M}_\gamma f(x) > A\lambda\big\}\Big). \qquad (4.241)
\end{aligned}$$

Recall that $\gamma \in (0, p-1)$ has been chosen so that $w \in A_{p/(1+\gamma)}(\partial\Omega, \sigma)$, hence $\mathcal{M}_\gamma$ is bounded on $L^p(\partial\Omega, w)$. Multiply the most extreme sides of (4.241) by $p\lambda^{p-1}$ and integrate over $\lambda \in (0, \infty)$. Bearing in mind that $A = \theta \cdot \phi(\delta)^{-1}$, after three natural changes of variables (namely, $\widetilde{\lambda} := 4\lambda$ in the first integral, $\widetilde{\lambda} := \frac{1}{2}\lambda$ in the second integral, and $\widetilde{\lambda} := \theta\phi(\delta)^{-1}\lambda$ in the third integral) we therefore obtain

$$\begin{aligned}\int_{I_0} |T_* f|^p \, dw &\leq \frac{1}{2}\int_{I_0} |T_* f|^p \, dw + \phi(\delta)^p \theta^{-p}\big(2^{2p} + 2^{-p-1}\big)\int_{I_0} (\mathcal{M}_\gamma f)^p \, dw \\ &\leq \frac{1}{2}\int_{I_0} |T_* f|^p \, dw + C\,\phi(\delta)^p \int_{\partial\Omega} |f|^p \, dw, \end{aligned} \tag{4.242}$$

for some constant $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, ϕ, and the Ahlfors regularity constant of $\partial\Omega$ (hence, in particular, independent of the function f, the quantity δ, as well as the parameters x_0, m defining the set I_0). Since $f \in L^p(\partial\Omega, w)$ and the operator T_* maps the space $L^p(\partial\Omega, w)$ into itself (cf. Proposition 3.4), it follows that $\int_{I_0} |T_* f|^p \, dw \leq \|T_* f\|^p_{L^p(\partial\Omega, w)} < \infty$. Hence, the first integral in the right-most side of (4.242) may be absorbed in the left-most side. By also taking into account (4.165), we therefore obtain

$$\int_{2\Delta_0} |T_* f|^p \, dw \leq \int_{I_0} |T_* f|^p \, dw \leq C\phi(\delta)^p \int_{\partial\Omega} |f|^p \, dw. \tag{4.243}$$

Recall that $2\Delta_0 = \Delta(x_0, 2^{-m+1})$ and the only constraint on the integer $m \in \mathbb{Z}$ has been that $\operatorname{supp} f \subseteq 2\Delta_0$. Upon letting $m \to -\infty$ and invoking Lebesgue's Monotone Convergence Theorem we arrive at the conclusion that, for some constant $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the Ahlfors regularity constant of $\partial\Omega$, we have the estimate

$$\begin{gathered}\int_{\partial\Omega} |T_* f|^p \, dw \leq C\phi(\delta)^p \int_{\partial\Omega} |f|^p \, dw, \\ \text{for every } f \in L^p(\partial\Omega, w) \text{ with compact support.}\end{gathered} \tag{4.244}$$

To treat the case when the function $f \in L^p(\partial\Omega, w)$ is now arbitrary, for each $j \in \mathbb{N}$ define $f_j := \mathbf{1}_{\Delta(x_0, j)} f$. Then Lebesgue's Dominated Convergence Theorem implies that $f_j \to f$ in $L^p(\partial\Omega, w)$ as $j \to \infty$, and since T_* is continuous on $L^p(\partial\Omega, w)$ we also have $T_* f_j \to T_* f$ in $L^p(\partial\Omega, w)$ as $j \to \infty$. Writing the estimate in (4.244) for f_j in place of f and passing to limit $j \to \infty$ then yields

$$\int_{\partial\Omega} |T_* f|^p \, dw \leq C\phi(\delta)^p \int_{\partial\Omega} |f|^p \, dw \quad \text{for each} \quad f \in L^p(\partial\Omega, w), \tag{4.245}$$

where $C \in (0, \infty)$ depends only on n, p, $[w]_{A_p}$, ψ, ϕ, and the Ahlfors regularity constant of $\partial\Omega$. Sending $\delta \searrow \|\nu\|_{[\mathrm{BMO}(\partial\Omega, \sigma)]^n}$ (cf. (4.152) and the second line in (4.121)), then finishes the proof of (4.151).

Finally, the very last claim in the statement of Theorem 4.2 follows from (4.153). The proof of Theorem 4.2 is therefore complete. □

Recall the notion of chord-arc domain introduced, in the two-dimensional setting, in Definition 2.16.

Corollary 4.1 *Fix $\varkappa_* \in (0, \infty)$ and let $\Omega \subseteq \mathbb{R}^2$ be a $\varkappa$-CAD for some $\varkappa \in [0, \varkappa_*)$. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. In addition, select some integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Consider next a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^2 \setminus \{0\})$, for a sufficiently large integer $N \in \mathbb{N}$, which is even and positive homogeneous of degree -2, and define the maximal operator T_* acting on each function $f \in L^p(\partial\Omega, w)$ according to*

$$T_* f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big| \quad \textit{for each} \quad x \in \partial\Omega, \tag{4.246}$$

where, for each $\varepsilon > 0$,

$$T_\varepsilon f(x) := \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \quad \textit{for all} \quad x \in \partial\Omega. \tag{4.247}$$

Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$, which depends only on m, $\varkappa_*$, p, and $[w]_{A_p}$ such that

$$\begin{aligned} \|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} &\leq C_m \Big(\sum_{|\alpha|\leq N} \sup_{S^1} |\partial^\alpha k| \Big) \times \\ &\quad \times \sqrt{\varkappa} \cdot \underbrace{\ln\Big(\cdots \ln\big(\ln}_{m \text{ natural logarithms}}(1/\min\{\varkappa, ({}^m e)^{-1}\})\big) \cdots \Big). \end{aligned} \tag{4.248}$$

Of course, the crux of the matter is the presence of $\sqrt{\varkappa}$ as a multiplicative factor in the right-hand side of (4.248). As a consequence, $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ is as small as we please if $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD whose constant $\varkappa \in (0, 1)$ is sufficiently small (relative to the integral exponent p, the characteristic $[w]_{A_p}$ of the Muckenhoupt weight, and the integral kernel k).

Proof of Corollary 4.1 From (2.229) and (2.118) we deduce that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \leq \min\Big\{1, 2\sqrt{\varkappa(2+\varkappa)}\Big\} \leq \sqrt{4+\sqrt{20}} \cdot \sqrt{\varkappa}. \tag{4.249}$$

Also, Proposition 2.10 implies that Ω is a UR domain, with the UR constants of $\partial\Omega$ controlled in terms of $\varkappa_*$. Granted these properties, Theorem 4.2 applies and (4.106) together with (4.100) give (4.248). □

Theorem 4.2 readily implies similar operator norm estimates for principal-value singular integral operators whose integral kernel has a special algebraic format, in that it involves the inner product between the outward unit normal and the chord, as a factor. This is made precise later on (see Theorem 4.7). Specifically, for a given second-order, homogeneous, constant complex coefficient system L with $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, and a given UR domain $\Omega \subseteq \mathbb{R}^n$, we shall employ Corollary 4.2 below with T either the boundary-to-boundary double layer potential operator K_A associated with a coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$ or its "transpose" version $K_A^\#$, acting on Muckenhoupt weighted Lebesgue spaces on $\partial\Omega$.

Corollary 4.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight w in $A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same symbol w for the measure associated with the given weight w as in (2.509). Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$ and suppose $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ is a complex-valued function which is even and positive homogeneous of degree $-n$. In this setting consider the principal-value singular integral operators $T, T^\#$ acting on each function $f \in L^p(\partial\Omega, w)$ according to*

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y), \tag{4.250}$$

and

$$T^\# f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\, d\sigma(y), \tag{4.251}$$

at σ-a.e. point $x \in \partial\Omega$. Then for each $m \in \mathbb{N}$ there exists a constant $C_m \in (0,\infty)$, which depends only on m, n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has

$$\|T\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \tag{4.252}$$

and

$$\Big\|T^\#\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.253}$$

Also, if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in (4.252)–(4.253) to depend only on said entities (i.e., n, p, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$) and m.

In addition, with $p' \in (1, \infty)$ *denoting the Hölder conjugate exponent of* p *and with* $w' := w^{1-p'} \in A_{p'}(\partial\Omega, \sigma)$*, it follows that*

$$\begin{aligned}&\textit{the (real) transpose of } T : L^p(\partial\Omega, w) \to L^p(\partial\Omega, w)\\ &\textit{is the operator } T^{\#} : L^{p'}(\partial\Omega, w') \to L^{p'}(\partial\Omega, w').\end{aligned} \tag{4.254}$$

Proof Fix $m \in \mathbb{N}$. In view of the fact that

$$\|T\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq \|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}, \tag{4.255}$$

the estimate claimed in (4.252) follows directly from (4.106). The claim in the subsequent paragraph in the statement follows from Theorem 2.3. Next, observe that (4.254) is implied by (4.250)–(4.251) and (3.83). To justify the claim made in (4.253), we write

$$\begin{aligned}\left\|T^{\#}\right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} &= \|T\|_{L^{p'}(\partial\Omega,w')\to L^{p'}(\partial\Omega,w')}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{4.256}$$

thanks to (4.252) used with p', w' in place of p, w. □

Remark 4.8 Of course, in the special case when $w \equiv 1$, Theorem 4.2 and Corollary 4.2 yield estimates on ordinary Lebesgue spaces, $L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. Via real interpolation, these further imply similar estimates on the scale of Lorentz spaces on $\partial\Omega$. Specifically, from (4.106), (4.252)–(4.253), and real interpolation (for sub-linear operators) we conclude that for each $m \in \mathbb{N}$, $p \in (1, \infty)$, and $q \in (0, \infty]$ there exists a constant $C_m \in (0, \infty)$, which depends only on m, n, p, q, and the UR constants of $\partial\Omega$, with the property that

$$\|T_*\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.257}$$

$$\|T\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.258}$$

and

$$\left\|T^{\#}\right\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.259}$$

More general results of this type are discussed later, in Theorem 8.8 (cf. also Examples 8.2 and 8.6).

Remark 4.9 In the context of Corollary 4.2, estimates (4.252)–(4.253) remain valid with a fixed constant $C_m \in (0, \infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are allowed to vary while retaining control. Concretely, Remark 4.3 implies that for each compact interval $I \subset (0, \infty)$ and each number $W \in (0, \infty)$ there exists a constant $C_m \in (0, \infty)$, which depends only on m, n, I, W, and the UR constants of $\partial\Omega$, with the property that (4.252)–(4.253) hold for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

Similar considerations apply to the estimates in (4.257)–(4.259).

4.3 Norm Estimates and Invertibility Results for Double Layers

We first recall a result (cf. [61, Theorem 2.16, p. 2603]) which is a combination of the extrapolation theorem of Rubio de Francia and the commutator theorem of Coifman et al., [31], suitably adapted to the setting of spaces of homogeneous type.

Theorem 4.3 *Make the assumption that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0 \in (1, \infty)$ along with some non-decreasing function $\Phi : (0, \infty) \to (0, \infty)$ and let T be a linear operator which is bounded on $L^{p_0}(\Sigma, w)$ for every $w \in A_{p_0}(\Sigma, \sigma)$, with operator norm $\leq \Phi\big([w]_{A_{p_0}}\big)$.*

Then for each integrability exponent $p \in (1, \infty)$ there exist $C_1, C_2 \in (0, \infty)$ which depend exclusively on the dimension n, the exponents p_0, p, and the Ahlfors regularity constant of Σ, such that for any Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ the operator

$$T : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{4.260}$$

is well defined, linear, and bounded, with operator norm

$$\|T\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi\Big(C_2 \cdot [w]_{A_p}^{1+(p_0-1)/(p-1)}\Big). \tag{4.261}$$

In addition, given any $p \in (1, \infty)$ along with some $w \in A_p(\Sigma, \sigma)$, there exists a constant $C = C(\Sigma, n, p_0, p, [w]_{A_p}) \in (0, \infty)$ with the property that for every complex-valued function $b \in L^\infty(\Sigma, \sigma)$ one has (with C_1 as before)

$$\|[M_b, T]\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi(C)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,, \tag{4.262}$$

where $[M_b, T]$ is the commutator of T considered as in (4.260) and the operator M_b of pointwise multiplication on $L^p(\Sigma, w)$ by the function b, i.e.,

$$[M_b, T]f := b(Tf) - T(bf) \quad \textit{for each} \quad f \in L^p(\Sigma, w). \tag{4.263}$$

In particular, from (4.262) *with* $w \equiv 1$ *and real interpolation it follows that, for any* $p \in (1, \infty)$ *and* $q \in (0, \infty]$, *there exists some* $C = C(\Sigma, n, p, q) \in (0, \infty)$ *with the property that for every complex-valued function* $b \in L^\infty(\Sigma, \sigma)$ *one has*

$$\|[M_b, T]\|_{L^{p,q}(\Sigma,\sigma)\to L^{p,q}(\Sigma,\sigma)} \leq C_1 \cdot \Phi(C)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)} \,. \tag{4.264}$$

Theorem 4.3 is a particular case of a more general result proved in Theorem 4.4, stated just after the following remark.

Remark 4.10 Even though Theorem 4.3 suffices for the purposes we have in mind, it is worth noting that there is a version of (4.262) in which the pointwise multiplier b is allowed to belong to the larger space $\mathrm{BMO}(\Sigma, \sigma)$. The price to pay is that we now no longer may regard $[M_b, T]$ as in (4.263) and, instead, have to interpret this as an abstract extension (by density) of a genuine commutator. Specifically, given a real-valued function $b \in \mathrm{BMO}(\Sigma, \sigma)$, for each $N \in \mathbb{N}$ define

$$b_N := \min\Big\{\max\{b, -N\}, N\Big\} = \max\Big\{\min\{b, N\}, -N\Big\}, \tag{4.265}$$

and note that there exists $C \in (0, \infty)$ such that

$$\begin{aligned}
&b_N \in L^\infty(\Sigma, \sigma), \ \text{ thus } \ b_N \in \mathrm{BMO}(\Sigma, \sigma), \ \text{ and} \\
&\|b_N\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq 2\|b\|_{\mathrm{BMO}(\Sigma,\sigma)} \ \text{ for all } \ N \in \mathbb{N}, \\
&|b_N(x)| \leq |b(x)| \ \text{ for all } \ x \in \Sigma \ \text{ and } \ N \in \mathbb{N}, \\
&\lim_{N\to\infty} b_N(x) = b(x) \ \text{ for each } x \text{ belonging to } \Sigma.
\end{aligned} \tag{4.266}$$

Fix an exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ and pick a function $f \in L^p(\Sigma, w)$ with the property that $bf \in L^p(\Sigma, w)$. Then Lebesgue's Dominated Convergence Theorem implies $b_N f \to bf$ in $L^p(\Sigma, w)$ as $N \to \infty$, hence also $T(b_N f) \to T(bf)$ in $L^p(\Sigma, w)$ as $N \to \infty$ by (4.260). Since we also have $b_N T(f) \to bT(f)$ at each point in Σ as $N \to \infty$, we ultimately conclude that

$$\begin{aligned}
&\text{for each function } f \in L^p(\Sigma, w) \text{ such that } bf \in L^p(\Sigma, w) \text{ there} \\
&\text{exists a strictly increasing sequence } \{N_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N} \text{ for which} \\
&\big[M_{b_{N_j}}, T\big]f \to \big[M_b, T\big]f \text{ at } \sigma\text{-a.e. point in } \Sigma \text{ as } j \to \infty.
\end{aligned} \tag{4.267}$$

For example, the fact that we have $\mathrm{BMO}(\Sigma, \sigma) \subseteq L^p_{\mathrm{loc}}(\Sigma, w)$ (cf. Lemma 2.13) means that the pointwise convergence result in (4.267) is valid for each function f belonging to $L^\infty_{\mathrm{comp}}(\Sigma, w) = L^\infty_{\mathrm{comp}}(\Sigma, \sigma)$, the space of essentially bounded functions with compact support in Σ.

Granted (4.267), for each such function $f \in L^p(\Sigma, w)$ such that $bf \in L^p(\Sigma, w)$ we may now write (bearing in mind that w and σ have the same nullsets)

$$\int_\Sigma \big|[M_b, T]f\big|^p\,\mathrm{d}w = \int_\Sigma \liminf_{j\to\infty}\Big|\big[M_{b_{N_j}}, T\big]f\Big|^p\,\mathrm{d}w$$
$$\leq \liminf_{j\to\infty}\int_\Sigma \Big|\big[M_{b_{N_j}}, T\big]f\Big|^p\,\mathrm{d}w$$
$$\leq \liminf_{j\to\infty}\Big(C_1\cdot\Phi(C)\,\|b_{N_j}\|_{\mathrm{BMO}(\Sigma,\sigma)}\Big)^p\|f\|^p_{L^p(\Sigma,w)}$$
$$\leq \Big(2C_1\cdot\Phi(C)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\Big)^p\|f\|^p_{L^p(\Sigma,w)}, \tag{4.268}$$

where the equality comes from (4.267), the first inequality is implied by Fatou's Lemma, the second inequality is a consequence of (4.262) (bearing in mind the first property in (4.266)), and the last inequality follows from the second line of (4.266).

In turn, (4.268) proves that

the operator $[M_b, T] := b(T\cdot) - T(b\cdot)$ maps the linear space $\big\{f \in L^p(\Sigma, w) : bf \in L^p(\Sigma, w)\big\}$ boundedly into $L^p(\Sigma, w)$. (4.269)

Given that $\big\{f \in L^p(\Sigma, w) : bf \in L^p(\Sigma, w)\big\}$ is dense in $L^p(\Sigma, w)$ (since, as already noted, this contains $L^\infty_{\mathrm{comp}}(\Sigma, w)$ which is itself dense in $L^p(\Sigma, w)$), we finally conclude that $[M_b, T]$, originally acting as a commutator in the manner described in (4.269), extends by density to a linear and bounded mapping from $L^p(\Sigma, w)$ into itself.

Here is a generalization of Theorem 4.3, involving the "maximal commutator" associated with a given family of linear and bounded operators.

Theorem 4.4 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0 \in (1,\infty)$ and let $\{T_j\}_{j\in\mathbb{N}}$ be a family of linear operators which are bounded on $L^{p_0}(\Sigma, w)$ for every $w \in A_{p_0}(\Sigma,\sigma)$. Define the action of the maximal operator associated with this family on each function $f \in L^{p_0}(\Sigma, w)$ with $w \in A_{p_0}(\Sigma,\sigma)$ as*

$$T_{\max} f(x) := \sup_{j\in\mathbb{N}} |T_j f(x)| \ \textit{ for each } \ x \in \Sigma. \tag{4.270}$$

Assume that for each $w \in A_{p_0}(\Sigma,\sigma)$ the sub-linear operator $T_{\max}$ maps $L^{p_0}(\Sigma, w)$ into itself, and that there exists some non-decreasing function $\Phi : (0,\infty) \to (0,\infty)$ with the property that

$$\|T_{\max}\|_{L^{p_0}(\Sigma,w)\to L^{p_0}(\Sigma,w)} \leq \Phi\big([w]_{A_{p_0}}\big) \ \textit{ for each } \ w \in A_{p_0}(\Sigma,\sigma). \tag{4.271}$$

Then the following statements are true.

(i) For each integrability exponent $p \in (1, \infty)$ there exist $C_1, C_2 \in (0, \infty)$ which depend exclusively on the dimension n, the exponents p_0, p, and the Ahlfors regularity constant of Σ, with the property that for any Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ the operator

$$T_{\max} : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{4.272}$$

is well defined, sub-linear, and bounded, with operator norm

$$\|T_{\max}\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi\Big(C_2 \cdot [w]_{A_p}^{1+(p_0-1)/(p-1)}\Big). \tag{4.273}$$

In particular, for each $j \in \mathbb{N}$ the operator T_j is a well-defined, linear, and bounded mapping on $L^p(\Sigma, w)$ with operator norm satisfying a similar estimate to (4.273).

(ii) Pick an arbitrary $p \in (1, \infty)$ along with some $w \in A_p(\Sigma, \sigma)$, and fix an arbitrary complex-valued function $b \in L^\infty(\Sigma, \sigma)$. Define the action of the "maximal commutator" (associated with the given function b and the family $\{T_j\}_{j\in\mathbb{N}}$) on each function $f \in L^p(\Sigma, w)$ as

$$C_{\max} f(x) := \sup_{j\in\mathbb{N}} \big|[M_b, T_j] f(x)\big| \ \ \text{for each} \ \ x \in \Sigma, \tag{4.274}$$

where M_b denotes the operator of pointwise multiplication by the function b. Then there exist two constants $C_i = C_i(\Sigma, n, p_0, p, [w]_{A_p}) \in (0, \infty)$, $i \in \{1, 2\}$, independent of the function b and the family $\{T_j\}_{j\in\mathbb{N}}$, with the property that

$$\|C_{\max}\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi(C_2)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)} \,. \tag{4.275}$$

The particular case when all operators in the family $\{T_j\}_{j\in\mathbb{N}}$ are identical to one another corresponds to Theorem 4.3.

Proof of Theorem 4.4 The fact that for each $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$ the sub-linear operator $T_{\max}$ induces a bounded mapping on $L^p(\Sigma, w)$ whose operator norm may be estimated as in (4.273) follows from Rubio de Francia's extrapolation theorem, in the format presented in [111, §7.7] (this is responsible for the specific format of the constant in (4.273); see also [34, Theorem 3.22, p.40] and [42, Theorem 3.2] for the Euclidean setting). This takes care of item *(i)*.

To deal with item *(ii)*, we shall adapt the argument in [31], [69], [61], [13]. First, from simple linearity and homogeneity considerations, there is no loss of generality in assuming that $b \in L^\infty(\Sigma, \sigma)$ is actually real-valued and satisfies $\|b\|_{\mathrm{BMO}(\Sigma,\sigma)} = 1$ (the case when b is constant is trivial). Fix now $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$. From item *(8)* of Proposition 2.20 we know that there exists some small $\varepsilon = \varepsilon(\Sigma, p, [w]_{A_p}) > 0$ with the property that for each complex number z with $|z| \leq \varepsilon$ we have

$$w \cdot \mathrm{e}^{(\mathrm{Re}\, z)b} \in A_p(\Sigma, \sigma) \ \text{ with } \ \big[w \cdot \mathrm{e}^{(\mathrm{Re}\, z)b}\big]_{A_p} \leq C, \tag{4.276}$$

where the constant $C = C(\Sigma, p, [w]_{A_p}) \in (0, \infty)$ is independent of z.

To proceed, denote by $\mathscr{L}(L^p_w)$ the space of all linear and bounded operators from $L^p(\Sigma, w)$ into itself, equipped with the operator norm $\|\cdot\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)}$. The idea is now to observe that, for each $j \in \mathbb{N}$,

$$\begin{aligned} &\Phi_j : \big\{z \in \mathbb{C} : |z| < \varepsilon/2\big\} \longrightarrow \mathscr{L}(L^p_w) \ \text{ defined as} \\ &\Phi_j(z) := M_{\mathrm{e}^{zb}} T_j M_{\mathrm{e}^{-zb}} \ \text{ for each } \ z \in \mathbb{C} \ \text{ with } \ |z| < \varepsilon/2 \end{aligned} \tag{4.277}$$

is an analytic map which, for each $z \in \mathbb{C}$ with $|z| < \varepsilon/2$ and each $f \in L^p(\Sigma, w)$, satisfies

$$\begin{aligned} &\int_\Sigma \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, w(x)\, \mathrm{d}\sigma(x) \\ &\quad = \int_\Sigma \sup_{j\in\mathbb{N}} \big|T_j(\mathrm{e}^{-zb} f)(x)\big|^p \, w(x) \cdot \mathrm{e}^{(\mathrm{Re}\, z)b(x)}\, \mathrm{d}\sigma(x) \\ &\quad = \int_\Sigma \big|T_{\max}(\mathrm{e}^{-zb} f)(x)\big|^p \, w(x) \cdot \mathrm{e}^{(\mathrm{Re}\, z)b(x)}\, \mathrm{d}\sigma(x) \\ &\quad \leq \|T_{\max}\|^p_{L^p(\Sigma, w\cdot \mathrm{e}^{(\mathrm{Re}\, z)b}) \to L^p(\Sigma, w\cdot \mathrm{e}^{(\mathrm{Re}\, z)b})} \times \\ &\qquad\quad \times \int_\Sigma \big|\mathrm{e}^{-zb(x)} f(x)\big|^p \, w(x) \cdot \mathrm{e}^{(\mathrm{Re}\, z)b(x)}\, \mathrm{d}\sigma(x) \\ &\quad \leq C_1^p \cdot \Phi\Big(C_2 \cdot C^{1+(p_0-1)/(p-1)}\Big)^p \|f\|^p_{L^p(\Sigma,w)}, \end{aligned} \tag{4.278}$$

thanks to (4.277), (4.270), (4.276), and (4.273). In addition, from (4.277) and Cauchy's reproducing formula for analytic functions we see that for each $j \in \mathbb{N}$ we have

$$[M_b, T_j] = \Phi_j'(0) = \frac{1}{2\pi \mathrm{i}} \int_{|z|=\varepsilon/4} \frac{\Phi_j(z)}{z^2}\, \mathrm{d}z. \tag{4.279}$$

Consequently, for each $f \in L^p(\Sigma, w)$ and $x \in \Sigma$, we have

$$C_{\max} f(x) = \sup_{j\in\mathbb{N}} \big|[M_b, T_j] f(x)\big| \leq \frac{8}{\pi\varepsilon^2}\int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big| \, d\mathcal{H}^1(z), \tag{4.280}$$

hence

$$\big|C_{\max} f(x)\big|^p \leq \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, d\mathcal{H}^1(z). \tag{4.281}$$

From the last property in item *(i)* and (4.274) we see that for each $f \in L^p(\Sigma, w)$ the function $C_{\max} f$ is σ-measurable. In concert with (4.281) and (4.278), this permits us to estimate

$$\begin{aligned}
&\int_{\Sigma} \big|C_{\max} f(x)\big|^p \, dw(x) \\
&\qquad \leq \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{\Sigma} \Big(\int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, d\mathcal{H}^1(z)\Big) \, dw(x) \\
&\qquad = \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{|z|=\varepsilon/4} \Big(\int_{\Sigma} \sup_{j\in\mathbb{N}} \big|\Phi_j(z) f(x)\big|^p \, dw(x)\Big) \, d\mathcal{H}^1(z) \\
&\qquad \leq \Big(\frac{2^{3p-1}}{\pi^{p-1}\varepsilon^{2p-1}}\Big) C_1^p \cdot \Phi\Big(C_2 \cdot C^{1+(p_0-1)/(p-1)}\Big)^p \|f\|^p_{L^p(\Sigma,w)},
\end{aligned} \tag{4.282}$$

and (4.275) readily follows from this. □

We next discuss a companion result to Theorem 4.2, the novelty being the consideration of a maximal "transpose" operator as defined below in (4.283).

Theorem 4.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same symbol w for the measure associated with the given weight w as in* (2.509)*. Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$. Given a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, consider the maximal operator $T_*^{\#}$ whose action on each given function $f \in L^p(\partial\Omega, w)$ is defined as*

$$T_*^{\#} f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon^{\#} f(x)\big| \quad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega, \tag{4.283}$$

where, for each $\varepsilon > 0$,

$$T_\varepsilon^\# f(x) := \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y) \quad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega. \tag{4.284}$$

Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0,\infty)$, which depends only on m, n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has

$$\left\| T_*^\# \right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \le C_m \Big(\sum_{|\alpha|\le N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.285}$$

Furthermore, when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in (4.285) to depend itself only on said entities (i.e., n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) and m.

In particular, Theorem 4.5 may be used to give a direct proof of (4.253), without having to rely on duality.

Proof of Theorem 4.5 To get started, we observe that if $\mathbb{Q}_+$ denotes the collection of all positive rational numbers, then for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ we have

$$(T_*^\# f)(x) = \sup_{\varepsilon\in\mathbb{Q}_+} \left|(T_\varepsilon^\# f)(x)\right| \quad \text{for every } \ x \in \partial^*\Omega. \tag{4.286}$$

To justify this, pick some $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$. We claim that if $x \in \partial^*\Omega$ is arbitrary and fixed then for each $\varepsilon \in (0,\infty)$ and each sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$ we have

$$\lim_{j\to\infty} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon_j}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y) = \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y). \tag{4.287}$$

To justify (4.287) note that

$$\{y \in \partial\Omega : |x-y| > \varepsilon_j\} \nearrow \{y \in \partial\Omega : |x-y| > \varepsilon\} \ \text{ as } \ j \to \infty, \tag{4.288}$$

in the sense that

$$\{y \in \partial\Omega : |x-y| > \varepsilon\} = \bigcup_{j\in\mathbb{N}} \{y \in \partial\Omega : |x-y| > \varepsilon_j\} \text{ and}$$
$$\{y \in \partial\Omega : |x-y| > \varepsilon_j\} \subseteq \{y \in \partial\Omega : |x-y| > \varepsilon_{j+1}\} \text{ for every } j \in \mathbb{N}. \tag{4.289}$$

Then (4.287) follows from (4.288), the properties of k, and Lebesgue's Dominated Convergence Theorem. What we have just proved amounts to saying that for every function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ we have

$$\lim_{j\to\infty} (T^{\#}_{\varepsilon_j} f)(x) = (T^{\#}_{\varepsilon} f)(x) \text{ for every } x \in \partial^*\Omega, \tag{4.290}$$

whenever $\varepsilon \in (0,\infty)$ and $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ are such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$. Having established this, (4.286) readily follows on account of the density of $\mathbb{Q}_+$ in $(0,\infty)$.

To proceed, let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}_+$. Also, bring back the operators (4.105) and observe that for each $j \in \mathbb{N}$, each $f \in L^p(\partial\Omega, w)$, and each $x \in \partial^*\Omega$ we have

$$T^{\#}_{\varepsilon_j} f(x) + T_{\varepsilon_j} f(x) = \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon_j}} \langle y-x, \nu(x)-\nu(y)\rangle k(x-y) f(y)\, d\sigma(y). \tag{4.291}$$

Write $(\nu_i)_{1\le i\le n}$ for the scalar components of the geometric measure theoretic outward unit normal ν to Ω and, for every $i \in \{1,\dots,n\}$, every $j \in \mathbb{N}$, and every $f \in L^p(\partial\Omega, w)$ set

$$\mathbb{T}^{(i)}_j f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon_j}} (y_i - x_i) k(x-y) f(y)\, d\sigma(y) \text{ for each } x \in \partial\Omega. \tag{4.292}$$

Then, for each $j \in \mathbb{N}$ and each $f \in L^p(\partial\Omega, w)$ we may recast (4.291) as

$$T^{\#}_{\varepsilon_j} f(x) + T_{\varepsilon_j} f(x) = \sum_{i=1}^{n} \big[M_{\nu_i}, \mathbb{T}^{(i)}_j\big] f(x) \text{ for each } x \in \partial^*\Omega. \tag{4.293}$$

If for each $i \in \{1,\dots,n\}$ and each $f \in L^p(\partial\Omega, w)$ we now define

$$C^{(i)}_{\max} f(x) := \sup_{j\in\mathbb{N}} \Big|\big[M_{\nu_i}, \mathbb{T}^{(i)}_j\big] f(x)\Big| \text{ for each } x \in \partial^*\Omega, \tag{4.294}$$

then, thanks to Proposition 3.4, for each $i \in \{1,\dots,n\}$ we may invoke Theorem 4.4 for the family $\big\{\mathbb{T}^{(i)}_j\big\}_{j\in\mathbb{N}}$ to conclude that

$$\left\| C^{(i)}_{\max} \right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.295}$$

where $C \in (0, \infty)$ depends only on n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$. Also, from (4.293), (4.294), (4.286), and (4.104) we deduce that for each $f \in L^p(\partial\Omega, w)$ we have

$$T^{\#}_* f(x) \leq T_* f(x) + \sum_{i=1}^{n} C^{(i)}_{\max} f(x) \quad \text{for each} \quad x \in \partial^*\Omega. \tag{4.296}$$

At this stage, the estimate in (4.285) becomes a consequence of (4.296), (4.106), (4.295), (4.98), and (2.118), keeping in mind that, as is apparent from (4.286), the function $T^{\#}_* f$ is σ-measurable, and that we currently have $\sigma\big(\partial\Omega \setminus \partial^*\Omega\big) = 0$ (cf. Definition 2.4 and (2.24)). Finally, the very last claim in the statement is seen from Theorem 2.3. □

To discuss a significant application of Theorem 4.3 let us first formally introduce the family of Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on the boundary a UR domain $\Omega \subseteq \mathbb{R}^n$. Specifically, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, for each $j \in \{1, \dots, n\}$ the j-th Riesz transform R_j acts on any given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y) \tag{4.297}$$

at σ-a.e. point $x \in \partial\Omega$.

Theorem 4.6 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator K_Δ from (3.29), the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ from (4.297), and for each index $k \in \{1, \dots, n\}$ denote by M_{ν_k} the operator of pointwise multiplication by ν_k, the k-th scalar component of the vector ν.*

Then there exists some $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ and, for each $m \in \mathbb{N}$, there exists some $C_m \in (0, \infty)$ which depends only on m, n, p, $[w]_{A_p}$, and the UR constants of $\partial\Omega$ with the property that, with the piece of notation introduced in (4.93), one has

$$\begin{aligned} &\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \quad \textit{and} \\ &\max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \end{aligned} \tag{4.298}$$

Also, when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ *is sufficiently small relative to n, p,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$ *one may take the constants* $C, C_m \in (0,\infty)$ *appearing in* (4.298) *to depend only on said entities (i.e., n, p,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$*) and, in the case of* C_m*, also on m.*

Proof The estimate claimed in (4.298) is implied by (3.29), Corollary 4.2, (4.297), Proposition 3.4, and Theorem 4.3. The very last claim in the statement is implied by Theorem 2.3. □

We shall, once again, see Theorem 4.3 in action shortly, in the proof of Theorem 4.7. In the latter result the focus is obtaining operator norm estimates for double layer potentials associated with distinguished coefficient tensors on Muckenhoupt weighted Lebesgue and Sobolev spaces, exhibiting explicit dependence on the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain.

Theorem 4.7 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a UR domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *and consider the boundary-to-boundary double layer potential operators* $K_A, K_A^{\#}$ *associated with* Ω *and the coefficient tensor A as in* (3.24) *and* (3.25)*, respectively. Finally, fix an integrability exponent* $p \in (1,\infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$*.*

Then for each $m \in \mathbb{N}$ *there exists some* $C_m \in (0,\infty)$ *which depends only on m, n, A, p,* $[w]_{A_p}$*, and the UR constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{4.299}$$

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M \to [L_1^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{4.300}$$

and

$$\left\|K_A^{\#}\right\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}. \tag{4.301}$$

In addition, when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ *is sufficiently small relative to n, p,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$ *one may take* $C_m \in (0,\infty)$ *appearing in* (4.299)–(4.301) *to depend itself only on said entities (i.e., n, p,* $[w]_{A_p}$*, the Ahlfors regularity constant of* $\partial\Omega$*) and m.*

Note that the estimate in (4.299) implies that the operator K_A becomes identically zero whenever Ω is a half-space in $\mathbb{R}^n$. From *(i)* ⇔ *(ii)* in Proposition 3.9 we know that this may only occur if $A \in \mathfrak{A}_L^{\mathrm{dis}}$. Hence, the assumption $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ is actually *necessary* in light of the conclusion in Theorem 4.7.

Proof of Theorem 4.7 The estimates claimed in (4.299) and (4.301) are direct consequences of Corollary 4.2 and Proposition 3.9, bearing in mind (3.24) and (3.25).

Turning to the task of proving (4.300), it is apparent from (3.35) that each U_{jk} is a sum of operators of commutator type. Then, given any integer $m \in \mathbb{N}$ along with any function $f \in \big[L^p_1(\partial\Omega, w)\big]^M$, based on (3.37), (4.299), Theorem 4.3, and (4.98) we may write

$$
\begin{aligned}
\|K_A f\|_{[L^p_1(\partial\Omega,w)]^M} &= \|K_A f\|_{[L^p(\partial\Omega,w)]^M} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}}(K_A f)\big\|_{[L^p(\partial\Omega,w)]^M} \\
&= \|K_A f\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + \sum_{j,k=1}^{n} \Big(\big\|K_A(\partial_{\tau_{jk}} f)\big\|_{[L^p(\partial\Omega,w)]^M} + \big\|U_{jk}(\nabla_{\tan} f)\big\|_{[L^p(\partial\Omega,w)]^M} \Big) \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, \|f\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^M} \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \, \|f\|_{[L^p_1(\partial\Omega,w)]^M}\,,
\end{aligned}
\tag{4.302}
$$

which establishes (4.300). The very last claim in the statement is a consequence of Theorem 2.3. □

Remark 4.11 The unweighted case (i.e., the scenario in which $w \equiv 1$) of Theorem 4.7 gives norm estimates for the double layer operator and its formal transpose on ordinary Lebesgue and Sobolev spaces. By relying on (4.258)–(4.259), Proposition 3.2, (4.264), and (2.589) we may also obtain similar estimates on Lorentz spaces and Lorentz-based Sobolev spaces (cf. (2.590)–(2.591)). Specifically, in the same setting as Theorem 4.7, the aforementioned results imply that for each $m \in \mathbb{N}$, $p \in (1, \infty)$ and $q \in (0, \infty]$ there exists some $C_m \in (0, \infty)$ which depends only on m, n, A, p, q, and UR constants of $\partial\Omega$, such that

$$\|K_A\|_{[L^{p,q}(\partial\Omega,\sigma)]^M \to [L^{p,q}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.303}$$

$$\|K_A\|_{[L^{p,q}_1(\partial\Omega,\sigma)]^M \to [L^{p,q}_1(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.304}$$

and

$$\left\| K_A^{\#} \right\|_{[L^{p,q}(\partial\Omega,\sigma)]^M \to [L^{p,q}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}. \tag{4.305}$$

More general results of this type are discussed later, in Theorem 8.9 (see also Examples 8.2 and 8.6).

Remark 4.12 By reasoning much as in the proof of Theorem 4.7, we may also obtain operator norm estimates for the double layer K_A with $A \in \mathfrak{A}_L^{\mathrm{dis}}$ on *off-diagonal* weighted Sobolev spaces, i.e., when the integrability exponents and the weights for the Lebesgue spaces to which the actual function and its tangential derivatives belong to are allowed to be different. Specifically, given two integrability exponents $p_1, p_2 \in (1, \infty)$ along with two Muckenhoupt weights $w_1 \in A_{p_1}(\partial\Omega, \sigma)$ and $w_2 \in A_{p_2}(\partial\Omega, \sigma)$, define the off-diagonal weighted Sobolev space

$$\begin{aligned} L_1^{p_1;p_2}(\partial\Omega, w_1; w_2) := \big\{ f \in & L^{p_1}(\partial\Omega, w_1) : \\ & \partial_{\tau_{jk}} f \in L^{p_2}(\partial\Omega, w_2),\ \ 1 \leq j,k \leq n \big\}, \end{aligned} \tag{4.306}$$

equipped with the natural norm defined for each $f \in L_1^{p_1;p_2}(\partial\Omega, w_1; w_2)$ as

$$\|f\|_{L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)} := \|f\|_{L^{p_1}(\partial\Omega,w_1)} + \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f \right\|_{L^{p_2}(\partial\Omega,w_2)}. \tag{4.307}$$

Then much as in (4.302), for each $m \in \mathbb{N}$ we now obtain

$$\|K_A\|_{[L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)]^M \to [L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{4.308}$$

for some $C_m \in (0, \infty)$ which depends only on m, n, A, p_1, p_2, $[w_1]_{A_{p_1}}$, $[w_2]_{A_{p_2}}$, and the UR constants of $\partial\Omega$.

Remark 4.13 In the setting of Theorem 4.7, estimates (4.299)–(4.301) continue to hold with a fixed constant $C_m \in (0, \infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are permitted to vary with control. Specifically, from Remark 4.9 and the proof of Theorem 4.7 we see that for each $m \in \mathbb{N}$, each compact interval $I \subset (0, \infty)$, and each number $W \in (0, \infty)$ there exists a constant $C_m \in (0, \infty)$, which depends only on n, I, W, and the UR constants of $\partial\Omega$, with the property that (4.299)–(4.301) hold for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

Having proved Theorem 4.7, we may now establish invertibility results for boundary double layer potentials associated with distinguished coefficient tensors, assuming Ω is a δ-flat Ahlfors regular domain with δ suitably small relative to n and the Ahlfors regularity constant of $\partial\Omega$. By means of counterexamples we show that assuming that the double layer potentials are associated with distinguished coefficient tensors is a hypothesis one cannot simply omit. Also, as explained a little later, in Remark 4.19, the flatness condition imposed on the domain is actually in the nature of best possible as far as the invertibility results from Theorem 4.8 are concerned.

Theorem 4.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in (3.24) and (3.25), respectively. Finally, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, \infty)$.*

Then there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, A, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following operators are linear, bounded, and invertible:

$$zI + K_A : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M, \tag{4.309}$$

$$zI + K_A : \big[L_1^p(\partial\Omega, w)\big]^M \longrightarrow \big[L_1^p(\partial\Omega, w)\big]^M, \tag{4.310}$$

$$zI + K_A^{\#} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M. \tag{4.311}$$

Furthermore, the above result is optimal in the sense that if $A \notin \mathfrak{A}_L^{\rm dis}$ then either of the operators (4.309)–(4.311) may fail to be invertible even when $z = 1/2$ and $\Omega = \mathbb{R}^n_+$.

Proof Let C denote the constant appearing in estimates (4.299)–(4.301), for the choice $m := 1$, and choose $t_\varepsilon \in (0, 1/e)$ small enough so that $t_\varepsilon \cdot \ln(1/t_\varepsilon) < \varepsilon/C$. To get going, pick $\delta \in (0, t_\varepsilon)$. By decreasing δ if necessary, we may insure that Ω is a UR domain, with the UR constants of $\partial\Omega$ controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$ (cf. Theorem 2.3). Granted this, Theorem 4.7 applies and gives that

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C\delta^{\langle 1\rangle} \leq C(t_\varepsilon)^{\langle 1\rangle} < \varepsilon. \tag{4.312}$$

Analogously,

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M \to [L_1^p(\partial\Omega,w)]^M} < \varepsilon, \tag{4.313}$$

$$\big\|K_A^{\#}\big\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} < \varepsilon. \tag{4.314}$$

In particular, the operators in (4.309)–(4.311) are invertible for each given $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ using a Neumann series, i.e.,

$$\big(zI + K_A\big)^{-1} = z^{-1} \sum_{m=0}^{\infty} \big(- z^{-1} K_A\big)^m \tag{4.315}$$

with convergence in the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$ as well as on $\big[L^p_1(\partial\Omega, w)\big]^M$, and

$$\big(zI + K_A^{\#}\big)^{-1} = z^{-1} \sum_{m=0}^{\infty} \big(- z^{-1} K_A^{\#}\big)^m \tag{4.316}$$

with convergence in the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$.

There remains to address the optimality claim in the last part of the statement. To this end, recall the second-order, weakly elliptic, constant (real) coefficient, symmetric, $n \times n$ system L_D defined in (3.371). From (3.23), (3.31), (2.575), (3.112), and (3.377) we see that if K_A, $K_A^{\#}$ are the boundary layer potential operators associated as in (3.24), (3.25) with $\Omega := \mathbb{R}^n_+$ and any coefficient tensor $A \in \mathfrak{A}_{L_D}$ then

$$\begin{aligned} &\Big\{\big(\tfrac{1}{2}I + K_A\big)f : f \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n\Big\} \\ &\qquad\qquad \subseteq \Big\{(f_1, \dots, f_n) \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n : f_n = \sum_{j=1}^{n-1} R_j f_j\Big\}. \end{aligned} \tag{4.317}$$

Thus, $\Big\{(0, \dots, 0, f) : f \in L^p(\mathbb{R}^{n-1}, w)\Big\}$ is an infinite dimensional subspace of $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ whose intersection with $\Big\{\big(\tfrac{1}{2}I + K_A\big)f : f \in \big[L^p(\mathbb{R}^{n-1}, w)\big]^n\Big\}$ is $\{0\}$. Consequently, $\tfrac{1}{2}I + K_A$ acting on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ has an infinite dimensional cokernel for each $p \in (1, \infty)$ and each $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. By duality (cf. (3.119)), it follows that $\tfrac{1}{2}I + K_A^{\#}$ acting on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^n$ has an infinite dimensional kernel for each $p \in (1, \infty)$ and each $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. In particular, the operators in (4.309), (4.311) corresponding to $z = 1/2$ and $\Omega = \mathbb{R}^n_+$ fail to be invertible in this case.

Likewise, from (3.23), (3.31), (2.575), (3.112), (3.113), and (3.378) it follows that if K_A is the double layer potential operator associated as in (3.24) with $\Omega := \mathbb{R}^n_+$ and any coefficient tensor $A \in \mathfrak{A}_{L_D}$ then

$$\Big\{\big(\tfrac{1}{2}I + K_A\big)f : f \in \big[L^p_1(\mathbb{R}^{n-1}, w)\big]^n\Big\}$$

$$\subseteq \Big\{(f_1,\dots,f_n)\in \big[L^p_1(\mathbb{R}^{n-1},w)\big]^n:\ f_n=\sum_{j=1}^{n-1}R_j f_j\Big\}. \tag{4.318}$$

Much as before, this shows that $\frac{1}{2}I+K_A$ acting on $\big[L^p_1(\mathbb{R}^{n-1},w)\big]^n$ has an infinite dimensional cokernel for each $p\in(1,\infty)$ and each $w\in A_p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. In particular, the operator in (4.310) corresponding to $z=1/2$ and $\Omega=\mathbb{R}^n_+$ also fails to be invertible in this case.

In all cases, the source of the failure for invertibility is the fact that any coefficient tensor $A\in\mathfrak{A}_{L_D}$ fails to be distinguished (cf. (3.406)). □

In Remarks 4.14–4.15 we continue to elaborate on the nature of the optimality claim in the last portion of the statement of Theorem 4.8.

Remark 4.14 Work with a scalar operator in the two-dimensional setting (i.e., when $n=2$ and $M=1$). Specifically, take $L:=\Delta$, the Laplacian in the plane, written as $\Delta=a_{jk}\partial_j\partial_k$ for the matrix $A=(a_{jk})_{1\le j,k\le 2}$ given by

$$A:=\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\in\mathbb{C}^{2\times 2}. \tag{4.319}$$

Then, as noted in (1.23)–(1.24), the boundary-to-boundary double layer potential operator K_A associated as in (3.24) with this coefficient tensor and the domain $\Omega:=\mathbb{R}^2_+$ is $K_A=(\mathrm{i}/2)H$ where H is the Hilbert transform on the real line. Fix an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\mathbb{R},\mathcal{L}^1)$. Since $-H^2=I$, the identity operator on $L^p(\mathbb{R},w)$, it follows that we currently have $\big(K_A\big)^2=4^{-1}I$ on $L^p(\mathbb{R},w)$. This further entails

$$\big(\tfrac{1}{2}I+K_A\big)\big(-\tfrac{1}{2}I+K_A\big)=0\ \text{ on }\ L^p(\mathbb{R},w) \tag{4.320}$$

which, in view of the fact that $K_A\neq\pm\frac{1}{2}I$, ultimately proves that the operator $\frac{1}{2}I+K_A$ is not invertible[3] on any Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R},w)$.

From what we have just proved and duality (cf. (3.119)) we then see that the operator $\frac{1}{2}I+K^{\#}_A$ fails to be invertible on any Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R}^{n-1},w)$ as well. Finally, given that (4.320) implies

$$\big(\tfrac{1}{2}I+K_A\big)\big(-\tfrac{1}{2}I+K_A\big)=0\ \text{ on }\ L^p_1(\mathbb{R},w), \tag{4.321}$$

we also infer that the operator $\frac{1}{2}I+K_A$ fails to be invertible when acting on any Muckenhoupt weighted Sobolev space $L^p_1(\mathbb{R},w)$.

[3] In fact, $\frac{1}{2}I+K_A$ acting on $L^p(\mathbb{R},w)$ has an infinite dimensional kernel and an infinite dimensional cokernel.

Since $A \neq I_{2\times 2}$ and $\mathfrak{A}_{\Delta}^{\text{dis}} = \{I_{2\times 2}\}$, the above analysis shows that for coefficient tensors $A \in \mathfrak{A}_{\Delta} \setminus \mathfrak{A}_{\Delta}^{\text{dis}}$ it may actually happen that the conclusions in Theorem 4.8 corresponding to $z := 1/2$ and $\Omega := \mathbb{R}^2_+$ fail.

The following is a higher-dimensional version of Remark 4.14.

Remark 4.15 Fix $n \in \mathbb{N}$ with $n \geq 2$ and define $M := 2^n$. Bring back the $M \times M$ second-order system $L := \Delta \cdot I_{M\times M}$ in $\mathbb{R}^n$ (cf. (1.31)). In particular, from (3.396) and Proposition 3.9 we see that $\mathfrak{A}_L^{\text{dis}} = \{I_{M\times M}\}$. Consequently, the coefficient tensor $A := \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$ with entries as in (1.33) satisfies

$$A \in \mathfrak{A}_L \setminus \mathfrak{A}_L^{\text{dis}}. \tag{4.322}$$

To proceed, let K_A be the boundary-to-boundary double layer potential operator associated as in (3.24) with the coefficient tensor (4.322) and the domain $\Omega := \mathbb{R}^n_+$. Given some arbitrary integrability exponent $p \in (1, \infty)$ along with some arbitrary Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, the same type of argument as in (1.39) gives

$$\big(K_A\big)^2 = \tfrac{1}{4} I \quad \text{on} \quad \big[L^p(\mathbb{R}^{n-1}, w)\big]^M, \tag{4.323}$$

where I is the identity operator on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$. Thus,

$$\big(\tfrac{1}{2} I + K_A\big)\big(-\tfrac{1}{2} I + K_A\big) = 0 \quad \text{on} \quad \big[L^p(\mathbb{R}^{n-1}, w)\big]^M. \tag{4.324}$$

In view of the fact that[4] $K_A \neq \pm\frac{1}{2} I$, the above identity ultimately proves that the operator $\frac{1}{2} I + K_A$ is not invertible[5] on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$.

Ultimately, this discussion shows that for coefficient tensors as in (4.322) it may well happen that the operator $\frac{1}{2} I + K_A$ is not invertible on any Muckenhoupt weighted Lebesgue space $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$. Via duality (cf. (3.119)) we conclude that the operator $\frac{1}{2} I + K_A^{\#}$ fails to be invertible on any Muckenhoupt weighted Lebesgue space $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$ as well. Finally, since (4.324) implies

$$\big(\tfrac{1}{2} I + K_A\big)\big(-\tfrac{1}{2} I + K_A\big) = 0 \quad \text{on} \quad \big[L^p_1(\mathbb{R}^{n-1}, w)\big]^M, \tag{4.325}$$

[4] Since K_A is a Fourier multiplier operator with symbol $m(\xi') := \frac{\xi_j}{2|\xi'|} E_n E_j$ for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$.

[5] In fact, $\frac{1}{2} I + K_A$ acting on $\big[L^p(\mathbb{R}^{n-1}, w)\big]^M$ has both an infinite dimensional kernel and an infinite dimensional cokernel.

we also conclude that the operator $\frac{1}{2}I + K_A$ is not invertible when acting on any Muckenhoupt weighted Sobolev space $\big[L^p_1(\mathbb{R}^{n-1}, w)\big]^M$. Hence, all conclusions in Theorem 4.8 corresponding to $z := 1/2$ and $\Omega := \mathbb{R}^2_+$ fail.

Remark 4.16 In view of (4.303)–(4.305), and (4.308), invertibility results which are similar to those proved in Theorem 4.8 may be established on Lorentz spaces and Lorentz-based Sobolev spaces, as well as on the brand of off-diagonal Muckenhoupt weighted Sobolev spaces defined as in (4.306)–(4.307).

Remark 4.17 It is of interest to contrast Theorem 4.8 with the precise invertibility results known in the particular case when Ω is an infinite sector in the plane, with opening angle $\theta \in (0, 2\pi)$ and when $L = \Delta$ (the two-dimensional Laplacian). In such a setting, it is known (cf. [48], [115, §4.2], [126, Theorem 5, p. 192]) that

given $p \in (1,\infty)$, the operators $\pm\frac{1}{2}I + K_\Delta$ are invertible on $L^p(\partial\Omega, \sigma)$ if and only if $p \neq 1 + |\pi - \theta|/\pi$ (which amounts to saying that necessarily $p \neq \frac{2\pi-\theta}{\pi}$ for $\theta \in (0,\pi)$ and $p \neq \frac{\theta}{\pi}$ for $\theta \in (\pi, 2\pi)$). (4.326)

When $\theta = \pi$ (i.e., when Ω is a half-plane) then, of course, any $p \in (1,\infty)$ will do. In this vein, see also [105, Lemma 4.5, p. 2042]. Consider next the case of the two-dimensional Lamé system in an infinite sector of aperture $\theta \in (0, 2\pi)$, and recall from the discussion at the end of Example 3.4 that the pseudo-stress double layer potential operator for the Lamé system is denoted by K_Ψ. Then there are two critical values of the integrability exponent $p \in (1,\infty)$, which depend on θ and a specific combination of the Lamé moduli, for which the invertibility of the operators $\pm\frac{1}{2}I + K_\Psi$ on $\big[L^p(\partial\Omega,\sigma)\big]^2$ fails. See [110, Theorem 1.1(A.2) on pp. 153-154, and Theorem 1.3 on pp. 157-158] for more precise information in this regard (including the location of these critical values, which are no longer as explicit as in the case of the Laplacian, and certain monotonicity properties with respect to the angle θ and the Lamé moduli). We shall revisit the case of the two-dimensional Lamé system in Sect. 4.5.

Remark 4.18 In the context of Theorem 4.8, the operators in (4.309)–(4.311) continue to be invertible when the integrability exponent and the corresponding Muckenhoupt weight are permitted to vary while retaining control. More specifically, from Remark 4.13 and the proof of Theorem 4.8 it follows that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a threshold $\delta \in (0,1)$, which depends only on n, I, W, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta \tag{4.327}$$

then the operators (4.309)–(4.311) are linear, bounded, and invertible for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

Remark 4.19 The more general version of Theorem 4.8 from Remark 4.18 is in the nature of best possible, in the sense that the simultaneous invertibility result described in Remark 4.18 forces $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ to be small (relative to the other geometric characteristics of Ω). To illustrate this, consider the case when $\Omega = \Omega_\theta$, an infinite sector in the plane with opening angle $\theta \in (0, 2\pi)$ (cf. (2.289)), and when $L = \Delta$, the two-dimensional Laplacian. We are interested in the geometric implications of having the operators $\pm\frac{1}{2}I + K_\Delta$ invertible on $L^p(\partial\Omega_\theta, \sigma_\theta)$ (where $\sigma_\theta := \mathcal{H}^1 \lfloor \partial\Omega_\theta$) for all p's belonging to a compact sub-interval of $(1, \infty)$.

Specifically, suppose said operators are invertible whenever $p \in I_\eta := [1+\eta, 2]$ for some fixed $\eta \in (0, 1)$. From (4.326) we see that this forces $\theta \neq \pi(2-p)$ if $\theta \in (0, \pi)$ and $\theta \neq \pi p$ if $\theta \in (\pi, 2\pi)$. As p swipes the interval $[1+\eta, 2]$, the set of prohibited values for the aperture θ becomes $\big(0, (1-\eta)\pi\big] \cup \big[(1+\eta)\pi, 2\pi\big)$. Hence, we necessarily have $\theta \in \big((1-\eta)\pi, (1+\eta)\pi\big)$ which further entails

$$-\sin\big(\eta\tfrac{\pi}{2}\big) = \cos\big((1+\eta)\tfrac{\pi}{2}\big) < \cos(\theta/2) < \cos\big((1-\eta)\tfrac{\pi}{2}\big) = \sin\big(\eta\tfrac{\pi}{2}\big). \tag{4.328}$$

If ν denotes the outward unit normal vector to Ω_θ, then from (4.328) and (2.290) we conclude that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^2} = |\cos(\theta/2)| < \sin\big(\eta\tfrac{\pi}{2}\big) \longrightarrow 0^+ \ \text{ as } \ \eta \to 0^+. \tag{4.329}$$

This goes to show that, in general, the smallness of the BMO semi-norm of the geometric measure theoretic outward unit normal stipulated in (4.327) *cannot* be dispensed with, as far as the invertibility of the operator in (4.309) (in this case, with $z \in \{\pm\frac{1}{2}\}$, $L = \Delta$, A the identity matrix, $M = 1$, and $w \equiv 1$) for each $p \in I_\eta$ is concerned.

The invertibility results from Theorem 4.8 may be further enhanced by allowing the coefficient tensor to be a small perturbation of any distinguished coefficient tensor of the given system. Concretely, by combining Theorem 4.7 with the continuity of the operator-valued assignments in (3.120)–(3.122), we obtain the following result.

Theorem 4.9 *Retain the original background assumptions on the set Ω from Theorem 4.8 and, as before, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, 1)$. Consider $L \in \mathfrak{L}^{\mathrm{dis}}$ (cf. (3.195)) and pick an arbitrary $A_o \in \mathfrak{A}^{\mathrm{dis}}_L$. Then there exist some small threshold $\delta \in (0, 1)$ along with some open neighborhood O of A_o in $\mathfrak{A}_{\mathrm{WE}}$, both of which depend only on n, p, $[w]_{A_p}$, A_o, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for each $A \in O$ and each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$, the operators (4.309)–(4.311) are linear, bounded, and invertible.*

In the last portion of this section we discuss the issue of the compatibility of the inverses of the integral operators from Theorem 4.8 when simultaneously considered on different spaces. This requires that we briefly digress for the purpose of bringing in useful language and basic results of general functional analytic nature. Specifically, call two linear normed spaces, $X_0 = \big(X_0, \|\cdot\|_{X_0}\big)$ and $X_1 = \big(X_1, \|\cdot\|_{X_1}\big)$, `compatible` if there exists a Hausdorff topological vector space $\mathcal{X}$ such that

$$X_i \hookrightarrow \mathcal{X} \text{ continuously, } i \in \{0, 1\}. \tag{4.330}$$

Note that, in this scenario, we can talk about the algebraic sum $X_0 + X_1 (\subseteq \mathcal{X})$. This becomes a linear normed space when equipped with

$$\|x\|_{X_0+X_1} := \inf_{\substack{x=x_0+x_1\\ x_0\in X_0,\, x_1\in X_1}} \big(\|x_0\|_{X_0} + \|x_1\|_{X_1}\big), \qquad \forall\, x \in X_0 + X_1, \tag{4.331}$$

and $X_0 + X_1 \hookrightarrow \mathcal{X}$ continuously. Furthermore, $X_i \hookrightarrow X_0 + X_1$ continuously, for $i \in \{0, 1\}$. One may check that if X_0, X_1 are complete then so is $X_0 + X_1$ equipped with $\|\cdot\|_{X_0+X_1}$. Hence, X_0+X_1 turns out to be a Banach space if X_0, X_1 are Banach spaces to begin with.

We continue by recording two useful basic results of functional analytic nature. To state the first such result, suppose $X_0 = \big(X_0, \|\cdot\|_{X_0}\big)$ and $X_1 = \big(X_1, \|\cdot\|_{X_1}\big)$ on the one hand, and $Y_0 = \big(Y_0, \|\cdot\|_{Y_0}\big)$ and $Y_1 = \big(Y_1, \|\cdot\|_{Y_1}\big)$ on the other hand, are two pairs of compatible linear normed spaces. Then

$$\begin{array}{l}\text{having a linear mapping } T : X_0+X_1 \to Y_0+Y_1 \text{ which satisfies}\\ TX_i \subseteq Y_i \text{ for } i \in \{0,1\} \text{ is equivalent to having two linear maps}\\ T_i : X_i \to Y_i \text{ for } i \in \{0,1\} \text{ that are compatible with one another,}\\ \text{in the sense that } T_0\big|_{X_0\cap X_1} = T_1\big|_{X_0\cap X_1}\text{; in this case one has}\\ \|T\|_{X_0+X_1\to Y_0+Y_1} \leq \max\big\{\|T_0\|_{X_0\to Y_0}, \|T_1\|_{X_1\to Y_1}\big\}.\end{array} \tag{4.332}$$

To state our second result alluded to above, assume now that X, Y are two Banach spaces with the property that $Y \subseteq X$. One may check without difficulty that

$$\begin{array}{l}\text{if } T : X \to X \text{ is a linear isomorphism with the property that}\\ T(Y) \subseteq Y \text{ and } T\big|_Y : Y \to Y \text{ is also an isomorphism, then}\\ T^{-1}(Y) \subseteq Y \text{ and } \big(T\big|_Y\big)^{-1} = T^{-1}\big|_Y \text{ as operators on } Y.\end{array} \tag{4.333}$$

We are now ready to establish norm estimates for double layer operators acting on sums of Muckenhoupt weighted Lebesgue and Sobolev spaces.

Proposition 4.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic*

$M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Pick a coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in (3.24) and (3.25), respectively. Finally, fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$.

Then for each $m \in \mathbb{N}$ there exists some constant $C \in (0,\infty)$ which depends only on m, n, A, p_0, p_1, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has

$$\|K_A\|_{[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M\to[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.334}$$

$$\|K_A\|_{[L_1^{p_0}(\partial\Omega,w_0)+L_1^{p_1}(\partial\Omega,w_1)]^M\to[L_1^{p_0}(\partial\Omega,w_0)+L_1^{p_1}(\partial\Omega,w_1)]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{4.335}$$

$$\big\|K_A^{\#}\big\|_{[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M\to[L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.336}$$

Also, if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in (4.334)–(4.336) to depend only on said entities (i.e., n, p, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$) and m.

Proof This is a consequence of Theorems 4.7, (4.332), and 2.3. In the case of (4.334) and (4.336) take $X_i := Y_i := \big[L^{p_i}(\partial\Omega, w_i)\big]^M$ for $i \in \{0,1\}$, in which case (4.330) is satisfied if we choose $\mathcal{X} := \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ (cf. (2.575)). Finally, for the estimate claimed in (4.335), take $X_i := Y_i := \big[L_1^{p_i}(\partial\Omega, w_i)\big]^M$ for each $i \in \{0,1\}$, so now the inclusion in (4.330) holds if $\mathcal{X} := \Big[L_1^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ where

$$L_1^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) := \Big\{f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) : \partial_{\tau_{jk}} f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \text{ for each } j,k \in \{1,\dots,n\}\Big\}, \tag{4.337}$$

equipped with the norm

$$\|f\|_{L^1_1\left(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\right)} := \|f\|_{L^1\left(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\right)} + \sum_{j,k=1}^{n} \left\|\partial_{\tau_{jk}} f\right\|_{L^1\left(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\right)} \tag{4.338}$$

for each $f \in L^1_1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\right)$. □

Here are the compatibility results for the inverses of integral operators from Theorem 4.8 when simultaneously considered on different Muckenhoupt weighted Lebesgue and Sobolev spaces.

Proposition 4.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \emptyset$. Pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in* (3.24) *and* (3.25)*, respectively. Finally, fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$, and some number $\varepsilon \in (0,1)$.*

Then there exists some small threshold $\delta \in (0,1)$ which depends only on n, p_0, p_1, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, A, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following properties hold:

the operator $zI + K_A$ is invertible both as a mapping from the space $\left[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\right]^M$ onto itself and also from the space $\left[L_1^{p_0}(\partial\Omega, w_0) + L_1^{p_1}(\partial\Omega, w_1)\right]^M$ onto itself; (4.339)

the operator $zI + K_A$ is invertible both as a mapping from $\left[L^{p_0}(\partial\Omega, w_0)\right]^M$ onto itself and also as a mapping from $\left[L^{p_1}(\partial\Omega, w_1)\right]^M$ onto itself, and the two inverses are in fact compatible with one another on the intersection; (4.340)

the operator $zI + K_A$ is invertible both as a mapping from $\left[L_1^{p_0}(\partial\Omega, w_0)\right]^M$ onto itself and also as a mapping from $\left[L_1^{p_1}(\partial\Omega, w_1)\right]^M$ onto itself, and the two inverses are in fact compatible with one another on the intersection; (4.341)

the operator $zI + K_A^{\#}$ is invertible both as a mapping from $\left[L^{p_0}(\partial\Omega, w_0)\right]^M$ onto itself and also as a mapping from $\left[L^{p_1}(\partial\Omega, w_1)\right]^M$ onto itself, and the two inverses are in fact compatible with one another on the intersection. (4.342)

Proof Bring in the constant C appearing in estimates (4.334)–(4.336) (corresponding to $m := 1$), and denote by $t_\varepsilon \in (0,1)$ the unique solution of the equation $t\cdot\ln(\mathrm{e}/t) = \varepsilon/\max\{C, 1\}$. Pick $\delta \in (0, t_\varepsilon)$ and, if necessary, further decrease δ as to insure that Ω is a UR domain, with the UR constants of $\partial\Omega$ controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$ (cf. Theorem 2.3).

Then, via a Neumann series argument (much as in the proof of Theorem 4.8) it follows that $zI + K_A$ is invertible when considered from $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ onto itself, from $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself, from $\big[L^{p_0}(\partial\Omega, w_0)+L^{p_1}(\partial\Omega, w_1)\big]^M$ onto itself, and also from $\big[L^{p_0}_1(\partial\Omega, w_0) + L^{p_1}_1(\partial\Omega, w_1)\big]^M$ onto itself. Invoking (4.333) with $X := \big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$ and with Y either $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ or $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$, then proves that both the inverse of $zI + K_A$ on $\big[L^{p_0}(\partial\Omega, w_0)\big]^M$ and the inverse of $zI + K_A$ on $\big[L^{p_1}(\partial\Omega, w_1)\big]^M$ arise as restrictions to these respective spaces of a common operator, namely the inverse of the operator $zI + K_A$ on the bigger space $\big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$. As such, they agree with one another so the conclusion in (4.340) follows. The claims in (4.341)–(4.342) are proved in a similar fashion. □

Remark 4.20 Compatibility results similar in spirit to the ones proved in Proposition 4.2 are also valid for other spaces of interest. For example, in the context of Proposition 4.2, taking the threshold $\delta \in (0, 1)$ sufficiently small ensures that the operator $zI + K_A$ is invertible on the hybrid space $\big[L^{p_1;p_2}_1(\partial\Omega, w_1; w_2)\big]^M$ (cf. Remark 4.12) and its inverse continues to be compatible with the inverse of $zI + K_A$ on any other (a priori) given Muckenhoupt weighted Lebesgue space or Sobolev space on $\partial\Omega$. In this vein, we also claim that there exists some constant $C \in (0, \infty)$ with the property that

$$
\begin{aligned}
&\text{whenever } \ f \in \big[L^{p_1;p_2}_1(\partial\Omega, w_1; w_2)\big]^M \\
&\text{and } \ g := (zI + K_A)^{-1} f \in \big[L^{p_1;p_2}_1(\partial\Omega, w_1; w_2)\big]^M \\
&\text{then } \ \|\nabla_{\tan} g\|_{[L^{p_2}(\partial\Omega, w_2)]^{n\cdot M}} \leq C\,\|\nabla_{\tan} f\|_{[L^{p_2}(\partial\Omega, w_2)]^{n\cdot M}}\,.
\end{aligned}
\tag{4.343}
$$

To justify this, use (3.37) to write, for each $j, k \in \{1, \dots, n\}$,

$$
\begin{aligned}
\partial_{\tau_{jk}} f &= \partial_{\tau_{jk}}\big[\big(zI + K_A\big)g\big] = \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}(\nabla_{\tan} g) \\
&= \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}\Big(\big(\nu_r \partial_{\tau_{rs}} g_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}\Big)
\end{aligned}
\tag{4.344}
$$

at σ-a.e. point on $\partial\Omega$, where $\nu = (\nu_r)_{1\leq r\leq n}$ is the geometric measure theoretic outward unit normal to Ω. Using the abbreviations

$$
\nabla_\tau f := \big(\partial_{\tau_{jk}} f\big)_{1\leq j,k\leq n}, \quad \nabla_\tau g := \big(\partial_{\tau_{jk}} g\big)_{1\leq j,k\leq n}, \tag{4.345}
$$

we find it convenient to recast the collection of all formulas as in (4.344), corresponding to all indices $j, k \in \{1, \dots, n\}$, simply as

$$\nabla_\tau f = \big(zI + R\big)(\nabla_\tau g), \tag{4.346}$$

where I is the identity and R is the operator acting from $\big[L^{p_2}(\partial\Omega, w_2)\big]^{M\cdot n^2}$ into itself according to

$$R := K_A + \Big(U_{jk} \circ \big(M_{\nu_r} \circ \pi^\alpha_{rs}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}\Big)_{1\leq j,k\leq n}. \tag{4.347}$$

Above, we let K_A act on each $F = \big(F^\alpha_{rs}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq r,s\leq n}} \in \big[L^{p_2}(\partial\Omega, w_2)\big]^{M\cdot n^2}$ by setting

$$K_A F := \Big(K_A\big(F^\alpha_{rs}\big)_{1\leq\alpha\leq M}\Big)_{1\leq r,s\leq n}. \tag{4.348}$$

Also recall that, much as in the past, each M_{ν_r} denotes the operator of pointwise multiplication by ν_r, the r-th scalar component of ν. Finally, in (4.347) we let each π^α_{rs} be the "coordinate-projection" operator which acts as $\pi^\alpha_{rs}(X) := X^\alpha_{rs}$ for every $X = \big(X^\alpha_{rs}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq r,s\leq n}} \in \mathbb{C}^{M\cdot n^2}$. From (4.347), (4.299), (3.35), Theorem 4.3, and (3.81), we then conclude that

$$\|R\|_{[L^{p_2}(\partial\Omega,w_2)]^{M\cdot n^2}\to[L^{p_2}(\partial\Omega,w_2)]^{M\cdot n^2}} \leq C\|\nu\|^{\langle 1\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \tag{4.349}$$

for some $C \in (0,\infty)$ which depends only on n, A, p_2, $[w_2]_{A_{p_2}}$, and the Ahlfors regularity constant of $\partial\Omega$. As a consequence of this, if we assume $\delta > 0$ to be sufficiently small to begin with, a Neumann series argument gives that

$$zI + R \text{ is invertible on } \big[L^{p_2}(\partial\Omega, w_2)\big]^{M\cdot n^2} \tag{4.350}$$

and provides an estimate for the norm of the inverse. At this stage, the estimate claimed in (4.343) follows from (4.346), (4.350), (4.345), and (2.585)–(2.586).

We conclude this section by proving estimates for the operator norm of the modified boundary-to-boundary double layer operator acting on homogeneous Muckenhoupt weighted Sobolev spaces in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain, complementing results in Theorem 4.7.

Theorem 4.10 *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided NTA domain whose boundary is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$. Pick $A \in \mathfrak{A}^{\mathrm{dis}}_L$ and consider the modified* *boundary-to-boundary double layer potential operator* $\big[K_{A,mod}\big]$ *associated with* Ω *and the coefficient tensor* A *as in* (3.142)*. Finally, fix an integrability exponent* $p \in (1, \infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$*.*

Then for each $m \in \mathbb{N}$ *there exists some* $C_m \in (0, \infty)$ *which depends only on* m*,* n*,* A*,* p*,* $[w]_{A_p}$*, and the Ahlfors regularity constant of* $\partial\Omega$*, such that*

$$\big\|\big[K_{A,mod}\big]\big\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M \to [\dot{L}^p_1(\partial\Omega,w)/\sim]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{4.351}$$

Furthermore, the above result is optimal in the sense that, given any $A \in \mathfrak{A}_L$*, having* (4.351) *valid for every half-space in* $\mathbb{R}^n$ *implies that actually* $A \in \mathfrak{A}_L^{\mathrm{dis}}$*.*

Proof From (2.88) we know that Ω satisfies a two-sided local John condition. Pick an arbitrary function $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$. In particular, from (2.598) and (2.576) we see that

$$f \in \big[L^q_{\mathrm{loc}}(\partial\Omega, \sigma)\big]^M \quad \text{for some} \quad q \in (1, \infty). \tag{4.352}$$

Keeping this in mind, we may rely on (3.142), Propositions 2.26, 3.3, (4.299), Theorem 4.3, and (4.98) to write, for each given $m \in \mathbb{N}$,

$$\begin{aligned}
\big\|\big[K_{A,\mathrm{mod}}\big][f]\big\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M} &= \big\|\big[K_{A,\mathrm{mod}} f\big]\big\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M} \\
&= \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}}(K_{A,\mathrm{mod}} f)\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\leq \sum_{j,k=1}^{n} \Big(\big\|K_A(\partial_{\tau_{jk}} f)\big\|_{[L^p(\partial\Omega,w)]^M} + \big\|U_{jk}(\nabla_{\tan} f)\big\|_{[L^p(\partial\Omega,w)]^M}\Big) \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\dot{M}}} \\
&\leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \|[f]\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M},
\end{aligned} \tag{4.353}$$

bearing in mind that each U_{jk} is a sum of operators of commutator type (cf. (3.35)).

There remain to address the optimality claim made in the last portion of the statement of the theorem. To this end, suppose $A \in \mathfrak{A}_L$ is such that (4.351) is valid in every half-space Ω in $\mathbb{R}^n$. In view of the fact that the BMO semi-norm of the normal vanishes in such cases, this amounts to having the modified boundary-to-boundary double layer operator $K_{A,\mathrm{mod}}$ map each function from $\big[\mathscr{C}^\infty_c(\partial\Omega)\big]^M$ into a constant in $\mathbb{C}^M$. Granted this, the implication *(iii')* $\Rightarrow$ *(i)* in Proposition 3.9 gives that actually $A \in \mathfrak{A}_L^{\text{dis}}$. □

4.4 Invertibility on Muckenhoupt Weighted Homogeneous Sobolev Spaces

Earlier in (3.132), we have considered the boundary-to-boundary single layer operator $\big[S_{\text{mod}}\big] : \big[L^p(\partial\Omega, w)\big]^M \to \big[\dot{L}_1^p(\partial\Omega, w)\big/ \sim\big]^M$. Its invertibility properties are going to be of basic importance in the context of boundary value problems for the system L in Ω. For example, under suitable geometric assumptions on the set Ω, if $\big[S_{\text{mod}}\big]$ is injective then the Homogeneous Regularity Problem for L in Ω has at most one solution, and if $\big[S_{\text{mod}}\big]$ is surjective then the Homogeneous Regularity Problem for L in Ω is solvable. In particular, having $\big[S_{\text{mod}}\big]$ bijective guarantees the well-posedness of the Homogeneous Regularity Problem for L in Ω. Lemma 4.3 and Proposition 4.3 elaborate on this topic.

Lemma 4.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture parameter $\kappa > 0$, an integrability exponent $p \in (1,\infty)$, and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Also, suppose L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider the Homogeneous Regularity Problem for L in Ω, with boundary data prescribed in homogeneous Muckenhoupt weighted Sobolev spaces, i.e.,*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M, \end{cases} \tag{4.354}$$

where $\dot{L}_1^p(\partial\Omega, w)$ is the homogeneous Muckenhoupt weighted boundary Sobolev space defined in (2.598). Also, consider the operator (cf. (3.132))

$$\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}_1^p(\partial\Omega, w)\big/ \sim\big]^M. \tag{4.355}$$

Then the following statements are true:

(a) If $\big[S_{mod}\big]$ as in (4.355) is surjective then the Homogeneous Regularity Problem (4.354) has a solution.

(b) If Ω is actually an NTA domain with an unbounded Ahlfors regular boundary and if $\big[S_{mod}\big]$ as in (4.355) is injective then the Homogeneous Regularity Problem (4.354) has at most one solution modulo constants.

Proof Suppose the operator in (4.355) is surjective and let $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ be arbitrary. Then there exists $g \in \big[L^p(\partial\Omega, w)\big]^M$ such that $S_{\rm mod}\, g = f + c$ for some $c \in \mathbb{C}^M$. If we now define $u := \mathscr{S}_{\rm mod}\, g - c$ then item *(c)* in Proposition 3.5, (3.47), and (2.575) imply that this is a solution of (4.354) for the boundary datum f.

To deal with the claim in item *(b)*, strengthen the original hypotheses on Ω by assuming now that Ω is actually an NTA domain with an unbounded Ahlfors regular boundary (in particular, Ω is connected; see (2.65)). Also, suppose $\big[S_{\rm mod}\big]$ defined as in (4.355) is an injective operator. To proceed, denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and pick an arbitrary coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}} \in \mathfrak{A}_L$. Let u be a solution of (4.354) corresponding to $f := c \in \mathbb{C}^M$. From the current assumptions and the Fatou-type result recalled in Theorem 3.4 (whose present applicability is ensured by (2.576)) we conclude that

$$\text{the trace } (\nabla u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exists and belongs to } \big[L^p(\partial\Omega, w)\big]^{M\times n}. \tag{4.356}$$

In view of this and (3.66), the conormal derivative

$$\begin{gathered}\partial^A_\nu u := \Big(\nu_r a^{\alpha\beta}_{rs}\big(\partial_s u_\beta\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq M} \text{ exists } \sigma\text{-a.e. on } \partial\Omega\\ \text{and belongs to } \big[L^p(\partial\Omega, w)\big]^M.\end{gathered} \tag{4.357}$$

Based on (4.354), (2.575), (3.54), Proposition 2.24, the fact that $u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = c$, the integral representation formula (3.69), and the fact that we are presently assuming that Ω is connected, we may write

$$u = -\mathscr{S}_{\rm mod}\big(\partial^A_\nu u\big) + c_u \quad \text{in } \Omega, \tag{4.358}$$

for some constant $c_u \in \mathbb{C}^M$ (depending on u). By taking the nontangential trace to the boundary (recall (3.47)) the latter implies $c = -S_{\rm mod}\big(\partial^A_\nu u\big) + c_u$, hence

$$\big[S_{\rm mod}\big]\big(\partial^A_\nu u\big) = 0. \tag{4.359}$$

Since $\partial^A_\nu u \in \big[L^p(\partial\Omega, w)\big]^M$ and since we are assuming that the operator $\big[S_{\rm mod}\big]$ is injective in the context of (4.355), this forces $\partial^A_\nu u = 0$. When used back in (4.358), this proves that u is constant in Ω. The claim in *(b)* is therefore established. □

Our next result builds on Lemma 4.3 by establishing a two-way street between invertibility of the single layer potential operator and the well-posedness of the Homogeneous Regularity Problem.

Proposition 4.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided NTA domain with an unbounded Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture* *parameter $\kappa > 0$, an integrability exponent $p \in (1, \infty)$, and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Next, assume L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ and denote by* (HRP$_+$) *and* (HRP$_-$) *the Homogeneous Regularity Problems formulated as in* (4.354) *corresponding to $\Omega_+ := \Omega$ and to $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$, respectively. Finally, recall the operator $\big[S_{mod}\big]$ from* (4.355)*. Then the following statements are true:*

(a) The operator $\big[S_{mod}\big]$ is injective in the context of (4.355) *if and only if* (HRP$_+$) *and* (HRP$_-$) *have at most one solution modulo constants.*
(b) The operator $\big[S_{mod}\big]$ is surjective in the context of (4.355) *if and only if* (HRP$_+$) *and* (HRP$_-$) *have a solution.*
(c) The operator $\big[S_{mod}\big]$ is an isomorphism in the context of (4.355) *if and only if* (HRP$_+$) *and* (HRP$_-$) *are well-posed.*

Proof Suppose (HRP$_+$) and (HRP$_-$) have at most one solution modulo constants and let $f \in \big[L^p(\partial\Omega, w)\big]^M$ be such that $S_{\rm mod} f = c \in \mathbb{C}^M$. Then $u^+ := \mathscr{S}_{\rm mod} f$ in Ω_+ and $u^- := \mathscr{S}_{\rm mod} f$ in Ω_- solve (HRP$_+$) and (HRP$_-$), respectively, for the boundary datum c (see item *(c)* in Proposition 3.5, (3.47), and (2.575)). In view of the current working hypothesis, this forces $u^\pm$ to be constant functions in $\Omega_\pm$. Picking $A \in \mathfrak{A}_L$ and invoking (3.126) as well as (6.191)–(6.192), we obtain that $f = \partial^A_\nu u^- - \partial^A_\nu u^+ = 0$, where the last equality is implied by the fact that the functions $u^\pm$ are constant in $\Omega_\pm$ and (3.66). Hence, $\big[S_{\rm mod}\big]$ is injective in the context of (4.355). The converse implication stated in *(a)* is a consequence of item *(b)* in Lemma 4.3 (used both for Ω_+ and Ω_-).

Moving on to the claim made in item *(b)*, suppose (HRP$_+$) and (HRP$_-$) are solvable and pick $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ arbitrary. Denote by u^+ and u^- a solution of (HRP$_+$) and of (HRP$_-$), respectively, for the boundary datum f. Also, fix a coefficient tensor $A \in \mathfrak{A}_L$. Collectively, the current assumptions, the Fatou-type result recalled in Theorem 3.4 (whose present applicability is ensured by (2.576)), (2.575), and Proposition 2.24 guarantee that the integral representation formula (3.69) holds both for u^+ in Ω_+ and for u^- in Ω_-. Specifically,

$$
\begin{aligned}
u^+ &= \mathcal{D}_{A,\rm mod}\big(u^+\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) - \mathscr{S}_{\rm mod}\big(\partial^A_\nu u^+\big) + c_+ \ \text{ in } \ \Omega_+,\\
u^- &= -\mathcal{D}_{A,\rm mod}\big(u^-\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) + \mathscr{S}_{\rm mod}\big(\partial^A_\nu u^-\big) + c_- \ \text{ in } \ \Omega_-,
\end{aligned}
\tag{4.360}
$$

for some constants $c_\pm \in \mathbb{C}^M$ (keep in mind that both Ω_+ and Ω_- are connected; cf. (2.65)). Taking nontangential boundary traces in (4.360) yields

$$
\begin{aligned}
f &= \big(\tfrac{1}{2}I + K_{A,\rm mod}\big) f - S_{\rm mod}\big(\partial^A_\nu u^+\big) + c_+ \ \text{ on } \ \partial\Omega,\\
f &= -\big(-\tfrac{1}{2}I + K_{A,\rm mod}\big) f + S_{\rm mod}\big(\partial^A_\nu u^-\big) + c_- \ \text{ on } \ \partial\Omega,
\end{aligned}
\tag{4.361}
$$

on account of (3.61) and (3.47). After adding the two equalities in (4.361) we arrive at

$$f = S_{\text{mod}}\big(- \partial_\nu^A u^+ + \partial_\nu^A u^- \big) + c_+ + c_- \ \text{ on } \ \partial\Omega, \tag{4.362}$$

hence $[f] = \big[S_{\text{mod}}\big]\big(- \partial_\nu^A u^+ + \partial_\nu^A u^-\big)$. The latter proves that the operator S_{mod} is surjective in the context of (4.355), since $-\partial_\nu^A u^+ + \partial_\nu^A u^- \in \big[L^p(\partial\Omega, w)\big]^M$. The converse implication stated in *(b)* is a consequence of Lemma 4.3 (used both for Ω_+ and Ω_-). Finally, the claim in item *(c)* follows from *(a)*-*(b)*, so the proof of the proposition is complete. □

We next turn our attention to the issue of invertibility (or lack thereof) for the operator $\big[S_{\text{mod}}\big]$ in the context of (4.355). We begin with the following proposition, which offers an example of the failure of the operator (4.355) to be Fredholm (in every single respect: $\big[S_{\text{mod}}\big]$ has an infinite dimensional kernel, as well as an infinite dimensional cokernel) even when the underlying domain is a half-space and when the system involved is symmetric. As we shall see a little later, in Theorem 4.11, the source of this failure is the lack of a distinguished coefficient tensor for said system.

Proposition 4.4 *Consider the second-order $n \times n$ system $L_D := \Delta - 2\nabla\text{div}$ in $\mathbb{R}^n$ with $n \geq 2$. Fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. Then the single layer potential operator $\big[S_{mod}\big]$, associated as in (3.42) with the system L_D and the domain $\Omega := \mathbb{R}^n_+$, acting in the context*

$$\big[S_{mod}\big] : \big[L^p(\mathbb{R}^{n-1}, w)\big]^n \longrightarrow \big[\dot{L}^p_1(\mathbb{R}^{n-1}, w)/\sim\big]^n \tag{4.363}$$

has an infinite dimensional kernel and an infinite dimensional cokernel.

Proof Denote by $\text{Ker}\big(\text{HRP}_{L_D}\big)$ the space of null-solutions of the Homogeneous Regularity Problem for the system L_D in the upper half-space, i.e., the space of functions u satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \\ L_D u = 0 \ \text{ in } \ \mathbb{R}^n_+, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}^{n-1}, w), \\ u\big|^{\kappa-\text{n.t.}}_{\mathbb{R}^{n-1}} = 0. \end{cases} \tag{4.364}$$

Also, denote by $\text{Ker}[S_{\text{mod}}]$ the kernel of the operator (4.363) and fix a coefficient tensor $A \in \mathfrak{A}_{L_D}$. Then, as seen from the proof of part *(b)* in Lemma 4.3 (see the reasoning leading up to (4.359)), the mapping

$$\text{Ker}\big(\text{HRP}_{L_D}\big) \ni u \longmapsto \partial_\nu^A u \in \text{Ker}[S_{\text{mod}}] \tag{4.365}$$

is well defined and injective. Being also linear, this entails

$$\dim\big(\text{Ker}[S_{\text{mod}}]\big) \geq \dim\big(\text{Ker}\big(\text{HRP}_{L_D}\big)\big). \tag{4.366}$$

The later when combined with (3.391) shows that $\dim\big(\mathrm{Ker}[S_{\mathrm{mod}}]\big) = +\infty$.

Also, much as in the proof of item *(a)* in Lemma 4.3, from item *(c)* in Proposition 3.5, (3.47), and (2.575) we see that $\mathrm{Im}[S_{\mathrm{mod}}]$, the image of the operator (4.363), is a subspace of

$$\Big\{u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n,\ L_D u = 0 \text{ in } \mathbb{R}^n_+,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}^{n-1}, w)\Big\}. \tag{4.367}$$

Recalling (3.385), this proves that $\dim\big(\mathrm{CoKer}[S_{\mathrm{mod}}]\big) = +\infty$, where $\mathrm{CoKer}[S_{\mathrm{mod}}]$ denotes the cokernel of the operator (4.363). □

We now turn our attention to the issue of identifying concrete algebraic and geometric conditions guaranteeing the injectivity, surjectivity, and the eventual invertibility of the modified single layer potential operator in the context of (3.132).

Theorem 4.11 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider the modified boundary-to-boundary single layer potential operator S_{mod} associated with Ω and the system L as in (3.42). Fix some exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$.*

Finally, recall that $\big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ denotes the M-th power of the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{L}^p_1(\partial\Omega, w)$, equipped with the semi-norm defined in (2.601) and, additionally, recall the operator $\big[S_{mod}\big] : \big[L^p(\partial\Omega, w)\big]^M \to \big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ defined as in (3.132). In relation to this, the following statements are valid.

(1) [Surjectivity] Whenever $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$, there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that (2.601) is a genuine norm and the operator (3.132) is surjective.

(2) [Injectivity] Whenever $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$, there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that the operator (3.132) is injective.

(3) [Isomorphism] Whenever both $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$ and $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$, there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that $\big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$ is a Banach space when equipped with the norm (2.601) and the operator (3.132) is an isomorphism.

(4) [Optimality] If $\mathfrak{A}^{\mathrm{dis}}_L = \varnothing$ then the operator (3.132) may fail to be surjective (in fact, may have an infinite dimensional cokernel) even in the case when Ω is a

half-space, and if $\mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$ *then the operator* (3.132) *may fail to be injective (in fact, may have an infinite dimensional kernel) even in the case when* Ω *is a half-space.*

We wish to note that, corresponding to the case when Ω is the upper-graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, the operator L is the Laplacian Δ in $\mathbb{R}^n$ (hence, $M = 1$), and for the integrability exponent $p = 2$, the invertibility of the harmonic single layer has been treated in [35, Lemma 3.1, p. 451] using Rellich estimates.

Proof of Theorem 4.11 To deal with item *(1)*, assume $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Pick a coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$ then select some threshold $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a condition which we shall henceforth assume) then

$$\Omega \text{ is a two-sided NTA domain with an unbounded boundary,} \tag{4.368}$$

and

$$\text{the operators } \pm\tfrac{1}{2}I + K_A \text{ are invertible on } \big[L_1^p(\partial\Omega, w)\big]^M. \tag{4.369}$$

Theorem 2.3 together with Theorems 2.4 and 4.8 ensure that this is indeed possible. To proceed, choose a scalar-valued function $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ on $B(0,1)$ and $\operatorname{supp}\phi \subseteq B(0,2)$. Having fixed a reference point $x_0 \in \partial\Omega$, for each scale $r \in (0,\infty)$ define

$$\phi_r(x) := \phi\Big(\frac{x - x_0}{r}\Big) \text{ for each } x \in \mathbb{R}^n, \tag{4.370}$$

and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. Suppose now some arbitrary function $g \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M$ has been given. Hence, from (2.598) we have

$$\begin{aligned} & g \in \big[L^p_{\rm loc}(\partial\Omega, w) \cap L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^M \text{ and} \\ & \partial_{\tau_{jk}} g \in \big[L^p(\partial\Omega, w)\big]^M \text{ for } 1 \leq j,k \leq n. \end{aligned} \tag{4.371}$$

For each $r \in (0,\infty)$ set $\Delta_r := \partial\Omega \cap B(x_0, r)$ and define $g_{\Delta_r} := ⨍_{\Delta_r} g \, d\sigma \in \mathbb{C}^M$ then set

$$g_r := \phi_r \cdot \big(g - g_{\Delta_{2r}}\big) \text{ on } \partial\Omega. \tag{4.372}$$

From Proposition 2.25 (whose applicability in the current setting is ensured by (4.368) and (4.371)) we know that there exists $C = C(\Omega, p, w, x_0) \in (0,\infty)$, independent of the function g, with the property that

$$\sup_{r>0} \frac{1}{r}\Big(\int_{\Delta_r} \big|g - g_{\Delta_r}\big|^p \, dw\Big)^{1/p} \leq C \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} g\big\|_{[L^p(\partial\Omega,w)]^M}. \tag{4.373}$$

Also, from (4.371)–(4.372) we see that for each radius $r \in (0, \infty)$ and all indices $j, k \in \{1, \dots, n\}$ we have

$$g_r \in \big[L^p_1(\partial\Omega, w)\big]^M \text{ and } \partial_{\tau_{jk}} g_r = \big(\partial_{\tau_{jk}}\phi_r\big)\cdot\big(g - g_{\Delta_{2r}}\big) + \phi_r \cdot \partial_{\tau_{jk}} g. \tag{4.374}$$

Since there exists a constant $C \in (0, \infty)$ such that for each $j, k \in \{1, \dots, n\}$ and each $r \in (0, \infty)$ we have

$$\operatorname{supp}\big(\partial_{\tau_{jk}}\phi_r\big) \subseteq \Delta_{2r} \text{ and } \big|\partial_{\tau_{jk}}\phi_r\big| \leq C/r \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{4.375}$$

it follows that for each $j, k \in \{1, \dots, n\}$ and each $r \in (0, \infty)$ we may estimate, making use of the version of the Poincaré inequality recorded in (4.373),

$$\begin{aligned}\Big\|\big(\partial_{\tau_{jk}}\phi_r\big)\cdot\big(g - g_{\Delta_{2r}}\big)\Big\|_{[L^p(\partial\Omega,w)]^M} &\leq C r^{-1}\Big(\int_{\Delta_{2r}} \big|g - g_{\Delta_{2r}}\big|^p \,\mathrm{d}w\Big)^{1/p} \\ &\leq C\sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} g\big\|_{[L^p(\partial\Omega,w)]^M},\end{aligned} \tag{4.376}$$

for some constant $C \in (0, \infty)$ independent of g and r. In turn, from (4.374), (4.376), (2.585)–(2.586), and (2.576) we conclude that

$$\big\|\nabla_{\tan} g_r\big\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \leq C\big\|\nabla_{\tan} g\big\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \tag{4.377}$$

for some $C \in (0, \infty)$ independent of g and r. If for each $r \in (0, \infty)$ we now define

$$h_r := \big(\tfrac{1}{2}I + K_A\big)^{-1}\big(-\tfrac{1}{2}I + K_A\big)^{-1} g_r \tag{4.378}$$

then from the membership in (4.374) and the invertibility results in (4.369) it follows that h_r is a meaningfully defined function which belongs to $\big[L^p_1(\partial\Omega, w)\big]^M$. In addition, from (4.378), (4.343), and (4.377) we conclude that there exists a constant $C \in (0, \infty)$, independent of g, such that

$$\|\nabla_{\tan} h_r\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \leq C\,\|\nabla_{\tan} g\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \text{ for each } r \in (0, \infty). \tag{4.379}$$

Going further, for each $r \in (0, \infty)$ define

$$f_r := \partial^A_\nu\big(\mathcal{D}_A h_r\big) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{4.380}$$

Since $h_r \in \big[L^p_1(\partial\Omega, w)\big]^M$, the boundedness result recorded in (3.115) together with (4.379) imply that $f_r \in \big[L^p(\partial\Omega, w)\big]^M$ and for each $r \in (0, \infty)$ we have

$$\|f_r\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\nabla_{\tan} h_r\|_{[L^p(\partial\Omega,w)]^{n\cdot M}} \leq C\,\|\nabla_{\tan} g\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}\,, \tag{4.381}$$

where $C \in (0, \infty)$ is independent of g and r. Collectively, (3.130), (4.378), (4.380), and Theorem 2.4 also ensure that for each $r \in (0, \infty)$ there exists some constant $c_r \in \mathbb{C}^M$ such that

$$S_{\text{mod}} f_r = g_r + c_r \quad \text{on} \quad \partial\Omega. \tag{4.382}$$

Select now a sequence $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, \infty)$ which converges to infinity. Since from (4.381) we know that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\big[L^p(\partial\Omega, w)\big]^M$, we may rely on the Banach–Alaoglu Theorem to assume, without loss of generality, that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is actually weak-$*$ convergent to some $f \in \big[L^p(\partial\Omega, w)\big]^M$. On account of (3.46), (4.382), and (4.372), for each test function $\psi \in \big[\text{Lip}(\partial\Omega)\big]^M$ with compact support we may write

$$\begin{aligned}
\int_{\partial\Omega} \big\langle S_{\text{mod}} f, \psi\big\rangle\, d\sigma &= \lim_{j\to\infty} \int_{\partial\Omega} \big\langle S_{\text{mod}} f_{r_j}, \psi\big\rangle\, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} \big\langle g_{r_j} + c_{r_j}, \psi\big\rangle\, d\sigma \\
&= \lim_{j\to\infty} \int_{\partial\Omega} \big\langle \phi_{r_j} \cdot \big(g - g_{\Delta_{2r_j}}\big) + c_{r_j}, \psi\big\rangle\, d\sigma \\
&= \lim_{j\to\infty} \int_{\partial\Omega} \big\langle g - g_{\Delta_{2r_j}} + c_{r_j}, \psi\big\rangle\, d\sigma \\
&= \int_{\partial\Omega} \langle g, \psi\rangle\, d\sigma + \lim_{j\to\infty} \Big\langle c_{r_j} - g_{\Delta_{2r_j}}, \int_{\partial\Omega} \psi\, d\sigma\Big\rangle.
\end{aligned} \tag{4.383}$$

In view of the arbitrariness of ψ, this forces the sequence $\big\{c_{r_j} - g_{\Delta_{2r_j}}\big\}_{j\in\mathbb{N}} \subseteq \mathbb{C}^M$ to converge to some constant $c \in \mathbb{C}^M$. Bearing this in mind, we may then conclude from (4.383) that

$$\int_{\partial\Omega} \big\langle S_{\text{mod}} f, \psi\big\rangle\, d\sigma = \int_{\partial\Omega} \langle g + c, \psi\rangle\, d\sigma \tag{4.384}$$

for each function $\psi \in \big[\text{Lip}(\partial\Omega)\big]^M$ with compact support. Ultimately, from (4.384) and (2.578) we obtain

$$S_{\text{mod}} f = g + c \quad \text{at } \sigma\text{-a.e. point on} \quad \partial\Omega. \tag{4.385}$$

Hence, $\big[S_{\text{mod}}\big] f = \big[S_{\text{mod}} f\big] = [g]$ and since $[g] \in \big[\dot{L}^p(\partial\Omega, w)/\sim\big]^M$ is arbitrary, it follows that the operator (3.132) is surjective. Moreover, from (4.381) we see that

$$\|f\|_{[L^p(\partial\Omega,w)]^M} \leq \limsup_{j\to\infty} \|f_{r_j}\|_{[L^p(\partial\Omega,w)]^M} \leq C\,\|\nabla_{\tan} g\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}$$

$$\leq C\,\|[g]\|_{[\dot{L}^p_1(\partial\Omega,w)/\sim]^M}, \tag{4.386}$$

for some constant $C \in (0,\infty)$ independent of g, so the surjectivity of (3.132) comes with quantitative control.

Let us also observe that the fact that (2.601) is, as claimed, a genuine norm is clear from (4.368) and Proposition 2.26.

Moving on, we treat item *(2)*, now working under the assumption that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Select a coefficient tensor $\widetilde{A} \in \mathfrak{A}_L$ such that $\widetilde{A}^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$, then choose $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (something we shall henceforth assume) then

$$\text{the operators } \pm\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top} \text{ are invertible on } \big[L^p(\partial\Omega,w)\big]^M. \tag{4.387}$$

That this is indeed possible is guaranteed by Theorem 4.8. The goal is to show that the operator (3.132) is injective. To this end, suppose $f \in \big[L^p(\partial\Omega,w)\big]^M$ is such that $\big[S_{\rm mod}\big]f = [0]$. Hence, $\big[S_{\rm mod} f\big] = [0]$ which implies that there exists some constant $c \in \mathbb{C}^M$ for which

$$S_{\rm mod} f = c \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{4.388}$$

In concert with (3.129), this further implies

$$\big(\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top}\big)f\Big) = 0 \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega \tag{4.389}$$

which, in view of (4.387), forces $f = 0$. Since the operator (3.132) is linear, it follows that this is indeed injective.

As far as the claims in item *(3)* are concerned, assume that $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Results established earlier then guarantee that the operator (3.132) is a continuous bijection. Since $\big[\dot{L}^p_1(\partial\Omega,w)\big/\sim\big]^M$ is a Banach space (cf. Proposition 2.26 and (4.368)) it follows that the operator (3.132) is a linear isomorphism.

Finally, the claims in item *(4)* are clear from Proposition 4.4 and (3.406). The proof of Theorem 4.11 is therefore complete. □

Here is a useful variant of Theorem 4.11:

Remark 4.21 Let Ω, L, be as in Theorem 4.11 and assume $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$. Fix some pair of integrability exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$. From (4.341) and the proof of Theorem 4.11 (cf. (4.378)) it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p_0, p_1, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for every given function g in $\big[\dot{L}^{p_0}_1(\partial\Omega, w_0) \cap \dot{L}^{p_1}_1(\partial\Omega, w_1)\big]^M$ there exist some function $f \in \big[L^{p_0}(\partial\Omega, w_0) \cap L^{p_1}(\partial\Omega, w_1)\big]^M$ and a constant $c \in \mathbb{C}^M$ such that $S_{\mathrm{mod}} f = g + c$.

As a consequence of Theorem 4.10 we shall prove the invertibility result contained in the next theorem, for modified boundary-to-boundary double layer operators associated with weakly elliptic systems possessing a distinguished coefficient tensor acting on homogeneous Muckenhoupt weighted Sobolev spaces on the boundary of sufficiently flat Ahlfors regular domains. Moreover, we show that this is optimal in the sense that in the absence of a distinguished coefficient tensor the modified boundary-to-boundary double layer operator may actually have an infinite dimensional cokernel, even when the underlying domain is a half-space.

Theorem 4.12 *Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$. Pick $A \in \mathfrak{A}^{\mathrm{dis}}_L$ and consider the modified boundary-to-boundary double layer potential operator $\big[K_{A,mod}\big]$ associated with Ω and the coefficient tensor A as in (3.142). Finally, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, \infty)$.*

Then there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, A, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the operator

$$zI + \big[K_{A,mod}\big] : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \tag{4.390}$$

is invertible. Moreover, this conclusion may fail when $\mathfrak{A}^{\mathrm{dis}}_L = \varnothing$ even when Ω is a half-space (in fact, in such a scenario it may happen that $\frac{1}{2}I + \big[K_{A,mod}\big]$ has an infinite dimensional cokernel when acting on the space $\big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M$).

Proof Theorems 2.3 and 2.4 imply that there exists some threshold $\delta \in (0, 1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then Ω is a two-sided NTA domain with an unbounded boundary. Granted this, the desired invertibility result pertaining to the operator (4.390) follows from Theorem 4.10, via a Neumann series argument.

In addition, from (3.133)–(3.134), (3.385), and (3.406) we conclude that the operator $\frac{1}{2}I + \big[K_{A,\mathrm{mod}}\big]$ associated with the $n \times n$ system L_D defined in (3.371) and the set $\Omega := \mathbb{R}^n_+$ has an infinite dimensional cokernel when acting on the space $\big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^n$. □

Here is another useful version of Theorem 4.12:

Remark 4.22 Let Ω, L, be as in Theorem 4.12 and assume $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$. Fix some pair of integrability exponents $p_0, p_1 \in (1, \infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega, \sigma)$ and $w_1 \in A_{p_1}(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, 1)$. From the proof of Theorem 4.12 (which produces a Neumann series representation for the inverse) we see that there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p_0, p_1, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, L, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ it follows that

$$\begin{array}{l}\text{the operator } zI + \big[K_{A,\mathrm{mod}}\big] \text{ is invertible both as a mapping}\\ \text{from } \big[\dot{L}_1^{p_0}(\partial\Omega, w_0)/\sim\big]^M \text{ onto itself and also as a mapping}\\ \text{from } \big[\dot{L}_1^{p_1}(\partial\Omega, w_1)/\sim\big]^M \text{ onto itself, and the two inverses}\\ \text{are in fact compatible with one another on the intersection.}\end{array} \tag{4.391}$$

See the proof of Proposition 4.2 for details in similar circumstances.

We next discuss invertibility results for the conormal of the double layer operator acting from homogeneous Muckenhoupt weighted Sobolev spaces.

Theorem 4.13 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Fix some exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Pick some coefficient tensor $A \in \mathfrak{A}_L$ and consider the modified conormal derivative of the modified double layer operator in the context of (3.138), i.e.,*

$$\begin{aligned}&\big[\partial_\nu^A \mathcal{D}_{A,mod}\big] : \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M \ \ \textit{defined as}\\ &\big[\partial_\nu^A \mathcal{D}_{A,mod}\big][f] := \partial_\nu^A(\mathcal{D}_{A,mod} f) \ \ \textit{for each} \ \ f \in \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M.\end{aligned} \tag{4.392}$$

From Theorem 3.5 this is known to be a well-defined, linear, and bounded operator when the quotient space is equipped with the norm (2.601). In relation to this, the following statements are valid.

(1) [Injectivity] Whenever $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and actually $A \in \mathfrak{A}_L^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator (4.392) is injective.

(2) [Surjectivity] Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ and actually $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ it follows that there exists a small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator (4.392) is surjective.

(3) [Isomorphism] If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, it follows that there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator (4.392) is an isomorphism.

Proof To deal with the claim made in item *(1)*, assume $A \in \mathfrak{A}_L^{\rm dis}$. From Theorems 2.3, 2.4, and 4.12 we know that it is possible to pick some threshold $\delta \in (0, 1)$ small enough so that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ then

$$\Omega \text{ is a two-sided NTA domain with an unbounded connected boundary,} \tag{4.393}$$

and

$$\pm\tfrac{1}{2}I + \big[K_{A,{\rm mod}}\big] \text{ are invertible operators on the Banach space } \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M. \tag{4.394}$$

Granted these, (3.149) then implies that the operator (4.392) is injective.

To justify the claim made in item *(2)*, suppose next that $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$. By relying on Theorems 2.3 and 4.8 we may choose $\delta \in (0, 1)$ small enough such that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ then Ω is a two-sided NTA domain with an unbounded boundary and

$$\pm\,\tfrac{1}{2}I + K^\#_{A^\top} \text{ are invertible operators on } \big[L^p(\partial\Omega, w)\big]^M. \tag{4.395}$$

Once these properties are satisfied, we may invoke (3.153) to conclude that the operator (4.392) is surjective. Finally, the claim made in item *(3)* is a direct consequence of the current items *(1)*-*(2)* and Theorem 3.9. □

Remark 4.23 Let Ω, L, be as in Theorem 4.13. Also, assume $A \in \mathfrak{A}_L^{\rm dis}$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$. Finally, fix some pair of exponents $p_0, p_1 \in (1,\infty)$ along with some pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ and $w_1 \in A_{p_1}(\partial\Omega,\sigma)$. From the proof of Theorem 4.13 (cf. (4.394), (4.395), Remark 4.22, and Proposition 4.2) it follows that there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p_0, p_1, $[w_0]_{A_{p_0}}$, $[w_1]_{A_{p_1}}$, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ then

$$\text{the operator } \big[\partial_\nu^A \mathcal{D}_{A,{\rm mod}}\big] \text{ is invertible both as a mapping from } \big[\dot{L}_1^{p_0}(\partial\Omega, w_0)/\sim\big]^M \text{ onto } \big[L_1^{p_0}(\partial\Omega, w_0)\big]^M \text{ and as a mapping from } \big[\dot{L}_1^{p_1}(\partial\Omega, w_1)/\sim\big]^M \text{ onto } \big[L^{p_1}(\partial\Omega, w_1)\big]^M, \text{ and these two inverses are compatible with one another on the intersection.} \tag{4.396}$$

Remark 4.24 An alternative proof of Theorem 4.11 can be obtained by taking collectively, (3.149), Theorem 4.12 (with $z = \pm\frac{1}{2}$), (3.153), Theorem 4.8 (with $z = \pm\frac{1}{2}$), (3.138), Theorems 2.3, and 2.4.

4.5 Another Look at Double Layers for the Two-Dimensional Lamé System

Throughout this section, we shall work in the two-dimensional case, i.e., in the case $n = 2$. As a preamble, we introduce a singular integral operator which is going to be relevant shortly. To set the stage, suppose $\Omega \subseteq \mathbb{R}^2$ is a UR domain, abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$, and denote by $\nu = (\nu_1, \nu_2)$ the geometric measure theoretic outward unit normal to Ω. Then for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|}\big)$ define

$$R_\Delta f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{\nu_1(y)(y_2 - x_2) - \nu_2(y)(y_1 - x_1)}{|x-y|^2} f(y)\, d\sigma(y), \tag{4.397}$$

at σ-a.e. point $x \in \partial\Omega$. Let us fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. It has been proved in [113, §2.5] that the singular integral operator R_Δ introduced in (4.397) is bounded on $L^p(\partial\Omega, w)$ and satisfies

$$(R_\Delta)^2 = \big(\tfrac{1}{2}I + K_\Delta\big)\big(-\tfrac{1}{2}I + K_\Delta\big) \text{ on } L^p(\partial\Omega, w), \tag{4.398}$$

$$K_\Delta R_\Delta + R_\Delta K_\Delta = 0 \text{ on } L^p(\partial\Omega, w), \tag{4.399}$$

where K_Δ is the harmonic double layer potential operator in this setting (i.e., K_Δ is as in (3.29) with $n := 2$).

Our main result in this section is Theorem 4.14 below, which elaborates on the spectra of double layer potential operators, associated with the two-dimensional complex Lamé system, when acting on Muckenhoupt weighted Lebesgue and Sobolev spaces on the boundary of a δ-AR unbounded domain in the plane.

Theorem 4.14 *Let $\Omega \subseteq \mathbb{R}^2$ be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix two Lamé moduli $\mu, \lambda \in \mathbb{C}$ satisfying*

$$\mu \neq 0, \qquad 2\mu + \lambda \neq 0, \tag{4.400}$$

and bring back the one-parameter family coefficient tensors from (3.226) *(corresponding to $n = 2$), i.e.,*

$$A(\zeta) = \big(a^{\alpha\beta}_{jk}(\zeta)\big)_{\substack{1\le j,k\le 2\\ 1\le\alpha,\beta\le 2}} \text{ defined for each } \zeta \in \mathbb{C} \text{ according to}$$
$$a^{\alpha\beta}_{jk}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu+\lambda-\zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \tag{4.401}$$
$$\text{for } 1 \le j,k,\alpha,\beta \le 2,$$

which allows to represent the 2×2 *Lamé system* $L_{\mu,\lambda} = \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$ *in* $\mathbb{R}^2$ *as*

$$L_{\mu,\lambda} = \Big(a^{\alpha\beta}_{jk}(\zeta)\partial_j\partial_k\Big)_{1\le\alpha,\beta\le 2} \text{ for each } \zeta\in\mathbb{C}. \tag{4.402}$$

Fix some integrability exponent $p \in (1,\infty)$ *along with some Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$*. Finally, suppose* $z,\zeta\in\mathbb{C}$ *are such that*

$$z \ne \pm\frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{4\mu(2\mu+\lambda)}, \tag{4.403}$$

and associate the double layer potential operator $K_{A(\zeta)}$ *with the coefficient tensor* $A(\zeta)$ *and the domain* Ω *as in* (3.24).

Then there exists some small threshold $\delta\in(0,1)$ *which depends only on* μ, λ, p, $[w]_{A_p}$, z, ζ, *and the Ahlfors regular constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta$ *it follows that*

$$\begin{aligned} &\text{the operator } zI_{2\times 2} + K_{A(\zeta)} \text{ is invertible}\\ &\text{both on } \big[L^p(\partial\Omega,w)\big]^2 \text{ and on } \big[L^p_1(\partial\Omega,w)\big]^2. \end{aligned} \tag{4.404}$$

Before presenting the proof of this theorem, a few clarifications are in order. From (4.309)–(4.310) in Theorem 4.8 and (3.228)–(3.229) we already know that, under suitable geometric assumptions, the conclusion in (4.404) holds (and this is true in all dimensions $n \ge 2$) when

$$3\mu+\lambda \ne 0 \text{ and } \zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}. \tag{4.405}$$

The point of Theorem 4.14 is that, for the two-dimensional Lamé system, the invertibility results from (4.309)–(4.310) holds with $A = A(\zeta)$ as in (3.226) for *a much larger range of* ζ's than the singleton in (4.405). (Parenthetically we wish to note that what is special about the scenario described in (4.405) is that this makes $\pm\frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{4\mu(2\mu+\lambda)}$ zero, so (4.403) simply reads $z\in\mathbb{C}\setminus\{0\}$ in this case, as was assumed in Theorem 4.8.) It should be also remarked that, in the setting on Theorem 4.14, the double layer $K_{A(\zeta)}$ does *not* necessarily have small operator norm, and this is in stark contrast with the case of the double layer operators considered in Theorem 4.8. References to other related results may be found in [82, Chapter 7]; in this vein, see also [99].

We are now ready to present the proof of Theorem 4.14.

Proof of Theorem 4.14 From Theorem 2.3 we know that it is possible to pick some threshold $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta$ then Ω is a UR domain, with the UR constants of $\partial\Omega$ controlled solely in terms of the Ahlfors regularity constant of $\partial\Omega$. Henceforth, assume this is the case.

Recall the numbers $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ associated with ζ, μ, λ as in (3.234). From (3.29), (3.235), (3.236), and (4.397) we see that for each $\zeta \in \mathbb{C}$ we have

$$K_{A(\zeta)} = C_1(\zeta)K_\Delta I_{2\times 2} - (1 - C_1(\zeta))Q + C_2(\zeta)\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix} \tag{4.406}$$

as operators on $\left[L^p(\partial\Omega, w)\right]^2$. Note that (4.398) implies

$$\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix}^2 = \big(\tfrac14 I - (K_\Delta)^2\big)I_{2\times 2} \ \text{ on } \ \left[L^p(\partial\Omega, w)\right]^2. \tag{4.407}$$

Staring with (4.406) and then using (4.407), (4.399) we may write, with all operators acting on the space $\left[L^p(\partial\Omega, w)\right]^2$,

$$\begin{aligned} \big(zI_{2\times 2} + K_{A(\zeta)}\big)\big(-zI_{2\times 2} + K_{A(\zeta)}\big) &= (K_{A(\zeta)})^2 - z^2 I_{2\times 2} \\ &= \big[\tfrac14 C_2(\zeta)^2 - z^2\big]I_{2\times 2} + T_\zeta, \end{aligned} \tag{4.408}$$

for all $z, \zeta \in \mathbb{C}$, where T_ζ is the operator

$$\begin{aligned} T_\zeta &= \big(C_1(\zeta)^2 - C_2(\zeta)^2\big)K_\Delta^2 I_{2\times 2} + (1 - C_1(\zeta))^2 Q^2 \\ &\quad - C_1(\zeta)(1 - C_1(\zeta))(K_\Delta I_{2\times 2})Q - C_1(\zeta)(1 - C_1(\zeta))Q(K_\Delta I_{2\times 2}) \\ &\quad - C_2(\zeta)(1 - C_1(\zeta))Q\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix} - C_2(\zeta)(1 - C_1(\zeta))\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix}Q. \end{aligned} \tag{4.409}$$

Fix now $\zeta \in \mathbb{C}$ along with $\varepsilon > 0$ arbitrary. Note that T_ζ in (4.409) is a finite linear combination of compositions of pairs of singular integral operators such that, in each case, at least one of them falls under the scope of Corollary 4.2. As a consequence of this and Proposition 3.4, it follows that there exists $\delta \in (0,1)$ small enough (relative to $\mu, \lambda, \zeta, \varepsilon, p, [w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$), matters may be arranged so that, under the additional assumption that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta, \tag{4.410}$$

we have

$$\|T_\zeta\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \leq \varepsilon^2/2. \tag{4.411}$$

Consider now

$$z \in \mathbb{C} \setminus \Big\{B\big(\tfrac{1}{2}C_2(\zeta),\varepsilon\big) \cup B\big(-\tfrac{1}{2}C_2(\zeta),\varepsilon\big)\Big\}, \tag{4.412}$$

which entails

$$\big|\tfrac{1}{4}C_2(\zeta)^2 - z^2\big| = \big|\tfrac{1}{2}C_2(\zeta) - z\big|\big|\tfrac{1}{2}C_2(\zeta) + z\big| \geq \varepsilon^2. \tag{4.413}$$

Then from (4.413), (4.411) it follows that

$$\begin{gathered}\big[\tfrac{1}{4}C_2(\zeta)^2 - z^2\big]I_{2\times 2} + T_\zeta \ \text{ is invertible on } \big[L^p(\partial\Omega,w)\big]^2 \\ \text{for each } z \text{ as in (4.412)},\end{gathered} \tag{4.414}$$

and

$$\begin{gathered}\Big\|\big(\big[\tfrac{1}{4}C_2(\zeta)^2 - z^2\big]I_{2\times 2} + T_\zeta\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \leq (\varepsilon^2/2)^{-1} \\ \text{for each } z \text{ as in (4.412)}.\end{gathered} \tag{4.415}$$

Since the operators $zI_{2\times 2} + K_{A(\zeta)}$ and $-zI_{2\times 2} + K_{A(\zeta)}$ commute with one another, from (4.408) and (4.414) we ultimately conclude that

$$zI_{2\times 2} + K_{A(\zeta)} \ \text{ is invertible on } \big[L^p(\partial\Omega,w)\big]^2 \ \text{ for each } z \text{ as in (4.412)}. \tag{4.416}$$

In relation to (4.416) we also claim that there exists some small number

$$c := c\big(\Omega,\varepsilon,\zeta,p,[w]_{A_p}\big) \in (0,1], \tag{4.417}$$

where the dependence of c on Ω manifests itself only through the Ahlfors regularity constant of $\partial\Omega$, with the property that

$$\begin{gathered}c\|f\|_{[L^p(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big\|_{[L^p(\partial\Omega,w)]^2} \\ \text{for each } z \text{ as in (4.412) and each } f \in \big[L^p(\partial\Omega,w)\big]^2.\end{gathered} \tag{4.418}$$

To prove this, first observe that

$$\begin{aligned}&\text{whenever } |z| > 1 + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} \text{ then}\\ &\quad zI_{2\times 2} + K_{A(\zeta)} \text{ is invertible on } \big[L^p(\partial\Omega, w)\big]^2 \text{ and}\\ &\quad \Big\| \big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1} \Big\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} < 1.\end{aligned} \tag{4.419}$$

Hence, as long as $|z| > 1 + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}$, the estimate in (4.418) is true for any choice of $c \in (0, 1]$. As such, there remains to study the case in which

$$\begin{aligned}&z \text{ is as in (4.412) and also satisfies}\\ &|z| \le 1 + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}.\end{aligned} \tag{4.420}$$

Henceforth assume z is as in (4.420). From (4.408) and (4.415) we know that

$$\begin{aligned}\Big\| \big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1} \big(-zI_{2\times 2} + K_{A(\zeta)}\big)^{-1} \Big\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}&\\ \le (\varepsilon^2/2)^{-1}.&\end{aligned} \tag{4.421}$$

Write $\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}$ as

$$\Big[\big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1} \big(-zI_{2\times 2} + K_{A(\zeta)}\big)^{-1}\Big] \big(-zI_{2\times 2} + K_{A(\zeta)}\big), \tag{4.422}$$

then use this formula and (4.421) to estimate

$$\begin{aligned}&\Big\| \big(zI_{2\times 2} + K_{A(\zeta)}\big)^{-1} \Big\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}\\ &\qquad \le (\varepsilon^2/2)^{-1} \| -zI_{2\times 2} + K_{A(\zeta)} \|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}\\ &\qquad \le (\varepsilon^2/2)^{-1} \Big(|z| + \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2} \Big)\\ &\qquad \le C\big(\Omega, \varepsilon, \zeta, p, [w]_{A_p}\big),\end{aligned} \tag{4.423}$$

where the last inequality comes from (4.420), and

$$C\big(\Omega, \varepsilon, \zeta, p, [w]_{A_p}\big) := 2\varepsilon^{-2} + 4\varepsilon^{-2} \|K_{A(\zeta)}\|_{[L^p(\partial\Omega,w)]^2 \to [L^p(\partial\Omega,w)]^2}. \tag{4.424}$$

Hence, if we define

$$c := c\big(\Omega, \varepsilon, \zeta, p, [w]_{A_p}\big) := \min\Big\{1, \big[C(\Omega, \varepsilon, \zeta, p, [w]_{A_p})\big]^{-1}\Big\} \in (0, 1], \tag{4.425}$$

we may rely on (4.423) to write

$$c\|f\|_{[L^p(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big\|_{[L^p(\partial\Omega,w)]^2},$$
$$\text{for all } f \in \big[L^p(\partial\Omega, w)\big]^2, \tag{4.426}$$

finishing the proof of (4.418).

We next claim that, if the threshold $\delta \in (0, 1)$ appearing in (4.410) is taken sufficiently small to begin with, we also have

$$zI_{2\times 2} + K_{A(\zeta)} \text{ invertible on } \big[L^p_1(\partial\Omega, w)\big]^2 \quad \text{for each } z \text{ as in (4.412).} \tag{4.427}$$

For starters, observe that for each point $z \in \mathbb{C}$, and each $f \in \big[L^p_1(\partial\Omega, w)\big]^2$, Proposition 3.2 gives

$$\partial_{\tau_{12}}\big[\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big] = \big(zI_{2\times 2} + K_{A(\zeta)}\big)(\partial_{\tau_{12}} f) + U^{\zeta}_{12}(\nabla_{\tan} f), \tag{4.428}$$

where the commutator U^{ζ}_{12} is defined as in (3.35) with $n = 2$, $j = 1$, $k = 2$, and the coefficient tensor $A(\zeta)$ as in (4.401). If z is as in (4.412) then, on account of (4.428), (4.418), and Theorem 4.3 (also keeping in mind Proposition 3.4) for each $f \in \big[L^p_1(\partial\Omega, w)\big]^2$ we may estimate

$$\begin{aligned} c\|\partial_{\tau_{12}} f\|_{[L^p(\partial\Omega,w)]^2} &\leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)(\partial_{\tau_{12}} f)\big\|_{[L^p(\partial\Omega,w)]^2} \\ &\leq \big\|\partial_{\tau_{12}}\big[\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big]\big\|_{[L^p(\partial\Omega,w)]^2} + \big\|U^{\zeta}_{12}(\nabla_{\tan} f)\big\|_{[L^p(\partial\Omega,w)]^2} \\ &\leq \big\|\big(zI_{2\times 2} + K_{A(\zeta)}\big)f\big\|_{[L^p_1(\partial\Omega,w)]^2} + C\delta\|\partial_{\tau_{12}} f\|_{[L^p(\partial\Omega,w)]^2}, \end{aligned} \tag{4.429}$$

(since we presently have $\partial_{\tau_{11}} = \partial_{\tau_{22}} = 0$ and $\partial_{\tau_{12}} = -\partial_{\tau_{21}}$), where $C \in (0, \infty)$ depends only on μ, λ, ζ, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Assuming $\delta < c/(2C)$ to begin with, the very last term above may be absorbed in the left-most side of (4.429). By combining the resulting inequality with (4.418) we therefore arrive at the conclusion that if δ in (4.410) is small enough then we may find some small $\eta > 0$ with the property that

$$\eta\|f\|_{[L^p_1(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2}+K_{A(\zeta)}\big)f\big\|_{[L^p_1(\partial\Omega,w)]^2}$$
$$\text{for each } z \text{ as in (4.412) and each } \ f\in\big[L^p_1(\partial\Omega,w)\big]^2. \tag{4.430}$$

In such a scenario, (4.430) implies that the operator $zI_{2\times 2}+K_{A(\zeta)}$ acting on $\big[L^p_1(\partial\Omega,w)\big]^2$ is injective and has closed range for each z as in (4.412). Consequently, the operator $zI_{2\times 2}+K_{A(\zeta)}$ acting on $\big[L^p_1(\partial\Omega,w)\big]^2$ is semi-Fredholm for each z as in (4.412). Since this depends continuously on z, the homotopic invariance of the index on connected sets then ensures that the index of $zI_{2\times 2}+K_{A(\zeta)}$ on $\big[L^p_1(\partial\Omega,w)\big]^2$ is independent of z in said range. Given that, via a Neumann series argument,

$$zI_{2\times 2}+K_{A(\zeta)} \ \text{ is invertible on } \ \big[L^p_1(\partial\Omega,w)\big]^2$$
$$\text{if } \ |z|>\big\|K_{A(\zeta)}\big\|_{[L^p_1(\partial\Omega,w)]^2\to[L^p_1(\partial\Omega,w)]^2}, \tag{4.431}$$

we may therefore conclude that the index of $zI_{2\times 2}+K_{A(\zeta)}$ on $\big[L^p_1(\partial\Omega,w)\big]^2$ is zero for each z as in (4.412). In view of the fact that, as already noted from (4.430), the operator $zI_{2\times 2}+K_{A(\zeta)}$ is injective on $\big[L^p_1(\partial\Omega,w)\big]^2$ for each z as in (4.412), this ultimately proves that $zI_{2\times 2}+K_{A(\zeta)}$ is invertible on $\big[L^p_1(\partial\Omega,w)\big]^2$ for each z as in (4.412). Hence, the claim made in (4.427) is true. At this stage, the claim made in (4.404) readily follows from (4.416) and (4.427). □

It is of interest to single out the case $z=\pm\frac{1}{2}$ in (4.404), and in Corollary 4.3 stated next we do just that.

Corollary 4.3 *Let $\Omega\subseteq\mathbb{R}^2$ be an Ahlfors regular domain. Abbreviate $\sigma:=\mathcal{H}^1\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix two Lamé moduli $\mu,\lambda\in\mathbb{C}$ satisfying*

$$\mu\neq 0, \qquad 2\mu+\lambda\neq 0, \qquad 3\mu+\lambda\neq 0, \tag{4.432}$$

and recall the one-parameter family coefficient tensors $A(\zeta)$ defined for each $\zeta\in\mathbb{C}$ as in (4.401). Fix an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$. Finally, pick some

$$\zeta\in\mathbb{C}\setminus\Big\{-\mu,\,\tfrac{\mu(5\mu+3\lambda)}{3\mu+\lambda}\Big\} \tag{4.433}$$

and associate double layer potential operator $K_{A(\zeta)}$ with the coefficient tensor $A(\zeta)$ and the domain Ω as in (3.24).

Then there exists some small threshold $\delta\in(0,1)$ which depends only on μ, λ, p, $[w]_{A_p}$, ζ, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2}<\delta$ it follows that

$$\begin{aligned}&\textit{the operators} \quad \pm\tfrac{1}{2}I_{2\times 2}+K_{A(\zeta)} \quad \textit{are invertible}\\&\textit{both on} \quad \big[L^p(\partial\Omega,w)\big]^2 \quad \textit{and on} \quad \big[L^p_1(\partial\Omega,w)\big]^2,\end{aligned} \tag{4.434}$$

and

$$\textit{the operators} \quad \pm\tfrac{1}{2}I_{2\times 2}+K^{\#}_{A(\zeta)} \quad \textit{are invertible on} \quad \big[L^p(\partial\Omega,w)\big]^2. \tag{4.435}$$

As seen from (4.433) (also keeping in mind (4.432)), under the additional assumption that $\mu+\lambda\neq 0$ the value $\zeta := \mu$ becomes acceptable in the formulation of the conclusions in (4.434)–(4.435). This special choice leads to the conclusion that, if Ω is sufficiently flat (relative to μ, λ, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) then the operators

$$\pm\tfrac{1}{2}I_{2\times 2}+K_{A(\mu)} : \big[L^p(\partial\Omega,w)\big]^2 \longrightarrow \big[L^p(\partial\Omega,w)\big]^2, \tag{4.436}$$

$$\pm\tfrac{1}{2}I_{2\times 2}+K_{A(\mu)} : \big[L^p_1(\partial\Omega,w)\big]^2 \longrightarrow \big[L^p_1(\partial\Omega,w)\big]^2, \tag{4.437}$$

$$\pm\tfrac{1}{2}I_{2\times 2}+K^{\#}_{A(\mu)} : \big[L^p(\partial\Omega,w)\big]^2 \longrightarrow \big[L^p(\partial\Omega,w)\big]^2, \tag{4.438}$$

are all invertible whenever

$$\mu\neq 0, \qquad \mu+\lambda\neq 0, \qquad 2\mu+\lambda\neq 0, \qquad 3\mu+\lambda\neq 0. \tag{4.439}$$

This is relevant in the context of Remark 6.10.

Proof of Corollary 4.3 The claim in (4.434) is a direct consequence of Theorem 4.14, upon observing that when $z=\pm 1/2$ the demand in (4.403) becomes equivalent to the condition stipulated in (4.433). The claim in (4.435) then follows from (4.434) and duality. □

Chapter 5
Controlling the BMO Semi-Norm of the Unit Normal

In the previous chapter we have succeeded in estimating the size of a certain brand of singular integrals operators (which includes the harmonic double layer operator; cf. Theorem 4.7) in terms of the geometry of the underlying "surface." A key characteristic of these estimates (originating with Theorem 4.2) is the presence of the BMO semi-norm of the unit normal to the surface as a factor in the right side. In particular, the flatter said surface, the smaller the norm of the singular integral operators in question. Similar results are also valid for a specific type of commutators, of the sort described in Theorem 4.3.

By way of contrast, the principal goal in this chapter is to proceed in the opposite direction, and control geometry in terms of analysis. More specifically, we seek to quantify flatness of a given "surface" (by estimating the BMO semi-norm of its unit normal) in terms of analytic entities, such as the operator norms of the harmonic double layer and the commutators of Riesz transforms with the operator of pointwise multiplication by the (scalar components of the) unit normals, or various natural algebraic combinations of Riesz transforms (where all singular integral operators just mentioned are intrinsically defined on the given "surface").

In this endeavor, the catalyst is the language of Clifford algebras which allows us to glue together singular integral operators of the sort described above into a single, Cauchy-like, singular integral which exhibits excellent non-degeneracy properties (i.e., up to normalization, such a Cauchy-Clifford operator is its own inverse; cf. (5.20)). We therefore begin with a brief tutorial about Clifford algebras, which are a highly non-commutative higher-dimensional version of the field of complex numbers, where some of the magic cancellations and algebraic miracles typically associated with the complex plane still occur. This chapter ends with Sect. 5.4 which contains results characterizing Muckenhoupt weights in terms of the boundedness Riesz transforms. The Clifford algebra formalism turns out to be useful in this regard, both as tool and as a mean to bring into play other types of operators, like the Cauchy–Clifford singular integral operator alluded to above.

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_5

5.1 Clifford Algebras and Cauchy–Clifford Operators

The Clifford algebra with n imaginary units is the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra $(\mathcal{C}\ell_n, +, \odot)$, which is not generated as an algebra by any proper subspace of $\mathbb{R}^n$ and such that

$$x \odot x = -|x|^2 \quad \text{for every} \quad x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{5.1}$$

In particular, with $\{\mathbf{e}_j\}_{1\leq j\leq n}$ denoting the standard orthonormal basis in $\mathbb{R}^n$, we have

$$\begin{aligned} \mathbf{e}_j \odot \mathbf{e}_j = -1 \quad \text{for all} \quad j \in \{1, \dots, n\} \quad \text{and} \\ \mathbf{e}_j \odot \mathbf{e}_k = -\mathbf{e}_k \odot \mathbf{e}_j \quad \text{for each distinct} \quad j, k \in \{1, \dots, n\}. \end{aligned} \tag{5.2}$$

This allows us to define an embedding $\mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$ by identifying

$$\mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv \sum_{j=1}^{n} x_j \mathbf{e}_j \in \mathcal{C}\ell_n. \tag{5.3}$$

In particular, $\{\mathbf{e}_j\}_{1\leq j\leq n}$ become n imaginary units in $\mathcal{C}\ell_n$, and (5.2) implies

$$a \odot b + b \odot a = -2\langle a, b\rangle \quad \text{for all} \quad a, b \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{5.4}$$

Moving on, any element $u \in \mathcal{C}\ell_n$ has a unique representation of the form

$$u = \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I \mathbf{e}_I, \qquad u_I \in \mathbb{R}, \tag{5.5}$$

where $\sum'$ indicates that the sum is performed only over strictly increasing multi-indices I, i.e., $I = (i_1, i_2, \dots, i_\ell)$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, and $\mathbf{e}_I$ denotes the Clifford algebra product $\mathbf{e}_I := \mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_\ell}$. Write $\mathbf{e}_0 := \mathbf{e}_\varnothing := 1$ for the multiplicative unit in $\mathcal{C}\ell_n$. For each $u \in \mathcal{C}\ell_n$ represented as in (5.5) define the vector part of u as

$$u_{\text{vect}} := \sum_{j=1}^{n} u_j \mathbf{e}_j \in \mathbb{R}^n, \tag{5.6}$$

and denote by

$$u_{\text{scal}} := u_\varnothing \mathbf{e}_\varnothing = u_\varnothing \in \mathbb{R}, \quad \text{the scalar part of} \quad u. \tag{5.7}$$

We endow $\mathcal{C}\ell_n$ with the natural Euclidean metric, hence

$$|u| := \left(\sum_{\ell=0}^{n} \sum_{|I|=\ell}' |u_I|^2 \right)^{1/2} \quad \text{for each} \ \ u = \sum_{\ell=0}^{n} \sum_{|I|=\ell}' u_I \mathbf{e}_I \in C\ell_n. \tag{5.8}$$

Next, define the conjugate of each $\mathbf{e}_I$ as the unique element $\overline{\mathbf{e}_I} \in C\ell_n$ such that $\mathbf{e}_I \odot \overline{\mathbf{e}_I} = \overline{\mathbf{e}_I} \odot \mathbf{e}_I = 1$. Thus, if $I = (i_1, \dots, i_\ell)$ with $1 \le i_1 < i_2 < \cdots < i_\ell \le n$, then the conjugate of $\mathbf{e}_I$ is given by $\overline{\mathbf{e}_I} = (-1)^\ell \mathbf{e}_{i_\ell} \odot \cdots \odot \mathbf{e}_2 \odot \mathbf{e}_1$. More generally, for an arbitrary element $u \in C\ell_n$ represented as in (5.5) we define

$$\overline{u} := \sum_{\ell=0}^{n} \sum_{|I|=\ell}' u_I \overline{\mathbf{e}_I}. \tag{5.9}$$

Note that $\overline{x} = -x$ for every $x \in \mathbb{R}^n \hookrightarrow C\ell_n$, and $|u| = |\overline{u}|$ for every $u \in C\ell_n$. One may also check that for any $u, v \in C\ell_n$ we have

$$|u \odot v| \le 2^{n/2} |u||v|, \qquad \overline{u \odot v} = \overline{v} \odot \overline{u}, \tag{5.10}$$

and, in fact,

$$\begin{gathered} |u \odot v| = |u||v| \ \ \text{if either} \\ u \in \mathbb{R}^n \hookrightarrow C\ell_n \ \ \text{or} \ \ v \in \mathbb{R}^n \hookrightarrow C\ell_n. \end{gathered} \tag{5.11}$$

For further details on Clifford algebras, the reader is referred to [101].

Consider an arbitrary UR domain $\Omega \subseteq \mathbb{R}^n$. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ its geometric measure theoretic outward unit normal. For the goals we have in mind, it is natural to identify ν with the Clifford algebra-valued function $\nu = \nu_1 \mathbf{e}_1 + \cdots + \nu_n \mathbf{e}_n$. Bearing this identification in mind, we then proceed to define the action of the boundary-to-boundary Cauchy–Clifford operator of any given $C\ell_n$-valued function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ as

$$\mathbf{C} f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y) \, d\sigma(y), \tag{5.12}$$

for σ-a.e. point $x \in \partial\Omega$. In particular, with Riesz transforms $\{R_j\}_{1 \le j \le n}$ on $\partial\Omega$ defined as in (4.297), for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ we have

$$\mathbf{C} f = \frac{1}{2} \sum_{1 \le j,k \le n} \mathbf{e}_j \odot \mathbf{e}_k \odot R_j(\nu_k f) \ \ \text{at } \sigma\text{-a.e. point on} \ \ \partial\Omega. \tag{5.13}$$

Another closely related integral operator which is of interest to us acts on each given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ according to

$$\mathbf{C}^{\#}f(x) := -\lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}\nu(x)\odot\frac{x-y}{|x-y|^n}\odot f(y)\,\mathrm{d}\sigma(y) \tag{5.14}$$

for σ-a.e. $x\in\partial\Omega$. Analogously to (5.13), for each $f\in L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\otimes C\ell_n$ we have

$$\mathbf{C}^{\#}f = -\frac{1}{2}\sum_{1\le j,k\le n}\mathbf{e}_k\odot\mathbf{e}_j\odot\nu_k R_j f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{5.15}$$

As is apparent from (5.13), (5.15), both $\mathbf{C}$ and $\mathbf{C}^{\#}$ are amenable to Proposition 3.4. Hence, whenever $p\in(1,\infty)$ and $w\in A_p(\partial\Omega,\sigma)$,

$$\mathbf{C}: L^p(\partial\Omega,w)\otimes C\ell_n \longrightarrow L^p(\partial\Omega,w)\otimes C\ell_n \tag{5.16}$$

and

$$\mathbf{C}^{\#}: L^p(\partial\Omega,w)\otimes C\ell_n \longrightarrow L^p(\partial\Omega,w)\otimes C\ell_n \tag{5.17}$$

are well-defined, linear, and bounded operators, with

$$\begin{gathered}\|\mathbf{C}\|_{L^p(\partial\Omega,w)\otimes C\ell_n\to L^p(\partial\Omega,w)\otimes C\ell_n},\ \ \|\mathbf{C}^{\#}\|_{L^p(\partial\Omega,w)\otimes C\ell_n\to L^p(\partial\Omega,w)\otimes C\ell_n}\\ \text{controlled in terms of } n,\ p,\ [w]_{A_p},\ \text{and the UR constants of } \partial\partial\Omega.\end{gathered} \tag{5.18}$$

In fact (see [61, Sections 4.6-4.7] and [114, §1.6]),

> the transpose of $\mathbf{C}$ from (5.16) is the operator $\mathbf{C}^{\#}$ acting in the context of (5.17) with the exponent p replaced by its Hölder conjugate $p'\in(1,\infty)$ and with the given weight w replaced by $w^{1-p'}\in A_{p'}(\partial\Omega,\sigma)$. (5.19)

For this reason, it is natural to refer to $\mathbf{C}^{\#}$ as the "transpose" Cauchy–Clifford operator. Moreover, with I denoting the identity operator, we have

$$\mathbf{C}^2 = \tfrac{1}{4}I \quad\text{and}\quad \big(\mathbf{C}^{\#}\big)^2 = \tfrac{1}{4}I, \tag{5.20}$$

on $L^p(\partial\Omega,\sigma)\otimes C\ell_n$ with $p\in(1,\infty)$ (cf. [61, Sections 4.6-4.7]). In view of (5.16)–(5.18), a standard density argument then shows that these formulas remain valid on $L^p(\partial\Omega,w)\otimes C\ell_n$ whenever $p\in(1,\infty)$ and $w\in A_p(\partial\Omega,\sigma)$.

Here we are interested in the difference $\mathbf{C}-\mathbf{C}^{\#}$ which, up to multiplication by 2^{-1}, may be thought of as the antisymmetric part of the Cauchy–Clifford operator $\mathbf{C}$. The following lemma elaborates on the relationship between the antisymmetric part of the Cauchy–Clifford operator, i.e., $\mathbf{C}-\mathbf{C}^{\#}$, and the harmonic boundary double layer potential (cf. (3.29)) together with commutators between Riesz transforms (cf. (4.297)) and operators of pointwise multiplication by scalar components of the unit vector. For a proof see [61, Lemma 4.45].

Lemma 5.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. For each index $j \in \{1, \dots, n\}$, denote by M_{ν_j} the operator of pointwise multiplication by ν_j. Also, recall the boundary-to-boundary harmonic double layer potential operator K_Δ from* (3.29) *and the family of Riesz transforms $\{R_j\}_{1\le j\le n}$ from* (4.297)*. Then*

$$(\mathbf{C} - \mathbf{C}^{\#})f = 2\sum_{\ell=0}^{n}\sideset{}{'}\sum_{|I|=\ell}(K_\Delta f_I)\mathbf{e}_I$$

$$+\sum_{\ell=0}^{n}\sideset{}{'}\sum_{|I|=\ell}\sum_{j,k=1}^{n}\big([M_{\nu_j}, R_k]f_I\big)\,\mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_I \tag{5.21}$$

for each $\mathcal{C}\ell_n$-valued function $f = \sum_{\ell=0}^{n}\sum'_{|I|=\ell} f_I \odot \mathbf{e}_I$ belonging to the weighted Lebesgue space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes \mathcal{C}\ell_n$.

In turn, the structural result from Lemma 5.1 is a basic ingredient in the proof of the following corollary.

Corollary 5.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then for each $m \in \mathbb{N}$ there exists some constant $C_m \in (0, \infty)$ which depends only on m, n, p, $[w]_{A_p}$, and the* UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n \to L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n} \le C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{5.22}$$

Moreover, if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0, \infty)$ appearing in (5.22) *to depend only on said entities (i.e., n, p, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$), and m.*

Proof This is a consequence of Lemmas 5.1, 2.15, (3.29), Corollary 4.2, (4.297), Proposition 3.4, Theorems 4.3, and 2.3. □

5.2 Estimating the BMO Semi-Norm of the Unit Normal

The next goal is to establish a bound from below for the operator norm of $\mathbf{C}-\mathbf{C}^{\#}$ on Muckenhoupt weighted Lebesgue spaces on the boundary of a UR domain in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal vector to said domain.

Theorem 5.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain such that $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Then there exists some $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\|\mathbf{C}-\mathbf{C}^{\#}\right\|_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n}. \tag{5.23}$$

A couple of comments are in order. First, as a consequence of (5.23), definitions, and a result from [111, §5.10] (based on work in [59]) to the effect that an Ahlfors regular domain is a half-space if and only if its geometric measure theoretic outward unit normal is a constant vector, we see that

given a UR domain $\Omega \subseteq \mathbb{R}^n$ such that $\partial\Omega$ is unbounded, and given $p \in (1,\infty)$ together with $w \in A_p(\partial\Omega,\sigma)$, we have $\mathbf{C} = \mathbf{C}^{\#}$ as operators on $L^p(\partial\Omega,w)\otimes C\ell_n$ if and only if Ω is a half-space. (5.24)

Second, estimate (5.23) may fail without the assumption that $\partial\Omega$ is unbounded. Indeed, from (5.12)–(5.14) one may easily check that $\mathbf{C} = \mathbf{C}^{\#}$ if Ω is an open ball, or the complement of a closed ball, in $\mathbb{R}^n$ and yet $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} > 0$ in either case. In fact, open balls, complements of closed balls, and half-spaces in $\mathbb{R}^n$ are the only UR domains for which $\mathbf{C} = \mathbf{C}^{\#}$ (see [60] for more on this).

We now turn to the task of presenting the proof of Theorem 5.1.

Proof of Theorem 5.1 Fix a location $x_0 \in \partial\Omega$ along with a scale $R > 0$. Also, pick a sufficiently large number $\Lambda \in (10,\infty)$, which ultimately will depend only on n and the Ahlfors regularity constant of $\partial\Omega$, in a manner to be specified later. Let $C \in [1,\infty)$ be the Ahlfors regularity constant of $\partial\Omega$ (cf. (2.32)) and choose

$$\lambda := (2C)^{2/(n-1)}. \tag{5.25}$$

We may then write (making use of the fact that no smallness condition on the scale is necessary since $\partial\Omega$ is unbounded)

$$\begin{aligned}\sigma\big(\Delta(x_0,\lambda(\Lambda R))\setminus\Delta(x_0,\Lambda R)\big) &= \sigma\big(\Delta(x_0,\lambda(\Lambda R))\big)-\sigma\big(\Delta(x_0,\Lambda R)\big)\\ &\geq \Big(\frac{1}{C}\lambda^{n-1}-C\Big)(\Lambda R)^{n-1} > 0.\end{aligned} \tag{5.26}$$

In turn, this guarantees that $\Delta(x_0, \lambda(\Lambda R)) \setminus \Delta(x_0, \Lambda R) \neq \varnothing$, hence we may choose some point

$$y_0 \in \Delta(x_0, \lambda(\Lambda R)) \setminus \Delta(x_0, \Lambda R). \tag{5.27}$$

As a consequence,

$$\Lambda R \leq |x_0 - y_0| < \lambda(\Lambda R). \tag{5.28}$$

Next, fix a point $x \in \Delta(x_0, R)$ and note that this entails $|x_0 - y| \geq (\Lambda - 1)R > 0$ for all $y \in \Delta(y_0, R)$. As such, we may write

$$\begin{aligned}
&\int_{\Delta(y_0,R)} \Big\{ \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) + \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n} \Big\} \, d\sigma(y) \\
&\quad = \int_{\Delta(y_0,R)} \Big\{ \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) - \frac{x - y}{|x - y|^n} \odot \nu(y) \Big\} \, d\sigma(y) \\
&\qquad + \int_{\Delta(y_0,R)} \Big\{ \frac{x - y}{|x - y|^n} \odot \nu(y) + \nu(x) \odot \frac{x - y}{|x - y|^n} \Big\} \, d\sigma(y) \\
&\qquad + \int_{\Delta(y_0,R)} \Big\{ \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n} - \nu(x) \odot \frac{x - y}{|x - y|^n} \Big\} \, d\sigma(y) \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{5.29}$$

Note that for each $y \in \Delta(y_0, R)$ we have

$$\Lambda R \leq |x_0 - y_0| \leq |x_0 - x| + |x - y| + |y - y_0| < |x - y| + 2R. \tag{5.30}$$

Based on definitions (cf. (5.12) and (5.14)), and the fact that, as seen from (5.30), we have $|x - y| > (\Lambda - 2)R$ for each $y \in \Delta(y_0, R)$, the second term in (5.29) is identified as

$$\mathrm{II} = \omega_{n-1} \, (\mathbf{C} - \mathbf{C}^{\#}) \mathbf{1}_{\Delta(y_0,R)}(x). \tag{5.31}$$

If for each $u, w, z \in \mathbb{R}^n$ with $z \notin \{u, w\}$ we now abbreviate

$$E(u, w; z) := \frac{u - z}{|u - z|^n} - \frac{w - z}{|w - z|^n}, \tag{5.32}$$

and if we set

$$\nu_{\Delta(z,r)} := \fint_{\partial\Omega \cap B(z,r)} \nu \, d\sigma \quad \text{for each} \quad z \in \partial\Omega \quad \text{and} \quad r > 0, \tag{5.33}$$

then, on account of (5.4),

$$\begin{aligned}
\mathrm{I}+\mathrm{III} &= \int_{\Delta(y_0,R)} \big\{E(x_0,x;y)\odot\nu(y)+\nu(x)\odot E(x_0,x;y)\big\}\,\mathrm{d}\sigma(y)\\
&= -2\int_{\Delta(y_0,R)} \big\langle E(x_0,x;y),\,\nu_{\Delta(x_0,R)}\big\rangle\,\mathrm{d}\sigma(y)\\
&\quad+\int_{\Delta(y_0,R)} E(x_0,x;y)\odot\big(\nu(y)-\nu_{\Delta(x_0,R)}\big)\,\mathrm{d}\sigma(y)\\
&\quad+\int_{\Delta(y_0,R)} \big(\nu(x)-\nu_{\Delta(x_0,R)}\big)\odot E(x_0,x;y)\,\mathrm{d}\sigma(y)\\
&=: \mathrm{IV}+\mathrm{V}+\mathrm{VI}.
\end{aligned}\tag{5.34}$$

Since

$$\begin{aligned}
E(x_0,x;y) &= \frac{x_0-y}{|x_0-y|^n}-\frac{(x-x_0)-(y-x_0)}{|x-y|^n}\\
&= -\frac{x-x_0}{|x-y|^n}+(x_0-y)\Big(\frac{1}{|x_0-y|^n}-\frac{1}{|x-y|^n}\Big)
\end{aligned}\tag{5.35}$$

for each $y\in\Delta(y_0,R)$, it follows that

$$\begin{aligned}
\mathrm{IV} &= 2\int_{\Delta(y_0,R)} \frac{\big\langle x-x_0,\,\nu_{\Delta(x_0,R)}\big\rangle}{|x-y|^n}\,\mathrm{d}\sigma(y)\\
&\quad+2\int_{\Delta(y_0,R)} \big\langle y-x_0,\,\nu_{\Delta(x_0,R)}\big\rangle\Big(\frac{1}{|x_0-y|^n}-\frac{1}{|x-y|^n}\Big)\,\mathrm{d}\sigma(y)\\
&=: \mathrm{IV}_a+\mathrm{IV}_b.
\end{aligned}\tag{5.36}$$

In view of (5.30) for each $y\in\Delta(y_0,R)$ we have $(\Lambda/2)R<(\Lambda-2)R<|x-y|$ which, together with Proposition 2.15, permits us to estimate

$$\begin{aligned}
|\mathrm{IV}_a| &= 2\big|\big\langle x-x_0,\,\nu_{\Delta(x_0,R)}\big\rangle\big|\int_{\Delta(y_0,R)} \frac{1}{|x-y|^n}\,\mathrm{d}\sigma(y)\\
&\le C\Lambda^{-n}\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},
\end{aligned}\tag{5.37}$$

where $C\in(0,\infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. Also, since the Mean Value Theorem gives that for each point $y\in\Delta(y_0,R)$ we have, for some purely dimensional constant $C\in(0,\infty)$,

$$\left|\frac{1}{|x_0 - y|^n} - \frac{1}{|x - y|^n}\right| \leq \frac{CR}{(\Lambda R)^{n+1}} = C\Lambda^{-n-1}R^{-n}, \tag{5.38}$$

we may use Proposition 2.15 and the fact that $y \in \Delta(y_0, R) \subseteq \Delta\big(x_0, (1 + \lambda\Lambda)R\big)$ to conclude that

$$\begin{aligned}\big|\mathrm{IV}_b\big| &\leq C(\Lambda R \ln \Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\Lambda^{-n-1}R^{-n}\sigma\big(\Delta(x_0, R)\big)\\ &\leq C(\Lambda^{-n} \ln \Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{5.39}$$

where $C \in (0, \infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. Next, the Mean Value Theorem shows that for each $y \in \Delta(y_0, R)$ we have

$$|E(x_0, x; y)| = \left|\frac{x_0 - y}{|x_0 - y|^n} - \frac{x - y}{|x - y|^n}\right| \leq \frac{CR}{(\Lambda R)^n} = C\Lambda^{-n}R^{1-n}, \tag{5.40}$$

for some purely dimensional constant $C \in (0, \infty)$. In addition, (2.104), (2.105), and (2.106) permit us to write

$$\begin{aligned}\big|\nu_{\Delta(x_0,R)} - \nu_{\Delta(y_0,R)}\big| &\leq \big|\nu_{\Delta(x_0,R)} - \nu_{\Delta(x_0,\lambda\Lambda R)}\big| + \big|\nu_{\Delta(x_0,\lambda\Lambda R)} - \nu_{\Delta(y_0,\lambda\Lambda R)}\big|\\ &\quad + \big|\nu_{\Delta(y_0,\lambda\Lambda R)} - \nu_{\Delta(y_0,R)}\big|\\ &\leq C(\ln \Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\end{aligned} \tag{5.41}$$

for some $C \in (0, \infty)$ which depends only on n and the Ahlfors regularity constant of $\partial\Omega$. Based on (5.40) and (5.41) we may then estimate

$$\begin{aligned}\big|\mathrm{V}\big| &\leq \int_{\Delta(y_0,R)} |E(x_0, x; y)|\big|\nu(y) - \nu_{\Delta(x_0,R)}\big|\, d\sigma(y)\\ &\leq C\Lambda^{-n} \fint_{\Delta(y_0,R)} \big|\nu(y) - \nu_{\Delta(x_0,R)}\big|\, d\sigma(y)\\ &\leq C\Lambda^{-n} \fint_{\Delta(y_0,R)} \big|\nu(y) - \nu_{\Delta(y_0,R)}\big|\, d\sigma(y) + C\Lambda^{-n}\big|\nu_{\Delta(x_0,R)} - \nu_{\Delta(y_0,R)}\big|\\ &\leq C(\Lambda^{-n} \ln \Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{5.42}$$

where $C \in (0, \infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. Finally, (5.40) implies that for some purely dimensional constant $C \in (0, \infty)$ we have

$$\big|\mathrm{VI}\big| \leq \int_{\Delta(y_0,R)} |E(x_0, x; y)|\big|\nu(x) - \nu_{\Delta(x_0,R)}\big|\, d\sigma(y)$$

$$\leq C\Lambda^{-n}\big|\nu(x) - \nu_{\Delta(x_0,R)}\big|. \tag{5.43}$$

For further use, let us note here that (2.538) plus the John-Nirenberg inequality (cf. (2.102)) allow to estimate (for some exponent $q' \in (1,\infty)$ which depends only on p, $[w]_{A_p}$, n, and the Ahlfors regularity constant of $\partial\Omega$)

$$\begin{aligned}
\fint_{\Delta(x_0,R)} \big|\nu(x) - \nu_{\Delta(x_0,R)}\big|^p \,\mathrm{d}w(x) &= \fint_{\Delta(x_0,R)} \Big|\nu - \fint_{\Delta(x_0,R)} \nu\,\mathrm{d}\sigma\Big|^p \,\mathrm{d}w \\
&\leq C\bigg(\fint_{\Delta(x_0,R)} \Big|\nu - \fint_{\Delta(x_0,R)} \nu\,\mathrm{d}\sigma\Big|^{pq'}\,\mathrm{d}\sigma\bigg)^{1/q'} \\
&\leq C\|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}
\end{aligned} \tag{5.44}$$

for some constant $C \in (0,\infty)$ of the same nature as before. It is also useful to note that we may use (2.535) to estimate

$$\begin{aligned}
&\fint_{\Delta(x_0,R)} |(\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|^p \,\mathrm{d}w(x) \\
&\quad\leq \frac{\|\mathbf{1}_{\Delta(y_0,R)}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n}}{w\big(\Delta(x_0,R)\big)} \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n} \\
&\quad= \frac{w\big(\Delta(y_0,R)\big)}{w\big(\Delta(x_0,R)\big)} \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n} \\
&\quad\leq \frac{w\big(\Delta(x_0,2\lambda\Lambda R)\big)}{w\big(\Delta(x_0,R)\big)} \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n} \\
&\quad\leq [w]_{A_p}\Big(\frac{\sigma\big(\Delta(x_0,2\lambda\Lambda R)\big)}{\sigma\big(\Delta(x_0,R)\big)}\Big)^p \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n} \\
&\quad\leq C\Lambda^{(n-1)p}\|\mathbf{C} - \mathbf{C}^{\#}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n},
\end{aligned} \tag{5.45}$$

where $C \in (0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$.

Altogether, from (5.29), (5.31), (5.34), (5.36), (5.37), (5.39), (5.42), (5.43), (5.44), and (5.45) we conclude that

$$\fint_{\Delta(x_0,R)} \bigg|\int_{\Delta(y_0,R)} \Big\{\frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) + \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n}\Big\}\,\mathrm{d}\sigma(y)\bigg|^p \,\mathrm{d}w(x)$$

$$
\begin{aligned}
&\leq C(\Lambda^{-n}\ln\Lambda)^p\|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + C_{n,p}⨍_{\Delta(x_0,R)}|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|^p\,\mathrm{d}w(x)\\
&\qquad + C\Lambda^{-np}⨍_{\Delta(x_0,R)}\big|\nu(x)-\nu_{\Delta(x_0,R)}\big|^p\,\mathrm{d}w(x)\\
&\leq C(\Lambda^{-n}\ln\Lambda)^p\|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\\
&\qquad + C\Lambda^{(n-1)p}\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n\to L^p(\partial\Omega,w)\otimes C\ell_n}
\end{aligned}
\tag{5.46}
$$

where $C\in(0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$.

Going further, define

$$
a := ⨍_{\Delta(y_0,R)}\frac{x_0-y}{|x_0-y|^n}\,\mathrm{d}\sigma(y)\in\mathbb{R}^n\hookrightarrow C\ell_n \tag{5.47}
$$

and

$$
b := ⨍_{\Delta(y_0,R)}\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)\,\mathrm{d}\sigma(y)\in C\ell_n. \tag{5.48}
$$

Note that

$$
a=\frac{x_0-y_0}{|x_0-y_0|^n}+⨍_{\Delta(y_0,R)}\Big(\frac{x_0-y}{|x_0-y|^n}-\frac{x_0-y_0}{|x_0-y_0|^n}\Big)\,\mathrm{d}\sigma(y) \tag{5.49}
$$

and observe that the Mean Value Theorem gives, for some purely dimensional constant $C\in(0,\infty)$,

$$
\Big|\frac{x_0-y}{|x_0-y|^n}-\frac{x_0-y_0}{|x_0-y_0|^n}\Big|\leq\frac{CR}{(\Lambda R)^n}=C\Lambda^{-n}R^{1-n}, \tag{5.50}
$$

for each $y\in\Delta(y_0,R)$. As a consequence of this and (5.28),

$$
\begin{aligned}
|a| &\geq \Big|\frac{x_0-y_0}{|x_0-y_0|^n}\Big| - ⨍_{\Delta(y_0,R)}\Big|\frac{x_0-y}{|x_0-y|^n}-\frac{x_0-y_0}{|x_0-y_0|^n}\Big|\,\mathrm{d}\sigma(y)\\
&\geq \frac{1}{|x_0-y_0|^{n-1}}-C\Lambda^{-n}R^{1-n}\geq(\Lambda R)^{1-n}-C\Lambda^{-n}R^{1-n}\\
&\geq 2^{-1}(\Lambda R)^{1-n},
\end{aligned}
\tag{5.51}
$$

if $\Lambda>2C$. Hence, if we also introduce

$$A := b \odot \Big(\frac{a}{|a|^2} \Big) \in C\ell_n, \tag{5.52}$$

we may now estimate, using (5.6), (5.52), (5.51), (5.11), (5.1), (5.47), (5.48), and (5.46),

$$\begin{aligned}
⨍_{\Delta(x_0,R)} |\nu(x) - A_{\rm vect}|^p \, dw(x) &\leq ⨍_{\Delta(x_0,R)} |\nu(x) - A|^p \, dw(x) \\
&= ⨍_{\Delta(x_0,R)} \Big| \nu(x) - b \odot (a/|a|^2) \Big|^p \, dw(x) \\
&\leq C(\Lambda R)^{(n-1)p} ⨍_{\Delta(x_0,R)} \Big| \nu(x) - b \odot (a/|a|^2) \Big|^p |a|^p \, dw(x) \\
&= C(\Lambda R)^{(n-1)p} ⨍_{\Delta(x_0,R)} \Big| (\nu(x) - b \odot (a/|a|^2)) \odot a \Big|^p \, dw(x) \\
&= C(\Lambda R)^{(n-1)p} ⨍_{\Delta(x_0,R)} |\nu(x) \odot a + b|^p \, dw(x) \\
&= C(\Lambda R)^{(n-1)p} ⨍_{\Delta(x_0,R)} \Big| \nu(x) \odot \Big(⨍_{\Delta(y_0,R)} \frac{x_0 - y}{|x_0 - y|^n} \, d\sigma(y) \Big) \\
&\qquad + \Big(⨍_{\Delta(y_0,R)} \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) \, d\sigma(y) \Big) \Big|^p \, dw(x) \\
&= C(\Lambda R)^{(n-1)p} ⨍_{\Delta(x_0,R)} \Big| ⨍_{\Delta(y_0,R)} \Big\{ \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n} \\
&\qquad + \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) \Big\} \, d\sigma(y) \Big|^p \, dw(x) \\
&\leq C\Lambda^{(n-1)p} ⨍_{\Delta(x_0,R)} \Big| \int_{\Delta(y_0,R)} \Big\{ \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n} \\
&\qquad + \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) \Big\} \, d\sigma(y) \Big|^p \, dw(x) \\
&\leq C(\Lambda^{-1} \ln \Lambda)^p \|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \\
&\qquad + C\Lambda^{2(n-1)p} \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n},
\end{aligned} \tag{5.53}$$

for some $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. From this, (2.109), and Lemma 2.14 we then deduce that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C(\Lambda^{-1}\ln\Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + C\Lambda^{2(n-1)}\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{L^p(\partial\Omega,w)\otimes C\ell_n\to L^p(\partial\Omega,w)\otimes C\ell_n}, \tag{5.54}$$

where $C \in (0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. By eventually further increasing the value of Λ as to ensure that we also have $\Lambda^{-1}\ln\Lambda < 1/(2C)$, we finally conclude from (5.54) that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{L^p(\partial\Omega,w)\otimes C\ell_n\to L^p(\partial\Omega,w)\otimes C\ell_n}, \tag{5.55}$$

where $C \in (0,\infty)$ depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. $\square$

Our next result contains estimates in the opposite direction to those given in Theorem 4.6.

Theorem 5.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\le k\le n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator K_Δ from (3.29), the Riesz transforms $\{R_j\}_{1\le j\le n}$ on $\partial\Omega$ from (4.297), and for each index $k \in \{1,\dots,n\}$ denote by M_{ν_k} the operator of pointwise multiplication by the k-th scalar component of ν.*

Then there exists some $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} + \max_{1\le j,k\le n}\big\|[M_{\nu_k},R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}\Big\}. \tag{5.56}$$

Proof If $\partial\Omega$ is unbounded, then the estimate claimed in (5.56) is a direct consequence of Theorem 5.1 and Lemma 5.1 (also bearing in mind Lemma 2.15). In the case when $\partial\Omega$ is bounded, we have $K_\Delta 1 = \pm\frac{1}{2}$ (cf. [114, §1.5]) with the sign plus if Ω is bounded, and the sign minus if Ω is unbounded, hence $\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \ge \frac{1}{2}$ in such a scenario. Since from (2.118) we know that we always have $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le 1$, the estimate claimed in (5.56) holds in this case if we take $C \ge 2$. $\square$

We conclude this section by presenting a characterization of δ-flat Ahlfors regular domains in terms of the size of the operator norms of the classical harmonic double layer and commutators of Riesz transforms with pointwise multiplication by the scalar components of the unit normal.

Corollary 5.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\le k\le n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight*

$w \in A_p(\partial\Omega, \sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator K_Δ on $\partial\Omega$ from (3.29), the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from (4.297), and for each $k \in \{1,\dots,n\}$ denote by M_{ν_k} the operator of pointwise multiplication by the k-th scalar component of ν.

Then there exists some $C \in (0,\infty)$ which depends only on n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if

$$\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} + \max_{1\leq j,k\leq n} \big\|[M_{\nu_k}, R_j]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} < \delta \tag{5.57}$$

then Ω is a $(C\delta)$–flat Ahlfors regular domain.

Proof All desired conclusions follow from Theorem 5.2 and Definition 2.15. □

5.3 Using Riesz Transforms to Quantify Flatness

Recall from (1.16) that for each $j \in \{1,\dots,n\}$ the j-th Riesz transform R_j associated with a UR domain $\Omega \subseteq \mathbb{R}^n$ is the formal convolution operator on $\partial\Omega$ with the kernel $k_j(x) := \frac{2}{\omega_{n-1}}\frac{x_j}{|x|^n}$ for $x \in \mathbb{R}^n \setminus \{0\}$. From Proposition 3.4 we know that these are bounded operators on $L^p(\partial\Omega, w)$ for each $p \in (1,\infty)$ and $w \in A_p(\partial\Omega,\sigma)$. The most familiar setting is when $\Omega = \mathbb{R}^n_+$, in which case it is well known that

$$\sum_{j=1}^{n} R_j^2 = -I \quad\text{and}\quad R_jR_k = R_kR_j \quad\text{for all}\quad j,k \in \{1,\dots,n\}, \tag{5.58}$$

when all operators are considered on Muckenhoupt weighted Lebesgue spaces. Indeed, in such a setting, for the integrability exponent $p = 2$ and the weight $w = 1$ these are immediate consequences of the fact that each R_j is a Fourier multiplier in $\partial\Omega \equiv \mathbb{R}^{n-1}$ corresponding to the symbol $i\xi_j/|\xi|$, then said identities extend to $L^p(\partial\Omega, w)$ via a density argument. For ease of reference, we shall refer to the formulas in (5.58) as being URTI, i.e., the `usual Riesz transform identities`.

Remarkably, Theorem 5.3 below provides a stability result to the effect that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain with an unbounded boundary for which the URTI are "almost" true in the context of a Muckenhoupt weighted Lebesgue space, then $\partial\Omega$ is "almost" flat, in that the BMO semi-norm of the outward unit normal to Ω is small.

Theorem 5.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain with an unbounded boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1,\infty)$ along with* *a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and recall the Riesz transforms $\{R_j\}_{1 \le j \le n}$ on $\partial\Omega$ from (4.297). Then there exists some $C \in (0, \infty)$ which depends only on n, p, $[w]_{A_p}$, and the* UR *constants of $\partial\Omega$ with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{\Big\|I + \sum_{j=1}^{n} R_j^2\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} + \max_{1\le j,k\le n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}\Big\}. \tag{5.59}$$

It is perhaps surprising (but nonetheless true; cf. [60]) that URTI are also valid in the context of Muckenhoupt weighted Lebesgue spaces when Ω is an open ball, or the complement of a closed ball in $\mathbb{R}^n$. This shows that, in the context of Theorem 5.3, our assumption that $\partial\Omega$ is unbounded is warranted, since otherwise (5.59) may fail.

Proof of Theorem 5.3 Formula [61, (4.6.46), p. 2752] (which is valid in any UR domain, irrespective of whether its boundary is compact or not) tells us that for each $f \in L^p(\partial\Omega, \sigma) \otimes C\ell_n$ we have

$$(\mathbf{C} - \mathbf{C}^{\#})f = \mathbf{C}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f + \sum_{1\le j<k\le n} \mathbf{C}\big([R_j, R_k](\mathbf{e}_j \odot \mathbf{e}_k \odot f)\big). \tag{5.60}$$

Since $\big(L^p(\partial\Omega, \sigma) \cap L^p(\partial\Omega, w)\big) \otimes C\ell_n$ is a dense subspace of $L^p(\partial\Omega, w) \otimes C\ell_n$ and since all operators involved are continuous on $L^p(\partial\Omega, w) \otimes C\ell_n$, we conclude that formula (5.60) continues to hold for each $f \in L^p(\partial\Omega, w) \otimes C\ell_n$. From this version of (5.60) we then see that

$$(\mathbf{C} - \mathbf{C}^{\#})f = \mathbf{C}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f + \sum_{1\le j<k\le n} \mathbf{C}\big([R_j, R_k](\mathbf{e}_j \odot \mathbf{e}_k \odot f)\big) \tag{5.61}$$

for each $f \in L^p(\partial\Omega, w) \otimes C\ell_n$. In concert with (5.18), this implies

$$\begin{aligned}\|\mathbf{C} - \mathbf{C}^{\#}\|_{L^p(\partial\Omega,w)\otimes C\ell_n \to L^p(\partial\Omega,w)\otimes C\ell_n} &\le C\Big\|I + \sum_{j=1}^{n} R_j^2\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \\ &\quad + C\sum_{1\le j<k\le n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}\end{aligned} \tag{5.62}$$

Then (5.59) becomes a consequence of (5.62) and Theorem 5.1. □

Our next result contains estimates in the opposite direction to those from Theorem 5.3. Collectively, Theorems 5.4 and 5.3 amount to saying that, under natural background geometric assumptions on the set Ω, the URTI are "almost" true in the context of a Muckenhoupt weighted Lebesgue space if and only if $\partial\Omega$ is "almost" flat (in that the BMO semi-norm of the outward unit normal to Ω is small).

Theorem 5.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and recall the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from* (4.297).

Then for each $m \in \mathbb{N}$ there exists some constant $C_m \in (0,\infty)$ which depends only on m, n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ such that, with the piece of notation introduced in (4.93)*, one has*

$$\Big\| I + \sum_{j=1}^{n} R_j^2 \Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}, \tag{5.63}$$

and

$$\max_{1\leq j<k\leq n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle}. \tag{5.64}$$

Furthermore, if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small relative to n, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ one may take $C_m \in (0,\infty)$ appearing in (5.63)–(5.64) *to depend only on said entities* (*i.e., n, p, $[w]_{A_p}$, the Ahlfors regularity constant of $\partial\Omega$*) *and m.*

Proof From the Muckenhoupt version of (5.20) and (5.61) we see that for each function $f \in L^p(\partial\Omega,\sigma)\otimes C\ell_n$ we have

$$\mathbf{C}(\mathbf{C}^{\#} - \mathbf{C})f = -\tfrac{1}{4}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f - \tfrac{1}{4}\sum_{1\leq j<k\leq n}[R_j, R_k](\mathbf{e}_j \odot \mathbf{e}_k \odot f). \tag{5.65}$$

Fix a scalar function $f \in L^p(\partial\Omega, w)$ normalized so that $\|f\|_{L^p(\partial\Omega,w)} = 1$. In particular, this shows that the function f belongs to the space $L^p(\partial\Omega,w)\otimes C\ell_n$ and $\|f\|_{L^p(\partial\Omega,w)\otimes C\ell_n} = 1$. Bearing this in mind, for each $m \in \mathbb{N}$ we may then write

$$\max\Bigg\{\Big\|\tfrac{1}{4}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f\Big\|_{L^p(\partial\Omega,w)}, \max_{1\leq j<k\leq n}\Big\|\tfrac{1}{4}[R_j,R_k]f\Big\|_{L^p(\partial\Omega,w)}\Bigg\}$$

$$\leq \Bigg\| \Bigg\{ \Big|\tfrac{1}{4}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f\Big|^2 + \sum_{1\leq j<k\leq n}\Big|\tfrac{1}{4}[R_j,R_k]f\Big|^2 \Bigg\}^{1/2} \Bigg\|_{L^p(\partial\Omega,w)}$$

$$= \Big\| \tfrac{1}{4}\Big(I + \sum_{j=1}^{n} R_j^2\Big) f + \tfrac{1}{4} \sum_{1 \le j < k \le n} ([R_j, R_k] f) \mathbf{e}_j \odot \mathbf{e}_k \Big\|_{L^p(\partial\Omega, w) \otimes C\ell_n}$$

$$= \|\mathbf{C}(\mathbf{C}^{\#} - \mathbf{C}) f\|_{L^p(\partial\Omega, w) \otimes C\ell_n}$$

$$\le \|\mathbf{C}\|_{L^p(\partial\Omega, w) \otimes C\ell_n \to L^p(\partial\Omega, w) \otimes C\ell_n} \Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{L^p(\partial\Omega, w) \otimes C\ell_n \to L^p(\partial\Omega, w) \otimes C\ell_n}$$

$$\le C_m \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega, \sigma)]^n}, \tag{5.66}$$

where the first inequality is trivial, the subsequent equality is implied by (5.8), the second equality is seen from formula (5.65) (since f is scalar-valued), the penultimate estimate uses the normalization of f, while the last inequality is provided by (5.18) and (5.22). With estimate (5.66) in hand, the claims in (5.63)–(5.64) readily follow (in view of the arbitrariness of the scalar-valued function $f \in L^p(\partial\Omega, w)$ with $\|f\|_{L^p(\partial\Omega, w)} = 1$). The final claim in the statement is a direct consequence of Theorem 2.3. □

5.4 Using Riesz Transforms to Characterize Muckenhoupt Weights

Assume $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \ge 2$, is a closed UR set and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. For $j \in \{1, \dots, n\}$, the j-th Riesz transform R_j on Σ is defined as the operator acting on each $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{n-1}} \int_{\substack{y \in \Sigma \\ |x-y| > \varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y) \, \mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \Sigma. \tag{5.67}$$

From Proposition 3.4 we know that these Riesz transforms are well defined in this context, and that for each integrability exponent $p \in (1, \infty)$ and Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ they induce linear and bounded mappings on $L^p(\Sigma, w)$. The goal in this section is to show that the class of Muckenhoupt weights is the largest class of weights for which the latter boundedness properties hold.

As a preamble, we note that for a variety of purposes it is convenient to glue together all Riesz transforms $\{R_j\}_{1 \le j \le n}$ from (5.67) into a unique operator now acting on Clifford algebra-valued functions $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ according to

$$Rf(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{y\in\Sigma \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot f(y)\,\mathrm{d}\sigma(y)$$

$$= \mathbf{e}_1 \odot R_1 f(x) + \cdots + \mathbf{e}_n \odot R_n f(x) \ \text{ for } \sigma\text{-a.e. } x \in \Sigma. \tag{5.68}$$

Theorem 5.5 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed* UR *set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p \in (1,\infty)$ and consider a weight w on Σ which belongs to $L^1_{\mathrm{loc}}(\Sigma,\sigma)$ and has the property that, for each $j \in \{1,\dots,n\}$, the j-th Riesz transform R_j on Σ originally defined as in (5.67) extends to a linear and bounded operator on $L^p(\Sigma,w)$. Then necessarily $w \in A_p(\Sigma,\sigma)$ and there exists $C \in (0,\infty)$ which depends only on the Ahlfors regularity constant of Σ, n, and p with the property that*

$$[w]_{A_p} \le C \begin{cases} \max_{1\le j\le n} \|R_j\|^{2p}_{L^p(\Sigma,w)\to L^p(\Sigma,w)} & \text{if } \Sigma \text{ unbounded,} \\ \max_{1\le j\le n} \|R_j\|^{5p}_{L^p(\Sigma,w)\to L^p(\Sigma,w)} & \text{if } \Sigma \text{ bounded.} \end{cases} \tag{5.69}$$

From assumptions and (2.508) we know that σ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on Σ. Since the weight w belongs to $L^1_{\mathrm{loc}}(\Sigma,\sigma)$, it follows that

$$\text{the measure } \mathrm{d}w := w\,\mathrm{d}\sigma \text{ is complete, locally finite (hence also sigma-finite), separable, and Borel-regular on } \Sigma. \tag{5.70}$$

Granted this, results in [7], [111, §3.7] then guarantee that the natural inclusion

$$\mathscr{X} := \big\{\phi\big|_\Sigma : \phi \in \mathscr{C}^\infty_0(\mathbb{R}^n)\big\} \hookrightarrow L^p(\Sigma,w) \ \text{ has dense range.} \tag{5.71}$$

From the preamble to Theorem 5.5 we know that the Riesz transforms (5.67) act in a meaningful fashion on $\mathscr{X}$, and this is the manner in which the R_j's are originally considered in the context of Theorem 5.5. The point of the latter theorem is that if the R_j's originally defined on $\mathscr{X}$ extend via density (cf. (5.71)) to linear and bounded operators on $L^p(\Sigma,w)$ then necessarily $w \in A_p(\Sigma,\sigma)$.

Let us now present the proof of Theorem 5.5.

Proof of Theorem 5.5 The fact that all Riesz transforms on Σ originally defined as in (5.67) on functions $f \in \mathscr{X} := \big\{\phi\big|_\Sigma : \phi \in \mathscr{C}^\infty_0(\mathbb{R}^n)\big\}$ induce (via density; cf. (5.71)) linear and bounded mappings on $L^p(\Sigma,w)$, implies that the operator R from (5.68), originally defined on functions $f \in \mathscr{X} \otimes C\ell_n$ induces (via density) a linear and bounded mapping on $L^p(\Sigma,w) \otimes C\ell_n$. Henceforth abbreviate

$$C_0 := \|R\|_{L^p(\Sigma,w)\otimes C\ell_n \to L^p(\Sigma,w)\otimes C\ell_n} \tag{5.72}$$

and note that there exists a dimensional constant $C_n \in (0, \infty)$ with the property that

$$C_0 \leq C_n \cdot \max_{1 \leq j \leq n} \|R_j\|_{L^p(\Sigma,w) \to L^p(\Sigma,w)}. \tag{5.73}$$

To proceed in earnest, denote by $C_{\text{AR}} \in [1, \infty)$ the Ahlfors regularity constant of Σ and fix a number $\lambda \in (1, \infty)$ which is sufficiently large relative to the Ahlfors regularity constant of Σ as to ensure that

$$\Delta(x, \lambda\rho) \setminus \Delta(x, \rho) \neq \varnothing \quad \text{for each} \quad x \in \Sigma \quad \text{and} \quad \rho \in \big(0, \operatorname{diam}(\Sigma)\big/\lambda\big). \tag{5.74}$$

For example, any $\lambda > C_{\text{AR}}^{2/(n-1)}$ will do. Fix $r \in \big(0, \operatorname{diam}(\Sigma)\big/(10\lambda)\big)$ and suppose $x_1, x_2 \in \Sigma$ are such that

$$10\, r \leq |x_1 - x_2| \leq 200\lambda r. \tag{5.75}$$

Abbreviate

$$\Delta_1 := \Delta(x_1, r) \quad \text{and} \quad \Delta_2 := \Delta(x_2, r). \tag{5.76}$$

Next, select a real-valued function $f \in \mathscr{X}$ and set $f_\pm := \max\{\pm f, 0\}$. We then have $0 \leq f_\pm \leq |f| = f_+ + f_-$ on Σ, and $f_\pm \in L^p(\Sigma, w)$ since $\mathscr{X} \subseteq L^p(\Sigma, w)$. For each $y \in \Sigma$ define

$$g_\pm(y) := \begin{cases} -\dfrac{x_2 - y}{|x_2 - y|} f_\pm(y) & \text{if } \ y \in \Delta_1, \\[2mm] \qquad 0 & \text{if } \ y \in \Sigma \setminus \Delta_1, \end{cases} \tag{5.77}$$

so $g_\pm$ belong to $L^p(\Sigma, w) \otimes C\ell_n$ and are supported in Δ_1. Consequently,

$$Rg_\pm(x) = \frac{2}{\omega_{n-1}} \int_{\Delta_1} \frac{x - y}{|x - y|^n} \odot \frac{-(x_2 - y)}{|x_2 - y|} f_\pm(y)\, d\sigma(y) \quad \text{for each} \quad x \in \Delta_2. \tag{5.78}$$

Recall that the scalar component u_{scal} of a Clifford algebra element $u \in C\ell_n$ is defined as in (5.7). For each $x \in \Delta_2$ and $y \in \Delta_1$ we may use (5.1), (5.8), (5.11), as well as (5.75) to compute

$$\begin{aligned} \Big(\frac{x - y}{|x - y|^n} \odot \frac{-(x_2 - y)}{|x_2 - y|}\Big)_{\text{scal}} &= \Big(\frac{x - y}{|x - y|^n} \odot \frac{-(x - y)}{|x_2 - y|}\Big)_{\text{scal}} \\ &\quad + \Big(\frac{x - y}{|x - y|^n} \odot \frac{x - x_2}{|x_2 - y|}\Big)_{\text{scal}} \\ &= \frac{1}{|x - y|^{n-2} \cdot |x_2 - y|} + \Big(\frac{x - y}{|x - y|^n} \odot \frac{x - x_2}{|x_2 - y|}\Big)_{\text{scal}} \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{|x-y|^{n-2}\cdot|x_2-y|} - \frac{|x-x_2|}{|x-y|^{n-1}\cdot|x_2-y|}\\
&= \frac{|x-y|-|x-x_2|}{|x-y|^{n-1}\cdot|x_2-y|}\\
&\geq \frac{7r}{(200\lambda r+2r)^{n-1}(200\lambda r+r)} = c_{n,\lambda}\cdot r^{1-n},
\end{aligned}
\tag{5.79}
$$

where the last equality defines $c_{n,\lambda}$. Based on (5.78) and (5.79) we then conclude that we have the pointwise lower bound

$$
\big|Rg_{\pm}\big| \geq \big(Rg_{\pm}\big)_{\rm scal} \geq c_{n,\lambda}\cdot C_{\rm AR}^{-1} ⨍_{\Delta_1} f_{\pm}\,{\rm d}\sigma \quad \text{on} \quad \Delta_2. \tag{5.80}
$$

In concert with the boundedness of R on $L^p(\Sigma, w)\otimes C\ell_n$ (mentioned in the first part of the proof) and the piece of notation introduced in (5.72), this permits us to estimate

$$
\begin{aligned}
c_{n,\lambda}^p\cdot C_{\rm AR}^{-p}\Big(⨍_{\Delta_1} f_{\pm}\,{\rm d}\sigma\Big)^p &\leq \frac{1}{w(\Delta_2)}\int_{\Delta_2}\big|Rg_{\pm}\big|^p\,{\rm d}w \leq \frac{1}{w(\Delta_2)}\int_{\Sigma}\big|Rg_{\pm}\big|^p\,{\rm d}w\\
&\leq \frac{C_0^p}{w(\Delta_2)}\int_{\Sigma}\big|g_{\pm}\big|^p\,{\rm d}w \leq \frac{C_0^p}{w(\Delta_2)}\int_{\Delta_1}|f|^p\,{\rm d}w.
\end{aligned}
\tag{5.81}
$$

Combining the two versions of (5.81), corresponding to f_+ and f_-, yields

$$
c_{n,\lambda}^p\cdot C_{\rm AR}^{-p}\Big(⨍_{\Delta_1}|f|\,{\rm d}\sigma\Big)^p \leq \frac{2^{p-1}\cdot C_0^p}{w(\Delta_2)}\int_{\Delta_1}|f|^p\,{\rm d}w. \tag{5.82}
$$

Specializing (5.82) to the case when the real-valued function $f\in\mathscr{X}$ is chosen such that $f\equiv 1$ on Δ_1 then yields

$$
c_{n,\lambda}^p\cdot C_{\rm AR}^{-p} \leq 2^{p-1}\cdot C_0^p\,\frac{w(\Delta_1)}{w(\Delta_2)}. \tag{5.83}
$$

Running the same type of argument as above but with the roles of x_1 and x_2 (which are interchangeable) reversed then produces, in place of (5.83),

$$
c_{n,\lambda}^p\cdot C_{\rm AR}^{-p} \leq 2^{p-1}\cdot C_0^p\,\frac{w(\Delta_2)}{w(\Delta_1)}. \tag{5.84}
$$

From (5.84) and (5.82) we then conclude that for each real-valued function $f \in \mathscr{X}$ we have

$$\fint_{\Delta_1} |f| \, d\sigma \le C_1 \Big(\fint_{\Delta_1} |f|^p \, dw \Big)^{1/p}, \tag{5.85}$$

with

$$C_1 := \big(2^{1-1/p} \cdot C_0 \cdot c_{n,\lambda}^{-1} \cdot C_{\text{AR}} \big)^2. \tag{5.86}$$

Consider now an arbitrary function $h \in L^p_{\text{loc}}(\Sigma, w)$. In particular, the extension of $h\big|_{\Delta_1}$ by zero to the rest of Σ belongs to $L^p(\Sigma, w)$. Granted this, (5.71) guarantees the existence of a sequence of functions $\{f_j\}_{j \in \mathbb{N}} \subseteq \mathscr{X}$ such that

$$f_j\big|_{\Delta_1} \to h\big|_{\Delta_1} \ \text{ in } \ L^p(\Delta_1, w) \ \text{ as } \ j \to \infty. \tag{5.87}$$

By eventually passing to sub-sequences there is no loss of generality in also assuming that $\lim\limits_{j \to \infty} f_j(x) = h(x)$ for σ-a.e. $x \in \Delta_1$. Based on this, Fatou's lemma, and (5.85) we may then write

$$\begin{aligned} \fint_{\Delta_1} |h| \, d\sigma &\le \liminf_{j \to \infty} \fint_{\Delta_1} |f_j| \, d\sigma \le C_1 \cdot \liminf_{j \to \infty} \Big(\fint_{\Delta_1} |f_j|^p \, dw \Big)^{1/p} \\ &\le C_1 \Big(\fint_{\Delta_1} |h|^p \, dw \Big)^{1/p}. \end{aligned} \tag{5.88}$$

Ultimately, this goes to show that for each $h \in L^p_{\text{loc}}(\Sigma, w)$ we have

$$\fint_{\Delta_1} |h| \, d\sigma \le C_1 \Big(\fint_{\Delta_1} |h|^p \, dw \Big)^{1/p}, \tag{5.89}$$

with $C_1 \in (0, \infty)$ as in (5.86) (hence, in particular, independent of h, x_1, and r).

Start now with an arbitrary point $x \in \Sigma$, and continue to assume that the scale r belongs to $\big(0, \text{diam}(\Sigma)/(10\lambda)\big)$. We may then employ (5.74) with $\rho := 10r$ to conclude that there exists some $\widetilde{x} \in \Delta(x, 10\lambda r) \setminus \Delta(x, 10r)$. For such a choice we then have $10r \le |x - \widetilde{x}| < 10\lambda r$ which, in light of (5.75), shows that we may run the argument so far with $x_1 := x$ and $x_2 := \widetilde{x}$. In place of (5.89) we then arrive at the conclusion that, with $C_1 \in (0, \infty)$ as in (5.86),

$$\begin{aligned} &\fint_{\Delta(x,r)} |h| \, d\sigma \le C_1 \Big(\fint_{\Delta(x,r)} |h|^p \, dw \Big)^{1/p} \quad \text{for each} \\ &h \in L^p_{\text{loc}}(\Sigma, w), \ \ x \in \Sigma, \ \ r \in \big(0, \text{diam}(\Sigma)/(10\lambda)\big). \end{aligned} \tag{5.90}$$

In the case when Σ is unbounded, from (5.90) (which now holds with no restriction on the size of the scale r since $\mathrm{diam}(\Sigma) = \infty$) and the second part of Lemma 2.12 we conclude that

$$w \in A_p(\Sigma, \sigma) \quad \text{and} \quad [w]_{A_p} \leq C_1^p. \tag{5.91}$$

There remains to treat the scenario in which Σ is bounded. When this is the case, starting with (5.90), the argument in the proof of Lemma 2.12 that has led to (2.529) presently gives (with p' denoting the Hölder conjugate exponent of p)

$$\begin{gathered}\Big(\fint_{\Delta(x,r)} w\,\mathrm{d}\sigma\Big)\Big(\fint_{\Delta(x,r)} w^{1-p'}\,\mathrm{d}\sigma\Big)^{p-1} \leq C_1^p \\ \text{for each } \ x \in \Sigma \ \text{ and } \ r \in \big(0, \mathrm{diam}(\Sigma)/(10\lambda)\big).\end{gathered} \tag{5.92}$$

To obtain a similar inequality in the regime

$$\mathrm{diam}(\Sigma)/(10\lambda) \leq r \leq \mathrm{diam}(\Sigma), \tag{5.93}$$

observe that for each $x \in \Sigma$ we may estimate, using the Ahlfors regularity of Σ and the fact that r is comparable with $\mathrm{diam}(\Sigma)$,

$$\begin{aligned}&\Big(\fint_{\Delta(x,r)} w\,\mathrm{d}\sigma\Big)\Big(\fint_{\Delta(x,r)} w^{1-p'}\,\mathrm{d}\sigma\Big)^{p-1} \\ &\qquad \leq C_{\mathrm{AR}}^{2p} \cdot (10\lambda)^{(n-1)p}\Big(\fint_{\Sigma} w\,\mathrm{d}\sigma\Big)\Big(\fint_{\Sigma} w^{1-p'}\,\mathrm{d}\sigma\Big)^{p-1}.\end{aligned} \tag{5.94}$$

At this stage, there remains to bound the right-hand side of (5.94) by a suitable finite constant which is independent of w. To this end, introduce the following threshold $r_0 := \mathrm{diam}(\Sigma)/(20\lambda)$. We claim that there exist an integer

$$N \in \mathbb{N} \quad \text{with} \quad N \leq C_{\mathrm{AR}}^2 \cdot (40\lambda)^{n-1} \tag{5.95}$$

along with a family of points $\{x_j\}_{j=1}^N \subseteq \Sigma$ satisfying

$$\begin{gathered}|x_j - x_k| \geq r_0 \text{ for every } j, k \in \{1, \dots, N\} \text{ with } j \neq k \\ \text{and} \quad \Sigma \subseteq \bigcup_{j=1}^N \Delta(x_j, r_0).\end{gathered} \tag{5.96}$$

To justify this claim, observe that

$$\mathcal{A} := \Big\{A \subseteq \Sigma : |x - x'| \geq r_0 \ \text{ for all } x, x' \in A \text{ with } x \neq x'\Big\} \tag{5.97}$$

is a partially ordered set with respect to the canonical inclusion of sets. It is also clear that any totally ordered subset $\mathcal{B}$ of $\mathcal{A}$ has an upper bound in $\mathcal{A}$, namely $\bigcup_{B\in\mathcal{B}} B$. By Zorn's lemma, there exists a maximal element $A_{\max}$ in $\mathcal{A}$. By maximality we necessarily have

$$\Sigma \subseteq \bigcup_{x\in A_{\max}} \Delta(x, r_0). \tag{5.98}$$

Since $\Sigma \subseteq \mathbb{R}^n$ is currently assumed to be compact, there exist $\{x_j\}_{j=1}^N \subseteq A_{\max}$ such that $\Sigma \subseteq \bigcup_{j=1}^N \Delta(x_j, r_0)$. This takes care of (5.96). To estimate N as in (5.95), start by observing that the balls $\big\{B(x_j, r_0/2)\big\}_{j=1}^N$ are, thanks to the first line in (5.96), mutually disjoint. Bearing this in mind, we may use the Ahlfors regularity of Σ to write

$$\begin{aligned} C_{\text{AR}} \cdot (\text{diam}(\Sigma))^{n-1} &\geq \sigma(\Sigma) \geq \sum_{j=1}^N \sigma\big(B(x_j, r_0/2) \cap \Sigma\big) \\ &\geq N \cdot C_{\text{AR}}^{-1} \cdot (r_0/2)^{n-1} = N \cdot C_{\text{AR}}^{-1} \cdot \big(\text{diam}(\Sigma)\big/(40\lambda)\big)^{n-1} \end{aligned} \tag{5.99}$$

from which (5.95) readily follows.

Moving on, note that for every $j, k \in \{1, \dots, N\}$ with $j \neq k$ one has

$$r_0 \leq |x_j - x_k| \leq \text{diam}(\Sigma) = 20\lambda r_0. \tag{5.100}$$

Thus, (5.75) holds with $r := r_0/10 = \text{diam}(\Sigma)/(200\lambda)$, and x_j, x_k playing the role of x_1 and x_2. As such, (5.76) and (5.83) yield

$$\frac{w\big(\Delta(x_k, r_0/10)\big)}{w\big(\Delta(x_j, r_0/10)\big)} \leq 2^{p-1} \cdot C_0^p \cdot c_{n,\lambda}^{-p} \cdot C_{\text{AR}}^p. \tag{5.101}$$

On the other hand, Ahlfors regularity and (5.90) applied to $\Delta(x_k, r_0)$ and the function $h = \mathbf{1}_{\Delta(x_k, r_0/10)}$ readily gives

$$C_{\text{AR}}^{-2p} \cdot 10^{-(n-1)p} \leq \left(\frac{\sigma\big(\Delta(x_k, r_0/10)\big)}{\sigma\big(\Delta(x_k, r_0)\big)}\right)^p \leq C_1^p \cdot \frac{w\big(\Delta(x_k, r_0/10)\big)}{w\big(\Delta(x_k, r_0)\big)}. \tag{5.102}$$

Collecting then (5.101) and (5.102) we conclude that

$$\frac{w\big(\Delta(x_k, r_0)\big)}{w\big(\Delta(x_j, r_0)\big)} \leq C_{\rm AR}^{2p} \cdot 10^{(n-1)p} \cdot C_1^p \cdot \frac{w\big(\Delta(x_k, r_0/10)\big)}{w\big(\Delta(x_j, r_0/10)\big)} \leq C_2, \tag{5.103}$$

with

$$C_2 := 2^{3p-3} \cdot 10^{(n-1)p} \cdot C_{\rm AR}^{5p} \cdot C_0^{3p} \cdot c_{n,\lambda}^{-3p}. \tag{5.104}$$

Since the latter estimate holds for every $j, k \in \{1, \dots, N\}$ with $j \neq k$ we obtain that for every $j \in \{1, \dots, N\}$

$$w(\Sigma) \leq \sum_{j=1}^{N} w\big(\Delta(x_k, r_0)\big) \leq N \cdot C_2 \cdot\, w\big(\Delta(x_j, r_0)\big) \leq C_3 \cdot w\big(\Delta(x_j, r_0)\big), \tag{5.105}$$

where

$$C_3 := 2^{3p-3} \cdot 10^{(n-1)p} \cdot (40\lambda)^{n-1} \cdot C_{\rm AR}^{5p+2} \cdot C_0^{3p} \cdot c_{n,\lambda}^{-3p}. \tag{5.106}$$

From (5.105) and (5.90) used with $r := r_0 \in \big(0, {\rm diam}(\Sigma)\big/(10\lambda)\big)$ we then obtain that for each $h \in L^p(\Sigma, w)$ we have

$$\int_{\Delta(x_j, r_0)} |h| \, d\sigma \leq \sigma(\Sigma) \cdot C_1 \cdot C_3^{1/p} \Big(⨍_{\Sigma} |h|^p \, dw \Big)^{1/p} \quad \text{for} \quad j \in \{1, \dots, N\}. \tag{5.107}$$

Summing up in j further yields

$$⨍_{\Sigma} |h| \, d\sigma \leq N \cdot C_1 \cdot C_3^{1/p} \Big(⨍_{\Sigma} |h|^p \, dw \Big)^{1/p} \quad \text{for each} \quad h \in L^p(\Sigma, w). \tag{5.108}$$

Having established (5.108), the argument in the proof of Lemma 2.12 that has produced (2.529) (used with $\Delta := \Sigma$) then currently gives

$$\begin{aligned} \Big(⨍_{\Sigma} w \, d\sigma \Big) \Big(⨍_{\Sigma} w^{1-p'} \, d\sigma \Big)^{p-1} &\leq (N \cdot C_1 \cdot C_3^{1/p})^p = N^p \cdot C_1^p \cdot C_3 \\ &\leq 2^{5p-5} \cdot 10^{(n-1)p} \cdot (40\lambda)^{(n-1)(p+1)} \cdot C_{\rm AR}^{9p+2} \cdot C_0^{5p} \cdot c_{n,\lambda}^{-5p}. \end{aligned} \tag{5.109}$$

Together with (5.94) this finally proves that $w \in A_p(\Sigma, \sigma)$ and that

$$[w]_{A_p} \leq C \|R\|^{5p}_{L^p(\Sigma,w)\otimes C\ell_n \to L^p(\Sigma,w)\otimes C\ell_n} \tag{5.110}$$

for $C \in (0, \infty)$ depending only on the Ahlfors regularity constant of Σ, n, and p.

Finally, from (5.91), (5.86), (5.110), and (5.73) we conclude that (5.69) holds. □

In concert with earlier results, Theorem 5.5 yields the following remarkable characterization of Muckenhoupt weights.

Theorem 5.6 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Fix a function* $w \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ *which is strictly positive* σ*-a.e. on* $\partial\Omega$*, along with an integrability exponent* $p \in (1, \infty)$. *Then the following statements are equivalent.*

(1) The weight w *belongs to the Muckenhoupt class* $A_p(\partial\Omega, \sigma)$.
(2) For each $j \in \{1, \dots, n\}$*, the* j*-th Riesz transform* R_j *on* $\partial\Omega$ *(cf.* (4.297)) *induces a linear and bounded operator on* $L^p(\partial\Omega, w)$.
(3) The Cauchy–Clifford operator $\mathbf{C}$ *from* (5.12) *induces a linear and bounded mapping on* $L^p(\partial\Omega, w) \otimes C\ell_n$.
(4) The "transpose" Cauchy–Clifford operator $\mathbf{C}^{\#}$ *from* (5.14) *induces a linear and bounded mapping on* $L^p(\partial\Omega, w) \otimes C\ell_n$.
(5) For each complex-valued function $k \in \mathscr{C}^\infty\big(\mathbb{R}^n \setminus \{0\}\big)$ *which is odd and positive homogeneous of degree* $1 - n$*, the integral operator originally defined for each function* $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ *as*

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k(x-y) f(y)\, \mathrm{d}\sigma(y) \quad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega \tag{5.111}$$

induces a linear and bounded mapping on $L^p(\partial\Omega, w)$.

Proof The implications *(1)* ⇒ *(2)* and *(1)* ⇒ *(5)* are direct consequences of Proposition 3.4 and (4.297). From (4.297) it is also clear that *(5)* ⇒ *(2)*. To proceed, let $\nu = (\nu_1, \dots, \nu_n)$ denote the geometric measure theoretic outward unit normal to Ω. Then (5.13) and (5.15) imply that the Cauchy–Clifford operator $\mathbf{C}$ from (5.12) as well as the "transpose" Cauchy–Clifford operator $\mathbf{C}^{\#}$ from (5.14) induce linear and bounded mappings on $L^p(\partial\Omega, w) \otimes C\ell_n$ whenever all Riesz transforms on $\partial\Omega$, i.e., R_j as in (4.297) with $1 \le j \le n$, induce linear and bounded operators on $L^p(\partial\Omega, w)$. This takes care of the implications *(2)* ⇒ *(3)* and *(2)* ⇒ *(4)*.

Going further, bring in the integral operator R defined as in (5.68) for $\Sigma := \partial\Omega$, i.e., $Rf = \mathbf{e}_1 \odot R_1 f + \dots + \mathbf{e}_n \odot R_n f$ for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$, where $\{R_j\}_{1\le j\le n}$ are Riesz transforms on $\partial\Omega$ defined in (4.297). From definitions and the fact that $\nu \odot \nu = -1$ at σ-a.e. point of $\partial\Omega$ (cf. (5.1)) we then see that for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ we have

$$\begin{aligned} &\nu \odot \mathbf{C}^{\#} f = \tfrac{1}{2} Rf, \quad -\mathbf{C}(\nu \odot f) = \tfrac{1}{2} Rf, \quad \mathbf{C} f = \nu \odot \mathbf{C}^{\#}(\nu \odot f), \\ &\mathbf{C}^{\#} f = -\tfrac{1}{2} \nu \odot Rf, \quad \mathbf{C} f = \tfrac{1}{2} R(\nu \odot f), \quad \mathbf{C}^{\#} f = \nu \odot \mathbf{C}(\nu \odot f). \end{aligned} \tag{5.112}$$

It is also clear that the statement in item *(2)* is equivalent to the demand that R induces a linear and bounded operator on $L^p(\partial\Omega, w) \otimes C\ell_n$. On account of this and (5.112) we then conclude that the implications *(3)* $\Rightarrow$ *(2)* and *(4)* $\Rightarrow$ *(2)* are valid. Finally, Theorem 5.5 gives the implication *(2)* $\Rightarrow$ *(1)*. The proof of Theorem 5.6 is therefore complete. □

Chapter 6
Boundary Value Problems in Muckenhoupt Weighted Spaces

This chapter is devoted to studying the Dirichlet, Regularity, Neumann, and Transmission boundary value problems in δ-AR domains with boundary data in Muckenhoupt weighted Lebesgue and Sobolev spaces. The technology that we bring to bear on such problems also allows us to deal with similar boundary value problems formulated in terms of ordinary Lorentz spaces and Lorentz-based Sobolev spaces.

As a preamble, in Theorem 6.1 below we recall from [113, §4.4] a Poisson integral representation formula for solutions of the Dirichlet Problem for a given weakly elliptic second-order system L, in domains of a very general geometric nature, which involves the conormal derivative of the Green function for the transpose system $L^{\top}$ as integral kernel. Stating this requires that we review a definition and a couple of related results. Specifically, following [111, §8.9] we shall say that a set Ω is `globally pathwise nontangentially accessible` provided Ω is an open nonempty proper subset of $\mathbb{R}^n$ such that:

given any $\kappa > 0$ there exist $\widetilde{\kappa} \geq \kappa$ along with $c \in [1, \infty)$ such that σ-a.e. point $x \in \partial\Omega$ has the property that any $y \in \Gamma_\kappa(x)$ may be joined by a rectifiable curve $\gamma_{x,y}$ such that $\gamma_{x,y} \setminus \{x\} \subset \Gamma_{\widetilde{\kappa}}(x)$ and whose length is $\leq c|x - y|$. (6.1)

It has been noted in [111, §8.9] that

any one-sided NTA domain with unbounded boundary is a globally pathwise nontangentially accessible set, (6.2)

and that

any semi-uniform set (in the sense of Aikawa-Hirata; cf. [4]) is a globally pathwise nontangentially accessible set. (6.3)

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_6

We are now ready to state the Poisson integral representation formula advertised earlier (for a proof see [113, §4.4]).

Theorem 6.1 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) which is globally pathwise nontangentially accessible (in the sense of (6.1)), and such that $\partial\Omega$ is unbounded and Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Next, suppose L is a weakly elliptic, homogeneous, constant (complex) coefficient, second-order, $M \times M$ system in $\mathbb{R}^n$. Fix a parameter $\kappa \in (0, \infty)$, along with an arbitrary point $x_0 \in \Omega$, and suppose $0 < \rho < \frac{1}{4}\,\mathrm{dist}(x_0, \partial\Omega)$. Finally, define $K := \overline{B(x_0, \rho)}$.*

Then there exists some $\widetilde{\kappa} > 0$, which depends only on Ω and κ, with the following significance. Assume G is a matrix-valued function satisfying

$$\begin{cases} G = \big(G_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^{M\times M}, \\ \big(L^\top G_{\cdot\beta}\big)_\alpha = -\delta_{x_0}\delta_{\alpha\beta} \ \ in\ \big[\mathcal{D}'(\Omega)\big]^M \ \ for\ all\ \ \alpha,\beta \in \{1,\dots,M\}, \\ \big(\nabla G\big)\Big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega} \ \ exists\ (in\ \mathbb{C}^{n\cdot M^2})\ at\ \sigma\text{-a.e. point on}\ \ \partial\Omega, \\ G\Big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega} = 0 \in \mathbb{C}^{M\times M} \ \ at\ \sigma-a.e.\ point\ on\ \ \partial\Omega, \\ \mathcal{N}^{\Omega\setminus K}_{\widetilde{\kappa}}(\nabla G) < +\infty \ \ at\ \sigma - a.e.\ point\ on\ \ \partial\Omega, \end{cases} \tag{6.4}$$

and assume $u = (u_\beta)_{1\leq\beta\leq M}$ is a $\mathbb{C}^M$-valued function in Ω satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \ in\ \ \Omega, \\ u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \ exists\ at\ \sigma\text{-a.e. point on}\ \ \partial\Omega, \\ \mathcal{N}_\kappa u < +\infty \ \ at\ \sigma - a.e.\ point\ on\ \ \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}^{\Omega\setminus K}_{\widetilde{\kappa}}(\nabla G)\, d\sigma < +\infty. \end{cases} \tag{6.5}$$

Then for any choice of a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}} \in \mathfrak{A}_L$ one has the Poisson integral representation formula

$$u_\beta(x_0) = -\int_{\partial_*\Omega} \Big\langle u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \partial^{A^\top}_\nu G_{\cdot\beta}\Big\rangle\, d\sigma, \qquad \forall\,\beta \in \{1,\dots,M\}, \tag{6.6}$$

where $\partial^{A^\top}_\nu$ stands for the conormal derivative associated with $A^\top$, acting on the columns of the matrix-valued function G according to (compare with (3.66))

$$\partial^{A^\top}_\nu G_{\cdot\beta} := \Big(\nu_r a^{\gamma\alpha}_{sr}\big(\partial_s G_{\gamma\beta}\big)\Big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq M} \ \ at\ \sigma\text{-a.e. point on}\ \ \partial_*\Omega, \tag{6.7}$$

for each $\beta \in \{1,\dots,M\}$.

One remarkable feature of this result is that the only quantitative aspect of the hypotheses made in its statement is the finiteness condition in the fourth line of (6.5). Not only is this most natural (in view of the conclusion in (6.6)), but avoiding to specify separate memberships of $\mathcal{N}_\kappa u$ and $\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)$ to concrete dual function spaces on $\partial\Omega$ gives Theorem 6.1 a wide range of applicability. In particular, the various Poisson integral representation formulas this provides in a multitude of contexts permit us to derive, rather painlessly, uniqueness results for the Dirichlet Problem.

6.1 The Dirichlet Problem in Weighted Lebesgue Spaces

Theorem 6.2 below describes solvability, regularity, and well-posedness results for the Dirichlet Problem in δ-AR domains $\Omega \subseteq \mathbb{R}^n$ with boundary data in Muckenhoupt weighted Lebesgue spaces for weakly elliptic second-order homogeneous constant coefficient systems L in $\mathbb{R}^n$ with the property that $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ and/or $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$. Examples of such systems include the Laplacian, all scalar weakly elliptic operators when $n \geq 3$, as well as the complex Lamé system given by $L_{\mu,\lambda} := \mu\Delta + (\lambda + \mu)\nabla{\rm div}$ with $\mu \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{-2\mu, -3\mu\}$. In particular, the well-posedness result described in item *(e)* of Theorem 6.2 holds in all these cases. Furthermore, we provide counterexamples showing that our results are optimal, in the sense that the aforementioned assumptions on the existence of distinguished coefficient tensors cannot be dispensed with.

Theorem 6.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa > 0$. Also, pick an exponent $p \in (1,\infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, consider the Dirichlet Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ \text{in} \ \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^p(\partial\Omega, w)\big]^M. \end{cases} \tag{6.8}$$

The following claims are true:

(a) [Existence, Estimates, and Integral Representation] *If $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\rm dis}$, then there exists $\delta \in (0,1)$ depending only on n, p, $[w]_{A_p}$, A, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a*

scenario which ensures that Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the operator* $\frac{1}{2}I + K_A$ *is invertible on the weighted Lebesgue space* $\big[L^p(\partial\Omega, w)\big]^M$ *and the function* $u : \Omega \to \mathbb{C}^M$ *defined as*

$$u(x) := \Big(\mathcal{D}_A \Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x) \ \ \textit{for all} \ \ x \in \Omega, \tag{6.9}$$

is a solution of the Dirichlet Problem (6.8). *Moreover, there exists some constant* $C \in (0, \infty)$ *independent of* f *with the property that*

$$\|f\|_{[L^p(\partial\Omega,w)]^M} \leq \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} \leq C\|f\|_{[L^p(\partial\Omega,w)]^M}. \tag{6.10}$$

(b) [Additional Integrability] *Under the background assumptions made in item (a), for the solution* u *of the Dirichlet Problem* (6.8) *defined in* (6.9)*, one has the following integrability result: For any given* $q \in (1, \infty)$ *and* $\omega \in A_q(\partial\Omega, \sigma)$*, after eventually further decreasing* $\delta \in (0, 1)$ (*relative to* q *and* $[\omega]_{A_q}$)*, one has*

$$\mathcal{N}_\kappa u \in L^q(\partial\Omega, \omega) \Longleftrightarrow f \in \big[L^q(\partial\Omega, \omega)\big]^M \tag{6.11}$$

and if either of these conditions holds then

$$\|\mathcal{N}_\kappa u\|_{L^q(\partial\Omega,\omega)} \approx \|f\|_{[L^q(\partial\Omega,\omega)]^M}. \tag{6.12}$$

(c) [Regularity] *Under the background assumptions made in item (a), for the solution* u *of the Dirichlet Problem* (6.8) *defined in* (6.9)*, one has the following regularity result: For any given* $q \in (1, \infty)$ *and* $\omega \in A_q(\partial\Omega, \sigma)$*, after eventually further decreasing* $\delta \in (0, 1)$ (*relative to* q *and* $[\omega]_{A_q}$)*, one has*

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega, \omega) \Longleftrightarrow \partial_{\tau_{jk}} f \in \big[L^q(\partial\Omega, \omega)\big]^M, \ \ 1 \leq j, k \leq n, \tag{6.13}$$

and if either of these conditions holds then

$$\begin{aligned} &(\nabla u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \ \textit{exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \ \partial\Omega, \\ &\textit{and} \ \ \|\mathcal{N}_\kappa(\nabla u)\|_{L^q(\partial\Omega,\omega)} \approx \|\nabla_{\tan} f\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}}. \end{aligned} \tag{6.14}$$

In particular, corresponding to $q := p$ *and* $\omega := w$*, if* $\delta \in (0, 1)$ *is sufficiently small to begin with then*

$$\begin{aligned} &\mathcal{N}_\kappa(\nabla u) \textit{ belongs to } L^p(\partial\Omega, w) \textit{ if and only if } f \textit{ belongs to} \\ &\big[L^p_1(\partial\Omega, w)\big]^M\textit{, and if either of these conditions holds then} \\ &\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} + \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p_1(\partial\Omega,w)]^M}. \end{aligned} \tag{6.15}$$

(d) [Uniqueness] *Whenever* $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$, *there exists* $\delta \in (0, 1)$ *which depends only on* n, p, $[w]_{A_p}$, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Dirichlet Problem* (6.8) *has at most one solution.*

(e) [Well-Posedness] *If* $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ *and* $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$ *then there exists* $\delta \in (0, 1)$ *which depends only on* n, p, $[w]_{A_p}$, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*in other words, if* Ω *is a* δ-AR *domain*) *then the Dirichlet Problem* (6.8) *is well posed* (*i.e., it is uniquely solvable and the solution satisfies the naturally accompanying estimate formulated in* (6.10)).

(f) [Sharpness] *If* $\mathfrak{A}_L^{\rm dis} = \varnothing$ *then the Dirichlet Problem* (6.8) *may not be solvable. Also, if* $\mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$ *then the Dirichlet Problem* (6.8) *may have more than one solution. In fact, there exists a homogeneous, second-order, constant real coefficient, weakly elliptic* $n \times n$ *system* L *in* $\mathbb{R}^n$ *with* $\mathfrak{A}_L^{\rm dis} = \mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$ *and which satisfies the following two properties: (i) the Dirichlet Problem formulated for this system as in* (6.8) *with* $\Omega := \mathbb{R}^n_+$ *fails to have a solution for each non-zero boundary datum belonging to an infinite dimensional linear subspace of* $\big[L^p(\partial\Omega, w)\big]^n$, *and (ii) the linear space of null-solutions for the Dirichlet Problem formulated for the system* L *as in* (6.8) *with* $\Omega := \mathbb{R}^n_+$ *is actually infinite dimensional.*

From Example 2.12 we know that, once a point $x_0 \in \partial\Omega$ has been fixed, then for each power $a \in \big(1 - n, (p - 1)(n - 1)\big)$ the function

$$w : \partial\Omega \to [0, \infty], \quad w(x) := |x - x_0|^a \ \text{ for } \ x \in \partial\Omega, \tag{6.16}$$

is a Muckenhoupt weight in the class $A_p(\partial\Omega, \sigma)$. Boundary value problems for a real constant coefficient system L satisfying the Legendre–Hadamard strong ellipticity condition in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with boundary data in weighted (Lebesgue and Sobolev) spaces on $\partial\Omega$ for a weight of the form (6.16) have been considered in [128].

More generally, Proposition 2.21 tells us that, for each d-set $E \subseteq \partial\Omega$ with $d \in [0, n - 1)$ and each power $a \in \big(d + 1 - n, (p - 1)(n - 1 - d)\big)$, the function $w := \big[\mathrm{dist}(\cdot, E)\big]^a$ is a Muckenhoupt weight in the class $A_p(\partial\Omega, \sigma)$. Theorem 6.2 may therefore be specialized to this type of weights. A natural choice corresponds to the case when E is a subset of the set of singularities of the "surface" $\partial\Omega$. Weighted boundary value problems in which the weight is a power of the distance to the singular set (of the boundary) have been studied extensively in the setting of conical and polyhedral domains, for which there is a vast amount of literature (see, e.g., [80, 81], and the references therein).

Finally, we wish to mention that, in the class of systems considered in Theorem 6.2, the ensuing solvability, regularity, uniqueness, and well-posedness results are new even in the standard case when $\Omega = \mathbb{R}^n_+$.

Here is the proof of Theorem 6.2.

Proof of Theorem 6.2 To deal with the claims made in item *(a)* assume $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L^{\rm dis}$. Then Theorems 2.3 and 4.8 guarantee the existence of some threshold $\delta \in (0, 1)$, whose nature is as specified in the statement of the theorem, such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then the set $\partial\Omega$ is unbounded, Ω satisfies a two-sided local John condition with constants which depend only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n (in particular, the UR constants of $\partial\Omega$ are also controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$), and the operator $\frac{1}{2}I + K_A$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$. Granted this, from (3.23) and Proposition 3.5 (also keeping in mind (2.575)) we conclude that the function u defined as in (6.9) solves the Dirichlet Problem (6.8) and satisfies (6.10).

Consider next the claim made in item *(b)*, regarding additional integrability properties for the solution constructed in (6.9). The right-pointing implication in (6.11) together with the right-pointing inequality in (6.12) are simple consequences of the fact that we have $|f| = \Big|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big| \leq \mathcal{N}_\kappa u$ at σ-a.e. point on $\partial\Omega$. The left-pointing implication in (6.11) along with the left-pointing inequality in (6.12) are seen from (6.9), (4.340), and Proposition 3.5.

Let us now prove the claims made in item *(c)* pertaining to the regularity of the solution u just constructed. Retain the background assumptions made in item *(a)* and fix some exponent $q \in (1, \infty)$ along with some weight $\omega \in A_q(\partial\Omega, \sigma)$. As regards the equivalence claimed in (6.13), assume first that $f \in \big[L^p(\partial\Omega, w)\big]^M$ is such that $\partial_{\tau_{jk}} f \in \big[L^q(\partial\Omega, \omega)\big]^M$ for each $j, k \in \{1, \dots, n\}$. To proceed, define $g := \Big(\frac{1}{2}I + K_A\Big)^{-1} f \in \big[L^p(\partial\Omega, w)\big]^M$ where the inverse is considered in the space $\big[L^p(\partial\Omega, w)\big]^M$. As noted in Remark 4.16 (assuming $\delta > 0$ is sufficiently small), the operator $\frac{1}{2}I + K_A$ is also invertible on the off-diagonal Muckenhoupt weighted Sobolev space $\big[L^{p;q}_1(\partial\Omega, w; \omega)\big]^M$ (cf. (4.306)–(4.307)). Moreover, since the latter is a subspace of $\big[L^p(\partial\Omega, w)\big]^M$, it follows that the inverse of $\frac{1}{2}I + K_A$ on $\big[L^{p;q}_1(\partial\Omega, w; \omega)\big]^M$ is compatible with the inverse of $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega, w)\big]^M$. In particular, since we are currently assuming that $f \in \big[L^{p;q}_1(\partial\Omega, w; \omega)\big]^M$, we conclude that $g \in \big[L^{p;q}_1(\partial\Omega, w; \omega)\big]^M$. As a consequence of this membership and (2.575), we have

$$
\begin{aligned}
& g = (g_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \quad \text{and} \\
& \partial_{\tau_{jk}} g \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \quad \text{for all } \ j, k \in \{1, \dots, n\}.
\end{aligned}
\tag{6.17}
$$

Granted these, we may invoke Proposition 3.1 and from (3.34) we conclude that the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\nabla\mathcal{D}_A g\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists (in $\mathbb{C}^{n\cdot M}$) at σ-a.e. point on $\partial\Omega$ (hence, the first property listed in (6.14) holds). Also, formula (3.33) gives that for each index $\ell \in \{1, \dots, n\}$ and each point $x \in \Omega$ we have

$$
\begin{aligned}
(\partial_\ell u)(x) &= \partial_\ell\big(\mathcal{D}_A g\big)(x) \\
&= \left(\int_{\partial\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} g_\alpha)(y)\,d\sigma(y)\right)_{1\leq\gamma\leq M}
\end{aligned}
\tag{6.18}
$$

if the coefficient tensor A is expressed as $\big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$, and if the fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ is as in Theorem 3.1. In concert with (3.85) and (2.586), this proves that

$$
\begin{aligned}
&\|\mathcal{N}_\kappa(\nabla u)\|_{L^q(\partial\Omega,\omega)} \leq C\|\nabla_{\tan} g\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}} \\
&\text{for some constant } C\in(0,\infty) \text{ independent of } g.
\end{aligned}
\tag{6.19}
$$

In particular, $\mathcal{N}_\kappa(\nabla u)$ belongs to the space $L^q(\partial\Omega,\omega)$, which finishes the justification of the right-to-left implication in (6.13). Also, from (4.343) we know that, for some constant $C\in(0,\infty)$ independent of f,

$$
\|\nabla_{\tan} g\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}} \leq C\|\nabla_{\tan} f\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}}. \tag{6.20}
$$

In light of (6.19), this justifies the left-pointing inequality in the equivalence claimed in (6.14). To complete the treatment of item *(b)*, there remains to observe that the right-pointing implication in (6.13) together with the right-pointing inequality in the equivalence claimed in (6.14) are consequences of Proposition 2.23 (bearing in mind (2.585)).

Consider next the uniqueness result claimed in item *(d)*. Suppose $\mathfrak{A}_{L^\top}^{\rm dis}\neq\varnothing$ and pick some $A\in\mathfrak{A}_L$ such that $A^\top\in\mathfrak{A}_{L^\top}^{\rm dis}$. Also, denote by $p'\in(1,\infty)$ the Hölder conjugate exponent of p, and set $w':=w^{1-p'}\in A_{p'}(\partial\Omega,\sigma)$. From Theorem 4.8, presently used with L replaced by $L^\top$, p' in place of p, and w' in place of w, we know that there exists $\delta\in(0,1)$, which depends only on n, p, $[w]_{A_p}$, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if Ω is a δ-AR domain then

$$
\tfrac{1}{2}I + K_{A^\top} : \big[L_1^{p'}(\partial\Omega,w')\big]^M \longrightarrow \big[L_1^{p'}(\partial\Omega,w')\big]^M \tag{6.21}
$$

is an invertible operator.

By eventually decreasing the value of $\delta\in(0,1)$ if necessary, we may ensure that Ω is an NTA domain with unbounded boundary (cf. Theorem 2.3). In such a case, (6.2) guarantees that Ω is globally pathwise nontangentially accessible.

To proceed, let $E=\big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ be the fundamental solution associated with the system L as in Theorem 3.1. Fix $x_\star\in\mathbb{R}^n\setminus\overline{\Omega}$ along with $x_0\in\Omega$, arbitrary. Also, pick $\rho\in\big(0,\frac{1}{4}\,{\rm dist}(x_0,\partial\Omega)\big)$ and define $K:=\overline{B(x_0,\rho)}$. Finally, recall the aperture parameter $\widetilde{\kappa}>0$ associated with Ω and κ as in Theorem 6.1. Next, for each fixed $\beta\in\{1,\dots,M\}$, consider the $\mathbb{C}^M$-valued function

$$f^{(\beta)}(x) := \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big)_{1\leq\alpha\leq M}, \qquad \forall\, x \in \partial\Omega. \tag{6.22}$$

From (6.22), (2.587), (2.579), (2.572), (3.16), and the Mean Value Theorem we then conclude that

$$f^{(\beta)} \in \big[L_1^{p'}(\partial\Omega, w')\big]^M. \tag{6.23}$$

As a consequence, with $\big(\frac{1}{2}I + K_{A^\top}\big)^{-1}$ denoting the inverse of the operator in (6.21),

$$v_\beta := \big(v_{\beta\alpha}\big)_{1\leq\alpha\leq M} := \mathcal{D}_{A^\top}\Big(\big(\tfrac{1}{2}I + K_{A^\top}\big)^{-1} f^{(\beta)}\Big) \tag{6.24}$$

is a well-defined $\mathbb{C}^M$-valued function in Ω which, thanks to Proposition 3.5, satisfies

$$\begin{gathered} v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_{\widetilde{\kappa}} v_\beta \in L^{p'}(\partial\Omega, w'), \quad \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in L^{p'}(\partial\Omega, w'), \\ \text{and } \ v_\beta\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{gathered} \tag{6.25}$$

Moreover, from (6.23)–(6.24) and (3.114) we see that

$$\big(\nabla v_\beta\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists (in} \mathbb{C}^{n\cdot M}) \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{6.26}$$

Subsequently, for each pair of indices $\alpha, \beta \in \{1, \dots, M\}$ define

$$G_{\alpha\beta}(x) := v_{\beta\alpha}(x) - \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big), \qquad \forall\, x \in \Omega \setminus \{x_0\}. \tag{6.27}$$

If we now consider $G := \big(G_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ regarded as a $\mathbb{C}^{M\times M}$-valued function defined $\mathcal{L}^n$-a.e. in Ω, then from (6.27) and Theorem 3.1 we see that G belongs to the space $\big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^{M\times M}$. Also, by design,

$$\begin{gathered} L^\top G = -\delta_{x_0} I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^{M\times M} \ \text{ and} \\ G\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \big(\nabla G\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{gathered} \tag{6.28}$$

while if $v := \big(v_{\beta\alpha}\big)_{1\leq\alpha,\beta\leq M}$ then from (2.8), (3.16), and the Mean Value Theorem it follows that at each point $x \in \partial\Omega$ we have

$$\begin{aligned}\big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K} G\big)(x) &\leq \big(\mathcal{N}_{\widetilde{\kappa}} v\big)(x) + C_{x_0,\rho}(1+|x|)^{1-n} \quad \text{and} \\ \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K} (\nabla G)\big)(x) &\leq \big(\mathcal{N}_{\widetilde{\kappa}} (\nabla v)\big)(x) + C_{x_0,\rho}(1+|x|)^{-n},\end{aligned} \tag{6.29}$$

where $C_{x_0,\rho} \in (0,\infty)$ is independent of x. In view of (6.25), (6.29), and (2.572) we see that the conditions listed in (6.4) are presently satisfied and, in fact,

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in L^{p'}(\partial\Omega, w') = \big(L^p(\partial\Omega, w)\big)^*. \tag{6.30}$$

Suppose now that $u = (u_\beta)_{1\leq\beta\leq M}$ is a $\mathbb{C}^M$-valued function in Ω satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \text{and } \ \mathcal{N}_\kappa u \ \text{ belongs to the space } \ L^p(\partial\Omega, w). \end{gathered} \tag{6.31}$$

Since (6.30) implies

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\, d\sigma < +\infty, \tag{6.32}$$

we may then invoke Theorem 6.1 to conclude that the Poisson integral representation formula (6.6) holds. In particular, this proves that whenever $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0$ at σ-a.e. point on $\partial\Omega$ we necessarily have $u(x_0) = 0$. Given that x_0 has been arbitrarily chosen in Ω, this ultimately shows such a function u is actually identically zero in Ω. This finishes the proof of the claim made in item *(d)*.

Next, the well-posedness claim in item *(e)* is a consequence of what we have proved in items *(a)* and *(d)*. Finally, the two optimality results formulated in item *(f)* are seen from (3.381), (3.393), and (3.406) (cf. also Proposition 3.10 and Example 3.5 in the two-dimensional setting). □

Remark 6.1 The approach used to prove Theorem 6.2 relies on mapping properties and invertibility results for boundary layer potentials on Muckenhoupt weighted Lebesgue and Sobolev spaces. Given that analogous of these results are also valid on Lorentz spaces and Lorentz-based Sobolev spaces (cf. Remark 4.16, and the Lorentz space version of (3.85) obtained via real interpolation), the type of argument used to establish Theorem 6.2 produces similar results for the Dirichlet Problem with data in *Lorentz spaces*, i.e., for

$$\begin{cases} u \in \left[\mathscr{C}^{\infty}(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p,q}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \left[L^{p,q}(\partial\Omega, \sigma)\right]^M. \end{cases} \tag{6.33}$$

More specifically, for this boundary problem existence holds in the setting of item *(a)* of Theorem 6.2 whenever $p \in (1, \infty)$ and $q \in (0, \infty]$, whereas uniqueness holds in the setting of item *(d)* of Theorem 6.2 provided $p \in (1, \infty)$ and $q \in (0, \infty]$ (see [55, Theorem 1.4.17, p. 52] for duality results for Lorentz spaces).

In particular, corresponding to $q = \infty$, whenever $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$ it follows that for each $p \in (1, \infty)$ the weak-L^p Dirichlet Problem

$$\begin{cases} u \in \left[\mathscr{C}^{\infty}(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p,\infty}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \left[L^{p,\infty}(\partial\Omega, \sigma)\right]^M \end{cases} \tag{6.34}$$

is well posed assuming Ω is a δ-AR domain for a sufficiently small $\delta \in (0, 1)$, relative to n, p, L, and the Ahlfors regularity constant of $\partial\Omega$. As in the proof of Theorem 6.2, uniqueness is obtained relying on the Poisson integral representation formula from Theorem 6.1. This requires checking that the Green function with components as in (6.27) is well defined and satisfies $\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in L^{p',1}(\partial\Omega, \sigma)$, where p' is the Hölder conjugate exponents of p. Once this task is accomplished, the fact that we presently have $\mathcal{N}_\kappa u \in L^{p,\infty}(\partial\Omega, \sigma) = \left(L^{p',1}(\partial\Omega, \sigma)\right)^*$ (cf. [55, Theorem 1.4.17(v), p. 52]) guarantees that the finiteness condition (6.32) presently holds, and the desired conclusion follows. In turn, the membership of $\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)$ to $L^{p',1}(\partial\Omega, \sigma)$ is seen from (6.29) and (6.24), keeping in mind that the operator $\frac{1}{2}I + K_{A^\top}$ (where $A \in \mathfrak{A}_L$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\text{dis}}$) is invertible on the Lorentz-based Sobolev space $\left[L_1^{p',1}(\partial\Omega, \sigma)\right]^M$ and, as seen from standard real interpolation inclusions, $(1 + |x|)^{-N} \in L^{p,q}(\partial\Omega, \sigma)$ whenever $N \geq n - 1$, $p \in (1, \infty)$, and $q \in (0, \infty]$.

See Theorem 8.18 (and also Examples 8.2, 8.6) for a more general perspective on this topic.

To offer an example, let $\Omega \subseteq \mathbb{R}^n$ be a δ-AR domain and fix an arbitrary aperture parameter $\kappa > 0$ along with a power $a \in (0, n - 1)$. Set $p := (n - 1)/a \in (1, \infty)$. Then, if $\delta \in (0, 1)$ is sufficiently small (relative to n, a, and the Ahlfors regularity constant of $\partial\Omega$, it follows that for each point $x_o \in \partial\Omega$ the Dirichlet Problem

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa u \in L^{p,\infty}(\partial\Omega, \sigma), \\ \Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = |x - x_o|^{-a} \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega, \end{cases} \tag{6.35}$$

is uniquely solvable. In addition, there exists a constant $C(\Omega, n, \kappa, a) \in (0, \infty)$ with the property that if u_{x_o} denotes the unique solution of (6.35) then we have the estimate $\|\mathcal{N}_\kappa u_{x_o}\|_{L^{p,\infty}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a)$ for each $x_o \in \partial\Omega$. Indeed, since the function $f_{x_o}(x) := |x - x_o|^{-a}$ for σ-a.e. point $x \in \partial\Omega$ belongs to the Lorentz space $L^{p,\infty}(\partial\Omega, \sigma)$ and $\sup_{x_o\in\partial\Omega} \|f_{x_o}\|_{L^{p,\infty}(\partial\Omega,\sigma)} < \infty$, the solvability result in Remark 6.1 applies. This example is particularly relevant in view of the fact that the boundary datum $|\cdot - x_o|^{-a}$ does *not* belong to any ordinary Lebesgue space on $\partial\Omega$ with respect to the "surface measure" σ. In addition, since for each $j, k \in \{1, \dots, n\}$ the boundary datum f_{x_o} satisfies

$$\begin{gathered} \partial_{\tau_{jk}} f_{x_o} \in L^{q,\infty}(\partial\Omega, \sigma) \text{ and } \sup_{x_o\in\partial\Omega} \big\|\partial_{\tau_{jk}} f_{x_o}\big\|_{L^{q,\infty}(\partial\Omega,\sigma)} < \infty, \\ \text{where } q := (n-1)/(a+1) \in (1, \infty), \end{gathered} \tag{6.36}$$

given that, if $(\nu_i)_{1\leq i\leq n}$ are the components of the geometric outward unit normal vector to Ω,

$$\big(\partial_{\tau_{jk}} f_{x_o}\big)(x) = a\frac{(x - x_o)_j \nu_k(x) - (x - x_o)_k \nu_j(x)}{|x - x_o|^{a+2}} \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{6.37}$$

then the analogues of (6.13)–(6.14) in the current setting imply that the unique solution u_{x_o} of the Dirichlet Problem (6.35) enjoys additional regularity. Specifically, if $\delta \in (0, 1)$ is sufficiently small to begin with, then

$$\begin{gathered} \text{for each } x_o \in \partial\Omega, \text{ the nontangential boundary trace} \\ (\nabla u_{x_o})\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{R}^n) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\ \text{and } \sup_{x_o\in\partial\Omega} \big\|\mathcal{N}_\kappa(\nabla u_{x_o})\big\|_{L^{q,\infty}(\partial\Omega,\sigma)} < +\infty \text{ if } q := \tfrac{n-1}{a+1}. \end{gathered} \tag{6.38}$$

In relation to the Dirichlet Problem with data in weak-Lebesgue spaces formulated in (6.34), we also wish to note that, in contrast to the well-posedness result in the range $p \in (1, \infty)$, uniqueness no longer holds in the limiting case when $p = 1$. Indeed, if we take $\Omega := \mathbb{R}^n_+$ and $u(x) := x_n/|x|^n$ for each $x = (x_1, \dots, x_n) \in \Omega$ then, since under the identification $\partial\Omega \equiv \mathbb{R}^{n-1}$ we have $\big(\mathcal{N}_\kappa u\big)(x') \approx |x'|^{1-n}$ uniformly for $x' \in \mathbb{R}^{n-1} \setminus \{0\}$, we see that

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \\ \Delta u = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{1,\infty}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point } \ x \in \partial\Omega, \end{cases} \tag{6.39}$$

and yet, obviously, $u \not\equiv 0$ in Ω.

We may also establish solvability results for the Dirichlet Problem formulated for boundary data belonging to sums of Muckenhoupt weighted Lebesgue spaces, of the sort described below.

Theorem 6.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick $p_0, p_1 \in (1, \infty)$ along with a pair of Muckenhoupt weights $w_0 \in A_{p_0}(\partial\Omega, \sigma)$ and $w_1 \in A_{p_1}(\partial\Omega, \sigma)$. Finally, consider a homogeneous, second-order, constant complex coefficient, $M \times M$ weakly elliptic system L in $\mathbb{R}^n$.*

Then similar results, concerning existence, integral representation formulas, estimates, additional integrability properties, regularity, uniqueness, well-posedness, and sharpness, as in Theorem 6.2, are valid for the Dirichlet Problem:

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M. \end{cases} \tag{6.40}$$

Proof Assume $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\text{dis}}$. Then, as noted in the proof of Proposition 4.2, if Ω is a δ-AR domain with $\delta \in (0, 1)$ small enough matters may be arranged so that Ω satisfies a two-sided local John condition with constants which depend only on the Ahlfors regularity constant of $\partial\Omega$ and the dimension n (in particular, the UR constants of $\partial\Omega$ are also controlled solely in terms of the dimension n and the Ahlfors regularity constant of $\partial\Omega$), and the operator $\frac{1}{2}I + K_A$ is invertible when acting on the space $\big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$. Granted this, we claim that the function $u : \Omega \to \mathbb{C}^M$ defined as in (6.9) (with this interpretation of the inverse and for the current boundary datum f) solves (6.40). Thanks to (3.23), (3.31), (2.575), this function u satisfies the conditions in the first, second, and last line of (6.40). To verify the condition stipulated in the penultimate line of (6.40), decompose

$$\big(\tfrac{1}{2}I + K_A\big)^{-1} f \in \big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M \tag{6.41}$$

as

$$\big(\tfrac{1}{2}I + K_A\big)^{-1} f = g_0 + g_1 \text{ with } g_i \in \big[L^{p_i}(\partial\Omega, w_i)\big]^M \text{ for } i \in \{0, 1\}. \tag{6.42}$$

Then $u = \mathcal{D}_A g_0 + \mathcal{D}_A g_1$ so $\mathcal{N}_\kappa u \leq \mathcal{N}_\kappa(\mathcal{D}_A g_0) + \mathcal{N}_\kappa(\mathcal{D}_A g_1)$ on $\partial\Omega$. Consequently,

$$U_0 := \big(\mathcal{N}_\kappa u\big) \cdot \mathbf{1}_{\{\mathcal{N}_\kappa(\mathcal{D}_A g_0) \geq \mathcal{N}_\kappa(\mathcal{D}_A g_1)\}} \in L^{p_0}(\partial\Omega, w_0), \tag{6.43}$$

$$U_1 := \big(\mathcal{N}_\kappa u\big) \cdot \mathbf{1}_{\{\mathcal{N}_\kappa(\mathcal{D}_A g_0) < \mathcal{N}_\kappa(\mathcal{D}_A g_1)\}} \in L^{p_1}(\partial\Omega, w_1), \tag{6.44}$$

and

$$\mathcal{N}_\kappa u = U_0 + U_1 \in L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1). \tag{6.45}$$

This establishes the membership in the third line of (6.40). Incidentally, the argument above also yields a naturally accompanying estimate, namely

$$\|\mathcal{N}_\kappa u\|_{L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)} \leq C\|f\|_{L^{p_0}(\partial\Omega,w_0)+L^{p_1}(\partial\Omega,w_1)} \tag{6.46}$$

for some $C \in (0, \infty)$ independent of f.

To prove uniqueness for the boundary problem (6.40) under the assumption that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ and Ω is a δ-AR domain with $\delta \in (0, 1)$ sufficiently small, we reason as in the proof of item *(d)* of Theorem 6.2. The chief novel aspect is that since for $f^{(\beta)}$ as in (6.22) we have

$$f^{(\beta)} \in \big[L_1^{p_0'}(\partial\Omega, w_0') \cap L_1^{p_1'}(\partial\Omega, w_1')\big]^M \tag{6.47}$$

(where p_0', p_1' are the Hölder conjugate exponents of p_0, p_1, and w_0', w_1' are the dual weights for w_0, w_1), from the compatibility property recorded in (4.341) we conclude that the function v_β defined as in (6.24) enjoys additional regularity/integrability properties compared to (6.25), namely:

$$\begin{gathered} v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_{\widetilde{\kappa}} v_\beta \in L^{p_0'}(\partial\Omega, w_0') \cap L^{p_1'}(\partial\Omega, w_1'), \\ \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in L^{p_0'}(\partial\Omega, w_0') \cap L^{p_1'}(\partial\Omega, w_1'), \\ \text{and} \ \ v_\beta\big|^{\widetilde{\kappa}-\text{n.t.}}_{\partial\Omega} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{gathered} \tag{6.48}$$

In turn, this permits us to improve (6.30) to

$$\begin{aligned} \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in L^{p_0'}(\partial\Omega, w_0') &\cap L^{p_1'}(\partial\Omega, w_1') \\ &\hookrightarrow \big(L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big)^* \end{aligned} \tag{6.49}$$

which ultimately goes to show that the finiteness condition from (6.32) continues to hold in the present setting. As such, we may once again rely on the Poisson integral representation formula from Theorem 6.1 to conclude that the solution u of (6.40) vanishes in Ω whenever $f = 0$.

All other claims in the statement of the present theorem have proofs very similar to their counterparts in Theorem 6.2. □

Moving on, it is remarkable that the solvability results described in Theorem 6.2 turn out to be stable under small perturbations. This is made precise in the next theorem.

Theorem 6.4 *Retain the original background assumptions on the set Ω from Theorem 6.2 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then the following statements are true.*

(a) [Existence] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ (cf. (3.195)) there exist some small threshold $\delta \in (0, 1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (6.8) formulated for L is solvable.*

(b) [Uniqueness] *For each given system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta \in (0, 1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (6.8) formulated for L has at most one solution.*

(c) [Well-Posedness] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta \in (0, 1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (6.8) formulated for L is well posed.*

Proof To deal with the claim made in item *(a)*, start by observing that the assumption $L_o \in \mathfrak{L}^{\rm dis}$ guarantees the existence of some $A_o \in \mathfrak{A}^{\rm dis}_{L_o}$. Theorem 4.9 (used with, say, $\varepsilon := 1/4$) ensures the existence of some small threshold $\delta \in (0, 1)$ along with some open neighborhood $\mathcal{O}$ of A_o in $\mathfrak{A}_{\rm WE}$, both of which depend only on n, p, $[w]_{A_p}$, A_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for each $\widetilde{A} \in \mathcal{O}$ the operator $\frac{1}{2}I + K_{\widetilde{A}}$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$. Pick $\varepsilon > 0$ such that $\{A \in \mathfrak{A} : \|A - A_o\| < \varepsilon\} \subseteq \mathcal{O}$, and define $\mathcal{U} := \{L \in \mathfrak{L} : \|L - L_o\| < \varepsilon\}$. Choose now an arbitrary system $L \in \mathcal{U}$. By design, there exist $A \in \mathfrak{A}_L$ and $B \in \mathfrak{A}^{\rm ant}$ such that $\|A - A_o - B\| < \varepsilon$. Hence, if we now introduce $\widetilde{A} := A - B$, then $\widetilde{A} \in \mathfrak{A}_L$ and the fact that $\|\widetilde{A} - A_o\| < \varepsilon$ implies that $\widetilde{A} \in \mathcal{O}$. In particular, the latter property permits us to conclude (in light of our earlier discussion) that the operator $\frac{1}{2}I + K_{\widetilde{A}}$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$. Given that we also have $\widetilde{A} \in \mathfrak{A}_L$, it follows (much as in the proof of Theorem 6.2) that the function $u : \Omega \to \mathbb{C}^M$ defined as

$$u(x) := \Big(\mathcal{D}_{\widetilde{A}}\Big(\tfrac{1}{2}I + K_{\widetilde{A}}\Big)^{-1} f\Big)(x) \quad \text{for all} \quad x \in \Omega \tag{6.50}$$

is a solution of the Dirichlet Problem (6.8) formulated for the current system L. This finishes the proof of the claim made in item *(a)*.

On to the claim in item *(b)*, pick some $A_o \in \mathfrak{A}_{L_o}$ with $A_o^\top \in \mathfrak{A}^{\rm dis}_{L_o^\top}$. Running the same argument as above (with $L_o^\top$ playing the role of L_o, $A_o^\top$ playing the role of A_o, and keeping in mind that transposition is an isometry) yields some small threshold $\delta \in (0,1)$ along with some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, $[w]_{A_p}$, A_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then for each system $L \in \mathcal{U}$ we may find a coefficient tensor $\widetilde{A} \in \mathfrak{A}_L$ with the property that the operator $\frac{1}{2}I + K_{(\widetilde{A})^\top}$ is invertible on the Muckenhoupt weighted Sobolev space $\big[L_1^{p'}(\partial\Omega, w')\big]^M$. This is a perturbation of the invertibility result in (6.21) and, once this has been established, the same argument as in the proof of item *(c)* of Theorem 6.2 applies and gives the conclusion we presently seek. Finally, the claim in item *(c)* is a direct consequence of what we have proved in items *(a)*–*(b)*. □

6.2 The Regularity Problem in Weighted Sobolev Spaces

Traditionally, the label "Regularity Problem" is intended for a version of the Dirichlet Problem in which both the boundary datum and the solution sought are more "regular" than in the standard formulation of the Dirichlet Problem. For us here, this means that we shall now select boundary data from Muckenhoupt weighted Sobolev spaces and also demand control of the nontangential maximal operator of the gradient of the solution. Given that this involves an inhomogeneous Sobolev space, we shall label it the `Inhomogeneous Regularity Problem`.

The specific manner in which we have formulated the solvability result for the Dirichlet Problem in Theorem 6.2, in particular, having already elaborated on how extra regularity of the boundary datum affects the regularity of the solution (cf. (6.13)), renders the Inhomogeneous Regularity Problem a "sub-problem" of the Dirichlet Problem. As seen below, this makes light work of the treatment of the Inhomogeneous Regularity Problem. Later on, in Theorem 6.8, we shall consider what we call the Homogeneous Regularity Problem which is related to, yet fundamentally distinct, from the Inhomogeneous Regularity Problem dealt with in the following theorem:

Theorem 6.5 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa > 0$. Also, pick an exponent $p \in (1,\infty)$ and a Muckenhoupt* *weight* $w \in A_p(\partial\Omega, \sigma)$. *Given a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system* L *in* $\mathbb{R}^n$, *consider the Inhomogeneous Regularity Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[L^p_1(\partial\Omega, w)\big]^M. \end{cases} \tag{6.51}$$

The following statements are true:

(a) [Existence, Estimates, and Integral Representation] *If* $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$ *and* $A \in \mathfrak{A}^{\mathrm{dis}}_L$, *then there exists* $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, A, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then* $\frac{1}{2}I + K_A$ *is an invertible operator on the Muckenhoupt weighted Sobolev space* $\big[L^p_1(\partial\Omega, w)\big]^M$ *and the function*

$$u(x) := \Big(\mathcal{D}_A\Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x), \qquad \forall\, x \in \Omega, \tag{6.52}$$

is a solution of the Inhomogeneous Regularity Problem (6.51). *In addition,*

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p(\partial\Omega,w)]^M}, \tag{6.53}$$

and

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}. \tag{6.54}$$

In particular,

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} + \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p_1(\partial\Omega,w)]^M}. \tag{6.55}$$

(b) [Uniqueness] *Whenever* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$, *there exists* $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, A, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e.,* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the Inhomogeneous Regularity Problem* (6.51) *has at most one solution.*

(c) [Well-Posedness] *If* $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$ *and* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$ *then there exists* $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, A, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*hence* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the Inhomogeneous Regularity Problem* (6.51) *is uniquely solvable and the solution satisfies* (6.53)–(6.55).

(d) [Sharpness] *If* $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ *the Inhomogeneous Regularity Problem* (6.51) *may fail to be solvable* (*actually for boundary data belonging to an infinite dimensional subspace of the corresponding weighted Sobolev space*) *even when* Ω *is a half-space, and if* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ *the Inhomogeneous Regularity Problem* (6.51) *may possess more than one solution* (*in fact, the linear space of null-solutions may actually be infinite dimensional*) *even when* Ω *is a half-space. In particular, if either* $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ *or* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$, *then the Inhomogeneous Regularity Problem* (6.51) *may fail to be well posed even when* Ω *is a half-space.*

Under the assumption that Ω is a δ-AR domain for some sufficiently small $\delta \in (0, 1)$ (which is in effect for items *(a)–(c)* of the theorem), it follows from Proposition 2.24, Theorem 2.3, Proposition 2.23, and (2.576) that the first three assumptions in (6.51) always imply that $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists and belongs to $\big[L^p_1(\partial\Omega, w)\big]^M$. It is therefore natural that the boundary datum f is currently taken from this Muckenhoupt weighted boundary Sobolev space.

Proof of Theorem 6.5 All claims made in items *(a)–(c)* are direct consequences of Theorem 4.8 and Theorem 6.2. As regards the sharpness results formulated in item *(d)*, the fact that the Inhomogeneous Regularity Problem (6.51) may fail to be solvable when $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ is seen from Proposition 3.11 and (3.268). Finally, that the Inhomogeneous Regularity Problem (6.51) for L may have more than one solution if $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ is seen from (3.383), (3.392), and (3.406) (cf. also Example 3.5 and Proposition 3.10 in the two-dimensional setting). □

Remark 6.2 From Remark 6.1 we see that the Inhomogeneous Regularity Problem with data in Lorentz-based Sobolev spaces, i.e.,

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u, \ \mathcal{N}_\kappa(\nabla u) \in L^{p,q}(\partial\Omega, \sigma), \\ u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = f \in \big[L^{p,q}_1(\partial\Omega, \sigma)\big]^M, \end{cases} \tag{6.56}$$

enjoys similar solvability and well-posedness results to those described in Theorem 6.5. Concretely, for this boundary problem we have existence in the setting of item *(a)* of Theorem 6.5 whenever $p \in (1, \infty)$ and $q \in (0, \infty]$, and we have uniqueness in the setting of item *(b)* of Theorem 6.5 whenever $p, q \in (1, \infty)$.

See Theorem 8.19 (as well as Examples 8.2 and 8.6) for more general results of this nature.

Remark 6.3 An inspection of the proof of Theorem 6.5 reveals that similar solvability and well-posedness results are valid in the case when the boundary data belong to the off-diagonal Muckenhoupt weighted Sobolev spaces discussed in (4.306)–(4.307). More specifically, given two integrability exponents $p_1, p_2 \in (1, \infty)$ along with two Muckenhoupt weights $w_1 \in A_{p_1}(\partial\Omega, \sigma)$ and $w_2 \in A_{p_2}(\partial\Omega, \sigma)$, the off-diagonal Inhomogeneous Regularity Problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p_1}(\partial\Omega, w_1), \\ \mathcal{N}_\kappa(\nabla u) \in L^{p_2}(\partial\Omega, w_2), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L_1^{p_1;p_2}(\partial\Omega, w_1; w_2)\big]^M, \end{cases} \tag{6.57}$$

continues to enjoy similar solvability and well-posedness results to those described in Theorem 6.5. Of course, this time, the a priori estimates (6.53)–(6.54) read

$$\|\mathcal{N}_\kappa u\|_{L^{p_1}(\partial\Omega, w_1)} \approx \|f\|_{[L^{p_1}(\partial\Omega, w_1)]^M}, \tag{6.58}$$

and

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^{p_2}(\partial\Omega, w_2)} \approx \|\nabla_{\tan} f\|_{[L^{p_2}(\partial\Omega, w_2)]^{n\cdot M}}. \tag{6.59}$$

Remark 6.4 Once again, in the class of systems considered in Theorem 6.5, the solvability, uniqueness, and well-posedness results for the Inhomogeneous Regularity Problem (6.51) are new even in the standard case when $\Omega = \mathbb{R}^n_+$.

As in the case of the Dirichlet Problem, it turns out that the solvability results presented in Theorem 6.5 are stable under small perturbations, of the sort described below.

Theorem 6.6 *Retain the original background assumptions on the set Ω from Theorem 6.5 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then the following statements are true.*

(a) [Existence] *Given any system $L_o \in \mathfrak{L}^{\text{dis}}$ (cf. (3.195)), there exist a threshold $\delta \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then for each system $L \in \mathcal{U}$ the Inhomogeneous Regularity Problem (6.51) formulated for L is solvable.*

(b) [Uniqueness] *Given any system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\text{dis}}$ there exist a threshold $\delta \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then for each system $L \in \mathcal{U}$ the Inhomogeneous Regularity Problem (6.51) formulated for L has at most one solution.*

(c) [Well-Posedness] *Given any system* $L_o \in \mathfrak{L}^{\rm dis}$ *with* $L_o^\top \in \mathfrak{L}^{\rm dis}$ *there exist a threshold* $\delta \in (0,1)$ *and an open neighborhood* $\mathcal{U}$ *of* L_o *in* $\mathfrak{L}$*, both of which depend only on* n*,* p*,* $[w]_{A_p}$*,* L_o*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ-AR *domain) then for each system* $L \in \mathcal{U}$ *the Inhomogeneous Regularity Problem* (6.51) *formulated for* L *is well posed.*

Proof The same type of argument used in the proof of Theorem 6.4 continues to work in this setting. □

The integral representation contained in the theorem below, itself proved in [113, §1.5], is going to be of great relevance in dealing with the issue of uniqueness in boundary value problems where only assumptions on the nontangential maximal operator of the gradient of the solution are made.

Theorem 6.7 *Let* $\Omega \subseteq \mathbb{R}^n$*, where* $n \geq 2$*, be an Ahlfors regular domain with* $\partial\Omega$ *unbounded. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu = (\nu_1, \dots, \nu_n)$ *the geometric measure theoretic outward unit normal to* Ω*. With the summation convention over repeated indices understood throughout, let*

$$L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{6.60}$$

be a homogeneous, weakly elliptic, second-order $M \times M$ *system in* $\mathbb{R}^n$*, with complex constant coefficients, and denote by* $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ *the matrix-valued fundamental solution associated with* L *as in Theorem 3.1.*

In this setting, assume $u = (u_\beta)_{1\leq\beta\leq M} \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ *is a vector-valued function which, for some* $\kappa > 0$*, satisfies*

$$\begin{gathered} Lu = 0 \ \text{ in } \ \Omega, \quad (\nabla u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \text{and } \ \mathcal{N}_\kappa(\nabla u) \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big). \end{gathered} \tag{6.61}$$

Then for each $\ell \in \{1, \dots, n\}$ *and each* $\gamma \in \{1, \dots, M\}$ *one has*

$$\begin{aligned} (\partial_\ell u_\gamma)(x) = \int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\Big\{\nu_\ell(y)\Big((\partial_s u_\alpha)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(y) \\ - \nu_s(y)\Big((\partial_\ell u_\alpha)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(y)\Big\}\,d\sigma(y) \\ - \int_{\partial\Omega}(\partial_\ell E_{\gamma\alpha})(x-y)\nu_r(y)a^{\alpha\beta}_{rs}\Big((\partial_s u_\beta)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(y)\,d\sigma(y) \end{aligned} \tag{6.62}$$

at every point $x \in \Omega$*, and*

$$0 = \int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\Big\{\nu_\ell(y)\Big((\partial_s u_\alpha)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(y) \tag{6.63}$$

$$- \nu_s(y)\big((\partial_\ell u_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\Big\}\, d\sigma(y)$$

$$- \int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\nu_r(y) a_{rs}^{\alpha\beta}\big((\partial_s u_\beta)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\, d\sigma(y)$$

at every point $x \in \mathbb{R}^n \setminus \overline{\Omega}$.

We are now ready to formulate and solve the `Homogeneous Regularity Problem`. Compared to its inhomogeneous counterpart, considered in (6.51), this boundary value problem involves boundary data from homogeneous Muckenhoupt weighted Sobolev spaces and only requires control of the nontangential maximal operator of the gradient of the solution. This being said, it turns out that the Homogeneous Regularity Problem "contains" the Inhomogeneous Regularity Problem in the sense that the latter becomes equivalent to the former whenever the boundary data are prescribed from the (smaller) inhomogeneous Muckenhoupt weighted Sobolev space. Here is a formal statement of our result, which sheds light on the issue singled out as Question 2.5 in [137]:

Theorem 6.8 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, *denote by* ν *the geometric measure theoretic outward unit normal to* Ω, *and fix an aperture parameter* $\kappa > 0$. *Also, pick an exponent* $p \in (1,\infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$. *Given a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system* L *in* $\mathbb{R}^n$, *consider the Homogeneous Regularity Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M, \end{cases} \tag{6.64}$$

where $\dot{L}_1^p(\partial\Omega, w)$ *is the homogeneous Muckenhoupt weighted boundary Sobolev space defined in* (2.598). *The following statements are true:*

(a) [Existence, Estimates, and Integral Representations] *If* $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ *then there exists* $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15) then the following properties are true. First, the operator*

$$\big[S_{\text{mod}}\big] : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M \tag{6.65}$$

is surjective and the Homogeneous Regularity Problem (6.64) *is solvable. More specifically, with* $[f] \in \big[\dot{L}_1^p(\partial\Omega, w)/\sim\big]^M$ *denoting the equivalence class* (*modulo constants*) *of the boundary datum* f, *and with*

$$g \in \big[L^p(\partial\Omega, w)\big]^M \ \text{ chosen so that } \ \big[S_{\rm mod}\big]g = [f], \tag{6.66}$$

there exists a constant $c \in \mathbb{C}^M$ *such that the function*

$$u := \mathscr{S}_{\rm mod}\, g + c \ \text{ in } \ \Omega \tag{6.67}$$

is a solution of the Homogeneous Regularity Problem (6.64)*. In addition, this solution satisfies* (*with implicit constants independent of* f)

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}. \tag{6.68}$$

Second, for each coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$ *the operator*

$$\tfrac12 I + \big[K_{A,\rm mod}\big] : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \tag{6.69}$$

is an isomorphism, and the Homogeneous Regularity Problem (6.64) *may be solved as*

$$u := \mathcal{D}_{A,\rm mod}\, h + c \ \text{ in } \ \Omega, \tag{6.70}$$

for a suitable constant $c \in \mathbb{C}^M$ *and with*

$$h \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \text{ such that } [h] = \Big(\tfrac12 I + \big[K_{A,\rm mod}\big]\Big)^{-1}[f]. \tag{6.71}$$

Moreover, this solution continues to satisfy (6.68).

(b) [Uniqueness] *Whenever* $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$, *there exists* $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Homogeneous Regularity Problem* (6.64) *has at most one solution.*

(c) [Well-Posedness and Additional Integrability/Regularity] *Whenever* $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ *and* $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$ *it follows that there exists* $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Homogeneous Regularity Problem* (6.64) *is uniquely solvable. Moreover, for each* $q \in (1,\infty)$ *and* $\omega \in A_q(\partial\Omega,\sigma)$, *the unique solution* u *of* (6.64) *satisfies* (*in a quantitative fashion*)

$$\mathcal{N}_\kappa u \in L^q(\partial\Omega,\omega) \Longleftrightarrow f \in \big[L_1^{q;p}(\partial\Omega,\omega;w)\big]^M \tag{6.72}$$

with the off-diagonal weighted Sobolev space $L_1^{q;p}(\partial\Omega,\omega;w)$ *defined as in* (4.306)*, as well as*

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega, \omega) \Longleftrightarrow f \in \big[\dot{L}^q_1(\partial\Omega, \omega)\big]^M, \tag{6.73}$$

provided $\delta \in (0, 1)$ *is sufficiently small to begin with, relative to* q *and* $[\omega]_{A_q}$.

In particular, corresponding to $q := p$*, the equivalence in* (6.72) *proves that the unique solution of the Homogeneous Regularity Problem* (6.64) *for a boundary datum* f *belonging to* $\big[L^p_1(\partial\Omega, w)\big]^M$ (*which is a subspace of* $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$; *cf.* (2.600)) *is actually the unique solution of the Inhomogeneous Regularity Problem* (6.51) *for the boundary datum* f.

(d) [Sharpness] *If* $\mathfrak{A}^{\rm dis}_L = \varnothing$ *the Homogeneous Regularity Problem* (6.64) *may fail to be solvable* (*actually for boundary data belonging to an infinite dimensional subspace of the corresponding weighted homogeneous Sobolev space*)*, and if* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ *the Homogeneous Regularity Problem* (6.64) *may possess more than one solution* (*in fact, the linear space of null-solutions may actually be infinite dimensional*)*, even in the case when* $\Omega = \mathbb{R}^n_+$*. In particular, if either* $\mathfrak{A}^{\rm dis}_L = \varnothing$ *or* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$*, then the Homogeneous Regularity Problem* (6.64) *may fail to be well posed, again, even in the case when* $\Omega = \mathbb{R}^n_+$.

In the context of the Homogeneous Regularity Problem (6.64) it is natural that the boundary datum is selected from a homogeneous Muckenhoupt weighted boundary Sobolev space. More concretely, from Proposition 2.24 we see that if $\Omega \subseteq \mathbb{R}^n$ is an NTA domain with an unbounded Ahlfors regular boundary then for any weight $w \in A_p(\partial\Omega, \sigma)$, with $p \in (1, \infty)$ and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, any aperture $\kappa \in (0, \infty)$, and any truncation parameter $\varepsilon \in (0, \infty)$ we have:

$$\left.\begin{array}{r} u \in \mathscr{C}^1(\Omega) \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w) \end{array}\right\} \Longrightarrow \begin{cases} u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ belongs to } \dot{L}^p_1(\partial\Omega, w), \\ \mathcal{N}^\varepsilon_\kappa u \text{ belongs to } L^p_{\rm loc}(\partial\Omega, w), \\ \Big\| u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{\dot{L}^p_1(\partial\Omega,w)} \leq C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,w)}, \end{cases} \tag{6.74}$$

for some dimensional constant $C \in (0, \infty)$. In particular, Theorem 2.3 gives that (6.74) holds whenever $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain with $\delta \in (0, 1)$ sufficiently small (relative to the dimension n and the Ahlfors regularity constant of $\partial\Omega$).

We now present the proof of Theorem 6.8.

Proof of Theorem 6.8 To deal with the claims in item *(a)*, work under the assumption that $\mathfrak{A}^{\rm dis}_L \neq \varnothing$. Theorem 4.11 then implies that there exists $\delta \in (0, 1)$ (whose nature is as in the statement of the theorem) such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (which we shall henceforth assume) then the operator (6.65) is onto. In particular, there exists a function $g \in \big[L^p(\partial\Omega, w)\big]^M$ as in (6.66). In fact (cf. (4.386)), matters may be arranged so that this function satisfies

$$\|g\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}, \tag{6.75}$$

for some $C \in (0, \infty)$ independent of f. Also, since $\big[S_{\rm mod}\, g\big] = \big[S_{\rm mod}\big]g = [f]$, it follows that $c := f - S_{\rm mod}\, g$ is a constant in $\mathbb{C}^M$ (since $\partial\Omega$ is a connected set; cf. Theorem 2.4). If we then define u as in (6.67) for this choice of c, from (3.124), (3.127), (3.47), and (2.575) we see that all conditions in (6.64) are satisfied. Collectively, (6.67), (3.127), (6.74), and (6.75) also guarantee that (6.68) holds.

If $A \in \mathfrak{A}_L^{\rm dis}$, then taking $\delta \in (0, 1)$ sufficiently small also allows us to invoke Theorem 4.12 which guarantees that the operator (6.69) is an isomorphism. In turn, this implies that there exists a unique function h as in (6.71). In particular, we have

$$[f] = \Big(\tfrac{1}{2}I + \big[K_{A,\rm mod}\big]\Big)[h] = \Big[\big(\tfrac{1}{2}I + K_{A,\rm mod}\big)h\Big] \tag{6.76}$$

so

$$c := f - \big(\tfrac{1}{2}I + K_{A,\rm mod}\big)h \ \text{ is a constant in } \ \mathbb{C}^M. \tag{6.77}$$

If we now define the function u as in (6.70), we conclude from Theorem 3.5 that u solves the Homogeneous Regularity Problem (6.64) and satisfies (6.68). This completes the treatment of item *(a)*.

To deal with the uniqueness issue claimed in item *(b)*, assume $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$. Let $u = (u_\gamma)_{1\leq\gamma\leq M}$ solve the version of the Homogeneous Regularity Problem (6.64) corresponding to $f = 0$. From Theorem 3.4, (2.48), and (2.576) we see that

$$\begin{array}{l} (\nabla u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \text{ and} \\ \text{is a } \sigma\text{-measurable function on } \partial\Omega. \end{array} \tag{6.78}$$

Granted this, if $\nu = (\nu_1, \dots, \nu_n)$ denotes the geometric measure theoretic outward unit normal to Ω, we may then invoke Proposition 2.22 (whose applicability in the present setting is ensured by Proposition 2.24) to write

$$\nu_j\Big((\partial_k u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) - \nu_k\Big((\partial_j u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) = \partial_{\tau_{jk}}\Big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) = 0, \tag{6.79}$$

for each $j, k \in \{1, \dots, n\}$.

To proceed, pick a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}} \in \mathfrak{A}_L$ such that

$$A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}. \tag{6.80}$$

Theorem 4.8 then ensures (cf. (4.311) with $z := 1/2$ and with A replaced by $A^\top$) that, if δ is sufficiently small to begin with, it follows that

$$\begin{array}{c} \tfrac{1}{2}I + K^{\#}_{A^\top} : \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M \\ \text{is an invertible operator.} \end{array} \tag{6.81}$$

From (6.78), (2.13), and (3.66) we also see that

$$\partial_\nu^A u \in \big[L^p(\partial\Omega, w)\big]^M. \tag{6.82}$$

Next, let $E = E_L$ be the fundamental solution associated with the system L as in Theorem 3.1. Keeping in mind (6.79) and (3.66), formula (6.62) implies that for each pair of indices, say $\ell \in \{1, \dots, n\}$ and $\gamma \in \{1, \dots, M\}$, we have

$$(\partial_\ell u_\gamma)(x) = -\int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial_\nu^A u\big)_\alpha(y)\, \mathrm{d}\sigma(y) \tag{6.83}$$

at every point $x \in \Omega$. Going nontangentially to the boundary in (6.83) then yields (on account of (3.86)) that at σ-a.e. $x \in \partial\Omega$ we have

$$\begin{aligned}\Big((\partial_\ell u_\gamma)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) &= -\frac{1}{2\mathrm{i}}\widehat{\partial_\ell E_{\gamma\alpha}}\big(\nu(x)\big)\big(\partial_\nu^A u\big)_\alpha(x)\\ &\quad - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} (\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial_\nu^A u\big)_\alpha(y)\, \mathrm{d}\sigma(y)\end{aligned} \tag{6.84}$$

for each $\ell \in \{1, \dots, n\}$ and $\gamma \in \{1, \dots, M\}$. Based on this and (3.66), at σ-a.e. point $x \in \partial\Omega$ we may then write

$$\begin{aligned}\big(\partial_\nu^A u\big)_\mu(x) &= \nu_r(x) a_{rs}^{\mu\beta}\Big(\big(\partial_s u_\beta\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x)\\ &= -\frac{1}{2\mathrm{i}}\widehat{\partial_s E_{\beta\alpha}}\big(\nu(x)\big)\big(\partial_\nu^A u\big)_\alpha(x)\nu_r(x) a_{rs}^{\mu\beta}\\ &\quad - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_r(x) a_{rs}^{\mu\beta}(\partial_s E_{\beta\alpha})(x-y)\big(\partial_\nu^A u\big)_\alpha(y)\, \mathrm{d}\sigma(y)\end{aligned} \tag{6.85}$$

for each $\mu \in \{1, \dots, M\}$. Note that, thanks to (3.17),

$$\begin{aligned}-\frac{1}{2\mathrm{i}}\widehat{\partial_s E_{\beta\alpha}}\big(\nu(x)\big)\big(\partial_\nu^A u\big)_\alpha(x)\nu_r(x) a_{rs}^{\mu\beta}\\ &= -\frac{1}{2}\big(a_{rs}^{\mu\beta}\nu_r(x)\nu_s(x)\big)\widehat{E_{\beta\alpha}}\big(\nu(x)\big)\big(\partial_\nu^A u\big)_\alpha(x)\\ &= \frac{1}{2}\big[L\big(\nu(x)\big)\big]_{\mu\beta}\big[L\big(\nu(x)\big)\big]^{-1}_{\beta\alpha}\big(\partial_\nu^A u\big)_\alpha(x)\\ &= \frac{1}{2}\delta_{\mu\alpha}\big(\partial_\nu^A u\big)_\alpha(x) = \frac{1}{2}\big(\partial_\nu^A u\big)_\mu(x),\end{aligned} \tag{6.86}$$

at σ-a.e. point $x \in \partial\Omega$, for each $\mu \in \{1, \dots, M\}$. Also, from (3.25) and the first equality in (3.20) we see that

$$\lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \nu_r(x) a_{rs}^{\mu\beta} (\partial_s E_{\beta\alpha})(x-y) \big(\partial_\nu^A u\big)_\alpha (y)\, d\sigma(y)$$

$$= \Big(K^{\#}_{A^\top} \big(\partial_\nu^A u\big)\Big)_\mu (x) \tag{6.87}$$

at σ-a.e. point $x \in \partial\Omega$, for each $\mu \in \{1, \dots, M\}$. Altogether, from (6.85), (6.86), and (6.87) we conclude that

$$\partial_\nu^A u = \frac{1}{2} \partial_\nu^A u - K^{\#}_{A^\top} \big(\partial_\nu^A u\big) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{6.88}$$

Hence,

$$\big(\tfrac{1}{2} I + K^{\#}_{A^\top}\big)\big(\partial_\nu^A u\big) = 0 \tag{6.89}$$

which, in view of (6.81) and (6.82), forces $\partial_\nu^A u = 0$. In concert with (6.83), this ultimately implies that

$$\nabla u = 0 \text{ in } \Omega. \tag{6.90}$$

Hence, u is a constant in Ω (since the latter is a connected set if $\delta \in (0, 1)$ is small enough; cf. Theorem 2.4). The fact that we are currently assuming $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0$ at σ-a.e. point on $\partial\Omega$ then allows us to conclude that $u \equiv 0$ in Ω. This proves the claim in item *(b)*.

Another proof of the claim made in item *(b)* is as follows. Pick a coefficient tensor $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\text{dis}}_{L^\top}$. Choosing $\delta \in (0, 1)$ small guarantees (cf. Theorem 2.3) that Ω is an NTA domain with an unbounded connected boundary. As such, Corollary 3.1 applies. In particular, for any null-solution u of the Homogeneous Regularity Problem (6.64) the conormal derivative $\partial_\nu^A u$ belongs to $\big[L^p(\partial\Omega, w)\big]^M$ and the integral representation formula (3.75) presently becomes

$$u = -\mathscr{S}_{\text{mod}}\big(\partial_\nu^A u\big) + c \text{ in } \Omega, \tag{6.91}$$

for some constant $c \in \mathbb{C}^M$. Taking the conormal derivative ∂_ν^A of both sides of (6.91) yields (in light of the jump-formula (3.126))

$$\partial_\nu^A u = -\big(-\tfrac{1}{2} I + K^{\#}_{A^\top}\big)\big(\partial_\nu^A u\big) \tag{6.92}$$

or, equivalently,

$$\big(\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\big(\partial^A_\nu u\big) = 0. \tag{6.93}$$

Since $\frac{1}{2}I+K^{\#}_{A^{\top}}$ is an invertible operator on $\big[L^p(\partial\Omega, w)\big]^M$ (cf. (6.81)), we conclude that $\partial^A_\nu u$. When used back in (6.91) this ultimately proves that $u = c$ in Ω, as wanted.

Next we turn attention to item *(c)*. Thus, we work under the assumption $\mathfrak{A}^{\rm dis}_L \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^{\top}} \neq \varnothing$. Choose $\delta \in (0, 1)$ small enough so that all the conclusions so far hold. Then from item *(a)–(b)* we conclude that the Homogeneous Regularity Problem (6.64) is uniquely solvable. Next, the right-pointing implication in (6.72) is a direct consequence of the last property in (6.64) and (2.13). As for the converse implication, start by assuming that $f \in \big[L^{q;p}_1(\partial\Omega, \omega; w)\big]^M$. Choose $A \in \mathfrak{A}^{\rm dis}_L$ and observe that if $\delta \in (0, 1)$ is small enough to begin with, then (see Remark 4.16)

$$\begin{gathered}\tfrac{1}{2}I + K_A : \big[L^{q;p}_1(\partial\Omega, \omega; w)\big]^M \longrightarrow \big[L^{q;p}_1(\partial\Omega, \omega; w)\big]^M \\ \text{is an invertible operator.}\end{gathered} \tag{6.94}$$

In particular, it is meaningful to consider

$$g := \big(\tfrac{1}{2}I + K_A\big)^{-1} \in \big[L^{q;p}_1(\partial\Omega, \omega; w)\big]^M. \tag{6.95}$$

Then (3.23), (2.575), (3.112), Propositions 3.1, 3.4, and (3.123) guarantee that the function $\widetilde{u} := \mathcal{D}_A g$ in Ω satisfies

$$\begin{cases} \widetilde{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ L\widetilde{u} = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa \widetilde{u} \in L^q(\partial\Omega, \omega), \\ \mathcal{N}_\kappa(\nabla\widetilde{u}) \in L^p(\partial\Omega, w), \\ \widetilde{u}\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{cases} \tag{6.96}$$

The uniqueness in the Homogeneous Regularity Problem established in item *(b)* then allows us to conclude that $u = \widetilde{u}$. Hence, $\mathcal{N}_\kappa u = \mathcal{N}_\kappa \widetilde{u} \in L^q(\partial\Omega, \omega)$, finishing the proof of (6.72). Finally, the right-pointing implication in (6.73) is a consequence of (6.74), while the left-pointing implication in (6.73) follows from Remark 4.21.

Lastly, the claims in item *(d)* are seen from (3.391), (3.385), and (3.406) (cf. also Proposition 3.12 and Example 3.5 in the two-dimensional setting). The proof of Theorem 6.8 is therefore complete. □

We next discuss a variant of the Homogeneous Regularity Problem (6.64), dubbed the Tangential Derivative Problem, which involves as boundary data tangential derivatives of functions from homogeneous Muckenhoupt weighted Sobolev spaces.

Theorem 6.9 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some $M \in \mathbb{N}$, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, fix an aperture parameter $\kappa > 0$, pick an integrability exponent $p \in (1, \infty)$, and select a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. In this setting, consider the Tangential Derivative Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \nu_j\Big((\partial_k u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \nu_k\Big((\partial_j u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = \partial_{\tau_{jk}} f \\ \sigma\text{-a.e. on } \partial\Omega, \text{ for each } \ j, k \in \{1, \dots, n\}, \end{cases} \tag{6.97}$$

where f belongs to $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$, the homogeneous Muckenhoupt weighted boundary Sobolev space defined in (2.598). The following statements are then valid:

(a) [Existence, Estimates, and Integral Representations] *If $\mathfrak{A}^{\text{dis}}_L \neq \varnothing$ then there exists $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a scenario which ensures that Ω is a δ-*AR *domain; cf. Definition 2.15) then the Tangential Derivative Problem (6.97) is solvable for each given $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$. Moreover, a solution u may be found so that*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}, \tag{6.98}$$

where the implicit constants are independent of f. Specifically, one may take u as in (6.66)–(6.67), or as in (6.70)–(6.71).

(b) [Uniqueness (modulo constants)] *Whenever $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$, there exists $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain; cf. Definition 2.15) then any two solutions of the Tangential Derivative Problem (6.97) differ by a constant (from $\mathbb{C}^M$).*

(c) [Well-Posedness and Additional Integrability/Regularity] *Whenever $\mathfrak{A}^{\text{dis}}_L \neq \varnothing$ and $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$ it follows that there exists $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., Ω is a δ-*AR *domain; cf. Definition 2.15) then*

the Homogeneous Regularity Problem (6.64) *is always solvable and any two solutions differ by a constant from* $\mathbb{C}^M$. *In addition, for each* $q \in (1, \infty)$ *and* $\omega \in A_q(\partial\Omega, \sigma)$, *if* $\delta \in (0, 1)$ *is sufficiently small relative to* q *and* $[\omega]_{A_q}$ *then any solution* u *of* (6.97) *satisfies* (*in a quantitative fashion*)

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega, \omega) \Longleftrightarrow f \in \big[\dot{L}^q_1(\partial\Omega, \omega)\big]^M, \tag{6.99}$$

as well as

there exists $c \in \mathbb{C}^M$ *such that* $\mathcal{N}_\kappa(u - c) \in L^q(\partial\Omega, \omega)$ *if and only if there exists* $c \in \mathbb{C}^M$ *such that* $f - c$ *belongs to the off-diagonal weighted Sobolev space* $\big[L^{q;p}_1(\partial\Omega, \omega; w)\big]^M$. (6.100)

(d) [Sharpness] *If* $\mathfrak{A}^{\rm dis}_L = \varnothing$ *the Tangential Derivative Problem* (6.97) *may fail to be solvable, whereas if* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ *the Tangential Derivative Problem* (6.97) *may possess infinitely many solutions, even when* $\Omega = \mathbb{R}^n_+$.

Thanks to Theorem 3.4, (2.576), and Theorem 2.3 we see that whenever Ω is a δ-*AR* domain with $\delta \in (0, 1)$ sufficiently small (as assumed in items *(a)–(c)* in the statement of the theorem) then the first three assumptions in (6.97) guarantee that the nontangential boundary trace $(\nabla u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial\Omega$. This ensures that in all these cases the boundary conditions in (6.97) are meaningfully formulated, without having to *a priori* demand that the first-order partial derivatives of u have nontangential traces at σ-a.e. point on $\partial\Omega$.

Proof of Theorem 6.9 To deal with the claims in item *(a)*, work under the assumption that $\mathfrak{A}^{\rm dis}_L \neq \varnothing$, and suppose $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ where $\delta \in (0, 1)$ is sufficiently small relative to n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$. Given $f \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ let u solve the Homogeneous Regularity Problem (6.64) constructed in (6.67). From (6.74) we see that $\mathcal{N}^\varepsilon_\kappa u \in L^p_{\rm loc}(\partial\Omega, w)$ for each truncation parameter $\varepsilon > 0$, the nontangential trace $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial\Omega$ and, in fact, $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$. We may then rely on Proposition 2.22 (bearing (2.576) in mind) and the last condition in (6.64) to write

$$\begin{gathered}\nu_j\Big((\partial_k u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) - \nu_k\Big((\partial_j u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) = \partial_{\tau_{jk}}\Big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) = \partial_{\tau_{jk}} f,\\ \text{at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } j, k \in \{1, \dots, n\}.\end{gathered} \tag{6.101}$$

Hence, the boundary conditions in (6.97) are satisfied, which goes to show that u is a solution of the Tangential Derivative Problem (6.97). That this solution satisfies (6.98) is then clear from (6.68).

Assume next that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Let u_1, u_2 be two solutions of the Tangential Derivative Problem (6.97) and set $u := u_1 - u_2$. Then the same proof which, starting with (6.79) has let to (6.90), shows that if $\delta \in (0, 1)$ is small enough then u is a constant in Ω. The claim in item *(b)* then follows from this. Finally, the claims in the current items *(c)–(d)* are consequences of items *(c)–(d)* in Theorem 6.8. □

Remark 6.5 Retain the background assumptions made in Theorem 6.9 and recall that the tangential gradient operator has been defined in (2.585). In light of (2.585)–(2.586) we may then equivalently reformulate the Tangential Derivative Problem (6.97) as

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ (\partial_j u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} - \nu_j \nu_k \Big((\partial_k u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \Big) = (\nabla_{\tan} f)_j \\ \sigma\text{-a.e. on } \partial\Omega, \text{ for each } \ j \in \{1, \dots, n\}, \end{cases} \tag{6.102}$$

where, as before, f belongs to $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$. Then, for this boundary value problem, the same results as in Theorem 6.9 are valid.

We continue by discussing the following notable consequence of Theorem 6.8:

Corollary 6.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an aperture parameter $\kappa > 0$. Next, suppose L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Finally, pick an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$.*

Then there exists $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, L, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-AR domain; cf. Definition 2.15) it follows that each function u satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \textit{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w) \end{cases} \tag{6.103}$$

may be represented as

$$u = \mathscr{S}_{\rm mod} f + c \ \textit{ in } \ \Omega \tag{6.104}$$

for some function $f \in \big[L^p(\partial\Omega, w)\big]^M$ *and some constant* $c \in \mathbb{C}^M$*. Moreover, both* f *and* c *are uniquely determined by* u*, and there exists* $C \in (0, \infty)$ *independent of* u *such that*

$$\|f\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)}. \tag{6.105}$$

Additionally, for any given coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$ *there exists some function* $h \in \big[\dot{L}^p(\partial\Omega, w)\big]^M$ *and some constant* $c \in \mathbb{C}^M$ *with the property that*

$$u = \mathcal{D}_{A,{\rm mod}} h + c \ \text{ in } \ \Omega. \tag{6.106}$$

Once again, both h *and* c *are uniquely determined by the function* u*, and there exists a constant* $C \in (0, \infty)$ *independent of* u *such that*

$$\|h\|_{[\dot{L}^p(\partial\Omega,w)]^M} \leq C\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)}. \tag{6.107}$$

Proof Assume $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$, for some threshold $\delta \in (0, 1)$ sufficiently small so that the conclusions in Theorem 4.11 and Theorem 6.8 hold in the current setting. From (6.74) we know that $g := u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists σ-a.e. on $\partial\Omega$ and belongs to $\big[\dot{L}^p_1(\partial\Omega, w)\big]^M$. Since, tautologically, u solves the Homogeneous Dirichlet Problem (6.64) with the boundary datum g, Theorem 6.8 implies that there exists a function $f \in \big[L^p(\partial\Omega, w)\big]^M$ along with a constant $c \in \mathbb{C}^M$ such that u may be represented as in (6.104). Note that (6.105) holds by virtue of (6.65)–(6.68). To show that f and c are uniquely determined by u, assume $f_1, f_2 \in \big[L^p(\partial\Omega, w)\big]^M$ and $c_1, c_2 \in \mathbb{C}^M$ are such that

$$\mathscr{S}_{\rm mod} f_1 + c_1 = \mathscr{S}_{\rm mod} f_2 + c_2 \ \text{ in } \ \Omega. \tag{6.108}$$

Then, with $f := f_1 - f_2 \in \big[L^p(\partial\Omega, w)\big]^M$ and $c := c_2 - c_1 \in \mathbb{C}^M$, we have

$$\mathscr{S}_{\rm mod} f = c \ \text{ in } \ \Omega. \tag{6.109}$$

From (6.109), (2.575), and (3.47) we next conclude that

$$S_{\rm mod} f = c \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \tag{6.110}$$

hence $\big[S_{\rm mod}\big]f = \big[S_{\rm mod} f\big] = [c] = [0] \in \big[\dot{L}^p_1(\partial\Omega, w)\big/\sim\big]^M$. Since $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$, by virtue of item *(2)* in Theorem 4.11 this implies that $f = 0$. Once this has been established then (6.110) gives that $c = 0$. Thus,

$$f = 0 \ \text{ and } \ c = 0, \tag{6.111}$$

from which we conclude that $f_1 = f_2$ and $c_1 = c_2$.

Finally, the fact that u solves the Homogeneous Dirichlet Problem (6.64) formulated for the boundary datum g implies, in light of (6.70)–(6.71) and Theorem 4.12 (with $z = \frac{1}{2}$), that u may be uniquely represented as in (6.106) for some constant $c \in \mathbb{C}^M$ and some function $h \in \big[\dot{L}^p(\partial\Omega, w)\big]^M$ satisfying (6.107). □

As with the Dirichlet Problem and the Inhomogeneous Regularity Problem (cf. Theorem 6.4 and Theorem 6.6), the solvability results derived in Theorem 6.8 are stable under small perturbations. We leave the formulation of such a result to the interested reader and, instead, prove the following brand of stability result, which does not require flatness for the underlying domain, nor does it explicitly ask for the existence of a distinguished coefficient tensor.

Theorem 6.10 *Let $\Omega \subseteq \mathbb{R}^n$ be an* NTA *domain with an unbounded Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick some integrability exponent $p \in (1, \infty)$ and some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L_o in $\mathbb{R}^n$ with the property that the Homogeneous Regularity Problem formulated for L_o in Ω as in* (6.64) *is solvable.*

Then there exists an open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$ which depends only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that for each system $L \in \mathcal{U}$ the Homogeneous Regularity Problem formulated for L in Ω as in (6.64) *continues to be solvable.*

Proof For each coefficient tensor $A \in \mathfrak{A}_{\rm WE}$ define the operator

$$T_A : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \oplus \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \tag{6.112}$$

given by

$$\begin{aligned} T_A([g], h) &:= \Big(\tfrac{1}{2}I + \big[K_{A,\rm mod}\big]\Big)[g] + \big[S_{\rm mod}\big]h \\ \text{for all } \ [g] &\in \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \text{ and } h \in \big[L^p(\partial\Omega, w)\big]^M. \end{aligned} \tag{6.113}$$

With the piece of notation introduced in (3.13), from (6.113) and (3.143) we see that

$$\begin{gathered} \text{the operator-valued assignment mapping each } A \in \mathfrak{A}_{\rm WE} \text{ into} \\ T_A \in \mathrm{Bd}\Big(\big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \oplus \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M\Big) \end{gathered} \tag{6.114}$$

is continuous. To proceed, pick an arbitrary $A_o \in \mathfrak{A}_{L_o}$. From Proposition 3.6 we see that the solvability of the Homogeneous Regularity Problem formulated for L_o in Ω as in (6.64) is equivalent to having T_{A_o} surjective. Since the set of linear bounded surjective operators between two Banach spaces is open (cf. [70, Lemma 2.4]), we conclude from (6.114) that there exists some small $\varepsilon > 0$ such that T_A in (6.112) is surjective whenever $A \in \mathfrak{A}$ satisfies $\|A - A_o\| < \varepsilon$. Having established this, another appeal to Proposition 3.6 then proves that there exists an open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, which depends only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that for each system $L \in \mathcal{U}$ the Homogeneous Regularity Problem formulated for L in Ω as in (6.64) continues to be solvable. □

6.3 The Neumann Problem in Weighted Lebesgue Spaces

To set the stage, recall the definition of the conormal derivative operator from (3.66).

Theorem 6.11 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1,\infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$.*

Suppose L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Select $A \in \mathfrak{A}_L$ and consider the Neumann Problem

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial_\nu^A u = f \in \left[L^p(\partial\Omega, w)\right]^M. \end{cases} \tag{6.115}$$

Then the following statements are valid:

(a) [Existence, Estimates, and Integral Representations] *Whenever $A^\top \in \mathfrak{A}^{\text{dis}}_{L^\top}$ there exists some threshold $\delta \in (0,1)$ which depends only on n, p, $[w]_{A_p}$, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a scenario which ensures that Ω is a δ-*AR *domain; cf. Definition 2.15) then $-\frac{1}{2}I + K^\#_{A^\top}$ is an invertible operator on the Muckenhoupt weighted Lebesgue space $\left[L^p(\partial\Omega, w)\right]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \left(\mathscr{S}_{\text{mod}}\left(-\tfrac{1}{2}I + K^\#_{A^\top}\right)^{-1} f\right)(x) \ \text{ for all } \ x \in \Omega \tag{6.116}$$

is a solution of the Neumann Problem (6.115) which satisfies

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p(\partial\Omega,w)]^M}, \tag{6.117}$$

where the implicit proportionality constants are independent of f. Also, the operator $\partial_\nu^A \mathcal{D}_{A,\text{mod}}$ in (4.392) is surjective which implies that, for some constant $C \in (0,\infty)$,

$$\begin{aligned}&\textit{there exists } g \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \textit{ with } \partial^A_\nu(\mathcal{D}_{A,\mathrm{mod}} g) = f\\ &\textit{and such that } \|g\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \leq C\|f\|_{[L^p(\partial\Omega,w)]^M}.\end{aligned} \tag{6.118}$$

Consequently, the function

$$u := \mathcal{D}_{A,\mathrm{mod}} g \ \textit{ in } \ \Omega \tag{6.119}$$

is a solution of the Neumann Problem (6.115) *which continues to satisfy* (6.117)*.*

(b) [Additional Integrability] *Under the background assumptions made in item (a), for the solution u of the Neumann Problem* (6.115) *defined in* (6.116)*, one has the following integrability result: For any given $q \in (1,\infty)$ and $\omega \in A_q(\partial\Omega,\sigma)$, further decreasing $\delta \in (0,1)$ (relative to q and $[\omega]_{A_q}$) one has*

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega,\omega) \Longleftrightarrow f \in \big[L^q(\partial\Omega,\omega)\big]^M \tag{6.120}$$

and if either of these conditions holds then

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^q(\partial\Omega,\omega)} \approx \|f\|_{[L^q(\partial\Omega,\omega)]^M}. \tag{6.121}$$

(c) [Uniqueness (modulo constants)] *Assume $A \in \mathfrak{A}^{\mathrm{dis}}_L$. Then there exists $\delta \in (0,1)$ which depends only on n, p, $[w]_{A_p}$, A, and the Ahlfors regularity constant of $\partial\Omega$ such that whenever $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence, whenever Ω is a δ-AR domain; cf. Definition 2.15) it follows that any two solutions of the Neumann Problem* (6.115) *differ by a constant from $\mathbb{C}^M$.*

(d) [Well-Posedness] *Whenever $A \in \mathfrak{A}^{\mathrm{dis}}_L$ and $A^\top \in \mathfrak{A}^{\mathrm{dis}}_{L^\top}$ there exists $\delta \in (0,1)$ which depends only on n, p, $[w]_{A_p}$, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain; cf. Definition 2.15) then the Neumann Problem* (6.115) *is solvable, the solution is unique modulo constants from $\mathbb{C}^M$, and each solution satisfies* (6.117)*.*

(e) [Sharpness] *If $A^\top \notin \mathfrak{A}^{\mathrm{dis}}_{L^\top}$ then the Neumann Problem* (6.115) *may not be solvable. In addition, if $A \notin \mathfrak{A}^{\mathrm{dis}}_L$ then the Neumann Problem* (6.115) *may have more than one solution. In fact, even the two-dimensional Laplacian may be written as $\Delta = \mathrm{div}\, A\nabla$ for some matrix $A \in \mathbb{C}^{2\times 2}$ (not belonging to $\mathfrak{A}^{\mathrm{dis}}_\Delta = \{I_{2\times 2}\}$) such that the Neumann Problem formulated for this as in* (6.115) *for this choice of A and with $\Omega := \mathbb{R}^2_+$ fails to have a solution for each non-zero boundary datum belonging to an infinite dimensional linear subspace of $L^p(\partial\Omega,w)$, and the linear space of null-solutions for the Neumann Problem formulated as in* (6.115) *for this choice of A and with $\Omega := \mathbb{R}^2_+$ is actually infinite dimensional.*

Remark 6.6 In view of (2.576), (3.66), and the Fatou-type result described in Theorem 3.4 it follows that the conormal derivative $\partial^A_\nu u$ is well defined in the context of (6.115).

Remark 6.7 In special circumstances, the statement of Theorem 6.11 may be further streamlined. For example, Theorem 3.8 gives that if the system L actually satisfies the strong Legendre–Hadamard ellipticity condition then for the well-posedness formulated in item *(d)* it suffices to assume that $A \in \mathfrak{A}_L^{\rm dis}$, and if $n \geq 3$, $M = 1$, it suffices to assume that the matrix $A \in \mathfrak{A}_L$ is symmetric.

Remark 6.8 The solvability result presented in Theorem 6.11 is relevant in relation to the issue singled out as Question 2.5 in [137].

We now turn to the task of presenting the proof of Theorem 6.11.

Proof of Theorem 6.11 Assume first that the coefficient tensor $A \in \mathfrak{A}_L$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$. From the current assumptions and Theorem 4.8 we know that there exists some threshold $\delta \in (0,1)$, whose nature is as specified in the statement of the theorem, such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then the operator $-\frac{1}{2}I + K^\#_{A^\top}$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$. Granted this, item *(c)* in Proposition 3.5 then guarantees that the function (6.116) solves the Neumann Problem (6.115) and satisfies (6.117).

Next, the claims in (6.118) are consequences of the surjectivity of the operator (4.392) (itself implied by item *(2)* of Theorem 4.13), and the Open Mapping Theorem. In turn, (6.118) and Theorem 3.5 guarantee that the function u in (6.119) solves the Neumann Problem (6.115) and satisfies (6.117). This takes care of the claims in item *(a)*.

Let us now turn our attention to the claim made in item *(b)*, concerning additional integrability properties for the solution constructed in (6.116). The right-pointing implication in (6.120) together with the right-pointing inequality in (6.121) are simple consequences of the fact that we have $|f| = \big|\partial^A_\nu u\big| \leq C\mathcal{N}_\kappa(\nabla u)$ at σ-a.e. point on $\partial\Omega$. The left-pointing implication in (6.120) along with the left-pointing inequality in (6.121) are seen from (6.116), (4.342), and Proposition 3.5.

To prove uniqueness modulo constants in the case when $A \in \mathfrak{A}_L^{\rm dis}$, suppose u solves the homogeneous version of the Neumann Problem (6.115) (corresponding to $f = 0$). Also, fix two arbitrary indices $\ell \in \{1, \dots, n\}$ and $\gamma \in \{1, \dots, M\}$. Since by (3.66) the second integral in (6.62) involves $\nu_r a^{\alpha\beta}_{rs}(\partial_s u_\beta)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = \big(\partial^A_\nu u\big)_\alpha = 0$ for each $\alpha \in \{1, \dots, M\}$, we conclude that we presently have

$$(\partial_\ell u_\gamma)(x) = \int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\Big\{\nu_\ell(y)\big((\partial_s u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(y) - \nu_s(y)\big((\partial_\ell u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(y)\Big\}\,d\sigma(y) \tag{6.122}$$

at every point $x \in \Omega$. On account of (3.86), going nontangentially to the boundary in (6.122) then yields

$$\Big((\partial_\ell u_\gamma)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x) = \frac{1}{2i} a^{\beta\alpha}_{rs}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)\Big\{\nu_\ell(x)\big((\partial_s u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x) \tag{6.123}$$

$$- \nu_s(x)\big((\partial_\ell u_\alpha)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big\}$$

$$+ \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\Big\{\nu_\ell(y)\big((\partial_s u_\alpha)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(y)$$

$$- \nu_s(y)\big((\partial_\ell u_\alpha)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(y)\Big\}\,\mathrm{d}\sigma(y)$$

at σ-a.e. $x \in \partial\Omega$. For each $r \in \{1,\dots,n\}$ and $\beta \in \{1,\dots,M\}$ we may rely on (3.17) to write

$$\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big) = \mathrm{i}\nu_r(x)\widehat{E_{\gamma\beta}}\big(\nu(x)\big) = \mathrm{i}\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta} \tag{6.124}$$

at σ-a.e. $x \in \partial\Omega$. For ease of notation, henceforth we agree to abbreviate

$$\begin{aligned} &\big(\partial_{T_{jk}} u_\alpha\big)(x) := \nu_j(x)\big((\partial_k u_\alpha)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x) - \nu_k(x)\big((\partial_j u_\alpha)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x) \\ &\text{for each } j,k \in \{1,\dots,n\},\ \alpha \in \{1,\dots,M\}, \text{ and } \sigma\text{-a.e. } x \in \partial\Omega. \end{aligned} \tag{6.125}$$

Bring in an additional index $t \in \{1,\dots,n\}$. If we now multiply (6.123) by $\nu_t(x)$ then subtract from the resulting formula its version with ℓ and t interchanged we then arrive, bearing in mind (6.124), (6.125), (3.2), at the identity

$$\big(\partial_{T_{t\ell}} u_\gamma\big)(x) = \frac{1}{2}\big[L\big(\nu(x)\big)\big]_{\beta\alpha}\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\big(\partial_{T_{t\ell}} u_\alpha\big)(x) \tag{6.126}$$

$$- \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_s(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{t\ell}} u_\alpha\big)(y)\,\mathrm{d}\sigma(y)$$

$$+ \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big(\nu_t(x) - \nu_t(y)\big) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{\ell s}} u_\alpha\big)(y)\,\mathrm{d}\sigma(y)$$

$$+ \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big(\nu_\ell(x) - \nu_\ell(y)\big) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{st}} u_\alpha\big)(y)\,\mathrm{d}\sigma(y),$$

valid for each $t, \ell \in \{1,\dots,n\}$, each $\gamma \in \{1,\dots,M\}$, and σ-a.e. $x \in \partial\Omega$. In relation to (6.126), we make several observations. For starters, the first line in the right-hand side of (6.126) is

$$\frac{1}{2}\big[L\big(\nu(x)\big)\big]_{\beta\alpha}\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\big(\partial_{T_{t\ell}}u_\alpha\big)(x)$$

$$= \frac{1}{2}\delta_{\gamma\alpha}\big(\partial_{T_{t\ell}}u_\alpha\big)(x) = \frac{1}{2}\big(\partial_{T_{t\ell}}u_\gamma\big)(x). \tag{6.127}$$

This may be absorbed in the left-hand side of (6.126), which subsequently becomes $\frac{1}{2}\big(\partial_{T_{t\ell}}u_\gamma\big)(x)$. The second observation is that, as is visible from (3.24), the second line in the right-hand side of (6.126) is precisely

$$\Big(K_A\big(\partial_{T_{t\ell}}u\big)\Big)_\gamma(x), \quad \text{where} \;\; \partial_{T_{t\ell}}u := \big(\partial_{T_{t\ell}}u_\alpha\big)_{1\leq\alpha\leq M}. \tag{6.128}$$

The final observation we wish to make with regard to (6.126) is that the third and fourth lines in the right-hand side of (6.126) are commutators of the form

$$\Big(\big[M_\nu, T\big]\big(\partial_T u\big)\Big)(x). \tag{6.129}$$

Above, M_ν denotes the operator of pointwise multiplication by generic scalar components of ν. Also, T stands for the principal-value singular integral operator of formal convolution type with a suitable matrix-valued kernel whose entries are linear combinations with coefficients which are entries from A of generic first-order partial derivatives of generic entries of the matrix E. Finally, $\partial_T u$ stands for generic functions of the form $\partial_{T_{t\ell}}u_\alpha$ with $1 \leq \ell, t \leq n$ and $1 \leq \alpha \leq M$.

In view of these observations, we may recast (6.126) as

$$\begin{gathered}\tfrac{1}{2}\partial_{T_{t\ell}}u = K_A\big(\partial_{T_{t\ell}}u\big) + \big[M_\nu, T\big]\big(\partial_T u\big)\\ \text{at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } t, \ell \in \{1, \dots, n\}.\end{gathered} \tag{6.130}$$

Since we are currently assuming that $A \in \mathfrak{A}^{\rm dis}_L$, from (6.130), (4.299), and Theorem 4.3 (whose applicability in the present context takes into account the format of T specified above as well as Proposition 3.4) we then conclude that for each $t, \ell \in \{1, \dots, n\}$ we have

$$\frac{1}{2}\big\|\partial_{T_{t\ell}}u\big\|_{[L^p(\partial\Omega,w)]^M} \leq C_\delta\big\|\partial_{T_{t\ell}}u\big\|_{[L^p(\partial\Omega,w)]^M} + C_\delta\sum_{j,k=1}^n\big\|\partial_{T_{jk}}u\big\|_{[L^p(\partial\Omega,w)]^M}$$

$$\text{where } C_\delta = o(1) \text{ as } \delta \to 0^+. \tag{6.131}$$

After summing up in all $t, \ell \in \{1, \dots, n\}$ we conclude from (6.131) that

$$\frac{1}{2}\sum_{t,\ell=1}^n\big\|\partial_{T_{t\ell}}u\big\|_{[L^p(\partial\Omega,w)]^M} \leq C_\delta\sum_{t,\ell=1}^n\big\|\partial_{T_{t\ell}}u\big\|_{[L^p(\partial\Omega,w)]^M}$$

$$\text{with } C_\delta = o(1) \text{ as } \delta \to 0^+. \tag{6.132}$$

Assuming $\delta \in (0, 1)$ is sufficiently small to begin with, it follows from (6.132) that

$$\sum_{t,\ell=1}^{n} \big\|\partial_{\tau_{t\ell}} u\big\|_{[L^p(\partial\Omega,w)]^M} \leq 0 \tag{6.133}$$

hence, necessarily,

$$\partial_{\tau_{t\ell}} u_\alpha = 0 \ \text{ for each } \ t, \ell \in \{1, \dots, n\} \ \text{ and } \ \alpha \in \{1, \dots, M\}. \tag{6.134}$$

In concert with (6.125) and (6.122) this ultimately shows that

$$\partial_\ell u_\gamma = 0 \ \text{ in } \ \Omega \ \text{ for each } \ \ell \in \{1, \dots, n\} \ \text{ and } \ \gamma \in \{1, \dots, M\}. \tag{6.135}$$

Thus, the function u is locally constant in Ω. Since the latter is a connected set (cf. Theorem 2.4), we conclude that there exists a constant $c \in \mathbb{C}^M$ such that $u \equiv c$ in Ω.

An alternative proof of uniqueness modulo constants in the case when $A \in \mathfrak{A}_L^{\text{dis}}$ goes as follows. Suppose $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ is a function satisfying $Lu = 0$ in Ω, as well as $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w)$, and $\partial_\nu^A u = 0$. Then Corollary 3.1 implies that $g := u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $\partial\Omega$, belongs to $\big[\dot{L}_1^p(\partial\Omega, w)\big]^M$, and

$$u = \mathcal{D}_{A,\text{mod}}\, g + c \ \text{ in } \ \Omega, \tag{6.136}$$

for some constant $c \in \mathbb{C}^M$ (recall that the present assumptions ensure that Ω is a connected set; cf. Theorem 2.4). In light of the jump-formula (3.134), going nontangentially to the boundary in (6.136) then yields $\big(-\frac{1}{2}I + K_{A,\text{mod}}\big)g = -c$, hence

$$[g] \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M/\sim \ \text{ satisfies } \ \big(-\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\big)[g] = 0. \tag{6.137}$$

Since we are currently assuming that $A \in \mathfrak{A}_L^{\text{dis}}$, from this and Theorem 4.12 (with $z = -\frac{1}{2}$) we conclude that $[g] = 0 \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M/\sim$, i.e., g is a constant on $\partial\Omega$. Having established this, from (6.136) and (3.54) we then conclude that u is a constant in Ω, as wanted.

Next, the claims in *(d)* are direct consequences of results established in items *(a)* and *(c)*. As regards the claims made in item *(e)*, consider the Laplacian Δ in $\mathbb{R}^2 \equiv \mathbb{C}$, written as $\Delta = a_{jk}\partial_j\partial_k$, where the coefficient tensor $A = \big(a_{jk}\big)_{1\leq j,k\leq 2}$ is the 2×2 complex matrix

$$A := \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}. \tag{6.138}$$

Fix an aperture parameter $\kappa \in (0, \infty)$, an integrability exponent $p \in (1, \infty)$, and a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. We claim that the space of admissible boundary data for the L^p-Neumann boundary value problem for the Laplacian in the upper-half plane where the prescribed conormal derivative is the one associated with the matrix A may be described as

$$\begin{aligned} &\Big\{\partial_\nu^A u : u \in \mathscr{C}^\infty(\mathbb{R}^2_+),\ \Delta u = 0 \text{ in } \mathbb{R}^2_+,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)\Big\} \\ &\qquad = \Big\{f \in L^p(\mathbb{R}, w) : Hf = -\mathrm{i}f\Big\}, \end{aligned} \tag{6.139}$$

where H is the Hilbert transform on the real line (cf. (1.24)). Given that the latter space has infinite codimension in $L^p(\mathbb{R}, w)$ (since $H^2 = -I$ on this space), the identification in (6.139) suits our present purposes.

To prove the left-to-right inclusion in (6.139), pick a complex-valued function u satisfying

$$u \in \mathscr{C}^\infty(\mathbb{R}^2_+), \quad \Delta u = 0 \text{ in } \mathbb{R}^2_+, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w). \tag{6.140}$$

On account of the Fatou-type result recalled in Theorem 3.4, these properties guarantee that $(\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ exists at $\mathcal{L}^1$-a.e. point on $\partial\mathbb{R}^2_+$. In particular, $f := \partial_\nu^A u$ is a well-defined function in $L^p(\mathbb{R}, w)$. More specifically, bearing in mind that the outward unit normal for the upper-half plane is $\nu = (\nu_1, \nu_2) = (0, -1) \equiv -\mathrm{i}$, from (3.66) we see that

$$\begin{aligned} f &= \partial_\nu^A u = \nu_r a_{rs}\big(\partial_s u\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \\ &= \nu_1\big(\partial_1 u\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} - \mathrm{i}\nu_1\big(\partial_2 u\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} + \mathrm{i}\nu_2\big(\partial_1 u\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} + \nu_2\big(\partial_2 u\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \\ &= (\nu_1 + \mathrm{i}\nu_2)\Big((\partial_1 u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} - \mathrm{i}(\partial_2 u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = 2\nu(\partial_z u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \\ &= -2\mathrm{i}(\partial_z u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \text{ at } \mathcal{L}^1\text{-a.e. point on } \partial\mathbb{R}^2_+ \equiv \mathbb{R}, \end{aligned} \tag{6.141}$$

where $\partial_z := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ is the complex conjugate of the Cauchy–Riemann operator $\partial_{\bar z} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$. Hence, if we define

$$U := 2\partial_z u \text{ in } \mathbb{R}^2_+, \tag{6.142}$$

upon recalling that $\Delta = 4\partial_{\bar{z}}\partial_z$, the properties in (6.140) imply

$$U \in \mathscr{C}^\infty(\mathbb{R}^2_+), \quad \partial_{\bar{z}} w = 0 \text{ in } \mathbb{R}^2_+, \quad \mathcal{N}_\kappa w \in L^p(\mathbb{R}, w). \tag{6.143}$$

These simply amount to stating that U is a holomorphic function belonging to the Muckenhoupt weighted Hardy space $\mathcal{H}^p(\mathbb{R}^2_+, w)$ associated with the Cauchy–Riemann operator in the upper-half plane. In addition, (6.141) tells us that

$$U\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \mathrm{i} f \text{ at } \mathcal{L}^1\text{-a.e. point on } \partial\mathbb{R}^2_+ \equiv \mathbb{R}. \tag{6.144}$$

Together with Cauchy's reproducing formula for holomorphic functions in the aforementioned Hardy space, this gives

$$U(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, \mathrm{d}t \quad \text{for each } z \in \mathbb{C}_+. \tag{6.145}$$

After taking the nontangential trace to the boundary in (6.145) we arrive at the conclusion that $\mathrm{i}f = \mathrm{i}\big(\frac{1}{2}I + -\frac{1}{2\mathrm{i}}H\big)f$ at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$. This ultimately proves that f must satisfy the compatibility condition

$$Hf = -\mathrm{i}f \text{ at } \mathcal{L}^1\text{-a.e. point in } \mathbb{R}. \tag{6.146}$$

The left-to-right inclusion in (6.139) is therefore established.

To justify the converse inclusion, consider $f \in L^p(\mathbb{R}, w)$ satisfying $Hf = -\mathrm{i}f$ at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$. Bring $\mathscr{S}_{\text{mod}}$, the modified boundary-to-domain harmonic single layer potential operator associated with the Laplacian in the upper-half plane (cf. (3.38)), and note that

$$2\mathrm{i}\partial_z(\mathscr{S}_{\text{mod}} f)(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, \mathrm{d}t \quad \text{for each } z \in \mathbb{C}_+. \tag{6.147}$$

If we define $u := \mathscr{S}_{\text{mod}} f$ in $\mathbb{R}^2_+$, then this function belongs to $\mathscr{C}^\infty(\mathbb{R}^2_+)$, satisfies $\Delta u = 0$ in $\mathbb{R}^2_+$, has $\mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$, and (6.147) permits us to compute

$$\begin{aligned} \partial^A_\nu u &= 2\mathrm{i}\big(\partial_z u\big)\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 2\mathrm{i}(\partial_z \mathscr{S}_{\text{mod}} f)\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \\ &= \tfrac{1}{2} f - \tfrac{1}{2\mathrm{i}} Hf = \tfrac{1}{2} f + \tfrac{1}{2} f = f, \end{aligned} \tag{6.148}$$

as wanted.

As regards the space of null-solutions for the L^p-Neumann Problem (6.115) in the case when $n = 2$, $M = 1$, $L = \Delta$ (the two-dimensional Laplacian), $\Omega = \mathbb{R}^2_+$, and A as in (6.138), we claim that

$$\Big\{u \in \mathscr{C}^\infty(\mathbb{R}^2_+) : \ \Delta u = 0 \ \text{ in } \ \mathbb{R}^2_+, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w), \ \partial^A_\nu u = 0\Big\} \qquad (6.149)$$
$$= \Big\{\overline{U} : \ U \text{ holomorphic in } \mathbb{R}^2_+, \ \text{with } \ \mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)\Big\}.$$

To justify this identification, pick an arbitrary function belonging to the space in the left side of (6.149). Then $\partial_z u$ is holomorphic in $\mathbb{R}^2_+$ (since $\partial_{\bar z}\partial_z = \frac{1}{4}\Delta$), and satisfies $\mathcal{N}_\kappa(\partial_z u) \in L^p(\mathbb{R}, w)$. As such, $\partial_z u$ belongs to $\mathcal{H}^p(\mathbb{R}^2_+, w)$, the Muckenhoupt weighted Hardy space in the upper-half plane for the Cauchy–Riemann operator. Since from (6.141) we have

$$(\partial_z u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \partial\mathbb{R}^2_+ \equiv \mathbb{R}, \qquad (6.150)$$

we may rely on Cauchy's reproducing formula to conclude that $\partial_z u$ vanishes identically in $\mathbb{R}^2_+$. Hence, $U := \overline{u}$ is a holomorphic function in $\mathbb{R}^2_+$. This places $\overline{U}$ (and, ultimately, u) in the space in the right side of (6.149). In the opposite direction, given any holomorphic function U in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$, the function $u := \overline{U}$ is harmonic in $\mathbb{R}^2_+$, has $\mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$ and, much as in (6.141), we see that

$$\partial^A_\nu u = -2\mathrm{i}(\partial_z u)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = -2\mathrm{i}\overline{(\partial_{\bar z}\overline{u})\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}}$$
$$= -2\mathrm{i}\overline{(\partial_{\bar z} U)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \partial\mathbb{R}^2_+ \equiv \mathbb{R}, \qquad (6.151)$$

given that U is holomorphic in $\mathbb{R}^2_+ \equiv \mathbb{C}_+$. This completes the proof of (6.149). The space in the right side of (6.149) is infinite dimensional since, for example, for each $m \in \mathbb{N}$ the function $\mathbb{C}_+ \ni z \mapsto (\bar z - \mathrm{i})^{-m} \in \mathbb{C}$ belongs to this space. We therefore conclude that the space of null-solutions for the L^p-Neumann Problem (6.115) is, as claimed, infinite dimensional. □

Remark 6.9 For similar reasons as in past situations, a solvability result which is analogous to the one described in Theorem 6.11 also holds for the Neumann Problem with data in Lorentz spaces, i.e., for

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^{p,q}(\partial\Omega, \sigma), \\ \partial^A_\nu u = f \in \big[L^{p,q}(\partial\Omega, \sigma)\big]^M, \end{cases} \qquad (6.152)$$

with $p \in (1, \infty)$ and $q \in (0, \infty]$.

See Theorem 8.21 (and also Examples 8.2, 8.6) for more general results of this flavor.

Remark 6.10 In light of the remarks made in (3.228)–(3.229), Theorem 6.11 applies in the case of the Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ in $\mathbb{R}^n$ with $n \geq 2$, assuming $\mu \neq 0$, $2\mu + \lambda \neq 0$, and $3\mu + \lambda \neq 0$. Specifically, if $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain, and $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, then if $\delta \in (0, 1)$ sufficiently small (relative to μ, λ, p, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) the Neumann Problem (6.115), which in this case reads

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \\ \mu\Delta u + (\lambda + \mu)\nabla\mathrm{div} u = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial_\nu^{A(\zeta)} u = \Big[\mu(\nabla u)^\top + \zeta(\nabla u)\Big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu + (\mu + \lambda - \zeta)(\mathrm{div} u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu = f \end{cases} \tag{6.153}$$

is solvable (in the explicit manner described in (6.116)) for each given function $f \in \big[L^p(\partial\Omega, w)\big]^n$, provided the coefficient tensor $A(\zeta)$ is as in (3.226) with

$$\zeta := \frac{\mu(\mu + \lambda)}{3\mu + \lambda}. \tag{6.154}$$

Moreover, the solution is unique modulo constants from $\mathbb{C}^n$ and each solution satisfies (6.117) (with $M := n$).

By way of contrast, in the two-dimensional case, Corollary 4.3 ensures that the Neumann Problem (6.153) is actually solvable (again, in the manner described in (6.116), the solution being unique modulo constants from $\mathbb{C}^2$ and each solution satisfying a naturally accompanying estimate) for each given function f in the space $\big[L^p(\partial\Omega, w)\big]^2$, in the larger range

$$\zeta \in \mathbb{C} \setminus \Big\{ -\mu, \tfrac{\mu(5\mu+3\lambda)}{3\mu+\lambda} \Big\}. \tag{6.155}$$

In particular, if we also demand that $\mu + \lambda \neq 0$ then $\zeta := \mu$ becomes an admissible value, as far as (6.155) is concerned, and from (4.438), (6.116) we see that the Neumann Problem (6.153) with $\zeta := \mu$ is solvable uniquely (modulo constants) for each given function $f \in \big[L^p(\partial\Omega, w)\big]^2$. This is of interest since said problem involves the so-called `traction conormal derivative`, i.e.,

$$\partial_\nu^{A(\mu)} u = \mu\Big[(\nabla u)^\top + (\nabla u)\Big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu + \lambda(\mathrm{div} u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu, \tag{6.156}$$

which is particularly relevant in physics and engineering.

It is also of interest to note that the solvability result from Theorem 6.11 is stable under small perturbations. Specifically, by reasoning similarly as in the proof of Theorem 6.4 (while also bearing in mind Theorem 3.9) yields the following theorem.

Theorem 6.12 *Retain the original background assumptions on the set Ω from Theorem 6.11 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then the following statements are true.*

(a) [Existence] *Given any system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\text{dis}}$ (cf. (3.195)), it follows that for each $A_o \in \mathfrak{A}_{L_o}$ with $A_o^\top \in \mathfrak{A}^{\text{dis}}_{L_o^\top}$ there exist a threshold $\delta \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of A_o in $\mathfrak{A}$, both of which depend only on n, p, $[w]_{A_p}$, A_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then for each coefficient tensor $A \in \mathcal{U}$ the Neumann Problem (6.115) formulated for the system L_A (cf. (3.7)) and the conormal derivative associated with A (cf. (3.66)) is actually solvable.*

(b) [Uniqueness] *Assume $L_o \in \mathfrak{L}^{\text{dis}}$ and fix some $A_o \in \mathfrak{A}^{\text{dis}}_{L_o}$. Then there exist a threshold $\delta \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of A_o in $\mathfrak{A}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the following significance: Whenever $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., Ω is a δ-*AR *domain) then for each coefficient tensor $A \in \mathcal{U}$ it follows that any two solutions of the Neumann Problem (6.115) formulated for the system L_A (cf. (3.7)) and the conormal derivative associated with A (cf. (3.66)) differ by a constant in $\mathbb{C}^M$.*

(c) [Well-Posedness] *Assuming $L_o \in \mathfrak{L}^{\text{dis}}$ and $L_o^\top \in \mathfrak{L}^{\text{dis}}$, fix some $A_o \in \mathfrak{A}^{\text{dis}}_{L_o}$. Then there exist a threshold $\delta \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of A_o in $\mathfrak{A}$, both of which depend only on n, p, $[w]_{A_p}$, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the following significance: Whenever $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., Ω is a δ-*AR *domain) then for each coefficient tensor $A \in \mathcal{U}$ it follows that any two solutions of the Neumann Problem (6.115) formulated for the system L_A (cf. (3.7)) and the conormal derivative associated with A (cf. (3.66)) is solvable, and any two solutions differ by a constant from $\mathbb{C}^M$.*

In addition to Theorem 6.12, there is yet another type of stability result for the Neumann problem which does not require flatness for the underlying domain, nor does it explicitly ask for the existence of a distinguished coefficient tensor (compare with Theorem 6.10).

Theorem 6.13 *Let $\Omega \subseteq \mathbb{R}^n$ be an* NTA *domain with an unbounded Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an aperture parameter $\kappa > 0$, pick an integrability exponent $p \in (1, \infty)$, and choose some arbitrary Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, consider a coefficient tensor $A_o \in \mathfrak{A}_{\text{WE}}$ with the property that the Neumann Problem formulated for the system $L := L_{A_o}$ (cf. (3.7)) and the conormal derivative associated with A_o (cf. (3.66)) as in (6.115) is solvable.*

Then there exists an open neighborhood $\mathcal{U}$ of A_o in $\mathfrak{A}$ which depends only on n, p, $[w]_{A_p}$, A_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that *for each coefficient tensor* $A \in \mathcal{U}$ *the Neumann Problem formulated for the system* $L := L_A$ (*cf.* (3.7)) *and the conormal derivative associated with* A (*cf.* (3.66)) *as in* (6.115) *continues to be solvable.*

Proof For each coefficient tensor $A \in \mathfrak{A}_{\rm WE}$ define the operator

$$Q_A : \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \oplus \big[L^p(\partial\Omega, w)\big]^M \longrightarrow \big[L^p(\partial\Omega, w)\big]^M \tag{6.157}$$

given by

$$\begin{aligned} Q_A([g], h) &:= \big[\partial^A_\nu \mathcal{D}_{A,\rm mod}\big][g] + \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)h \\ \text{for all } \ [g] &\in \big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \ \text{ and } \ h \in \big[L^p(\partial\Omega, w)\big]^M. \end{aligned} \tag{6.158}$$

Recall the piece of notation introduced in (3.13). From (6.158), (3.139), and (3.122) we see that

$$\begin{gathered} \text{the operator-valued assignment mapping each } A \in \mathfrak{A}_{\rm WE} \text{ into} \\ Q_A \in {\rm Bd}\Big(\big[\dot{L}^p_1(\partial\Omega, w)/\sim\big]^M \oplus \big[L^p(\partial\Omega, w)\big]^M \to \big[L^p(\partial\Omega, w)\big]^M\Big) \end{gathered} \tag{6.159}$$

is continuous. To proceed, fix $A_o \in \mathfrak{A}_{\rm WE}$ as in the statement. From Proposition 3.7 it follows that Q_{A_o} is surjective. Since the set of linear bounded surjective operators between two Banach spaces is open (cf. [70, Lemma 2.4]), we conclude from (6.159) that there exists an open neighborhood $\mathcal{U}$ of A_o in $\mathfrak{A}$ (whose nature is as in the statement of the theorem) with the property that Q_A continues to be surjective in the context of (6.157) for each $A \in \mathcal{U}$. We may then once again employ Proposition 3.7 to conclude that the Neumann Problem formulated for the system $L := L_A$ and the conormal derivative associated with A as in (6.115) is solvable. □

Solvability results for the Neumann Problem formulated for boundary data belonging to sums of Muckenhoupt weighted Lebesgue spaces are described in the theorem below.

Theorem 6.14 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and fix an aperture parameter* $\kappa > 0$. *Also, pick* $p_0, p_1 \in (1, \infty)$ *along with a pair of Muckenhoupt weights* $w_0 \in A_{p_0}(\partial\Omega, \sigma)$ *and* $w_1 \in A_{p_1}(\partial\Omega, \sigma)$. *Finally, consider a homogeneous, second-order, constant complex coefficient,* $M \times M$ *weakly elliptic system* L *in* $\mathbb{R}^n$, *and select some coefficient tensor* $A \in \mathfrak{A}_L$

Then similar results, concerning existence, integral representation formulas, estimates, additional integrability properties, and well-posedness, as in Theorem 6.11, are valid for the Neumann Problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1), \\ \partial_\nu^A u = f \in \big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M. \end{cases} \tag{6.160}$$

Proof This is seen by reasoning as in the proof of Theorem 6.11, now making use of (4.336) and bearing in mind that the commutator estimates from Theorem 4.3 also extend to sums of Muckenhoupt weighted Lebesgue spaces (cf. (4.332)). □

We conclude with a result to the effect that solvability of the Neumann problem for a system L implies uniqueness (modulo locally constant functions) for the Neumann problem formulated for the transpose system $L^\top$.

Proposition 6.1 *Let $\Omega \subseteq \mathbb{R}^n$, with $n \geq 3$, be an* NTA *domain with an unbounded Ahlfors regular boundary. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an aperture parameter $\kappa > 0$ and consider two integrability exponents*

$$p, q \in (1, n-1) \ \ satisfying \ \ \tfrac{1}{p} + \tfrac{1}{q} = 1 + \tfrac{1}{n-1}. \tag{6.161}$$

Finally, pick a coefficient tensor $A \in \mathfrak{A}_{\rm WE}$ with the property that the Neumann Problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ L_A u = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \ \ at\ \sigma\text{-a.e. point on} \ \ \partial\Omega \end{cases} \tag{6.162}$$

is solvable for each $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$. Then having

$$\begin{cases} w \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ L_{A^\top} w = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla w) \in L^q(\partial\Omega, \sigma), \\ \partial_\nu^{A^\top} w = 0 \ \ at\ \sigma\text{-a.e. point on} \ \ \partial\Omega \end{cases} \tag{6.163}$$

forces w to be a locally constant function in Ω.

Proof Fix an arbitrary $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ and assume that u solves (6.162) for this choice of boundary datum. Also, let w be as in (6.163). Granted the present assumptions, Proposition 2.24 implies that the nontangential boundary traces

$$u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},\ \ w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\ \ \text{exist } \sigma\text{-a.e. on } \ \partial\Omega. \tag{6.164}$$

Also, work in [114, §2.2] guarantees that there exist two constants $c, \widetilde{c} \in \mathbb{C}^M$ such that

$$\begin{aligned} &u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c \in \big[L^{p^*}(\partial\Omega,\sigma)\big]^M \ \text{ and } \ \mathcal{N}_\kappa(u-c) \in L^{p^*}(\partial\Omega,\sigma) \\ &\qquad\qquad \text{where } \ p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1} \in (1,\infty), \end{aligned} \tag{6.165}$$

as well as

$$\begin{aligned} &w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \widetilde{c} \in \big[L^{q^*}(\partial\Omega,\sigma)\big]^M \ \text{ and } \ \mathcal{N}_\kappa(w-\widetilde{c}) \in L^{q^*}(\partial\Omega,\sigma) \\ &\qquad\qquad \text{where } \ q^* := \big(\tfrac{1}{q} - \tfrac{1}{n-1}\big)^{-1} \in (1,\infty). \end{aligned} \tag{6.166}$$

Let $\big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le j,k\le n\\ 1\le\alpha,\beta\le M}}$ be the entries of the given coefficient tensor $A \in \mathfrak{A}_{\mathrm{WE}}$. Also, denote by $(u_\alpha)_{1\le\alpha\le M}$ and $(w_\beta)_{1\le\beta\le M}$, respectively, the scalar components of the vector-valued functions u, w. Define the vector field

$$\vec{F} := \Big(a^{\alpha\beta}_{jk}(\partial_k u_\beta)(w-\widetilde{c})_\alpha - a^{\alpha\beta}_{kj}(u-c)_\beta(\partial_k w_\alpha)\Big)_{1\le j\le n}, \tag{6.167}$$

where the summation convention over repeated indices is in effect. Then

$$\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n \tag{6.168}$$

and

$$\begin{aligned} \operatorname{div}\vec{F} &= a^{\alpha\beta}_{jk}(\partial_j\partial_k u_\beta)(w-\widetilde{c})_\alpha + a^{\alpha\beta}_{jk}(\partial_k u_\beta)(\partial_j w_\alpha) \\ &\quad - a^{\alpha\beta}_{kj}(\partial_j u_\beta)(\partial_k w_\alpha) - a^{\alpha\beta}_{kj}(u-c)_\beta(\partial_j\partial_k w_\alpha) \\ &= (L_A u)_\alpha (w-\widetilde{c})_\alpha - (u-c)_\beta (L_{A^\top} w)_\beta \\ &= 0 - 0 = 0 \ \text{ in } \ \Omega, \end{aligned} \tag{6.169}$$

thanks to (6.162) and (6.163). Also, from (6.167), (6.165), (6.166), and the fact that, as seen from (6.161), we have

$$\tfrac{1}{p^*} + \tfrac{1}{q} = 1 \ \text{ and } \ \tfrac{1}{p} + \tfrac{1}{q^*} = 1, \tag{6.170}$$

we conclude that

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma). \tag{6.171}$$

Finally, from (6.167) and (6.164) we see that the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and, in fact,

$$\begin{aligned}\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \Big(a_{jk}^{\alpha\beta}(\partial_k u_\beta)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \Big(w\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \widetilde{c}\Big)_\alpha \\ - a_{kj}^{\alpha\beta}\Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - c\Big)_\beta (\partial_k w_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)_{1\leq j\leq n}.\end{aligned} \tag{6.172}$$

In particular, (6.172) and (3.66) imply that at σ-a.e. point on $\partial\Omega$ we have

$$\begin{aligned}\nu\cdot\Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) &= \nu_j a_{jk}^{\alpha\beta}(\partial_k u_\beta)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \Big(w\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \widetilde{c}\Big)_\alpha \\ &\quad - \nu_j a_{kj}^{\alpha\beta}\Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - c\Big)_\beta (\partial_k w_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \\ &= \Big\langle \partial_\nu^A u, w\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \widetilde{c}\Big\rangle - \Big\langle u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - c, \partial_\nu^{A^\top} w\Big\rangle \\ &= \Big\langle f, w\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \widetilde{c}\Big\rangle,\end{aligned} \tag{6.173}$$

where the last equality takes into account the boundary conditions in (6.162) and (6.163). Granted (6.168), (6.169), (6.170), (6.172), and the current assumptions on Ω, a version of the Divergence Theorem proved in [111, §1.2] applies and gives

$$\int_{\partial\Omega} \nu\cdot\Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\,d\sigma = 0. \tag{6.174}$$

In concert with (6.173) this further implies

$$\int_{\partial\Omega} \Big\langle f, w\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \widetilde{c}\Big\rangle\,d\sigma = 0 \tag{6.175}$$

which, in view of the arbitrariness of $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ forces $w\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \widetilde{c}$ at σ-a.e. point on $\partial\Omega$. With this in hand, the integral representation formula from (3.75) gives that, for some $\mathbb{C}^M$-valued locally constant function c_w in Ω, we have

$$\begin{aligned}w &= \mathcal{D}_{A^\top,\text{mod}}\Big(w\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathscr{S}_{\text{mod}}\big(\partial_\nu^{A^\top} w\big) + c_w \\ &= \mathcal{D}_{A^\top,\text{mod}}(\widetilde{c}) + c_w \ \text{ in } \ \Omega.\end{aligned} \tag{6.176}$$

Thanks to (3.54) we then conclude that w is indeed a locally constant function in the set Ω. □

6.4 The Transmission Problem in Weighted Lebesgue Spaces

The trademark characteristic of a Transmission Problem is the fact that one now seeks two functions, defined on either side of an interface, whose traces and conormal derivatives couple in a specific fashion along the common interface.

Theorem 6.15 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$, and set*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{6.177}$$

In addition, pick an integrability exponent $p \in (1, \infty)$, some Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, an aperture parameter $\kappa \in (0, \infty)$, and a transmission (or coupling) parameter $\eta \in \mathbb{C}$.

Next, assume L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some coefficient tensor $A \in \mathfrak{A}_L$ and consider the Transmission Problem:

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^{\pm} = 0 \ \text{ in } \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^{\pm}) \in L^p(\partial\Omega, w), \\ u^+\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = g \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M, \\ \partial_\nu^A u^+ - \eta \cdot \partial_\nu^A u^- = f \in \big[L^p(\partial\Omega, w)\big]^M. \end{cases} \tag{6.178}$$

Then, in relation to this, the following statements are valid:

(a) [Uniqueness (modulo constants)] *Assume that either*

$$A^\top \in \mathfrak{A}^{\text{dis}}_{L^\top} \text{ and } \eta \in \mathbb{C} \setminus \{-1\} \tag{6.179}$$

or

$$A \in \mathfrak{A}^{\text{dis}}_{L} \text{ and } \eta \in \mathbb{C} \setminus \{0, -1\}. \tag{6.180}$$

Then there exists $\delta \in (0, 1)$ which depends only on n, p, $[w]_{A_p}$, A, η, and the Ahlfors regularity constant of $\partial\Omega$ so that whenever $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a *scenario which renders* Ω *a* δ-AR *domain; cf. Definition 2.15) it follows any two solutions of the Transmission Problem* (6.178) *differ by a constant (from* $\mathbb{C}^M$*).*

(b) [Well-Posedness, Integral Representations, and Additional Regularity] *Assume* [1]

$$A \in \mathfrak{A}_L^{\rm dis},\ A^\top \in \mathfrak{A}_{L^\top}^{\rm dis},\ \textit{and}\ \eta \in \mathbb{C} \setminus \{-1\}. \tag{6.181}$$

Then there exists some $\delta \in (0,1)$ *which depends only on* n, p, $[w]_{A_p}$, A, η, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15) it follows that the Transmission Problem* (6.178) *is solvable. Specifically, in the scenario described in* (6.181)*, the operator* $-\frac{\eta+1}{2}I + (1-\eta)K^\#_{A^\top}$ *is invertible on the Muckenhoupt weighted Lebesgue space* $\big[L^p(\partial\Omega,w)\big]^M$*, the operator* $\big[S_{\rm mod}\big]$ *is invertible from* $\big[L^p(\partial\Omega,w)\big]^M$ *onto the space* $\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M$*, and the functions* $u^\pm : \Omega_\pm \to \mathbb{C}^M$ *defined as*

$$\begin{aligned} u^+ &:= \mathscr{S}^+_{\rm mod} h_0 + \mathscr{S}^+_{\rm mod} h_1 - c \ \ \textit{in}\ \ \Omega_+, \\ u^- &:= \mathscr{S}^-_{\rm mod} h_0 \ \ \textit{in}\ \ \Omega_-, \end{aligned} \tag{6.182}$$

where the superscripts $\pm$ *indicate that the modified single layer potentials are associated with the sets* $\Omega_\pm$ *and*

$$\begin{aligned} & h_1 := \big[S_{\rm mod}\big]^{-1}[g] \in \big[L^p(\partial\Omega,w)\big]^M, \quad c := S_{\rm mod} h_1 - g \in \mathbb{C}^M, \\ & h_0 := \Big(-\tfrac{\eta+1}{2}I + (1-\eta)K^\#_{A^\top}\Big)^{-1}\Big(f - \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)h_1\Big), \end{aligned} \tag{6.183}$$

solve the Transmission Problem (6.178) *and satisfy, for a finite constant* $C > 0$ *independent of* f *and* g,

$$\big\|\mathcal{N}_\kappa(\nabla u^\pm)\big\|_{L^p(\partial\Omega,w)} \le C\Big(\|f\|_{[L^p(\partial\Omega,w)]^M} + \|g\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}\Big). \tag{6.184}$$

Moreover, any two solutions of the Transmission Problem (6.178) *differ by a constant (from* $\mathbb{C}^M$*). In particular, any solution of the Transmission Problem* (6.178) *satisfies* (6.184).

Alternatively, under the conditions imposed in (6.181) *and, again, assuming* Ω *is a* δ-AR *domain with* $\delta \in (0,1)$ *sufficiently small, a solution of the Transmission Problem* (6.178) *may also be found in the form*

[1] According to Theorem 3.9, the set of demands made in (6.181) is further equivalent to $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$, and $\eta \in \mathbb{C} \setminus \{-1\}$, and also equivalent to $A \in \mathfrak{A}_L^{\rm dis}$, $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$, and $\eta \in \mathbb{C} \setminus \{-1\}$.

$$
\begin{aligned}
u^{+} &:= \mathcal{D}^{+}_{A,\mathrm{mod}}\psi_0 + c \quad in \quad \Omega_+,\\
u^{-} &:= \mathcal{D}^{-}_{A,\mathrm{mod}}\psi_1 \quad in \quad \Omega_-,
\end{aligned}
\tag{6.185}
$$

where the superscripts $\pm$ *indicate that the modified double layer potentials are associated with the sets* $\Omega_\pm$, *where* $c \in \mathbb{C}^M$ *is a suitable constant, and where* $\psi_0, \psi_1 \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ *are two suitable functions satisfying*

$$
\begin{aligned}
&\|\psi_0\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} + \|\psi_1\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}\\
&\qquad\qquad \le C\Big(\|f\|_{[L^p(\partial\Omega,w)]^M} + \|g\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}\Big),
\end{aligned}
\tag{6.186}
$$

for some constant $C \in (0,\infty)$ *independent of* f *and* g. *In particular,* $u^\pm$ *in* (6.185) *also satisfy* (6.184).

Finally, for any given $q \in (1,\infty)$ *and* $\omega \in A_q(\partial\Omega,\sigma)$ (*after possibly further decreasing* $\delta \in (0,1)$ *relative to* q *and* $[\omega]_{A_q}$) *one has*

$$
\begin{aligned}
&\mathcal{N}_\kappa(\nabla u^+), \mathcal{N}_\kappa(\nabla u^-) \in L^q(\partial\Omega,\omega)\\
&\qquad\qquad \Longleftrightarrow f \in \big[L^q(\partial\Omega,\omega)\big]^M \ \ and \ \ g \in \big[\dot{L}^q_1(\partial\Omega,\omega)\big]^M,
\end{aligned}
\tag{6.187}
$$

and if either of these conditions holds then

$$
\begin{aligned}
&\big\|\mathcal{N}_\kappa(\nabla u^+)\big\|_{L^q(\partial\Omega,\omega)} + \big\|\mathcal{N}_\kappa(\nabla u^-)\big\|_{L^q(\partial\Omega,\omega)}\\
&\qquad\qquad \approx \|f\|_{[L^q(\partial\Omega,\omega)]^M} + \|g\|_{[\dot{L}^q_1(\partial\Omega,\omega)]^M}.
\end{aligned}
\tag{6.188}
$$

(c) [Sharpness] *Fix some transmission parameter* $\eta \in \mathbb{C}\setminus\{-1\}$. *Then even for* $L = \Delta$ *and* $\Omega = \mathbb{R}^n_+$, *if* $A \notin \mathfrak{A}^{\mathrm{dis}}_L$ *it may happen that the Transmission Problem* (6.178) *fails to be solvable when* $p = 2$ *and* $w \equiv 1$.

(d) [Well-Posedness for $\eta = 1$] *In the case when*

$$
\eta = 1 \ \textit{and}\ \Omega\ \textit{is a two-sided}\ \mathrm{NTA}\ \textit{domain with an unbounded Ahlfors regular boundary}
\tag{6.189}
$$

the Transmission Problem (6.178) *is solvable, and any two solutions of the Transmission Problem* (6.178) *differ by a constant. Any solution is given by*

$$
\begin{aligned}
u^{+} &:= \mathcal{D}^{+}_{A,\mathrm{mod}} g - \mathscr{S}^{+}_{\mathrm{mod}} f + c \quad in \quad \Omega_+,\\
u^{-} &:= -\mathcal{D}^{-}_{A,\mathrm{mod}} g - \mathscr{S}^{-}_{\mathrm{mod}} f + c \quad in \quad \Omega_-,
\end{aligned}
\tag{6.190}
$$

for some $c \in \mathbb{C}^M$, where the superscripts $\pm$ indicate that the modified layer potentials are associated with the sets $\Omega_\pm$ introduced in (6.177)*. In addition, any solution satisfies* (6.184).

A few clarifications pertaining to the nature of the above theorem are in order here. First, Lemma 2.3 and definitions imply that

$$\begin{aligned}&\Omega_- \text{ is a UR domain whose topological boundary actually coincides with } \partial\Omega, \text{ and whose geometric measure theoretic boundary}\\&\text{agrees with that of } \Omega \text{ (hence, } \partial(\Omega_-) = \partial\Omega \text{ and } \partial_*(\Omega_-) = \partial_*\Omega);\\&\text{also, the geometric measure theoretic outward unit normal to } \Omega_-\\&\text{is } -\nu \text{ at } \sigma\text{-a.e. point on } \partial\Omega.\end{aligned} \tag{6.191}$$

In particular, this makes it meaningful to talk about the nontangential boundary trace $u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$, here understood as $u^-\big|_{\partial(\Omega_-)}^{\kappa-\text{n.t.}}$. Second, the existence of $u^\pm\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ at σ-a.e. point on $\partial\Omega$ is an implicit demand in the formulation of the Transmission Problem (6.178). Third, the conormal derivative $\partial_\nu^A u^+$ is defined as in (3.66), while in light of the last property in (6.191) we take $\partial_\nu^A u^-$ to be the opposite of (i.e.,-1 times) the conormal derivative operator from (3.66) for the domain Ω_- acting on the function u^-, i.e.,

$$\partial_\nu^A u^- := -\partial_{(-\nu)}^A u^-. \tag{6.192}$$

Collectively, (2.576), (2.48), (3.66), and the Fatou-type result from Theorem 3.4 imply that the conormal derivatives $\partial_\nu^A u^\pm$ are well defined in the context of (6.178).

We now turn to the task of proving Theorem 6.15.

Proof of Theorem 6.15 As regards item *(a)*, we need to address the issue of uniqueness (modulo constants) in either of the scenarios specified in (6.179)–(6.180), assuming that Ω is a δ-AR domain for some sufficiently small $\delta \in (0, 1)$. In all cases, the goal is to show that

$$\begin{aligned}&\text{if } u^\pm \text{ solve the homogeneous version of the Transmission Problem (6.178) (corresponding to having } f = 0 \text{ and } g = 0) \text{ then}\\&\text{there exists a constant } c \in \mathbb{C}^M \text{ with the property that } u^\pm = c \text{ in}\\&\Omega_\pm.\end{aligned} \tag{6.193}$$

Let us first justify (6.193) in the case when (6.179) holds. Suppose $u^\pm$ solve the homogeneous version of the Transmission Problem (6.178). Assuming that Ω is a δ-AR domain with $\delta \in (0, 1)$ sufficiently small, Theorem 2.3, Propositions 2.24, 2.22 (keeping in mind (2.576)), and the homogeneous version of the first boundary condition in (6.178), to the effect that

$$u^+\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \tag{6.194}$$

for each $j, k \in \{1, \dots, n\}$, allow us to write

$$\nu_j\Big((\partial_k u^+)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \nu_k\Big((\partial_j u^+)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = \partial_{\tau_{jk}}\Big(u^+\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)$$
$$= \partial_{\tau_{jk}}\Big(u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = \nu_j\Big((\partial_k u^-)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \nu_k\Big((\partial_j u^-)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) \tag{6.195}$$

at σ-a.e. point on $\partial\Omega$. In terms of the abbreviation introduced in (6.125) we agree to recast this as

$$\begin{aligned} &\partial_{T_{jk}} u_\alpha^+ = \partial_{T_{jk}} u_\alpha^- \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \\ &\text{for each } j, k \in \{1, \dots, n\} \text{ and } \alpha \in \{1, \dots, M\}. \end{aligned} \tag{6.196}$$

Also, from (6.62) (written for u^+ and Ω_+), (6.196), the fact that we are presently assuming

$$\partial_\nu^A u^+ = \eta \cdot \partial_\nu^A u^-, \tag{6.197}$$

and (6.63) (written for u^- and Ω_-) we see that for each integer $\ell \in \{1, \dots, n\}$ and each $\gamma \in \{1, \dots, M\}$, and each point $x \in \Omega$ we have

$$\begin{aligned} (\partial_\ell u_\gamma^+)(x) &= \int_{\partial\Omega} a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{\ell s}} u_\alpha^+\big)(y)\, d\sigma(y) \\ &\quad - \int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial_\nu^A u^+\big)_\alpha(y)\, d\sigma(y) \\ &= \int_{\partial\Omega} a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{\ell s}} u_\alpha^-\big)(y)\, d\sigma(y) \\ &\quad - \eta \int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial_\nu^A u^-\big)_\alpha(y)\, d\sigma(y) \\ &= (1-\eta) \int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial_\nu^A u^-\big)_\alpha(y)\, d\sigma(y). \end{aligned} \tag{6.198}$$

Granted this, the same type of argument which, starting with (6.83), has produced (6.88) presently yields

$$\partial_\nu^A u^+ = (1-\eta)\big(-\tfrac{1}{2}I + K_{A^\top}^{\#}\big)\big(\partial_\nu^A u^-\big) \tag{6.199}$$

which, given that we are currently assuming $\partial_\nu^A u^+ = \eta \cdot \partial_\nu^A u^-$ (cf. (6.197)), further implies

$$\Big(-\tfrac{\eta+1}{2}I+(1-\eta)K^{\#}_{A^\top}\Big)\big(\partial^A_\nu u^-\big)=0. \tag{6.200}$$

Since we are presently assuming $A^\top\in\mathfrak{A}^{\rm dis}_{L^\top}$ and $\eta\in\mathbb{C}\setminus\{-1\}$, Theorem 4.8 ensures (taking $\delta\in(0,1)$ sufficiently small, to begin with) that $-\frac{\eta+1}{2}I+(1-\eta)K^{\#}_{A^\top}$ is an invertible operator on $\big[L^p(\partial\Omega,w)\big]^M$. Together with (6.200) this forces

$$\partial^A_\nu u^-=0. \tag{6.201}$$

Going back with this in (6.198) then yields $\nabla u^+=0$ in Ω_+. In concert with Theorem 2.4 this goes to show that u^+ is a constant in Ω_+, say $u^+\equiv c\in\mathbb{C}^M$ in Ω_+. Based on this and (6.194) we then conclude that

$$u^-\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}=u^+\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}=c\ \text{ at }\sigma\text{-a.e. point on }\ \partial\Omega, \tag{6.202}$$

hence also

$$\nu_j\Big((\partial_k u^-)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)-\nu_k\Big((\partial_j u^-)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)=\partial_{\tau_{jk}}\Big(u^-\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)=0 \tag{6.203}$$

for each $j,k\in\{1,\dots,n\}$ (cf. (6.195)). Keeping (6.201) and (6.203) in mind and writing (6.62) for u^- and Ω_-, we then see that $\nabla u^-=0$ in Ω_-. By once again relying on Theorem 2.4, we infer that u^- is a constant in Ω_-. In concert with (6.202) this shows that $u^-\equiv c$ in Ω_-, finishing the proof of (6.193) under the assumptions made in (6.179).

Going further, the goal is to prove (6.193) when Ω is a δ-AR domain for some sufficiently small $\delta\in(0,1)$, under the assumptions made in (6.180). As before, (6.194)–(6.196) and (6.197) are presently true. Also, from (6.62) (written for u^+ and Ω_+), (6.196), (6.197), and (6.63) (written for u^- and Ω_-) we see that for each pair of indices, $\ell\in\{1,\dots,n\}$ and $\gamma\in\{1,\dots,M\}$, and each point $x\in\Omega$ we have

$$\begin{aligned}(\partial_\ell u^+_\gamma)(x)&=\int_{\partial\Omega}a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{\ell s}}u^+_\alpha\big)(y)\,d\sigma(y)\\&\quad-\int_{\partial\Omega}(\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial^A_\nu u^+\big)_\alpha(y)\,d\sigma(y)\\&=\int_{\partial\Omega}a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{\ell s}}u^-_\alpha\big)(y)\,d\sigma(y)\\&\quad-\eta\int_{\partial\Omega}(\partial_\ell E_{\gamma\alpha})(x-y)\big(\partial^A_\nu u^-\big)_\alpha(y)\,d\sigma(y)\end{aligned}$$

$$= (1-\eta)\int_{\partial\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{\ell s}} u_\alpha^-\big)(y)\,d\sigma(y)$$
$$= (1-\eta)\int_{\partial\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{T_{\ell s}} u_\alpha^+\big)(y)\,d\sigma(y). \tag{6.204}$$

Having established this, the same type of argument which, starting with (6.122), has produced (6.130) currently gives (with the factor $1-\eta$ absorbed in T)

$$\partial_{T_{t\ell}} u^+ = (1-\eta)\big(\tfrac{1}{2}I + K_A\big)\big(\partial_{T_{t\ell}} u^+\big) + \big[M_\nu, T\big]\big(\partial_T u^+\big)$$
$$\text{at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } t,\ell \in \{1,\dots,n\}. \tag{6.205}$$

Hence, for each $t,\ell \in \{1,\dots,n\}$ we have

$$\Big(\tfrac{1+\eta}{2}\Big)\partial_{T_{t\ell}} u^+ = (1-\eta)K_A\big(\partial_{T_{t\ell}} u^+\big) + \big[M_\nu, T\big]\big(\partial_T u^+\big) \ \text{ on } \ \partial\Omega. \tag{6.206}$$

Since $\eta \neq -1$ and $A \in \mathfrak{A}_L^{\rm dis}$, much as in (6.131)–(6.134) this forces

$$\partial_{T_{t\ell}} u_\alpha^+ = 0 \ \text{ for each } \ t,\ell \in \{1,\dots,n\} \ \text{ and } \ \alpha \in \{1,\dots,M\} \tag{6.207}$$

if $\delta \in (0,1)$ is sufficiently small to begin with. Feeding this back into (6.204) then proves that $\nabla u^+ = 0$ in Ω_+, hence (cf. Theorem 2.4), u^+ is a constant in Ω_+, say

$$u^+ \equiv c^+ \in \mathbb{C}^M \ \text{ in } \ \Omega_+. \tag{6.208}$$

Based on this, (6.197), (6.195), and keeping in mind that $\eta \neq 0$, we then obtain

$$\partial_\nu^A u^- = \eta^{-1}\cdot\partial_\nu^A u^+ = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega,$$
$$\text{and } \ \partial_{T_{jk}} u_\alpha^- = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \tag{6.209}$$
$$\text{for each } j,k \in \{1,\dots,n\} \text{ and } \alpha \in \{1,\dots,M\}.$$

With this in hand, the integral representation formula (6.62) written for u^- in Ω_-, then shows that $\nabla u^- = 0$ in Ω_- thus, as before, the function u^- is a constant in Ω_-, say $u^- = c^- \in \mathbb{C}^M$ in Ω_-. The final step is to invoke equality (6.194) to write $c^+ = u^+\big|_{\partial\Omega}^{\kappa-\rm n.t.} = u^-\big|_{\partial\Omega}^{\kappa-\rm n.t.} = c^-$, which completes the proof of (6.193) under the assumptions made in (6.180). This completes the treatment of item *(a)*.

To deal with the claims in item *(b)*, work under the assumptions made in (6.181), i.e., $A \in \mathfrak{A}_L^{\rm dis}$, $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$, and $\eta \in \mathbb{C}\setminus\{-1\}$. Then Theorems 4.8 and 4.11 ensure the existence of some threshold $\delta \in (0,1)$, whose nature is as specified in the statement of the present theorem, such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ it follows that the operators

$$-\tfrac{\eta+1}{2}I+(\eta-1)K^{\#}_{A^\top}:\big[L^p(\partial\Omega,w)\big]^M\longrightarrow\big[L^p(\partial\Omega,w)\big]^M,\tag{6.210}$$

and

$$\big[S_{\rm mod}\big]:\big[L^p(\partial\Omega,w)\big]^M\longrightarrow\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M,\tag{6.211}$$

are invertible. Assuming this is the case, it is meaningful to define $u^\pm$ as in (6.182)–(6.183). In view of (6.191) and item *(c)* in Proposition 3.5 (used both for Ω_+ and Ω_-), these functions satisfy the first three conditions in (6.178), the estimates claimed in (6.184), and we have (keeping (6.192) and (6.191) in mind)

$$\begin{aligned}\partial^A_\nu u^+-\eta\cdot\partial^A_\nu u^- &= \big(-\tfrac12 I+K^{\#}_{A^\top}\big)h_0+\big(-\tfrac12 I+K^{\#}_{A^\top}\big)h_1\\ &\quad-\eta(-1)\big(-\tfrac12 I-K^{\#}_{A^\top}\big)h_0\\ &=\Big(-\tfrac{\eta+1}{2}I+(1-\eta)K^{\#}_{A^\top}\Big)h_0+\big(-\tfrac12 I+K^{\#}_{A^\top}\big)h_1\\ &=f-\big(-\tfrac12 I+K^{\#}_{A^\top}\big)h_1+\big(-\tfrac12 I+K^{\#}_{A^\top}\big)h_1\\ &=f\ \text{ at }\sigma\text{-a.e. point on }\ \partial\Omega.\end{aligned}\tag{6.212}$$

Finally, thanks to (3.42)–(3.47), (2.575), and (6.191), we see that

$$\begin{aligned}u^+\big|^{\kappa-\text{n.t.}}_{\partial\Omega}-u^-\big|^{\kappa-\text{n.t.}}_{\partial\Omega}&=S_{\rm mod}h_0+S_{\rm mod}h_1+c-S_{\rm mod}h_0\\ &=S_{\rm mod}h_1+c=g\ \text{ at }\sigma\text{-a.e. point on }\ \partial\Omega.\end{aligned}\tag{6.213}$$

Hence, the functions $u^\pm$ defined as in (6.182)–(6.183) solve the Transmission Problem (6.178) and satisfy the estimates demanded in (6.184).

An alternative proof of the solvability of the Transmission Problem (6.178) in the case when $A\in\mathfrak{A}^{\rm dis}_L$, $A^\top\in\mathfrak{A}^{\rm dis}_{L^\top}$, and $\eta\in\mathbb{C}\setminus\{-1\}$, which now employs double layers in the integral representation of the solution, goes as follows. First, item *(2)* in Theorem 4.13 guarantees that the operator (4.392) is surjective. Together with the Open Mapping Theorem this implies that, for some constant $C\in(0,\infty)$,

$$\begin{aligned}&\text{there exists } k\in\big[\dot{L}^p_1(\partial\Omega,w)\big]^M \text{ with } \partial^A_\nu(\mathcal{D}_{A,{\rm mod}}k)=f \text{ and such}\\ &\text{that } \|k\|_{[\dot{L}^p_1(\partial\Omega,w)]^M}\leq C\|f\|_{[L^p(\partial\Omega,w)]^M}.\end{aligned}\tag{6.214}$$

Also, since $A\in\mathfrak{A}^{\rm dis}_L$ and $\eta\in\mathbb{C}\setminus\{-1\}$, from Theorem 4.12 we see that

$$-\tfrac{\eta+1}{2}I+(1-\eta)\big[K_{A,{\rm mod}}\big]:\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M\longrightarrow\big[\dot{L}^p_1(\partial\Omega,w)/\sim\big]^M\tag{6.215}$$

is an invertible operator. Consequently, there exists $\psi_1 \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M$ such that

$$\big(-\tfrac{\eta+1}{2}I + (1-\eta)K_{A,\mathrm{mod}}\big)\psi_1 = g - \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)k - c \tag{6.216}$$

for some constant $c \in \mathbb{C}^M$, and

$$\|\psi_1\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \leq C\Big\|g - \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)k\Big\|_{[\dot{L}^p_1(\partial\Omega,w)]^M} \tag{6.217}$$

for some constant $C \in (0,\infty)$ independent of f, g. To proceed, introduce

$$\psi_0 := k - \eta \cdot \psi_1 \in \big[\dot{L}^p_1(\partial\Omega, w)\big]^M \tag{6.218}$$

and, finally, define the functions $u^\pm$ as in (6.185) for these choices of ψ_0, ψ_1, and c. Then Theorem 3.5 gives that $u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M$ satisfy $Lu^\pm = 0$ in $\Omega_\pm$ and $\mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega, w)$. Moreover,

$$\begin{aligned}
u^+\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - u^-\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} &= \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)\psi_0 + c - \big(\tfrac{1}{2}I - K_{A,\mathrm{mod}}\big)\psi_1 \\
&= \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)(k - \eta\cdot\psi_1) + c - \big(\tfrac{1}{2}I - K_{A,\mathrm{mod}}\big)\psi_1 \\
&= \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)k + \big(-\tfrac{\eta+1}{2}I + (1-\eta)K_{A,\mathrm{mod}}\big)\psi_1 + c \\
&= (g - c) + c = g,
\end{aligned} \tag{6.219}$$

by (6.185), (3.134), and (6.216) (keeping in mind (6.191)). In addition,

$$\begin{aligned}
\partial^A_\nu u^+ - \eta\cdot\partial^A_\nu u^- &= \big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)\psi_0 + \eta\big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)\psi_1 \\
&= \big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)(\psi_0 + \eta\cdot\psi_1) = \big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)k = f,
\end{aligned} \tag{6.220}$$

thanks to (6.192), (3.135), and (6.214). This goes to show that (u^+, u^-) is, as claimed, a solution of the Transmission Problem (6.178). Furthermore, the estimate recorded in (6.186) is a consequence of (6.214), (6.217), (6.218), and Theorem 3.6.

At this stage, all claims pertaining to existence and estimates in item *(b)* have been established. The fact that, in the current setting, any two solutions of the Transmission Problem (6.178) differ by a constant is a consequence of the assumptions in (6.181) and item *(a)*. As regards additional integrability properties for the solution of the Transmission Problem (6.178), the right-pointing implication in (6.187) together with the right-pointing inequality in (6.188) are consequences of (6.74) and the fact that we have

$$|f| = \left|\partial_\nu^A u^+ - \eta \cdot \partial_\nu^A u^-\right| \le C\big(\mathcal{N}_\kappa(\nabla u^+) + \mathcal{N}_\kappa(\nabla u^-)\big) \\ \text{at } \sigma - \text{a.e. point on } \partial\Omega. \tag{6.221}$$

The left-pointing implication in (6.187) along with the left-pointing inequality in (6.188) are seen from (7.290), (6.182), (4.342), Remarks 4.21, 4.22, Theorem 3.5, and Proposition 3.5.

Let us now justify the claim made in item *(c)*. Fix some arbitrary transmission parameter $\eta \in \mathbb{C} \setminus \{-1\}$. Also, pick a coefficient matrix $A = (a_{jk})_{1\le j,k\le n} \in \mathbb{C}^{n\times n}$ whose entries satisfy

$$a_{jk} + a_{kj} = 2\delta_{jk} \quad \text{for each} \quad j, k \in \{1, \dots, n\}. \tag{6.222}$$

This condition simply ensures that

$$\Delta = a_{jk}\partial_j\partial_k. \tag{6.223}$$

The goal is to show that we may choose a coefficient matrix A as above together with some boundary datum $f \in L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ such that the Transmission Problem

$$\begin{cases} u^\pm \in \mathscr{C}^\infty(\mathbb{R}^n_\pm), \\ \Delta u^\pm = 0 \;\text{ in }\; \mathbb{R}^n_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \\ u^+\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} - u^-\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_-} = 0 \;\text{ at } \mathcal{L}^{n-1}\text{-a.e. point on }\; \mathbb{R}^{n-1}, \\ \partial_\nu^A u^+ - \eta \cdot \partial_\nu^A u^- = f \;\text{ at } \mathcal{L}^{n-1}\text{-a.e. point on }\; \mathbb{R}^{n-1} \end{cases} \tag{6.224}$$

does not have a solution. To this end, observe that the first three conditions above guarantee that there exists a function $h \in L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ such that

$$u^\pm = \mathscr{S}_{\text{mod}} h \;\text{ in }\; \mathbb{R}^n_\pm. \tag{6.225}$$

Indeed, if $A_o := I_{n\times n}$, then the function $f_o := \partial_\nu^{A_o} u^+ - \eta \cdot \partial_\nu^{A_o} u^-$ belongs to $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ and $u^\pm$ solve the Transmission problem (6.178) in the case when $L = \Delta$, $\Omega = \mathbb{R}^n_+$, $p = 2$, $w \equiv 1$, and corresponding to the boundary data $g := 0$ and $f := f_o$. Then what we have proved in item *(b)* (cf. (6.182)–(6.183)) implies (6.225). Granted (6.225), using the last boundary condition in (6.224) and reasoning as in (6.212) shows that we have

$$\begin{aligned} f &= \partial_\nu^A u^+ - \eta \cdot \partial_\nu^A u^- = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)h - \eta(-1)\big(-\tfrac{1}{2}I - K^\#_{A^\top}\big)h \\ &= \Big(-\tfrac{\eta+1}{2}I + (1-\eta)K^\#_{A^\top}\Big)\,h \;\text{ at } \mathcal{L}^{n-1}\text{-a.e. point on }\; \mathbb{R}^{n-1}. \end{aligned} \tag{6.226}$$

Thus, in order for the Transmission Problem (6.224) to be solvable, f must necessarily be in the range of the operator $\frac{\eta+1}{2(\eta-1)}I+K^{\#}_{A^\top}$ acting on $L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. As such, in order to find an example for which the Transmission Problem (6.224) may not be solvable for arbitrary boundary data f in $L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$, it suffices to produce an example of a coefficient matrix $A=(a_{jk})_{1\leq j,k\leq n}\in\mathbb{C}^{n\times n}$ whose entries satisfy (6.222) for which the operator $\frac{\eta+1}{2(\eta-1)}I+K^{\#}_{A^\top}$ fails to be surjective on $L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. In this regard, first note that, straight from definitions, for any function $\phi\in L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ we have

$$
\begin{aligned}
\big(K^{\#}_{A^\top}\phi\big)(x') &= \lim_{\varepsilon\to0^+}\int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} a_{jn}(\partial_j E_\Delta)(x'-y',0)\phi(y')\,\mathrm{d}y' \\
&= \frac{1}{2}\sum_{j=1}^{n-1}a_{jn}(R_j\phi)(x') \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. } x'\in\mathbb{R}^{n-1}, \qquad (6.227)
\end{aligned}
$$

where E_Δ is the standard fundamental solution for the Laplacian (cf. (3.404)), and where R_j is the j-th Riesz transform in $\mathbb{R}^{n-1}$. In view of this, we may reformulate our goal as the task of finding a coefficient matrix $A=(a_{jk})_{1\leq j,k\leq n}\in\mathbb{C}^{n\times n}$ whose entries satisfy (6.222) for which the operator

$$
T:=\frac{\eta+1}{\eta-1}I+\sum_{j=1}^{n-1}a_{jn}R_j \qquad (6.228)
$$

fails to be surjective on $L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. Bring in the Fourier transform $\mathcal{F}'$ in $\mathbb{R}^{n-1}$. Since, as is well known (see, e.g., [102, (4.9.15), p. 183]), for each given function $\phi\in L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ and each $j\in\{1,\dots,n-1\}$ we have

$$
\mathcal{F}'\big(R_j\phi\big)(\xi')=(-\mathrm{i})\frac{\xi_j}{|\xi'|}\big(\mathcal{F}'\phi\big)(\xi'),\qquad \xi'\in\mathbb{R}^{n-1}\setminus\{0\}, \qquad (6.229)
$$

it follows that

$$
\mathcal{F}'\big(T\phi\big)=m\mathcal{F}'\phi \ \text{ for each } \ \phi\in L^2(\mathbb{R}^{n-1},\mathcal{L}^{n-1}), \qquad (6.230)
$$

where, for each $xi'=(\xi_1,\dots,\xi_{n-1})\in\mathbb{R}^{n-1}\setminus\{0\}$, we have set

$$
m(\xi'):=\frac{\eta+1}{\eta-1}+(-\mathrm{i})\sum_{j=1}^{n-1}\frac{a_{jn}\xi_j}{|\xi'|}. \qquad (6.231)
$$

Thanks to (6.230) and Plancherel's theorem, the operator T is surjective if and only if m only vanishes of a set of Lebesgue measure zero in $\mathbb{R}^{n-1}$ and $1/m$ is essentially bounded in $\mathbb{R}^{n-1}$. To prevent T from being surjective, it therefore suffices to choose A so that m vanishes somewhere in $\mathbb{R}^{n-1}\setminus\{0\}$. For example, this is the case whenever

$$\begin{aligned} &A = I_{n\times n} + C \quad\text{with}\quad C = (c_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n} \\ &\text{satisfying}\quad C^{\top} = -C \quad\text{as well as}\quad (c_{jn})_{1\leq j\leq n-1} \in i\,S^{n-2}. \end{aligned} \tag{6.232}$$

In particular, this precludes A from being the identity, hence from being a distinguished coefficient tensor for the Laplacian. Ultimately, the conclusion is that, even for $L = \Delta$ and $\Omega = \mathbb{R}^n_+$, if $A \notin \mathfrak{A}^{\rm dis}_L$ then the Transmission Problem (6.178) may fail to be solvable when $p = 2$ and $w \equiv 1$. This concludes the treatment of item *(c)*.

To deal with the claims in item *(d)*, suppose for the remainder of the proof that $\eta = 1$ and that Ω is a two-sided NTA domain with an unbounded Ahlfors regular boundary. Consider $u^{\pm}$ defined as in (6.190). Since we are presently assuming that Ω is a UR domain, from Theorem 3.5 and item *(c)* in Proposition 3.5 we see that $u^{\pm} \in \big[\mathscr{C}^{\infty}(\Omega_{\pm})\big]^M$ satisfy $Lu^{\pm} = 0$ in $\Omega_{\pm}$ as well as $\mathcal{N}_{\kappa}(\nabla u^{\pm}) \in L^p(\partial\Omega, w)$. In addition,

$$\begin{aligned} u^{+}\big|^{\kappa-\text{n.t.}}_{\partial\Omega} - u^{-}\big|^{\kappa-\text{n.t.}}_{\partial\Omega} &= \big(\tfrac12 I + K_{A,\text{mod}}\big)g - S_{A,\text{mod}}f \\ &\quad - \big(-\tfrac12 I + K_{A,\text{mod}}\big)g + S_{A,\text{mod}}f = g, \end{aligned} \tag{6.233}$$

by (3.134) and (3.47) (also keeping in mind (2.575)). Also,

$$\begin{aligned} \partial^A_\nu u^{+} - \partial^A_\nu u^{-} &= \big(\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big)g - \big(-\tfrac12 I + K^{\#}_{A^\top}\big)f \\ &\quad - \big(\partial^A_\nu \mathcal{D}_{A,\text{mod}}\big)g + \big(\tfrac12 I + K^{\#}_{A^\top}\big)f = f, \end{aligned} \tag{6.234}$$

thanks to (6.192), (3.126), and (3.135). The conclusion is that (u^+, u^-) is indeed a solution of the Transmission Problem (6.178).

Let us next justify (6.193) in the case when (6.189) holds (hence $\eta = 1$ and Ω is a two-sided NTA domain with an unbounded Ahlfors regular boundary). To this end, assume $u^{\pm}$ solve the homogeneous version of the Transmission Problem (6.178) formulated with $\eta = 1$. The off-diagonal Carleson measure estimate of reverse Hölder type from Proposition 2.5 ensures the existence of a constant $C \in (0,\infty)$ with the property that for every point $x \in \partial\Omega$ and every radius $r \in (0,\infty)$ we have

$$\Big(\fint_{\Omega_{\pm}\cap B(x,r)} |\nabla u^{\pm}|^{\frac{np}{n-1}}\,d\mathcal{L}^n\Big)^{\frac{n-1}{np}} \leq C\Big(\fint_{\partial\Omega\cap B(x,Cr)} \big(\mathcal{N}_{\kappa}(\nabla u^{\pm})\big)^p\,d\sigma\Big)^{\frac1p}. \tag{6.235}$$

In concert with (2.525), this permits us to estimate

$$\Big(\fint_{\Omega_\pm\cap B(x,r)}|\nabla u^\pm|^{\frac{np}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}}\leq C[w]_{A_p}^{1/p}\Big(\fint_{\partial\Omega\cap B(x,Cr)}\big(\mathcal{N}_\kappa(\nabla u^\pm)\big)^p\,\mathrm{d}w\Big)^{\frac{1}{p}}$$
$$\leq\frac{C[w]_{A_p}^{1/p}}{w\big(\partial\Omega\cap B(x,Cr)\big)^{\frac{1}{p}}}\cdot\big\|\mathcal{N}_\kappa(\nabla u^\pm)\big\|_{L^p(\partial\Omega,w)} \tag{6.236}$$

for every $x\in\partial\Omega$ and every $r\in(0,\infty)$; in particular,

$$\nabla u^\pm\in\big[L^{np/(n-1)}(\Omega_\pm\cap B(x,r),\mathcal{L}^n)\big]^{M\cdot n}\quad\text{for each } x\in\partial\Omega \text{ and } r\in(0,\infty). \tag{6.237}$$

Likewise, from Proposition 2.5 and (2.525) we see that there exists some constant $C\in(0,\infty)$ such that for every point $x\in\partial\Omega$ and every radius $r\in(0,\infty)$ we have

$$\Big(\fint_{\Omega_\pm\cap B(x,r)}|u^\pm|^{\frac{np}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}}$$
$$\leq C[w]_{A_p}^{1/p}\Big(\fint_{\partial\Omega\cap B(x,Cr)}\big(\mathcal{N}_\kappa^{Cr}u^\pm\big)^p\,\mathrm{d}w\Big)^{\frac{1}{p}}<+\infty, \tag{6.238}$$

since (cf. (6.178) and (6.74))

$$\mathcal{N}_\kappa^{Cr}u\in L^p_{\mathrm{loc}}(\partial\Omega,w). \tag{6.239}$$

In particular,

$$u^\pm\in\big[L^{np/(n-1)}(\Omega_\pm\cap B(x,r),\mathcal{L}^n)\big]^{M}\quad\text{for each } x\in\partial\Omega \text{ and } r\in(0,\infty). \tag{6.240}$$

Also, if we consider the function defined $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$ as

$$u:=\begin{cases}u^+ & \text{in } \ \Omega_+,\\ u^- & \text{in } \ \Omega_-,\end{cases} \tag{6.241}$$

then u is $\mathcal{L}^n$-measurable and (6.238) implies that

$$u\in\big[L^{np/(n-1)}_{\mathrm{loc}}(\mathbb{R}^n,\mathcal{L}^n)\big]^M\hookrightarrow\big[L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathcal{L}^n)\big]^M. \tag{6.242}$$

Let $(\nu_1, \ldots, \nu_n)$ denote the scalar components of the geometric measure theoretic outward unit normal ν to Ω. Then for each index $j \in \{1, \ldots, n\}$ and each vector-valued test function $\varphi \in \big[\mathscr{C}_0^\infty(\mathbb{R}^n)\big]^M$, we may compute (with the first two pairings considered in the sense of distributions in $\mathbb{R}^n$)

$$\begin{aligned}
\langle \partial_j u, \varphi\rangle &= -\langle u, \partial_j \varphi\rangle = -\int_{\mathbb{R}^n} \langle u, \partial_j\varphi\rangle \, \mathrm{d}\mathcal{L}^n \\
&= -\int_{\Omega_+} \langle u^+, \partial_j \varphi\rangle \, \mathrm{d}\mathcal{L}^n - \int_{\Omega_-} \langle u^-, \partial_j\varphi\rangle \, \mathrm{d}\mathcal{L}^n \\
&= \int_{\Omega_+} \langle \partial_j u^+, \varphi\rangle \, \mathrm{d}\mathcal{L}^n - \int_{\partial\Omega} \nu_j \Big\langle \Big(u^+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big), \varphi\Big\rangle \, \mathrm{d}\sigma \\
&\quad + \int_{\Omega_-} \langle \partial_j u^-, \varphi\rangle \, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} \nu_j \Big\langle \Big(u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big), \varphi\Big\rangle \, \mathrm{d}\sigma \\
&= \int_{\Omega_+} \langle \partial_j u^+, \varphi\rangle \, \mathrm{d}\mathcal{L}^n + \int_{\Omega_-} \langle \partial_j u^-, \varphi\rangle \, \mathrm{d}\mathcal{L}^n.
\end{aligned} \tag{6.243}$$

Above, the fourth equality is provided by the integration by parts formula proved in [111, §1.7], whose present applicability is ensured by (6.178), (6.74), (6.237), (6.240), and the fact that (6.239) together with (2.576) imply

$$\mathcal{N}_\kappa^{Cr} u \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma) \quad \text{for each} \quad r \in (0, \infty). \tag{6.244}$$

Also, the last equality in (6.243) uses (6.191) and the fact that we are currently assuming $u^+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$. In turn, from (6.243) and (6.237) we conclude that, with the derivatives computed in the sense of distributions, for each $j \in \{1, \ldots, n\}$ we have

$$\partial_j u \in \big[L^{np/(n-1)}_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)\big]^M \tag{6.245}$$

and, in fact,

$$\partial_j u = \begin{cases} \partial_j u^+ & \text{in } \ \Omega_+, \\ \partial_j u^- & \text{in } \ \Omega_-. \end{cases} \tag{6.246}$$

Moreover, combining (6.246) with (6.236) shows that there exists some constant $C \in (0, \infty)$ with the property that for every $x \in \partial\Omega$ and every $r \in (0, \infty)$ we have

$$\begin{aligned}
&\Big(\fint_{B(x,r)} |\nabla u|^{\frac{np}{n-1}} \, \mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}} \\
&\qquad \leq C\Big(\fint_{\Omega_+\cap B(x,r)} |\nabla u^+|^{\frac{np}{n-1}} \, \mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}} \\
&\qquad\quad + C\Big(\fint_{\Omega_-\cap B(x,r)} |\nabla u^-|^{\frac{np}{n-1}} \, \mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}} \qquad (6.247) \\
&\qquad \leq \frac{C[w]_{A_p}^{1/p}}{w\big(\partial\Omega \cap B(x, Cr)\big)^{\frac{1}{p}}} \Big(\big\|\mathcal{N}_\kappa(\nabla u^+)\big\|_{L^p(\partial\Omega,w)} + \big\|\mathcal{N}_\kappa(\nabla u^-)\big\|_{L^p(\partial\Omega,w)}\Big).
\end{aligned}$$

To proceed, consider now an arbitrary point $x \in \mathbb{R}^n$ and pick some $x_* \in \partial\Omega$ such that $\mathrm{dist}(x, \partial\Omega) = |x - x_*|$. Since $B(x, r) \subseteq B(x_*, 2r)$ for each $r > \mathrm{dist}(x, \partial\Omega)$, we conclude from (6.247) that there exists $C \in (0, \infty)$ such that

$$\begin{aligned}
&\Big(\fint_{B(x,r)} |\nabla u|^{\frac{np}{n-1}} \, \mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}} \\
&\qquad \leq C\Big(\fint_{B(x_*,2r)} |\nabla u|^{\frac{np}{n-1}} \, \mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}} \qquad (6.248) \\
&\qquad \leq \frac{C[w]_{A_p}^{1/p}}{w\big(\partial\Omega \cap B(x_*, Cr)\big)^{\frac{1}{p}}} \Big(\big\|\mathcal{N}_\kappa(\nabla u^+)\big\|_{L^p(\partial\Omega,w)} + \big\|\mathcal{N}_\kappa(\nabla u^-)\big\|_{L^p(\partial\Omega,w)}\Big)
\end{aligned}$$

for every point $x \in \mathbb{R}^n$ and every radius $r > \mathrm{dist}(x, \partial\Omega)$, where $x_* \in \partial\Omega$ is such that $\mathrm{dist}(x, \partial\Omega) = |x - x_*|$.

We next claim that

$$Lu = 0 \text{ in the sense of distributions in } \mathbb{R}^n. \qquad (6.249)$$

To justify this, pick an arbitrary vector-valued test function $\varphi \in \big[\mathscr{C}_0^\infty(\mathbb{R}^n)\big]^M$ and write (with the first two pairings considered in the sense of distributions in $\mathbb{R}^n$)

$$\begin{aligned}
\langle Lu, \varphi\rangle &= \langle u, L^\top \varphi\rangle = \int_{\mathbb{R}^n} \langle u, L^\top\varphi\rangle \, \mathrm{d}\mathcal{L}^n \\
&= \int_{\Omega_+} \langle u^+, L^\top\varphi\rangle \, \mathrm{d}\mathcal{L}^n + \int_{\Omega_-} \langle u^-, L^\top\varphi\rangle \, \mathrm{d}\mathcal{L}^n
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_+} \langle Lu^+, \varphi\rangle \, \mathrm{d}\mathcal{L}^n + \int_{\partial\Omega} \Big\langle u^+\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \partial_\nu^{A^\top}\varphi\Big\rangle \, \mathrm{d}\sigma - \int_{\partial\Omega} \big\langle \partial_\nu^A u^+, \varphi\big\rangle \, \mathrm{d}\sigma \\
&\quad + \int_{\Omega_-} \langle Lu^-, \varphi\rangle \, \mathrm{d}\mathcal{L}^n - \int_{\partial\Omega} \Big\langle u^-\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \partial_\nu^{A^\top}\varphi\Big\rangle \, \mathrm{d}\sigma + \int_{\partial\Omega} \big\langle \partial_\nu^A u^-, \varphi\big\rangle \, \mathrm{d}\sigma \\
&= 0.
\end{aligned}
\tag{6.250}
$$

The fourth equality in (6.250) is a consequence of the Green type formula for second-order systems established in [113, §1.7], whose present applicability is guaranteed by (6.178), (6.74), Theorem 3.4, (6.237), (6.240), and the fact that (6.239) together with (2.576) entail

$$
\mathcal{N}_\kappa(\nabla u) \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma) \ \text{ and } \ \mathcal{N}_\kappa^{Cr} u \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma) \ \text{ for all } \ r \in (0, \infty).
\tag{6.251}
$$

In addition, the last equality in (6.250) uses (6.178), (6.191), plus the fact that we are now assuming $u^+\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ and $\partial_\nu^A u^+ = \partial_\nu^A u^-$. This establishes (6.250) which, in turn, proves (6.249).

As a consequence of (6.249) and elliptic regularity, $u \in \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^M$. In particular, for each index $j \in \{1, \dots, n\}$ we have $\partial_j u \in \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^M$ as well as $L(\partial_j u) = \partial_j(Lu) = 0$, since L has constant coefficients. Bearing this in mind, interior estimates for weakly elliptic systems proved in [102, Theorem 11.12, p. 415] give

$$
|(\nabla u)(x)| \leq C\Big(\fint_{B(x,r)} |\nabla u|^{\frac{np}{n-1}} \, \mathrm{d}\mathcal{L}^n \Big)^{\frac{n-1}{np}}
\tag{6.252}
$$

for every point $x \in \mathbb{R}^n$ and every radius $r \in (0, \infty)$. Together with (6.248) this implies

$$
\begin{aligned}
&|(\nabla u)(x)| \\
&\leq \frac{C[w]_{A_p}^{1/p}}{w\big(\partial\Omega \cap B(x_*, Cr)\big)^{\frac{1}{p}}} \Big(\big\|\mathcal{N}_\kappa(\nabla u^+)\big\|_{L^p(\partial\Omega, w)} + \big\|\mathcal{N}_\kappa(\nabla u^-)\big\|_{L^p(\partial\Omega, w)} \Big)
\end{aligned}
\tag{6.253}
$$

for every point $x \in \mathbb{R}^n$ and every radius $r > \operatorname{dist}(x, \partial\Omega)$, where $x_* \in \partial\Omega$ is such that $\operatorname{dist}(x, \partial\Omega) = |x - x_*|$. At this stage, upon recalling (2.540) and the fact that $\mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega, w)$ (cf. (6.178)), after passing to limit $r \to \infty$ in (6.253) we arrive at the conclusion that

$$
(\nabla u)(x) = 0 \ \text{ for each point } \ x \in \mathbb{R}^n.
\tag{6.254}
$$

Hence, u is constant in $\mathbb{R}^n$, from which (6.193) readily follows on account of (6.241). This finishes the proof of (6.193) under the assumption made in (6.189).

The proof of Theorem 6.15 is therefore complete. □

We continue by making a series of remarks aimed at further exploring the nature of Theorem 6.15.

Remark 6.11 In various special circumstances, the statement of Theorem 6.15 may be further streamlined. For example, Theorem 3.8 gives that if the system L actually satisfies the strong Legendre–Hadamard ellipticity condition then in place of either set of conditions specified in (6.179), (6.180), (6.181) we may simply assume

$$A \in \mathfrak{A}_L^{\mathrm{dis}} \text{ and } \eta \in \mathbb{C} \setminus \{-1\}. \tag{6.255}$$

Also, if $n \geq 3$, $M = 1$, and the matrix $A \in \mathfrak{A}_L$ is symmetric then, thanks to (3.223), either set of conditions specified in (6.179), (6.180), (6.255) may simply be replaced by just the demand that $\eta \in \mathbb{C} \setminus \{-1\}$.

Remark 6.12 There is another boundary value problem, closely related to the Transmission Problem (6.178), in which the transmission parameter shows up in the formulation of the Dirichlet boundary condition (as opposed to the Neumann boundary condition, as was the case in (6.178)). Specifically, retaining the background assumptions made in Theorem 6.15 now consider

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^{\infty}(\Omega_{\pm})\big]^M, \\ Lu^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathcal{N}_{\kappa}(\nabla u^{\pm}) \in L^p(\partial\Omega, w), \\ u^{+}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \eta \cdot u^{-}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = g \in \big[\dot{L}_1^p(\partial\Omega, w)\big]^M, \\ \partial_\nu^A u^{+} - \partial_\nu^A u^{-} = f \in \big[L^p(\partial\Omega, w)\big]^M. \end{cases} \tag{6.256}$$

When $\eta \neq 0$, working with the functions $v^{+} := u^{+}$ in Ω_{+} and $v^{-} := \eta \cdot u^{-}$ in Ω_{-}, matters are readily reduced to the "standard" Transmission Problem (6.178) written with η^{-1} in place of η. When $\eta = 0$ it follows that (6.256) decouples into a Homogeneous Regularity Problem for the function u^{+} in Ω_{+}, and a Neumann Problem for the function u^{-} in Ω_{-} with boundary datum $\partial_\nu^A u^{+} - f$. In particular, we have solvability results for (6.256) which are similar to those in Theorem 6.15.

Remark 6.13 Much as in the case of the Tangential Derivative Problem (6.97), we may re-fashion the Transmission Problem (6.178) as

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^{\infty}(\Omega_{\pm})\big]^{M}, \\ Lu^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathcal{N}_{\kappa}(\nabla u^{\pm}) \in L^{p}(\partial\Omega, w), \\ \Big\{\nu_j\Big((\partial_k u^{+})\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \nu_k\Big((\partial_j u^{+})\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\Big\} \\ \quad -\Big\{\nu_j\Big((\partial_k u^{-})\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \nu_k\Big((\partial_j u^{-})\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\Big\} = \partial_{\tau_{jk}} g, \\ \quad \text{at } \sigma - \text{a.e. point on } \partial\Omega, \ \text{ for each } \ j,k \in \{1,\dots,n\}, \\ \partial_{\nu}^{A} u^{+} - \eta \cdot \partial_{\nu}^{A} u^{-} = f \in \big[L^{p}(\partial\Omega, w)\big]^{M}, \end{cases} \tag{6.257}$$

where the function g is arbitrarily specified in $\big[\dot{L}_1^{p}(\partial\Omega, w)\big]^{M}$, the homogeneous Muckenhoupt weighted boundary Sobolev space defined in (2.598). For this boundary value problem, similar results as in Theorem 6.15 continue to be valid.

Remark 6.14 Under the same background assumptions made in Theorem 6.15 (and with the same conventions adopted there), it is of interest to single out the special case corresponding to having $g = 0$ in (6.178), i.e., consider the following `Reduced Transmission Problem`:

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^{\infty}(\Omega_{\pm})\big]^{M}, \\ Lu^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathcal{N}_{\kappa}(\nabla u^{\pm}) \in L^{p}(\partial\Omega, w), \\ u^{+}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = u^{-}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \partial_{\nu}^{A} u^{+} - \eta \cdot \partial_{\nu}^{A} u^{-} = f \in \big[L^{p}(\partial\Omega, w)\big]^{M}. \end{cases} \tag{6.258}$$

Running the same argument as in the proof of Theorem 6.15, this time we no longer need to assume that the operator in (6.211) is an isomorphism, ultimately allows us to impose lighter demands on the nature of the system L and the coefficient tensor A. Specifically, now working under the sole assumption that $A^{\top} \in \mathfrak{A}_{L^{\top}}^{\text{dis}}$ and $\eta \in \mathbb{C} \setminus \{-1\}$, the same proof as before shows that there exists $\delta \in (0,1)$ which depends only on n, p, $[w]_{A_p}$, A, η, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence the set Ω is a δ-AR domain) then the operator $-\frac{\eta+1}{2} I + (1-\eta) K^{\#}_{A^{\top}}$ is invertible on the Muckenhoupt weighted Lebesgue space $\big[L^{p}(\partial\Omega, w)\big]^{M}$ and the functions $u^{\pm} : \Omega_{\pm} \to \mathbb{C}^{M}$ defined as

$$u^{\pm}(x) := \Big(\mathscr{S}_{\text{mod}}\Big(-\tfrac{\eta+1}{2} I + (1-\eta) K^{\#}_{A^{\top}}\Big)^{-1} f\Big)(x) \ \text{ for } \ x \in \Omega_{\pm} \tag{6.259}$$

solve the Reduced Transmission Problem formulated in (6.258) and satisfy, for some constant $C \in (0, \infty)$ independent of f,

$$\big\|\mathcal{N}_\kappa(\nabla u^{\pm})\big\|_{L^p(\partial\Omega, w)} \leq C\|f\|_{[L^p(\partial\Omega,w)]^M}. \tag{6.260}$$

Moreover, the result established in item *(a)* of Theorem 6.15 working under the hypotheses in (6.179) gives uniqueness (modulo constants) for the Reduced Transmission Problem (6.258). Hence, well posedness follows by simply assuming that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$.

Remark 6.15 Once again, for familiar reasons, a similar solvability result to the one established in Theorem 6.15 turns out to be true for the Transmission Problem with data in Lorentz spaces, i.e., for

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^\infty(\Omega_{\pm})\big]^M, \\ Lu^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathcal{N}_\kappa(\nabla u^{\pm}) \in L^{p,q}(\partial\Omega, \sigma), \\ u^+\big|^{\kappa-\text{n.t.}}_{\partial\Omega} - u^-\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = g \in \big[\dot{L}^{p,q}_1(\partial\Omega, \sigma)\big]^M, \\ \partial^A_\nu u^+ - \eta\cdot\partial^A_\nu u^- = f \in \big[L^{p,q}(\partial\Omega, \sigma)\big]^M, \end{cases} \tag{6.261}$$

with $p \in (1, \infty)$ and $q \in (0, \infty]$, where $\dot{L}^{p,q}_1(\partial\Omega, \sigma)$ is the Lorentz-based homogeneous Sobolev space defined in an analogous fashion to (2.598). The reader is referred to Theorem 7.23 (and also Examples 8.2, 8.6) for more general results of this type.

Remark 6.16 Thanks to (3.228)–(3.229), Theorem 6.15 is applicable to the Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ in $\mathbb{R}^n$ with $n \geq 2$, assuming $\mu \neq 0$, $2\mu + \lambda \neq 0$, $3\mu + \lambda \neq 0$, provided we work with the coefficient tensor $A(\zeta)$ defined as in (3.226) for the choice $\zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}$. In addition, when $n = 2$, we may rely on the invertibility result from Theorem 4.14 (and duality) to conclude that the transmission boundary problem for the two-dimensional Lamé system in sufficiently flat Ahlfors regular domains in the plane is solvable when formulated in a similar fashion to (6.178) with $A := A(\zeta)$ and $\eta \in \mathbb{C} \setminus \{\pm 1\}$, for a larger range of ζ's, namely

$$\zeta \in \mathbb{C} \setminus \Big\{ \pm\frac{\eta+1}{\eta-1}\Big[\frac{2\mu(2\mu+\lambda)}{3\mu+\lambda}\Big] + \frac{\mu(\mu+\lambda)}{3\mu+\lambda}\Big\}. \tag{6.262}$$

Remark 6.17 The case of the Transmission Problem for the Laplacian in upper-graph Lipschitz domains in $\mathbb{R}^n$, with $n \geq 2$ arbitrary, has been treated in [46]. In the two-dimensional setting, for $L = \Delta$ the Laplacian and Ω an infinite sector in the plane, counterexamples to the well-posedness of the Transmission Problem (6.178) for certain values of p (related to the aperture of Ω and the transmission parameter appearing in the formulation of the problem) have been given in [105].

Remark 6.18 It is of interest to observe that

$$\text{lack of uniqueness (modulo constants) for the Homogeneous Regularity Problem for the system } L \text{ in } \Omega_- \text{ (cf. (6.64)) implies lack of uniqueness (modulo constants) for the Transmission Problem (6.178) in the case when } \eta = 0. \tag{6.263}$$

Indeed, if u^- is such that

$$\begin{cases} u^- \in \big[\mathscr{C}^\infty(\Omega_-)\big]^M, \\ Lu^- = 0 \ \text{ in } \ \Omega_-, \\ \mathcal{N}_\kappa(\nabla u^-) \in L^p(\partial\Omega, w), \\ u^-\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = c \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{cases} \tag{6.264}$$

for some constant $c \in \mathbb{C}^M$, then setting $u^+ := c$ in Ω_+ yields a pair (u^+, u^-) which is a null-solution of the Transmission Problem (6.178) formulated for $\eta = 0$.

There are two scenarios under which uniqueness (modulo constants) for the Transmission Problem (6.178) has been established in item *(a)* of Theorem 6.15. First, it was assumed that (6.179) holds and, in this case, condition $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$ alone ensures uniqueness (modulo constants) for the Homogeneous Regularity Problem for the system L in Ω_-, as noted in item *(b)* of Theorem 6.8. Second, when (6.180) is assumed, in principle it may happen that the Homogeneous Regularity Problem for the system L in Ω_- lacks uniqueness (modulo constants). However, this time (as opposed to (6.179)), we are asking that $\eta \neq 0$, so the issue singled out in (6.263) becomes a moot point. This is a heuristic explanation of the perceived asymmetry in the manner in which the sets of hypotheses (6.179) and (6.180) have been formulated.

It is possible to enhance the solvability result from Theorem 6.15 via perturbations, and our next theorem elaborates on this aspect.

Theorem 6.16 *Retain the original background assumptions on the set Ω from Theorem 6.15 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ and a transmission parameter $\eta \in \mathbb{C} \setminus \{-1\}$. Consider a system $L_o \in \mathfrak{L}^{\text{dis}}$ with $L_o^\top \in \mathfrak{L}^{\text{dis}}$ (cf. (3.195)), and fix some $A_o \in \mathfrak{A}^{\text{dis}}_{L_o}$.*

Then there exist a threshold $\delta \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of A_o in $\mathfrak{A}$, both of which depend only on n, η, p, $[w]_{A_p}$, A_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each coefficient tensor $A \in \mathcal{U}$ the Transmission Problem (6.178) *formulated for the system* L_A (*cf.* (3.7)) *and the conormal derivative associated with* A (*cf.* (3.66)) *is actually solvable, and any two solutions differ by a constant from* $\mathbb{C}^M$.

Proof This is seen by reasoning as in the proofs of Theorems 6.4 and 6.15, keeping in mind Theorem 3.9. □

We may also establish solvability results for the version of the Reduced Transmission Problem (6.258) now formulated for boundary data belonging to sums of Muckenhoupt weighted Lebesgue spaces.

Theorem 6.17 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and fix an aperture parameter* $\kappa > 0$. *Also, pick a pair of integrability exponents* $p_0, p_1 \in (1,\infty)$ *along with a pair of Muckenhoupt weights* $w_0 \in A_{p_0}(\partial\Omega,\sigma)$ *and* $w_1 \in A_{p_1}(\partial\Omega,\sigma)$. *Finally, consider a homogeneous, second-order, constant complex coefficient,* $M \times M$ *weakly elliptic system* L *in* $\mathbb{R}^n$, *and select some coefficient tensor* $A \in \mathfrak{A}_L$.

Then similar results, concerning existence, integral representation formulas, estimates, additional integrability properties, and well-posedness, as in Theorem 6.15, are valid for the Reduced Transmission Problem

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^{\pm} = 0 \ \ in \ \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^{\pm}) \in L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1), \\ u^+\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = u^-\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \ at\ \sigma\text{-a.e. point on } \ \partial\Omega, \\ \partial^A_\nu u^+ - \eta\cdot\partial^A_\nu u^- = f \in \big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M. \end{cases} \tag{6.265}$$

Proof Existence, estimates, and an integral representation formula are all established reasoning as in the proof of Theorem 6.15, using the fact that the operator $-\frac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^\top}$ is invertible on the space $\big[L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)\big]^M$ under the assumption made in item *(b)* of Theorem 6.15 (see Proposition 4.2). For uniqueness (modulo constants), we reason much as in the treatment of item *(a)* in Theorem 6.15, working under the hypotheses in (6.179). Specifically, (6.194)–(6.200) goes through since $L^{p_0}(\partial\Omega, w_0) + L^{p_1}(\partial\Omega, w_1)$ embeds into the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (cf. (2.575)), and then (4.336) used in concert with (6.200) gives (6.201). The rest is as before, and the conclusion is that any null-solution of (6.265) is a pair of identical constants. □

Chapter 7
Singular Integrals and Boundary Problems in Morrey and Block Spaces

The spaces which bear the name of Morrey have been introduced by C. Morrey in 1930s in relation to regularity problems for solutions to partial differential equations in the Euclidean setting. Membership of a function to a Morrey space amounts to a bound on the size of the L^p-integral average of said function over an arbitrary ball in terms of a fixed power of its radius. Since these are all measure-metric considerations, this brand of space naturally adapts to the more general setting of spaces of homogeneous type. Here we are concerned with the scale of Morrey spaces when the ambient is the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$. We make use of the Calderón–Zygmund theory for singular integral operators acting on Morrey spaces in such a setting as a platform that allows us to build in the direction of solving boundary value problems for weakly elliptic systems in δ-AR domains with boundary data in Morrey spaces (and their pre-duals).

7.1 Boundary Layer Potentials on Morrey and Block Spaces

The material in this section closely follows [113, §2.6]. We begin by discussing the scale of Morrey spaces on Ahlfors regular sets. To set the stage, assume $\Sigma \subseteq \mathbb{R}^n$ (where, as in the past, $n \in \mathbb{N}$ with $n \geq 2$) is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Given $p \in (0, \infty)$ and $\lambda \in (0, n-1)$, we then define the `Morrey space` $M^{p,\lambda}(\Sigma, \sigma)$ as

$$M^{p,\lambda}(\Sigma, \sigma) := \Big\{ f : \Sigma \to \mathbb{C} : \ f \text{ is } \sigma\text{-measurable and } \ \|f\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty \Big\}, \tag{7.1}$$

where, for each σ-measurable function f on Σ, we have set

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_7

$$\|f\|_{M^{p,\lambda}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ 0<R<2\,\mathrm{diam}(\Sigma)}} \left\{ R^{\frac{n-1-\lambda}{p}} \left(⨍_{\Sigma\cap B(x,R)} |f|^p \, d\sigma \right)^{\frac{1}{p}} \right\}. \tag{7.2}$$

The space $M^{p,\lambda}(\Sigma, \sigma)$ is complete, hence Banach (though not separable) when equipped with the norm (7.2), and (cf. [112, §6.2] for a proof)

$$\begin{aligned} &M^{p,\lambda}(\Sigma,\sigma) \hookrightarrow L^p_{\mathrm{loc}}(\Sigma,\sigma) \cap L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\Big) \\ &\text{if } p \in [1,\infty),\ \lambda \in (0, n-1), \text{ and } 0 \le \varepsilon < \tfrac{n-1-\lambda}{p}. \end{aligned} \tag{7.3}$$

As may be seen from (7.1)–(7.2) and Hölder's inequality, we also have

$$L^s(\Sigma,\sigma) \hookrightarrow M^{p,\lambda}(\Sigma,\sigma) \ \text{ continuously, with } \ s := \tfrac{p(n-1)}{n-1-\lambda} \in (p,\infty). \tag{7.4}$$

In particular, there exists some $C \in (0,\infty)$ which depends only on n, p, λ, and the Ahlfors regularity constant of Σ, with the property that for each σ-measurable set $E \subseteq \Sigma$ we have

$$\big\|\mathbf{1}_E\big\|_{M^{p,\lambda}(\Sigma,\sigma)} \le C\big\|\mathbf{1}_E\big\|_{L^s(\Sigma,\sigma)} = C \cdot \sigma(E)^{(n-1-\lambda)/[p(n-1)]}. \tag{7.5}$$

As a consequence, $\mathbf{1}_E$ belongs to $M^{p,\lambda}(\Sigma, \sigma)$ whenever $E \subseteq \Sigma$ is a σ-measurable set with $\sigma(E) < +\infty$. Other examples of functions belonging to Morrey spaces are presented below (see [112, §6.2]).

Example 7.1 Let Σ, σ be as above, and for each fixed point $x_o \in \Sigma$ consider the function $f_{x_o} : \Sigma \to \mathbb{R}$ defined for each $x \in \Sigma\setminus\{x_o\}$ as $f_{x_o}(x) := |x-x_o|^{-(n-1-\lambda)/p}$. Then each f_{x_o} belongs to the Morrey space $M^{p,\lambda}(\Sigma, \sigma)$ and, in fact,

$$\sup_{x_o\in\Sigma} \|f_{x_o}\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty. \tag{7.6}$$

This being said, each f_{x_o} fails to be in $L^s(\Sigma, \sigma)$ with $s := \frac{p(n-1)}{n-1-\lambda}$, so the inclusion in (7.4) is strict.

In view of (7.4) it is of interest to define the space

$$\mathring{M}^{p,\lambda}(\Sigma,\sigma) := \text{ the closure of } L^s(\Sigma,\sigma) \text{ with } s := \tfrac{p(n-1)}{n-1-\lambda} \text{ in } M^{p,\lambda}(\Sigma,\sigma). \tag{7.7}$$

Hence, by design,

$$\begin{aligned} &\mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ is a closed linear subspace of } M^{p,\lambda}(\Sigma,\sigma) \\ &\text{such that } L^s(\Sigma,\sigma) \hookrightarrow \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ continuously and} \\ &\text{densely.} \end{aligned} \tag{7.8}$$

Thus, when equipped with the norm inherited from the larger ambient $M^{p,\lambda}(\Sigma,\sigma)$, the space $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ is complete (hence Banach). As a consequence of (7.8) and (2.508) we also see that

$$\text{the space } \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ is separable.} \tag{7.9}$$

As noted in [112, §6.2],

$$\begin{array}{l}\text{the operator of pointwise multiplication by any given function } b \in L^\infty(\Sigma,\sigma) \text{ is a bounded mapping from the space} \\ \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ into itself, with operator norm} \leq \|b\|_{L^\infty(\Sigma,\sigma)},\end{array} \tag{7.10}$$

and

$$\begin{array}{l}\text{if } f, g : \Sigma \to \mathbb{C} \text{ are two } \sigma\text{-measurable functions with the property that } |g| \leq |f| \text{ at } \sigma\text{-a.e. point on } \Sigma \text{ and } f \in \mathring{M}^{p,\lambda}(\Sigma,\sigma), \\ \text{then } g \text{ also belongs to the space } \mathring{M}^{p,\lambda}(\Sigma,\sigma).\end{array} \tag{7.11}$$

In relation to the space introduced in (7.7), we also wish to remark that since $\mathrm{Lip}_{\mathrm{comp}}(\Sigma)$ (the space of Lipschitz functions with compact support on Σ) is dense in $L^s(\Sigma,\sigma)$ and since, according to (7.4), the latter space embeds continuously into $M^{p,\lambda}(\Sigma,\sigma)$, we have

$$\mathring{M}^{p,\lambda}(\Sigma,\sigma) = \text{ the closure of } \mathrm{Lip}_{\mathrm{comp}}(\Sigma) \text{ in } M^{p,\lambda}(\Sigma,\sigma). \tag{7.12}$$

An immediate corollary of the latter description of the space $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ worth mentioning is that functions f belonging to $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ enjoy the "vanishing" property

$$\lim_{\rho\to 0^+} \sup_{\substack{x\in\Sigma \text{ and} \\ R\in(0,\rho)}} \left\{ R^{\frac{n-1-\lambda}{p}} \left(\fint_{\Sigma\cap B(x,R)} |f|^p \, d\sigma \right)^{\frac{1}{p}} \right\} = 0. \tag{7.13}$$

As such, it is natural to refer to $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ as being a `vanishing Morrey space`.

The topic addressed next pertains to the pre-duals of Morrey spaces, and the duals of vanishing Morrey spaces. Continue to assume that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. To set the stage, given an integrability exponent $q \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$, a function $b \in L^q(\Sigma,\sigma)$ is said to be a $\mathcal{B}^{q,\lambda}$-`block` on Σ (or, simply, a block) provided there exist some point $x_o \in \Sigma$ and some radius $R \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ such that

$$\mathrm{supp}\, b \subseteq B(x_o, R) \cap \Sigma \quad\text{and}\quad \|b\|_{L^q(\Sigma,\sigma)} \leq R^{\lambda\left(\frac{1}{q}-1\right)}. \tag{7.14}$$

With $r := \frac{q(n-1)}{n-1+\lambda(q-1)} \in (1, q)$ we then define the block space

$$\mathcal{B}^{q,\lambda}(\Sigma,\sigma) := \Big\{ f \in L^r(\Sigma,\sigma) : \text{ there exist a numerical sequence } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and a family } \{b_j\}_{j\in\mathbb{N}} \text{ of } \mathcal{B}^{q,\lambda}\text{-blocks on } \Sigma \text{ with } f = \sum_{j=1}^{\infty} \lambda_j b_j \text{ in } L^r(\Sigma,\sigma) \Big\}, \tag{7.15}$$

and for each $f \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ define

$$\|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} := \inf \Big\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j b_j \text{ in } L^r(\Sigma,\sigma) \text{ with } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and each } b_j \text{ a } \mathcal{B}^{q,\lambda}\text{-block on } \Sigma \Big\}. \tag{7.16}$$

Work in [112, §6.2] gives that

$$\Big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma),\ \|\cdot\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}\Big) \text{ is a separable Banach space, and } \mathcal{B}^{q,\lambda}(\Sigma,\sigma) \hookrightarrow L^r(\Sigma,\sigma) \text{ with } r := \tfrac{q(n-1)}{n-1+\lambda(q-1)} \in (1,q) \tag{7.17}$$

and

the operator of pointwise multiplication by any given function $b \in L^\infty(\Sigma,\sigma)$ is a linear and bounded mapping from the space $\mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ into itself, with operator norm $\le \|b\|_{L^\infty(\Sigma,\sigma)}$. (7.18)

Note that the latter property further implies that

if $f, g : \Sigma \longrightarrow \mathbb{C}$ are two σ-measurable functions such that $|g| \le |f|$ at σ-a.e. point on Σ and $f \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma)$, then we have $g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ as well as $\|g\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} \le \|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}$. (7.19)

Examples of functions in the block space (7.15) may be produced using the following result from [112, §6.2].

Proposition 7.1 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix an exponent $q \in (1,\infty)$ along with $\lambda \in (0, n-1)$. Then for each $a > \lambda$ one has the continuous and dense embedding*

$$L^q\Big(\Sigma, (1+|x|)^{a(q-1)}\sigma(x)\Big) \hookrightarrow \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{7.20}$$

In particular,

$$\begin{aligned}&\text{if } N > \tfrac{\lambda(q-1)+n-1}{q} \text{ and } f_N(x) := (1+|x|)^{-N} \text{ for } x \in \Sigma,\\ &\text{then the function } f_N \text{ belongs to the space } \mathcal{B}^{q,\lambda}(\Sigma,\sigma).\end{aligned} \tag{7.21}$$

Our primary interest in the space (7.15) stems from the fact that this turns out to be the pre-dual of a Morrey space. In turn, vanishing Morrey spaces are pre-duals of block spaces. Specifically, we have the following result proved in [112, §6.2].

Proposition 7.2 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix two exponents $p, q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$. Then there exists $C \in (0,\infty)$ which depends only on the Ahlfors regularity constant of Σ, n, p, and λ, with the property that*

$$\begin{gathered}\int_\Sigma |f||g|\,\mathrm{d}\sigma \le C\|f\|_{M^{p,\lambda}(\Sigma,\sigma)}\|g\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}\\ \text{for all } \ f \in M^{p,\lambda}(\Sigma,\sigma) \ \text{ and } \ g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma).\end{gathered} \tag{7.22}$$

In addition, the mapping

$$\begin{aligned}&M^{p,\lambda}(\Sigma,\sigma) \ni f \longmapsto \Lambda_f \in \big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\big)^* \ \text{ given by}\\ &\Lambda_f(g) := \int_\Sigma fg\,\mathrm{d}\sigma \ \text{ for each } \ g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma)\end{aligned} \tag{7.23}$$

is a well-defined, linear, bounded isomorphism, with bounded inverse. Simply put, the integral pairing yields the quantitative identification

$$\big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\big)^* = M^{p,\lambda}(\Sigma,\sigma). \tag{7.24}$$

Furthermore, regarding $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ as a Banach space equipped with the norm inherited from $M^{p,\lambda}(\Sigma,\sigma)$, the mapping

$$\begin{aligned}&\mathcal{B}^{q,\lambda}(\Sigma,\sigma) \ni g \longmapsto \Lambda_g \in \big(\mathring{M}^{p,\lambda}(\Sigma,\sigma)\big)^* \ \text{ given by}\\ &\Lambda_g(f) := \int_\Sigma fg\,\mathrm{d}\sigma \ \text{ for each } \ f \in \mathring{M}^{p,\lambda}(\Sigma,\sigma)\end{aligned} \tag{7.25}$$

is a well-defined, linear, bounded isomorphism, with bounded inverse. As such, the integral pairing yields the identification

$$\big(\mathring{M}^{p,\lambda}(\Sigma,\sigma)\big)^* = \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{7.26}$$

In the setting of Proposition 7.2, from (7.24), (7.17), and the Sequential Banach–Alaoglu Theorem we conclude that any bounded sequence in $M^{p,\lambda}(\Sigma, \sigma)$ has a sub-sequence which is weak-$*$ convergent. (7.27)

A result in this spirit in which a stronger conclusion is reached, provided one assumes more than mere boundedness for said sequence, has been proved in [112, §6.2].

Proposition 7.3 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix two exponents $p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$. Suppose $\{f_j\}_{j\in\mathbb{N}} \subseteq M^{p,\lambda}(\Sigma, \sigma)$ is a sequence of functions with the property that*

$$\begin{aligned} &f(x) := \lim_{j\to\infty} f_j(x) \ \ \textit{exists for } \sigma\textit{-a.e. } \ x \in \Sigma, \quad \textit{and} \\ &\textit{there exists some } g \in M^{p,\lambda}(\Sigma, \sigma) \textit{ such that for each} \\ &j \in \mathbb{N} \ \ \textit{one has } \ |f_j(x)| \leq |g(x)| \ \ \textit{for } \sigma\textit{-a.e. } \ x \in \Sigma. \end{aligned} \tag{7.28}$$

Then $f \in M^{p,\lambda}(\Sigma, \sigma)$ and $f_j \to f$ as $j \to \infty$ weak-$*$ in $M^{p,\lambda}(\Sigma, \sigma)$, i.e.,

$$\lim_{j\to\infty} \int_\Sigma f_j h \, \mathrm{d}\sigma = \int_\Sigma f h \, \mathrm{d}\sigma \ \ \textit{for each} \ \ h \in \mathcal{B}^{q,\lambda}(\Sigma, \sigma). \tag{7.29}$$

Remarkably, certain types of estimates on Muckenhoupt weighted Lebesgue space imply estimates on Morrey spaces. Here is a basic result of this flavor from [112, §6.2] (cf. also [43] for related results in the Euclidean setting).

Proposition 7.4 *Let $\Sigma \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) be a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix an integrability exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Finally, let $\mathcal{F}$ be a family of pairs (f, g) of σ-measurable functions defined on Σ such that*

for each Muckenhoupt weight $w \in A_1(\Sigma, \sigma)$ there exists some constant $C_w = C([w]_{A_1}) \in (0, \infty)$, which stays bounded as $[w]_{A_1}$ stays bounded, and with the property that for each pair $(f, g) \in \mathcal{F}$ one has $\|f\|_{L^p(\Sigma,w)} \leq C_w \|g\|_{L^p(\Sigma,w)}$. (7.30)

Then there exist two constants $C_{\Sigma,p} \in (0, \infty)$ (depending only on p and the Ahlfors regularity constant of Σ) and $Q_{n,\lambda} \in (0, \infty)$ (depending only on n and λ) such that, with

$$C := C_{\Sigma,p} \cdot \sup_{\substack{w\in A_1(\Sigma,\sigma) \\ [w]_{A_1}\leq Q_{n,\lambda}}} C_w, \tag{7.31}$$

one has

$$\|f\|_{M^{p,\lambda}(\Sigma,\sigma)} \leq C \|g\|_{M^{p,\lambda}(\Sigma,\sigma)} \ \ \textit{for each pair} \ \ (f, g) \in \mathcal{F}. \tag{7.32}$$

Based on Propositions 7.4, 3.4, 7.2 (as well as Coltar's inequality and boundedness results for the Hardy–Littlewood maximal operator on Morrey and block spaces), the following result has been established in [113, §2.6].

Proposition 7.5 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $\partial\Omega$ is a UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N\big(\mathbb{R}^n \setminus \{0\}\big)$ which is odd and positive homogeneous of degree $1-n$. Also, fix two integrability exponents $p, q \in (1,\infty)$ with $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$, and pick an aperture parameter $\kappa > 0$. In this setting, for each f belonging to either $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ define*

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k(x-y)f(y)\,\mathrm{d}\sigma(y) \ \ \text{for each} \ \ x \in \partial\Omega, \tag{7.33}$$

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \ \ \text{for each} \ \ x \in \partial\Omega, \tag{7.34}$$

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \ \ \text{for } \sigma\text{-a.e.} \ \ x \in \partial\Omega, \tag{7.35}$$

$$\mathcal{T} f(x) := \int_{\partial\Omega} k(x-y)f(y)\,\mathrm{d}\sigma(y) \ \ \text{for each} \ \ x \in \Omega. \tag{7.36}$$

Then there exists a constant $C \in (0,\infty)$ which depends exclusively on n, p, λ, and the UR constants of $\partial\Omega$ with the property that for each $f \in M^{p,\lambda}(\partial\Omega,\sigma)$ one has

$$\|T_* f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\Big(\sum_{|\alpha|\le N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.37}$$

$$\|\mathcal{N}_\kappa(\mathcal{T} f)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\Big(\sum_{|\alpha|\le N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.38}$$

for each $f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ one has

$$\|T_* f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \le C\Big(\sum_{|\alpha|\le N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.39}$$

$$\|\mathcal{N}_\kappa(\mathcal{T} f)\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \le C\Big(\sum_{|\alpha|\le N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}, \tag{7.40}$$

and for each $f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *one has*

$$\|T_* f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}, \tag{7.41}$$

$$\|\mathcal{N}_\kappa(\mathcal{T} f)\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.42}$$

Also, for each function f *belonging to either* $M^{p,\lambda}(\partial\Omega,\sigma)$*,* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$*, or* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *the limit defining* $Tf(x)$ *in* (7.35) *exists at* σ*-a.e.* $x \in \partial\Omega$ *and the operators*

$$T : M^{p,\lambda}(\partial\Omega,\sigma) \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma), \tag{7.43}$$

$$T : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma), \tag{7.44}$$

$$T : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \tag{7.45}$$

are well defined, linear, and bounded. In addition,

the (*real*) *transpose of the operator* (7.44) *is the operator* $-T$ *with* T *as in* (7.45)*, and the* (*real*) *transpose of the operator* (7.45) *is the operator* $-T$ *with* T *as in* (7.43)*.* (7.46)

Thus, the results from Proposition 7.5 are applicable to the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ defined as in (4.297) on the boundary of a UR domain $\Omega \subseteq \mathbb{R}^n$. This proves that, in such a setting, for each $p, q \in (1,\infty)$ and $\lambda \in (0, n-1)$

the operators $\{R_j\}_{1\leq j\leq n}$ are well defined, linear, and bounded on the spaces $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, and $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$. (7.47)

In concert with Theorem 4.3, (7.7), and duality (cf. Proposition 7.2), Proposition 7.4 also yields the following version of the commutator theorem from [31], in Morrey and block spaces.

Theorem 7.1 *Make the assumption that* $\Sigma \subseteq \mathbb{R}^n$ *be a closed Ahlfors regular set, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$*. Fix* $p_0 \in (1,\infty)$ *along with some non-decreasing function* $\Phi : (0,\infty) \to (0,\infty)$ *and let* T *be a linear operator which is bounded on* $L^{p_0}(\Sigma, w)$ *for every* $w \in A_{p_0}(\Sigma,\sigma)$*, with operator norm* $\leq \Phi\big([w]_{A_{p_0}}\big)$*.*

Then for each exponent $p \in (1,\infty)$ *and each parameter* $\lambda \in (0, n-1)$ *the operator* T *induces well-defined, linear, and bounded mappings in the contexts*

$$T : M^{p,\lambda}(\Sigma,\sigma) \longrightarrow M^{p,\lambda}(\Sigma,\sigma), \tag{7.48}$$

$$T : \mathring{M}^{p,\lambda}(\Sigma, \sigma) \longrightarrow \mathring{M}^{p,\lambda}(\Sigma, \sigma). \tag{7.49}$$

In addition, given any integrability exponent $p \in (1, \infty)$ *along with some parameter* $\lambda \in (0, n-1)$*, there exist two constants,* $C_1 = C_1(\Sigma, n, p_0, p, \lambda) \in (0, \infty)$ *and* $C_2 = C_2(\Sigma, n, p_0, p, \lambda) \in (0, \infty)$*, with the property that for every complex-valued function* $b \in L^\infty(\Sigma, \sigma)$ *one has*

$$\begin{aligned}\|[M_b, T]\|_{\mathring{M}^{p,\lambda}(\Sigma,\sigma)\to\mathring{M}^{p,\lambda}(\Sigma,\sigma)} &\leq \|[M_b, T]\|_{M^{p,\lambda}(\Sigma,\sigma)\to M^{p,\lambda}(\Sigma,\sigma)}\\ &\leq C_1\Phi(C_2)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,,\end{aligned} \tag{7.50}$$

where $[M_b, T] := bT(\cdot) - T(b\,\cdot)$ *is the commutator of* T (*considered either as in* (7.48) *or as in* (7.49)) *and the operator* M_b *of pointwise multiplication* (*either on* $M^{p,\lambda}(\Sigma, \sigma)$ *or on* $\mathring{M}^{p,\lambda}(\Sigma, \sigma)$) *by the function* b*.*

Moreover, if $T^\top$ *denotes the* (*real*) *transpose of the original operator* T*, then for each* $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *the operator* $T^\top$ *induces a well-defined, linear, and bounded mapping*

$$T^\top : \mathcal{B}^{q,\lambda}(\Sigma, \sigma) \longrightarrow \mathcal{B}^{q,\lambda}(\Sigma, \sigma). \tag{7.51}$$

Finally, for each $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *there exist two positive finite constants,* $C_1 = C_1(\Sigma, n, p_0, q, \lambda)$ *and* $C_2 = C_2(\Sigma, n, p_0, q, \lambda)$*, with the property that for every complex-valued function* $b \in L^\infty(\Sigma, \sigma)$ *one has*

$$\Big\|[M_b, T^\top]\Big\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\to\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} \leq C_1\Phi(C_2)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{7.52}$$

For example, if $\Omega \subseteq \mathbb{R}^n$ is a UR domain then, for each complex-valued function $k \in \mathscr{C}^N\big(\mathbb{R}^n \setminus \{0\}\big)$ (where $N = N(n) \in \mathbb{N}$ is sufficiently large) which is odd and positive homogeneous of degree $1-n$, Theorem 7.1 applies with $\Sigma := \partial\Omega$ and T as in (7.35). In such a scenario, from (7.52) and (7.46) we see that for each $b \in L^\infty(\partial\Omega, \sigma)$, $q \in (1, \infty)$, and $\lambda \in (0, n-1)$, the following commutator estimate holds:

$$\|[M_b, T]\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\,\|b\|_{\mathrm{BMO}(\partial\Omega,\sigma)}\,, \tag{7.53}$$

where $C \in (0, \infty)$ depends exclusively on n, q, λ, and the UR constants of $\partial\Omega$.

Following [112, §11.7], we may also consider `Morrey-based Sobolev spaces` on the boundaries of Ahlfors regular domains. Specifically, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then for each $p \in (1, \infty)$ and $\lambda \in (0, n-1)$ we define

$$M_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in M^{p,\lambda}(\partial\Omega,\sigma) \cap L^1_{1,\mathrm{loc}}(\partial\Omega,\sigma) : \qquad (7.54)$$
$$\partial_{\tau_{jk}} f \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ for each } \ j,k \in \{1,\dots,n\} \Big\},$$

equipped with the natural norm

$$M_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \qquad (7.55)$$

A significant closed subspace of $M_1^{p,\lambda}(\partial\Omega,\sigma)$ is the vanishing Morrey-based Sobolev space

$$\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } \ j,k \in \{1,\dots,n\} \qquad (7.56)$$
$$\text{one has } \ \partial_{\tau_{jk}} f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \Big\}.$$

In the same vein, for each $q \in (1,\infty)$ let us also define the block-based Sobolev space

$$\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) : \text{ for each } \ j,k \in \{1,\dots,n\} \qquad (7.57)$$
$$\text{one has } \ \partial_{\tau_{jk}} f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \Big\},$$

and endowed with the norm

$$\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \qquad (7.58)$$

It has been noted in [114, §3.3] that by combining the extrapolation result from Proposition 7.4 with Proposition 3.5 (while also keeping in mind Proposition 3.2, (7.3), (7.8), (7.17), Proposition 7.5, and (7.18)) one obtains the following result pertaining to the action of boundary layer potentials associated with weakly elliptic second-order systems in UR domains, on the scales of spaces discussed earlier.

Theorem 7.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let L be a homogeneous, weakly elliptic, constant complex coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Pick a coefficient tensor $A \in \mathfrak{A}_L$ and consider the double layer potential operators $\mathcal{D}_A$, K_A, $K_A^{\#}$ associated with the coefficient tensor A and the set Ω as in* (3.22), (3.24)*, and* (3.25)*, respectively. Finally, select $p \in (1,\infty)$ along with $\lambda \in (0,n-1)$ and some aperture parameter $\kappa > 0$.*

Then the operators

$$K_A, K_A^{\#} : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{7.59}$$

are well defined, linear, and bounded. Additionally, the operators $K_A, K_A^{\#}$ *in the context of* (7.59) *depend continuously on the underlying coefficient tensor A. Specifically, with the piece of notation introduced in* (3.13)*, the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in {\rm Bd}\Big(\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M\Big), \tag{7.60}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A^{\#} \in {\rm Bd}\Big(\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M\Big). \tag{7.61}$$

Furthermore, there exists a constant $C \in (0,\infty)$*, depending only on the* UR *constants of* $\partial\Omega$*, L, n,* κ*, p, and* λ*, with the property that*

$$\begin{gathered}\big\|\mathcal{N}_\kappa\big(\mathcal{D}_A f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \\ \text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{7.62}$$

Moreover, for each given function f in the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *the following nontangential boundary trace formula holds* (*with I denoting the identity operator*)

$$\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} = \big(\tfrac{1}{2}I + K_A\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.63}$$

In addition, for each function f belonging to the Morrey-based Sobolev space $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *it follows that*

$$\begin{gathered}\text{the nontangential boundary trace } \big(\partial_\ell\mathcal{D}_A f\big)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \text{ exists (in } \mathbb{C}^M) \\ \text{at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } \ell \in \{1,\dots,n\},\end{gathered} \tag{7.64}$$

and there exits some finite constant $C > 0$*, depending only on* $\partial\Omega$*, L, n,* κ*, p,* λ*, such that*

$$\begin{aligned}\big\|\mathcal{N}_\kappa(\mathcal{D}_A f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} &+ \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}_A f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \\ &\leq C\|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{7.65}$$

In fact, similar results are valid with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced throughout by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ (*defined as in* (7.7) *with*

$\Sigma := \partial\Omega$*), or by the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$ *(defined as in* (7.15)–(7.16) *with* $\Sigma := \partial\Omega$*).*

Next, the operators

$$K_A : \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{7.66}$$

$$K_A : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{7.67}$$

are well defined, linear, bounded and, for each $q \in (1,\infty)$*, so is*

$$K_A : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.68}$$

Also, much as in (7.60)–(7.61)*, the operator* K_A *in the context of* (7.66)–(7.68) *depends in a continuous fashion on the underlying coefficient tensor* A*.*

Next we introduce the `homogeneous Morrey-based Sobolev spaces`. Consider an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Given an integrability exponent $p \in (1,\infty)$ and $\lambda \in (0, n-1)$ let us define the space

$$\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in L^1\big(\partial\Omega\,,\tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma) : \\ \partial_{\tau_{jk}} f \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ for each } \ j,k \in \{1,\dots,n\}\Big\} \tag{7.69}$$

and equip it with the semi-norm

$$\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.70}$$

Then (7.3) ensures that we have the following continuous embedding

$$M_1^{p,\lambda}(\partial\Omega,\sigma) \hookrightarrow \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma). \tag{7.71}$$

It is also clear that constant functions on $\partial\Omega$ belong to $\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)$ and have vanishing semi-norm. We shall occasionally work with $\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim$, the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)$, equipped with the semi-norm

$$\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\, \ni [f] \mapsto \big\|[f]\big\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.72}$$

To proceed, choose a scalar-valued function $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ in $B(0,1)$ and $\operatorname{supp}\phi \subseteq B(0,2)$. Having fixed a reference point $x_0 \in \partial\Omega$, for each scale $r \in (0,\infty)$ define

$$\phi_r(x) := \phi\Big(\frac{x-x_0}{r}\Big) \text{ for each } x \in \mathbb{R}^n, \tag{7.73}$$

and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. For each $r \in (0,\infty)$ set $\Delta_r := \partial\Omega \cap B(x_0,r)$. Given any $f \in L^1_{\rm loc}(\partial\Omega,\sigma)$, define

$$f_r := \phi_r \cdot \big(f - f_{\Delta_{2r}}\big) \text{ on } \partial\Omega, \quad \text{where } f_{\Delta_{2r}} := \fint_{\Delta_{2r}} f\,d\sigma. \tag{7.74}$$

Lemma 7.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain with the property that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix some reference point $x_0 \in \partial\Omega$, along with some integrability exponent $p \in (1,\infty)$ and a parameter $\lambda \in (0,n-1)$. Finally, pick a function f which belongs to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\rm loc}(\partial\Omega,\sigma)$ and, for each radius $r \in (0,\infty)$, define the surface ball $\Delta_r := B(x_0,r)\cap\partial\Omega$ and $f_{\Delta_r} := \fint_{\Delta_r} f\,d\sigma$. Then the following statements are true.*

(i) There exists a constant $C = C(\Omega,p,\lambda,x_0) \in (0,\infty)$, independent of the function f, such that

$$\sup_{r>0}\frac{1}{r}\big\||f-f_{\Delta_r}|\cdot\mathbf{1}_{\Delta_r}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\sum_{j,k=1}^{n}\big\|\partial_{\tau_{jk}}f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.75}$$

(ii) For each $r \in (0,\infty)$ there exists a constant $C_r \in (0,\infty)$ which depends on Ω, p, λ, x_0, and r, but is independent of f, such that

$$\int_{\partial\Omega}\frac{|f(x)-f_{\Delta_r}|}{1+|x|^n}\,d\sigma(x) \leq \frac{C_r}{\|\mathbf{1}_{\Delta_r}\|_{M^{p,\lambda}(\partial\Omega,\sigma)}}\sum_{j,k=1}^{n}\big\|\partial_{\tau_{jk}}f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.76}$$

(iii) There exists a constant $C = C(\Omega,p,\lambda,x_0) \in (0,\infty)$, independent of the function f, such that with the notation introduced in (7.74) one has

$$\sup_{r>0}\big\|\nabla_{\tan}f_r\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^n} \leq C\big\|\nabla_{\tan}f\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^n}. \tag{7.77}$$

Proof We shall prove all claims using extrapolation (cf. Proposition 7.4). Consider first the task of establishing *(i)*. Recall (2.585) and define

$$\mathcal{F}_1 := \Big\{\big(\tfrac{|f-f_{\Delta_r}|}{r}\mathbf{1}_{\Delta_r}, |\nabla_{\tan}f|\big) : \\ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\cap L^1_{1,\rm loc}(\partial\Omega,\sigma),\ r>0\Big\}. \tag{7.78}$$

We claim that for the given integrability exponent $p \in (1, \infty)$ and for every weight $w \in A_p(\partial\Omega, \sigma)$ there exists a constant $C = C(\Omega, p, [w]_{A_p}, x_0) \in (0, \infty)$ such that

$$\|F_1\|_{L^p(\partial\Omega,w)} \leq C\|F_2\|_{L^p(\partial\Omega,w)} \tag{7.79}$$

for all $(F_1, F_2) \in \mathcal{F}_1$. Indeed, this inequality is trivial if $\|F_2\|_{L^p(\partial\Omega,w)} = \infty$, whereas if $\|F_2\|_{L^p(\partial\Omega,w)} < \infty$ we may rely on (7.78) and (2.586) to invoke Proposition 2.25 to obtain (2.618). This, in turn, gives (7.79) on account of (2.586). Moreover, the intervening constant C stays bounded if $[w]_{A_p}$ stays bounded. In particular, in view of item *(2)* from Proposition 2.20, the argument so far shows that (7.79) holds for every $w \in A_1(\partial\Omega, \sigma)$ and that the intervening constant stays bounded if $[w]_{A_1}$ stays bounded. We may then invoke Theorem 7.4 to conclude that for each given number $\lambda \in (0, n-1)$ we have $\|F_1\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|F_2\|_{M^{p,\lambda}(\partial\Omega,\sigma)}$ for each $(F_1, F_2) \in \mathcal{F}_1$. This and (2.585) then imply (7.75), finishing the proof of *(i)*.

Let us now address the claim made in item *(ii)*. Fix $r \in (0, \infty)$ and define

$$\mathcal{F}_2 := \Big\{\Big(\|f - f_{\Delta_r}\|_{L^1\left(\partial\Omega,\, \frac{\sigma(x)}{1+|x|^n}\right)} \mathbf{1}_{\Delta_r}, |\nabla_{\tan} f|\Big) : \\ f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)\Big\}. \tag{7.80}$$

As before, the goal is to check that (7.79) holds for all weights $w \in A_p(\partial\Omega, \sigma)$ and all pairs $(F_1, F_2) \in \mathcal{F}_2$ (where now the constant C is allowed to depend on the scale r, which has been fixed). This may be seen reasoning much as before, applying Proposition 2.25, but this time the relevant estimate is (2.620). Granted (7.79), we may then apply Theorem 7.4 to the family $\mathcal{F}_2$ and, as desired, conclude that (7.76) holds.

To justify the claim made in item *(iii)*, we introduce

$$\mathcal{F}_3 := \Big\{\big(|\nabla_{\tan} f_r|, |\nabla_{\tan} f|\big) : f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma),\ r > 0\Big\}. \tag{7.81}$$

In line with what we have done in the previous cases, we now wish to show that (7.79) holds for all weights $w \in A_p(\partial\Omega, \sigma)$ and all pairs $(F_1, F_2) \in \mathcal{F}_3$. Again, it suffices to consider the case when $\|F_2\|_{L^p(\partial\Omega,w)} < \infty$. By definition, we have $(F_1, F_2) = \big(|\nabla_{\tan} g_r|, |\nabla_{\tan} g|\big)$ for some $g \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ and some $r > 0$. This, the assumption $\|F_2\|_{L^p(\partial\Omega,w)} < \infty$, (2.586), and Proposition 2.25 then guarantee that $g \in \dot{L}^p_1(\partial\Omega, w)$. We may therefore proceed as in (4.370)–(4.377) in the proof of Theorem 4.11 to conclude that (4.377) holds. Equivalently, this proves (7.79) for the given choice of (F_1, F_2). Moreover, a careful examination of the proof shows that the intervening constant $C \in (0, \infty)$ stays bounded if $[w]_{A_p}$

stays bounded. We have therefore shown that (7.79) holds for each $(F_1, F_2) \in \mathcal{F}_3$ and each $w \in A_p(\partial\Omega, \sigma)$. In particular (cf. item *(2)* in Proposition 2.20), this is the case for every $w \in A_1(\partial\Omega, \sigma)$ and the intervening constant $C \in (0, \infty)$ stays bounded if $[w]_{A_1}$ stays bounded. As such, we may avail ourselves of Theorem 7.4 to conclude that, given any $\lambda \in (0, n-1)$, one has $\|F_1\|_{M_1^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|F_2\|_{M_1^{p,\lambda}(\partial\Omega,\sigma)}$ for every $(F_1, F_2) \in \mathcal{F}_3$. Hence, there exists $C = C(\Omega, p, \lambda, x_0) \in (0, \infty)$ such that

$$\big\||\nabla_{\tan} f_r|\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\big\||\nabla_{\tan} f|\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{7.82}$$

for every $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ and every $r > 0$. This completes the proof of (7.77). □

It turns out that, when considered on the boundaries of two-sided NTA domains, the quotient space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ is actually a Banach space.

Proposition 7.6 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided* NTA *domain with an unbounded Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Pick some integrability exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Finally, recall that $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ denotes the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$, equipped with the semi-norm* (7.72)*.*

Then (7.72) *is a genuine norm on $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$, and $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ is a Banach space when equipped with the norm* (7.72)*.*

Proof Let us first observe from (7.76) that the semi-norm (7.72) is indeed a norm on the space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$. We shall next show that $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$ is complete when equipped with the norm (7.72). With this goal in mind, let $\{f_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$ be such that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the quotient space $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/\sim$. This means that $\big\{\partial_{\tau_{jk}} f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $M^{p,\lambda}(\partial\Omega, \sigma)$, for any two fixed indices $j, k \in \{1, \ldots, n\}$. Using the fact that $M^{p,\lambda}(\partial\Omega, \sigma)$ is a Banach space, we then conclude that for each $j, k \in \{1, \ldots, n\}$ there exists $g_{jk} \in M^{p,\lambda}(\partial\Omega, \sigma)$ such that

$$\partial_{\tau_{jk}} f_\alpha \to g_{jk} \quad \text{in} \quad M^{p,\lambda}(\partial\Omega, \sigma) \quad \text{as} \quad \alpha \to \infty. \tag{7.83}$$

Fix a reference point $x_0 \in \partial\Omega$ and, for each $r \in (0, \infty)$, set $\Delta_r := B(x_0, r) \cap \partial\Omega$. Also, set $f_{\alpha,\Delta_r} := ⨍_{\Delta_r} f_\alpha \, d\sigma$ for each $r \in (0, \infty)$ and each $\alpha \in \mathbb{N}$. Applying (7.76) to $f := f_\alpha - f_\beta$ we obtain that for any radius $r \in (0, \infty)$ there exists some constant $C_r \in (0, \infty)$ which depends only on Ω, p, λ, r, and x_0, such that that for all indices $\alpha, \beta \in \mathbb{N}$ we have

$$\big\|\big(f_\alpha - f_{\alpha,\Delta_r}\big) - \big(f_\beta - f_{\beta,\Delta_r}\big)\big\|_{L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\right)}$$

$$\leq \frac{C_r}{\|\mathbf{1}_{\Delta_r}\|_{M^{p,\lambda}(\partial\Omega,\sigma)}} \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta \right\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.84}$$

Since $\{\partial_{\tau_{jk}} f_\alpha\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $M^{p,\lambda}(\partial\Omega,\sigma)$, it then follows that for each fixed $r \in (0,\infty)$ the sequence $\{f_\alpha - f_{\alpha,\Delta_r}\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Hence, for each fixed $r \in (0,\infty)$ there exists $h_r \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ such that

$$f_\alpha - f_{\alpha,\Delta_r} \to h_r \text{ in } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \text{ as } \alpha\to\infty. \tag{7.85}$$

On the other hand, by (7.75) (applied to the difference $f := f_\alpha - f_\beta$), there exists some constant $C = C(\Omega, p, \lambda, x_0) \in (0,\infty)$ such that for each fixed $r \in (0,\infty)$ we have

$$\begin{aligned} &\big\| |(f_\alpha - f_{\alpha,\Delta_r}) - (f_\beta - f_{\beta,\Delta_r})| \cdot \mathbf{1}_{\Delta_r} \|_{M^{p,\lambda}(\partial\Omega,\sigma)} \\ &\qquad \leq C\, r \sum_{j,k=1}^{n} \left\| \partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\alpha \right\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \end{aligned} \tag{7.86}$$

Hence, the sequence $\{(f_\alpha - f_{\alpha,\Delta_r})\mathbf{1}_{\Delta_r}\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $M^{p,\lambda}(\partial\Omega,\sigma)$ for each fixed $r \in (0,\infty)$. As a result, for each fixed $r \in (0,\infty)$ it follows that

$$\begin{gathered} \text{there exists a function } k_r \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ such that} \\ \big(f_\alpha - f_{\alpha,\Delta_r}\big)\mathbf{1}_{\Delta_r} \to k_r \text{ in } M^{p,\lambda}(\partial\Omega,\sigma) \text{ as } \alpha\to\infty. \end{gathered} \tag{7.87}$$

Note that convergence in $M_1^{p,\lambda}(\partial\Omega,\sigma)$ implies convergence in $L^p(\Delta_r,\sigma)$ and, after eventually passing to a sub-sequence, pointwise a.e. convergence. Thus (7.85) and (7.87) immediately give

$$h_r\big|_{\Delta_r} = k_r \in M^{p,\lambda}(\partial\Omega,\sigma) \text{ for each } r \in (0,\infty). \tag{7.88}$$

Additionally, for each fixed $r_1, r_2 \in (0,\infty)$ the convergence recorded in (7.85) also yields

$$f_{\alpha,\Delta_{r_2}} - f_{\alpha,\Delta_{r_1},w} \to h_{r_1} - h_{r_2} \text{ in } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \text{ as } \alpha\to\infty. \tag{7.89}$$

Thus $h_{r_1} - h_{r_2}$ must be constant. This, (7.85), (7.88), and (7.3) eventually lead to

$$h_r \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each } r \in (0,\infty). \tag{7.90}$$

To continue we simply write h for h_r with $r = 1$, and c_α for f_{α,Δ_r} with $r = 1$. Then, as seen from (7.90),

$$h \text{ belongs to } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma), \tag{7.91}$$

and the sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \mathbb{C}$ is such that

$$f_\alpha - c_\alpha \to h \text{ in } L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \text{ as } \alpha \to \infty. \tag{7.92}$$

For each $j, k \in \{1, \ldots, n\}$ and each test function $\varphi \in \mathscr{C}^\infty_0(\mathbb{R}^n)$ we may then write

$$\begin{aligned}\int_{\partial\Omega} h(\partial_{\tau_{jk}}\varphi)\, d\sigma &= \lim_{\alpha\to\infty}\int_{\partial\Omega}(f_\alpha - c_\alpha)(\partial_{\tau_{jk}}\varphi)\, d\sigma \\ &= -\lim_{\alpha\to\infty}\int_{\partial\Omega}\partial_{\tau_{jk}}(f_\alpha - c_\alpha)\varphi\, d\sigma = -\lim_{\alpha\to\infty}\int_{\partial\Omega}(\partial_{\tau_{jk}} f_\alpha)\varphi\, d\sigma \\ &= \int_{\partial\Omega} g_{jk}\varphi\, d\sigma, \end{aligned}\tag{7.93}$$

thanks to (7.92), (2.583), (7.83), and (7.3). From this and (2.581)–(2.582) we then conclude that

$$\partial_{\tau_{jk}} h = g_{jk} \in M^{p,\lambda}(\partial\Omega, \sigma) \text{ for each } j, k \in \{1, \ldots, n\}. \tag{7.94}$$

Collectively, (7.91) and (7.94) prove that $h \in \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)$. Finally, from (7.83), (7.94), and (7.72) we conclude that the sequence $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ converges to $[h]$, the class of h, in the quotient space $\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big/\sim$. □

We continue by making the following definition, which should be compared with (7.69).

Definition 7.1 Assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and pick an exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. In this context, define the `vanishing Morrey-based homogeneous Sobolev space` of order one on $\partial\Omega$ as

$$\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega, \sigma) := \Big\{ f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega, \sigma) : \\ \partial_{\tau_{jk}} f \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \text{ for each } j, k \in \{1, \ldots, n\}\Big\} \tag{7.95}$$

and equip this space with the semi-norm

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.96}$$

As seen from of Definition 7.1, all constant functions on $\partial\Omega$ belong to $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ and their respective semi-norms vanish. It is also apparent from (7.95)–(7.96) and (7.69)–(7.70) that

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) = \Big\{ f \in \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \quad \text{for all } \; j,k \in \{1,\dots,n\} \Big\} \tag{7.97}$$

and

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \text{ is a closed subspace of } \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma). \tag{7.98}$$

Moreover, we have the continuous embedding

$$\mathring{\dot{M}}{}_1^{p,\lambda}(\partial\Omega,\sigma) \hookrightarrow \dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{7.99}$$

Much as in Proposition 7.6, if $\Omega \subseteq \mathbb{R}^n$ is a two-sided NTA domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, then

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\; \ni [f] \longmapsto \big\|[f]\big\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{7.100}$$

is a genuine norm on $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim$, and $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim$ is a Banach space when equipped with the norm (7.100).

In a similar fashion, we introduce the following brand of block-based homogeneous Sobolev spaces:

Definition 7.2 Suppose that $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an integrability exponent $q \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$. Also, introduce

$$q_\lambda := \frac{q(n-1)}{n-1+\lambda(q-1)} \in (1,q). \tag{7.101}$$

In this context, define the `block-based homogeneous Sobolev space` of order one on $\partial\Omega$ as

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^{q_\lambda}_{\mathrm{loc}}(\partial\Omega,\sigma) : \tag{7.102}$$

$$\partial_{\tau_{jk}} f \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \text{ for each } \; j, k \in \{1, \dots, n\}\Big\}$$

and equip this space with the semi-norm

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma) \ni f \mapsto \|f\|_{\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.103}$$

It turns out that we have the continuous embeddings

$$\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma) \hookrightarrow \dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big), \tag{7.104}$$

and

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma) \hookrightarrow \dot{L}_1^{q_\lambda}(\partial\Omega, \sigma). \tag{7.105}$$

In the context of Definition 7.2 it follows that all constant functions on $\partial\Omega$ belong to $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)$ and their respective semi-norms vanish. We shall occasionally work with the space $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big/\sim$, the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)$, which we equip with the semi-norm

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big/\sim\; \ni [f] \mapsto \big\|[f]\big\|_{\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{7.106}$$

Analogously to Proposition 7.6, we have the following completeness result (see [112, §11.13] for a proof).

Proposition 7.7 *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and pick an integrability exponent $q \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Then* (7.106) *is a genuine norm on $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big/\sim$, and $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big/\sim$ is a Banach space when equipped with the norm* (7.106).

We continue by recording the following remarkable trace result proved in [112, §11.13].

Proposition 7.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an* NTA *domain such that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa \in (0, \infty)$ along with some integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. In this setting, the following statements are true.*

(1) For each function $u : \Omega \to \mathbb{C}$ satisfying

$$u \in \mathscr{C}^1(\Omega) \;\; \textit{and} \;\; \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \tag{7.107}$$

the nontangential trace

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ belongs to } \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma), \\ \text{and } \big\|u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)} \le C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{7.108}$$

for some constant $C \in (0,\infty)$ independent of u.

(2) For each function $u : \Omega \to \mathbb{C}$ satisfying

$$u \in \mathscr{C}^1(\Omega) \text{ and } \mathcal{N}_\kappa(\nabla u) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \tag{7.109}$$

the nontangential trace

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ belongs to } \dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma), \\ \text{and } \big\|u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big\|_{\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)} \le C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \tag{7.110}$$

for some constant $C \in (0,\infty)$ independent of u.

(3) For each function $u \in \mathscr{C}^1(\Omega)$ satisfying

$$\mathcal{N}_\kappa(\nabla u) \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \text{ and } (\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \tag{7.111}$$

the nontangential trace

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ belongs to } \dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma), \\ \text{and } \big\|u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)} \le C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{7.112}$$

for some constant $C \in (0,\infty)$ independent of u.

It has also been noted in [114, §3.3] that Theorems 3.3, 3.4, and Proposition 7.8 imply the following Fatou-type results and integral representation formulas.

Theorem 7.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ (where $M \in \mathbb{N}$) be a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,\text{mod}}$ from (3.49), and the modified version of the single layer operator $\mathscr{S}_{\text{mod}}$ from (3.38). Fix an aperture parameter $\kappa \in (0,\infty)$ along with some integrability exponents $p, q \in (1,\infty)$ and a number $\lambda \in (0, n-1)$. Finally, consider a function $u : \Omega \to \mathbb{C}^M$ satisfying*

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \ and \ \ Lu = 0 \ \ in \ \ \Omega. \tag{7.113}$$

(1) If $\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma)$ *then*

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exists\ \sigma\text{-a.e. on} \ \ \partial\Omega \ \ and\ belongs\ to \ \ \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exists\ \sigma\text{-a.e. on} \ \ \partial\Omega \ \ and \ \ \partial_\nu^A u \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{7.114}$$

and there exists some $\mathbb{C}^M$*-valued locally constant function* c_u *in* Ω *with the property that*

$$u = \mathcal{D}_{A,\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathscr{S}_{\text{mod}}\big(\partial_\nu^A u\big) + c_u \ \ in \ \ \Omega. \tag{7.115}$$

(2) If $\mathcal{N}_\kappa(\nabla u) \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *then*

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exists\ \sigma\text{-a.e. on} \ \ \partial\Omega \ \ and\ belongs\ to \ \ \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exists\ \sigma\text{-a.e. on} \ \ \partial\Omega \ \ and \ \ \partial_\nu^A u \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{7.116}$$

and (7.115) *continues to hold.*

(3) If $\mathcal{N}_\kappa(\nabla u) \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ *then*

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exists\ \sigma\text{-a.e. on} \ \ \partial\Omega \ \ and\ belongs\ to \ \ \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \ exists\ \sigma\text{-a.e. on} \ \ \partial\Omega \ \ and \ \ \partial_\nu^A u \in \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{7.117}$$

and (7.115) *once again continues to hold.*

We wish to augment Theorem 7.2 with a series of results dealing with modified boundary layer potentials.

Theorem 7.4 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Denote by* $\nu = (\nu_1, \dots, \nu_n)$ *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, for some* $M \in \mathbb{N}$, *let* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n \\ 1\le\alpha,\beta\le M}}$ *be a complex coefficient tensor with the property that* $L := L_A$ *is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$. *Recall the modified boundary-to-boundary single layer operator* S_{mod} *associated with* L *and* Ω *as in* (3.42). *Finally, fix two exponents* $p, q \in (1, \infty)$ *along with a parameter* $\lambda \in (0, n-1)$. *Then the following properties are true.*

(1) The modified boundary-to-boundary single layer operator induces a mapping

$$S_{\text{mod}} : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{7.118}$$

which is well defined, linear, and bounded, when the target space is endowed with the semi-norm (7.70). *In particular,*

$$\text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and pair of indices } j,k \in \{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}}\big(S_{\text{mod}} f\big) \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.119}$$

Also, for each function $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, *at* σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\begin{aligned}&\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big)(x) \\ &= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} \nu_i(x)a^{\mu\gamma}_{ij}a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{js}}\big(S_{\text{mod}} f\big)_\alpha(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\le\mu\le M},\end{aligned} \tag{7.120}$$

where $K^{\#}_{A^\top}$ *is the singular integral operator associated as in* (3.25) *with the set* Ω *and the transpose coefficient tensor* $A^\top$. *Finally,*

$$\begin{aligned}&\text{for each sequence of functions } \{f_j\}_{j\in\mathbb{N}} \subseteq \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ &\text{which is weak-}*\text{ convergent to some } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and} \\ &\text{for each test function } \phi \in \big[\mathrm{Lip}(\partial\Omega)\big]^M \text{ with compact support} \\ &\text{one has } \lim_{j\to\infty}\int_{\partial\Omega}\big\langle S_{\text{mod}} f_j,\phi\big\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega}\big\langle S_{\text{mod}} f,\phi\big\rangle\,\mathrm{d}\sigma.\end{aligned} \tag{7.121}$$

(2) As a consequence of (7.118), *the following is a well-defined linear operator:*

$$\begin{aligned}\big[S_{\text{mod}}\big] : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M \text{ defined as}& \\ \big[S_{\text{mod}}\big]f := \big[S_{\text{mod}} f\big] \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M,& \\ \text{for all } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M.&\end{aligned} \tag{7.122}$$

Moreover, if actually $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator* (7.122) *is also bounded when the quotient space is endowed with the norm introduced in* (7.72).

(3) With $\mathscr{S}_{\text{mod}}$ *denoting the modified version of the single layer operator acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (3.38), *for each given aperture parameter* $\kappa > 0$ *there exists some constant* $C = C(\Omega, L, n, p, \lambda, \kappa) \in (0,\infty)$ *with the property that for each given function* $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\begin{aligned} &\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \ in \ \ \Omega, \\ &\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big) \ \ belongs\ to \ \ M^{p,\lambda}(\partial\Omega, \sigma) \ \ and \\ &\big\|\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}, \\ &\Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|_{\partial\Omega}^{\kappa-\rm n.t.}\Big)(x) = (S_{\rm mod} f)(x) \ \ at\ \sigma\text{-}a.e.\ point \ \ x \in \partial\Omega. \end{aligned} \tag{7.123}$$

Moreover, for each given function f in the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ the following jump formula holds (with I denoting the identity operator)

$$\partial_\nu^A \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f \ \ at\ \sigma\text{-}a.e.\ point\ in \ \ \partial\Omega, \tag{7.124}$$

where $K^\#_{A^\top}$ is the singular integral operator associated as in (3.25) with the set Ω and the transpose coefficient tensor $A^\top$.

(4) Similar properties to those described in items (1)–(3) are valid for block spaces (and block-based homogeneous Sobolev spaces) in place of Morrey spaces (and homogeneous Morrey-based Sobolev spaces). More specifically, the operator

$$S_{\rm mod} : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \tag{7.125}$$

is well defined, linear, and bounded, when the target space is endowed with the semi-norm (7.103). Also,

$$\begin{aligned} &\big[S_{\rm mod}\big] : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big/\sim\big]^M \ \ defined\ as \\ &\big[S_{\rm mod}\big]f := \big[S_{\rm mod} f\big] \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega, \sigma)\big/\sim\big]^M, \\ &for\ all\ f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \end{aligned} \tag{7.126}$$

is a well-defined linear operator, which is also bounded in the case when $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set (assuming the quotient space is endowed with the norm introduced in (7.106)). Finally, for each aperture parameter $\kappa > 0$ there exists $C = C(\Omega, L, n, q, \lambda, \kappa) \in (0, \infty)$ with the property that for each function $f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ one has

$$\begin{aligned}
&\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \ in \ \ \Omega, \\
&\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big) \ \ belongs\ to \ \ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \\
&\big\|\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}, \\
&\Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x) = (S_{\rm mod} f)(x) \ \ at\ \sigma\text{-}a.e.\ point \ \ x \in \partial\Omega, \\
&and \ \ \partial^A_\nu \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \ \ at\ \sigma\text{-}a.e.\ point\ in \ \ \partial\Omega.
\end{aligned} \tag{7.127}$$

(5) Analogous properties to those presented in items (1)–(3) above are also valid for vanishing Morrey spaces $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ *(cf.* (7.7)*) and homogeneous vanishing Morrey-based Sobolev spaces* $\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)$ *(cf. Definition 7.1) in place of Morrey spaces and homogeneous Morrey-based Sobolev spaces, respectively.*

This theorem has been established in [114, §3.3, §1.5]. Here we wish to note that an alternative argument may be given along the lines of the proof of item *(2)* in Theorem 8.5 (where a more general result of this flavor is obtained).

Some of the main properties of the modified boundary-to-domain double layer potential operators and their conormal derivatives acting on homogeneous Morrey-based and block-based Sobolev spaces on boundaries of UR domains are collected in the next theorem from [114, §3.3].

Theorem 7.5 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a* UR *domain. Denote by* $\nu = (\nu_1,\dots,\nu_n)$ *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *In addition, for some* $M \in \mathbb{N}$, *let* $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ *be a complex coefficient tensor with the property that* $L := L_A$ *is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$. *Also, let* $E = (E_{\gamma\beta})_{1\le\gamma,\beta\le M}$ *be the matrix-valued fundamental solution associated with* L *as in Theorem 3.1. In this setting, recall the modified version of the double layer operator* $\mathcal{D}_{A,{\rm mod}}$ *acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (3.49). *Finally, fix some integrability exponents* $p, q \in (1,\infty)$ *along with a number* $\lambda \in (0, n-1)$, *and an aperture parameter* $\kappa \in (0,\infty)$. *Then the following statements are true.*

(1) There exists some constant $C = C(\Omega, A, n, p, \lambda, \kappa) \in (0,\infty)$ *with the property that for each function* $f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ *it follows that*

$$\begin{aligned}
&\mathcal{D}_{A,{\rm mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,{\rm mod}} f\big) = 0 \ \ in \ \ \Omega, \\
&\big(\mathcal{D}_{A,{\rm mod}} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}, \ \ \big(\nabla\mathcal{D}_{A,{\rm mod}} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \ \ exist\ \sigma\text{-}a.e.\ on \ \ \partial\Omega, \\
&\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,{\rm mod}} f\big) \ \ belongs\ to \ \ M^{p,\lambda}(\partial\Omega,\sigma) \ \ and \\
&\big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,{\rm mod}} f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{7.128}$$

In fact, for each function $f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ *one has*

$$(\mathcal{D}_{A,\mathrm{mod}} f)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big) f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \tag{7.129}$$

where I is the identity operator on $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$, and $K_{A,\mathrm{mod}}$ is the modified boundary-to-boundary double layer potential operator from (3.50) and (3.48).

(2) Given any function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the homogeneous Morrey-based Sobolev space $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$, at σ-a.e. point $x \in \partial\Omega$ one has

$$\big(\partial^A_\nu(\mathcal{D}_{A,\mathrm{mod}} f)\big)(x) = \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) \times \times \big(\partial_{\tau_{js}} f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\mu\leq M}, \tag{7.130}$$

where the conormal derivative is considered as in (3.66).

(3) The operator

$$\begin{aligned} &\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}} : \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \quad \text{defined as} \\ &\big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big) f := \partial^A_\nu(\mathcal{D}_{A,\mathrm{mod}} f) \quad \text{for each} \quad f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \end{aligned} \tag{7.131}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm (7.70). As a consequence of (7.131), the following is a well-defined linear operator:

$$\begin{aligned} \big[\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big] : \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M &\longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ \text{given by } \big[\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big][f] &:= \partial^A_\nu(\mathcal{D}_{A,\mathrm{mod}} f) \\ \text{for each function } f &\in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.132}$$

If, in fact, $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator (7.132) is also bounded when the quotient space is equipped with the norm (7.72).

(4) With $K^{\#}_{A^\top}$ denoting the singular integral operator associated as in (3.25) with the set Ω and the transpose coefficient tensor $A^\top$, one has

$$\begin{aligned} \big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) &= \big[\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big]\big[S_{\mathrm{mod}}\big] \\ \text{as mappings acting from } &\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{7.133}$$

and

$$\begin{aligned}&\big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big]\big[K_{A,\text{mod}}\big] = K^{\#}_{A^\top}\big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big]\\ &\textit{as mappings acting from } \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M.\end{aligned} \tag{7.134}$$

Moreover, if $\partial\Omega$ is connected then also

$$\begin{aligned}&\big(\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\big)\big(-\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\big) = \big[S_{\text{mod}}\big]\big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big]\\ &\textit{as mappings acting from } \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M,\end{aligned} \tag{7.135}$$

and

$$\begin{aligned}&\big[S_{\text{mod}}\big]K^{\#}_{A^\top} = \big[K_{A,\text{mod}}\big]\big[S_{\text{mod}}\big]\\ &\textit{as mappings acting from } \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{aligned} \tag{7.136}$$

(5) *Similar properties to those described in items (1)–(4) above are also valid for block spaces (and block-based homogeneous Sobolev spaces) in place of Morrey spaces (and homogeneous Morrey-based Sobolev spaces). Concretely, there exists a constant $C = C(\Omega, A, n, q, \lambda, \kappa) \in (0,\infty)$ with the property that for each function $f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ one has*

$$\begin{aligned}&\mathcal{D}_{A,\text{mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,\text{mod}} f\big) = 0 \ \textit{ in } \ \Omega,\\ &\big(\mathcal{D}_{A,\text{mod}} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{A,\text{mod}}\big)f \ \textit{ at } \sigma\textit{-a.e. point on } \ \partial\Omega,\\ &\big(\nabla\mathcal{D}_{A,\text{mod}} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \textit{ exist at } \sigma\textit{-a.e. point on } \ \partial\Omega,\\ &\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\text{mod}} f\big) \ \textit{ belongs to } \ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \ \textit{ and}\\ &\big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\text{mod}} f\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{7.137}$$

Also, formula (7.130) remains true for each function $f = (f_\alpha)_{1\le\alpha\le M}$ belonging to the space $\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, and the operator

$$\begin{aligned}&\partial_\nu^A \mathcal{D}_{A,\text{mod}} : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \ \textit{ defined as}\\ &\big(\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big) f := \partial_\nu^A(\mathcal{D}_{A,\text{mod}} f) \ \textit{ for each } \ f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M\end{aligned} \tag{7.138}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm (7.103). Furthermore,

$$\begin{aligned}&\big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M\\ &\textit{defined as } \big[\partial_\nu^A \mathcal{D}_{A,\text{mod}}\big][f] := \partial_\nu^A(\mathcal{D}_{A,\text{mod}} f)\\ &\textit{for each } \ f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M\end{aligned} \tag{7.139}$$

is a well-defined linear operator, which is also bounded when the quotient space is equipped with the norm (7.106) *if, in fact,* $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set. Finally, the operator identities in* (7.133)–(7.135) *are valid for functions in* $\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M$.

(6) *Analogous properties to those presented in items (1)–(4) above are also valid for homogeneous vanishing Morrey-based Sobolev spaces* $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ (*cf. Definition 7.1*) *in place of homogeneous Morrey-based Sobolev spaces.*

We next study mapping properties for modified boundary-to-boundary double layer potential operators acting on homogeneous Morrey-based and block-based Sobolev spaces on boundaries of UR domains.

Theorem 7.6 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$ *satisfies* $n \geq 2$) *be an NTA domain such that* $\partial\Omega$ *is an Ahlfors regular set, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, let* $L = \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ *be a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order* $M \times M$ *system in* $\mathbb{R}^n$ (*for some integer* $M \in \mathbb{N}$). *In this context, consider the modified boundary-to-boundary double layer potential operator* $K_{A,\mathrm{mod}}$ *from* (3.50). *Finally, select some exponents* $p, q \in (1,\infty)$ *along with a parameter* $\lambda \in (0, n-1)$. *Then the following statements are valid.*

(1) *The modified boundary-to-boundary double layer potential operator induces a mapping*

$$K_{A,\mathrm{mod}} : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{7.140}$$

which is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (7.70). *As a corollary of* (7.140), *the following operator is well defined and linear:*

$$\begin{aligned} &\big[K_{A,\mathrm{mod}}\big] : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \\ &\textit{given by } \big[K_{A,\mathrm{mod}}\big][f] := \big[K_{A,\mathrm{mod}}f\big] \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \\ &\qquad\qquad \textit{for each function } \ f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.141}$$

Moreover, if actually $\Omega \subseteq \mathbb{R}^n$ *is a two-sided* NTA *domain whose boundary is an unbounded Ahlfors regular set then the operator* (7.141) *is also bounded when all quotient spaces are endowed with the norm introduced in* (7.72).

(2) *If* U_{jk} *with* $j, k \in \{1,\dots,n\}$ *is the family of singular integral operators defined in* (3.35), *then*

$$\partial_{\tau_{jk}}\big(K_{A,\mathrm{mod}} f\big) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \;\; \textit{at } \sigma\textit{-a.e. point on } \;\partial\Omega$$
$$\textit{for each } f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \textit{ and each } j,k \in \{1,\dots,n\}. \tag{7.142}$$

(3) Similar properties to those described in items (1)–(2) are valid for block-based homogeneous Sobolev spaces in place of homogeneous Morrey-based Sobolev spaces. Specifically,

$$K_{A,\mathrm{mod}} : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \tag{7.143}$$

is a well-defined, linear, and bounded operator when the spaces involved are endowed with the semi-norm (7.103). Also,

$$\big[K_{A,\mathrm{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M$$
$$\textit{given by } \;\big[K_{A,\mathrm{mod}}\big][f] := \big[K_{\mathrm{mod}} f\big] \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \tag{7.144}$$
$$\textit{for each function } \; f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$$

is a well-defined linear mapping, which is also bounded when all quotient spaces are endowed with the norm introduced in (7.106) if in fact $\Omega \subseteq \mathbb{R}^n$ is a two-sided NTA *domain whose boundary is an unbounded Ahlfors regular set. Finally,*

$$\partial_{\tau_{jk}}\big(K_{A,\mathrm{mod}} f\big) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \;\; \textit{at } \sigma\textit{-a.e. point on } \;\partial\Omega$$
$$\textit{for each } f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \textit{ and each } j,k \in \{1,\dots,n\}. \tag{7.145}$$

(4) Analogous properties to those presented in items (1)–(2) above are also valid for homogeneous vanishing Morrey-based Sobolev spaces $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ (cf. Definition 7.1) in place of homogeneous Morrey-based Sobolev spaces.

7.2 Inverting Double Layer Operators on Morrey and Block Spaces

The starting point is deriving estimates for the operator norms of singular integral operators whose integral kernels contain, as a factor, the crucial inner product between the unit normal and the “chord” (cf. (7.146), (7.147)), of the sort obtained earlier in Theorem 4.2 and Corollary 4.2 in the context of Muckenhoupt weighted Lebesgue spaces, but now working in the framework of Morrey spaces, vanishing Morrey spaces, and block spaces. We carry out this task in Theorem 7.7 below.

Theorem 7.7 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Fix an arbitrary integrability exponent $p \in (1, \infty)$ along with some parameter $\lambda \in (0, n-1)$. Also, consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ (for some sufficiently large integer $N = N(n) \in \mathbb{N}$) which is even and positive homogeneous of degree $-n$. In this setting consider the principal-value singular integral operators T, $T^\#$ acting on each given function $f \in M^{p,\lambda}(\partial\Omega, \sigma)$ according to*

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, \mathrm{d}\sigma(y), \tag{7.146}$$

and

$$T^\# f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\, \mathrm{d}\sigma(y), \tag{7.147}$$

at σ-a.e. point $x \in \partial\Omega$. Also, define the action of the maximal operator T_* on each given function $f \in M^{p,\lambda}(\partial\Omega, \sigma)$ as

$$T_* f(x) := \sup_{\varepsilon>0} \Big| \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, \mathrm{d}\sigma(y) \Big| \text{ for each } x \in \partial\Omega, \tag{7.148}$$

and its companion

$$T_*^\# f(x) := \sup_{\varepsilon>0} \Big| \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(x)\rangle k(x-y) f(y)\, \mathrm{d}\sigma(y) \Big| \text{ for each } x \in \partial\Omega. \tag{7.149}$$

Then the following are well-defined, bounded operators

$$T_*, T_*^\#, T, T^\# : M^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow M^{p,\lambda}(\partial\Omega, \sigma), \tag{7.150}$$

$$T_*, T_*^\#, T, T^\# : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma), \tag{7.151}$$

and for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$, which depends only on m, n, p, λ, and the UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has*

$$\|T_*\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \leq \|T_*\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}$$

$$\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.152}$$

$$\begin{aligned}\Big\|T^{\#}_{*}\Big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} &\leq \Big\|T^{\#}_{*}\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{7.153}$$

$$\begin{aligned}\|T\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} &\leq \|T\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{7.154}$$

$$\begin{aligned}\Big\|T^{\#}\Big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} &\leq \Big\|T^{\#}\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\\ &\leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}.\end{aligned} \tag{7.155}$$

Furthermore, for each $q\in(1,\infty)$ *the operators*

$$T,\,T^{\#}:\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\longrightarrow\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \tag{7.156}$$

are well defined, linear, bounded, and for each $m\in\mathbb{N}$ *there exists some constant* $C_m\in(0,\infty)$*, which depends only on* m*,* n*,* q*,* λ*, and the* UR *constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\begin{aligned}\max\Big\{\|T\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\,,\,\Big\|T^{\#}\Big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\Big\}&\\ \leq C_m\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}.&\end{aligned} \tag{7.157}$$

Proof The claims made in (7.150)–(7.155) follow from Theorem 4.2, Corollary 4.2, and Proposition 7.4 (also keeping in mind (7.3) and (7.7)). Then the claims in (7.156)–(7.157) become consequences of what we have just proved and duality (cf. Proposition 7.2 and (7.46)). □

In concert with the commutator estimates discussed earlier (cf. Theorem 7.1), Theorem 7.7 implies the following result, which is the Morrey space (respectively, vanishing Morrey space, and block space) counterpart of Theorem 4.6.

Corollary 7.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix two arbitrary integrability exponents $p, q \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator K_Δ from* (3.29)*, the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ from* (4.297)*, and for each index $k \in \{1,\dots,n\}$ denote by M_{ν_k} the operator of pointwise multiplication by the k-th scalar component of ν.*

Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0,\infty)$ which depends only on m, n, p, q, λ, and the UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_\Delta\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n}\big\|[M_{\nu_k}, R_j]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.158}$$

$$\|K_\Delta\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n}\big\|[M_{\nu_k}, R_j]\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.159}$$

and

$$\|K_\Delta\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n}\big\|[M_{\nu_k}, R_j]\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.160}$$

Proof The estimates claimed in (7.158)–(7.160) are implied by (3.29), Theorem 7.7, (4.297), Proposition 3.4, and Theorem 7.1. □

We shall revisit Corollary 7.1 later, in Theorem 7.15, which contains estimates in the opposite direction to those obtained in (7.158)–(7.160).

For the time being, we take up the task of establishing estimates akin to those obtained in Theorem 4.7 for Muckenhoupt weighted Lebesgue and Sobolev spaces, now working in the setting of Morrey spaces, vanishing Morrey spaces, block spaces, as well as the brands of Sobolev spaces naturally associated with these scales.

Theorem 7.8 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$*

system in $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *and consider the boundary-to-boundary double layer potential operators* K_A, $K_A^{\#}$ *associated with* Ω *and the coefficient tensor* A *as in* (3.24) *and* (3.25)*, respectively. Finally, fix two integrability exponents* $p, q \in (1, \infty)$ *and a parameter* $\lambda \in (0, n-1)$.

Then for each $m \in \mathbb{N}$ *there exists some constant* $C_m \in (0, \infty)$ *which depends only on* m*,* n*,* A*,* p*,* q*,* λ*, and the* UR *constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_A\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.161}$$

$$\|K_A\|_{[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.162}$$

$$\|K_A\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.163}$$

$$\|K_A\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.164}$$

$$\|K_A\|_{[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.165}$$

$$\|K_A\|_{[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.166}$$

as well as

$$\left\|K_A^{\#}\right\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.167}$$

$$\left\|K_A^{\#}\right\|_{[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.168}$$

$$\left\|K_A^{\#}\right\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.169}$$

Proof All claims are justified as in the proof of Theorem 4.7, now making use of Theorem 7.7, Proposition 3.2, Theorem 7.1, (7.54)–(7.58), as well as (7.3), (7.8), (7.10), (7.17), (7.18). □

Remark 7.1 Similar estimates to those established in Theorem 7.8 are valid for the double layer operators acting on sums of Morrey spaces, vanishing Morrey spaces, and block spaces (cf. (4.332)).

The stage is now set for obtaining invertibility results for certain types of double layer potential operators acting on Morrey spaces, vanishing Morrey spaces, block spaces, as well as on the brands of Sobolev spaces naturally associated with these scales.

Theorem 7.9 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\rm dis}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in* (3.24) *and* (3.25)*, respectively. Finally, fix two integrability exponents $p, q \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$, and some number $\varepsilon \in (0, \infty)$.*

Then there exists some small threshold $\delta \in (0, 1)$ which depends only on n, p, q, λ, A, ε, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if Ω is a δ-AR domain; cf. Definition 2.15*) *it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following operators are invertible:*

$$zI + K_A : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.170}$$

$$zI + K_A : \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.171}$$

$$zI + K_A : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.172}$$

$$zI + K_A : \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.173}$$

$$zI + K_A : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.174}$$

$$zI + K_A : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.175}$$

$$zI + K_A^{\#} : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.176}$$

$$zI + K_A^{\#} : \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{7.177}$$

$$zI + K_A^{\#} : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M. \tag{7.178}$$

In addition, the inverses in (7.170)–(7.175) *are compatible with one another and also with the inverses of* (4.309)–(4.310)*. Also, the inverses in* (7.176)–(7.178) *are compatible with one another and also with the inverse of* (4.311).

Proof All claims are consequence of Theorem 7.8, reasoning as in the proof of Theorem 4.8 and Proposition 4.2. □

Remark 7.2 The conclusions in Theorem 7.9 may fail when $A \notin \mathfrak{A}_L^{\rm dis}$ even when Ω is a half-space. For example, from Proposition 3.13 and Theorem 7.2 we see that in such a scenario it may happen that $\frac{1}{2}I + K_A$ has an infinite dimensional cokernel when acting on Morrey and block spaces.

The operators in Remarks 4.14-4.15 (now considered on Morrey and block spaces) also offer counter-examples for the conclusions in Theorem 7.9 in the case when $A \notin \mathfrak{A}_L^{\rm dis}$ even when Ω is a half-space.

Remark 7.3 In the context of Theorem 7.9, if the threshold $\delta \in (0,1)$ is taken sufficiently small in such a way that the operator $zI + K_A$ is invertible on the space $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ we also claim that there exists some constant $C \in (0,\infty)$ with the property that

$$\begin{aligned} &\text{whenever} \quad f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ &\text{and} \quad g := (zI + K_A)^{-1} f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ &\text{then} \quad \|\nabla_{\tan} g\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \leq C\,\|\nabla_{\tan} f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}. \end{aligned} \tag{7.179}$$

To justify this, use (3.37) to write, for each $j,k \in \{1,\dots,n\}$,

$$\begin{aligned} \partial_{\tau_{jk}} f &= \partial_{\tau_{jk}}\big[(zI + K_A)g\big] = \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}(\nabla_{\tan} g) \\ &= \big(zI + K_A\big)(\partial_{\tau_{jk}} g) + U_{jk}\Big(\big(\nu_r \partial_{\tau_{rs}} g_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}\Big) \end{aligned} \tag{7.180}$$

at σ-a.e. point on $\partial\Omega$, where $\nu = (\nu_r)_{1\leq r\leq n}$ is the geometric measure theoretic outward unit normal to Ω. Using the abbreviations introduced in (4.345), the formulas in (7.180), corresponding to all indices $j,k \in \{1,\dots,n\}$, may be collectively re-fashioned as

$$\nabla_\tau f = \big(zI + R\big)(\nabla_\tau g), \tag{7.181}$$

where I is the identity and R is the operator acting from $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^{M\cdot n^2}$ into itself much as in (4.347)–(4.348). From these, (7.161), (3.35), Theorem 7.1, and (3.81), we then conclude that for each $m \in \mathbb{N}$ we have

$$\|R\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{M\cdot n^2} \to [M^{p,\lambda}(\partial\Omega,\sigma)]^{M\cdot n^2}} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m\rangle} \tag{7.182}$$

for some $C_m \in (0,\infty)$ which depends only on m, n, A, p, λ, and the UR constants of $\partial\Omega$. As a consequence of this, if we assume $\delta > 0$ to be sufficiently small to begin with, a Neumann series argument gives that

$$zI + R \ \text{ is invertible on } \ \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^{M\cdot n^2} \tag{7.183}$$

and provides an estimate for the norm of the inverse. At this stage, the estimate claimed in (7.179) follows from (7.181), (7.183), (4.345), and (2.585)–(2.586).

We may be further enhance the invertibility results from Theorem 7.9 by allowing the coefficient tensor to be a small perturbation of any distinguished coefficient tensor of the given system. Specifically, Theorem 7.8 in concert with the continuity of the operator-valued assignments $\mathfrak{A}_{\rm WE} \ni A \mapsto K_A$ and $\mathfrak{A}_{\rm WE} \ni A \mapsto K_A^{\#}$, considered in all contexts discussed in Theorem 7.2, yield the following result.

Theorem 7.10 *Retain the original background assumptions on the set* Ω *from Theorem 7.9 and, as before, fix some integrability exponents* $p, q \in (1, \infty)$, *a parameter* $\lambda \in (0, n-1)$, *and some number* $\varepsilon \in (0, \infty)$. *Consider* $L \in \mathfrak{L}^{\rm dis}$ *(cf. (3.195)) and pick an arbitrary* $A_o \in \mathfrak{A}_L^{\rm dis}$. *Then there exist some small threshold* $\delta \in (0, 1)$ *along with some open neighborhood* $\mathcal{O}$ *of* A_o *in* $\mathfrak{A}_{\rm WE}$, *both of which depend only on* n, p, q, λ, A_o, ε, *and the Ahlfors regularity constant of* $\partial\Omega$, *with the property that if* $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(hence,* Ω *is a* δ*-AR domain; cf. Definition 2.15) then for each* $A \in \mathcal{O}$ *and each spectral parameter* $z \in \mathbb{C}$ *with* $|z| \geq \varepsilon$, *the operators (7.170)–(7.178) are invertible.*

We close this section with the following remark.

Remark 7.4 In the two-dimensional setting, more can be said about the Lamé system. Specifically, the versions of Theorem 4.14 and Corollary 4.3 naturally formulated in terms of Morrey spaces, vanishing Morrey spaces, block spaces, as well as their associated Sobolev spaces, continue to hold, virtually with the same proofs (now making use of Proposition 7.5, Theorems 7.1, 7.2, and 7.7).

7.3 Invertibility on Morrey/Block-Based Homogeneous Sobolev Spaces

The starting point in this section is the following counterpart of Theorem 4.10 containing operator norm estimates for double layer potentials associated with distinguished coefficient tensors on Morrey-based and block-based Sobolev spaces. As in the past, the key feature of said estimates is the explicit dependence on the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain.

Theorem 7.11 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a two-sided* NTA *domain whose boundary is an unbounded Ahlfors regular set. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Also, fix some integrability exponents* $p, q \in (1, \infty)$ *and a parameter* $\lambda \in (0, n-1)$. *Next, let* L *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. *Finally, pick* $A \in \mathfrak{A}_L^{\rm dis}$ *and consider the modified boundary-to-boundary double layer potential operator* $\big[K_{A,{\rm mod}}\big]$ *associated with* Ω *and the coefficient tensor* A *as in Theorem 7.6.*

Then for each $m \in \mathbb{N}$ *there exists some* $C_m \in (0, \infty)$ *which depends only on* m, n, A, p, q, λ, *the two-sided* NTA *constants of* Ω, *and the Ahlfors regularity constant* *of* $\partial\Omega$*, such that, with the piece of notation introduced in* (4.93)*, one has*

$$\big\|\big[K_{A,\mathrm{mod}}\big]\big\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M\to[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.184}$$

$$\big\|\big[K_{A,\mathrm{mod}}\big]\big\|_{[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim]^M\to[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.185}$$

$$\big\|\big[K_{A,\mathrm{mod}}\big]\big\|_{[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M\to[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.186}$$

Proof The estimate claimed in (7.184) is justified much as in the proof of Theorem 4.10, making use of (7.141), (7.142), Theorem 7.7, and Theorem 7.1. For the estimate in (7.185), use (7.144), (7.145), Theorem 7.7, and Theorem 7.1. Finally, the estimate in (7.186) is dealt with similarly, relying on item *(4)* in Theorem 7.6. □

Having established Theorem 7.11, we now arrive at the first main result in this section concerning invertibility properties of boundary-to-boundary double layer potential operators associated with distinguished coefficient tensors on Morrey-based and block-based Sobolev spaces.

Theorem 7.12 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. Assume* L *is a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *and consider the modified boundary-to-boundary double layer potential operator* $\big[K_{A,\mathrm{mod}}\big]$ *associated with* Ω *and the coefficient tensor* A *as in Theorem 7.6. Finally, fix some integrability exponents* $p, q \in (1,\infty)$*, a parameter* $\lambda \in (0, n-1)$*, and some number* $\varepsilon \in (0,\infty)$*.*

Then there exists some small threshold $\delta \in (0,1)$ *which depends only on* n*,* p*,* q λ*,* A*,* ε*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(hence* Ω *is a* δ-AR *domain; cf. Definition 2.15) it follows that for each spectral parameter* $z \in \mathbb{C}$ *with* $|z| \geq \varepsilon$ *the operators*

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \tag{7.187}$$

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \tag{7.188}$$

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \tag{7.189}$$

are all invertible.

Proof Pick $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then Ω is a two-sided NTA domain with an unbounded boundary. That this is possible is guaranteed by Theorem 2.3. Then all desired invertibility result follow (via a Neumann series argument) from Theorem 7.11. □

Remark 7.5 The conclusions in Theorem 7.12 may fail when $A \notin \mathfrak{A}_L^{\text{dis}}$ even when Ω is a half-space. For example, Proposition 3.13 and Theorem 7.5 imply that in such a case it may happen that $\frac{1}{2}I + \big[K_{A,\text{mod}}\big]$ has an infinite dimensional cokernel when acting on homogeneous Morrey-based and block-based Sobolev spaces.

Our next main result in this section, concerning the invertibility of S_{mod} in quotient Morrey/block spaces, reads as follows:

Theorem 7.13 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider the modified boundary-to-boundary single layer potential operator S_{mod} associated with Ω and the system L as in* (3.42)*. Fix some exponent $p \in (1, \infty)$ and $\lambda \in (0, n-1)$. Finally, use $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M$ to denote the M-th power of the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)$, equipped with the semi-norm* (7.72)*.*

Then the following statements are valid.

(1) [Boundedness] *If Ω satisfying a two-sided local John condition then the operator*

$$
\begin{gathered}
\big[S_{\text{mod}}\big] : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \\
\textit{defined as } \big[S_{\text{mod}}\big]f := \big[S_{\text{mod}}f\big] \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \\
\textit{for all } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M
\end{gathered}
\tag{7.190}
$$

is well defined, linear, and bounded.

(2) [Surjectivity] *Whenever $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain; cf. Definition 2.15) it follows that* (7.72) *is a genuine norm and the operator* (7.190) *is surjective.*

(3) [Injectivity] *Whenever $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain) it follows that the operator* (7.190) *is injective.*

(4) [Isomorphism] *Whenever both $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence the domain Ω is a δ-*AR *domain) it follows that $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M$ is a Banach space when equipped with the norm* (7.72) *and the operator* (7.190) *is an isomorphism.*

(5) [Other spaces] *For each given $q \in (1,\infty)$, similar results to those described in items (1)–(4) are valid for the operator*

$$\begin{aligned}&\big[S_{\rm mod}\big]:\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M\\ &\textit{defined as }\big[S_{\rm mod}\big]f:=\big[S_{\rm mod}f\big]\in\big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M,\\ &\textit{for all } f\in\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M,\end{aligned}\tag{7.191}$$

as well as the operator

$$\begin{aligned}&\big[S_{\rm mod}\big]:\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M\\ &\textit{defined as }\big[S_{\rm mod}\big]f:=\big[S_{\rm mod}f\big]\in\big[\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M,\\ &\textit{for all } f\in\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M.\end{aligned}\tag{7.192}$$

(6) [Optimality] *If $\mathfrak{A}^{\rm dis}_L=\varnothing$ then the operator $\big[S_{\rm mod}\big]$ may fail to be surjective (in fact, may have an infinite dimensional cokernel) in all settings considered above even in the case when Ω is a half-space, and if $\mathfrak{A}^{\rm dis}_{L^\top}=\varnothing$ then the operator $\big[S_{\rm mod}\big]$ may fail to be injective (in fact, may have an infinite dimensional kernel) in all settings considered above even in the case when Ω is a half-space.*

Proof That the operator (7.190) is well defined, linear, and bounded follows from item *(2)* in Theorem 7.4, bearing in mind (2.87) and (2.48). This takes care of item *(1)*.

To deal with the claims in item *(2)*, pick a coefficient tensor $A\in\mathfrak{A}^{\rm dis}_L$. Together, Theorems 2.3, 7.9, and 4.8 guarantee that we may choose a threshold $\delta\in(0,1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}<\delta$ (a condition which we shall henceforth assume) then

$$\Omega \text{ is a two-sided NTA domain with an unbounded boundary,}\tag{7.193}$$

and

$$\begin{aligned}&\text{the operators }\pm\tfrac12 I+K_A\text{ are invertible on}\\ &\big[M^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M\ \text{ and on }\ \big[L^p_1(\partial\Omega,\sigma)\big]^M.\end{aligned}\tag{7.194}$$

To proceed, choose a scalar-valued function $\phi\in\mathscr{C}^\infty_0(\mathbb{R}^n)$ with $\phi\equiv 1$ on $B(0,1)$ and $\operatorname{supp}\phi\subseteq B(0,2)$. Having fixed a reference point $x_0\in\partial\Omega$, for each scale $r\in(0,\infty)$ define ϕ_r as in (7.73) and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. Suppose now some arbitrary function $g\in\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ has been given, and for each $r\in(0,\infty)$ define g_r as in (7.74). Thanks to (7.69) we may invoke item *(iii)* in Lemma 7.1 which gives

$$\big\|\nabla_{\tan}g_r\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}\leq C\big\|\nabla_{\tan}g\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}\tag{7.195}$$

for some $C \in (0, \infty)$ independent of g and r. For each $r \in (0, \infty)$ let us now define h_r as in (4.378) (here it helps to note that $\pm\frac{1}{2}I + K_A$ are invertible both on $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ and on $\big[L_1^p(\partial\Omega, \sigma)\big]^M$, and the two inverses are compatible). Using the formula $\partial_{\tau_{jk}} g_r = \big(\partial_{\tau_{jk}}\phi_r\big) \cdot \big(g - g_{\Delta_{2r}}\big) + \phi_r \cdot \partial_{\tau_{jk}} g$, the fact that the function g belongs to the space $\big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, and (7.69) it is straightforward to show that $g_r \in \big[M_1^{p,\lambda}(\partial\Omega, \sigma) \cap L_1^p(\partial\Omega, \sigma)\big]^M$. Hence, h_r is a meaningfully defined function which belongs to $\big[M_1^{p,\lambda}(\partial\Omega, \sigma) \cap L_1^p(\partial\Omega, \sigma)\big]^M$. Moreover, from the definition of h_r (cf. (4.378)), (7.179), and (7.195) we conclude that there exists a constant $C \in (0, \infty)$, independent of g and r, such that

$$\|\nabla_{\tan} h_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \leq C\,\|\nabla_{\tan} g_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \quad \text{for each} \ \ r \in (0, \infty). \tag{7.196}$$

Going further, for each $r \in (0, \infty)$ abbreviate

$$f_r := \partial_\nu^A\big(\mathcal{D}_A h_r\big) \ \ \text{at } \sigma\text{-a.e. point on} \ \ \partial\Omega. \tag{7.197}$$

Since $h_r \in \big[M_1^{p,\lambda}(\partial\Omega, \sigma) \cap L_1^p(\partial\Omega, \sigma)\big]^M$, the boundedness result recorded in (3.115) implies that $f_r \in \big[L^p(\partial\Omega, w)\big]^M$ and for each $r \in (0, \infty)$ we have

$$\|f_r\|_{[L^p(\partial\Omega,w)]^M} \leq C\|\nabla_{\tan} h_r\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}, \tag{7.198}$$

where $C \in (0, \infty)$ is independent of g and r. Moreover, (7.64), (3.33), (3.66), (2.586), Proposition 7.5, and (7.196) permit us to write

$$\begin{aligned}\|f_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} &\leq C\|\nabla_{\tan} h_r\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \\ &\leq C\|\nabla_{\tan} g\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}.\end{aligned} \tag{7.199}$$

We use next that $h_r \in \big[L_1^p(\partial\Omega, \sigma)\big]^M$, (3.130), (4.378), (7.197), and Theorem 2.4 to ensure that for each $r \in (0, \infty)$ there exists some constant $c_r \in \mathbb{C}^M$ such that

$$S_{\text{mod}} f_r = g_r + c_r \ \ \text{on} \ \ \partial\Omega. \tag{7.200}$$

Select now a sequence $\{r_j\}_{j\in\mathbb{N}} \subseteq (0, \infty)$ which converges to infinity. Since from (7.199) we know that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, we can rely on the Banach–Alaoglu Theorem (cf. (7.27)) and (7.24) to assume, without loss of generality, that $\{f_{r_j}\}_{j\in\mathbb{N}}$ is actually weak-$*$ convergent to some function $f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$. On account of (7.121), (7.200), and the definition of g_r given in (7.74), for each test function $\psi \in \big[\mathrm{Lip}(\partial\Omega)\big]^M$ with compact support we may write

$$\begin{aligned}\int_{\partial\Omega} \langle S_{\rm mod} f, \psi\rangle \, d\sigma &= \lim_{j\to\infty} \int_{\partial\Omega} \langle S_{\rm mod} f_{r_j}, \psi\rangle \, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} \langle g_{r_j} + c_{r_j}, \psi\rangle \, d\sigma \\ &= \lim_{j\to\infty} \int_{\partial\Omega} \langle \phi_{r_j} \cdot (g - g_{\Delta_{2r_j}}) + c_{r_j}, \psi\rangle \, d\sigma \\ &= \lim_{j\to\infty} \int_{\partial\Omega} \langle g - g_{\Delta_{2r_j}} + c_{r_j}, \psi\rangle \, d\sigma \\ &= \int_{\partial\Omega} \langle g, \psi\rangle \, d\sigma + \lim_{j\to\infty} \Big\langle c_{r_j} - g_{\Delta_{2r_j}}, \int_{\partial\Omega} \psi \, d\sigma \Big\rangle. \end{aligned} \tag{7.201}$$

Since ψ is arbitrary, we conclude that the sequence $\{c_{r_j} - g_{\Delta_{2r_j}}\}_{j\in\mathbb{N}} \subseteq \mathbb{C}^M$ converges to some constant $c \in \mathbb{C}^M$. Hence, we may then conclude from (7.201) that

$$\int_{\partial\Omega} \langle S_{\rm mod} f, \psi\rangle \, d\sigma = \int_{\partial\Omega} \langle g + c, \psi\rangle \, d\sigma \tag{7.202}$$

for each function $\psi \in [\mathrm{Lip}(\partial\Omega)]^M$ with compact support. Eventually, from (7.202) we obtain (see [111, §3.7] for a general measure theoretic result of this nature)

$$S_{\rm mod} f = g + c \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.203}$$

Hence, $[S_{\rm mod}]f = [S_{\rm mod} f] = [g]$ and since $[g] \in [\dot{L}^p_1(\partial\Omega, w)/\sim]^M$ is arbitrary, it follows that the operator (7.190) is surjective. Moreover, from (7.199) we see that

$$\begin{aligned}\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} &\leq \limsup_{j\to\infty} \|f_{r_j}\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\, \|\nabla_{\tan} g\|_{M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}} \\ &\leq C\, \|[g]\|_{[\dot{M}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M}, \end{aligned} \tag{7.204}$$

for some constant $C \in (0, \infty)$ independent of g, so the surjectivity of the operator in (7.190) comes with quantitative control.

Let us also observe that the fact that (7.72) is, as claimed, a genuine norm is clear from (7.193) and Proposition 7.6.

Moving on, let us now deal with item *(3)*. Pick a coefficient tensor $\widetilde{A} \in \mathfrak{A}_L$ such that $\widetilde{A}^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$. By Theorem 7.9 we may then choose $\delta \in (0, 1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (something we shall henceforth assume) then

$$\text{the operators } \pm\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top} \text{ are invertible on } [M^{p,\lambda}(\partial\Omega,\sigma)]^M. \tag{7.205}$$

The goal is to show that the operator (7.190) is injective. To this end, suppose the function $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ is such that $\big[S_{\rm mod}\big]f = [0]$. Hence, $\big[S_{\rm mod} f\big] = [0]$ which implies that there exists some constant $c \in \mathbb{C}^M$ for which

$$S_{\rm mod} f = c \quad \text{at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{7.206}$$

This, together with (7.120), allows us to obtain

$$\big(\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^{\top}}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^{\top}}\big)f\Big) = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega \tag{7.207}$$

which, by (7.205), leads to $f = 0$. Since the operator (7.190) is linear, it follows that this is indeed injective.

Next, to treat the claims in item *(4)*, assume that $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^{\top}} \neq \varnothing$. Then, by the previous items the operator (7.190) is a continuous bijection. Moreover, Proposition 7.6 and (7.193) imply that $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M$ is a Banach space, hence the operator (7.190) is a linear isomorphism.

Considered now the claims made in item *(5)*. First, the fact that the operator (7.191) is well defined, linear, and bounded is seen from item *(4)* in Theorem 7.4, keeping in mind (2.87) and (2.48). Second, that the operator (7.191) satisfies the properties described in items *(2)–(3)* of Theorem 7.13 is a consequence of the operator identities

$$\begin{gathered}\big(\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big) = \big[\partial^A_\nu \mathcal{D}_{A,{\rm mod}}\big]\big[S_{\rm mod}\big] \\ \text{as mappings acting from } \ \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M,\end{gathered} \tag{7.208}$$

and

$$\begin{gathered}\big(\tfrac{1}{2}I + \big[K_{A,{\rm mod}}\big]\big)\big(-\tfrac{1}{2}I + \big[K_{A,{\rm mod}}\big]\big) = \big[S_{\rm mod}\big]\big[\partial^A_\nu \mathcal{D}_{A,{\rm mod}}\big] \\ \text{as mappings acting from } \ \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M,\end{gathered} \tag{7.209}$$

both of which are contained in Theorem 7.5, (7.178) in Theorem 7.9 (specialized to $z = \pm\frac{1}{2}$), (7.188) in Theorem 7.12 (again with $z = \pm\frac{1}{2}$), as well as (7.139), Theorem 2.3, and Theorem 2.4. The case of the operator $\big[S_{\rm mod}\big]$ in (7.192) is handled analogously.

Finally, the optimality results in item *(6)* are seen from (3.406) and the natural version of Proposition 4.4 for Morrey and block spaces. □

Remark 7.6 Together, (7.133), Theorem 7.9 (with $z = \pm\frac{1}{2}$), (7.135), Theorem 7.12 (with $z = \pm\frac{1}{2}$), (7.132), Theorems 2.3, and 2.4 provide an alternative proof of items *(2)–(3)* in Theorem 7.13.

We conclude this section with the following theorem addressing the issue of invertibility for the conormal of the double layer operator acting from homogeneous Morrey-based and block-spaces Sobolev spaces.

Theorem 7.14 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Fix some exponent $p \in (1, \infty)$ along with some parameter $\lambda \in (0, n-1)$. Pick some coefficient tensor $A \in \mathfrak{A}_L$ and consider the modified conormal derivative of the modified double layer operator in the context of (7.132), i.e.,*

$$\begin{aligned} &\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}} : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \ \text{ defined as} \\ &\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)[f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \text{ for each } \ f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.210}$$

From Theorem 7.5 this is known to be a well-defined, linear, and bounded operator when the quotient space is equipped with the norm (7.72). In relation to this, the following statements are valid.

(1) [*Injectivity*] *Whenever $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and actually $A \in \mathfrak{A}_L^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-AR domain; cf. Definition 2.15) then the operator (7.210) is injective.*

(2) [*[*Surjectivity] *Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ and actually $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain) then the operator (7.210) is surjective.*

(3) [Isomorphism] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, and $A \in \mathfrak{A}_L^{\mathrm{dis}}$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence Ω is a δ-*AR *domain) then the operator (7.210) is an isomorphism.*

(4) [Other spaces] *For each $q \in (1,\infty)$, similar results to those described in items (1)–(3) above are valid for the modified conormal derivative of the modified double layer operator in the context of block and vanishing Morrey spaces, i.e.,*

$$\begin{aligned} &\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big] : \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \ \text{ given by} \\ &\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big][f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \text{ for each } \ f \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{7.211}$$

and

$$\begin{aligned} &\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}} : \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big/\sim\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \ \text{ given by} \\ &\big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big)[f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \text{ for each } \ f \in \big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{7.212}$$

Proof All claims may be established by arguing as in the proof of Theorem 4.13, now making use of Theorems 7.5, 7.9, and 7.12. □

7.4 Characterizing Flatness in Terms of Morrey and Block Spaces

How do the quantitative aspects of the analysis of a certain geometric environment affect the very geometric features of said environment? Here we address a specific aspect of this general question by characterizing the flatness of a "surface" in terms of the size of the norms of certain singular integral operators acting on Morrey and block spaces considered on this surface.

In order be able to elaborate on this topic, we need some notation. Given a UR domain $\Omega \subseteq \mathbb{R}^n$, denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. From Proposition 7.4 and (5.16)–(5.18) we then conclude that whenever $p \in (1,\infty)$ and $\lambda \in (0, n-1)$, the operators

$$\mathbf{C} : M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{7.213}$$

$$\mathbf{C} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{7.214}$$

and

$$\mathbf{C}^{\#} : M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{7.215}$$

$$\mathbf{C}^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \tag{7.216}$$

are all well defined, linear, and continuous, with

$$\begin{aligned}
&\|\mathbf{C}\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\\
&\|\mathbf{C}^{\#}\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\\
&\|\mathbf{C}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\\
&\|\mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},
\end{aligned} \tag{7.217}$$

bounded exclusively in terms of n, p, λ, and the UR constants of $\partial\Omega$.

Granted these, via duality (cf. (5.19) and Proposition 7.2) we also obtain that for each $q \in (1,\infty)$ and $\lambda \in (0, n-1)$ the operators

$$\mathbf{C} : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{7.218}$$

$$\mathbf{C}^{\#} : \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n \longrightarrow \mathcal{B}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n \tag{7.219}$$

are all well defined, linear, and bounded, with

$$\begin{aligned} &\|\mathbf{C}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n}, \\ &\|\mathbf{C}^{\#}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \end{aligned} \tag{7.220}$$

controlled only in terms of n, q, λ, and the UR constants of $\partial\Omega$.

In addition, from (5.20) and duality (cf. (5.19) and Proposition 7.2) we conclude that, for each $p, q \in (1, \infty)$ and $\lambda \in (0, n-1)$,

the operator identities $\mathbf{C}^2 = \frac{1}{4}I$ and $\big(\mathbf{C}^{\#}\big)^2 = \frac{1}{4}I$ are valid on either of the spaces $M^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$, $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$, and $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$. (7.221)

More delicate estimates than (7.217), (7.220) turn out to hold for the antisymmetric part of the Cauchy–Clifford operator, i.e., for the difference $\mathbf{C} - \mathbf{C}^{\#}$, of the sort described in the proposition below.

Proposition 7.9 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix two integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Then for each $m \in \mathbb{N}$ there exists some constant $C_m \in (0, \infty)$ which depends only on m, n, p, q, λ, and the* UR *constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has*

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.222}$$

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.223}$$

$$\Big\|\mathbf{C} - \mathbf{C}^{\#}\Big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\!\ell_n} \leq C_m \|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{7.224}$$

Proof This is implied by the structural result from Lemma 5.1 (bearing in mind (7.3), (7.8), (7.17)), together with Theorems 7.1, 7.7, and (3.29). □

Remarkably, it is also possible to establish bounds from below for the operator norm of $\mathbf{C} - \mathbf{C}^{\#}$ on Morrey spaces and their pre-duals, considered on the boundary of a UR domain, in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal vector to the said domain.

Proposition 7.10 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain such that $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an integrability exponent $p \in (1, \infty)$ along* *with a parameter* $\lambda \in (0, n-1)$. *Then there exists some* $C \in (0, \infty)$ *which depends only on* n, p, λ, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}$$

$$\leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}. \tag{7.225}$$

Furthermore, for each $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *there exists some constant* $C \in (0, \infty)$ *which depends only on* n, q, λ, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}. \tag{7.226}$$

Proof The argument largely follows the proof of the unweighted version of Theorem 5.1 (i.e., when $w \equiv 1$), so we will only indicate the main changes. First, in place of (5.45) we now write (making use of (7.2), the fact that $\mathbf{1}_{\Delta(y_0,R)} \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$, and (7.5))

$$\begin{aligned}
&⨍_{\Delta(x_0,R)} |(\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|^p \, d\sigma(x) \\
&\quad \leq R^{-(n-1-\lambda)} \left\| (\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)} \right\|^p_{M^{p,\lambda}(\partial\Omega,\sigma)} \\
&\quad \leq R^{-(n-1-\lambda)} \left\| \mathbf{1}_{\Delta(y_0,R)} \right\|^p_{M^{p,\lambda}(\partial\Omega,\sigma)} \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n} \\
&\quad \leq C R^{-(n-1-\lambda)} \sigma\big(\Delta(y_0, R)\big)^{(n-1-\lambda)/(n-1)} \times \\
&\qquad \times \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n} \\
&\quad \leq C \|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},
\end{aligned} \tag{7.227}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$.

Second, thanks to (7.227), in place of (5.46) we have

$$⨍_{\Delta(x_0,R)} \left| \int_{\Delta(y_0,R)} \left\{ \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) + \nu(x) \odot \frac{x_0 - y}{|x_0 - y|^n} \right\} d\sigma(y) \right|^p d\sigma(x)$$

$$\leq C(\Lambda^{-n} \ln \Lambda)^p \|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + C_{n,p} ⨍_{\Delta(x_0,R)} |(\mathbf{C} - \mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|^p \, d\sigma(x)$$

$$+ C\Lambda^{-np} \fint_{\Delta(x_0,R)} \left|\nu(x) - \nu_{\Delta(x_0,R)}\right|^p \, d\sigma(x)$$

$$\leq C(\Lambda^{-n} \ln \Lambda)^p \|\nu\|^p_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$$

$$+ C\|\mathbf{C} - \mathbf{C}^{\#}\|^p_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \tag{7.228}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$.

Third, with (7.228) in hand, the same type of argument as in the end-game of the proof of Theorem 5.1 (cf. (5.47)–(5.54)) presently gives

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C(\Lambda^{-1} \ln \Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$$
$$+ C\Lambda^{n-1}\|\mathbf{C} - \mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \tag{7.229}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$. By eventually further increasing Λ as to ensure that $\Lambda^{-1} \ln \Lambda < 1/(2C)$, we finally conclude from (7.229) that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\|\mathbf{C} - \mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}, \tag{7.230}$$

where $C \in (0, \infty)$ depends only on n, p, λ, and the Ahlfors regularity constant of $\partial\Omega$. This establishes the first estimate claimed in (7.225). The second estimate in (7.225) is a direct consequence of (7.8).

Finally, the estimate claimed in (7.226) follows from the first inequality in (7.225), plus the fact that whenever $p, q \in (1, \infty)$ are such that $1/p + 1/q = 1$ then the (real) transpose of

$$\mathbf{C} - \mathbf{C}^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \tag{7.231}$$

is the operator

$$\mathbf{C}^{\#} - \mathbf{C} : \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n. \tag{7.232}$$

See (5.19) and Proposition 7.2 in this regard. □

Our next result contains estimates in the opposite direction to those presented in Corollary 7.1.

Theorem 7.15 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix two arbitrary integrability exponents $p, q \in (1, \infty)$ along with some parameter $\lambda \in (0, n-1)$. Finally, recall the boundary-to-boundary harmonic double* *layer potential operator* K_Δ *from* (3.29)*, the Riesz transforms* $\{R_j\}_{1\le j\le n}$ *on* $\partial\Omega$ *from* (4.297)*, and for each index* $k\in\{1,\dots,n\}$ *denote by* M_{ν_k} *the operator of pointwise multiplication by the* k*-th scalar component of* ν*. Then there exists some* $C\in(0,\infty)$ *which depends only on* n*,* p*,* q*,* λ*, and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\le C\Big\{\|K_\Delta\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\le j,k\le n}\big\|[M_{\nu_k},R_j]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\Big\}, \tag{7.233}$$

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\le C\Big\{\|K_\Delta\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\le j,k\le n}\big\|[M_{\nu_k},R_j]\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}\Big\}, \tag{7.234}$$

and

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\le C\Big\{\|K_\Delta\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \max_{1\le j,k\le n}\big\|[M_{\nu_k},R_j]\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\Big\}. \tag{7.235}$$

Proof If $\partial\Omega$ is unbounded then all estimates are implied by Proposition 7.10 and the structural result from Lemma 5.1 (keeping in mind (7.3), (7.8), (7.17)). When $\partial\Omega$ is bounded, we have $K_\Delta 1=\pm\frac{1}{2}$ (cf. [114, §1.5]) with the sign plus if Ω is bounded, and the sign minus if Ω is unbounded, hence the norm of K_Δ on either $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ or $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ is $\ge\frac{1}{2}$ in such a case. Given that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\le 1$ (cf. (2.118)), the estimates claimed in (7.233)–(7.235) are valid in this case if we take $C\ge 2$. □

In turn, the results established in Theorem 7.15 may be used to characterize the class of δ-AR domains in $\mathbb{R}^n$, in the spirit of Corollary 5.2, using Morrey spaces and their pre-duals.

By way of contrast, Theorem 7.16 discussed next is a stability result stating that if $\Omega\subseteq\mathbb{R}^n$ is a UR domain with an unbounded boundary for which the URTI (cf. (5.58)) are "almost" true in the context of either Morrey or block spaces, then $\partial\Omega$ is "almost" flat, in that the BMO semi-norm of the outward unit normal to Ω is small.

Theorem 7.16 *Let* $\Omega\subseteq\mathbb{R}^n$ *be a* UR *domain with an unbounded boundary. Abbreviate* $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, fix* $p,q\in(1,\infty)$ *along with* $\lambda\in(0,n-1)$*, and recall the Riesz transforms* $\{R_j\}_{1\le j\le n}$ *on* $\partial\Omega$ *from* (4.297)*. Then there exists some* $C\in(0,\infty)$ *which depends only on* n*,* p*,* q*,* λ*, and the* UR *constants of* $\partial\Omega$

with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{\Big\|I+\sum_{j=1}^{n}R_j^2\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n}\big\|[R_j,R_k]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\Big\}, \tag{7.236}$$

plus similar estimates with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, *or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$.

Proof A key ingredient is the fact that we have the operator identities

$$\mathbf{C}-\mathbf{C}^{\#}=\mathbf{C}\Big(I+\sum_{j=1}^{n}R_j^2\Big)+\sum_{1\leq j<k\leq n}\mathbf{C}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k \tag{7.237}$$
$$\text{on } M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n, \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n.$$

These are proved much like formula [61, (4.6.46), p. 2752], now making use of (7.221). Once (7.237) has been established, Proposition 7.10 and (7.213)–(7.220) to conclude (much as in the proof of Theorem 5.3) that the estimate claimed in (7.236) as well as its related versions on vanishing Morrey spaces and block spaces are all true. □

The last result in this section contains estimates in the opposite direction to those from Theorem 7.16. Together, Theorems 7.17 and 7.16 amount to saying that, under natural background geometric assumptions on the set Ω, the URTI are "almost" true on Morrey spaces or block spaces if and only if ∂Ω is "almost" flat (in that the BMO semi-norm of the outward unit normal to Ω is small).

Theorem 7.17 *Let* $\Omega\subseteq\mathbb{R}^n$ *be a* UR *domain. Abbreviate* $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Also, fix* $p,q\in(1,\infty)$ *along with* $\lambda\in(0,n-1)$, *and recall the Riesz transforms* $\{R_j\}_{1\leq j\leq n}$ *on* ∂Ω *from* (4.297).

Then for each $m\in\mathbb{N}$ *there exists some constant* $C_m\in(0,\infty)$ *which depends only on m, n, p, q,* λ, *and the* UR *constants of* ∂Ω *such that, with the piece of notation introduced in* (4.93), *one has*

$$\Big\|I+\sum_{j=1}^{n}R_j^2\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.238}$$

$$\max_{1\leq j<k\leq n}\big\|[R_j,R_k]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{7.239}$$

plus similar estimates with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, *or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$.

Proof The starting point is to observe that we have the operator identities

$$\mathbf{C}(\mathbf{C}^{\#}-\mathbf{C})=-\tfrac{1}{4}\Big(I+\sum_{j=1}^{n}R_j^2\Big)-\tfrac{1}{4}\sum_{1\leq j<k\leq n}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k,$$
$$\text{on } M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n, \tag{7.240}$$

which are themselves consequences of (7.237) and (7.221). With (7.240) in hand, the estimates claimed in the statement of the theorem may then be justified via an estimate similar in spirit to (5.66), and also invoking Proposition 7.9 (as well as (7.217), (7.220)) in the process. □

7.5 Boundary Value Problems in Morrey and Block Spaces

We begin by discussing the Dirichlet Problem for weakly elliptic systems in δ-AR domains with boundary data in ordinary Morrey spaces, vanishing Morrey spaces, and block spaces.

Theorem 7.18 *Let $\Omega\subseteq\mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa>0$. Also, pick an exponent $p\in(1,\infty)$ and a parameter $\lambda\in(0,n-1)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system L in $\mathbb{R}^n$, consider the Dirichlet Problem*

$$\begin{cases} u\in\big[\mathscr{C}^\infty(\Omega)\big]^M,\\ Lu=0 \ \ in \ \ \Omega,\\ \mathcal{N}_\kappa u\in M^{p,\lambda}(\partial\Omega,\sigma),\\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}=f\in\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{cases} \tag{7.241}$$

The following claims are true:

(a) [Existence, Regularity, and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}}\neq\varnothing$ and $A\in\mathfrak{A}_L^{\mathrm{dis}}$, then there exists $\delta\in(0,1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}<\delta$ (hence Ω is a δ-AR domain; cf. Definition 2.15) then $\frac{1}{2}I+K_A$ is an invertible operator on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and the function $u:\Omega\to\mathbb{C}^M$ defined as*

$$u(x):=\Big(\mathcal{D}_A\Big(\tfrac{1}{2}I+K_A\Big)^{-1}f\Big)(x)\ \ for\ all\ \ x\in\Omega, \tag{7.242}$$

is a solution of the Dirichlet Problem (7.241). Moreover,

$$\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{7.243}$$

Furthermore, the function u defined in (7.242) *satisfies the following regularity result*

$$\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma) \Longleftrightarrow f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{7.244}$$

and if either of these conditions holds then

$$\begin{gathered}(\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{C}^{n\cdot M}\text{) at } \sigma\text{-a.e. point on } \partial\Omega \text{ and}\\ \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}.\end{gathered} \tag{7.245}$$

(b) [Uniqueness] *Whenever* $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$*, there exists some* $\delta \in (0,1)$ *which depends only on* n*,* p*,* $[w]_{A_p}$*,* L*,* η*, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*hence* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the Dirichlet Problem* (7.241) *has at most one solution.*

(c) [Well-Posedness] *If* $\mathfrak{A}^{\text{dis}}_{L} \neq \varnothing$ *and* $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$ *then there exists some* $\delta \in (0,1)$ *which depends only on* n*,* p*,* $[w]_{A_p}$*,* A*,* η*, and the Ahlfors regularity constant of* $\partial\Omega$ *such that whenever* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the Dirichlet Problem* (7.241) *is uniquely solvable and the solution satisfies* (7.243).

(d) [Sharpness] *If* $\mathfrak{A}^{\text{dis}}_{L} = \varnothing$ *then the Dirichlet Problem* (7.241) *may fail to be solvable* (*actually for boundary data belonging to an infinite dimensional subspace of the corresponding Morrey space*)*. Also, if* $\mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing$ *then the Dirichlet Problem* (7.241) *may have more than one solution* (*in fact, the linear space of null-solutions may actually be infinite dimensional*).

(e) [Other Spaces of Boundary Data] *Similar results to those described in items (a)–(d) above hold with the Morrey space* $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$*, or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$.

In addition, given any pair of integrability exponents $p_0, p_1 \in (1,\infty)$ *along with any pair of parameters* $\lambda_0, \lambda_1 \in (0, n-1)$*, similar results are valid for the Dirichlet Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa u \in M^{p_0,\lambda_0}(\partial\Omega,\sigma) + M^{p_1,\lambda_1}(\partial\Omega,\sigma), \\ u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[M^{p_0,\lambda_0}(\partial\Omega,\sigma) + M^{p_1,\lambda_1}(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{7.246}$$

as well as for its versions with the Morrey spaces replaced by vanishing Morrey space or block spaces.

To give an example, suppose $\Omega \subseteq \mathbb{R}^n$ is a δ-AR domain and fix an arbitrary aperture parameter $\kappa > 0$ along with some power $a \in (0, n-1)$. In addition, choose a number $\lambda \in (0, n-1-a)$ and define $p := (n-1-\lambda)/a \in (1, \infty)$. Then, if $\delta > 0$ is sufficiently small (relative to n, a, λ, and the Ahlfors regularity constant of $\partial\Omega$), it follows that for each point $x_o \in \partial\Omega$ the Dirichlet Problem

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \ \text{ in } \ \Omega, \quad \mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega, \sigma), \\ \Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = |x - x_o|^{-a} \ \text{ at } \sigma\text{-a.e. point } \ x \in \partial\Omega \end{cases} \tag{7.247}$$

has a unique solution. Moreover, there exists a constant $C(\Omega, n, \kappa, a, \lambda) \in (0, \infty)$ with the property that said solution satisfies $\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a, \lambda)$. The reason is that, as seen from Example 7.1, the function $f_{x_o}(x) := |x - x_o|^{-a}$ for σ-a.e. point $x \in \partial\Omega$ belongs to the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ and we have $\sup_{x_o\in\partial\Omega} \|f_{x_o}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty$. As such, the result in item *(c)* of Theorem 7.18 applies and yields the desired conclusion.

In addition, there is a naturally accompanying regularity result. To formulate it, assume $q \in (1, \infty)$ and $\mu \in (0, n-1)$ are such that $a + 1 = (n - 1 - \mu)/q$. Starting from the realization that the boundary datum f_{x_o} actually belongs to a suitably defined off-diagonal Morrey-based Sobolev space on $\partial\Omega$, from (6.37) and Example 7.1 we see that there exists $C(\Omega, n, \kappa, a, q, \mu) \in (0, \infty)$ independent of $x_o \in \partial\Omega$ such that, if $\delta > 0$ is sufficiently small to begin with, then the unique solution of the Dirichlet Problem (7.247) satisfies the following additional regularity properties

$$\begin{aligned} &(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists (in } \mathbb{R}^n) \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ &\text{and } \ \|\mathcal{N}_\kappa(\nabla u)\|_{M^{q,\mu}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a, q, \mu). \end{aligned} \tag{7.248}$$

To wrap up the discussion about (7.247) we wish to note that since the inverse of $\frac{1}{2}I + K_\Delta$ on $M^{p,\lambda}(\partial\Omega, \sigma)$ is compatible with the inverse of $\frac{1}{2}I + K_\Delta$ on $L^{p,\infty}(\partial\Omega, \sigma)$ (as alluded to in Remark 4.20), we conclude (from the manner in which the solution is constructed; cf. (7.242)) that the solution u of the Dirichlet Problem (7.247) actually coincides with the solution u of the Dirichlet Problem (6.35).

In closing, let us also mention that boundary value problems in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with boundary data with components in the Morrey spaces $M^{2,\lambda}(\partial\Omega, \sigma)$ (with λ belonging to a certain sub-interval of $(0, n-1)$) for symmetric, homogeneous, second-order, systems with constant real coefficients satisfying the Legendre–Hadamard strong ellipticity condition have been considered in [127].

After this digression we turn to the task of giving the proof of Theorem 7.18.

Proof of Theorem 7.18 The argument parallels the proof of Theorem 6.2. First, Theorem 7.9 shows that there exists some number $\delta \in (0, 1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $\frac{1}{2}I + K_A$ is invertible on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. Hence, the function u in (7.242) is meaningfully defined, and according to (3.23), (7.3), and Theorem 7.2, we have $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, $Lu = 0$ in Ω, $\mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega,\sigma)$, and (7.243) holds. Concerning the equivalence claimed in (7.244), if $f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ then Theorem 7.9 gives (assuming $\delta > 0$ is sufficiently small) that $\Big(\frac{1}{2}I + K_A\Big)^{-1} f$ belongs to $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. With this in hand, (7.64)–(7.65) then imply that the function u defined as in (7.242) satisfies $\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma)$, the nontangential boundary trace $\big(\nabla u\big)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $\partial\Omega$, and the left-pointing inequality in the equivalence claimed in (7.245) holds. In particular, this justifies the left-pointing implication in (7.244). The right-pointing implication in (7.244) together with the right-pointing inequality in the equivalence claimed in (7.245) are consequences of (7.3) and Proposition 2.22.

Turning our attention to the uniqueness result claimed in item *(b)*, make the assumption that $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\text{dis}}_{L^\top}$. Also, denote by $q \in (1,\infty)$ the Hölder conjugate exponent of p. From Theorem 7.9, presently used with L replaced by $L^\top$, we know that there exists $\delta \in (0,1)$, which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if Ω is a δ-AR domain then the following operator is invertible:

$$\tfrac{1}{2}I + K_{A^\top} : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.249}$$

Also, decreasing the value of $\delta \in (0,1)$ if necessary guarantees that Ω is an NTA domain with unbounded boundary (cf. Theorem 2.3). In such a case, (6.2) ensures that Ω is globally pathwise nontangentially accessible.

Moving on, recall the fundamental solution $E = \big(E_{\alpha\beta}\big)_{1\le\alpha,\beta\le M}$ associated with the system L as in Theorem 3.1. Pick $x_\star \in \mathbb{R}^n\setminus\overline{\Omega}$ along with $x_0 \in \Omega$, arbitrary. Also, fix $\rho \in \big(0, \frac{1}{4}\,\text{dist}(x_0,\partial\Omega)\big)$ and define $K := \overline{B(x_0,\rho)}$. Finally, recall the aperture parameter $\widetilde{\kappa} > 0$ associated with Ω and κ as in Theorem 6.1. To proceed, for each fixed index $\beta \in \{1,\dots,M\}$, consider the $\mathbb{C}^M$-valued function

$$f^{(\beta)}(x) := \big(E_{\beta\alpha}(x-x_0) - E_{\beta\alpha}(x-x_\star)\big)_{1\le\alpha\le M}, \qquad \forall\, x \in \partial\Omega. \tag{7.250}$$

Based on (7.19), (7.250), (7.57), (2.579), (7.21), (3.16), and the Mean Value Theorem we then conclude that

$$f^{(\beta)} \in \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.251}$$

Consequently, with $\Big(\frac{1}{2}I + K_{A^\top}\Big)^{-1}$ denoting the inverse of the operator in (7.249),

$$v_\beta := \big(v_{\beta\alpha}\big)_{1\le\alpha\le M} := \mathcal{D}_{A^\top}\Big(\Big(\tfrac{1}{2}I + K_{A^\top}\Big)^{-1} f^{(\beta)}\Big) \tag{7.252}$$

is a well-defined $\mathbb{C}^M$-valued function in Ω which, by virtue of Theorem 7.2, satisfies

$$\begin{gathered} v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_{\widetilde{\kappa}} v_\beta \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \quad \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \\ \text{and } \ v_\beta\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{gathered} \tag{7.253}$$

In addition, from (7.251)–(7.252) and (7.64) we see that

$$(\nabla v_\beta)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists (in } \mathbb{C}^{n\cdot M}\text{) at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{7.254}$$

For each pair of indices $\alpha, \beta \in \{1, \dots, M\}$ let us now define

$$G_{\alpha\beta}(x) := v_{\beta\alpha}(x) - \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big), \qquad \forall\, x \in \Omega\setminus\{x_0\}. \tag{7.255}$$

Regarding $G := \big(G_{\alpha\beta}\big)_{1\le\alpha,\beta\le M}$ as a $\mathbb{C}^{M\times M}$-valued function defined $\mathcal{L}^n$-a.e. in Ω, from (7.255) and Theorem 3.1 we then see that $G \in \big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^{M\times M}$. Furthermore, by design,

$$\begin{gathered} L^\top G = -\delta_{x_0} I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^{M\times M} \ \text{ and} \\ G\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \big(\nabla G\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{gathered} \tag{7.256}$$

while if $v := \big(v_{\beta\alpha}\big)_{1\le\alpha,\beta\le M}$ then from (2.8), (3.16), and the Mean Value Theorem it follows that at each point $x \in \partial\Omega$ we have

$$\begin{gathered} \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K} G\big)(x) \le \big(\mathcal{N}_{\widetilde{\kappa}} v\big)(x) + C_{x_0,\rho}(1+|x|)^{1-n} \ \text{ and} \\ \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big)(x) \le \big(\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\big)(x) + C_{x_0,\rho}(1+|x|)^{-n}, \end{gathered} \tag{7.257}$$

where $C_{x_0,\rho} \in (0,\infty)$ is independent of x. From (7.253), (7.257), (7.21), and (7.19) we see that the conditions listed in (6.4) are presently satisfied and, in fact,

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma). \tag{7.258}$$

Assume now that $u = (u_\beta)_{1\le\beta\le M}$ is a $\mathbb{C}^M$-valued function in Ω satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \text{and } \ \mathcal{N}_\kappa u \ \text{ belongs to the space } \ M^{p,\lambda}(\partial\Omega,\sigma). \end{gathered} \tag{7.259}$$

Since (7.258) and (7.22) imply

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\,\mathrm{d}\sigma$$
$$\leq C\big\|\mathcal{N}_\kappa u\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}\big\|\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} < \infty, \tag{7.260}$$

we may rely on Theorem 6.1 to conclude that the Poisson integral representation formula (6.6) holds. In particular, said formula proves that whenever $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = 0$ at σ-a.e. point on $\partial\Omega$ we necessarily have $u(x_0) = 0$. Given that x_0 has been arbitrarily chosen in Ω, this ultimately shows such a function u is actually identically zero in Ω. This finishes the proof of the uniqueness claim made in item *(b)*. The well-posedness claim in item *(c)* is a consequence of what we have already proved in items *(a)–(b)*.

Going further, the first claim in item *(d)*, regarding the potential failure of solvability of the Dirichlet Problem (7.241), is a consequence of Proposition 3.10 formulated for Morrey spaces. Its proof goes through virtually unchanged, with one caveat. Specifically, to justify (3.308), instead of Lebesgue's Dominated Convergence Theorem on Muckenhoupt weighted Lebesgue spaces we now use the weak-$*$ convergence on Morrey spaces from Proposition 7.3 (bearing in mind the continuity and skew-symmetry of the Hilbert transform on Morrey and block spaces on the real line). For higher dimensions, see Proposition 3.13. Also, the second claim in item *(d)*, regarding the potential failure of uniqueness for the Dirichlet Problem (7.241), is a consequence of Example 3.5 (keeping in mind (3.258) and (7.4)). Again, for higher dimensions see Proposition 3.13.

Consider next the claim made in item *(e)*. When the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ in the formulation of (7.241), virtually the same proof goes through, given that matters may be arranged (by taking $\delta > 0$ sufficiently small) so that the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and $\big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 7.9). In the scenario in which the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ for some given $q \in (1,\infty)$ in the formulation of (7.241), the same line of reasoning applies, with a few notable changes. First, if p is the Hölder conjugate exponent of q, then taking δ sufficiently small we may ensure that the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, $\big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, and $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 7.9). Second, with $f^{(\beta)}$ as in (7.250), thanks to (7.4) in place of (7.251) we now have

$$f^{(\beta)} \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.261}$$

In place of (7.258), this eventually implies

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in M^{p,\lambda}(\partial\Omega,\sigma), \tag{7.262}$$

so in place of (7.260) we now have (again, thanks to (7.22))

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\,\mathrm{d}\sigma \leq C\big\|\mathcal{N}_\kappa u\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\big\|\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty. \tag{7.263}$$

As before, this allows us to invoke Theorem 6.1 to conclude that the Poisson integral representation formula (6.6) holds. Ultimately, this readily implies the uniqueness result we presently seek. The versions of the claims in item *(d)* for vanishing Morrey spaces and block spaces are dealt with much as before (for the former scale, use (7.8); in the case of block spaces, it is useful to observe that (7.17) and Lebesgue's Dominated Convergence Theorem yield, in place of (3.308), that $\lim_{\varepsilon\to 0^+} h_\varepsilon = f_1 + \mathrm{i} f_2$ in $L^r(\mathbb{R}, \mathcal{L}^1)$ where r is as in (7.17), and this suffices to conclude that (3.309) holds in this case). Once more, for higher dimensions see Proposition 3.13. Finally, one deals with (7.246) and its related versions along the lines of the proof of Theorem 6.3. The proof of Theorem 7.18 is therefore complete. □

It turns out that the solvability results established in Theorem 7.18 may be further enhanced, via perturbation arguments, as described in our next theorem.

Theorem 7.19 *Retain the original background assumptions on the set Ω from Theorem 7.18 and, as before, fix two integrability exponents $p, q \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$. Then the following statements are true.*

(a) [Existence] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ (cf. (3.195)) there exist some small threshold $\delta \in (0,1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, q, λ, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (7.241), along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$, are all solvable.*

(b) [Uniqueness] *For each given system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta \in (0,1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, q, λ, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (7.241) along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$, have at most one solution.*

(c) [Well-Posedness] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta \in (0,1)$ and some open neighborhood $\mathcal{U}$ of L_o in $\mathfrak{L}$, both of which depend only on n, p, q, λ, L_o, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for each system $L \in \mathcal{U}$ the Dirichlet Problem (7.241) along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$, are all well posed.*

Proof This may be justified by reasoning as in the proof of Theorem 6.4, now making use of the invertibility results from Theorem 7.10. □

We continue by discussing the Inhomogeneous Regularity Problem for weakly elliptic systems in δ-*AR* domains with boundary data in Morrey-based Sobolev spaces, vanishing Morrey-based Sobolev spaces, as well as block-based Sobolev spaces.

Theorem 7.20 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa > 0$. Also, pick an exponent $p \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, consider the Inhomogeneous Regularity Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa u, \ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M. \end{cases} \tag{7.264}$$

The following statements are true:

(a) [Existence and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, then there exists $\delta \in (0, 1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then $\frac{1}{2}I + K_A$ is an invertible operator on the Morrey-based Sobolev space $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ and the function*

$$u(x) := \Big(\mathcal{D}_A\Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x), \qquad \forall\, x \in \Omega \tag{7.265}$$

is a solution of the Inhomogeneous Regularity Problem (7.264). In addition,

$$\begin{aligned} &\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \quad and \\ &\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}. \end{aligned} \tag{7.266}$$

(b) [Uniqueness] *Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, there exists $\delta \in (0, 1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then the Inhomogeneous Regularity Problem (7.264) has at most one solution.*

(c) [Well-Posedness] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ then there exists $\delta \in (0, 1)$ which depends only on n, p, λ, L, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then*

the Inhomogeneous Regularity Problem (7.264) *is uniquely solvable and the solution satisfies* (7.266).

(d) [Other Spaces of Boundary Data] *Analogous results to those described in items (a)–(c) above are also valid for the Inhomogeneous Regularity Problem formulated with boundary data in the vanishing Morrey-based Sobolev space* $\big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$*, or the block-based Sobolev space* $\big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ *with* $q\in(1,\infty)$.

(e) [Perturbation Results] *In each of the cases considered in items (a)–(d), there are naturally accompanying perturbation results of the sort described in Theorem 7.19.*

(f) [Sharpness] *If* $\mathfrak{A}_L^{\rm dis}=\varnothing$ *the Regularity Problem* (7.264) (*and its variants involving vanishing Morrey-based Sobolev spaces, or block-based Sobolev spaces*) *may fail to be solvable, and if* $\mathfrak{A}_{L^\top}^{\rm dis}=\varnothing$ *the Inhomogeneous Regularity Problem* (7.264) (*along with its aforementioned variants*) *may possess more than one solution.*

Proof The claims in items *(a)–(d)* are implied by Theorems 7.9 and 7.18, while the claim in item *(e)* may be justified by reasoning as in the proof of Theorem 6.4, now making use of the invertibility results from Theorem 7.10. Finally, the claims in item *(f)* are consequences of the versions of Example 3.5 and Proposition 3.11 formulated for Morrey spaces, as well as vanishing Morrey spaces and block spaces (whose proofs naturally adapt to these spaces; see the discussion in the proof of item *(d)* in Theorem 7.18). For higher dimensions see Proposition 3.13. □

Remark 7.7 Much as indicated in Remark 6.3, similar solvability and well-posedness results as in Theorem 7.20 hold for the versions of the Regularity Problem (7.264) formulated with boundary data belonging to suitably defined off-diagonal Morrey-based Sobolev spaces (as well as off-diagonal vanishing Morrey-based Sobolev spaces, and off-diagonal block-based Sobolev spaces).

The next goal is to formulate and solve the Homogeneous Regularity Problem with boundary data from homogeneous Morrey-based Sobolev spaces. This augments solvability results established earlier in Theorems 7.18 and 7.20.

Theorem 7.21 *Assume* $\Omega\subseteq\mathbb{R}^n$ *is an Ahlfors regular domain. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and set* $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, fix an aperture parameter* $\kappa>0$ *and pick some exponent* $p\in(1,\infty)$ *along with a number* $\lambda\in(0,n-1)$. *For a given homogeneous, second-order, constant complex coefficient, weakly elliptic* $M\times M$ *system* L *in* $\mathbb{R}^n$*, consider the Homogeneous Regularity Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \end{cases} \tag{7.267}$$

where $\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$ *is the homogeneous Morrey-based boundary Sobolev space defined in* (7.69). *In relation to this, the following statements are valid:*

(a) [Existence, Estimates, and Integral Representations] *If* $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ *then there exists* $\delta \in (0, 1)$ *which depends only on* n, p, λ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *then the following properties are true. First, the operator*

$$\big[S_{\mathrm{mod}}\big] : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/ \sim\big]^M \tag{7.268}$$

is surjective and the Homogeneous Regularity Problem (7.267) *is solvable. More specifically, with* $[f] \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big/ \sim\big]^M$ *denoting the equivalence class* (*modulo constants*) *of the boundary datum* f, *and with*

$$g \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \ \ selected\ so\ that \ \ \big[S_{\mathrm{mod}}\big]g = [f], \tag{7.269}$$

there exists a constant $c \in \mathbb{C}^M$ *such that the function*

$$u := \mathscr{S}_{\mathrm{mod}}\, g + c \ \ in \ \ \Omega \tag{7.270}$$

is a solution of the Homogeneous Regularity Problem (7.267). *In addition, this solution satisfies* (*with implicit constants independent of* f)

$$\|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|\nabla_{\tan} f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^{n\cdot M}}\,. \tag{7.271}$$

Second, for each coefficient tensor $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *the operator*

$$\tfrac{1}{2}I + \big[K_{A,\mathrm{mod}}\big] : \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)/ \sim\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)/ \sim\big]^M \tag{7.272}$$

is an isomorphism, and the Homogeneous Regularity Problem (7.267) *may be solved as*

$$u := \mathcal{D}_{A,\mathrm{mod}}\, h + c \ \ in \ \ \Omega, \tag{7.273}$$

for a suitable constant $c \in \mathbb{C}^M$ *and with*

$$h \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ such that } [h] = \Big(\tfrac{1}{2}I + \big[K_{A,\text{mod}}\big]\Big)^{-1}[f]. \tag{7.274}$$

Moreover, this solution continues to satisfy (7.271).

(b) [Uniqueness] *Whenever* $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$, *there exists* $\delta \in (0,1)$ *which depends only on* n, p, λ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Homogeneous Regularity Problem* (7.267) *has at most one solution.*

(c) [Well-Posedness and Additional Integrability/Regularity] *Whenever* $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ *and* $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$ *it follows that there exists* $\delta \in (0,1)$ *which depends only on* n, p, λ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Homogeneous Regularity Problem* (7.267) *is uniquely solvable. Moreover, the unique solution* u *of* (7.267) *satisfies* (*in a quantitative fashion*)

$$\mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega,\sigma) \Longleftrightarrow f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{7.275}$$

In particular, the equivalence in (7.275) *proves that the unique solution of the Homogeneous Regularity Problem* (7.267) *for a boundary datum* f *belonging to* $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (*which is a subspace of* $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$*; cf.* (7.71)) *is actually the unique solution of the Inhomogeneous Regularity Problem* (7.264) *for the boundary datum* f.

(d) [Other Spaces of Boundary Data] *Analogous results to those described in items (a)–(c) above are also valid for the Homogeneous Regularity Problem formulated with boundary data in homogeneous vanishing Morrey-based Sobolev spaces, or homogeneous block-based Sobolev spaces.*

(e) [Perturbation Results] *In each of the scenarios considered in items (a)–(d), there are naturally accompanying perturbation results of the sort described in Theorem 7.19.*

(f) [Sharpness] *If* $\mathfrak{A}_L^{\text{dis}} = \varnothing$ *the Homogeneous Regularity Problem* (7.267) *may fail to be solvable* (*actually for boundary data belonging to an infinite dimensional subspace of the corresponding weighted homogeneous Sobolev space*)*, and if* $\mathfrak{A}_{L^\top}^{\text{dis}} = \varnothing$ *the Homogeneous Regularity Problem* (7.267) *may possess more than one solution* (*in fact, the linear space of null-solutions may actually be infinite dimensional*)*, even in the case when* $\Omega = \mathbb{R}_+^n$. *In particular, if either* $\mathfrak{A}_L^{\text{dis}} = \varnothing$ *or* $\mathfrak{A}_{L^\top}^{\text{dis}} = \varnothing$, *then the Homogeneous Regularity Problem* (7.267) *may fail to be well posed, again, even in the case when* $\Omega = \mathbb{R}_+^n$.

Proof All claims are established by reasoning along the lines of the proof of Theorem 6.8, now making use of Proposition 7.8, Theorems 7.4, 7.5, 7.9, 7.12, 7.13, and 7.3. □

We next treat the Neumann Problem for weakly elliptic systems in δ-AR domains with boundary data in Morrey spaces, vanishing Morrey spaces, and block spaces.

Theorem 7.22 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Next, suppose L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some coefficient tensor $A \in \mathfrak{A}_L$ and consider the Neumann Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M. \end{cases} \tag{7.276}$$

Then the following statements are valid:

(a) [Existence, Estimates, and Integral Representation] *If $A^\top \in \mathfrak{A}_{L^\top}^{\text{dis}}$ then there exists $\delta \in (0, 1)$, depending only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then the operator $-\frac{1}{2}I + K^\#_{A^\top}$ is invertible on the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \Big(\mathscr{S}_{\text{mod}}\Big(-\tfrac{1}{2}I + K^\#_{A^\top}\Big)^{-1} f\Big)(x) \ \ for\ all \ \ x \in \Omega, \tag{7.277}$$

is a solution of the Neumann Problem (7.276) *which satisfies*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\,\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \tag{7.278}$$

for some constant $C \in (0, \infty)$ independent of f. Also, the operator (7.210) *is surjective which implies that, for some constant $C \in (0, \infty)$,*

$$\begin{aligned} &there\ exists\ g \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega, w)\big]^M \ with\ \partial_\nu^A(\mathcal{D}_{A,\text{mod}}\,g) = f \\ &and\ such\ that\ \|g\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,w)]^M} \leq C\,\|f\|_{[M^{p,\lambda}(\partial\Omega,w)]^M}. \end{aligned} \tag{7.279}$$

Consequently, the function

$$u := \mathcal{D}_{A,\text{mod}}\,g \ \ in \ \ \Omega \tag{7.280}$$

is a solution of the Neumann Problem (7.276) *which continues to satisfy* (7.278).

(b) [Uniqueness (modulo constants)] *Whenever $A \in \mathfrak{A}_L^{\text{dis}}$ there exists $\delta \in (0, 1)$ which depends only on n, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-*AR *domain) then any two solutions of the Neumann Problem* (7.276) *differ by a constant from $\mathbb{C}^M$.*

(c) [Well-Posedness] *Whenever* $A \in \mathfrak{A}_L^{\rm dis}$ *and* $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$ *then there exists* $\delta \in (0,1)$ *which depends only on* n, p, λ, A, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*i.e., if* Ω *is a* δ-AR *domain*) *then the Neumann Problem* (7.276) *is solvable, the solution is unique modulo constants from* $\mathbb{C}^M$, *and each solution satisfies* (7.278).

(d) [Other Spaces of Boundary Data and Perturbation Results] *Similar results as in items (a)–(c) are valid with the Morrey space* $M^{p,\lambda}(\partial\Omega,\sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, *or the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$. *In each of these cases there are naturally accompanying perturbation results of the sort described in Theorem 7.19. Finally, given any pair of integrability exponents* $p_0, p_1 \in (1,\infty)$ *along with any pair of parameters* $\lambda_0, \lambda_1 \in (0, n-1)$, *similar results are valid for the Neumann Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p_0,\lambda_0}(\partial\Omega,\sigma) + M^{p_1,\lambda_1}(\partial\Omega,\sigma), \\ \partial_\nu^A u = f \in \big[M^{p_0,\lambda_0}(\partial\Omega,\sigma) + M^{p_1,\lambda_1}(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{7.281}$$

as well as for its versions with the Morrey spaces replaced by vanishing Morrey space or block spaces.

(e) [Sharpness] *If* $A^\top \notin \mathfrak{A}_{L^\top}^{\rm dis}$ *then the Neumann Problem* (7.276) *may not be solvable. In addition, if* $A \notin \mathfrak{A}_L^{\rm dis}$ *then the Neumann Problem* (7.276) *may have more than one solution. In fact, even the two-dimensional Laplacian may be written as* $\Delta = \operatorname{div} A\nabla$ *for some matrix* $A \in \mathbb{C}^{2\times 2}$ (*not belonging to* $\mathfrak{A}_\Delta^{\rm dis} = \{I_{2\times 2}\}$) *such that the Neumann Problem formulated for this as in* (7.276) *for this choice of A and with* $\Omega := \mathbb{R}^2_+$ *fails to have a solution for each non-zero boundary datum belonging to an infinite-dimensional linear subspace of the full space of boundary data, and the linear space of null-solutions for the Neumann Problem formulated as in* (7.276) *for this choice of A and with* $\Omega := \mathbb{R}^2_+$ *is actually infinite dimensional. The aforementioned lack of Fredholm solvability is also present for the Neumann Problem formulated in other function spaces, like those considered in item (d).*

Proof Theorem 7.9 guarantees the existence of some threshold $\delta \in (0,1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $-\frac{1}{2}I + K^\#_{A^\top}$ is invertible on $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, and $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ (assuming $q \in (1,\infty)$ has been fixed to begin with). Granted this, all conclusions, save for the very last claim in item *(d)*, follow from Theorems 7.4, 7.9, 7.10, and 7.14 by reasoning as in the proof of Theorem 6.11. The claims pertaining to the Neumann Problem (7.281) are dealt with much as in the proof of Theorem 6.14. Finally, the sharpness aspect highlighted in item *(e)* may be justified by reasoning much as in the proof of Theorem 6.11. □

In relation to Theorem 7.22, we wish to note that in the formulation of the Neumann Problem (7.276) for the two-dimensional Lamé system we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (4.401) for any ζ as in (6.155) (see Remark 7.4 and Remark 6.10 in this regard).

Finally, we formulate and solve the Transmission Problem for weakly elliptic systems in δ-AR domains with boundary data in Morrey spaces, vanishing Morrey spaces, and block spaces. In the formulation on this problem, the clarifications made right after the statement of Theorem 6.15 continue to remain relevant.

Theorem 7.23 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and set*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{7.282}$$

Also, pick an exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$, an aperture parameter $\kappa > 0$, and a transmission (or coupling) parameter $\eta \in \mathbb{C}$. Next, assume L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some $A \in \mathfrak{A}_L$ and consider the Transmission Problem

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^{\pm} = 0 \ \ in \ \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ u^+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = g \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega, w)\big]^M, \\ \partial_\nu^A u^+ - \eta \cdot \partial_\nu^A u^- = f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M. \end{cases} \tag{7.283}$$

In relation to this, the following statements are valid:

(a) [Uniqueness (modulo constants)] *Suppose either*

$$A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}} \ and \ \eta \in \mathbb{C} \setminus \{-1\}, \tag{7.284}$$

or

$$A \in \mathfrak{A}_L^{\mathrm{dis}} \ and \ \eta \in \mathbb{C} \setminus \{0, -1\}. \tag{7.285}$$

*Then there exists $\delta \in (0, 1)$ which depends only on n, η, p, λ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that whenever $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a scenario which renders Ω a δ-*AR *domain; cf. Definition 2.15) it follows any two solutions of the Transmission Problem (7.283) differ by a constant (from $\mathbb{C}^M$).*

(b) [Well-Posedness, Integral Representations, and Additional Regularity] *Assume*

$$A \in \mathfrak{A}_L^{\mathrm{dis}}, \ A^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}, \ and \ \eta \in \mathbb{C} \setminus \{-1\}. \tag{7.286}$$

Then there exists some small $\delta \in (0,1)$ *which depends only on* n, p, λ, A, η, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (*a scenario which ensures that* Ω *is a* δ-AR *domain; cf. Definition 2.15*) *it follows that the Transmission Problem* (7.283) *is solvable. Specifically, in the scenario described in* (7.286), *the operator* $-\frac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^\top}$ *is invertible on the Morrey space* $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, *the operator* $\big[S_{\mathrm{mod}}\big]$ *is invertible from* $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *onto the space* $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big/\sim\big]^M$, *and the functions* $u^\pm : \Omega_\pm \to \mathbb{C}^M$ *defined as*

$$
\begin{aligned}
u^+ &:= \mathscr{S}^+_{\mathrm{mod}} h_0 + \mathscr{S}^+_{\mathrm{mod}} h_1 - c \ \ \textit{in} \ \ \Omega_+,\\
u^- &:= \mathscr{S}^-_{\mathrm{mod}} h_0 \ \ \textit{in} \ \ \Omega_-,
\end{aligned}
\tag{7.287}
$$

where the superscripts $\pm$ *indicate that the modified single layer potentials are associated with the sets* $\Omega_\pm$ *and*

$$
\begin{aligned}
& h_1 := \big[S_{\mathrm{mod}}\big]^{-1}[g] \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \quad c := S_{\mathrm{mod}} h_1 - g \in \mathbb{C}^M,\\
& h_0 := \Big(-\tfrac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^\top}\Big)^{-1}\Big(f - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)h_1\Big),
\end{aligned}
\tag{7.288}
$$

solve the Transmission Problem (7.283) *and satisfy, for a finite constant* $C > 0$ *independent of* f *and* g,

$$
\big\|\mathcal{N}_\kappa(\nabla u^\pm)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} + \|g\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}\Big).
\tag{7.289}
$$

Moreover, any two solutions of the Transmission Problem (7.283) *differ by a constant (from* $\mathbb{C}^M$*). In particular, any solution of the Transmission Problem* (7.283) *satisfies* (7.289).

Alternatively, under the conditions imposed in (7.286) *and, again, assuming* Ω *is a* δ-AR *domain with* $\delta \in (0,1)$ *sufficiently small, a solution of the Transmission Problem* (7.283) *may also be found in the form*

$$
\begin{aligned}
u^+ &:= \mathcal{D}^+_{A,\mathrm{mod}} \psi_0 + c \ \ \textit{in} \ \ \Omega_+,\\
u^- &:= \mathcal{D}^-_{A,\mathrm{mod}} \psi_1 \ \ \textit{in} \ \ \Omega_-,
\end{aligned}
\tag{7.290}
$$

where the superscripts $\pm$ *indicate that the modified double layer potentials are associated with the sets* $\Omega_\pm$, *where* $c \in \mathbb{C}^M$ *is a suitable constant, and where* $\psi_0, \psi_1 \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ *are two suitable functions satisfying*

$$
\begin{aligned}
& \|\psi_0\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M} + \|\psi_1\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}\\
& \qquad \leq C\Big(\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} + \|g\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}\Big),
\end{aligned}
\tag{7.291}
$$

for some constant $C \in (0, \infty)$ *independent of* f *and* g*. In particular,* $u^{\pm}$ *in* (7.290) *also satisfy* (7.289).

(c) [Well-Posedness for $\eta = 1$] *In the case when*

$$\eta = 1 \text{ and } \Omega \text{ is a two-sided NTA domain with an unbounded Ahlfors regular boundary} \tag{7.292}$$

the Transmission Problem (7.283) *is solvable, and any two solutions of the Transmission Problem* (7.283) *differ by a constant. Any solution is given by*

$$\begin{aligned} u^+ &:= \mathcal{D}^+_{A,\mathrm{mod}}\, g - \mathscr{S}^+_{\mathrm{mod}} f + c \ \ in \ \ \Omega_+, \\ u^- &:= -\mathcal{D}^-_{A,\mathrm{mod}}\, g - \mathscr{S}^-_{\mathrm{mod}} f + c \ \ in \ \ \Omega_-, \end{aligned} \tag{7.293}$$

for some $c \in \mathbb{C}^M$*, where the superscripts* $\pm$ *indicate that the modified layer potentials are associated with the sets* $\Omega_{\pm}$ *introduced in* (7.282)*. In addition, any solution satisfies* (7.289).

(d) [Other Spaces of Boundary Data and Perturbation Results] *Analogous results hold with the Morrey space* $M^{p,\lambda}(\partial\Omega, \sigma)$ *replaced by the vanishing Morrey space* $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$*, the block space* $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ *with* $q \in (1, \infty)$*, or by sums of such spaces. In addition, in each of these cases there are naturally accompanying perturbation results of the sort described in Theorem 7.19.*

Proof For each fixed $\eta \in \mathbb{C} \setminus \{-1\}$, $p, q \in (1, \infty)$, and $\lambda \in (0, n-1)$, Theorem 7.9 guarantees that there exists some threshold $\delta \in (0, 1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $-\frac{\eta+1}{2} I + (1-\eta) K^{\#}_{A^{\top}}$ is invertible on $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, $\big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, and $\big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$. With this in hand, the same type of argument as in the proof of Theorem 6.15 (which now relies on Theorems 7.2, 7.4, 7.5, 7.9, 7.12, 7.13) and the proof of Theorem 6.4 (which now makes use of Theorem 7.10) yields all desired conclusions. □

We close by noting that, in the formulation of the Transmission Problem (7.283) for the two-dimensional Lamé system, we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (4.401) for any ζ as in (6.262) (see Remarks 7.4 and 6.16 in this regard).

Chapter 8
Singular Integrals and Boundary Problems in Weighted Banach Function Spaces

In this chapter we shall show that singular integral operators remain effective tools in proving well-posedness results for boundary problems for second-order systems formulated in sufficiently flat Ahlfors regular domains and with boundary data in weighted Banach function spaces (aka, Köthe function spaces). In the first part we develop the theory of boundary layer potentials and boundary value problems in such a general functional analytic setting then, in the last part of this chapter, we shall specialize this discussion to the case of rearrangement invariant Banach function spaces (RIBFS for short), including Orlicz spaces, Zygmund space, Lorentz spaces, and their weighted versions.

8.1 Basic Properties and Extrapolation in Banach Function Spaces

In this section we consider abstract Banach function spaces on Ahlfors regular sets. To get started, we shall assume that[1]

$$\begin{array}{l}(\Sigma, \mathfrak{M}) \text{ is a measurable space, and } \mu \text{ is a positive, non-atomic,}\\ \text{sigma-finite measure on the sigma-algebra } \mathfrak{M}, \text{ with } \mu(\Sigma) > 0.\end{array} \tag{8.1}$$

For example, if $\Sigma \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a nonempty closed Ahlfors regular set and $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ then we may take $\mu := \sigma$, or $\mu := w\sigma$ with the weight $w \in A_\infty(\Sigma, \sigma)$. More generally,

[1] Recall that a measure μ is said to be *non-atomic* provided for any μ-measurable set A with $\mu(A) > 0$ there exists a μ-measurable subset B of A such that $\mu(A) > \mu(B) > 0$.

J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4_8

if $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed Ahlfors regular set and we abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$, for each function $v \in L^1_{\rm loc}(\Sigma, \sigma)$ with $v > 0$ at σ-a.e. point on Σ it follows that $\mu := v\sigma$ is a positive, non-atomic, sigma-finite measure on Σ with $\mu(\Sigma) > 0$. (8.2)

The fact that μ is non-atomic in this scenario follows from Lebesgue's Differentiation Theorem (cf. [7], [111, §7.4]). Specifically, if $A \subseteq \Sigma$ is a σ-measurable set with $\mu(A) > 0$ then $\sigma(A) > 0$ so there exists a point $x_* \in A$ such that $0 < v(x_*) < \infty$ and

$$\lim_{r\to 0^+} \frac{\mu\big(A \cap \Delta(x_*, r)\big)}{\sigma\big(\Delta(x_*, r)\big)} = \lim_{r\to 0^+} ⨍_{\Delta(x_*,r)} \mathbf{1}_A \cdot v \, d\sigma = v(x_*) \in (0, \infty). \tag{8.3}$$

As a consequence, $\mu\big(A \cap \Delta(x_*, r)\big) > 0$ whenever $r > 0$ is sufficiently small, and $\lim_{r\to 0^+} \mu\big(A \cap \Delta(x_*, r)\big) = 0$. This proves that $0 < \mu\big(A \cap \Delta(x_*, r)\big) < \mu(A)$ if $r > 0$ is small enough, so the measure μ is, as claimed, non-atomic. In the scenario described in (8.2) we agree to identify the weight function v with the weighted measure

$$dv := v \, d\sigma \tag{8.4}$$

so (8.2) implies that, as a measure, v is non-atomic.

Definition 8.1 Let $\mathbb{M}_\mu$ be the set of all complex-valued μ-measurable functions on Σ. A mapping $\|\cdot\| : \mathbb{M}_\mu \to [0, \infty]$ is called a `function norm` provided the following properties are satisfied for all $f, g \in \mathbb{M}_\mu$:

1. $\|f\| = \||f|\|$ and $\|f\| = 0$ if and only if $f = 0$ at μ-a.e. point on Σ;
2. $\|f + g\| \leq \|f\| + \|g\|$;
3. $\|\lambda f\| = |\lambda|\|f\|$ for all $\lambda \in \mathbb{C}$;
4. if $|f| \leq |g|$ at μ-a.e. point on Σ, then $\|f\| \leq \|g\|$;
5. if $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathbb{M}_\mu$ is a sequence such that $|f_i|$ increases to $|f|$ as $i \to \infty$ at μ-a.e. point on Σ, then $\|f_i\|$ increases to $\|f\|$ as $i \to \infty$;
6. if $E \subseteq \Sigma$ is a μ-measurable set such that $\mu(E) < \infty$, then $\|\mathbf{1}_E\| < \infty$;
7. for each μ-measurable set $E \subseteq \Sigma$ with $\mu(E) < \infty$ there exists some constant $C_E \in (0, \infty)$ (independent of f) such that $\int_E |f| \, d\mu \leq C_E \|f\|$.

The space associated with a function norm $\|\cdot\|$, i.e.,

$$\mathbb{X} := \{f \in \mathbb{M}_\mu : \|f\| < +\infty\}, \tag{8.5}$$

is called a `Banach function space` over (Σ, μ) (referred to as a `Köthe function space`). In such a scenario, it is agreed to write $\|\cdot\|_{\mathbb{X}}$ in place of $\|\cdot\|$ in order to emphasize the connection between the function norm $\|\cdot\|$ and its associated Köthe function space $\mathbb{X}$.

The latter piece of terminology is justified since, as is well known, $\big(\mathbb{X}, \|\cdot\|_{\mathbb{X}}\big)$ is a complete normed vector subspace of $\mathbb{M}_\mu$, hence a Banach space.

Starting with a Banach function space $\mathbb{X}$ over (Σ, μ), we can define its Köthe dual (also known as its `associated space` in the terminology of [15]) according to

$$\begin{aligned} &\mathbb{X}' := \big\{f \in \mathbb{M}_\mu : \|f\|_{\mathbb{X}'} < \infty\big\} \text{ where, for each } f \in \mathbb{M}_\mu, \\ &\|f\|_{\mathbb{X}'} := \sup\left\{\int_\Sigma |f(x)\,g(x)|\,\mathrm{d}\mu(x) : g \in \mathbb{X},\ \|g\|_{\mathbb{X}} \le 1\right\}. \end{aligned} \tag{8.6}$$

One can check that $\|\cdot\|_{\mathbb{X}'}$ is indeed a function norm, hence $\mathbb{X}'$ is itself a Banach function space over (Σ, μ). In addition, $\|\cdot\|_{\mathbb{X}'}$ satisfies the generalized Hölder inequality

$$\int_\Sigma |f(x)g(x)|\,\mathrm{d}\mu(x) \le \|f\|_{\mathbb{X}}\|g\|_{\mathbb{X}'}, \tag{8.7}$$

for every $f, g \in \mathbb{M}_\mu$. Every Banach function space $\mathbb{X}$ over (Σ, μ) coincides with the associated space of $\mathbb{X}'$, that is (cf. [15, Theorem 2.7, p. 10]),

$$\mathbb{X} = \mathbb{X}''. \tag{8.8}$$

Hence, for every $f \in \mathbb{M}_\mu$ we have

$$\|f\|_{\mathbb{X}} = \sup\left\{\int_\Sigma |f(x)g(x)|\,\mathrm{d}\mu(x) : g \in \mathbb{X}',\ \|g\|_{\mathbb{X}'} \le 1\right\}. \tag{8.9}$$

For future reference, let us also observe here that property (g) in Definition 8.1 implies

$$\text{if } f \in \mathbb{X} \text{ then } |f| < \infty \text{ at } \mu\text{-a.e. point in } \Sigma. \tag{8.10}$$

The following theorem does not appear explicitly in [34, Section 4.2] but its proof follows the lines of [34, Theorem 4.10, pp. 75–76] without the assumption that the Banach function space is rearrangement invariant (see Definition 8.2 below). The statement is quite general so it can accommodate a multitude of relevant particular cases. We shall work in the context of weighted Banach function spaces. To be more specific, assume $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Having fixed some $v \in L^1_{\rm loc}(\Sigma, \sigma)$ with $v > 0$ at σ-a.e. point on Σ, we shall let $\mathbb{X}_v$ be a Banach function space over $(\Sigma, v\sigma)$. The first main example pertains to the case when $v \equiv 1$ and $\mathbb{X}$ is a generic Banach function space, a scenario in which condition (8.11) below reduces to having the Hardy–Littlewood maximal operator $\mathcal{M}$ on Σ bounded both on $\mathbb{X}$ and on $\mathbb{X}'$. In the second main example, we demand that $v = w \in A_\infty(\Sigma, \sigma)$ and further assume that $\mathbb{X}$ is a rearrangement invariant Banach function space (see Definition 8.2 below). Thus, $\mathbb{X}_v = \mathbb{X}(w)$ (see (8.392)), and Corollary 8.2 shows that (8.11) holds provided $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$ and $w \in A_{p_{\mathbb{X}}}(\Sigma, \sigma)$, where $p_{\mathbb{X}}, q_{\mathbb{X}}$ are, respectively, the lower and upper Boyd indices (see Definition 8.3). For the moment we would like to keep the discussion general, and later on we will specialize matters to the aforementioned examples (see Sect. 8.7).

Theorem 8.1 *Assume $\Sigma \subseteq \mathbb{R}^n$ is an arbitrary closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Suppose $\mathcal{F}$ is a family of pairs of σ-measurable functions on Σ. Pick an exponent $p_0 \in [1, \infty)$, denote by p_0' its Hölder conjugate exponent, and consider a non-decreasing function $\Phi : (0, \infty) \to (0, \infty)$. Also, fix a function $v \in L^1_{\rm loc}(\Sigma, \sigma)$ with $v > 0$ at σ-a.e. point on Σ. Let $\mathbb{X}_v$ be a Banach function space over $(\Sigma, v\sigma)$ and let $\mathbb{X}_v'$ be its Köthe dual. Finally, with $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on (Σ, σ) and with $\mathcal{M}'f := \mathcal{M}(fv)/v$ for each σ-measurable function f on Σ, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}_v'. \tag{8.11}$$

If for each $w \in A_{p_0}(\Sigma, \sigma)$ with $[w]_{A_{p_0}} \le 2^{p_0}\|\mathcal{M}\|^{p_0-1}_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}_v'\to\mathbb{X}_v'}$ one has

$$\|f\|_{L^{p_0}(\Sigma,w)} \le \Phi([w]_{A_{p_0}})\|g\|_{L^{p_0}(\Sigma,w)} \text{ for every } (f,g) \in \mathcal{F}, \tag{8.12}$$

then one may conclude that

$$\|f\|_{\mathbb{X}_v} \le 2^{2+1/p_0'}\,\Phi\big(2^{p_0}\|\mathcal{M}\|^{p_0-1}_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}_v'\to\mathbb{X}_v'}\big)\|g\|_{\mathbb{X}_v} \quad \text{for every } (f,g) \in \mathcal{F}. \tag{8.13}$$

The main ingredient in the proof of this result is contained in the following proposition.

Proposition 8.1 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix a function $v \in L^1_{\rm loc}(\Sigma, \sigma)$ with $v > 0$ at σ-a.e. point on Σ. Let $\mathbb{X}_v$ be a Banach function space over $(\Sigma, v\sigma)$ and let $\mathbb{X}_v'$ be its Köthe dual. With $\mathcal{M}$ denoting the standard Hardy–Littlewood maximal operator on (Σ, σ) and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for each σ-measurable function f on Σ, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}_v'. \tag{8.14}$$

Then for every $f, g \in \mathbb{X}_v$ and every $p_0 \in [1, \infty)$ there exists a Muckenhoupt weight $w = w(f, g) \in A_{p_0}(\Sigma, \sigma)$ satisfying $[w]_{A_{p_0}} \le 2^{p_0}\|\mathcal{M}\|^{p_0-1}_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}_v'\to\mathbb{X}_v'}$ and such that

$$\|f\|_{\mathbb{X}_v} \le 2^{1+2/p_0'}\|f\|_{L^{p_0}(\Sigma,w)} \text{ and } \|g\|_{L^{p_0}(\Sigma,w)} \le 2^{1/p_0}\|g\|_{\mathbb{X}_v}, \tag{8.15}$$

where p_0' is the Hölder conjugate exponent of p_0.

In particular, for every $f \in \mathbb{X}_v$ *and every* $p_0 \in [1,\infty)$ *there exists a Muckenhoupt weight*

$$w = w(f) \in A_{p_0}(\Sigma,\sigma) \text{ with } \\ [w]_{A_{p_0}} \le W_{p_0,\mathbb{X}_v} := 2^{p_0}\|\mathcal{M}\|^{p_0-1}_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v} \tag{8.16}$$

such that the function f *belongs to* $L^{p_0}(\Sigma,w)$ *and*

$$2^{-1-2/p_0'}\|f\|_{\mathbb{X}_v} \le \|f\|_{L^{p_0}(\Sigma,w)} \le 2^{1/p_0}\|f\|_{\mathbb{X}_v}. \tag{8.17}$$

As a corollary of these considerations, for every $p_0 \in [1,\infty)$ *one has*

$$\mathbb{X}_v \subseteq \bigcup_{w\in A_{p_0}(\Sigma,\sigma)[w]_{A_{p_0}}\le W_{p_0,\mathbb{X}_v}} L^{p_0}(\Sigma,w). \tag{8.18}$$

Assuming this result for the time being, we can painlessly deal with Theorem 8.1.

Proof of Theorem 8.1 Given $(f,g)\in\mathcal{F}$, we may assume without loss of generality that $\|g\|_{\mathbb{X}_v} < \infty$, otherwise there is nothing to prove. Note that this entails $|f| < \infty$ at σ-a.e. point on Σ, since otherwise (8.12) and the fact that Muckenhoupt weights are strictly positive σ-a.e. on Σ would force $\|g\|_{L^{p_0}(\Sigma,w)} = \infty$ for every weight $w \in A_{p_0}(\Sigma,\sigma)$. This, however, contradicts (8.17) in Proposition 8.1 applied to $g \in \mathbb{X}_v$.

For every $N \ge 1$, let $E_N := \{x \in \Sigma \cap B(0,N) : |f(x)| \le N\}$ and define $f_N := f\mathbf{1}_{E_N}$. Properties (d) and (f) in Definition 8.1, the notation introduced in (8.4) and the fact that $v \in L^1_{\rm loc}(\Sigma,\sigma)$ imply $\|f_N\|_{\mathbb{X}_v} \le N\|\mathbf{1}_{\Sigma\cap B(0,N)}\|_{\mathbb{X}_v} < \infty$. We therefore have $f_N \in \mathbb{X}_v$. Apply now Proposition 8.1 to find a Muckenhoupt weight $w_N := w(f_N,g)$ in $A_{p_0}(\Sigma,\sigma)$ such that $[w_N]_{A_{p_0}} \le 2^{p_0}\|\mathcal{M}\|^{p_0-1}_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$ and so that (8.15) holds for this weight and the functions f_N, g. We may then invoke (8.12) to write

$$\begin{aligned}\|f_N\|_{\mathbb{X}_v} &\le 2^{1+2/p_0'}\|f_N\|_{L^{p_0}(\Sigma,w)} \le 2^{1+2/p_0'}\|f\|_{L^{p_0}(\Sigma,w)}\\ &\le 2^{1+2/p_0'}\Phi([w_N]_{A_{p_0}})\|g\|_{L^{p_0}(\Sigma,w)}\\ &\le 2^{2+1/p_0'}\Phi\big(2^{p_0}\|\mathcal{M}\|^{p_0-1}_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}\big)\|g\|_{\mathbb{X}_v}.\end{aligned} \tag{8.19}$$

To conclude, observe that $|f_N|$ increases to $|f|$ as $N\to\infty$ at σ-a.e. point on Σ, since $|f| < \infty$ at σ-a.e. point on Σ. As such, property (e) in Definition 8.1 implies that $\|f_N\|_{\mathbb{X}_v}$ increases to $\|f\|_{\mathbb{X}_v}$, and we readily obtain (8.13) from (8.19). □

We now turn to the proof of Proposition 8.1.

Proof of Proposition 8.1 We use some ideas from [34, Theorem 4.10, pp. 75–76] but keeping in mind that $\mathbb{X}$ is not necessarily rearrangement invariant. For any two non-negative, non-trivial, functions $h_1 \in \mathbb{X}_v$ and $h_2 \in \mathbb{X}'_v$ bring in the Rubio de Francia iteration algorithms:

$$\mathcal{R}h_1 := \sum_{k=0}^{\infty} \frac{\mathcal{M}^k h_1}{2^k \|\mathcal{M}\|^k_{\mathbb{X}_v \to \mathbb{X}_v}}, \qquad \mathcal{R}'h_2 := \sum_{k=0}^{\infty} \frac{(\mathcal{M}')^k h_2}{2^k \|\mathcal{M}'\|^k_{\mathbb{X}'_v \to \mathbb{X}'_v}}, \tag{8.20}$$

where $\mathcal{M}^0$ and $(\mathcal{M}')^0$ stand for the identity operator, while $\mathcal{M}^k := \mathcal{M} \circ \cdots \circ \mathcal{M}$ (respectively, $(\mathcal{M}')^k := \mathcal{M}' \circ \cdots \circ \mathcal{M}'$) is the k-th iteration of $\mathcal{M}$ (respectively, $\mathcal{M}'$) for each pair of integer $k \geq 1$. Based on (8.20), (8.14), and (2.523) we see that

$$h_1 \leq \mathcal{R}h_1 \quad \text{on} \quad \Sigma, \qquad h_2 \leq \mathcal{R}'h_2 \quad \text{on} \quad \Sigma, \tag{8.21}$$

$$\|\mathcal{R}h_1\|_{\mathbb{X}_v} \leq 2\|h_1\|_{\mathbb{X}_v}, \qquad \|\mathcal{R}'h_2\|_{\mathbb{X}'_v} \leq 2\|h_2\|_{\mathbb{X}'_v}, \tag{8.22}$$

$$[\mathcal{R}h_1]_{A_1} \leq 2\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v}, \qquad [(\mathcal{R}'h_2)\, v]_{A_1} \leq 2\|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}. \tag{8.23}$$

Fix next $f, g \in \mathbb{X}_v$. We make the claim that it suffices to consider the case when both $\|f\|_{\mathbb{X}_v} \neq 0$ and $\|g\|_{\mathbb{X}_v} \neq 0$. Indeed, when either $\|f\|_{\mathbb{X}_v} = 0$, or $\|g\|_{\mathbb{X}_v} = 0$, we just use the claim with f, or g, replaced by $\mathbf{1}_{B(x_0,r_0)\cap\Sigma}$, for some fixed $x_0 \in \Sigma$ and $0 < r_0 < \operatorname{diam}(\Sigma)$. Note that by property property (g) in Definition 8.1 with $d\mu = dv = v\, d\sigma$ and the fact that $v > 0$ at σ-a.e. point on Σ, one obtains that $\|\mathbf{1}_{B(x_0,r_0)\cap\Sigma}\|_{\mathbb{X}_v} \neq 0$.

To proceed, assume that $\|f\|_{\mathbb{X}_v} \neq 0$ and $\|g\|_{\mathbb{X}_v} \neq 0$. By (8.9) there exists some non-negative function $h \in \mathbb{X}'_v$ with $\|h\|_{\mathbb{X}'_v} \leq 1$ such that

$$2^{-1}\|f\|_{\mathbb{X}_v} \leq \int_\Sigma |f|\, h\, v\, d\sigma. \tag{8.24}$$

Note that since $\|f\|_{\mathbb{X}_v} \neq 0$ then the above estimate implies that $\sigma\big(\{h > 0\}\big) > 0$, hence h is not zero σ-a.e. on Σ.

Consider first the case when $p_0 > 1$. Let $\mathcal{R}$ and $\mathcal{R}'$ be as in (8.20) so that (8.21)–(8.23) hold. Define

$$\widetilde{g}(x) := \frac{|g(x)|}{\|g\|_{\mathbb{X}_v}} \quad \text{for each} \quad x \in \Sigma, \tag{8.25}$$

and set

$$w := (\mathcal{R}\widetilde{g})^{1-p_0}(\mathcal{R}'h)\, v. \tag{8.26}$$

From (8.23) and the factorization of weights (described in item *(4)* of Proposition 2.20) we have $w \in A_{p_0}(\Sigma, \sigma)$ and

$$[w]_{A_{p_0}} \le [\mathcal{R}\widetilde{g}]_{A_1}^{p_0-1}[(\mathcal{R}'h)v]_{A_1} \le 2^{p_0}\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}^{p_0-1}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}. \tag{8.27}$$

Also, (8.7) (with $d\mu = dv = v d\sigma$) and (8.22) yield

$$\int_\Sigma \mathcal{R}\widetilde{g}\,\mathcal{R}'h\,dv \le \|\mathcal{R}\widetilde{g}\|_{\mathbb{X}_v}\big\|\mathcal{R}'h\big\|_{\mathbb{X}'_v} \le 4\,\|\widetilde{g}\|_{\mathbb{X}_v}\,\|h\|_{\mathbb{X}'_v} \le 4. \tag{8.28}$$

All these, the notation introduced in (8.4), (8.24), and (8.21) allow us to estimate

$$\begin{aligned}
2^{-1}\|f\|_{\mathbb{X}_v} &\le \int_\Sigma |f|\,h\,v\,d\sigma \le \int_\Sigma |f|(\mathcal{R}'h)\,v\,d\sigma \\
&= \int_\Sigma |f|\,(\mathcal{R}\widetilde{g})^{-1/p_0'}\,(\mathcal{R}\widetilde{h})^{1/p_0'}\,(\mathcal{R}'h)\,v\,d\sigma \\
&\le \left(\int_\Sigma |f|^{p_0}\,(\mathcal{R}\widetilde{g})^{1-p_0}\,\mathcal{R}'h\,dv\right)^{1/p_0}\left(\int_\Sigma \mathcal{R}\widetilde{g}\,\mathcal{R}'h\,dv\right)^{1/p_0'} \\
&\le 4^{1/p_0'}\|f\|_{L^{p_0}(\Sigma,w)}.
\end{aligned} \tag{8.29}$$

This argument gives the first estimate claimed in (8.15). On the other hand, since (8.21) implies $|g|/\|g\|_{\mathbb{X}_v} = \widetilde{g} \le \mathcal{R}\widetilde{g}$, and since $1 - p_0 < 0$, we may estimate $\big(\mathcal{R}\widetilde{g}\big)^{1-p_0} \le \|g\|_{\mathbb{X}_v}^{p_0-1}|g|^{1-p_0}$. From this, the definition of the weight w, the generalized Hölder inequality recorded from (8.7) (presently used with $d\mu = dv = v\,d\sigma$), and (8.22) we therefore obtain

$$\begin{aligned}
\|g\|_{L^{p_0}(\Sigma,w)} &= \left(\int_\Sigma |g|^{p_0}\,(\mathcal{R}\widetilde{g})^{1-p_0}\,(\mathcal{R}'h)\,v\,d\sigma\right)^{1/p_0} \\
&\le \|g\|_{\mathbb{X}_v}^{\frac{p_0-1}{p_0}}\left(\int_\Sigma |g|\,(\mathcal{R}'h)\,v\,d\sigma\right)^{1/p_0} \\
&\le \|g\|_{\mathbb{X}_v}\big\|\mathcal{R}'h\big\|_{\mathbb{X}'_v}^{1/p_0} \le 2^{1/p_0}\|g\|_{\mathbb{X}_v}.
\end{aligned} \tag{8.30}$$

This gives the second estimate claimed in (8.15).

Consider next the case when $p_0 = 1$. In this scenario, define $w := (\mathcal{R}'h)\,v$. From (8.23) we have $w \in A_1(\Sigma,\sigma)$ with $[w]_{A_1} \le 2\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. This, (8.24), and (8.21) give

$$2^{-1}\|f\|_{\mathbb{X}_v} \le \int_\Sigma |f|\,h\,v\,d\sigma \le \int_\Sigma |f|(\mathcal{R}'h)\,v\,d\sigma = \|f\|_{L^1(\Sigma,w)}. \tag{8.31}$$

On the other hand, (8.7) (with $d\mu = dv = v\,d\sigma$) and (8.22) yield

$$\|g\|_{L^1(\Sigma,w)} = \int_\Sigma |g|\,(\mathcal{R}'h)\,v\,\mathrm{d}\sigma \leq \|g\|_{\mathbb{X}_v}\|\mathcal{R}'h\|_{\mathbb{X}'_v} \leq 2\|g\|_{\mathbb{X}_v}. \tag{8.32}$$

This finishes the justification of (8.15).

To complete the proof of Proposition 8.1 we just need to observe that (8.17) follows at once from (8.15) by simply taking $g := f$. □

We next discuss the following version of the commutator theorem from [31] in the setting of Banach function spaces.

Theorem 8.2 *Make the assumption that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0 \in (1,\infty)$ along with some non-decreasing function $\Phi : (0,\infty) \to (0,\infty)$, and let T be a linear operator which is bounded on $L^{p_0}(\Sigma, w)$ for every $w \in A_{p_0}(\Sigma, w)$, with operator norm $\leq \Phi\big([w]_{A_{p_0}}\big)$. Fix a function $v \in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ with $v > 0$ at σ-a.e. point on Σ. Let $\mathbb{X}_v$ be a Banach function space over $(\Sigma, v\sigma)$ and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the standard Hardy–Littlewood maximal operator on (Σ,σ) and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on Σ, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v \tag{8.33}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$.

Then T maps the space $\mathbb{X}_v$ boundedly into itself and one may find two constants $C, C_1 \in (0,\infty)$ depending only on Σ, n, p_0, and Ξ, with the property that for every complex-valued function $b \in L^\infty(\Sigma,\sigma)$ one has

$$\|[M_b, T]\|_{\mathbb{X}_v\to\mathbb{X}_v} \leq C\cdot\Phi(C_1)\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}, \tag{8.34}$$

where $[M_b, T]$ is the commutator of $T : \mathbb{X}_v \to \mathbb{X}_v$ and the operator M_b of pointwise multiplication on $\mathbb{X}$ by the function b, i.e.,

$$[M_b, T]f := bT(f) - T(bf) \ \text{ for each } \ f \in \mathbb{X}. \tag{8.35}$$

Proof All claims are direct consequences of Theorem 4.3 and Theorem 8.1, presently used with $\mathcal{F} := \big\{(Tf, f) : f \in \bigcup_{w\in A_{p_0}(\Sigma,w)} L^{p_0}(\Sigma, w)\big\}$, bearing in mind (8.18). □

Remark 8.1 The reader may well wonder why in the previous result we have not simply taken $\Xi = 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. The reason for this will become clear later but for the moment recall that one of our goals is to consider $\mathbb{X}_v = \mathbb{X}(w)$ with $\mathbb{X}$ being a RIBFS, and $v = w \in A_{p_{\mathbb{X}}}(\Sigma,\sigma)$. In such a scenario, it is desirable to obtain estimates which are uniform with respect to $[w]_{A_{p_{\mathbb{X}}}}$. That is, given any threshold $\lambda \in [1,\infty)$, we wish to show that the previous estimates hold for all Muckenhoupt weights w with $[w]_{A_{p_{\mathbb{X}}}} \leq \lambda$, and with all constants involved controlled in terms of λ. Working with the choice $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$ will enable us to pick

$\Xi = \Xi(\lambda, \mathbb{X})$ which ultimately will allow us to obtain the aforementioned uniform bounds.

The following result establishes a basic inclusion of abstract Banach function spaces on a closed Ahlfors regular set into weighted Lebesgue spaces on said set. This will then permit us to employ results from previous chapters in the current analysis.

Proposition 8.2 *Work under the assumption that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Having fixed a function $v \in L^1_{\text{loc}}(\Sigma, \sigma)$ with $v > 0$ at σ-a.e. point on Σ, let $\mathbb{X}_v$ be a Banach function space over $(\Sigma, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on (Σ, σ) and with $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on Σ, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v. \tag{8.36}$$

Then one has the continuous inclusion

$$\mathbb{X}_v \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{8.37}$$

In addition, given any $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$ there exists some number $r = r(\Sigma, \Xi) \in (1,2)$ such that one has a continuous inclusion

$$\mathbb{X}_v \hookrightarrow L^q\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{(n-1)\theta}}\Big) \text{ whenever } 1 \leq q \leq r \text{ and } \theta > r^{-1}. \tag{8.38}$$

Remark 8.2 We wish to note that (8.42), (8.44), (8.45) below, and item *(7)* in Proposition 2.20 imply that if $0 < \theta < 1$ then there exists $C_\theta \in (0,\infty)$ such that

$$\big(1+|x|^{(n-1)\theta}\big)^{-1} \in A_1(\Sigma,\sigma) \text{ with } \Big[\big(1+|x|^{(n-1)\theta}\big)^{-1}\Big]_{A_1} \leq C_\theta. \tag{8.39}$$

Hence, (8.38) implies

$$\mathbb{X}_v \hookrightarrow L^q(\Sigma, w) \text{ for some } w \in A_1(\Sigma,\sigma) \text{ and any } q \in [1,r]. \tag{8.40}$$

In particular, from (8.40) and item *(2)* in Proposition 2.20 we see that for any integrability exponent $q \in [1,r]$ there exists a weight $w \in A_q(\Sigma,\sigma)$ such that $\mathbb{X}_v \hookrightarrow L^q(\Sigma, w)$. Compared to the result in Proposition 8.1, here we have been able to contain the entire space $\mathbb{X}_v$ into a unique Muckenhoupt weighted Lebesgue space (albeit with a small integrability exponent).

Here is the proof of Proposition 8.2.

Proof of Proposition 8.2 Although (8.38) contains (8.37), it is worth presenting a direct, simple proof of the latter. Specifically, having fixed a reference point $z_0 \in \Sigma$, for every location $x \in \Sigma$ we may estimate

$$1 + |x - z_0| \leq 1 + |x| + |z_0| \leq (1 + |z_0|)(1 + |x|). \tag{8.41}$$

Hence, on the one hand,

$$\begin{aligned} 1 + |x - z_0|^{n-1} &\leq (1 + |x - z_0|)^{n-1} \leq (1 + |z_0|)^{n-1}(1 + |x|)^{n-1} \\ &\leq 2^{n-2}(1 + |z_0|)^{n-1}(1 + |x|^{n-1}). \end{aligned} \tag{8.42}$$

On the other hand,

$$1 + |x| \leq 1 + |x - z_0| + |z_0| \leq (1 + |z_0|)(1 + |x - z_0|), \tag{8.43}$$

therefore

$$\begin{aligned} 1 + |x|^{n-1} &\leq (1 + |x|)^{n-1} \leq (1 + |z_0|)^{n-1}(1 + |x - z_0|)^{n-1} \\ &\leq 2^{n-2}(1 + |z_0|)^{n-1}(1 + |x - z_0|^{n-1}). \end{aligned} \tag{8.44}$$

We next claim that

$$\mathcal{M}\big(\mathbf{1}_{\Delta(z_0,1)}\big)(x) \approx \frac{1}{1 + |x - z_0|^{n-1}}, \quad \text{uniformly in } \ x \in \Sigma, \tag{8.45}$$

with implicit constants that depend only on n and the Ahlfors regularity constant of Σ. To see this, fix an arbitrary point $x \in \Sigma$. Note that $x \in \Delta\big(z_0, 1 + |x - z_0|\big)$ and $\Delta(z_0, 1) \subset \Delta\big(z_0, 1 + |x - z_0|\big)$, hence

$$\begin{aligned} \mathcal{M}\big(\mathbf{1}_{\Delta(z_0,1)}\big)(x) &\geq \fint_{\Delta(z_0,1+|x-z_0|)} \mathbf{1}_{\Delta(z_0,1)}\, d\sigma = \frac{\sigma\big(\Delta(z_0, 1)\big)}{\sigma\big(\Delta(z_0, 1 + |x - z_0|)\big)} \\ &\geq \frac{C}{1 + |x - z_0|^{n-1}}, \end{aligned} \tag{8.46}$$

by (2.522) and the Ahlfors regularity of Σ. Since we also always have

$$\mathcal{M}\big(\mathbf{1}_{\Delta(z_0,1)}\big)(x) \leq \big\|\mathbf{1}_{\Delta(z_0,1)}\big\|_{L^\infty(\Sigma,\sigma)} = 1, \tag{8.47}$$

it follows that $\mathcal{M}\big(\mathbf{1}_{\Delta(z_0,1)}\big)(x) \approx \frac{1}{1+|x-z_0|^{n-1}}$ uniformly in $x \in \Delta(z_0, 2)$. There remains to prove the right-pointing inequality in (8.45) when $x \in \Sigma \setminus \Delta(z_0, 2)$. Assume this is the case and pick an arbitrary location $y \in \Sigma$ along with some scale

$r \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ such that $x \in \Delta(y, r)$ (recall from (2.522) that we are dealing with the non-centered Hardy–Littlewood maximal operator). Note that, on the one hand,

$$\fint_{\Delta(y,r)} \mathbf{1}_{\Delta(z_0,1)}\,\mathrm{d}\sigma = 0 \quad \text{whenever} \quad \Delta(y,r) \cap \Delta(z_0,1) = \varnothing. \tag{8.48}$$

On the other hand, if $\Delta(y,r) \cap \Delta(z_0,1) \neq \varnothing$ then

$$2 \leq |x - z_0| \leq |x - y| + |y - z_0| < r + (r + 1) = 2r + 1, \tag{8.49}$$

hence $r \geq 1/2$. When used back in (8.49), this further implies $|x - z_0| \leq 4r$. Based on this and the Ahlfors regularity of Σ we may then estimate

$$\begin{aligned}\fint_{\Delta(y,r)} \mathbf{1}_{\Delta(z_0,1)}\,\mathrm{d}\sigma &\leq \frac{\sigma\big(\Delta(z_0,1)\big)}{\sigma\big(\Delta(y,r)\big)} \leq \frac{C}{r^{n-1}} \leq \frac{C}{|x - z_0|^{n-1}} \\ &\leq \frac{C}{1 + |x - z_0|^{n-1}}.\end{aligned} \tag{8.50}$$

Collectively, (8.48) and (8.50) prove that in the regime $x \in \Sigma \setminus \Delta(z_0, 2)$ we have

$$\begin{aligned}\mathcal{M}\big(\mathbf{1}_{\Delta(z_0,1)}\big)(x) &= \sup_{\Delta \ni x} \fint_{\Delta} \mathbf{1}_{\Delta(z_0,1)}\,\mathrm{d}\sigma = \sup_{\substack{0<r<2\mathrm{diam}(\Sigma)\\ y\in\Sigma,\ \Delta(y,r)\ni x}} \fint_{\Delta(y,r)} \mathbf{1}_{\Delta(z_0,1)}\,\mathrm{d}\sigma \\ &\leq \frac{C}{1 + |x - z_0|^{n-1}},\end{aligned} \tag{8.51}$$

where the first supremum is taken over all surface balls $\Delta \subseteq \Sigma$ containing x. This finishes the proof of (8.45). On account of (8.41), (8.43), (8.45), and (2.531), for every σ-measurable function f on Σ and every weight $w \in A_p(\Sigma, \sigma)$ with exponent $p \in (1, \infty)$ we may estimate

$$\begin{aligned}\int_\Sigma \frac{|f(x)|}{1 + |x|^{n-1}}\,\mathrm{d}\sigma(x) &\approx \int_\Sigma \frac{|f(x)|}{1 + |x - z_0|^{n-1}}\,\mathrm{d}\sigma(x) \\ &\approx \int_\Sigma |f(x)|\,\mathcal{M}\big(\mathbf{1}_{\Delta(z_0,1)}\big)(x)\,\mathrm{d}\sigma(x) \\ &\leq C\|f\|_{L^p(\Sigma,w)} \big\|\mathcal{M}(\mathbf{1}_{\Delta(z_0,1)})\big\|_{L^{p'}(\Sigma,w^{1-p'})} \\ &\leq C[w]_{A_p}\|f\|_{L^p(\Sigma,w)} \Big(\fint_{\Delta(z_0,1)} w^{1-p'}\,\mathrm{d}\sigma\Big)^{1/p'} \sigma\big(\Delta(z_0,1)\big)^{1/p'}\end{aligned}$$

$$\leq C\,[w]_{A_p}^{1+1/p}\|f\|_{L^p(\Sigma,w)}w\big(\Delta(z_0,1)\big)^{-1/p}, \tag{8.52}$$

where $C \in (0,\infty)$ depends only on n, p, z_0, and Σ. For each σ-measurable function f on Σ we next introduce

$$F := \Big(\int_\Sigma \frac{|f(x)|}{1+|x|^{n-1}}\,\mathrm{d}\sigma(x)\Big)\mathbf{1}_{\Delta(z_0,1)} \quad\text{and}\quad G := |f|. \tag{8.53}$$

Then (8.52) implies that for every σ-measurable function f on Σ and every weight $w \in A_p(\Sigma,\sigma)$ with $p \in (1,\infty)$ we have

$$\begin{aligned}\|F\|_{L^p(\Sigma,w)} &= \Big(\int_\Sigma \frac{|f(x)|}{1+|x|^{n-1}}\,\mathrm{d}\sigma(x)\Big)\,w\big(\Delta(z_0,1)\big)^{1/p}\\ &\leq C\,[w]_{A_p}^{1+1/p}\|G\|_{L^p(\Sigma,w)}.\end{aligned} \tag{8.54}$$

Fix now $p \in (1,\infty)$. Using Theorem 8.1 (with $\mathcal{F} := \{(F,G)\}$, $\Phi(t) := Ct^{1+1/p}$ for each $t > 0$, and $p_0 := p$), we conclude that

$$\|F\|_{\mathbb{X}_v} \leq C2^{p+4}\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}^{(p-1)(1+1/p)}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}^{(1+1/p)}\|G\|_{\mathbb{X}_v} \tag{8.55}$$

for some constant $C \in (0,\infty)$ independent of f. In turn, this translates into saying that

$$\begin{aligned}&\Big(\int_\Sigma \frac{|f(x)|}{1+|x|^{n-1}}\,\mathrm{d}\sigma(x)\Big)\,\big\|\mathbf{1}_{\Delta(z_0,1)}\big\|_{\mathbb{X}_v}\\ &\qquad\qquad \leq C2^{p+4}\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}^{(p-1)(1+1/p)}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}^{(1+1/p)}\|f\|_{\mathbb{X}_v}.\end{aligned} \tag{8.56}$$

We finally observe that

$$0 < \big\|\mathbf{1}_{\Delta(z_0,1)}\big\|_{\mathbb{X}_v} < \infty. \tag{8.57}$$

Indeed, the lower bound follows from property (g) in Definition 8.1 and the fact that $v > 0$ at σ-a.e. point on Σ, while the upper bound is a consequence of property (f) in Definition 8.1 along with the assumption that $v \in L^1_{\rm loc}(\Sigma,\sigma)$. This proves (8.37).

Let us next consider the embedding claimed in (8.38). With this goal in mind, pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$ and let $w \in A_2(\Sigma,\sigma)$ be an arbitrary weight such that $[w]_{A_2} \leq \Xi$. By items *(3)* and *(5)* in Proposition 2.20 and [65, Theorem 1.1] one can find $\tau = \tau(\Sigma,\Xi) \in (1,\infty)$ such that for every surface ball $\Delta \subseteq \Sigma$ we have

$$\Big(⨍_\Delta w^\tau \, \mathrm{d}\sigma\Big)^{1/\tau} \leq C[w]_{A_2} ⨍_\Delta w \, \mathrm{d}\sigma, \tag{8.58}$$

$$\Big(⨍_\Delta w^{-\tau} \, \mathrm{d}\sigma\Big)^{1/\tau} \leq C[w]_{A_2} ⨍_\Delta w^{-1} \, \mathrm{d}\sigma, \tag{8.59}$$

where $C \in (0, \infty)$ depends just on Σ and n. Note that for every integer $k \geq 0$ such that $k \leq \max\{1 + \log_2\big(\mathrm{diam}(\Sigma)\big), 0\}$ we may use Hölder's inequality, (8.58), and the lower Ahlfors regularity of Σ to estimate

$$\begin{aligned}
\frac{w\big(\Delta(z_0, 1)\big)}{\sigma\big(\Delta(z_0, 2^k)\big)} &= ⨍_{\Delta(z_0,2^k)} \mathbf{1}_{\Delta(z_0,1)} \, w \, \mathrm{d}\sigma \leq \left(\frac{\sigma\big(\Delta(z_0, 1)\big)}{\sigma\big(\Delta(z_0, 2^k)\big)}\right)^{\frac{1}{\tau'}} \Big(⨍_{\Delta(z_0,2^k)} w^\tau \, \mathrm{d}\sigma\Big)^{\frac{1}{\tau}} \\
&\leq C\,[w]_{A_2}\, 2^{-\frac{k\,(n-1)}{\tau'}} \frac{w\big(\Delta(z_0, 2^k)\big)}{\sigma\big(\Delta(z_0, 2^k)\big)},
\end{aligned} \tag{8.60}$$

where $\tau' := \big(1 - \frac{1}{\tau}\big)^{-1}$. In turn, (8.60) and the upper Ahlfors regularity of Σ imply

$$\begin{aligned}
\frac{\sigma\big(\Delta(z_0, 2^k)\big)}{w\big(\Delta(z_0, 2^k)\big)} &\leq C\,[w]_{A_2}\, 2^{-\frac{k\,(n-1)}{\tau'}} \frac{\sigma\big(\Delta(z_0, 2^k)\big)}{w\big(\Delta(z_0, 1)\big)}. \\
&\leq C\,[w]_{A_2} 2^{\frac{k\,(n-1)}{\tau}} w\big(\Delta(z_0, 1)\big)^{-1}.
\end{aligned} \tag{8.61}$$

Introduce $r := 2\tau/(\tau + 1) \in (1, 2)$, assume that $1 \leq q \leq r$, and fix some $\theta > r^{-1}$. With "prime" indicating Hölder conjugation (of exponents), these choices ensure that we have

$$\Big(\frac{2}{q}\Big)' - 1 \leq \Big(\frac{2}{r}\Big)' - 1 = \tau, \tag{8.62}$$

and

$$\begin{aligned}
\theta\Big(\frac{2}{q}\Big)' - \frac{1}{\tau}\Big(\Big(\frac{2}{q}\Big)' - 1\Big) - 1 &= \Big(\frac{2}{q}\Big)'\Big(\theta - \frac{1}{\tau}\frac{q}{2} - 1 + \frac{q}{2}\Big) = \Big(\frac{2}{q}\Big)'\Big(\theta - 1 + \frac{1}{\tau'}\frac{q}{2}\Big) \\
&> \Big(\frac{2}{q}\Big)'\Big(\frac{1}{r} - 1 + \frac{1}{\tau'}\frac{1}{2}\Big) = 0.
\end{aligned} \tag{8.63}$$

Then Jensen's inequality (which uses (8.62)), (8.59), (2.517) (applied with the exponent $p := 2$), and (8.61) imply that for every integer $k \geq 0$ with the property that $k \leq \max\{1 + \log_2\big(\mathrm{diam}(\Sigma)\big), 0\}$ we have

$$
\begin{aligned}
\Big(\fint_{\Delta(z_0,2^k)} w^{1-(\frac{2}{q})'}\,\mathrm{d}\sigma\Big)^{\frac{1}{(\frac{2}{q})'-1}} &\leq \Big(\fint_{\Delta(z_0,2^k)} w^{-\tau}\,\mathrm{d}\sigma\Big)^{\frac{1}{\tau}} \\
&\leq C\,[w]_{A_2} \fint_{\Delta(z_0,2^k)} w^{-1}\,\mathrm{d}\sigma \\
&\leq C\,[w]_{A_2}^2 \Big(\fint_{\Delta(z_0,2^k)} w\,\mathrm{d}\sigma\Big)^{-1} \\
&= C\,[w]_{A_2}^2 \frac{\sigma\big(\Delta(z_0,2^k)\big)}{w\big(\Delta(z_0,2^k)\big)} \\
&\leq C\,[w]_{A_2}^3 2^{\frac{k\,(n-1)}{\tau}} w\big(\Delta(z_0,1)\big)^{-1}.
\end{aligned}
\tag{8.64}
$$

This, (8.61), and (8.63) then permit us to estimate

$$
\begin{aligned}
&\int_\Sigma \frac{w(x)^{1-(\frac{2}{q})'}}{(1+|x-z_0|^{(n-1)\,\theta})^{(\frac{2}{q})'}}\,\mathrm{d}\sigma(x) \\
&\quad\leq C \int_{\Delta(z_0,1)} w^{1-(\frac{2}{q})'}\,\mathrm{d}\sigma \\
&\qquad + \sum_{0\leq k\leq \log_2(\mathrm{diam}(\Sigma))} 2^{-k\,(n-1)\,\theta\,(\frac{2}{q})'} \int_{\Delta(z_0,2^{k+1})\setminus\Delta(z_0,2^k)} w^{1-(\frac{2}{q})'}\,\mathrm{d}\sigma \\
&\quad\leq C\,[w]_{A_2}^{3((\frac{2}{q})'-1))} w\big(\Delta(z_0,1)\big)^{1-(\frac{2}{q})'} \sum_{k=0}^{\infty} 2^{-k\,(n-1)\,(\theta\,(\frac{2}{q})'-\frac{1}{\tau}((\frac{2}{q})'-1)-1)} \\
&\quad\leq C\,[w]_{A_2}^{3((\frac{2}{q})'-1))} w\big(\Delta(z_0,1)\big)^{1-(\frac{2}{q})'},
\end{aligned}
\tag{8.65}
$$

with the understanding that if $\mathrm{diam}(\Sigma) < 1$ the first sum above is void. Next, given any σ-measurable function f on Σ, Hölder's inequality yields

$$
\begin{aligned}
&\Big(\int_\Sigma \frac{|f(x)|^q}{1+|x-z_0|^{(n-1)\,\theta}}\,\mathrm{d}\sigma(x)\Big)^{\frac{1}{q}} \\
&\qquad\leq \|f\|_{L^2(\Sigma,w)} \left(\int_\Sigma \frac{w(x)^{1-(\frac{2}{q})'}}{(1+|x-z_0|^{(n-1)\,\theta})^{(\frac{2}{q})'}}\,\mathrm{d}\sigma(x)\right)^{\frac{1}{q\,(\frac{2}{q})'}}. \\
&\qquad\leq C\,[w]_{A_2}^{\frac{3}{2}} \|f\|_{L^2(\Sigma,w)} w\big(\Delta(z_0,1)\big)^{-\frac{1}{2}},
\end{aligned}
\tag{8.66}
$$

where $C \in (0, \infty)$ depends only on n, q, θ, Ξ, and Σ. For every σ-measurable function f on Σ define

$$F := \Big(\int_\Sigma \frac{|f(x)|^q}{1 + |x - z_0|^{(n-1)\theta}} \, d\sigma(x) \Big)^{\frac{1}{q}} \mathbf{1}_{\Delta(z_0,1)} \quad \text{and} \quad G := |f|. \tag{8.67}$$

Then (8.66) implies that for every σ-measurable function f on Σ and for every weight $w \in A_2(\Sigma, \sigma)$ such that $[w]_{A_2} \leq \Xi$ we have

$$\begin{aligned} \|F\|_{L^2(\Sigma,w)} &= \Big(\int_\Sigma \frac{|f(x)|^q}{1 + |x|^{(n-1)q}} \, d\sigma(x) \Big)^{\frac{1}{q}} w\big(\Delta(z_0, 1)\big)^{\frac{1}{2}} \\ &\leq C\, [w]_{A_2}^{\frac{3}{2}} \|G\|_{L^p(\Sigma,w)}. \end{aligned} \tag{8.68}$$

Recalling that $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$, the above estimate holds, in particular, for every $w \in A_2(\Sigma, \sigma)$ such that $[w]_{A_2} \leq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$. Using Theorem 8.1 (presently with $\mathcal{F} := \{(F, G)\}$, $\Phi(t) := Ct^{3/2}$ for each $t > 0$, and $p_0 := 2$), we conclude that

$$\|F\|_{\mathbb{X}_v} \leq C \|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v}^{\frac{3}{2}} \|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}^{\frac{3}{2}} \|G\|_{\mathbb{X}_v} \leq C\, \Xi^{\frac{3}{2}} \|G\|_{\mathbb{X}_v} \tag{8.69}$$

for some constant $C \in (0, \infty)$ independent of f. In turn, this translates into saying that

$$\Big(\int_\Sigma \frac{|f(x)|^q}{1 + |x - z_0|^{(n-1)\theta}} \, d\sigma(x) \Big)^{\frac{1}{q}} \big\|\mathbf{1}_{\Delta(z_0,1)}\big\|_{\mathbb{X}_v} \leq C\, \Xi^{\frac{3}{2}} \|f\|_{\mathbb{X}_v}. \tag{8.70}$$

The proof of (8.38) is then completed by invoking (8.57) and (8.42). □

8.2 Boundary Layer Potentials on Weighted Banach Function Spaces

We begin the study of boundary layer potentials on weighted Banach function spaces with the following basic result.

Proposition 8.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $\partial\Omega$ is a UR set and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N\big(\mathbb{R}^n \setminus \{0\}\big)$ which is odd and positive homogeneous of degree $1 - n$. Also, having fixed a function $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal* *operator on* $(\partial\Omega, \sigma)$ *and* $\mathcal{M}'f := \mathcal{M}(fv)/v$ *for any* σ*-measurable function* f *on* $\partial\Omega$*, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}_v'. \tag{8.71}$$

Pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}_v' \to \mathbb{X}_v'}$ *and fix an aperture parameter* $\kappa > 0$*. In this setting, for each* $f \in \mathbb{X}$*, define*

$$T_\varepsilon f(x) := \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} k(x-y) f(y)\, d\sigma(y) \ \text{ for each } \ x \in \partial\Omega, \tag{8.72}$$

$$T_* f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \ \text{ for each } \ x \in \partial\Omega, \tag{8.73}$$

$$Tf(x) := \lim_{\varepsilon \to 0^+} T_\varepsilon f(x) \ \text{ for } \sigma\text{-a.e. } \ x \in \partial\Omega, \tag{8.74}$$

$$\mathcal{T} f(x) := \int_{\partial\Omega} k(x-y) f(y)\, d\sigma(y) \ \text{ for each } \ x \in \Omega. \tag{8.75}$$

Then there exists a constant $C \in (0, \infty)$ *which depends exclusively on* n*,* Ξ*, and the UR constants of* $\partial\Omega$ *with the property that for each* $f \in \mathbb{X}_v$ *one has*

$$\|T_* f\|_{\mathbb{X}_v} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{\mathbb{X}_v}, \tag{8.76}$$

$$\|\mathcal{N}_\kappa(\mathcal{T} f)\|_{\mathbb{X}_v} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{\mathbb{X}_v}. \tag{8.77}$$

Also, for each $f \in \mathbb{X}_v$*, the limit defining* $Tf(x)$ *in* (8.74) *exists at* σ*-a.e.* $x \in \partial\Omega$ *and the operator*

$$T : \mathbb{X}_v \longrightarrow \mathbb{X}_v \tag{8.78}$$

is well defined, linear, and bounded.

Proof This follows by combining Proposition 3.4, Theorem 8.1 (used for the families of pairs $(|T_* f|, |f|)$, $(\mathcal{N}_\kappa(\mathcal{T} f), |f|)$, or $(|Tf|, |f|)$, with $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ and $p_0 = 2$), and Proposition 8.2. □

We also consider `weighted Banach function-based Sobolev spaces` on the boundaries of Ahlfors regular domains. Specifically, let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Pick some

$v \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, and let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$. In this setting, define

$$(\mathbb{X}_v)_1 := \big\{ f \in \mathbb{X}_v : \partial_{\tau_{jk}} f \in \mathbb{X}_v,\ 1 \le j,k \le n \big\}, \tag{8.79}$$

equipped with the natural norm

$$(\mathbb{X}_v)_1 \ni f \mapsto \|f\|_{(\mathbb{X}_v)_1} := \|f\|_{\mathbb{X}_v} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathbb{X}_v}. \tag{8.80}$$

In the following theorem we study boundedness properties of layer potential operators on Banach function spaces.

Theorem 8.3 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a UR domain. Define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let L be a homogeneous, weakly elliptic, constant complex coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Also, pick a coefficient tensor $A \in \mathfrak{A}_L$ and consider the double layer potential operators $\mathcal{D}_A$, K_A, $K_A^{\#}$ associated with the coefficient tensor A and the set Ω as in (3.22), (3.24), and (3.25), respectively. Next, having fixed $v \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}_v'$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}_v'. \tag{8.81}$$

Pick some $\Xi \ge 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}_v'\to\mathbb{X}_v'}$ and fix some aperture parameter $\kappa > 0$. Then the operators

$$K_A, K_A^{\#} : \big[\mathbb{X}_v\big]^M \longrightarrow \big[\mathbb{X}_v\big]^M \tag{8.82}$$

are well defined, linear, and bounded.

Furthermore, there exists a constant $C \in (0,\infty)$, depending only on the UR constants of $\partial\Omega$, L, n, κ, and Ξ, with the property that

$$\begin{gathered} \big\|\mathcal{N}_\kappa\big(\mathcal{D}_A f\big)\big\|_{\mathbb{X}_v} \le C\|f\|_{[\mathbb{X}_v]^M} \\ \text{for each function } f \in \big[\mathbb{X}_v\big]^M. \end{gathered} \tag{8.83}$$

Moreover, for each given function $f \in \big[\mathbb{X}_v\big]^M$ the following nontangential boundary trace formulas hold (with I denoting the identity operator)

$$\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_A\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{8.84}$$

In addition, for each function f *in the weighted Banach functions-based Sobolev space* $\big[(\mathbb{X}_v)_1\big]^M$ *it follows that*

$$\text{the nontangential boundary trace } \big(\partial_\ell \mathcal{D}_A f\big)\Big|_{\partial\Omega}^{\kappa-n.t.} \text{ exists (in } \mathbb{C}^M) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } \ell \in \{1,\dots,n\}, \tag{8.85}$$

and there exits some finite constant $C > 0$*, depending only on* $\partial\Omega$*,* L*,* n*,* κ*, and* Ξ*, such that*

$$\big\|\mathcal{N}_\kappa(\mathcal{D}_A f)\big\|_{\mathbb{X}_v} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}_A f)\big\|_{\mathbb{X}_v} \leq C\|f\|_{[(\mathbb{X}_v)_1]^M}. \tag{8.86}$$

Finally, the operator

$$K_A : \big[(\mathbb{X}_v)_1\big]^M \longrightarrow \big[(\mathbb{X}_v)_1\big]^M \tag{8.87}$$

is well defined, linear, and bounded.

Proof All claims follow from (3.31), Propositions 3.1, 8.2, 3.5, and Theorem 8.1. □

We introduce the homogeneous weighted Banach function-based Sobolev spaces on the boundaries of Ahlfors regular domains. Specifically, let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, having fixed $v \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$ and define (see also Remark 8.3 in this regard)

$$(\dot{\mathbb{X}}_v)_1 := \Big\{ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap (\mathbb{X}_v)_{\rm loc} : \partial_{\tau_{jk}} f \in \mathbb{X}_v \quad \text{for each } \ j,k \in \{1,\dots,n\}\Big\}, \tag{8.88}$$

where the membership $f \in (\mathbb{X}_v)_{\rm loc}$ means that $f\mathbf{1}_K \in \mathbb{X}_v$ for every compact set $K \subseteq \partial\Omega$. Equipped this space with the semi-norm

$$(\dot{\mathbb{X}}_v)_1 \ni f \longmapsto \|f\|_{(\dot{\mathbb{X}}_v)_1} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathbb{X}_v}. \tag{8.89}$$

Proposition 8.2 implies that if (8.36) holds, we have the following continuous embedding

$$(\mathbb{X}_v)_1 \hookrightarrow (\dot{\mathbb{X}}_v)_1. \tag{8.90}$$

Note also that constant functions on $\partial\Omega$ belong to $(\dot{\mathbb{X}}_v)_1$ and have vanishing semi-norm. We shall occasionally work with the quotient space $(\dot{\mathbb{X}}_v)_1\big/\sim$ of classes $[\,\cdot\,]$

of equivalence modulo constants of functions in $(\dot{\mathbb{X}}_v)_1$, equipped with the seminorm

$$(\dot{\mathbb{X}}_v)_1 \big/ \sim \ni [f] \mapsto \big\|[f]\big\|_{(\dot{\mathbb{X}}_v)_1/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{(\dot{\mathbb{X}}_v)_1}. \tag{8.91}$$

Choose next a scalar-valued function $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ satisfying $\phi \equiv 1$ in $B(0,1)$ and such that $\operatorname{supp}\phi \subseteq B(0,2)$. Having fixed a reference point $x_0 \in \partial\Omega$, for each scale $r \in (0,\infty)$ define

$$\phi_r(x) := \phi\Big(\frac{x - x_0}{r}\Big) \text{ for each } x \in \mathbb{R}^n, \tag{8.92}$$

and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. For each $r \in (0,\infty)$ set $\Delta_r := \partial\Omega \cap B(x_0, r)$ and, given any $f \in L^1_{\rm loc}(\partial\Omega, \sigma)$, define

$$f_r := \phi_r \cdot \big(f - f_{\Delta_{2r}}\big) \text{ on } \partial\Omega, \quad \text{where } f_{\Delta_{2r}} := \fint_{\Delta_{2r}} f \, d\sigma. \tag{8.93}$$

Lemma 8.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided NTA domain such that $\partial\Omega$ is an unbounded Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Having picked some function $v \in L^1_{\rm loc}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v. \tag{8.94}$$

Pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \big\|\mathcal{M}'\big\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$ and fix some reference point $x_0 \in \partial\Omega$. Finally, pick a function f which belongs to $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\rm loc}(\partial\Omega, \sigma)$ and, for each $r \in (0,\infty)$, define $\Delta_r := B(x_0, r) \cap \partial\Omega$ and $f_{\Delta_r} := \fint_{\Delta_r} f \, d\sigma$. Then the following statements are true.

(i) There exists a constant $C = C(\Omega, \Xi, x_0) \in (0,\infty)$, independent of the function f, such that

$$\sup_{r>0} \frac{1}{r} \big\| |f - f_{\Delta_r}| \cdot \mathbf{1}_{\Delta_r} \|_{\mathbb{X}_v} \leq C \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathbb{X}_v}. \tag{8.95}$$

(ii) For each $r \in (0,\infty)$ there exists a constant $C_r \in (0,\infty)$ which depends on Ω, Ξ, x_0, and r, but is independent of f, such that

$$\int_{\partial\Omega} \frac{|f(x) - f_{\Delta_r}|}{1 + |x|^n} \, d\sigma(x) \leq \frac{C_r}{\|\mathbf{1}_{\Delta_r}\|_{\mathbb{X}_v}} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathbb{X}_v}. \tag{8.96}$$

(iii) There exists a constant $C = C(\Omega, \Xi, x_0) \in (0, \infty)$*, independent of the function* f*, such that with the notation introduced in* (8.93) *one has*

$$\sup_{r>0} \big\| \nabla_{\tan} f_r \big\|_{[\mathbb{X}_v]^n} \leq C \big\| \nabla_{\tan} f \big\|_{[\mathbb{X}_v]^n}. \tag{8.97}$$

Proof We shall prove all claims using extrapolation (cf. Theorem 8.1). Let us first establish *(i)*. Recall (2.585) and define

$$\mathcal{F}_1 := \Big\{ \big(\tfrac{|f - f_{\Delta_r}|}{r} \mathbf{1}_{\Delta_r}, |\nabla_{\tan} f| \big) : \\ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma),\ r > 0 \Big\}. \tag{8.98}$$

Keeping this mind, we claim that for every weight $w \in A_2(\partial\Omega, \sigma)$ there exists a constant $C = C(\Omega, [w]_{A_2}, x_0) \in (0, \infty)$ such that

$$\|F_1\|_{L^2(\partial\Omega, w)} \leq C \|F_2\|_{L^2(\partial\Omega, w)} \tag{8.99}$$

for all pairs $(F_1, F_2) \in \mathcal{F}_1$. Indeed, this inequality is trivial if $\|F_2\|_{L^2(\partial\Omega, w)} = \infty$, whereas if $\|F_2\|_{L^2(\partial\Omega, w)} < \infty$ we may rely on (2.576), (8.98) and (2.586) to invoke Proposition 2.25 to obtain (2.618). This, in turn, gives (8.99) on account of (2.586). Moreover, the intervening constant C stays bounded if $[w]_{A_2}$ stays bounded. We may then apply Theorem 8.1 to conclude that $\|F_1\|_{\mathbb{X}_v} \leq C\|F_2\|_{\mathbb{X}_v}$ for every $(F_1, F_2) \in \mathcal{F}_1$. This and (2.585) then imply (8.95), completing the proof of *(i)*.

Let us now justify *(ii)*. Fix $r \in (0, \infty)$ and define

$$\mathcal{F}_2 := \Big\{ \Big(\|f - f_{\Delta_r}\|_{L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)} \mathbf{1}_{\Delta_r}, |\nabla_{\tan} f| \Big) : \\ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma) \Big\}. \tag{8.100}$$

As before, the goal is to check that (8.99) holds for all weights $w \in A_2(\partial\Omega, \sigma)$ and all pairs $(F_1, F_2) \in \mathcal{F}_2$ (where now the constant C is allowed to depend on the scale r, which has been fixed). This may be seen reasoning much as before, applying Proposition 2.25, but this time the relevant estimate is (2.620). Granted (8.99), we may then apply Theorem 8.1 to the family $\mathcal{F}_2$ and, as desired, conclude that (8.96) holds.

We finally address *(iii)*. Introduce

$$\mathcal{F}_3 := \Big\{ \big(|\nabla_{\tan} f_r|, |\nabla_{\tan} f| \big) : \\ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma),\ r > 0 \Big\}. \tag{8.101}$$

In line with what we have done in the previous cases, we now wish to show that (8.99) holds for all weights $w \in A_2(\partial\Omega, \sigma)$ and all pairs $(F_1, F_2) \in \mathcal{F}_3$. Again, it suffices to consider the case when $\|F_2\|_{L^2(\partial\Omega,w)} < \infty$. By definition, we have $(F_1, F_2) = \big(|\nabla_{\tan} g_r|, |\nabla_{\tan} g|\big)$ for some $g \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ and some $r > 0$. This, the assumption $\|F_2\|_{L^2(\partial\Omega,w)} < \infty$, (2.586), and Proposition 2.25 then guarantee that $g \in \dot{L}^2_1(\partial\Omega, w)$. We may therefore proceed as in (4.370)–(4.377) in the proof of Theorem 4.11 to conclude that (4.377) holds. This amounts to having (8.99) for the current choice of (F_1, F_2). Moreover, a careful examination of the proof shows that the intervening constant $C \in (0, \infty)$ stays bounded if $[w]_{A_2}$ stays bounded. Thus, we have shown that (8.99) holds for each $(F_1, F_2) \in \mathcal{F}_3$ and each $w \in A_2(\partial\Omega, \sigma)$. As such, we may invoke Theorem 8.1 to conclude that $\|F_1\|_{\mathbb{X}_v} \leq C\|F_2\|_{\mathbb{X}_v}$ for every $(F_1, F_2) \in \mathcal{F}_3$. In other words, there exists a constant $C = C(\Omega, \Xi, x_0) \in (0, \infty)$ such that

$$\big\| |\nabla_{\tan} f_r| \big\|_{\mathbb{X}_v} \leq C \big\| |\nabla_{\tan} f| \big\|_{\mathbb{X}_v}, \tag{8.102}$$

for every $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \cap L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$ and every $r > 0$. This completes the proof of (8.97). □

Remark 8.3 In the same setting of Lemma 8.1, it possible to provide a more convenient description of the space $(\dot{\mathbb{X}}_v)_1$. More precisely, one has

$$(\dot{\mathbb{X}}_v)_1 = \Big\{ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) : \partial_{\tau_{jk}} f \in \mathbb{X}_v \text{ for each } \ j, k \in \{1, \dots, n\} \Big\}. \tag{8.103}$$

As is apparent from definitions, this comes down to showing that any f belonging to the right-hand side satisfies $f \in (\mathbb{X}_v)_{\mathrm{loc}}$. To see this, note that Proposition 8.2 and the assumption $\partial_{\tau_{jk}} f \in \mathbb{X}_v$ yield $f \in L^1_{1,\mathrm{loc}}(\partial\Omega, \sigma)$. Consequently, (8.95) holds, and for each fixed $z_0 \in \partial\Omega$ and every $r > 0$ one has

$$\begin{aligned} \|f \mathbf{1}_{\Delta_r}\|_{\mathbb{X}_v} &\leq \big\| |f - f_{\Delta_r}| \cdot \mathbf{1}_{\Delta_r} \|_{\mathbb{X}_v} + |f_{\Delta_r}| \, \|\mathbf{1}_{\Delta_r} \big\|_{\mathbb{X}_v} \\ &\leq C \sum_{j,k=1}^{n} \big\| \partial_{\tau_{jk}} f \big\|_{\mathbb{X}_v} + |f_{\Delta_r}| \, \|\mathbf{1}_{\Delta_r}\|_{\mathbb{X}_v} < \infty, \end{aligned} \tag{8.104}$$

where $\Delta_r = \Delta(z_0, r)$ and we have used Definition 8.1 with $\mathrm{d}\mu = \mathrm{d}v = v\,\mathrm{d}\sigma$, and the fact that $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$. This eventually shows that $f \in (\mathbb{X}_v)_{\mathrm{loc}}$, as desired.

It turns out that, when considered on the boundaries of two-sided NTA domains, the quotient space $(\dot{\mathbb{X}}_v)_1 \big/ \sim$ is actually a Banach space.

Proposition 8.4 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a two-sided NTA domain with an unbounded Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Fix $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$. Let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$*

and let $\mathbb{X}_v'$ be its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, suppose that

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}_v'. \tag{8.105}$$

Finally, recall that $(\dot{\mathbb{X}}_v)_1 \big/ \sim$ denotes the quotient space of classes $[\,\cdot\,]$ of equivalence modulo constants of functions in $(\dot{\mathbb{X}}_v)_1$, equipped with the semi-norm (8.91).

Then (8.91) is a genuine norm on $(\dot{\mathbb{X}}_v)_1 \big/ \sim$, and $(\dot{\mathbb{X}}_v)_1 \big/ \sim$ is a Banach space when equipped with the norm (8.91).

Proof Let us first observe from (8.96) that the semi-norm (8.91) is indeed a norm on the space $(\dot{\mathbb{X}}_v)_1 \big/ \sim$. We shall next show that $(\dot{\mathbb{X}}_v)_1 \big/ \sim$ is complete when equipped with the norm (8.91). With this goal in mind, let $\{f_\alpha\}_{\alpha\in\mathbb{N}} \subseteq (\dot{\mathbb{X}}_v)_1$ be such that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the quotient space $(\dot{\mathbb{X}}_v)_1 \big/ \sim$. This means that $\big\{\partial_{\tau_{jk}} f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{X}_v$, for any two fixed indices $j, k \in \{1, \dots, n\}$. Using the fact that $\mathbb{X}_v$ is a Banach space, we then conclude that for each pair of indices $j, k \in \{1, \dots, n\}$ there exists $g_{jk} \in \mathbb{X}_v$ such that

$$\partial_{\tau_{jk}} f_\alpha \to g_{jk} \ \text{ in } \ \mathbb{X}_v \ \text{ as } \ \alpha \to \infty. \tag{8.106}$$

Fix a reference point $x_0 \in \partial\Omega$ and, for each $r \in (0, \infty)$, set $\Delta_r := B(x_0, r) \cap \partial\Omega$. Also, define $f_{\alpha,\Delta_r} := \fint_{\Delta_r} f_\alpha \, d\sigma$ for each $r \in (0, \infty)$ and each $\alpha \in \mathbb{N}$. Applying (8.96) to $f := f_\alpha - f_\beta$ we obtain that for any $r \in (0, \infty)$ there exists some constant $C_r \in (0, \infty)$, which depends on Ω, $\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v}$ $\|\mathcal{M}'\|_{\mathbb{X}_v' \to \mathbb{X}_v'}$, r, and x_0, such that for each $\alpha, \beta \in \mathbb{N}$ we have

$$\begin{aligned} &\big\|\big(f_\alpha - f_{\alpha,\Delta_r}\big) - \big(f_\beta - f_{\beta,\Delta_r}\big)\big\|_{L^1\left(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\right)} \\ &\qquad \le \frac{C_r}{\|\mathbf{1}_{\Delta_r}\|_{\mathbb{X}_v}} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\beta\big\|_{\mathbb{X}_v}. \end{aligned} \tag{8.107}$$

Since $\big\{\partial_{\tau_{jk}} f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the space $\mathbb{X}_v$, it follows that for each fixed radius $r \in (0, \infty)$ the sequence $\big\{f_\alpha - f_{\alpha,\Delta_r}\big\}_{\alpha\in\mathbb{N}}$ happens to be Cauchy in the Banach space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Hence, for each fixed $r \in (0, \infty)$ there exists $h_r \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ such that

$$f_\alpha - f_{\alpha,\Delta_r} \to h_r \ \text{ in } \ L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \ \text{ as } \ \alpha \to \infty. \tag{8.108}$$

On the other hand, by (8.95) (applied to the difference $f := f_\alpha - f_\beta$), there exists some constant $C = C\big(\Omega, \|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}_v' \to \mathbb{X}_v'}, x_0\big) \in (0, \infty)$ such that for each fixed $r \in (0, \infty)$ we have

$$\big\| |(f_\alpha - f_{\alpha,\Delta_r}) - (f_\beta - f_{\beta,\Delta_r})| \cdot \mathbf{1}_{\Delta_r} \big\|_{\mathbb{X}_v} \leq C\, r \sum_{j,k=1}^{n} \big\| \partial_{\tau_{jk}} f_\alpha - \partial_{\tau_{jk}} f_\alpha \big\|_{\mathbb{X}_v}. \tag{8.109}$$

Hence, the sequence $\big\{\big(f_\alpha - f_{\alpha,\Delta_r}\big)\mathbf{1}_{\Delta_r}\big\}_{\alpha\in\mathbb{N}}$ is Cauchy in the Banach space $\mathbb{X}_v$ for each fixed $r \in (0,\infty)$. As a result, for each fixed $r \in (0,\infty)$ it follows that

$$\begin{gathered}\text{there exists a function } k_r \in \mathbb{X}_v \text{ such that}\\ \big(f_\alpha - f_{\alpha,\Delta_r}\big)\mathbf{1}_{\Delta_r} \to k_r \ \text{ in } \ \mathbb{X}_v \ \text{ as } \ \alpha \to \infty.\end{gathered} \tag{8.110}$$

Note that convergence in $\mathbb{X}_v$ implies, after eventually passing to a sub-sequence, pointwise v-a.e. convergence (see [15, Theorem 1.4, p. 3]). Thus (8.108) and (8.110) immediately give

$$h_r\big|_{\Delta_r} = k_r \in \mathbb{X}_v \ \text{ for each } \ r \in (0,\infty). \tag{8.111}$$

Additionally, for each fixed $r_1, r_2 \in (0,\infty)$ the convergence recorded in (8.108) also yields

$$f_{\alpha,\Delta_{r_2}} - f_{\alpha,\Delta_{r_1},w} \to h_{r_1} - h_{r_2} \ \text{ in } \ L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \ \text{ as } \ \alpha \to \infty. \tag{8.112}$$

Thus $h_{r_1} - h_{r_2}$ must be constant. This, (8.108), and (8.111) eventually lead to

$$h_r \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap (\mathbb{X}_v)_{\mathrm{loc}} \ \text{ for each } \ r \in (0,\infty). \tag{8.113}$$

To continue, we simply write h for h_r with $r = 1$, and c_α for f_{α,Δ_r} with $r = 1$. Then, as seen from (8.113) and (8.108),

$$h \ \text{ belongs to } \ L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap (\mathbb{X}_v)_{\mathrm{loc}}, \tag{8.114}$$

and the sequence $\{c_\alpha\}_{\alpha\in\mathbb{N}} \subseteq \mathbb{C}$ is such that

$$f_\alpha - c_\alpha \to h \ \text{ in } \ L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \ \text{ as } \ \alpha \to \infty. \tag{8.115}$$

For each $j, k \in \{1, \dots, n\}$ and each test function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ we may then write

$$\begin{aligned}\int_{\partial\Omega} h(\partial_{\tau_{jk}}\varphi)\,\mathrm{d}\sigma &= \lim_{\alpha\to\infty}\int_{\partial\Omega}(f_\alpha - c_\alpha)(\partial_{\tau_{jk}}\varphi)\,\mathrm{d}\sigma\\ &= -\lim_{\alpha\to\infty}\int_{\partial\Omega}\partial_{\tau_{jk}}(f_\alpha - c_\alpha)\varphi\,\mathrm{d}\sigma = -\lim_{\alpha\to\infty}\int_{\partial\Omega}(\partial_{\tau_{jk}} f_\alpha)\varphi\,\mathrm{d}\sigma\\ &= \int_{\partial\Omega} g_{jk}\varphi\,\mathrm{d}\sigma,\end{aligned} \tag{8.116}$$

thanks to (8.115), (2.583), (8.106), and Proposition 8.2. From this and (2.581)–(2.582) we then conclude that

$$\partial_{\tau_{jk}} h = g_{jk} \in \mathbb{X}_v \quad \text{for each} \quad j,k \in \{1,\dots,n\}. \tag{8.117}$$

Collectively, (8.114) and (8.117) prove that $h \in (\dot{\mathbb{X}}_v)_1$. Finally, from (8.106), (8.117), and (8.91) we conclude that the sequence $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ converges to $[h]$, the class of h, in the quotient space $(\dot{\mathbb{X}}_v)_1 \big/ \sim$. □

We next state a trace result in the setting of weighted Banach functions spaces:

Proposition 8.5 *Let $\Omega \subseteq \mathbb{R}^n$ be an NTA domain with the property that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa > 0$. Select $v \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$. Also, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$ and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, make the assumption that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.118}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Finally, pick a function $u : \Omega \to \mathbb{C}$ satisfying $u \in \mathscr{C}^1(\Omega)$ and $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$.

Then the nontangential trace $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$, belongs to the space $(\dot{\mathbb{X}}_v)_1$, and

$$\Big\| u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \Big\|_{(\dot{\mathbb{X}}_v)_1} \leq C \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{(\mathbb{X}_v)_1} \tag{8.119}$$

for some constant $C = C(\Omega,\Xi) \in (0,\infty)$ independent of the function u.

Proof For an arbitrary u as in the statement, apply the second part of Proposition 8.1 with $p_0 = 2$ to the function $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$ to obtain a weight $w_u \in A_2(\partial\Omega,\sigma)$, which is allowed to depend on u, with the property that $\mathcal{N}_\kappa(\nabla u) \in L^2(\partial\Omega, w_u)$. Granted this, we are in a position to apply Proposition 2.24 to obtain that the nontangential trace $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists σ-a.e. and belongs to $\dot{L}^2_1(\partial\Omega, w_u)$. In particular, by (2.576) there exists an integrability exponent $q \in (1,2)$ (once again, dependent on u) such that

$$\begin{gathered} u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \quad \text{and} \\ \partial_{\tau_{jk}}\Big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) \in L^q_{\rm loc}(\partial\Omega,\sigma) \quad \text{for each} \quad j,k \in \{1,\dots n\}. \end{gathered} \tag{8.120}$$

We claim that for every weight $w \in A_2(\partial\Omega,\sigma)$ we actually have

$$\big\|\nabla_{\tan}\Big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)\big\|_{[L^2(\partial\Omega,w)]^n} \leq C\|\mathcal{N}_\kappa(\nabla u)\|_{L^2(\partial\Omega,w)}. \tag{8.121}$$

Indeed, such an estimate is trivial if $\mathcal{N}_\kappa(\nabla u)$ does not belong to $L^2(\partial\Omega, w)$. In the case when $\mathcal{N}_\kappa(\nabla u) \in L^2(\partial\Omega, w)$, simply use (2.585) and (2.611) in Proposition 2.24, where the constant C depends on $[w]_{A_2}$ via an increasing function. Having established this, we may apply Theorem 8.1 with $p_0 = 2$ to the family of pairs of functions

$$\mathcal{F} := \Big\{ \Big(\big| \nabla_{\tan} \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \big) \big|, \mathcal{N}_\kappa(\nabla u) \Big) : u \in \mathscr{C}^1(\Omega),\ \mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v \Big\} \tag{8.122}$$

and obtain that $\|F\|_{\mathbb{X}_v} \leq C\|G\|_{\mathbb{X}_v}$ for every $(F, G) \in \mathcal{F}$. That is, for every function $u \in \mathscr{C}^1(\Omega)$ with $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$ there holds

$$\big\| \nabla_{\tan} \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \big) \big\|_{[\mathbb{X}_v]^n} \leq C \|\mathcal{N}_\kappa(\nabla u)\|_{\mathbb{X}_v}. \tag{8.123}$$

This, (8.120), (2.586), and Remark 8.3 eventually yield that $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in (\dot{\mathbb{X}}_v)_1$ and satisfies (8.119). □

We next present a basic Fatou-type result and integral representation formula of the following sort:

Theorem 8.4 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an NTA domain such that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ (where $M \in \mathbb{N}$) be a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,\text{mod}}$ from (3.49), and the modified version of the single layer operator $\mathscr{S}_{\text{mod}}$ from (3.38). Choose $v \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. Also, with $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and with $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, suppose that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v. \tag{8.124}$$

Finally, fix an aperture parameter $\kappa \in (0, \infty)$ and consider a function $u : \Omega \to \mathbb{C}^M$ satisfying

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v. \tag{8.125}$$

Then

$$\begin{aligned} &u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \big[(\dot{\mathbb{X}}_v)_1\big]^M, \\ &(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ and } \partial_\nu^A u \in \big[\mathbb{X}_v\big]^M, \end{aligned} \tag{8.126}$$

and there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{A,\mathrm{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathscr{S}_{\mathrm{mod}}\big(\partial_\nu^A u\big) + c_u \ \text{ in } \ \Omega. \tag{8.127}$$

Proof From Proposition 8.5 we know that $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and belongs to $\big[(\dot{\mathbb{X}}_v)_1\big]^M$. In addition, the second part of Proposition 8.1 applied with $p_0 = 2$ to the function $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$ yields a weight $w_u \in A_2(\partial\Omega, \sigma)$, depending on u, with the property that $\mathcal{N}_\kappa(\nabla u) \in L^2(\partial\Omega, w_u)$. We may then invoke Corollary 3.1 to obtain that the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists (in $\mathbb{C}^{M\cdot n}$) at σ-a.e. point on $\partial\Omega$, the conormal derivative $\partial_\nu^A u$ belongs to $\big[L^2(\partial\Omega, w_u)\big]^M$, and (8.127) holds.

We are left with showing that $\partial_\nu^A u \in \big[\mathbb{X}_v\big]^M$. Observe that the present hypotheses on Ω ensure (cf. (2.48)) that Ω is a UR domain. Also, recall from (8.38) that there exists $q \in (1, 2)$ (depending on u) such that $\mathcal{N}_\kappa(\nabla u) \in L^q_{\mathrm{loc}}(\partial\Omega, \sigma)$. These properties allow us to invoke Theorem 3.4 to conclude that (3.72) holds, which by (3.66), readily gives $\partial_\nu^A u \in \big[\mathbb{X}_v\big]^M$. □

Our next goal is to extend Theorem 8.3 to modified boundary layer potentials.

Theorem 8.5 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a UR domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Recall the modified boundary-to-boundary single layer operator S_{mod} associated with L and Ω as in (3.42). Having fixed $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, make the assumption that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.128}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Then the following properties are true.

(1) The modified boundary-to-boundary single layer operator induces a mapping

$$S_{\mathrm{mod}} : \big[\mathbb{X}_v\big]^M \longrightarrow \big[(\dot{\mathbb{X}}_v)_1\big]^M \tag{8.129}$$

which is well defined, linear, and bounded, when the target space is endowed with the semi-norm (8.89). In particular,

$$\text{for each given function } f \in \big[\mathbb{X}_v\big]^M \text{ and each pair of indices } j, k \in \{1, \dots, n\} \text{ one has } \partial_{\tau_{jk}}\big(S_{\rm mod} f\big) \in \big[\mathbb{X}_v\big]^M. \tag{8.130}$$

Also, for each function $f \in \big[\mathbb{X}_v\big]^M$*, at* σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\begin{aligned}&\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big)(x) \\ &= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{js}}\big(S_{\rm mod} f\big)_\alpha(y)\,d\sigma(y)\Bigg)_{1\le\mu\le M},\end{aligned} \tag{8.131}$$

where $K^{\#}_{A^\top}$ *is the singular integral operator associated as in* (3.25) *with the set* Ω *and the transpose coefficient tensor* $A^\top$.

(2) As a consequence of (8.129)*, the following is a well-defined linear operator:*

$$\begin{aligned}&\big[S_{\rm mod}\big] : \big[\mathbb{X}_v\big]^M \longrightarrow \big[(\dot{\mathbb{X}}_v)_1\big/\sim\big]^M \ \text{ defined as} \\ &\big[S_{\rm mod}\big]f := \big[S_{\rm mod} f\big] \in \big[(\dot{\mathbb{X}}_v)_1\big/\sim\big]^M, \quad \forall\, f \in \big[\mathbb{X}_v\big]^M.\end{aligned} \tag{8.132}$$

Moreover, if actually $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator* (8.132) *is also bounded when the quotient space is endowed with the norm introduced in* (8.91).

(3) With $\mathscr{S}_{\rm mod}$ *denoting the modified version of the single layer operator acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (3.38)*, for each given aperture parameter* $\kappa > 0$ *there exists some constant* $C = C(\Omega, L, n, \Xi, \kappa) \in (0, \infty)$ *with the property that for each given function* $f \in \big[\mathbb{X}_v\big]^M$ *one has*

$$\begin{gathered}\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa\big(\nabla\mathscr{S}_{\rm mod} f\big) \text{ belongs to } \mathbb{X}_v \text{ and} \\ \big\|\mathcal{N}_\kappa\big(\nabla\mathscr{S}_{\rm mod} f\big)\big\|_{\mathbb{X}_v} \le C\|f\|_{[\mathbb{X}_v]^M}, \\ \Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = (S_{\rm mod} f)(x) \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega.\end{gathered} \tag{8.133}$$

Moreover, for each given function f *in the weighted Banach functions space* $\big[\mathbb{X}_v\big]^M$ *the following jump formula holds (with* I *denoting the identity operator)*

$$\partial^A_\nu \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \ \text{ at } \sigma\text{-a.e. point in } \partial\Omega, \tag{8.134}$$

where $K^{\#}_{A^\top}$ *is the singular integral operator associated as in* (3.25) *with the set* Ω *and the transpose coefficient tensor* $A^\top$.

Before proving this result we would like to call attention to the fact that, as opposed to Theorem 7.4, the current statement does not contain a version of (7.121) (see also (3.46)) since it is not clear whether $\mathbb{X}_v$ is the dual of some other Banach function space. This will affect the proof of the surjectivity of $\big[S_{\rm mod}\big]$ as we will not be able to invoke the Banach–Alaoglu Theorem (see the proof of Theorems 4.11 and 7.13 for the case of weighted Lebesgue and Morrey spaces).

Proof We start with item *(1)*. Recalling the definition of $S_{\rm mod}$ in (3.42), from (3.43) and Proposition 8.2 we see that

$$S_{\rm mod} : [\mathbb{X}_v]^M \longrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \tag{8.135}$$

is well defined and linear. By Proposition 8.1, for every $f \in [\mathbb{X}_v]^M$ it is possible to find a Muckenhoupt weight $w_f \in A_2(\partial\Omega, \sigma)$ such that $f \in \big[L^2(\partial\Omega, w_f)\big]^M$. Granted this, items *(d)* and *(f)* in Proposition 3.5, respectively, yield (8.131) and $S_{\rm mod} f \in \big[\dot{L}^2_1(\partial\Omega, w_f)\big]^M$. Thus, whenever $1 \leq j, k \leq n$, it follows that $\partial_{\tau_{jk}} S_{\rm mod} f$ belongs to $\big[L^2(\partial\Omega, w_f)\big]^M \subseteq \big[L^1_{\rm loc}(\partial\Omega, \sigma)\big]^M$, by Lemma 2.15. In view of this and (2.585), we then proceed to consider

$$\mathcal{F}_1 := \Big\{\big(|\nabla_{\tan} S_{\rm mod} f|, |f|\big) : f \in \big[\mathbb{X}_v\big]^M\Big\}, \tag{8.136}$$

which is a well-defined set. We claim that $\|F_1\|_{L^2(\partial\Omega,w)} \leq C\|F_2\|_{L^2(\partial\Omega,w)}$ for every pair $(F_1, F_2) \in \mathcal{F}_1$ and every weight $w \in A_2(\partial\Omega, \sigma)$. Indeed, we may assume without loss of generality that the right-hand side is finite, in which case the desired estimates follow from item *(f)* in Proposition 3.5 and (2.585). Invoking next Theorem 8.1 shows that $\|F_1\|_{\mathbb{X}_v} \leq C\|F_2\|_{\mathbb{X}_v}$ for every pair $(F_1, F_2) \in \mathcal{F}_1$. This, Proposition 8.2, and (2.586), then readily give (8.130) and (8.129).

Item *(2)* follows at once from item *(1)*. As regards the properties listed in *(3)*, we first observe that, thanks to Proposition 8.1, for every $f \in [\mathbb{X}_v]^M$ it is possible to find some $w_f \in A_2(\partial\Omega, \sigma)$ such that $f \in \big[L^2(\partial\Omega, w_f)\big]^M$. We may then invoke part *(c)* in Proposition 3.5 to immediately obtain that $\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, that $L\big(\mathscr{S}_{\rm mod} f\big) = 0$ in Ω, and also that (8.134) holds. Note that the last property in (8.133) follows from (3.47) and Proposition 8.2. To proceed, introduce

$$\mathcal{F}_2 := \Big\{\big(\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big), |f|\big) : f \in \big[\mathbb{X}_v\big]^M\Big\}. \tag{8.137}$$

We claim that $\|F_1\|_{L^2(\partial\Omega,w)} \leq C\|F_2\|_{L^2(\partial\Omega,w)}$ for every pair $(F_1, F_2) \in \mathcal{F}_2$ and every weight $w \in A_2(\partial\Omega, \sigma)$. Without loss of generality, assume the right-hand side is finite, in which case the desired estimates follow from item *(c)* in Proposition 3.5

(cf. (3.127)). Another use of Theorem 8.1 gives the estimate $\|F_1\|_{\mathbb{X}_v} \leq C\|F_2\|_{\mathbb{X}_v}$ for every pair (F_1, F_2) belonging to $\mathcal{F}$. This corresponds to the last two properties in (8.133), finishing the proof of Theorem 8.5. □

We next present some fundamental properties of the modified boundary-to-domain double layer potential operators and their conormal derivatives acting on homogeneous weighted Banach function-based Sobolev spaces.

Theorem 8.6 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a UR domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In addition, for some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that $L := L_A$ is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Also, let $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ be the matrix-valued fundamental solution associated with L as in Theorem 3.1. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{A,\mathrm{mod}}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (3.49). Also, having fixed $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.138}$$

pick some number $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$, and fix some aperture parameter $\kappa \in (0, \infty)$.

Then the following statements are true.

(1) There exists some constant $C = C(\Omega, A, n, \Xi, \kappa) \in (0, \infty)$ with the property that for each function $f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$ one has

$$\begin{gathered}
\mathcal{D}_{A,\mathrm{mod}}f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{A,\mathrm{mod}}f\big) = 0 \ \text{ in } \ \Omega,\\
\big(\mathcal{D}_{A,\mathrm{mod}}f\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \ \big(\nabla\mathcal{D}_{A,\mathrm{mod}}f\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exist } \sigma\text{-a.e. on } \ \partial\Omega,\\
\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\mathrm{mod}}f\big) \ \text{ belongs to } \ \mathbb{X}_v \ \text{ and}\\
\big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\mathrm{mod}}f\big)\big\|_{\mathbb{X}_v} \leq C\|f\|_{[(\dot{\mathbb{X}}_v)_1]^M}.
\end{gathered} \tag{8.139}$$

In fact, for each function $f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$ one has

$$(\mathcal{D}_{A,\mathrm{mod}}f)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_{A,\mathrm{mod}}\big)f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \tag{8.140}$$

where I is the identity operator on $\big[(\dot{\mathbb{X}}_v)_1\big]^M$, and $K_{A,\mathrm{mod}}$ is the modified boundary-to-boundary double layer potential operator from (3.50) and (3.48).

(2) Given an arbitrary function $f = (f_\alpha)_{1\leq\alpha\leq M}$ *belonging to the homogeneous weighted Banach function-based boundary Sobolev space* $\big[(\dot{\mathbb{X}}_v)_1\big]^M$*, it follows that at* σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\big(\partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f)\big)(x) = \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a_{ij}^{\mu\gamma} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\times \times \big(\partial_{\tau_{js}} f_\alpha\big)(y)\,d\sigma(y)\Bigg)_{1\leq\mu\leq M}, \tag{8.141}$$

where the conormal derivative is considered as in (3.66)*.*

(3) The operator

$$\begin{gathered}\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}} : \big[(\dot{\mathbb{X}}_v)_1\big]^M \longrightarrow \big[\mathbb{X}_v\big]^M \ \ \textit{defined as}\\ \big(\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big) f := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f) \ \ \textit{for each} \ \ f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M\end{gathered} \tag{8.142}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm (8.89)*. As a consequence of* (8.142)*, the following is a well-defined linear operator:*

$$\begin{gathered}\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big] : \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M \longrightarrow \big[\mathbb{X}_v\big]^M\\ \textit{given by } \big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big][f] := \partial_\nu^A(\mathcal{D}_{A,\mathrm{mod}} f)\\ \textit{for each function } \ f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M.\end{gathered} \tag{8.143}$$

If, in fact, $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then the operator* (8.143) *is also bounded when the quotient space is equipped with the norm* (8.91)*.*

(4) With $K^\#_{A^\top}$ *denoting the singular integral operator associated as in* (3.25) *with the set* Ω *and the transpose coefficient tensor* $A^\top$*, one has*

$$\begin{gathered}\big(\tfrac12 I + K^\#_{A^\top}\big)\big(-\tfrac12 I + K^\#_{A^\top}\big) = \big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big]\big[S_{\mathrm{mod}}\big]\\ \textit{as mappings acting from } \ \big[\mathbb{X}_v\big]^M,\end{gathered} \tag{8.144}$$

and

$$\begin{gathered}\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big]\big[K_{A,\mathrm{mod}}\big] = K^\#_{A^\top}\big[\partial_\nu^A \mathcal{D}_{A,\mathrm{mod}}\big]\\ \textit{as mappings acting from } \ \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M.\end{gathered} \tag{8.145}$$

Moreover, if $\partial\Omega$ *is connected then also*

$$\begin{gathered}\big(\tfrac{1}{2}I+\big[K_{A,\mathrm{mod}}\big]\big)\big(-\tfrac{1}{2}I+\big[K_{A,\mathrm{mod}}\big]\big)=\big[S_{\mathrm{mod}}\big]\big[\partial_\nu^A\mathcal{D}_{A,\mathrm{mod}}\big]\\ \textit{as mappings acting from }\ \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M,\end{gathered}\tag{8.146}$$

and

$$\begin{gathered}\big[S_{\mathrm{mod}}\big]K^{\#}_{A^\top}=\big[K_{A,\mathrm{mod}}\big]\big[S_{\mathrm{mod}}\big]\\ \textit{as mappings acting from }\ \big[\mathbb{X}_v\big]^M.\end{gathered}\tag{8.147}$$

Proof We begin by observing that for any given function $f\in\big[(\dot{\mathbb{X}}_v)_1\big]^M$ we may apply Proposition 8.1 to the function $\sum_{j,k=1}^n|\partial_{\tau_{jk}}f|\in\mathbb{X}_v$ to obtain a Muckenhoupt weight $w_f\in A_2(\partial\Omega,\sigma)$ such that $\sum_{j,k=1}^n|\partial_{\tau_{jk}}f|\in L^2(\partial\Omega,w_f)$. In concert with Remark 2.4, this implies that $f\in\big[\dot{L}^2_1(\partial\Omega,w_f)\big]^M$. The membership just established, Theorems 3.5, 3.6, and 3.7 then immediately imply the properties stated in the first and second lines of (8.139), (8.140), (8.141), and all the equalities in item *(4)*. Also, this allows us to introduce the following families which are well defined:

$$\mathcal{F}_1:=\Big\{\Big(\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{A,\mathrm{mod}}f\big),\sum_{j,k=1}^n|\partial_{\tau_{jk}}f|\Big):\ f\in\big[(\dot{\mathbb{X}}_v)_1\big]^M\Big\},$$

$$\mathcal{F}_2:=\Big\{\Big(\big|(\partial_\nu^A\mathcal{D}_{A,\mathrm{mod}})f\big|,\sum_{j,k=1}^n|\partial_{\tau_{jk}}f|\Big):\ f\in\big[(\dot{\mathbb{X}}_v)_1\big]^M\Big\}.\tag{8.148}$$

In relation to these, we claim that $\|F_1\|_{L^2(\partial\Omega,w)}\leq C\|F_2\|_{L^2(\partial\Omega,w)}$ for every $(F_1,F_2)\in\mathcal{F}_j$ and every $w\in A_2(\partial\Omega,\sigma)$, with $j=1,2$. To justify this claim, there is no loss of generality in assuming that the right-hand side is finite, in which case we have $f\in\big[\dot{L}^2_1(\partial\Omega,w)\big]^M$ by Remark 2.4. As such, we may use (3.133) (respectively, (3.136)) in Theorem 3.5 for $\mathcal{F}_1$ (respectively, $\mathcal{F}_2$). We can then invoke Theorem 8.1 to conclude that $\|F_1\|_{\mathbb{X}_v}\leq C\|F_2\|_{\mathbb{X}_v}$ for every pair $(F_1,F_2)\in\mathcal{F}_j$, with $j=1,2$. These estimates correspond precisely to the third and fourth lines in (8.139) when $j=1$, and to (8.142) when $j=2$. Note that the latter implies (8.143) on account of (3.137), and the corresponding boundedness claim follows from Proposition 8.4. This completes the proof. □

We next study mapping properties for modified boundary-to-boundary double layer potential operators acting on homogeneous weighted Banach function-based Sobolev spaces.

Theorem 8.7 *Let $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) be an NTA domain such that $\partial\Omega$ is an Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L=\big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M\times M$ system in $\mathbb{R}^n$ (for some integer $M\in\mathbb{N}$). In* *this context, consider the modified boundary-to-boundary double layer potential operator* $K_{A,\text{mod}}$ *from* (3.50)*. Having fixed some function* $v \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ *with* $v > 0$ *at* σ*-a.e. point on* $\partial\Omega$*, let* $\mathbb{X}_v$ *be a Banach function space over* $(\partial\Omega, v\sigma)$*, and let* $\mathbb{X}'_v$ *be its Köthe dual. With* $\mathcal{M}$ *denoting the Hardy–Littlewood maximal operator on* $(\partial\Omega, \sigma)$ *and with* $\mathcal{M}'f := \mathcal{M}(fv)/v$ *for any* σ*-measurable function* f *on* $\partial\Omega$*, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v. \tag{8.149}$$

Then the following statements are valid.

(1) The modified boundary-to-boundary double layer potential operator induces a mapping

$$K_{A,\text{mod}} : \big[(\dot{\mathbb{X}}_v)_1\big]^M \longrightarrow \big[(\dot{\mathbb{X}}_v)_1\big]^M \tag{8.150}$$

which is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (8.89)*. As a corollary of* (8.150)*, the following operator is well defined and linear:*

$$\begin{gathered}\big[K_{A,\text{mod}}\big] : \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M \longrightarrow \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M\\ \text{given by } \big[K_{A,\text{mod}}\big][f] := \big[K_{A,\text{mod}}f\big] \in \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M,\\ \text{for each function } f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M.\end{gathered} \tag{8.151}$$

Moreover, if actually $\Omega \subseteq \mathbb{R}^n$ *is a two-sided NTA domain whose boundary is an unbounded Ahlfors regular set then the operator* (8.151) *is also bounded when all quotient spaces are endowed with the norm introduced in* (8.91)*.*

(2) If U_{jk} *with* $j, k \in \{1, \dots, n\}$ *is the family of singular integral operators defined in* (3.35)*, then*

$$\begin{gathered}\partial_{\tau_{jk}}\big(K_{A,\text{mod}}f\big) = K_A(\partial_{\tau_{jk}}f) + U_{jk}(\nabla_{\tan}f) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega\\ \text{for each } f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M \text{ and each } j, k \in \{1, \dots, n\}.\end{gathered} \tag{8.152}$$

Proof To get started, given any $f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$, we may apply Proposition 8.1 to the function $\sum_{j,k=1}^n |\partial_{\tau_{jk}}f| \in \mathbb{X}_v$ to obtain a weight $w_f \in A_2(\partial\Omega, \sigma)$ with the property that $\sum_{j,k=1}^n |\partial_{\tau_{jk}}f| \in L^2(\partial\Omega, w_f)$. Together with Remark 2.4, this guarantees that $f \in \big[\dot{L}^2_1(\partial\Omega, w_f)\big]^M$. Granted the latter membership, Theorem 3.6 implies that $K_{A,\text{mod}}f \in \big[\dot{L}^2_1(\partial\Omega, w_f)\big]^M$. This observation allows us to introduce the well-defined family

$$\mathcal{F} := \Big\{ \Big(\sum_{j,k=1}^{n} |\partial_{\tau_{jk}} K_{A,\mathrm{mod}} f|, \sum_{j,k=1}^{n} |\partial_{\tau_{jk}} f| \Big) : f \in \big[(\dot{\mathbb{X}}_v)_1 \big]^M \Big\}. \tag{8.153}$$

We claim that $\|F_1\|_{L^2(\partial\Omega,w)} \leq C\|F_2\|_{L^2(\partial\Omega,w)}$ for every pair $(F_1, F_2) \in \mathcal{F}$ and every weight $w \in A_2(\partial\Omega, \sigma)$. As in the past, we may assume without loss of generality that the right-hand side is finite, in which case we have that the function $f \in \big[\dot{L}^2_1(\partial\Omega, w)\big]^M$ by Remark 2.4. Granted this, we may invoke Theorem 3.6 and the desired estimate follows. Also, by virtue of Theorem 8.1 we can conclude that $\|F_1\|_{\mathbb{X}_v} \leq C\|F_2\|_{\mathbb{X}_v}$ for every pair $(F_1, F_2) \in \mathcal{F}$. That is, the operator in (8.150) is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (8.89). This easily implies (8.151). Finally, (8.152) follows at once from Proposition 3.3, (8.88), and Proposition 8.2. □

8.3 Inverting Double Layer Operators on Weighted Banach Function Spaces

The following result is the counterpart of Theorem 4.2 in the more general setting of Banach function spaces.

Theorem 8.8 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Having picked some $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.154}$$

and pick some number $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$. Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$. Given a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, consider the principal-value singular integral operators $T, T^{\#}$ acting on each function $f \in \mathbb{X}_v$ according to

$$Tf(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y), \tag{8.155}$$

and

$$T^{\#} f(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle y - x, \nu(x) \rangle k(x-y) f(y) \, d\sigma(y), \tag{8.156}$$

at σ-a.e. point $x \in \partial\Omega$. In addition, define the action of the maximal operator T_* on each given function $f \in \mathbb{X}(w)$ as

$$T_* f(x) := \sup_{\varepsilon > 0} \Bigg| \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \langle x - y, \nu(y) \rangle k(x-y) f(y) \, d\sigma(y) \Bigg| \text{ for each } x \in \partial\Omega. \tag{8.157}$$

Then the following are well-defined, bounded operators

$$T_*, T, T^{\#} : \mathbb{X}_v \longrightarrow \mathbb{X}_v, \tag{8.158}$$

and for each $m \in \mathbb{N}$ there exists some $C_m \in (0, \infty)$, which depends only on m, n, Ξ, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has

$$\|T_*\|_{\mathbb{X}_v \to \mathbb{X}_v} \leq C_m \Big(\sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{8.159}$$

$$\|T\|_{\mathbb{X}_v \to \mathbb{X}_v} \leq C_m \Big(\sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{8.160}$$

$$\Big\| T^{\#} \Big\|_{\mathbb{X}_v \to \mathbb{X}_v} \leq C_m \Big(\sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|^{\langle m \rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{8.161}$$

Proof All claims are consequences of Theorem 4.2, Corollary 4.2, and Theorem 8.1 with $p_0 := 2$. □

In concert with the commutator estimates discussed earlier (cf. Theorem 8.2), Theorem 8.8 implies the following result, which is the weighted Banach function space counterpart of Theorem 4.6.

Corollary 8.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by $\nu = (\nu_k)_{1 \leq k \leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, having fixed $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. If $\mathcal{M}$ stands for the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}' f := \mathcal{M}(f v)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.162}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. *Finally, recall the boundary-to-boundary harmonic double layer potential operator* K_Δ *from* (3.29)*, the Riesz transforms* $\{R_j\}_{1\leq j\leq n}$ *from* (4.297)*, and for each* $k \in \{1,\dots,n\}$ *denote by* M_{ν_k} *the operator of pointwise multiplication by the* k*-th scalar component of* ν.

Then for each $m \in \mathbb{N}$ *there exists some* $C_m \in (0,\infty)$ *which depends only on* n, m, Ξ, *and the UR constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_\Delta\|_{\mathbb{X}_v\to\mathbb{X}_v} + \max_{1\leq j,k\leq n}\big\|[M_{\nu_k}, R_j]\big\|_{\mathbb{X}_v\to\mathbb{X}_v} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{8.163}$$

Proof We simply apply Theorem 8.8, (3.29), (4.297), Proposition 3.4, and Theorem 8.2. □

We shall revisit Corollary 8.1 later, in Theorem 8.15, which contains an estimate in the opposite direction to that obtained in (8.163).

In the next theorem we obtain operator norm estimates for double layer potentials associated with distinguished coefficient tensors on Banach function spaces, involving the BMO semi-norm of the unit normal to the boundary of the underlying domain as a factor.

Theorem 8.9 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a UR domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Also, let* L *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$. *Pick* $A \in \mathfrak{A}^{\mathrm{dis}}_L$ *and consider the boundary-to-boundary double layer potential operators* K_A, $K^\#_A$ *associated with* Ω *and the coefficient tensor* A *as in* (3.24) *and* (3.25)*, respectively. Also, having selected some* $v \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ *with* $v > 0$ *at* σ*-a.e. point on* $\partial\Omega$, *let* $\mathbb{X}_v$ *be a Banach function space over* $(\partial\Omega, v\sigma)$*, and let* $\mathbb{X}'_v$ *be its Köthe dual. With* $\mathcal{M}$ *denoting the Hardy–Littlewood maximal operator on* $(\partial\Omega,\sigma)$ *and* $\mathcal{M}'f := \mathcal{M}(fv)/v$ *for any* σ*-measurable function* f *on* $\partial\Omega$, *assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.164}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$.

Then for each $m \in \mathbb{N}$ *there exists some* $C_m \in (0,\infty)$ *which depends only on* m, n, A, Ξ, *and the UR constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\|K_A\|_{[\mathbb{X}_v]^M\to[\mathbb{X}_v]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{8.165}$$

$$\|K_A\|_{[(\mathbb{X}_v)_1]^M\to[(\mathbb{X}_v)_1]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{8.166}$$

and

$$\|K_A^{\#}\|_{[\mathbb{X}_v]^M \to [\mathbb{X}_v]^M} \leq C_m \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}^{\langle m \rangle}. \tag{8.167}$$

Proof All claims are justified as in the proof of Theorem 4.7, now making use of Theorem 8.8, Proposition 3.2, Theorem 8.2, (8.79)–(8.80), and Proposition 8.2. Another approach consists of extrapolating from Theorem 4.7 by means of Theorem 8.1 with $p_0 := 2$. □

We finish this section with a result that establishes invertibility for boundary-to-boundary double layer potential operators in Banach function spaces.

Theorem 8.10 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and consider the boundary-to-boundary double layer potential operators K_A, $K_A^{\#}$ associated with Ω and the coefficient tensor A as in (3.24) and (3.25), respectively. Next, having fixed some $v \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}_v'$ be its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, suppose that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}_v'. \tag{8.168}$$

Finally, pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}_v' \to \mathbb{X}_v'}$, and fix some number $\varepsilon \in (0,\infty)$.

Then there exists a threshold $\delta \in (0,1)$ which depends only on n, Ξ, A, ε, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following operators are invertible:

$$zI + K_A : [\mathbb{X}_v]^M \longrightarrow [\mathbb{X}_v]^M, \tag{8.169}$$

$$zI + K_A : [(\mathbb{X}_v)_1]^M \longrightarrow [(\mathbb{X}_v)_1]^M, \tag{8.170}$$

$$zI + K_A^{\#} : [\mathbb{X}_v]^M \longrightarrow [\mathbb{X}_v]^M. \tag{8.171}$$

Proof This is a direct consequence of Theorem 8.9, reasoning as in the proof of Theorem 4.8. □

Remark 8.4 In the context of Theorem 8.10, if the threshold $\delta \in (0,1)$ is taken sufficiently small so that the operator $zI + K_A$ is invertible on the space $[(\mathbb{X}_v)_1]^M$ we also claim that there exists some constant $C \in (0,\infty)$ with the property that

$$\begin{aligned} &\text{whenever } \ f \in \left[(\mathbb{X}_\nu)_1\right]^M \\ &\text{and } \ g := (zI + K_A)^{-1} f \in \left[(\mathbb{X}_\nu)_1\right]^M \\ &\text{then } \ \|\nabla_{\tan} g\|_{[\mathbb{X}_\nu]^{n\cdot M}} \leq C \|\nabla_{\tan} f\|_{[\mathbb{X}_\nu]^{n\cdot M}}. \end{aligned} \tag{8.172}$$

To justify this, we introduce the family

$$\mathcal{F} := \left\{ \left(|\nabla_{\tan} g|, |\nabla_{\tan} f|\right) : \ f \in \left[(\mathbb{X}_\nu)_1\right]^M \right\}, \tag{8.173}$$

and then invoke Theorem 8.1 with $p_0 := 2$ along with (4.333) and (4.343) with $p_1 = p_2 = 2$ and $w = w_1 = w_2 \in A_2(\partial\Omega, \sigma)$ (see also the proof of Theorem 4.2). There is, however, a subtle point here. Specifically, it is implicit in (4.343) that the choice of δ depends, among other thing, on $[w]_{A_2}$. Hence, without further provisions, asking for (4.343) to be valid for all weights in $A_2(\partial\Omega, \sigma)$ may result in the degenerate case $\delta = 0$. To avoid this undesirable outcome, let us recall that the hypothesis formulated in (8.12) as part of Theorem 8.1 needs, in fact, to hold only for those weights $w \in A_2(\partial\Omega, \sigma)$ with the property that $[w]_{A_2} \leq 4\|\mathcal{M}\|_{\mathbb{X}_\nu \to \mathbb{X}_\nu} \|\mathcal{M}'\|_{\mathbb{X}'_\nu \to \mathbb{X}'_\nu}$. In light of our choice $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_\nu \to \mathbb{X}_\nu} \|\mathcal{M}'\|_{\mathbb{X}'_\nu \to \mathbb{X}'_\nu}$, this means that it suffices to know that (4.343) is valid with $p_1 = p_2 = 2$ and $w = w_1 = w_2 \in A_2(\partial\Omega, \sigma)$ with $[w]_{A_2} \leq \Xi$. In turn, Proposition 4.2 (or Theorem 4.8 for this matter) allows us to choose δ depending on Ξ (and the other various parameters) so that (4.343) holds in the context just described.

We conclude this section with the following observation.

Remark 8.5 In the two-dimensional setting, more can be said about the Lamé system. Specifically, the versions of Theorem 4.14 and Corollary 4.3 naturally formulated in terms of weighted Banach function spaces, as well as their associated Sobolev spaces, continue to hold, virtually with the same proofs (now making use of Theorem 8.2, Propositions 8.2, 8.3, Theorems 8.3, and 8.8).

8.4 Invertibility on Homogeneous Weighted Banach Function-Based Sobolev Spaces

Our next goal is to present a version of Theorem 4.10 valid in the context of homogeneous weighted Banach function-based Sobolev spaces, where again the key feature is the explicit dependence on the BMO semi-norm of the geometric measure theoretic outward unit normal to the underlying domain.

Theorem 8.11 *Let $\Omega \subseteq \mathbb{R}^n$ be a two-sided NTA domain whose boundary is an unbounded Ahlfors regular set. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, having chosen some*

$\upsilon \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $\upsilon > 0$ at σ-a.e. point on $\partial\Omega$, consider a Banach function space $\mathbb{X}_\upsilon$ over $(\partial\Omega, \upsilon\sigma)$ and let $\mathbb{X}'_\upsilon$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f := \mathcal{M}(f\upsilon)/\upsilon$ for any σ-measurable function f on $\partial\Omega$, assume that

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_\upsilon \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_\upsilon, \tag{8.174}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_\upsilon\to\mathbb{X}_\upsilon}\|\mathcal{M}'\|_{\mathbb{X}'_\upsilon\to\mathbb{X}'_\upsilon}$. Next, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}^{\rm dis}_L \neq \varnothing$. Finally, pick $A \in \mathfrak{A}^{\rm dis}_L$ and consider the modified boundary-to-boundary double layer potential operator $\big[K_{A,{\rm mod}}\big]$ associated with Ω and the coefficient tensor A as in Theorem 8.7.

Then for each $m \in \mathbb{N}$ there exists some $C_m \in (0,\infty)$ which depends only on m, n, Ξ, A, the two-sided NTA constants of Ω, and the Ahlfors regularity constant of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has

$$\big\|\big[K_{A,{\rm mod}}\big]\big\|_{[(\dot{\mathbb{X}}_\upsilon)_1/\sim]^M\to[(\dot{\mathbb{X}}_\upsilon)_1/\sim]^M} \leq C_m\|\nu\|^{\langle m\rangle}_{[{\rm BMO}(\partial\Omega,\sigma)]^n}. \tag{8.175}$$

Proof We proceed much as in the proof of Theorem 4.10, making use of (8.151), (8.152), Theorems 8.8 and 8.2. We can also use another approach, using Theorem 8.1 with $p_0 := 2$, the family $\mathcal{F}$ consisting of the pairs

$$\Big(\sum_{j,k=1}^n |\partial_{\tau_{jk}}K_{A,{\rm mod}}f|, \sum_{j,k=1}^n |\partial_{\tau_{jk}}f|\Big), \quad \text{with } f \in \big[(\dot{\mathbb{X}}_\upsilon)_1\big]^M, \tag{8.176}$$

and where (8.12) now comes from Theorem 4.10. □

At this stage, we are ready to state our main result concerning the invertibility properties of boundary-to-boundary double layer potential operators associated with distinguished coefficient tensors on homogeneous weighted Banach function-based Sobolev spaces.

Theorem 8.12 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}^{\rm dis}_L \neq \varnothing$. Pick $A \in \mathfrak{A}^{\rm dis}_L$ and consider the modified boundary-to-boundary double layer potential operator $\big[K_{A,{\rm mod}}\big]$ associated with Ω and the coefficient tensor A as in Theorem 7.6. Also, having fixed $\upsilon \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $\upsilon > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_\upsilon$ be a Banach function space over $(\partial\Omega,\upsilon\sigma)$, and let $\mathbb{X}'_\upsilon$ be its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f := \mathcal{M}(f\upsilon)/\upsilon$ for any σ-measurable function f on $\partial\Omega$, make the assumption that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_\upsilon \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_\upsilon. \tag{8.177}$$

Finally, pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$*, and fix some number* $\varepsilon \in (0,\infty)$.

Then there exists some small threshold $\delta \in (0,1)$ *which depends only on* n*,* Ξ*,* A*,* ε*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) it follows that for each spectral parameter* $z \in \mathbb{C}$ *with* $|z| \geq \varepsilon$ *the operator*

$$zI + \big[K_{A,\mathrm{mod}}\big] : \big[[(\dot{\mathbb{X}}_v)_1/\sim\big]^M \longrightarrow \big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M \tag{8.178}$$

is invertible.

Proof By Theorem 2.3 we can pick $\delta \in (0,1)$ small enough so that if one assumes that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then Ω is a two-sided NTA domain with an unbounded boundary. Then the desired invertibility result follows (via a Neumann series argument) from Theorem 8.11. □

Remark 8.6 The conclusions in Theorem 8.12 may fail when $A \notin \mathfrak{A}_L^{\mathrm{dis}}$ even when Ω is a half-space. For example, Proposition 3.13, Theorem 8.6, and Proposition 8.2 imply that in such a case it may happen that $\frac{1}{2}I+\big[K_{A,\mathrm{mod}}\big]$ has an infinite dimensional cokernel when acting on homogeneous weighted Banach function-based Sobolev spaces.

To continue our discussion we next consider the invertibility of the modified boundary-to-boundary single layer potential operator in quotient homogeneous weighted Banach function-based Sobolev spaces.

Theorem 8.13 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, let* L *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$*. Consider the modified boundary-to-boundary single layer potential operator* S_{mod} *associated with* Ω *and the system* L *as in* (3.42)*. Choose* $v \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ *with* $v > 0$ *at* σ*-a.e. point on* $\partial\Omega$*. Let* $\mathbb{X}_v$ *be a Banach function space over* $(\partial\Omega, v\sigma)$ *and let* $\mathbb{X}'_v$ *be its Köthe dual. With* $\mathcal{M}$ *denoting the Hardy–Littlewood maximal operator on* $(\partial\Omega,\sigma)$ *and* $\mathcal{M}'f := \mathcal{M}(fv)/v$ *for any* σ*-measurable function* f *on* $\partial\Omega$*, assume that*

$$mathcal M \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.179}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$*. Finally, let* $\big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M$ *denote the* M*-th power of the quotient space of classes* $[\,\cdot\,]$ *of equivalence modulo constants of functions in* $(\dot{\mathbb{X}}_v)_1$*, equipped with the semi-norm* (8.91).

Then the following statements are valid.

(1) [Boundedness] If Ω *satisfies a two-sided local John condition then the operator*

$$\begin{gathered}\big[S_{\rm mod}\big] : \big[\mathbb{X}_v\big]^M \longrightarrow \big[(\dot{\mathbb{X}}_v)_1 / \sim\big]^M \ \ \textit{defined as} \\ \big[S_{\rm mod}\big] f := \big[S_{\rm mod} f\big] \in \big[(\dot{\mathbb{X}}_v)_1 / \sim\big]^M, \quad \forall\, f \in \big[\mathbb{X}_v\big]^M\end{gathered} \tag{8.180}$$

is well defined, linear, and bounded.

(2) [Surjectivity] Whenever $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, Ξ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) it follows that (8.91) is a genuine norm and the operator (8.180) is surjective.

(3) [Injectivity] Whenever $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, Ξ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) it follows that the operator (8.180) is injective.

(4) [Isomorphism] Whenever both $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$, there exists some small threshold $\delta \in (0,1)$ which depends only on n, Ξ, L, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) it follows that $\big[(\dot{\mathbb{X}}_v)_1 / \sim\big]^M$ is a Banach space when equipped with the norm (8.91) and the operator (8.180) is an isomorphism.

(5) [Optimality] If $\mathfrak{A}_L^{\rm dis} = \varnothing$ then the operator $\big[S_{\rm mod}\big]$ may fail to be surjective (in fact, may have an infinite dimensional cokernel) in all settings considered above even in the case when Ω is a half-space, and if $\mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$ then the operator $\big[S_{\rm mod}\big]$ may fail to be injective (in fact, may have an infinite dimensional kernel) in all settings considered above even in the case when Ω is a half-space.

Proof We first note that the operator (8.180) is well defined, linear, and bounded by item *(2)* in Theorem 8.5, bearing in mind (2.87) and (2.48). This takes care of item *(1)*.

To deal with the claims in item *(2)*, pick a coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$. Having fixed some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}_v' \to \mathbb{X}_v'}$, Theorems 2.3, 4.8, and 8.10 guarantee that we may choose $\delta \in (0,1)$ small enough so that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a condition which we shall henceforth assume) then

$$\Omega \text{ is a two-sided NTA domain with an unbounded boundary} \tag{8.181}$$

(hence, in particular, Ω satisfies a two-sided local John condition), and

$$\begin{gathered}\text{the operators } \pm\tfrac{1}{2}I + K_A \text{ are invertible on } \big[(\mathbb{X}_v)_1\big]^M \text{ and on} \\ \big[L_1^2(\partial\Omega, w)\big]^M \ \text{ for all } \ w \in A_2(\partial\Omega,\sigma) \ \text{ with } \ [w]_{A_2} \leq \Xi.\end{gathered} \tag{8.182}$$

This choice of δ allows us to run the argument in the proof of Theorem 4.11 for any $g \in \big[\dot{L}_1^2(\partial\Omega, w)\big]^M$ and any $w \in A_2(\partial\Omega,\sigma)$ with $[w]_{A_2} \leq \Xi$. We also wish to note that (8.91) is a genuine norm, thanks to (8.181) and Proposition 8.4.

To proceed, choose a scalar-valued function $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $\phi \equiv 1$ on $B(0,1)$ and $\operatorname{supp}\phi \subseteq B(0,2)$. Having fixed a reference point $x_0 \in \partial\Omega$, for each scale

$r \in (0, \infty)$ define ϕ_r as in (8.92) and use the same notation to denote the restriction of ϕ_r to $\partial\Omega$. Assume next that some arbitrary function $g \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$ has been given, and for each $r \in (0, \infty)$ define g_r as in (8.93).

On the other hand, we apply Proposition 8.1 to the function $\sum_{j,k=1}^n |\partial_{\tau_{jk}} g| \in \mathbb{X}_v$ to obtain $w_g \in A_2(\partial\Omega, \sigma)$ with $[w]_{A_2} \leq \Xi$ so that $\sum_{j,k=1}^n |\partial_{\tau_{jk}} g| \in L^2(\partial\Omega, w_g)$. This and Remark 2.4 imply that $g \in \big[\dot{L}^2_1(\partial\Omega, w_g)\big]^M$. As alluded to above, we proceed as in the proof of Theorem 4.11 and define

$$f_r^g := \partial_\nu^A \Big(\mathcal{D}_A\big(\tfrac{1}{2}I + K_A\big)^{-1}\big(-\tfrac{1}{2}I + K_A\big)^{-1} g_r\Big) \in \big[L^2(\partial\Omega, w_g)\big]^M. \tag{8.183}$$

One can then find a sequence $\{r_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ with $r_j \to \infty$ as $j \to \infty$ along with some $f^g \in \big[L^2(\partial\Omega, w_g)\big]^M$ having the following properties. First, $\{f_{r_j}^g\}_{j\in\mathbb{N}}$ is weak-$*$ convergent to f^g. Second, there exists a constant $c^g \in \mathbb{C}^M$ such that

$$S_{\text{mod}} f^g = g_r + c^g \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{8.184}$$

Third,

$$\|f^g\|_{[L^2(\partial\Omega, w_g)]^M} \leq C\|\nabla_{\tan} g\|_{[L^2(\partial\Omega, w_g)]^{n\cdot M}} \tag{8.185}$$

for some constant $C \in (0, \infty)$ independent of g. We next claim that

$$\begin{gathered} \|f^g\|_{[L^2(\partial\Omega, w)]^M} \leq C\|\nabla_{\tan} g\|_{[L^2(\partial\Omega, w)]^{n\cdot M}}, \quad \text{for every} \quad g \in \big[(\dot{\mathbb{X}}_v)_1\big]^M \\ \text{and for every} \quad w \in A_2(\partial\Omega, \sigma) \quad \text{with} \quad [w]_{A_2} \leq \Xi. \end{gathered} \tag{8.186}$$

In this vein, observe that (8.185) guarantees that the estimate in the first line of (8.186) holds with $w = w_g$. Our goal here is to extend that estimate to a bigger family of weights. To this end, fix $w \in A_2(\partial\Omega, \sigma)$ with $[w]_{A_2} \leq \Xi$. We may assume that $\|\nabla_{\tan} g\|_{[L^2(\partial\Omega, w)]^{n\cdot M}} < \infty$, otherwise there is nothing to prove. In such a case (8.88) and Remark 2.4 imply that $g \in \big[\dot{L}^2_1(\partial\Omega, w)\big]^M$. Since $[w]_{A_2} \leq \Xi$, we may repeat the argument in the proof of Theorem 4.11. Specifically, we start by defining

$$\widetilde{f}_r^g := \partial_\nu^A \Big(\mathcal{D}_A\big(\tfrac{1}{2}I + K_A\big)^{-1}\big(-\tfrac{1}{2}I + K_A\big)^{-1} g_r\Big) \in \big[L^2(\partial\Omega, w)\big]^M. \tag{8.187}$$

Next, we extract a sub-sequence of $\{r_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ (for which we retain the same notation) which converges to ∞ and has the property that $\{\widetilde{f}_{r_j}^g\}_{j\in\mathbb{N}}$ is weak-$*$ convergent to some $\widetilde{f}^g \in \big[L^2(\partial\Omega, w)\big]^M$ satisfying

$$\|\widetilde{f}^g\|_{[L^2(\partial\Omega, w)]^M} \leq C\|\nabla_{\tan} g\|_{[L^2(\partial\Omega, w)]^{n\cdot M}} \tag{8.188}$$

for some constant $C \in (0, \infty)$ independent of g. Proposition 4.2 then allows us to conclude that $f_r^g = \widetilde{f}_r^g$. Recall also that $\{f_{r_j}^g\}_{j\in\mathbb{N}}$ is weak-$*$ convergent to some f^g in $\big[L^2(\partial\Omega, w_g)\big]^M$ and that $\{\widetilde{f}_{r_j}^g\}_{j\in\mathbb{N}}$ is weak-$*$ convergent to some $\widetilde{f}^g$ in $\big[L^2(\partial\Omega, w)\big]^M$. Collectively, these properties permit us to write, for each test function $\psi \in \big[\mathrm{Lip}(\partial\Omega)\big]^M$ with compact support,

$$\int_{\partial\Omega} \langle f^g, \psi\rangle \, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} \langle f_{r_j}, \psi\rangle \, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} \langle \widetilde{f}_{r_j}, \psi\rangle \, d\sigma$$

$$= \int_{\partial\Omega} \langle \widetilde{f}^g, \psi\rangle \, d\sigma. \tag{8.189}$$

This readily yields $f^g = \widetilde{f}^g$ at σ-a.e. point on $\partial\Omega$. In concert with (8.188), this gives the desired estimate in (8.186).

Once this has been established, we are in a position to invoke Theorem 8.1 to obtain

$$\|f^g\|_{[\mathbb{X}_v]^M} \leq C\|\nabla_{\tan} g\|_{[\mathbb{X}_v]^{n\cdot M}}. \tag{8.190}$$

In particular, f^g belongs to $[\mathbb{X}_v]^M$ which, in light of (8.184), goes to show that the operator (8.180) is indeed surjective.

Consider next the claim in item *(3)*. Pick a coefficient tensor $\widetilde{A} \in \mathfrak{A}_L$ such that $\widetilde{A}^\top \in \mathfrak{A}_{L^\top}^{\mathrm{dis}}$. By Theorem 8.10 we may then choose $\delta \in (0, 1)$ small enough so that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (something we shall henceforth assume) then

$$\text{the operators } \pm\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top} \text{ are invertible on } \big[\mathbb{X}_v\big]^M. \tag{8.191}$$

To show that the operator (8.180) is injective, let $f \in \big[\mathbb{X}_v\big]^M$ be a function with the property that $\big[S_{\mathrm{mod}}\big]f = [0]$. Hence, $\big[S_{\mathrm{mod}} f\big] = [0]$ which implies that there exists some constant $c \in \mathbb{C}^M$ for which

$$S_{\mathrm{mod}} f = c \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{8.192}$$

This, together with (8.131), allows us to obtain

$$\big(\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{\widetilde{A}^\top}\big)f\Big) = 0 \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega \tag{8.193}$$

which, by (8.191), leads to $f = 0$. Since the operator (8.180) is linear, it follows that this is indeed injective.

For item *(4)*, we just observe that if $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, the previous items show that the operator (8.180) is a continuous bijection. Moreover, Proposition 8.4

and (8.181) imply that $\big[(\dot{\mathbb{X}}_v)_1\big/\sim\big]^M$ is a Banach space, hence the operator (8.180) is a linear isomorphism.

Finally, the optimality results in item *(5)* are seen from (3.406) and the natural version of Proposition 4.4 for weighted Banach function spaces. □

We conclude this section with the following theorem addressing the issue of invertibility for the conormal of the double layer operator acting from homogeneous weighted Banach function-based Sobolev spaces.

Theorem 8.14 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Select $v \in L^1_{\rm loc}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$. Let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$ and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.194}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Pick some coefficient tensor $A \in \mathfrak{A}_L$ and consider the modified conormal derivative of the modified double layer operator in the context of (8.143)*, i.e.,*

$$\begin{gathered} \partial^A_\nu \mathcal{D}_{A,\mathrm{mod}} : \big[(\dot{\mathbb{X}}_v)_1\big/\sim\big]^M \longrightarrow \big[\mathbb{X}_v\big]^M \ \ \text{defined as} \\ \big(\partial^A_\nu \mathcal{D}_{A,\mathrm{mod}}\big)[f] := \partial^A_\nu(\mathcal{D}_{A,\mathrm{mod}} f) \ \ \text{for each} \ \ f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M. \end{gathered} \tag{8.195}$$

From Theorem 8.6 this is known to be a well-defined, linear, and bounded operator when the quotient space is equipped with the norm (8.91)*. In relation to this, the following statements are valid.*

(1) [Injectivity] Whenever $\mathfrak{A}^{\rm dis}_L \neq \varnothing$ and actually $A \in \mathfrak{A}^{\rm dis}_L$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, Ξ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then the operator (8.195) *is injective.*

(2) [Surjectivity] Whenever $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ and actually $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, Ξ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then the operator (8.195) *is surjective.*

(3) [Isomorphism] If $\mathfrak{A}^{\rm dis}_L \neq \varnothing$, $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$, and $A \in \mathfrak{A}^{\rm dis}_L$ is such that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$ it follows that there exists some small threshold $\delta \in (0,1)$ which depends only on n, Ξ, A, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then the operator (8.195) *is an isomorphism.*

Proof All claims may be established by arguing as in the proof of Theorem 4.13, now making use of Theorems 8.6, 8.10, and 8.12. □

8.5 Characterizing Flatness in Terms of Weighted Banach Functions Spaces

In this section we characterize the flatness of a "surface" in terms of the size of the norms of certain singular integral operators acting on weighted Banach functions spaces on this surface.

In order to be able to elaborate on this topic, we need some notation. Given a UR domain $\Omega \subseteq \mathbb{R}^n$, denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, having fixed $v \in L^1_{\rm loc}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.196}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. From Theorem 8.1 with $p_0 := 2$ and (5.16)–(5.18) we then conclude that

$$\mathbf{C} : \mathbb{X}_v \otimes \mathcal{C}\!\ell_n \longrightarrow \mathbb{X}_v \otimes \mathcal{C}\!\ell_n, \tag{8.197}$$

$$\mathbf{C}^{\#} : \mathbb{X}_v \otimes \mathcal{C}\!\ell_n \longrightarrow \mathbb{X}_v \otimes \mathcal{C}\!\ell_n, \tag{8.198}$$

are all well defined, linear, and continuous, with

$$\begin{gathered} \|\mathbf{C}\|_{\mathbb{X}_v\otimes\mathcal{C}\!\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\!\ell_n},\ \ \|\mathbf{C}^{\#}\|_{\mathbb{X}_v\otimes\mathcal{C}\!\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\!\ell_n} \text{ bounded} \\ \text{exclusively in terms of } n,\ \Xi, \text{ and the UR constants of } \partial\Omega. \end{gathered} \tag{8.199}$$

In addition, from (5.20), the explanation right after it, and Proposition 8.2 we conclude that

$$\text{the operator identities } \mathbf{C}^2 = \tfrac{1}{4}I \text{ and } \big(\mathbf{C}^{\#}\big)^2 = \tfrac{1}{4}I \text{ are valid on } \mathbb{X}_v \otimes \mathcal{C}\!\ell_n. \tag{8.200}$$

Our next goal is to establish an estimate of the antisymmetric part of the Cauchy–Clifford operator, i.e., for the difference $\mathbf{C} - \mathbf{C}^{\#}$, in terms of the oscillation of the outer unit normal.

Proposition 8.6 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, having* *picked some $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. If $\mathcal{M}$ stands for the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.201}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Then for each $m \in \mathbb{N}$ there exists some constant $C_m \in (0,\infty)$ which depends only on m, n, Ξ, and the UR constants of $\partial\Omega$ such that, with the piece of notation introduced in (4.93), one has

$$\left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{\mathbb{X}_v\otimes\mathcal{C}\!\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\!\ell_n} \leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{8.202}$$

Proof The desired estimate follows from Lemma 5.1 (bearing in mind Proposition 8.2), together with Corollary 8.1. □

We can also establish bounds from below for the operator norm of $\mathbf{C} - \mathbf{C}^{\#}$ on weighted Banach function spaces, considered on the boundary of a UR domain, in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal vector to the said domain.

Proposition 8.7 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain such that $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Choose $v \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$. Let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$ and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and with $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.203}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Then there exists some $C \in (0,\infty)$ which depends only on n, Ξ, and the Ahlfors regularity constant of $\partial\Omega$ with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{\mathbb{X}_v\otimes\mathcal{C}\!\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\!\ell_n}. \tag{8.204}$$

Proof The argument largely follows the proof Theorem 5.1 but some changes are required. To proceed, we begin by observing that, for every σ-measurable function h on $\partial\Omega$, Lemma 2.12 implies that for every surface ball $\Delta \subset \partial\Omega$ and every weight $w \in A_2(\partial\Omega, \sigma)$ we have

$$\left\|\left(\fint_\Delta |h|\,d\sigma\right)\mathbf{1}_\Delta\right\|_{L^2(\partial\Omega,w)} \leq [w]^{1/2}_{A_2}\|h\,\mathbf{1}_\Delta\|_{L^2(\partial\Omega,w)}. \tag{8.205}$$

Invoking Theorem 8.1, we therefore obtain

$$\Big(\fint_{\Delta} |h|\,\mathrm{d}\sigma\Big)\|\mathbf{1}_\Delta\|_{\mathbb{X}_v} \leq 8\,\Xi^{1/2}\|h\,\mathbf{1}_\Delta\|_{\mathbb{X}_v}. \tag{8.206}$$

Thus, by (8.7) and (8.6) (with $\mathrm{d}\mu = \mathrm{d}v = v\,\mathrm{d}\sigma$) we may write

$$\begin{aligned}
\sigma(\Delta) &= \int_{\partial\Omega} \mathbf{1}_\Delta\, v^{-1}\, v\,\mathrm{d}\sigma \leq \|\mathbf{1}_\Delta\|_{\mathbb{X}_v}\|v^{-1}\mathbf{1}_\Delta\|_{\mathbb{X}'_v} \\
&= \|\mathbf{1}_\Delta\|_{\mathbb{X}_v}\sup_h\Big(\int_{\partial\Omega} v^{-1}\,\mathbf{1}_\Delta\,|h|\,v\,\mathrm{d}\sigma\Big) \\
&\leq 8\,\Xi^{1/2}\sigma(\Delta)\sup_h\|h\,\mathbf{1}_\Delta\|_{\mathbb{X}_v} \leq 8\,\Xi^{1/2}\sigma(\Delta),
\end{aligned} \tag{8.207}$$

where the suprema are taken over all function $h \in \mathbb{X}_v$ with $\|h\|_{\mathbb{X}_v} \leq 1$.

Bearing in mind (5.25)–(5.28), we may reason as in (5.29)–(5.43) and, for every point $x \in \Delta(x_0, R)$, write

$$\begin{aligned}
&\Big|\int_{\Delta(y_0,R)}\Big\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x_0-y}{|x_0-y|^n}\Big\}\,\mathrm{d}\sigma(y)\Big| \\
&\qquad\leq C\Lambda^{-n}\,\ln\Lambda\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}+\omega_{n-1}\,|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)| \\
&\qquad\qquad + C\Lambda^{-n}\big|\nu(x)-\nu_{\Delta(x_0,R)}\big|.
\end{aligned} \tag{8.208}$$

Consequently,

$$\begin{aligned}
&\fint_{\Delta(x_0,R)}\Big|\int_{\Delta(y_0,R)}\Big\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x_0-y}{|x_0-y|^n}\Big\}\,\mathrm{d}\sigma(y)\Big|\,\mathrm{d}\sigma(x) \\
&\leq C(\Lambda^{-n}\ln\Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}+C\fint_{\Delta(x_0,R)}|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|\,\mathrm{d}\sigma(x) \\
&\quad + C\Lambda^{-n}\fint_{\Delta(x_0,R)}\big|\nu(x)-\nu_{\Delta(x_0,R)}\big|\,\mathrm{d}\sigma(x) \\
&\leq C(\Lambda^{-n}\ln\Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \\
&\quad + C\fint_{\Delta(x_0,R)}|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|\,\mathrm{d}\sigma(x),
\end{aligned} \tag{8.209}$$

where $C \in (0, \infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. Based on (8.206) and (8.207) we may estimate the last term in the right-hand side as follows:

$$\begin{aligned}
&\int_{\Delta(x_0,R)} |(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}(x)|\,d\sigma(x) \\
&\quad \leq 8\,\Xi^{1/2}\big\|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,R)}\big\|_{\mathbb{X}_v}\frac{\sigma\big(\Delta(x_0,R)\big)}{\|\mathbf{1}_{\Delta(x_0,R)}\|_{\mathbb{X}_v}} \\
&\quad \leq 64\,\Xi\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n}\,\|\mathbf{1}_{\Delta(y_0,R)}\|_{\mathbb{X}_v}\,\|v^{-1}\,\mathbf{1}_{\Delta(x_0,R)}\|_{\mathbb{X}'_v} \\
&\quad \leq 64\,\Xi\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n}\,\|\mathbf{1}_{\Delta(x_0,2\lambda\Lambda R)}\|_{\mathbb{X}_v}\,\|v^{-1}\,\mathbf{1}_{\Delta(x_0,2\lambda\Lambda R)}\|_{\mathbb{X}'_v} \\
&\quad \leq 512\,\Xi^{3/2}\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n}\,\sigma\big(\Delta(x_0,2\lambda\Lambda R)\big) \\
&\quad \leq C\,\Xi^{3/2}\,\Lambda^{n-1}\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n}\,\sigma\big(\Delta(x_0,R)\big),
\end{aligned} \tag{8.210}$$

where $C \in (0, \infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. Collecting (8.209) and (8.210) we arrive at

$$\begin{aligned}
&⨍_{\Delta(x_0,R)}\bigg|\int_{\Delta(y_0,R)}\bigg\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x_0-y}{|x_0-y|^n}\bigg\}\,d\sigma(y)\bigg|\,d\sigma(x) \\
&\leq C(\Lambda^{-n}\ln\Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \\
&\quad + C\,\Xi^{3/2}\,\Lambda^{n-1}\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n},
\end{aligned} \tag{8.211}$$

where $C \in (0, \infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. With (8.211) in hand, the same type of argument as in the end-game of the proof of Theorem 5.1 (cf. (5.47)–(5.54)) with $p = 1$ gives

$$\begin{aligned}
\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} &\leq C(\Lambda^{-1}\ln\Lambda)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \\
&\quad + C\,\Xi^{3/2}\Lambda^{2(n-1)}\|\mathbf{C}-\mathbf{C}^{\#}\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n},
\end{aligned} \tag{8.212}$$

where $C \in (0, \infty)$ depends only on n and the Ahlfors regularity constant of $\partial\Omega$. By eventually further increasing the value of Λ as to ensure that $\Lambda^{-1}\ln\Lambda < 1/(2C)$, we finally conclude from (8.212) that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\big\|\mathbf{C}-\mathbf{C}^{\#}\|_{\mathbb{X}_v\otimes\mathcal{C}\ell_n\to\mathbb{X}_v\otimes\mathcal{C}\ell_n}, \tag{8.213}$$

where $C \in (0, \infty)$ depends only on n, Ξ, and the Ahlfors regularity constant of $\partial\Omega$. □

The next result contains estimates in the opposite direction to those presented in Corollary 8.1.

Theorem 8.15 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, having fixed $v \in L^1_{\rm loc}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the standard Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and with $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.214}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Finally, recall the boundary-to-boundary harmonic double layer potential operator K_Δ from (3.29), the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from (4.297), and for each index $k \in \{1,\dots,n\}$ denote by M_{ν_k} the operator of pointwise multiplication by the k-th scalar component of ν. Then there exists some $C \in (0,\infty)$ which depends only on n, Ξ, and the Ahlfors regularity constant of $\partial\Omega$ with the property that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{\|K_\Delta\|_{\mathbb{X}_v\to\mathbb{X}_v} + \max_{1\leq j,k\leq n}\big\|[M_{\nu_k}, R_j]\big\|_{\mathbb{X}_v\to\mathbb{X}_v}\Big\}. \tag{8.215}$$

Proof If $\partial\Omega$ is unbounded then all estimates are implied by Proposition 8.7 and the structural result from Lemma 5.1 (keeping in mind Proposition 8.2). When the set $\partial\Omega$ is bounded, we have $K_\Delta 1 = \pm\frac{1}{2}$ (cf. [114, §1.5]) with the sign plus if Ω is bounded, and the sign minus if Ω is unbounded, hence $\|K_\Delta\|_{\mathbb{X}_v\to\mathbb{X}_v} \geq \frac{1}{2}$ in such a case. Given that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1$ (cf. (2.118)), the desired estimate is valid in this case if we take $C \geq 2$. □

In turn, the results established in Theorem 8.15 may be used to characterize the class of δ-AR domains in $\mathbb{R}^n$, in the spirit of Corollary 5.2, now using weighted Banach function spaces.

Theorem 8.16 discussed next may be regarded as a stability result stating that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain with an unbounded boundary for which the URTI (cf. (5.58)) are "almost" true in the context of weighted Banach function spaces, then $\partial\Omega$ is "almost" flat, in the sense that the BMO semi-norm of the outward unit normal to Ω is small.

Theorem 8.16 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain with an unbounded boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Having chosen some $v \in L^1_{\rm loc}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.216}$$

pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$*, and recall from* (4.297) *the Riesz transforms* $\{R_j\}_{1\leq j\leq n}$ *on* $\partial\Omega$*. Then there exists some* $C\in(0,\infty)$ *which depends only on* n*,* Ξ*, and the UR constants of* $\partial\Omega$ *with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\leq C\Big\{\Big\|I+\sum_{j=1}^n R_j^2\Big\|_{\mathbb{X}_v\to\mathbb{X}_v}+\max_{1\leq j,k\leq n}\big\|[R_j,R_k]\big\|_{\mathbb{X}_v\to\mathbb{X}_v}\Big\}. \tag{8.217}$$

Proof A key ingredient is the operator identity

$$\mathbf{C}-\mathbf{C}^{\#}=\mathbf{C}\Big(I+\sum_{j=1}^n R_j^2\Big)+\sum_{1\leq j<k\leq n}\mathbf{C}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k \tag{8.218}$$

on the space $\mathbb{X}_v\otimes\mathcal{C}\!\ell_n$. This has been obtained in the proof of Theorem 5.3 for each function $f\in L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n$ with $1<p<\infty$ and $w\in A_p(\partial\Omega,\sigma)$. In turn Proposition 8.2 readily imply the desired equality in $\mathbb{X}_v\otimes\mathcal{C}\!\ell_n$. Once (8.218) has been established, we may rely on Proposition 8.7 and (8.197) to conclude (much as in the proof of Theorem 5.3) that the estimate claimed in (8.217) is true. □

The last result in this section contains estimates in the opposite direction to those from Theorem 8.16. Together, Theorem 8.17 and Theorem 8.16 amount to saying that, under natural background geometric assumptions on the set Ω, the URTI are "almost" true on weighted Banach function spaces if and only if $\partial\Omega$ is "almost" flat (in that the BMO semi-norm of the outward unit normal to Ω is small).

Theorem 8.17 *Let* $\Omega\subseteq\mathbb{R}^n$ *be a UR domain. Abbreviate* $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Fix a function* $v\in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ *with* $v>0$ *at* σ*-a.e. point on* $\partial\Omega$*. Let* $\mathbb{X}_v$ *be a Banach function space over* $(\partial\Omega,v\sigma)$ *and let* $\mathbb{X}'_v$ *be its Köthe dual. With* $\mathcal{M}$ *denoting the Hardy–Littlewood maximal operator on* $(\partial\Omega,\sigma)$ *and* $\mathcal{M}'f:=\mathcal{M}(fv)/v$ *for any* σ*-measurable function* f *on* $\partial\Omega$*, assume that*

$$\mathcal{M}\text{ is bounded on }\mathbb{X}_v\text{ and }\mathcal{M}'\text{ is bounded on }\mathbb{X}'_v, \tag{8.219}$$

pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$*, and recall from* (4.297) *the Riesz transforms* $\{R_j\}_{1\leq j\leq n}$ *on* $\partial\Omega$*.*

Then for each $m\in\mathbb{N}$ *there exists some constant* $C_m\in(0,\infty)$ *which depends only on* m*,* n*,* Ξ*, and the UR constants of* $\partial\Omega$ *such that, with the piece of notation introduced in* (4.93)*, one has*

$$\Big\|I+\sum_{j=1}^n R_j^2\Big\|_{\mathbb{X}_v\to\mathbb{X}_v}\leq C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{8.220}$$

and

$$\max_{1\le j<k\le n}\big\|[R_j,R_k]\big\|_{\mathbb{X}_v\to\mathbb{X}_v}\le C_m\|\nu\|^{\langle m\rangle}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{8.221}$$

Proof The starting point is to observe that we have the operator identity

$$\mathbf{C}(\mathbf{C}^{\#}-\mathbf{C})=-\tfrac14\Big(I+\sum_{j=1}^{n}R_j^2\Big)-\tfrac14\sum_{1\le j<k\le n}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k, \tag{8.222}$$

on $\mathbb{X}_v$ which is itself a consequence of (8.218) and (8.200). With (8.222) in hand, the estimates claimed in the statement of the theorem may then be justified via an estimate similar in spirit to (5.66), and also invoking Proposition 8.6 as well as (8.199). □

8.6 Boundary Value Problems in Weighted Banach Function Spaces

This section is devoted to studying boundary value problems for weakly elliptic systems in δ-AR domains with boundary data in weighted Banach function spaces. We start by discussing the Dirichlet Problem.

Theorem 8.18 *Let $\Omega\subseteq\mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa>0$. Having selected some $v\in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ with $v>0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega,v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f:=\mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M}\text{ is bounded on }\mathbb{X}_v\text{ and }\mathcal{M}'\text{ is bounded on }\mathbb{X}'_v, \tag{8.223}$$

and fix some $\Xi\ge 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathcal{M}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system L in $\mathbb{R}^n$, consider the Dirichlet Problem

$$\begin{cases} u\in\big[\mathscr{C}^\infty(\Omega)\big]^M,\\ Lu=0 \ \ in\ \ \Omega,\\ \mathcal{N}_\kappa u\in\mathbb{X}_v,\\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}=f\in\big[\mathbb{X}_v\big]^M. \end{cases} \tag{8.224}$$

Then the following claims are true:

(a) *[Existence, Regularity, and Estimates] If* $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ *and* $A \in \mathfrak{A}_L^{\text{dis}}$*, then there exists* $\delta \in (0,1)$ *depending only on n,* Ξ*, A, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then* $\frac{1}{2}I + K_A$ *is an invertible operator on* $\big[\mathbb{X}_v\big]^M$ *and the function* $u : \Omega \to \mathbb{C}^M$ *defined as*

$$u(x) := \Big(\mathcal{D}_A\Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x) \ \text{ for all } \ x \in \Omega \tag{8.225}$$

is a solution of the Dirichlet Problem (8.224)*. Moreover,*

$$\|\mathcal{N}_\kappa u\|_{\mathbb{X}_v} \approx \|f\|_{[\mathbb{X}_v]^M}. \tag{8.226}$$

Furthermore, the function u defined in (8.225) *satisfies the following regularity result*

$$\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v \Longleftrightarrow f \in \big[(\mathbb{X}_v)_1\big]^M, \tag{8.227}$$

and if either of these conditions holds then

$$\begin{gathered}(\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{exists (in } \mathbb{C}^{n\cdot M}\text{) at } \sigma\text{-a.e. point on } \partial\Omega \text{ and} \\ \|\mathcal{N}_\kappa u\|_{\mathbb{X}_v} + \|\mathcal{N}_\kappa(\nabla u)\|_{\mathbb{X}_v} \approx \|f\|_{[(\mathbb{X}_v)_1]^M}.\end{gathered} \tag{8.228}$$

(b) *[Uniqueness] Whenever* $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$*, there exists* $\delta \in (0,1)$ *depending only on n,* Ξ*, L, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Dirichlet Problem* (8.224) *has at most one solution.*

(c) *[Well-Posedness] If* $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ *and* $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$ *then there exists* $\delta \in (0,1)$ *which depends only on n,* Ξ*, L, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Dirichlet Problem* (8.224) *is uniquely solvable and the solution satisfies* (8.226)*.*

(d) *[Sharpness] If* $\mathfrak{A}_L^{\text{dis}} = \varnothing$ *then the Dirichlet Problem* (8.224) *may fail to be solvable (actually for boundary data belonging to an infinite dimensional subspace of the corresponding weighted Banach function space). Also, if there holds* $\mathfrak{A}_{L^\top}^{\text{dis}} = \varnothing$ *then the Dirichlet Problem* (8.224) *may have more than one solution (in fact, the linear space of null-solutions may actually be infinite dimensional).*

Proof In broad outline, we follow the same steps as in the proof of Theorem 6.2. For the existence of solutions, note that Theorem 8.10 guarantees the existence of some $\delta \in (0,1)$, whose nature is as specified in the statement of the theorem, with the property that whenever Ω is a δ-AR domain the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathbb{X}_v\big]^M$. Hence, the function u in (8.225) is meaningfully defined and, according to (3.23), Proposition 8.2, and Theorem 8.3, we have $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, $Lu = 0$ in Ω, $\mathcal{N}_\kappa u \in \mathbb{X}_v$, and (8.226) holds. Concerning the equivalence claimed in (8.227), if $f \in \big[(\mathbb{X}_v)_1\big]^M$ then Theorem 8.10 gives (assuming $\delta > 0$ is sufficiently small) that $\big(\frac{1}{2}I + K_A\big)^{-1} f \in \big[(\mathbb{X}_v)_1\big]^M$. With this in hand, (8.85)–(8.86) then imply that the function u defined as in (8.225) satisfies $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$, the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $\partial\Omega$, and the left-pointing inequality in the equivalence claimed in (8.228) holds. In particular, this justifies the left-pointing implication in (8.227). The right-pointing implication in (8.227) together with the right-pointing inequality in the equivalence claimed in (8.228) are consequences of Propositions 8.2 and 2.22.

Consider next the uniqueness result claimed in item *(b)*. We have two different and independent arguments, each of them interesting in its own right. To describe the first approach, suppose $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\text{dis}}_{L^\top}$. Also, consider $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$. Let $\widetilde{w} \in A_2(\partial\Omega, \sigma)$ be a weight with the property that $[\widetilde{w}]_{A_2} \leq \Xi$ and note that this entails $[\widetilde{w}^{-1}]_{A_2} \leq \Xi$. From Theorem 4.8, presently used with L replaced by $L^\top$, $p = 2$, and $\widetilde{w}^{-1}$ in place of w, we know that there exists $\delta \in (0, 1)$, which depends only on n, p, Ξ, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ then

$$\tfrac{1}{2}I + K_{A^\top} : \big[L^2_1(\partial\Omega, \widetilde{w}^{-1})\big]^M \longrightarrow \big[L^2_1(\partial\Omega, \widetilde{w}^{-1})\big]^M \tag{8.229}$$

is an invertible operator. By eventually decreasing the value of $\delta \in (0, 1)$ if necessary, we may ensure that Ω is an NTA domain with unbounded boundary (cf. Theorem 2.3). In such a case, (6.2) guarantees that Ω is globally pathwise nontangentially accessible.

To proceed, let $E = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ be the fundamental solution associated with the system L as in Theorem 3.1. Fix $x_\star \in \mathbb{R}^n \setminus \overline{\Omega}$ along with $x_0 \in \Omega$, arbitrary. Also, pick $\rho \in \big(0, \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega)\big)$ and define $K := \overline{B(x_0, \rho)}$. Finally, recall the aperture parameter $\widetilde{\kappa} > 0$ associated with Ω and κ as in Theorem 6.1. Next, for each fixed $\beta \in \{1, \dots, M\}$, consider the $\mathbb{C}^M$-valued function

$$f^{(\beta)}(x) := \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big)_{1\leq\alpha\leq M}, \qquad \forall\, x \in \partial\Omega. \tag{8.230}$$

From (8.230), (3.16), and the Mean Value Theorem we then conclude that

$$|f^{(\beta)}(x)| \leq \frac{C}{1 + |x|^{n-1}}, \qquad |\nabla f^{(\beta)}(x)| \leq \frac{C}{1 + |x|^n}. \tag{8.231}$$

Hence, by (2.587), (2.579), and (2.572),

$$f^{(\beta)} \in \big[L^2_1(\partial\Omega, \widetilde{w}^{-1})\big]^M. \tag{8.232}$$

It is useful to quantify this membership. Specifically, having fixed a point $z_0 \in \partial\Omega$ and arguing as in (8.41)–(8.52), we may write

$$\begin{aligned}\big\|f^{(\beta)}\big\|_{[L^2_1(\partial\Omega,\widetilde{w}^{-1})]^M} &\leq C\Big\|\frac{1}{1+|\cdot|^{n-1}}\Big\|_{L^2(\partial\Omega,\widetilde{w}^{-1})} \leq C\big\|\mathcal{M}(\mathbf{1}_{\Delta(z_0,1)})\big\|_{L^2(\partial\Omega,\widetilde{w}^{-1})}\\ &\leq C\,[\widetilde{w}^{-1}]_{A_2}\Big(\int_{\Delta(z_0,1)}\widetilde{w}^{-1}\,d\sigma\Big)^{1/2}\\ &\leq C\,[\widetilde{w}]_{A_2}^{3/2}\,\frac{\sigma\big(\Delta(z_0,1)\big)}{\widetilde{w}\big(\Delta(z_0,1)\big)^{1/2}}<\infty,\end{aligned} \tag{8.233}$$

for some geometric constant $C \in (0,\infty)$. As a consequence, with $\big(\tfrac{1}{2}I + K_{A^\top}\big)^{-1}$ denoting the inverse of the operator in (8.229),

$$v_\beta := \big(v_{\beta\alpha}\big)_{1\leq\alpha\leq M} := \mathcal{D}_{A^\top}\Big(\big(\tfrac{1}{2}I + K_{A^\top}\big)^{-1} f^{(\beta)}\Big) \tag{8.234}$$

is a well-defined $\mathbb{C}^M$-valued function in Ω which, thanks to Proposition 3.5, satisfies

$$\begin{aligned} &v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega,\\ &\mathcal{N}_{\widetilde{\kappa}}v_\beta \in L^2(\partial\Omega,\widetilde{w}^{-1}), \quad \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in L^2(\partial\Omega,\widetilde{w}^{-1}),\\ &\text{and } \ v_\beta\big|^{\widetilde{\kappa}-\text{n.t.}}_{\partial\Omega} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega.\end{aligned} \tag{8.235}$$

Moreover, from (8.232)–(8.234) and (3.114) we see that

$$\big(\nabla v_\beta\big)\big|^{\widetilde{\kappa}-\text{n.t.}}_{\partial\Omega} \ \text{ exists (in } \mathbb{C}^{n\cdot M}\text{) at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{8.236}$$

Subsequently, for each pair of indices $\alpha, \beta \in \{1,\dots,M\}$ define

$$G_{\alpha\beta}(x) := v_{\beta\alpha}(x) - \big(E_{\beta\alpha}(x-x_0) - E_{\beta\alpha}(x-x_\star)\big), \qquad \forall\, x \in \Omega\setminus\{x_0\}. \tag{8.237}$$

If we now consider $G := \big(G_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ regarded as a $\mathbb{C}^{M\times M}$-valued function defined $\mathcal{L}^n$-a.e. in Ω, then from (8.237) and Theorem 3.1 we see that G belongs to the space $\big[L^1_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^{M\times M}$. Also, by design,

$$\begin{aligned} &L^\top G = -\delta_{x_0} I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^{M\times M} \ \text{ and}\\ &G\big|^{\widetilde{\kappa}-\text{n.t.}}_{\partial\Omega} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega,\\ &\big(\nabla G\big)\big|^{\widetilde{\kappa}-\text{n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega,\end{aligned} \tag{8.238}$$

while if $v := \big(v_{\beta\alpha}\big)_{1\leq\alpha,\beta\leq M}$ then from (2.8), (3.16), and the Mean Value Theorem it follows that at each point $x \in \partial\Omega$ we have

$$\begin{aligned} \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K} G\big)(x) &\leq \big(\mathcal{N}_{\widetilde{\kappa}} v\big)(x) + C_{x_0,\rho}(1+|x|)^{1-n} \quad \text{and} \\ \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big)(x) &\leq \big(\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\big)(x) + C_{x_0,\rho}(1+|x|)^{-n}, \end{aligned} \tag{8.239}$$

where the constant $C_{x_0,\rho} \in (0,\infty)$ is independent of x.

To proceed, suppose now that $u = (u_\beta)_{1\leq\beta\leq M}$ is some $\mathbb{C}^M$-valued function in Ω satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \\ \text{and } \mathcal{N}_\kappa u \text{ belongs to the space } \mathbb{X}(w). \end{gathered} \tag{8.240}$$

In view of (8.234)–(8.235), (3.113), and (8.233) we see that

$$\begin{aligned} \|\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\|_{L^2(\partial\Omega,\widetilde{w}^{-1})} &\leq C\Big\| f^{(\beta)}\Big\|_{[L^2_1(\partial\Omega,\widetilde{w}^{-1})]^M} \\ &\leq C\,[\widetilde{w}\,]_{A_2}^{3/2}\,\frac{\sigma\big(\Delta(z_0,1)\big)}{\widetilde{w}\big(\Delta(z_0,1)\big)^{1/2}} \end{aligned} \tag{8.241}$$

for some constant $C \in (0,\infty)$, hence

$$\int_{\partial\Omega} \mathcal{N}_\kappa u\cdot\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\,d\sigma \leq C\,[\widetilde{w}\,]_{A_2}^{3/2}\,\frac{\sigma\big(\Delta(z_0,1)\big)}{\widetilde{w}\big(\Delta(z_0,1)\big)^{1/2}}\,\|\mathcal{N}_\kappa u\|_{L^2(\partial\Omega,\widetilde{w})}. \tag{8.242}$$

If we now define

$$F := \int_{\partial\Omega} \mathcal{N}_\kappa u\cdot\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\,d\sigma\,\frac{\mathbf{1}_{\Delta(z_0,1)}}{\sigma\big(\Delta(z_0,1)\big)} \quad \text{and} \quad H := \mathcal{N}_\kappa u, \tag{8.243}$$

then we may recast (8.242) as $\|F\|_{L^2(\partial\Omega,\widetilde{w})} \leq C\,[\widetilde{w}\,]_{A_2}^{3/2}\|H\|_{L^2(\partial\Omega,\widetilde{w})}$. Remember that this is valid for every $\widetilde{w} \in A_2(\partial\Omega,\sigma)$ such that $[\widetilde{w}\,]_{A_2} \leq \Xi$, and the choice of Ξ allows us to invoke Theorem 8.1 and obtain that $\|F\|_{\mathbb{X}_v} \leq 2^3\,C\,\Xi^{3/2}\|H\|_{\mathbb{X}_v}$. This further yields

$$\int_{\partial\Omega} \mathcal{N}_\kappa u\cdot\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\,d\sigma \leq 2^3\,C\,\Xi^{3/2}\,\frac{\sigma(\Delta(z_0,1))^{1/2}}{\big\|\mathbf{1}_{\Delta(z_0,1)}\big\|_{\mathbb{X}_v}}\,\|\mathcal{N}_\kappa u\|_{\mathbb{X}_v}, \tag{8.244}$$

where, as observed in the proof of Proposition 8.2, our current assumptions guarantee that $0 < \|\mathbf{1}_{\Delta(z_0,1)}\|_{\mathbb{X}_v} < \infty$. From (8.239), (8.244), and Proposition 8.2 we then conclude that

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\,\mathrm{d}\sigma \tag{8.245}$$

$$\leq \int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}(\nabla v)\,\mathrm{d}\sigma + C_{x_0,\rho}\int_{\partial\Omega} \frac{\mathcal{N}_\kappa u(x)}{(1+|x|)^n}\,\mathrm{d}\sigma(x) \leq C\|\mathcal{N}_\kappa u\|_{\mathbb{X}_v} < \infty.$$

To summarize, the argument so far shows that there exists $\delta \in (0,1)$ depending only on n, p, Ξ, A, and the Ahlfors regularity constant of $\partial\Omega$, such that if one assumes $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (hence, if Ω is a δ-AR domain) then (8.245) holds. We may then invoke Theorem 6.1 to conclude that the Poisson integral representation formula (6.6) holds. In particular, this proves that whenever $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = 0$ at σ-a.e. point on $\partial\Omega$ we necessarily have $u(x_0) = 0$. Given that x_0 has been arbitrarily chosen in Ω, this ultimately shows such a function u is actually identically zero in Ω. This finishes our first proof of the uniqueness result claimed in item *(b)*.

Our second argument for the uniqueness result claimed in item *(b)* is as follows. Suppose that $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$ and, with Ξ as in the statement, invoke Theorem 6.2 with $p := 2$ to conclude that there exists $\delta \in (0,1)$ which depends only on n, Ξ, L, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then for every $w \in A_2(\partial\Omega,\sigma)$ such that $[w]_{A_2} \leq \Xi$ and for every function $f \in \big[L^2(\partial\Omega, w)\big]^M$ the Dirichlet Problem (6.8) with $p := 2$ has at most one solution. Assume next that (8.224) has two solutions u_1, u_2 associated with the same boundary datum $f_0 \in \big[\mathbb{X}_v\big]^M$. This readily gives that $u := u_1 - u_2$ solves (8.224) with boundary datum $f = f_0 - f_0 = 0$ at σ-a.e. point on $\partial\Omega$. Using that $\mathcal{N}_\kappa u \in \mathbb{X}_v$ and Proposition 8.1 we conclude that there exists some Muckenhoupt weight $w_u \in A_2(\partial\Omega,\sigma)$, with the property that $[w_u]_{A_2} \leq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$ and $\mathcal{N}_\kappa u \in L^2(\partial\Omega, w_u)$. All these imply that u solves the Dirichlet Problem (6.8) with $p = 2$, $w = w_u$, $f = 0 \in \big[L^2(\partial\Omega, w_u)\big]^M$. As a consequence of the fact that $[w_u]_{A_2} \leq \Xi$ and the choice of δ we can then conclude that, by uniqueness in Theorem 6.2, u is necessarily zero. Hence, $u_1 \equiv u_2$, as desired.

Moving on, the claim in item *(c)* is a direct consequence of what we have already proved in items *(a)*–*(b)*. At this stage, we are left with justifying the claims in item *(d)*. Recall the second-order, weakly elliptic, constant (real) coefficient, symmetric, $n\times n$ system L_D defined in (3.371). As noted in (3.406), this system has the property that $\mathfrak{A}^{\mathrm{dis}}_{L_D} = \mathfrak{A}^{\mathrm{dis}}_{L_D^\top} = \varnothing$. Henceforth, assume $\Omega := \mathbb{R}^n_+$ and canonically identify $\partial\Omega = \mathbb{R}^{n-1}$, and $\sigma = \mathcal{L}^{n-1}$. Pick an arbitrary $g \in \mathbb{X}_v$, with $g \not\equiv 0$ at $\mathcal{L}^{n-1}$-a.e. point on $\mathbb{R}^{n-1}$, and define $f := (0,\dots,0,g) \in [\mathbb{X}_v]^n$. Suppose there exists a solution u of the Dirichlet Problem formulated as in (8.224) with $\Omega := \mathbb{R}^n_+$ and for the boundary datum f. Proposition 8.1 applied to $\mathcal{N}_\kappa u \in \mathbb{X}_v$ guarantees the existence of a weight $w_u \in A_2(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$, with $[w_u]_{A_2} \leq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\|\mathcal{M}'\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$ and such that $\mathcal{N}_\kappa u \in L^2(\mathbb{R}^{n-1}, w_u)$. Thus, (3.373) and (3.374) in Proposition 3.13 hold with $p := 2$ and $w := w_u$. As such, (3.375) yields

$$g = f_n = \sum_{j=1}^{n-1} R_j f_j = 0 \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \mathbb{R}^{n-1}, \tag{8.246}$$

which contradicts our assumption that $g \not\equiv 0$ at $\mathcal{L}^{n-1}$-a.e. point on $\mathbb{R}^{n-1}$. This proves that the boundary datum $f = (0, \dots, 0, g) \in [\mathbb{X}_v]^n$ with $g \in \mathbb{X}_v$ satisfying $g \not\equiv 0$ at $\mathcal{L}^{n-1}$-a.e. point on $\mathbb{R}^{n-1}$ yields a Dirichlet Problem formulated as in (8.224) for $\Omega := \mathbb{R}^n_+$ which does not have a solution.

To deal with the lack of uniqueness for the same system L_D in the upper half-space, we are going to prove that for any scalar function

$$\omega \in \mathscr{C}^\infty(\mathbb{R}^n_+) \ \text{ with } \ \Delta\omega = 0 \ \text{ in } \ \mathbb{R}^n_+ \ \text{ and } \ \mathcal{N}_\kappa\omega, \mathcal{N}_\kappa(\nabla\omega) \in \mathbb{X}_v, \tag{8.247}$$

the vector-valued function

$$\begin{gathered} \vec{u} : \mathbb{R}^n_+ \longrightarrow \mathbb{C}^n \ \text{ given by} \\ \vec{u}(x) := x_n(\nabla\omega)(x) \ \text{ for each } \ x = (x_1, \dots, x_n) \in \mathbb{R}^n_+ \end{gathered} \tag{8.248}$$

satisfies

$$\begin{gathered} \vec{u} \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^n, \quad L_D\vec{u} = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad \mathcal{N}_\kappa\vec{u}, \ \mathcal{N}_\kappa(\nabla\vec{u}) \in \mathbb{X}_v, \\ \text{and } \ \vec{u}\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1}. \end{gathered} \tag{8.249}$$

Assuming this momentarily, let us show that the space of solutions for the Dirichlet Problem formulated as in (8.224) with $\Omega := \mathbb{R}^n_+$ and zero boundary data is infinite dimensional. To this end, for each $k \in \mathbb{Z}$ pick a non-trivial function $\phi_k \in \mathscr{C}^\infty_0(\mathbb{R}^n)$ with $\operatorname{supp}\phi_k \subset B\big((k, 0, \dots, 0), 1/4\big)$. Then $\phi_k\big|_{\partial\mathbb{R}^n_+} \in (\mathbb{X}_v)_1$ for every $k \in \mathbb{Z}$, by properties (d) and (g) in Definition 8.1, the fact that $v \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, and (2.579). We may then rely on part *(c)* in Theorem 8.19 (formulated a little later below) with L being the Laplacian in $\mathbb{R}^n$ (a choice ensuring that $\mathfrak{A}^{\text{dis}}_L \neq \varnothing$ and $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$) and $\Omega := \mathbb{R}^n_+$ (so that $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} = 0$) to see that the problem (8.256) with boundary datum $f := \phi_k\big|_{\partial\mathbb{R}^n_+}$ has a unique solution ω_k (given by (8.257)) for every $k \in \mathbb{Z}$. Associated with each ω_k define $\vec{u}_k$ as in (8.248). In particular, (8.249) holds for each $\vec{u}_k$. To proceed, given $N \in \mathbb{N}$ and $\{\alpha_k\}_{j=-N}^N \subset \mathbb{C}$ assume that $\sum_{k=-N}^N \alpha_k\vec{u}_k \equiv 0$ in $\mathbb{R}^n_+$. In turn, this implies $\sum_{k=-N}^N \alpha_k\nabla\omega_k \equiv 0$ in $\mathbb{R}^n_+$, hence $\sum_{k=-N}^N \alpha_k\omega_k \equiv c$ in $\mathbb{R}^n_+$ for some $c \in \mathbb{C}$. Taking nontangential traces then leads to $\sum_{k=-N}^N \alpha_k\omega_k = c$ at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. In view of the fact that the supports of ϕ_k's are pairwise disjoint, each ϕ_k is non-trivial, and $\phi_k \equiv 0$ in $B\big((0, 1, 0, \dots, 0), 1/4\big)$, we may then conclude that $c = 0$ and $\alpha_k = 0$ for

$-N \leq k \leq N$. Thus, $\vec{u}_{-N}, \dots, \vec{u}_N$ are linearly independent for every $N \in \mathbb{N}$. Ultimately, the above argument shows that the space of solutions of the Dirichlet Problem formulated as in (8.224) with $\Omega := \mathbb{R}^n_+$ and zero boundary data is infinite dimensional.

There remains to justify (8.249). In this regard, observe that the first two properties and the last property listed there follow from (8.247)–(8.248) and Proposition 8.4 applied to the Laplacian. Thus, we are left with showing that

$$\|\mathcal{N}_\kappa \vec{u}\|_{\mathbb{X}_v} + \|\mathcal{N}_\kappa(\nabla\vec{u})\|_{\mathbb{X}_v} \leq \|\mathcal{N}_\kappa \omega\|_{\mathbb{X}_v} + \|\mathcal{N}_\kappa(\nabla\omega)\|_{\mathbb{X}_v}. \tag{8.250}$$

With this goal in mind, set

$$\mathcal{F} := \Big\{\big(\mathcal{N}_\kappa \vec{u} + \mathcal{N}_\kappa(\nabla\vec{u}),\, \mathcal{N}_\kappa\omega + \mathcal{N}_\kappa(\nabla\omega)\big) : \omega \in \mathscr{C}^\infty(\mathbb{R}^n_+)$$
$$\text{with } \Delta\omega = 0 \text{ in } \mathbb{R}^n_+ \text{ and } \mathcal{N}_\kappa\omega,\ \mathcal{N}_\kappa(\nabla\omega) \in \mathbb{X}_v\Big\}. \tag{8.251}$$

Let us fix some $(F, G) \in \mathcal{F}$, say $F = \mathcal{N}_\kappa\vec{u} + \mathcal{N}_\kappa(\nabla\vec{u})$ and $G = \mathcal{N}_\kappa\omega + \mathcal{N}_\kappa(\nabla\omega)$, with ω as in (8.247) and $\vec{u}$ as in (8.249). Let us also pick an arbitrary Muckenhoupt weight $w \in A_2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ and introduce the aperture $\kappa' := 3 + 2\kappa$. Then $B(y, y_n/2) \subset \Gamma_{\kappa'}(x', 0)$ for every $y = (y', y_n) \in \Gamma_\kappa(x', 0)$ and every $x' \in \mathbb{R}^{n-1}$. This, the fact that ω is harmonic, and interior estimates for harmonic functions, then yield

$$\mathcal{N}_\kappa(\vec{u})(x', 0) \leq C_n \mathcal{N}_{\kappa'}(\omega)(x', 0) \quad \text{and} \quad \mathcal{N}_\kappa(\nabla\vec{u})(x', 0) \leq C_n \mathcal{N}_{\kappa'}(\nabla\omega)(x', 0) \tag{8.252}$$

for each $x' \in \mathbb{R}^{n-1}$, where $C_n \in (0, \infty)$ depends only on the dimension n. Combining (8.252) with [92, Proposition A.6] (the reader is advised that the dependence of the constant in terms of $[w]_{A_2}$ is not explicitly stated in [92], but a cursory inspection shows that $[w]_{A_2}^2$ makes the argument work) gives

$$\begin{aligned}
\|F\|_{L^2(\mathbb{R}^{n-1},w)} &\leq \|\mathcal{N}_\kappa\vec{u}\|_{L^2(\mathbb{R}^{n-1},w)} + \|\mathcal{N}_\kappa(\nabla\vec{u})\|_{L^2(\mathbb{R}^{n-1},w)} \\
&\leq C_n\|\mathcal{N}_{\kappa'}\omega\|_{L^2(\mathbb{R}^{n-1},w)} + C_n\|\mathcal{N}_{\kappa'}(\nabla\omega)\|_{L^2(\mathbb{R}^{n-1},w)} \\
&\leq C_{n,\kappa}[w]_{A_2}^2\|\mathcal{N}_\kappa\omega\|_{L^2(\mathbb{R}^{n-1},w)} + C_{n,\kappa}[w]_{A_2}^2\|\mathcal{N}_\kappa(\nabla\omega)\|_{L^2(\mathbb{R}^{n-1},w)} \\
&\leq C_{n,\kappa}[w]_{A_2}^2\|G\|_{L^2(\mathbb{R}^{n-1},w)},
\end{aligned} \tag{8.253}$$

where $C_{n,\kappa}$ depends only on n and κ. We then invoke Theorem 8.1 with $p_0 = 2$ to obtain

$$\|\mathcal{N}_\kappa\vec{u}\|_{\mathbb{X}_v} + \|\mathcal{N}_\kappa(\nabla\vec{u})\|_{\mathbb{X}_v} \leq 2\|F\|_{\mathbb{X}_v} \leq C_{n,\kappa}\,\Xi^2\|G\|_{\mathbb{X}_v}$$

$$\leq C_{n,\kappa}\,\Xi^2\|\mathcal{N}_\kappa\omega\|_{\mathbb{X}_v} + C_{n,\kappa}\,\Xi^2\|\mathcal{N}_\kappa(\nabla\omega)\|_{\mathbb{X}_v}. \tag{8.254}$$

This gives (8.250) which, in turn, finishes the proof of (8.249). The proof of Theorem 8.18 is therefore complete. □

We continue with the Inhomogeneous Regularity Problem for weakly elliptic systems in δ-AR domains with boundary data in weighted Banach function spaces.

Theorem 8.19 *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and fix some aperture parameter $\kappa \in (0,\infty)$. Also, pick $v \in L^1_{\rm loc}(\partial\Omega,\sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$. Let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$ and denote by $\mathbb{X}'_v$ its Köthe dual. If $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $(\partial\Omega,\sigma)$ and $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.255}$$

and fix some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\left\|\mathcal{M}'\right\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system L in $\mathbb{R}^n$, consider the Inhomogeneous Regularity Problem

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v, \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \left[(\mathbb{X}_v)_1\right]^M. \end{cases} \tag{8.256}$$

In relation to this, the following statements are true:

(a) [Existence and Estimates] If $\mathfrak{A}^{\rm dis}_L \neq \varnothing$ and $A \in \mathfrak{A}^{\rm dis}_L$, then there exists some $\delta \in (0,1)$ which depends only on n, Ξ, A, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if Ω is a δ-AR domain) then the operator $\frac{1}{2}I + K_A$ is invertible on the weighted Banach function-based Sobolev space $\left[(\mathbb{X}_v)_1\right]^M$ and the function

$$u(x) := \left(\mathcal{D}_A\left(\tfrac{1}{2}I + K_A\right)^{-1} f\right)(x), \qquad \forall\, x \in \Omega \tag{8.257}$$

is a solution of the Inhomogeneous Regularity Problem (8.256). In addition,

$$\begin{aligned} &\|\mathcal{N}_\kappa u\|_{\mathbb{X}_v} \approx \|f\|_{[\mathbb{X}_v]^M} \ \ and \\ &\|\mathcal{N}_\kappa u\|_{\mathbb{X}_v} + \|\mathcal{N}_\kappa(\nabla u)\|_{\mathbb{X}_v} \approx \|f\|_{[(\mathbb{X}_v)_1]^M}. \end{aligned} \tag{8.258}$$

(b) *[Uniqueness] Whenever* $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$, *there exists* $\delta \in (0,1)$ *which depends only on* n, Ξ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Inhomogeneous Regularity Problem* (8.256) *has at most one solution.*

(c) *[Well-Posedness] If* $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ *and* $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ *then there exists* $\delta \in (0,1)$ *which depends only on* n, Ξ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Inhomogeneous Regularity Problem* (8.256) *is uniquely solvable and the solution satisfies* (8.258).

(d) *[Sharpness] If* $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ *the Inhomogeneous Regularity Problem* (8.256) *may fail to be solvable (actually for boundary data belonging to an infinite dimensional subspace of the corresponding weighted Banach function-based Sobolev space) even when* Ω *is a half-space, and if* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ *the Inhomogeneous Regularity Problem* (8.256) *may possess more than one solution (in fact, the linear space of null-solutions may actually be infinite dimensional) even when* Ω *is a half-space. In particular, if either* $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ *or* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$, *then the Inhomogeneous Regularity Problem* (8.256) *may fail to be well posed even when* Ω *is a half-space.*

Proof All desired conclusions follow from Theorems 8.10, 8.18, and 6.5. Item *(d)* follow easily from the proof of *(d)* of Theorem 8.18. Specifically, we have shown that the Dirichlet Problem formulated as in (8.224) with $\Omega := \mathbb{R}^n_+$, the $n \times n$ system $L := L_D$, and any boundary datum of the form $f := (0,\dots,0,g) \in [(\mathbb{X}_v)_1]^n$ with $g \in (\mathbb{X}_v)_1$ satisfying $g \not\equiv 0$ at $\mathcal{L}^{n-1}$-a.e. point on $\mathbb{R}^{n-1}$ is not solvable. As such, the Inhomogeneous Regularity Problem does not have a solution in this context either. Concerning the lack of uniqueness, the proof of *(d)* of Theorem 8.18 actually gives that there exist infinitely many linear independent $\vec{u}_k$'s as in (8.247)–(8.248) so that (8.249) holds for each of them. The latter implies that each $\vec{u}_k$ is actually a solution of the Inhomogeneous Regularity Problem with zero boundary trace. □

The next goal is to formulate and solve the Homogeneous Regularity Problem with boundary data from homogeneous weighted Banach function-based Sobolev spaces. This augments solvability results established earlier in Theorem 8.18 and Theorem 8.19.

Theorem 8.20 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Define* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, *denote by* ν *the geometric measure theoretic outward unit normal to* Ω, *and fix an aperture parameter* $\kappa > 0$. *In addition, pick* $v \in L^1_{\rm loc}(\partial\Omega,\sigma)$ *with* $v > 0$ *at* σ*-a.e. point on* $\partial\Omega$. *Let* $\mathbb{X}_v$ *be a Banach function space over* $(\partial\Omega, v\sigma)$ *and let* $\mathbb{X}'_v$ *be its Köthe dual. With* $\mathcal{M}$ *denoting the Hardy–Littlewood maximal operator on* $(\partial\Omega,\sigma)$ *and* $\mathcal{M}'f := \mathcal{M}(fv)/v$ *for any* σ*-measurable function* f *on* $\partial\Omega$, *assume that*

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.259}$$

and fix some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. *For a given homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system* L *in* $\mathbb{R}^n$, *consider the Homogeneous Regularity Problem*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v, \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \left[(\dot{\mathbb{X}}_v)_1\right]^M, \end{cases} \tag{8.260}$$

where $(\dot{\mathbb{X}}_v)_1$ *is the homogeneous weighted Banach function-based Sobolev space defined in* (8.88)*. In relation to this, the following statements are valid:*

(a) [Existence, Estimates, and Integral Representations] If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ *then there exists* $\delta \in (0,1)$ *which depends only on* n*,* Ξ*,* L*, and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(a scenario which ensures that* Ω *is a* δ*-AR domain; cf. Definition* 2.15*) then the following properties are true. First, the operator*

$$\left[S_{\mathrm{mod}}\right] : \left[\mathbb{X}_v\right]^M \longrightarrow \left[(\dot{\mathbb{X}}_v)_1 \big/ \sim\right]^M \tag{8.261}$$

is surjective and the Homogeneous Regularity Problem (8.260) *is solvable. More specifically, with* $[f] \in \left[(\dot{\mathbb{X}}_v)_1 \big/ \sim\right]^M$ *denoting the equivalence class (modulo constants) of the boundary datum* f*, and with*

$$g \in \left[\mathbb{X}_v\right]^M \ \ \textit{selected so that} \ \ \left[S_{\mathrm{mod}}\right]g = [f], \tag{8.262}$$

there exists a constant $c \in \mathbb{C}^M$ *such that the function*

$$u := \mathscr{S}_{\mathrm{mod}}\, g + c \ \ in \ \ \Omega \tag{8.263}$$

is a solution of the Homogeneous Regularity Problem (8.260)*. In addition, this solution satisfies (with implicit constants independent of* f*)*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{\mathbb{X}_v} \approx \|\nabla_{\tan} f\|_{[\mathbb{X}_v]^{n\cdot M}}. \tag{8.264}$$

Second, for each coefficient tensor $A \in \mathfrak{A}_L^{\mathrm{dis}}$ *the operator*

$$\tfrac{1}{2}I + \left[K_{A,\mathrm{mod}}\right] : \left[(\dot{\mathbb{X}}_v)_1 / \sim\right]^M \longrightarrow \left[(\dot{\mathbb{X}}_v)_1 / \sim\right]^M \tag{8.265}$$

is an isomorphism, and the Homogeneous Regularity Problem (8.260) *may be solved as*

$$u := \mathcal{D}_{A,\mathrm{mod}} h + c \ \ in \ \ \Omega, \tag{8.266}$$

for a suitable constant $c \in \mathbb{C}^M$ *and with*

$$h \in \big[(\dot{\mathbb{X}}_v)_1\big]^M \text{ such that } [h] = \Big(\tfrac{1}{2}I + \big[K_{A,\mathrm{mod}}\big]\Big)^{-1}[f]. \tag{8.267}$$

Moreover, this solution continues to satisfy (8.264).

(b) [Uniqueness] Whenever $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$*, there exists* $\delta \in (0,1)$ *which depends only on* n, Ξ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Homogeneous Regularity Problem* (8.260) *has at most one solution.*

(c) [Well-Posedness and Additional Integrability/Regularity] Whenever $\mathfrak{A}^{\mathrm{dis}}_{L} \neq \varnothing$ *and* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$ *it follows that there exists* $\delta \in (0,1)$ *which depends only on* n, Ξ, L, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Homogeneous Regularity Problem* (8.260) *is uniquely solvable. Moreover, the unique solution* u *of* (8.260) *satisfies (in a quantitative fashion)*

$$\mathcal{N}_\kappa u \in \mathbb{X}_v \Longleftrightarrow f \in \big[(\mathbb{X}_v)_1\big]^M. \tag{8.268}$$

In particular, the equivalence in (8.268) *proves that the unique solution of the Homogeneous Regularity Problem* (8.260) *for a boundary datum* f *belonging to* $\big[(\mathbb{X}_v)_1\big]^M$ *(which is a subspace of* $\big[(\dot{\mathbb{X}}_v)_1\big]^M$*; cf.* (8.90)*) is actually the unique solution of the Inhomogeneous Regularity Problem* (8.256) *for the boundary datum* f.

(d) [Sharpness] If $\mathfrak{A}^{\mathrm{dis}}_{L} = \varnothing$ *the Homogeneous Regularity Problem* (8.260) *may fail to be solvable (actually for boundary data belonging to an infinite dimensional subspace of the corresponding homogeneous weighted Banach function-based Sobolev space), and if* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$ *the Homogeneous Regularity Problem* (8.260) *may possess more than one solution (in fact, the linear space of null-solutions may actually be infinite dimensional), even in the case when* $\Omega = \mathbb{R}^n_+$*. In particular, if either* $\mathfrak{A}^{\mathrm{dis}}_{L} = \varnothing$ *or* $\mathfrak{A}^{\mathrm{dis}}_{L^\top} = \varnothing$*, then the Homogeneous Regularity Problem* (8.260) *may fail to be well posed, again, even in the case when* $\Omega = \mathbb{R}^n_+$.

Proof All claims are established by reasoning along the lines of the proof of Theorem 6.8, now making use of Propositions 8.2, 8.5, Theorems 8.4, 8.5, 8.6, 8.10, 8.12, and 8.13.

One can alternatively prove all the claims in items *(a)–(c)* via extrapolation. For example, to justify the claims in item *(a)*, for every $f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$ we may apply Proposition 8.1 to the function $\sum_{j,k=1}^n |\partial_{\tau_{jk}} f| \in \mathbb{X}_v$ to produce a weight $w_f \in A_2(\partial\Omega,\sigma)$ with the property that $\sum_{j,k=1}^n |\partial_{\tau_{jk}} f| \in L^2(\partial\Omega, w_f)$. This and Remark 2.4 imply that $f \in \big[\dot{L}^2_1(\partial\Omega, w_f)\big]^M$. We may then rely on part *(a)* in Theorem 6.8 to solve (6.64) with $p := 2$ and $w := w_f$. As we can do this with any $f \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$, we can then extrapolate the estimates in (6.68) by means of Theorem 8.1 to show that (8.264) holds.

Uniqueness can be established much as in the second argument used in the proof of item *(b)* of Theorem 8.18. More specifically, assume the function u solves the boundary problem (8.260) with datum $f = 0$, and apply Proposition 8.1 to $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$ to find $w_u \in A_2(\partial\Omega, \sigma)$ so that $\mathcal{N}_\kappa(\nabla u) \in L^2(\partial\Omega, w_u)$. Then part *(b)* in Theorem 6.8 shows that u must be zero. While carrying out this argument, one needs to use that $[w_u]_{A_2} \leq \Xi$, and also that Theorem 6.8 part *(b)* holds for some δ (depending on Ξ) and for all $w \in A_2(\partial\Omega, \sigma)$ with $[w]_{A_2} \leq \Xi$.

To justify *(d)* recall the second-order, weakly elliptic, constant (real) coefficient, symmetric, $n \times n$ system L_D defined in (3.371), for which $\mathfrak{A}^{\text{dis}}_{L_D} = \mathfrak{A}^{\text{dis}}_{L_D^\top} = \varnothing$ (cf. (3.406)). Pick an arbitrary non-constant function $g \in (\dot{\mathbb{X}}_v)_1$ and use it to define $f := (0, \dots, 0, g) \in [(\dot{\mathbb{X}}_v)_1]^n$. Suppose there exists a solution u of the Homogeneous Regularity Problem (8.260) formulated in $\Omega := \mathbb{R}^n_+$ for $L := L_D$ and the boundary datum f. We agree to identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$ and σ with $\mathcal{L}^{n-1}$. By Proposition 8.1 applied to $\mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v$, there exists some Muckenhoupt weight $w_u \in A_2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, with $[w_u]_{A_2} \leq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$ and such that $\mathcal{N}_\kappa(\nabla u) \in L^2(\mathbb{R}^{n-1}, w_u)$. Thus, (3.379) with $p := 2$ gives that

$$g = f_n - \sum_{j=1}^{n-1} R_j^{\text{mod}} f_j \text{ is constant in } \mathbb{R}^{n-1}, \tag{8.269}$$

which contradicts our assumption that g is non-constant. This argument shows that the boundary datum of the form $f = (0, \dots, 0, g) \in [(\dot{\mathbb{X}}_v)_1]^n$ with $g \in (\dot{\mathbb{X}}_v)_1$ non-constant cannot have a solution. On the other hand, as in the proof of item (d) of Theorem 8.18, any $\vec{u}$ as in (8.248) satisfies (8.249), hence $\vec{u}$ is a null-solution of the Homogeneous Regularity Problem (8.260) formulated in the domain $\Omega := \mathbb{R}^n_+$ and for the operator $L := L_D$. Finally, the argument there already gives that one can find a family $\{\vec{u}_k\}_{k\in\mathbb{Z}}$ of such solutions which are linearly independent. □

We next formulate and solve the Neumann Problem for weakly elliptic systems in δ-AR domains with boundary data in weighted Banach function spaces.

Theorem 8.21 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa > 0$. Choose $v \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$. Let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$ and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and with $\mathcal{M}' f := \mathcal{M}(f v)/v$ for any σ-measurable function f on $\partial\Omega$, assume that*

$$\mathcal{M} \textit{ is bounded on } \mathbb{X}_v \textit{ and } \mathcal{M}' \textit{ is bounded on } \mathbb{X}'_v, \tag{8.270}$$

and fix some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v \to \mathbb{X}_v} \|\mathcal{M}'\|_{\mathbb{X}'_v \to \mathbb{X}'_v}$. Next, suppose L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some coefficient tensor $A \in \mathfrak{A}_L$ and consider the Neumann Problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ in \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in \mathbb{X}_v, \\ \partial_\nu^A u = f \in \big[\mathbb{X}_v\big]^M. \end{cases} \tag{8.271}$$

Then the following statements are valid:

(a) [Existence, Estimates, and Integral Representation] If $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$ *then there exists* $\delta \in (0,1)$ *which depends only on n,* Ξ*, A, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then* $-\frac{1}{2}I + K^\#_{A^\top}$ *is an invertible operator on* $\big[\mathbb{X}_v\big]^M$ *and the function* $u : \Omega \to \mathbb{C}^M$ *defined as*

$$u(x) := \Big(\mathscr{S}_{\rm mod}\Big(-\tfrac{1}{2}I + K^\#_{A^\top}\Big)^{-1} f\Big)(x) \ \ for\ all \ \ x \in \Omega, \tag{8.272}$$

is a solution of the Neumann Problem (8.271) *which satisfies*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{\mathbb{X}_v} \leq C\|f\|_{[\mathbb{X}_v]^M} \tag{8.273}$$

for some constant $C \in (0,\infty)$ *independent of f. Also, the operator* (8.195) *is surjective which implies that, for some constant* $C \in (0,\infty)$*,*

$$\text{there exists } g \in [(\dot{\mathbb{X}}_v)_1]^M \text{ with } \partial_\nu^A(\mathcal{D}_{A,\rm mod} g) = f \text{ and such that } \|g\|_{[(\dot{\mathbb{X}}_v)_1]^M} \leq C\|f\|_{[\mathbb{X}_v]^M}. \tag{8.274}$$

Consequently, the function

$$u := \mathcal{D}_{A,\rm mod} g \ \ in \ \ \Omega \tag{8.275}$$

is a solution of the Neumann Problem (8.271) *which continues to satisfy* (8.273)*.*

(b) [Uniqueness (modulo constants)] Whenever $A \in \mathfrak{A}_L^{\rm dis}$ *there exists* $\delta \in (0,1)$ *which depends only on n,* Ξ*, A, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then any two solutions of the Neumann Problem* (8.271) *differ by a constant from* $\mathbb{C}^M$*.*

(c) [Well-Posedness] Whenever $A \in \mathfrak{A}_L^{\rm dis}$ *and* $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$ *there exists* $\delta \in (0,1)$ *which depends only on n,* Ξ*, A, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* Ω *is a* δ*-AR domain) then the Neumann Problem* (8.271) *is solvable, the solution is unique modulo constants from* $\mathbb{C}^M$*, and each solution satisfies* (8.273)*.*

Proof If $A^\top \in \mathfrak{A}_{L^\top}^{\rm dis}$ then Theorem 8.10 guarantees the existence of some threshold $\delta \in (0,1)$, whose nature is as specified in the statement of the theorem, with the property that if Ω is a δ-AR domain then the operator $-\frac{1}{2}I + K^{\#}_{A^\top}$ is invertible on $\big[\mathbb{X}_v\big]^M$. With this in hand, all claims in *(a)* up to, and including, (8.273) follow from Proposition 8.2 and Theorem 8.5.

Next, the claims in (8.274) are consequences of the surjectivity of the operator (8.195) (itself implied by item *(2)* of Theorem 8.14), and the Open Mapping Theorem. In turn, (8.274) and Theorem 8.6 guarantee that the function u in (8.275) solves the Neumann Problem (8.271) and satisfies (8.273). This takes care of the claims in item *(a)*.

The uniqueness result in *(b)* is proved as in Theorem 6.11, now relying on Proposition 8.2 and Theorems 8.9, 8.2, 8.4, 8.6, 8.12. We additionally note that there is an alternative argument to obtain the uniqueness based on Proposition 8.2, Remark 8.2, and the uniqueness modulo constants of the Neumann problem in weighted Lebesgue spaces (cf. part *(c)* in Theorem 6.11, plus the observation that the dependence of the parameter δ on the weight is via the quantity Ξ). □

As in the case of the Neumann problem for weighted Lebesgue spaces we observe that the Neumann Problem (8.271) for the two-dimensional Lamé system allows for conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (4.401) for any ζ as in (6.155) (see Remarks 8.5 and 6.10 in this regard).

Finally, we formulate and solve the Transmission Problem for weakly elliptic systems in δ-AR domains with boundary data in weighted Banach function spaces. In the formulation on this problem, the clarifications made right after the statement of Theorem 6.15 continue to remain relevant.

Theorem 8.22 *Let $\Omega \subseteq \mathbb{R}^n$ be a UR domain. Denote by ν the geometric measure theoretic outward unit normal to Ω, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and set*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{8.276}$$

Also, having fixed some function $v \in L^1_{\rm loc}(\partial\Omega, \sigma)$ with $v > 0$ at σ-a.e. point on $\partial\Omega$, let $\mathbb{X}_v$ be a Banach function space over $(\partial\Omega, v\sigma)$, and let $\mathbb{X}'_v$ be its Köthe dual. With $\mathcal{M}$ denoting the Hardy–Littlewood maximal operator on $(\partial\Omega, \sigma)$ and with $\mathcal{M}'f := \mathcal{M}(fv)/v$ for any σ-measurable function f on $\partial\Omega$, assume that

$$\mathcal{M} \text{ is bounded on } \mathbb{X}_v \text{ and } \mathcal{M}' \text{ is bounded on } \mathbb{X}'_v, \tag{8.277}$$

and pick some $\Xi \geq 4\|\mathcal{M}\|_{\mathbb{X}_v\to\mathbb{X}_v}\big\|\mathcal{M}'\big\|_{\mathbb{X}'_v\to\mathbb{X}'_v}$. Also, fix an aperture parameter $\kappa > 0$, and a transmission (or coupling) parameter $\eta \in \mathbb{C}$. Next, assume L is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Finally, select some $A \in \mathfrak{A}_L$ and consider the Transmission Problem

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^{\infty}(\Omega_{\pm})\big]^M, \\ Lu^{\pm} = 0 \ \ in \ \ \Omega_{\pm}, \\ \mathcal{N}_{\kappa}(\nabla u^{\pm}) \in \mathbb{X}_v, \\ u^{+}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - u^{-}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = g \in \big[(\dot{\mathbb{X}}_v)_1\big]^M, \\ \partial_\nu^A u^{+} - \eta \cdot \partial_\nu^A u^{-} = f \in \big[\mathbb{X}_v\big]^M. \end{cases} \tag{8.278}$$

In relation to this, the following statements are valid:

(a) [Uniqueness (modulo constants)] Suppose either

$$A^{\top} \in \mathfrak{A}_{L^{\top}}^{\text{dis}} \ and \ \eta \in \mathbb{C} \setminus \{-1\} \tag{8.279}$$

or

$$A \in \mathfrak{A}_{L}^{\text{dis}} \ and \ \eta \in \mathbb{C} \setminus \{0, -1\}. \tag{8.280}$$

Then there exists $\delta \in (0,1)$ *which depends only on* n*,* η*,* Ξ*,* A*, and the Ahlfors regularity constant of* $\partial\Omega$ *such that whenever* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(a scenario which renders* Ω *a* δ*-AR domain; cf. Definition 2.15) it follows any two solutions of the Transmission Problem (8.278) differ by a constant (from* $\mathbb{C}^M$*).*

(b) [Well-Posedness, Integral Representations, and Additional Regularity] Assume

$$A \in \mathfrak{A}_{L}^{\text{dis}},\ A^{\top} \in \mathfrak{A}_{L^{\top}}^{\text{dis}},\ and \ \eta \in \mathbb{C} \setminus \{-1\}. \tag{8.281}$$

Then there exists some small $\delta \in (0,1)$ *which depends only on* n*,* Ξ*,* A*,* η*, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\|\nu\|_{[\text{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(a scenario which ensures that* Ω *is a* δ*-AR domain; cf. Definition 2.15) it follows that the Transmission Problem (8.278) is solvable. Specifically, in the scenario described in (8.281), the operator* $-\frac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^{\top}}$ *is invertible on the weighted Banach function space* $\big[\mathbb{X}_v\big]^M$*, the operator* $\big[S_{\text{mod}}\big]$ *is invertible from* $\big[\mathbb{X}_v\big]^M$ *onto the space* $\big[(\dot{\mathbb{X}}_v)_1/\sim\big]^M$*, and the functions* $u^{\pm} : \Omega_{\pm} \to \mathbb{C}^M$ *defined as*

$$\begin{aligned} u^{+} &:= \mathscr{S}^{+}_{\text{mod}} h_0 + \mathscr{S}^{+}_{\text{mod}} h_1 - c \ \ in \ \ \Omega_{+}, \\ u^{-} &:= \mathscr{S}^{-}_{\text{mod}} h_0 \ \ in \ \ \Omega_{-}, \end{aligned} \tag{8.282}$$

where the superscripts $\pm$ *indicate that the modified single layer potentials are associated with the sets* $\Omega_{\pm}$ *and*

$$
\begin{aligned}
&h_1 := \big[S_{\rm mod}\big]^{-1}[g] \in \big[\mathbb{X}_v\big]^M, \quad c := S_{\rm mod} h_1 - g \in \mathbb{C}^M, \\
&h_0 := \Big(-\tfrac{\eta+1}{2} I + (1-\eta) K^{\#}_{A^\top}\Big)^{-1} \Big(f - \big(-\tfrac{1}{2} I + K^{\#}_{A^\top}\big) h_1\Big),
\end{aligned}
\tag{8.283}
$$

solve the Transmission Problem (8.278) *and satisfy, for some finite constant* $C > 0$ *independent of* f *and* g,

$$
\big\|\mathcal{N}_\kappa(\nabla u^\pm)\big\|_{\mathbb{X}_v} \leq C\Big(\|f\|_{[\mathbb{X}_v]^M} + \|g\|_{[(\dot{\mathbb{X}}_v)_1]^M}\Big). \tag{8.284}
$$

Moreover, any two solutions of the Transmission Problem (8.278) *differ by a constant (from* $\mathbb{C}^M$*). In particular, any solution of the Transmission Problem* (8.278) *satisfies* (8.284).

Alternatively, under the conditions imposed in (8.281) *and, again, assuming* Ω *is a* δ*-AR domain with* $\delta \in (0,1)$ *sufficiently small, a solution of the Transmission Problem* (8.278) *may also be found in the form*

$$
\begin{aligned}
&u^+ := \mathcal{D}^+_{A,\rm mod}\psi_0 + c \quad in \quad \Omega_+, \\
&u^- := \mathcal{D}^-_{A,\rm mod}\psi_1 \quad in \quad \Omega_-,
\end{aligned}
\tag{8.285}
$$

where the superscripts $\pm$ *indicate that the modified double layer potentials are associated with the sets* $\Omega_\pm$*, where* $c \in \mathbb{C}^M$ *is a suitable constant, and where* $\psi_0, \psi_1 \in \big[(\dot{\mathbb{X}}_v)_1\big]^M$ *are two suitable functions satisfying*

$$
\|\psi_0\|_{[(\dot{\mathbb{X}}_v)_1]^M} + \|\psi_1\|_{[(\dot{\mathbb{X}}_v)_1]^M} \leq C\Big(\|f\|_{[\mathbb{X}_v]^M} + \|g\|_{[(\dot{\mathbb{X}}_v)_1]^M}\Big), \tag{8.286}
$$

for some constant $C \in (0,\infty)$ *independent of* f *and* g*. In particular,* $u^\pm$ *in* (8.285) *also satisfy* (8.284).

(c) [Well-Posedness for $\eta = 1$*] In the case when*

$$
\eta = 1 \text{ and } \Omega \text{ is a two-sided NTA domain with an unbounded Ahlfors regular boundary} \tag{8.287}
$$

the Transmission Problem (8.278) *is solvable, and any two solutions of the Transmission Problem* (8.278) *differ by a constant. Any solution is given by*

$$
\begin{aligned}
&u^+ := \mathcal{D}^+_{A,\rm mod} g - \mathscr{S}^+_{\rm mod} f + c \quad in \quad \Omega_+, \\
&u^- := -\mathcal{D}^-_{A,\rm mod} g - \mathscr{S}^-_{\rm mod} f + c \quad in \quad \Omega_-,
\end{aligned}
\tag{8.288}
$$

for some $c \in \mathbb{C}^M$*, where the superscripts* $\pm$ *indicate that the modified layer potentials are associated with the sets* $\Omega_\pm$ *introduced in* (8.276)*. In addition, any solution satisfies* (8.284).

Proof The proofs of *(a)*, *(b)*, and the solvability stated in *(c)* follow much as in the corresponding items in Theorem 6.15 upon invoking Proposition 8.2, Theorems 8.5, 8.6, 8.7, 8.10, 8.12, 8.13, and 8.14.

Regarding the uniqueness stated in *(c)* we just need to justify that, under the assumption 8.287, the following property holds:

$$\begin{aligned}&\text{if } u^{\pm} \text{ solve the homogeneous version of the Transmission Problem}\\&\text{(8.278) (corresponding to having } f=0 \text{ and } g=0\text{) then there}\\&\text{exists some } c\in\mathbb{C}^M \text{ with the property that } u^{\pm}=c \text{ in } \Omega_{\pm}.\end{aligned} \tag{8.289}$$

To justify this, we note that (8.40) in Remark 8.2 (see also Proposition 8.2) implies that $\mathcal{N}_\kappa(\nabla u^{\pm}) \in \mathbb{X}_v \hookrightarrow L^r(\partial\Omega, w)$ for some integrability exponent $r \in (1,\infty)$ and some weight $w \in A_1(\partial\Omega,\sigma) \subseteq A_r(\partial\Omega,\sigma)$, both depending only on n, Ξ, and the Ahlfors regularity constant of $\partial\Omega$, with $[w]_{A_1} \leq C_\Xi$. In particular, $u^{\pm}$ solve the homogeneous version of the Transmission Problem (6.178) with $f = 0$ and $g = 0$, and with $L^r(\partial\Omega, w)$ in place of $L^p(\partial\Omega, w)$. We can then invoke part *(c)* in Theorem 6.15 plus the fact that we are currently assuming 8.287 and conclude from (6.190) that $u^{\pm} = c$ for some $c \in \mathbb{C}^M$ in $\Omega_{\pm}$. This completes the proof. □

We would like to observe that much as in Remark 6.12 one can also consider a weighted Banach function space version of (6.256) where the transmission parameter shows up in the formulation of the Dirichlet boundary condition (as opposed to the Neumann boundary condition, as was the case in (8.278)). Also, as in Remark 6.14 one can consider the Reduced Transmission Problem which corresponds to having $g = 0$ in (8.278) (cf. (6.258)). For example, under the assumption $A^\top \in \mathfrak{A}^{\text{dis}}_{L^\top}$ and $\eta \in \mathbb{C}\setminus\{-1\}$, it follows that there exists some $\delta \in (0,1)$ which depends on the same parameters as before such that $-\frac{\eta+1}{2}I + (1-\eta)K^{\#}_{A^\top}$ is an invertible operator on $\big[\mathbb{X}_v\big]^M$ and the functions $u^{\pm} : \Omega_{\pm} \to \mathbb{C}^M$ defined as in (6.259) solve the associated Reduced Transmission Problem formulated (8.278) with $g = 0$ (cf. (6.258)) and satisfy, for some constant $C \in (0,\infty)$ independent of f,

$$\big\|\mathcal{N}_\kappa(\nabla u^{\pm})\big\|_{\mathbb{X}_v} \leq C\|f\|_{[\mathbb{X}_v]^M}. \tag{8.290}$$

Moreover, the result established in item *(a)* of Theorem 8.22 working under the hypotheses in (8.279) gives uniqueness (modulo constants) for the associated Reduced Transmission Problem (8.278) with $g = 0$ (cf. (6.258)).

We close by making two remarks. First, in the formulation of the Transmission Problem (8.278) for the two-dimensional Lamé system, we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (4.401) for any ζ as in (6.262) (see Remark 8.5 and Remark 6.16 in this regard). Second, much as in §6, all solvability results established in this section turn out to be stable under small perturbations of the coefficient tensors involved.

8.7 Examples of Weighted Banach Function Spaces

The goal in this section is to provide relevant, concrete examples, of weighted (and unweighted) Banach function spaces for which our previous results pertaining to the solvability of boundary value problems in such functional analytic settings apply. We shall start by considering the case of general (unweighted) Banach function spaces. Next, we will restrict ourselves to the smaller collection of rearrangement invariant Banach function spaces, in which case we will be able to work with weighted spaces where the weights belong to some appropriate Muckenhoupt class.

8.7.1 Unweighted Banach Function Spaces

Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. As in the past, we denote by $\mathcal{M}$ the Hardy–Littlewood maximal operator on (Σ, σ) and set $\mathcal{M}'f := \mathcal{M}(fv)/v$ for each σ-measurable function f on Σ. In this section we take $\mathbb{X}$ to be a Banach function space over (Σ, σ) with the property that

$$\mathcal{M} \text{ is bounded both on } \mathbb{X} \text{ and } \mathbb{X}\supsetneq, \tag{8.291}$$

where $\mathbb{X}'$ is the Köthe dual of $\mathbb{X}$. To frame this example into the template used in previous sections, take $v \equiv 1$, so $\mathbb{X}_v = \mathbb{X}$ and $\mathcal{M}' = \mathcal{M}$. As such, Theorem 8.1 and Proposition 8.1 stated in the current context simply involve $\mathbb{X}_v = \mathbb{X}$. The same applies to Theorem 8.2, Proposition 8.2, and Remark 8.2 with the natural choice $\Xi := 4\|\mathcal{M}\|_{\mathbb{X}\to\mathbb{X}}\|\mathcal{M}\|_{\mathbb{X}'\to\mathbb{X}'}$.

Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set such that $\partial\Omega$ is a UR set and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Specialize the earlier discussion to the case when we have $\Sigma := \partial\Omega$. Specifically, assume $\mathbb{X}$ is a Banach function space over $(\partial\Omega, \sigma)$ so that (8.291) holds. If we pick $\Xi := 4\|\mathcal{M}\|_{\mathbb{X}\to\mathbb{X}}\|\mathcal{M}\|_{\mathbb{X}'\to\mathbb{X}'}$ we have versions of Propositions 8.3, 8.4, 8.5, 8.6, 8.7, Theorems 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13, 8.14, 8.15, 8.16, and 8.17, Lemma 8.1, and Corollary 8.1. All of them are valid for $\mathbb{X}$ and the implicit constants will depend on $\mathbb{X}$ via the quantity $\Xi := 4\|\mathcal{M}\|_{\mathbb{X}\to\mathbb{X}}\|\mathcal{M}\|_{\mathbb{X}'\to\mathbb{X}'}$. Concerning the topic of boundary value problems, we have versions of Theorems 8.18, 8.19, 8.20, 8.21, and 8.22, which give existence, estimates uniqueness, integral representation, uniqueness, and ultimately well-posedness for the Dirichlet Problem, the Inhomogeneous Regularity Problem, the Homogeneous Regularity Problem, Neumann Problem, and the Transmission Problem for the Banach function space $\mathbb{X}$, where the implicit constants and δ depend on $\mathbb{X}$ only via the quantity $\Xi := 4\|\mathcal{M}\|_{\mathbb{X}\to\mathbb{X}}\|\mathcal{M}\|_{\mathbb{X}'\to\mathbb{X}'}$ (more specifically, they depend on the operator norms of $\mathcal{M}$ on $\mathbb{X}$ and $\mathbb{X}'$).

We close by mentioning some relevant examples of spaces to which the results in this chapter apply (later on we will consider weighted versions).

Example 8.1 If $\mathbb{X} := L^p(\partial\Omega, \sigma)$ is a Lebesgue space with $p \in (1, \infty)$ then $\mathcal{M}$ is bounded on $\mathbb{X}$ and $\mathbb{X}' = L^{p'}(\partial\Omega, \sigma)$ where p' is the Hölder conjugate exponent of p. Hence, we recover the results from previous chapters dealing with unweighted Lebesgue spaces.

Example 8.2 If $\mathbb{X} := L^{p,q}(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ and $q \in [1, \infty]$ is a Lorentz space, then by real interpolation the operator $\mathcal{M}$ is bounded both on the space $\mathbb{X}$ and on $\mathbb{X}' = L^{p',q'}(\partial\Omega, \sigma)$, where p', q' are the Hölder conjugate exponents of p, q. In this case, we recover the results in Remarks 4.8, 4.11, 4.16, 6.1, 6.2, 6.9, and 6.15 in the regime $q \in [1, \infty]$.

Example 8.3 Given a measurable function $p(\cdot) : \partial\Omega \to (1, \infty)$, the variable Lebesgue space $L^{p(\cdot)}(\partial\Omega, \sigma)$ is defined as the collection of all measurable functions f such that, for some $\lambda > 0$,

$$\int_{\partial\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}\sigma(x) < \infty. \tag{8.292}$$

Here and elsewhere, we follow the custom of writing $p(\cdot)$ instead of p in order to emphasize that the exponent is a function and not necessarily a constant. The set $L^{p(\cdot)}(\partial\Omega, \sigma)$ becomes a Banach function space when equipped with the function norm

$$\|f\|_{L^{p(\cdot)}(\partial\Omega,\sigma)} := \inf \left\{ \lambda > 0 : \int_{\partial\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} \mathrm{d}\sigma(x) \leq 1 \right\}. \tag{8.293}$$

This family of spaces generalizes the scale of ordinary Lebesgue spaces. Indeed, if $p(x) \equiv p_0$, then $L^{p(\cdot)}(\partial\Omega, \sigma)$ equals $L^{p_0}(\partial\Omega, \sigma)$. The Köthe dual space of $L^{p(\cdot)}(\partial\Omega, \sigma)$ is $L^{p'(\cdot)}(\partial\Omega, \sigma)$, where the conjugate exponent function $p'(\cdot)$ is uniquely defined by the demand that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad \forall\, x \in \partial\Omega. \tag{8.294}$$

In this setting, we shall consider boundary value problems in variable Lebesgue spaces $L^{p(\cdot)}(\partial\Omega, \sigma)$ working under the assumption that

$$\mathcal{M} \text{ is bounded on both } L^{p(\cdot)}(\partial\Omega, \sigma) \text{ and } L^{p'(\cdot)}(\partial\Omega, \sigma). \tag{8.295}$$

As such, it is of interest to find exponent functions for which our working assumptions are satisfied. In this regard, it is shown in [1, Corollary 1.8] that (8.295) holds provided that

$$1 < \operatorname*{ess\,inf}_{\partial\Omega} p(\cdot) \leq \operatorname*{ess\,sup}_{\partial\Omega} p(\cdot) < \infty \tag{8.296}$$

and there exist constants $C \in (0, \infty)$ and $p_\infty \in (0, \infty)$, along with some point $x_0 \in \partial\Omega$, such that

$$|p(x) - p(y)| \le \frac{C}{\log(\mathrm{e} + |x - y|^{-1})}, \quad |p(x) - p_\infty| \le \frac{C}{\log(\mathrm{e} + |x_0 - y|)}, \tag{8.297}$$

for every $x, y \in \partial\Omega$.

8.7.2 *Rearrangement Invariant Banach Function Spaces*

In this section we consider the subclass of Banach function spaces which are rearrangement invariant. To set the stage, assume that

$$\begin{aligned} &(\Sigma, \mathfrak{M}) \text{ is a measurable space, and } \mu \text{ is a positive, non-atomic,} \\ &\text{sigma-finite measure on the sigma-algebra } \mathfrak{M}, \text{ with } \mu(\Sigma) > 0. \end{aligned} \tag{8.298}$$

Let $\mathbb{M}_\mu$ be the set of all complex-valued μ-measurable functions on Σ and let μ_f denote the `distribution function` of $f \in \mathbb{M}_\mu$, that is,

$$\mu_f(\lambda) = \mu(\{x \in \Sigma : |f(x)| > \lambda\}), \qquad \lambda \in [0, \infty). \tag{8.299}$$

Definition 8.2 Assume $\mathbb{X}$ is a Banach function space over (Σ, μ). One says that $\mathbb{X}$ is `rearrangement invariant` (or RIBFS) if $\|f\|_{\mathbb{X}} = \|g\|_{\mathbb{X}}$ for every pair of functions $f, g \in \mathbb{X}$ satisfying $\mu_f = \mu_g$.

Lebesgue spaces, Lorentz spaces, and Orlicz spaces are examples of rearrangement invariant function spaces. Given $f \in \mathbb{M}_\mu$, the `decreasing rearrangement` of f with respect to μ is the function f_μ^* (or simply f^*) defined as[2]

$$f_\mu^*(t) := \inf\{\lambda \ge 0 : \mu_f(\lambda) \le t\} \text{ for each } t \in [0, \mu(\Sigma)). \tag{8.300}$$

Note that f and f_μ^* have the same distribution function. As a consequence we have the Luxemburg Representation Theorem (cf. [15, Theorem 4.10, p. 62], [15, Theorem 2.7, p. 51], as well as the comment on [15, p. 64] and [15, Exercise 15, p. 90], bearing in mind (8.298)) stating that:

[2] One may define $f_\mu^*(t)$ as in (8.300) for each $t \ge 0$, though now $f_\mu^*(t) = 0$ whenever $t > \mu(\Sigma)$.

$$\text{If } \mathbb{X} \text{ is a RIBFS over the measure space } (\Sigma, \mu) \text{ then there exists a unique RIBFS } \overline{\mathbb{X}} \text{ over } \Big([0, \mu(\Sigma)), \mathcal{L}^1\Big) \text{ such that for each function } f \in \mathbb{M}_\mu \text{ one has } f \in \mathbb{X} \text{ if and only if } f^*_\mu \in \overline{\mathbb{X}}, \text{ as well as } \|f^*_\mu\|_{\overline{\mathbb{X}}} = \|f\|_{\mathbb{X}}. \text{ Also, the associated space } \mathbb{X}' \text{ of } \mathbb{X} \text{ is itself a RIBFS over } (\Sigma, \mu) \text{ and one has } \overline{\mathbb{X}'} = \overline{\mathbb{X}}' := (\overline{\mathbb{X}})'. \tag{8.301}$$

For the goals we have in mind we shall need to define Boyd indices, which contain information about how to interpolate operators in this context.

Definition 8.3 Let $\mathbb{X}$ be a RIBFS over (Σ, μ). The lower and upper Boyd indices are defined by

$$p_{\mathbb{X}} := \lim_{t \to \infty} \frac{\ln t}{\ln \|D_t\|_{\overline{\mathbb{X}} \to \overline{\mathbb{X}}}}, \qquad q_{\mathbb{X}} := \lim_{t \to 0^+} \frac{\ln t}{\ln \|D_t\|_{\overline{\mathbb{X}} \to \overline{\mathbb{X}}}}, \tag{8.302}$$

where $D_t : \overline{\mathbb{X}} \to \overline{\mathbb{X}}$ is the dilation operator defined for each $t > 0$, $f \in \overline{\mathbb{X}}$, and $s \in [0, \mu(\Sigma))$ by

$$(D_t f)(s) := \begin{cases} f(s/t) & \text{if } s \in \big[0, \ \mu(\Sigma) \cdot \min\{1, t\}\big), \\ 0 & \text{if } s \in \big[\mu(\Sigma) \cdot \min\{1, t\}, \ \mu(\Sigma)\big). \end{cases} \tag{8.303}$$

It is well known that

$$\text{if } \mathbb{X} \text{ is a RIBFS over } (\Sigma, \mu) \text{ then } 1 \le p_{\mathbb{X}} \le q_{\mathbb{X}} \le \infty. \tag{8.304}$$

Furthermore, for each $f \in \overline{\mathbb{X}}$, $g \in \overline{\mathbb{X}}'$, and $t > 0$,

$$\begin{aligned} \int_0^{\mu(\Sigma)} \big|(D_t f)(s) g(s)\big| \, \mathrm{d}s &= \int_0^{\mu(\Sigma) \cdot \min\{1,t\}} |f(s/t) g(s)| \, \mathrm{d}s \\ &= t \int_0^{\mu(\Sigma) \cdot \min\{1/t, 1\}} |f(s) g(st)| \, \mathrm{d}s \\ &= t \int_0^{\mu(\Sigma)} \big|f(s)(D_{1/t} g)(s)\big| \, \mathrm{d}s. \end{aligned} \tag{8.305}$$

As a consequence of this, (8.9), and (8.7) we then obtain

$$\|D_t\|_{\overline{\mathbb{X}} \to \overline{\mathbb{X}}} \le t \big\|D_{1/t}\big\|_{\overline{\mathbb{X}}' \to \overline{\mathbb{X}}'} \quad \text{for each} \quad t > 0. \tag{8.306}$$

Writing the version of (8.306) with t replaced by $1/t$ and with $\mathbb{X}$ replaced by $\mathbb{X}'$ then yields (bearing in mind (8.301) and (8.8)) the reverse inequality. We therefore arrive at the conclusion that

$$\|D_t\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}} = t\,\|D_{1/t}\|_{\overline{\mathbb{X}}'\to\overline{\mathbb{X}}'} \quad \text{for each} \quad t > 0. \tag{8.307}$$

As a consequence of (8.307), for each $t > 0$ we have

$$\frac{\ln \|D_t\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}}}{\ln t} = 1 - \frac{\ln \|D_{1/t}\|_{\overline{\mathbb{X}}'\to\overline{\mathbb{X}}'}}{\ln(1/t)}, \tag{8.308}$$

which ultimately implies

$$p_{\mathbb{X}'} = (q_{\mathbb{X}})' \quad \text{and} \quad q_{\mathbb{X}'} = (p_{\mathbb{X}})'. \tag{8.309}$$

We next prove an interpolation result along the lines of the classical Boyd's interpolation theorem (cf., e.g., [15, Theorem 5.16, p. 153]), allowing one to pass from Lorentz spaces on the measure space (Σ, μ) to estimates on $\mathbb{X}$, a RIBFS over (Σ, μ), with control over the norm of the interpolated operator. The reader is reminded that the scale of Lorentz spaces is monotonic in the second index and reduces to ordinary Lebesgue spaces on the diagonal, i.e.,

$$\begin{gathered} L^{p,r}(\Sigma,\mu) \hookrightarrow L^{p,q}(\Sigma,\mu) \ \text{ if } \ p \in (0,\infty] \ \text{ and } \ 0 < q < r \leq \infty, \\ \text{and } L^{p,p}(\Sigma,\mu) = L^p(\Sigma,\mu) \text{ for each } p \in (0,\infty]. \end{gathered} \tag{8.310}$$

Cf., e.g., [56, Proposition 1.4.10, p. 49], [15, Proposition 1.8, p. 43].

Theorem 8.23 *Assume $(\Sigma, \mathfrak{M})$ is a measurable space, and μ is a positive, non-atomic, sigma-finite measure on the sigma-algebra $\mathfrak{M}$, with $\mu(\Sigma) > 0$. Let $\mathbb{X}$ be a RIBFS over (Σ, μ). Denote by $p_{\mathbb{X}}$, $q_{\mathbb{X}}$ its lower and upper Boyd indices and pick two integrability exponents $p, q \in (0, \infty]$. Make the assumption that*

$$\begin{gathered} \textit{either} \ \ 0 < p < p_{\mathbb{X}} \leq q_{\mathbb{X}} < q < \infty \ \ \textit{or} \\ 0 < p < p_{\mathbb{X}} \ \ \textit{and} \ \ q = \infty. \end{gathered} \tag{8.311}$$

Suppose

$$T : L^{p,1}(\Sigma,\mu) + L^{q,1}(\Sigma,\mu) \longrightarrow \mathbb{M}_\mu \tag{8.312}$$

is a quasi-subadditive operator[3] *such that*

$$\|Tf\|_{L^{p,\infty}(\Sigma,\mu)} \leq M_p \|f\|_{L^{p,1}(\Sigma,\mu)} \ \ \textit{for every} \ \ f \in L^{p,1}(\Sigma,\mu), \tag{8.313}$$

[3] I.e., there exists a constant $C \in (0, \infty)$ with the property that for any two functions f, g in the domain of T one has $|T(f+g)| \leq C(|Tf| + |Tg|)$ at μ-a.e. point.

and[4]

$$\|Tf\|_{L^{q,\infty}(\Sigma,\mu)} \leq M_q \|f\|_{L^{q,1}(\Sigma,\mu)} \text{ for every } f \in L^{q,1}(\Sigma,\mu), \tag{8.314}$$

for some $M_p, M_q \in (0,\infty)$. *Then*

$$\mathbb{X} \subseteq L^{p,1}(\Sigma,\mu) + L^{q,1}(\Sigma,\mu) \tag{8.315}$$

and there exists some $C \in (0,\infty)$*, which depends only on the exponents* p*,* q*, and the quasi-triangle inequality constant for* T*, such that*

$$\|Tf\|_{\mathbb{X}} \leq C\,(M_p + M_q)\|f\|_{\mathbb{X}} \text{ for every } f \in \mathbb{X}. \tag{8.316}$$

As a corollary, if in place of (8.311) one now assumes

$$\begin{aligned} &\text{either } \ 0 < p < p_{\mathbb{X}} \leq q_{\mathbb{X}} < q < \infty \ \text{ or} \\ &0 < p < p_{\mathbb{X}} \ \text{ and } \ q = \infty \end{aligned} \tag{8.317}$$

and if in place of (8.312)–(8.314) one now assumes

$$T : L^p(\Sigma,\mu) + L^q(\Sigma,\mu) \longrightarrow \mathbb{M}_\mu \tag{8.318}$$

is a quasi-subadditive operator such that

$$\|Tf\|_{L^{p,\infty}(\Sigma,\mu)} \leq M_p \|f\|_{L^p(\Sigma,\mu)} \text{ for every } f \in L^p(\Sigma,\mu), \tag{8.319}$$

$$\|Tf\|_{L^{q,\infty}(\Sigma,\mu)} \leq M_q \|f\|_{L^q(\Sigma,\mu)} \text{ for every } f \in L^q(\Sigma,\mu), \tag{8.320}$$

then $\mathbb{X} \subseteq L^p(\Sigma,\mu) + L^q(\Sigma,\mu)$ and (8.316) continues to hold. Indeed, this is an immediate consequence of Theorem 8.23 and (8.310).

We now turn to the proof of Theorem 8.23.

Proof of Theorem 8.23 To get started, for every $a \in (0,\infty)$ define Calderón's (sub-linear) operators P_a, Q_a acting on each real-valued $\mathcal{L}^1$-measurable function h on $(0,\mu(\Sigma))$ at every point $t \in (0,\mu(\Sigma))$ according to

$$(P_a h)(t) := t^{-a} \int_0^t s^a |h(s)| \frac{\mathrm{d}s}{s} \in [0,+\infty], \tag{8.321}$$

[4] With the convention that $L^{\infty,1}(\Sigma,\mu) := L^\infty(\Sigma,\mu)$ and $L^{\infty,\infty}(\Sigma,\mu) := L^\infty(\Sigma,\mu)$.

$$(Q_a h)(t) := t^{-a} \int_t^{\mu(\Sigma)} s^a |h(s)| \frac{ds}{s} \in [0, +\infty]. \tag{8.322}$$

In relation to these, we make five claims. First, we claim that for each $a \in (0, \infty)$ and each $\mathcal{L}^1$-measurable function h on $(0, \mu(\Sigma))$ it follows that

$P_a h$ is the product of two monotonic functions, hence is an $\mathcal{L}^1$-measurable function on $(0, \mu(\Sigma))$, while the function $Q_a h$ is non-increasing, hence also an $\mathcal{L}^1$-measurable function on the interval $(0, \mu(\Sigma))$. (8.323)

In addition, for each $a \in (0, \infty)$ and each $\mathcal{L}^1$-measurable function h on $(0, \mu(\Sigma))$,

$$\text{if } (P_a h)(t_0) = \infty \text{ for some } t_0 \in (0, \mu(\Sigma)) \text{ then } (P_a h)(t) = \infty \text{ for every } t \in (t_0, \mu(\Sigma)), \tag{8.324}$$

and

$$\text{if } (Q_a h)(t_0) = \infty \text{ for some } t_0 \in (0, \mu(\Sigma)) \text{ then then } (Q_a h)(t) = \infty \text{ for every } t \in (0, t_0). \tag{8.325}$$

Indeed, properties (8.323)–(8.325) are immediate from definitions. Second, we claim that

$$\text{if } 1/p_{\mathbb{X}} < a < \infty \text{ then } P_a \text{ is bounded on } \overline{\mathbb{X}}. \tag{8.326}$$

To prove this, fix $a > 1/p_{\mathbb{X}}$ along with two functions, $h \in \overline{\mathbb{X}}$ and $g \in \overline{\mathbb{X}}'$, such that $\|h\|_{\overline{\mathbb{X}}} \leq 1$ and $\|g\|_{\overline{\mathbb{X}}'} \leq 1$. Using (8.321), a natural change of variables, (8.303), and (8.7) (for $\overline{\mathbb{X}}$ in place of $\mathbb{X}$) we may write

$$\begin{aligned} \int_0^{\mu(\Sigma)} \big|(P_a h)(t) g(t)\big| \, dt &= \int_0^{\mu(\Sigma)} \Big(\int_0^1 |h(st)| \, s^a \, \frac{ds}{s} \Big) |g(t)| \, dt \\ &= \int_0^1 \Big(\int_0^{\mu(\Sigma)} |h(st)| \, |g(t)| \, dt \Big) s^a \, \frac{ds}{s} \\ &= \int_0^1 \Big(\int_0^{\mu(\Sigma)} \big|(D_{1/s}|h|)(t) \, g(t)\big| \, dt \Big) s^a \, \frac{ds}{s} \\ &\leq \int_0^1 \big\| D_{1/s} \big\|_{\overline{\mathbb{X}} \to \overline{\mathbb{X}}} \, s^a \, \frac{ds}{s} \end{aligned}$$

$$= \int_1^\infty \|D_s\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}}\, s^{-a}\,\frac{\mathrm{d}s}{s}. \tag{8.327}$$

Take $\varepsilon \in \big(0, a - 1/p_{\mathbb{X}}\big)$. From the definition of the Boyd index $p_{\mathbb{X}}$ (cf. (8.302)) it follows that there exists $s_0 \in (1, \infty)$ such that

$$\frac{\ln \|D_s\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}}}{\ln s} < a - \varepsilon \quad \text{for every} \quad s \in [s_0, \infty), \tag{8.328}$$

that is, $\|D_s\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}} < s^{a-\varepsilon}$ for each $s \in [s_0, \infty)$. Based on this, (8.9), (8.327), and the fact that $\|D_s\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}} \leq \big\|D_{s_0}\big\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}}$ if $s \leq s_0$ (see [15, Proposition 5.11 and (5.24)–(5.25), p. 148]) we then conclude that

$$\|P_a\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}} \leq \big\|D_{s_0}\big\|_{\overline{\mathbb{X}}\to\overline{\mathbb{X}}} \int_1^{s_0} s^{-a}\,\frac{\mathrm{d}s}{s} + \int_{s_0}^\infty s^{-\varepsilon}\,\frac{\mathrm{d}s}{s} < \infty, \tag{8.329}$$

establishing the claim made in (8.326). The third claim we make in relation to Calderón's operators is that

$$\begin{gathered}\text{for each } a \in (0,1) \text{ the operator } Q_a \text{ is bounded on } \overline{\mathbb{X}} \\ \text{if and only if } P_{1-a} \text{ is bounded on } \overline{\mathbb{X}}'.\end{gathered} \tag{8.330}$$

To justify this, fix $a \in (0, 1)$ and observe that for each $h \in \overline{\mathbb{X}}$ and $g \in \overline{\mathbb{X}}'$ we have

$$\begin{aligned}\int_0^{\mu(\Sigma)} (Q_a h)(t)\,|g(t)|\,\mathrm{d}t &= \int_0^{\mu(\Sigma)} \Big(t^{-a}\int_t^{\mu(\Sigma)} s^a|h(s)|\,\frac{\mathrm{d}s}{s}\Big)|g(t)|\,\mathrm{d}t \\ &= \int_0^{\mu(\Sigma)} \Big(s^{a-1}\int_0^{s} t^{1-a}|g(t)|\,\frac{\mathrm{d}t}{t}\Big)|h(s)|\,\mathrm{d}s \\ &= \int_0^{\mu(\Sigma)} |h(s)|\,(P_{1-a}g)(s)\,\mathrm{d}s,\end{aligned} \tag{8.331}$$

thanks to (8.321)–(8.322) and Tonelli's Theorem. Together with (8.9) and (8.7), this readily shows that (8.330) holds. Our fourth claim is that

$$\text{the operator } Q_a \text{ is bounded on } \overline{\mathbb{X}} \text{ if } 0 < a < 1/q_{\mathbb{X}}. \tag{8.332}$$

This is seen from (8.330), (8.326), (8.309), and the fact that $\overline{\mathbb{X}}' = \overline{(\mathbb{X}')}$. The fifth (and final) claim in relation to Calderón's operators is that

$$\begin{gathered}\text{for any } h \in \overline{\mathbb{X}} \text{ one has } (P_a h)(t) < \infty \\ \text{for each } t \in (0, \mu(\Sigma)) \text{ if } 1/p_{\mathbb{X}} < a < \infty,\end{gathered} \tag{8.333}$$

and

$$\begin{aligned}&\text{for any } h \in \overline{\mathbb{X}} \text{ one has } (Q_a h)(t) < \infty\\&\text{for each } t \in (0, \mu(\Sigma)) \text{ if } 0 < a < 1/q_{\mathbb{X}}.\end{aligned} \tag{8.334}$$

To prove (8.333), recall from (8.326) that if $h \in \overline{\mathbb{X}}$ and $1/p_{\mathbb{X}} < a < \infty$ then $P_a h \in \overline{\mathbb{X}}$. Then (8.10) (written for $\overline{\mathbb{X}}$ in place of $\mathbb{X}$) implies that $(P_a h)(t) < \infty$ at $\mathcal{L}^1$-a.e. point $t \in \big(0, \mu(\Sigma)\big)$. Granted this, (8.324) ensures that $(P_a h)(t) < \infty$ for each $t \in (0, \mu(\Sigma))$. The proof of (8.334) is similar, now making use of (8.332).

Moving on, given $f \in \mathbb{M}_\mu$, for each $t \in (0, \mu(\Sigma))$ define

$$f_t(x) := \begin{cases} f(x) & \text{if } \ |f(x)| \leq f^*_\mu(t), \\ 0 & \text{if } \ |f(x)| > f^*_\mu(t), \end{cases} \tag{8.335}$$

and

$$f^t(x) := \begin{cases} 0 & \text{if } \ |f(x)| \leq f^*_\mu(t), \\ f(x) & \text{if } \ |f(x)| > f^*_\mu(t), \end{cases} \tag{8.336}$$

at each $x \in \Sigma$. In particular,

$$f = f_t + f^t \quad \text{for each} \quad t \in \big(0, \mu(\Sigma)\big). \tag{8.337}$$

We also claim that for each $t, s \in (0, \mu(\Sigma))$ we have

$$0 \leq (f_t)^*_\mu(s) \leq \begin{cases} f^*_\mu(t) & \text{if } \ s \in (0, t), \\ f^*_\mu(s) & \text{if } \ s \in [t, \mu(\Sigma)), \end{cases} \tag{8.338}$$

$$0 \leq \big(f^t\big)^*_\mu(s) \leq \begin{cases} f^*_\mu(s) & \text{if } \ s \in (0, t), \\ 0 & \text{if } \ s \in [t, \mu(\Sigma)). \end{cases} \tag{8.339}$$

To verify (8.338)–(8.339), fix an arbitrary $t \in \big(0, \mu(\Sigma)\big)$. Since $|f_t(x)| \leq f^*_\mu(t)$ for each point $x \in \Sigma$, it follows that $\big\{x \in \Sigma : |f_t(x)| > f^*_\mu(t)\big\} = \varnothing$ which in view of (8.300) forces $(f_t)^*_\mu(s) \leq f^*_\mu(t)$ for each $s \in \big(0, \mu(\Sigma)\big)$. Also, the fact that $|f_t| \leq |f|$ everywhere on Σ implies $(f_t)^*_\mu(s) \leq f^*_\mu(s)$ for each $s \in \big(0, \mu(\Sigma)\big)$. Thus, ultimately, for any two numbers $t, s \in \big(0, \mu(\Sigma)\big)$ we have $(f_t)^*_\mu(s) \leq \min\big\{f^*_\mu(t), f^*_\mu(s)\big\}$, so (8.338) becomes a consequence of this and the fact that f^*_μ is decreasing. The claim in (8.339) is clear from (8.336) if $f^*_\mu(t) = \infty$ (since this forces $f^t \equiv 0$ on Σ), while if $f^*_\mu(t) < \infty$ then for each $\lambda \geq 0$ we may estimate

$$\mu\big(\{x \in \Sigma : |f^t(x)| > \lambda\}\big) \le \mu\big(\{x \in \Sigma : |f^t(x)| > 0\}\big)$$
$$\le \mu\big(\{x \in \Sigma : |f(x)| > f^*_\mu(t)\}\big) = \mu_f\big(f^*_\mu(t)\big) \le t, \tag{8.340}$$

with the last inequality coming from [15, (1.18), p. 41]. In turn, from (8.340) and (8.300) we see that $\big(f^t\big)^*_\mu(s) = 0$ whenever $s \in [t, \mu(\Sigma))$, which suits our purposes. Finally, the fact that $|f^t| \le |f|$ everywhere on Σ guarantees that $\big(f^t\big)^*_\mu(s) \le f^*_\mu(s)$ for each $s \in \big(0, \mu(\Sigma)\big)$, finishing the proof of (8.339).

The stage has been set to consider the first case in (8.311); in particular, $q < \infty$. Given any function $f \in \mathbb{M}_\mu$, for each number $t \in (0, \mu(\Sigma))$ we may write

$$\begin{aligned} t^{-1/q}\|f_t\|_{L^{q,1}(\Sigma,\mu)} &= t^{-1/q}\int_0^{\mu(\Sigma)} s^{1/q}(f_t)^*_\mu(s)\frac{\mathrm{d}s}{s} \\ &\le q \cdot f^*_\mu(t) + t^{-1/q}\int_t^{\mu(\Sigma)} s^{1/q} f^*_\mu(s)\frac{\mathrm{d}s}{s} \\ &= q \cdot f^*_\mu(t) + Q_{1/q}(f^*_\mu)(t), \end{aligned} \tag{8.341}$$

where the first equality is simply the definition of the quasi-norm $\|\cdot\|_{L^{q,1}(\Sigma,\mu)}$, the subsequent inequality is based on (8.338), and the final equality comes from (8.322). Also, for each $t \in \big(0, \mu(\Sigma)\big)$ we have

$$\begin{aligned} t^{-1/p}\big\|f^t\big\|_{L^{p,1}(\Sigma,\mu)} &= t^{-1/p}\int_0^{\mu(\Sigma)} s^{1/p}(f^t)^*_\mu(s)\frac{\mathrm{d}s}{s} \\ &\le t^{-1/p}\int_0^t s^{1/p} f^*_\mu(s)\frac{\mathrm{d}s}{s} = P_{1/p}(f^*_\mu)(t), \end{aligned} \tag{8.342}$$

by the definition of the Lorentz quasi-norm $\|\cdot\|_{L^{p,1}(\Sigma,\mu)}$, (8.339), and (8.321).

We next fix a function $f \in \mathbb{X}$. Then (8.301) guarantees that $f^*_\mu \in \overline{\mathbb{X}}$. Since $0 < p < p_{\mathbb{X}}$ and $q_{\mathbb{X}} < q < \infty$, from (8.333)–(8.334) we know that

$$\max\big\{P_{1/p}(f^*_\mu)(t),\, Q_{1/q}(f^*_\mu)(t)\big\} < \infty \quad \text{for each} \quad t \in \big(0, \mu(\Sigma)\big). \tag{8.343}$$

In addition, the fact that f^*_μ is a monotonic function belonging to $\overline{\mathbb{X}}$ plus (8.10) (written for $\overline{\mathbb{X}}$ in place of $\mathbb{X}$) imply that

$$f^*_\mu(t) < \infty \quad \text{for each} \quad t \in \big(0, \mu(\Sigma)\big). \tag{8.344}$$

From (8.337), (8.341), (8.342), and (8.343)–(8.344) it follows that for each number $t \in \big(0, \mu(\Sigma)\big)$ we may estimate

$$\|f\|_{L^{p,1}(\Sigma,\mu)+L^{q,1}(\Sigma,\mu)} \leq \|f^t\|_{L^{p,1}(\Sigma,\mu)} + \|f_t\|_{L^{q,1}(\Sigma,\mu)} \qquad (8.345)$$
$$\leq q \cdot t^{1/q} f^*_\mu(t) + t^{1/p}\, P_{1/p}(f^*_\mu)(t) + t^{1/q}\, Q_{1/q}(f^*_\mu)(t) < \infty.$$

In view of the arbitrariness of the function $f \in \mathbb{X}$, we therefore have the inclusion $\mathbb{X} \subseteq L^{p,1}(\Sigma,\mu) + L^{q,1}(\Sigma,\mu)$, proving (8.315) in the current case. As a consequence, it makes sense to consider the action of the operator T on the space $\mathbb{X}$.

From (8.345) we also see that for each $t \in \big(0,\mu(\Sigma)\big)$ we have $f^t \in L^{p,1}(\Sigma,\mu)$ and $f_t \in L^{q,1}(\Sigma,\mu)$. Using this, the fact that T is a quasi-subadditive mapping, the properties of decreasing rearrangements (cf. [55, Proposition 1.4.5, p. 47], [15, Proposition 1.7, p. 41]), (8.313)–(8.314), and (8.341)–(8.342), for $t \in (0,\mu(\Sigma))$ we may now estimate

$$(Tf)^*_\mu(t) \leq C(|Tf_t| + |Tf^t|)^*_\mu(t) \leq C(Tf_t)^*_\mu(t/2) + C(Tf^t)^*_\mu(t/2)$$
$$\leq C\left(\frac{2}{t}\right)^{1/q} \sup_{t/2\leq s<\mu(\Sigma)} s^{1/q}(Tf_t)^*_\mu(s) + C\left(\frac{2}{t}\right)^{1/p} \sup_{t/2\leq s<\mu(\Sigma)} s^{1/p}(Tf^t)^*_\mu(s)$$
$$\leq C\left(\frac{2}{t}\right)^{1/q} \|Tf_t\|_{L^{q,\infty}(\Sigma,\mu)} + C\left(\frac{2}{t}\right)^{1/p} \|Tf^t\|_{L^{p,\infty}(\Sigma,\mu)}$$
$$\leq C\left(\frac{2}{t}\right)^{1/q} M_q\|f_t\|_{L^{q,1}(\Sigma,\mu)} + C\left(\frac{2}{t}\right)^{1/p} M_p\|f^t\|_{L^{p,1}(\Sigma,\mu)}$$
$$\leq C\, M_q\, f^*_\mu(t) + C\, M_q\, Q_{1/q}(f^*_\mu)(t) + C\, M_p\, P_{1/p}(f^*_\mu)(t). \qquad (8.346)$$

In concert, (8.346), the monotonicity, homogeneity, and triangle inequality satisfied by the function norm $\|\cdot\|_{\overline{\mathbb{X}}}$ (cf. Definition 8.1), as well as the boundedness results for $P_{1/p}$ and $Q_{1/q}$ on $\overline{\mathbb{X}}$ (since we presently assume $0 < p < p_{\mathbb{X}} \leq q_{\mathbb{X}} < q < \infty$; cf. (8.326) and (8.332)) imply

$$\big\|(Tf)^*_\mu\big\|_{\overline{\mathbb{X}}} \leq C(M_p + M_q)\big\|f^*_\mu\big\|_{\overline{\mathbb{X}}} \qquad (8.347)$$

for some $C \in (0,\infty)$ which depends only on p, q, and the quasi-triangle inequality constant for T. At this stage, we may rely on (8.301) and (8.347) to write, for each $f \in \mathbb{X}$,

$$\|Tf\|_{\mathbb{X}} = \big\|(Tf)^*_\mu\big\|_{\overline{\mathbb{X}}} \leq C(M_p + M_q)\big\|f^*_\mu\big\|_{\overline{\mathbb{X}}} = C(M_p + M_q)\|f\|_{\mathbb{X}} \qquad (8.348)$$

which finishes the proof of (8.316) corresponding to the first case in (8.311).

There remains to deal with the situation described in the second line in (8.311) (when $q = \infty$). Fix an arbitrary function $f \in \mathbb{X}$. We first observe that

$$\|f_t\|_{L^\infty(\Sigma,\mu)} \le f_\mu^*(t) < \infty \quad \text{for every} \quad t \in \big(0, \mu(\Sigma)\big), \tag{8.349}$$

by (8.335) and (8.344). Recalling from (8.343) that $P_{1/p}(f_\mu^*)(t) < \infty$ for every $t \in \big(0, \mu(\Sigma)\big)$, we next invoke (8.337) and (8.342) to obtain (with $t \in \big(0, \mu(\Sigma)\big)$ arbitrary)

$$\begin{aligned} \|f\|_{L^{p,1}(\Sigma,\mu)+L^\infty(\Sigma,\mu)} &\le \big\|f^t\big\|_{L^{p,1}(\Sigma,\mu)} + \|f_t\|_{L^\infty(\Sigma,\mu)} \le \big\|f^t\big\|_{L^{p,1}(\Sigma,\mu)} + f_\mu^*(t) \\ &\le t^{1/p}\, P_{1/p}(f_\mu^*)(t) + f_\mu^*(t) < \infty. \end{aligned} \tag{8.350}$$

Bearing in mind the arbitrariness of the function $f \in \mathbb{X}$, we see that the embedding $\mathbb{X} \subseteq L^{p,1}(\Sigma,\mu) + L^\infty(\Sigma,\mu)$ holds, so (8.315) continues to hold in the present case. In particular, T is well defined on the space $\mathbb{X}$. Based on the properties of decreasing rearrangements (cf. [55, Proposition 1.4.5, p. 47], [15, Proposition 1.7, p. 41]), (8.349), (8.313), (8.314) (with $q := \infty$), and (8.342), we may now estimate, bearing in mind that T is a quasi-subadditive mapping, for every $t \in \big(0, \mu(\Sigma)\big)$

$$\begin{aligned} (Tf)_\mu^*(t) &\le C(|Tf_t| + |Tf^t|)_\mu^*(t) \le C(Tf_t)_\mu^*(t/2) + C(Tf^t)_\mu^*(t/2) \\ &\le C(Tf_t)_\mu^*(t/2) + C\left(\frac{2}{t}\right)^{1/p} \sup_{t/2 \le s < \mu(\Sigma)} s^{1/p}(Tf^t)_\mu^*(s) \\ &\le C\|Tf_t\|_{L^\infty(\Sigma,\mu)} + C\left(\frac{2}{t}\right)^{1/p} \big\|Tf^t\big\|_{L^{p,\infty}(\Sigma,\mu)} \\ &\le C\, M_\infty \|f_t\|_{L^\infty(\Sigma,\mu)} + C\left(\frac{2}{t}\right)^{1/p} M_p \big\|f^t\big\|_{L^{p,1}(\Sigma,\mu)} \\ &\le C\, M_\infty\, f_\mu^*(t) + C\, M_p\, P_{1/p}(f_\mu^*)(t). \end{aligned} \tag{8.351}$$

From (8.351), the monotonicity, homogeneity, and triangle inequality satisfied by the function norm $\|\cdot\|_{\overline{\mathbb{X}}}$ (cf. Definition 8.1), as well as the boundedness of $P_{1/p}$ on $\overline{\mathbb{X}}$ (since $0 < p < p_{\mathbb{X}}$; cf. (8.326)) we conclude that

$$\big\|(Tf)_\mu^*\big\|_{\overline{\mathbb{X}}} \le C(M_p + M_\infty)\big\|f_\mu^*\big\|_{\overline{\mathbb{X}}} \tag{8.352}$$

for some $C \in (0,\infty)$ which depends only on p and the quasi-triangle inequality constant for T. Granted this, we may appeal to (8.301) to write

$$\|Tf\|_{\mathbb{X}} = \big\|(Tf)_\mu^*\big\|_{\overline{\mathbb{X}}} \le C(M_p + M_\infty)\big\|f_\mu^*\big\|_{\overline{\mathbb{X}}} = C(M_p + M_\infty)\|f\|_{\mathbb{X}} \tag{8.353}$$

for each $f \in \mathbb{X}$. This gives the estimate claimed in (8.316), in the case described in the second line in (8.311). The proof of Theorem 8.23 is therefore complete. □

While Theorem 8.23 is already a useful, versatile tool, there is a more general phenomenon at play here, involving two measures, of the sort brought forth in our next result.

Theorem 8.24 *Suppose $(\Sigma, \mathfrak{M})$ is a measurable space, and denote by $\mathbb{M}_{\mathfrak{M}}$ the space of $\mathfrak{M}$-measurable functions*[5] *on Σ. Assume μ is a positive, non-atomic, sigma-finite measure on the sigma-algebra $\mathfrak{M}$, with $\mu(\Sigma) > 0$. Let $\mathbb{X}$ be a RIBFS over (Σ, μ). Denote by $p_{\mathbb{X}}$, $q_{\mathbb{X}}$ its lower and upper Boyd indices and pick two additional integrability exponents $p, q \in (0, \infty]$. Make the assumption that*

$$\begin{aligned} &\textit{either} \;\; 0 < p < p_{\mathbb{X}} \leq q_{\mathbb{X}} < q < \infty \;\; \textit{or} \\ &0 < p < p_{\mathbb{X}} \;\; \textit{and} \;\; q = \infty. \end{aligned} \tag{8.354}$$

Consider next another measure $\widetilde{\mu} : \mathfrak{M} \to [0, \infty]$ such that $\widetilde{\mu}(\Sigma) = \mu(\Sigma)$ and define

$$\begin{aligned} &\|f\|_{\widetilde{\mathbb{X}}} := \|f^*_{\widetilde{\mu}}\|_{\overline{\mathbb{X}}} \in [0, \infty] \;\; \textit{for each} \;\; f \in \mathbb{M}_{\mathfrak{M}}, \\ &\textit{and} \;\; \widetilde{\mathbb{X}} := \big\{ f \in \mathbb{M}_{\mathfrak{M}} : \|f\|_{\widetilde{\mathbb{X}}} < \infty \big\}, \end{aligned} \tag{8.355}$$

where $\overline{\mathbb{X}}$ is constructed in relation to $\mathbb{X}$ as in (8.301).

Suppose

$$T : L^{p,1}(\Sigma, \widetilde{\mu}) + L^{q,1}(\Sigma, \widetilde{\mu}) \longrightarrow \mathbb{M}_{\widetilde{\mu}} = \mathbb{M}_{\mathfrak{M}} \tag{8.356}$$

is a quasi-subadditive operator[6] *such that*

$$\|Tf\|_{L^{p,\infty}(\Sigma,\widetilde{\mu})} \leq M_p \|f\|_{L^{p,1}(\Sigma,\widetilde{\mu})} \;\; \textit{for every} \;\; f \in L^{p,1}(\Sigma, \widetilde{\mu}), \tag{8.357}$$

and[7]

$$\|Tf\|_{L^{q,\infty}(\Sigma,\widetilde{\mu})} \leq M_q \|f\|_{L^{q,1}(\Sigma,\widetilde{\mu})} \;\; \textit{for every} \;\; f \in L^{q,1}(\Sigma, \widetilde{\mu}), \tag{8.358}$$

for some $M_p, M_q \in (0, \infty)$. Then

$$\widetilde{\mathbb{X}} \subseteq L^{p,1}(\Sigma, \widetilde{\mu}) + L^{q,1}(\Sigma, \widetilde{\mu}) \tag{8.359}$$

and there exists some $C \in (0, \infty)$, which depends only on p, q, and the quasi-triangle inequality constant for T, such that

[5] I.e., the collection of functions $f : \Sigma \to \mathbb{R}$ such that $f^{-1}(O) \in \mathfrak{M}$ for each $O \subseteq \mathbb{R}$ open set.

[6] I.e., there exists a constant $C \in (0, \infty)$ with the property that for any two functions f, g in the domain of T one has $|T(f + g)| \leq C(|Tf| + |Tg|)$ at μ-a.e. point.

[7] With the convention that $L^{\infty,1}(\Sigma, \widetilde{\mu}) := L^{\infty}(\Sigma, \widetilde{\mu})$ and $L^{\infty,\infty}(\Sigma, \widetilde{\mu}) := L^{\infty}(\Sigma, \widetilde{\mu})$.

$$\|Tf\|_{\widetilde{\mathbb{X}}} \leq C\,(M_p + M_q)\|f\|_{\widetilde{\mathbb{X}}} \ \textit{for every} \ \ f \in \widetilde{\mathbb{X}}. \tag{8.360}$$

In view of (8.355) and the result recalled in (8.301), it follows that $\widetilde{\mathbb{X}} = \mathbb{X}$ whenever $\widetilde{\mu} = \mu$, hence Theorem 8.24 is indeed a generalization of Theorem 8.23.

Proof of Theorem 8.24 Throughout, we agree to abbreviate

$$\tau := \widetilde{\mu}(\Sigma) = \mu(\Sigma) \in (0, \infty]. \tag{8.361}$$

The argument below largely parallels the proof of Theorem 8.23, with some natural alterations. To indicate these, for every $a \in (0, \infty)$ define the Calderón's (sub-linear) operators P_a, Q_a acting on each real-valued $\mathcal{L}^1$-measurable function h on $(0, \tau)$ at every point $t \in (0, \tau)$ according to

$$(P_a h)(t) := t^{-a} \int_0^t s^a |h(s)| \frac{\mathrm{d}s}{s} \in [0, +\infty], \tag{8.362}$$

$$(Q_a h)(t) := t^{-a} \int_t^\tau s^a |h(s)| \frac{\mathrm{d}s}{s} \in [0, +\infty]. \tag{8.363}$$

As in the proof of Theorem 8.24, these operators enjoy a number of useful properties. First, for each $a \in (0, \infty)$ and each $\mathcal{L}^1$-measurable function h on $(0, \tau)$ it follows that $P_a h$ and $Q_a h$ are $\mathcal{L}^1$-measurable functions on $(0, \tau)$ satisfying the following properties (see (8.323)–(8.325)):

$$\begin{gathered}\text{if } (P_a h)(t_0) = \infty \text{ for some } t_0 \in (0, \tau) \text{ then} \\ (P_a h)(t) = \infty \text{ for every } t \in (t_0, \tau),\end{gathered} \tag{8.364}$$

and

$$\begin{gathered}\text{if } (Q_a h)(t_0) = \infty \text{ for some } t_0 \in (0, \tau) \text{ then} \\ \text{then } (Q_a h)(t) = \infty \text{ for every } t \in (0, t_0).\end{gathered} \tag{8.365}$$

Second, much as in (8.326), we have that

$$\text{if } 1/p_{\mathbb{X}} < a < \infty \text{ then } P_a \text{ is bounded on } \overline{\mathbb{X}}. \tag{8.366}$$

The third useful property we wish to single out in relation to Calderón's operators is that (see (8.330))

$$\begin{gathered}\text{for each } a \in (0, 1) \text{ the operator } Q_a \text{ is bounded on } \overline{\mathbb{X}} \\ \text{if and only if } P_{1-a} \text{ is bounded on } \overline{\mathbb{X}}'.\end{gathered} \tag{8.367}$$

Our fourth observation is that (see (8.332))

$$\text{the operator } Q_a \text{ is bounded on } \overline{\mathbb{X}} \text{ if } 0 < a < 1/q_{\mathbb{X}}. \tag{8.368}$$

The fifth (and final) property in relation to Calderón's operators is that (see (8.333)–(8.334))

$$\begin{aligned} &\text{for any } h \in \overline{\mathbb{X}} \text{ one has } (P_a h)(t) < \infty \\ &\text{for each } t \in (0,\tau) \text{ if } 1/p_{\mathbb{X}} < a < \infty, \end{aligned} \tag{8.369}$$

and

$$\begin{aligned} &\text{for any } h \in \overline{\mathbb{X}} \text{ one has } (Q_a h)(t) < \infty \\ &\text{for each } t \in (0,\tau) \text{ if } 0 < a < 1/q_{\mathbb{X}}. \end{aligned} \tag{8.370}$$

Going further, given $f \in \mathbb{M}_{\mathfrak{M}} = \mathbb{M}_{\widetilde{\mu}}$, for each $t \in (0,\tau)$ define

$$\widetilde{f}_t(x) := \begin{cases} f(x) & \text{if } |f(x)| \leq f^*_{\widetilde{\mu}}(t), \\ 0 & \text{if } |f(x)| > f^*_{\widetilde{\mu}}(t), \end{cases} \tag{8.371}$$

and

$$\widetilde{f}^t(x) := \begin{cases} 0 & \text{if } |f(x)| \leq f^*_{\widetilde{\mu}}(t), \\ f(x) & \text{if } |f(x)| > f^*_{\widetilde{\mu}}(t), \end{cases} \tag{8.372}$$

at every $x \in \Sigma$. It is then apparent from definitions that

$$f = \widetilde{f}_t + \widetilde{f}^t \quad \text{for each } \ t \in (0,\tau). \tag{8.373}$$

Moreover, (8.338)–(8.339) written for $\widetilde{\mu}$ in place of μ imply that for $t, s \in (0,\tau)$ we have

$$0 \leq \big(\widetilde{f}_t\big)^*_{\widetilde{\mu}}(s) \leq \begin{cases} f^*_{\widetilde{\mu}}(t) & \text{if } s \in (0,t), \\ f^*_{\widetilde{\mu}}(s) & \text{if } s \in [t,\tau), \end{cases} \tag{8.374}$$

$$0 \leq \big(\widetilde{f}^t\big)^*_{\widetilde{\mu}}(s) \leq \begin{cases} f^*_{\widetilde{\mu}}(s) & \text{if } s \in (0,t), \\ 0 & \text{if } s \in [t,\tau). \end{cases} \tag{8.375}$$

We are now prepared to consider the first case in (8.354); in particular, $q < \infty$. Having fixed an arbitrary function function $f \in \mathbb{M}_{\mathfrak{M}} = \mathbb{M}_{\widetilde{\mu}}$, for each $t \in (0,\tau)$ we may write

$$
\begin{aligned}
t^{-1/q} \| \widetilde{f}_t \|_{L^{q,1}(\Sigma,\widetilde{\mu})} &= t^{-1/q} \int_0^{\widetilde{\mu}(\Sigma)} s^{1/q} \big(\widetilde{f}_t\big)^*_{\widetilde{\mu}}(s) \, \frac{ds}{s} = t^{-1/q} \int_0^{\tau} s^{1/q} \big(\widetilde{f}_t\big)^*_{\widetilde{\mu}}(s) \, \frac{ds}{s} \\
&\leq q \cdot f^*_{\widetilde{\mu}}(t) + t^{-1/q} \int_t^{\tau} s^{1/q} f^*_{\widetilde{\mu}}(s) \, \frac{ds}{s} \\
&= q \cdot f^*_{\widetilde{\mu}}(t) + Q_{1/q}\big(f^*_{\widetilde{\mu}}\big)(t), \qquad (8.376)
\end{aligned}
$$

where the first equality is simply the definition of the quasi-norm $\| \cdot \|_{L^{q,1}(\Sigma,\widetilde{\mu})}$, the second equality uses (8.361), the subsequent inequality follows by breaking up the integral and appealing to (8.374), while the final equality comes from (8.363). In a similar fashion, for each $t \in (0, \tau)$ we have

$$
\begin{aligned}
t^{-1/p} \| \widetilde{f}^t \|_{L^{p,1}(\Sigma,\widetilde{\mu})} &= t^{-1/p} \int_0^{\widetilde{\mu}(\Sigma)} s^{1/p} \big(\widetilde{f}^t\big)^*_{\widetilde{\mu}}(s) \, \frac{ds}{s} \\
&= t^{-1/p} \int_0^{\tau} s^{1/p} \big(\widetilde{f}^t\big)^*_{\widetilde{\mu}}(s) \, \frac{ds}{s} \\
&\leq t^{-1/p} \int_0^{t} s^{1/p} f^*_{\widetilde{\mu}}(s) \, \frac{ds}{s} \\
&= P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t), \qquad (8.377)
\end{aligned}
$$

by the definition of the Lorentz quasi-norm $\| \cdot \|_{L^{p,1}(\Sigma,\widetilde{\mu})}$, (8.361), (8.375), and (8.362).

Let us now fix a function $f \in \widetilde{\mathbb{X}}$. Then (8.355) ensures that $f^*_{\widetilde{\mu}} \in \overline{\mathbb{X}}$. In view of the fact that $0 < p < p_{\mathbb{X}}$ and $q_{\mathbb{X}} < q < \infty$, from (8.369)–(8.370) we conclude that

$$
\max \big\{ P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t), \, Q_{1/q}\big(f^*_{\widetilde{\mu}}\big)(t) \big\} < \infty \quad \text{for each} \quad t \in (0, \tau). \qquad (8.378)
$$

Also, the fact that $f^*_{\widetilde{\mu}}$ is a non-decreasing function belonging to $\overline{\mathbb{X}}$ together with (8.10) (written for $\overline{\mathbb{X}}$ in place of $\mathbb{X}$) ensure that

$$
f^*_{\widetilde{\mu}}(t) < \infty \quad \text{for each} \quad t \in (0, \tau). \qquad (8.379)
$$

From (8.373), (8.376), (8.377), and (8.378)–(8.379) it follows that for each number $t \in (0, \tau)$ we may estimate

$$
\begin{aligned}
\| f \|_{L^{p,1}(\Sigma,\widetilde{\mu}) + L^{q,1}(\Sigma,\widetilde{\mu})} &\leq \| \widetilde{f}^t \|_{L^{p,1}(\Sigma,\widetilde{\mu})} + \| \widetilde{f}_t \|_{L^{q,1}(\Sigma,\widetilde{\mu})} \qquad (8.380) \\
&\leq q \cdot t^{1/q} f^*_{\widetilde{\mu}}(t) + t^{1/p} \, P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t) + t^{1/q} \, Q_{1/q}\big(f^*_{\widetilde{\mu}}\big)(t) < \infty.
\end{aligned}
$$

Given that $f \in \widetilde{\mathbb{X}}$ is arbitrary, we conclude that $\widetilde{\mathbb{X}} \subseteq L^{p,1}(\Sigma, \widetilde{\mu}) + L^{q,1}(\Sigma, \widetilde{\mu})$, which establishes (8.359) in the current case. In particular, it is meaningful to consider the action of the operator T on the space $\widetilde{\mathbb{X}}$.

It is also implicit in (8.380) that $\widetilde{f}^t \in L^{p,1}(\Sigma, \widetilde{\mu})$ and $\widetilde{f}_t \in L^{q,1}(\Sigma, \widetilde{\mu})$ for each number $t \in (0, \tau)$. Based on this, the fact that T is a quasi-subadditive mapping, the properties of decreasing rearrangements (cf. [55, Proposition 1.4.5, p. 47], [15, Proposition 1.7, p. 41]), (8.357)–(8.358), and (8.376)–(8.377), for every $t \in (0, \tau)$ we may now estimate

$$\begin{aligned}
\big(Tf\big)^*_{\widetilde{\mu}}(t) &\leq C\big(|T\widetilde{f}_t| + |T\widetilde{f}^t|\big)^*_{\widetilde{\mu}}(t) \leq C\big(T\widetilde{f}_t\big)^*_{\widetilde{\mu}}(t/2) + C\big(T\widetilde{f}^t\big)^*_{\widetilde{\mu}}(t/2) \\
&\leq C\left(\frac{2}{t}\right)^{1/q} \sup_{t/2 \leq s < \tau} s^{1/q}\big(T\widetilde{f}_t\big)^*_{\widetilde{\mu}}(s) + C\left(\frac{2}{t}\right)^{1/p} \sup_{t/2 \leq s < \tau} s^{1/p}\big(T\widetilde{f}^t\big)^*_{\widetilde{\mu}}(s) \\
&\leq C\left(\frac{2}{t}\right)^{1/q} \big\|T\widetilde{f}_t\big\|_{L^{q,\infty}(\Sigma,\widetilde{\mu})} + C\left(\frac{2}{t}\right)^{1/p} \big\|T\widetilde{f}^t\big\|_{L^{p,\infty}(\Sigma,\widetilde{\mu})} \\
&\leq C\left(\frac{2}{t}\right)^{1/q} M_q \big\|\widetilde{f}_t\big\|_{L^{q,1}(\Sigma,\widetilde{\mu})} + C\left(\frac{2}{t}\right)^{1/p} M_p \big\|\widetilde{f}^t\big\|_{L^{p,1}(\Sigma,\widetilde{\mu})} \\
&\leq C\, M_q\, f^*_{\widetilde{\mu}}(t) + C\, M_q\, Q_{1/q}\big(f^*_{\widetilde{\mu}}\big)(t) + C\, M_p\, P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t).
\end{aligned} \tag{8.381}$$

Collectively, (8.381), the monotonicity, homogeneity, and triangle inequality satisfied by the function norm $\|\cdot\|_{\overline{\mathbb{X}}}$ (cf. in Definition 8.1), as well as the boundedness results for the operators $P_{1/p}$ and $Q_{1/q}$ on $\overline{\mathbb{X}}$ (given that we are presently assuming $0 < p < p_{\mathbb{X}} \leq q_{\mathbb{X}} < q < \infty$; cf. (8.366) and (8.368)) imply

$$\Big\|\big(Tf\big)^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} \leq C(M_p + M_q)\Big\|f^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} \tag{8.382}$$

for some $C \in (0, \infty)$ which depends only on p, q, and the quasi-triangle inequality constant for T. We may now invoke (8.355) and (8.382) to write

$$\|Tf\|_{\widetilde{\mathbb{X}}} = \Big\|(Tf)^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} \leq C(M_p + M_q)\Big\|f^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} = C(M_p + M_q)\|f\|_{\widetilde{\mathbb{X}}} \tag{8.383}$$

for each function $f \in \widetilde{\mathbb{X}}$. This finishes the proof of (8.360) corresponding to the first case in (8.354).

We are left with treating the case recorded in the second line in (8.354) (when $q = \infty$). To this end, fix an arbitrary function $f \in \widetilde{\mathbb{X}}$ and first note that

$$\big\|\widetilde{f}_t\big\|_{L^\infty(\Sigma,\widetilde{\mu})} \leq f^*_{\widetilde{\mu}}(t) < \infty \quad \text{for every} \quad t \in (0, \tau), \tag{8.384}$$

by (8.371) and (8.379). Upon recalling from (8.378) that $P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t) < \infty$ for every $t \in (0, \tau)$, we next rely on (8.373) and (8.377) to write

$$\|f\|_{L^{p,1}(\Sigma,\widetilde{\mu})+L^\infty(\Sigma,\widetilde{\mu})} \le \|\widetilde{f}^t\|_{L^{p,1}(\Sigma,\widetilde{\mu})} + \|\widetilde{f}_t\|_{L^\infty(\Sigma,\widetilde{\mu})} \le \|\widetilde{f}^t\|_{L^{p,1}(\Sigma,\widetilde{\mu})} + f^*_{\widetilde{\mu}}(t)$$
$$\le t^{1/p}\, P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t) + f^*_{\widetilde{\mu}}(t) < \infty \tag{8.385}$$

for each number $t \in (0,\tau)$. Keeping in mind the arbitrariness of the function $f \in \widetilde{\mathbb{X}}$, we deduce that $\widetilde{\mathbb{X}} \subseteq L^{p,1}(\Sigma,\widetilde{\mu}) + L^\infty(\Sigma,\widetilde{\mu})$, so (8.359) continues to hold in the present case. As such, the operator T is well defined on the space $\widetilde{\mathbb{X}}$. Thanks to the fact that T is a quasi-subadditive mapping, the properties of decreasing rearrangements (cf. [55, Proposition 1.4.5, p. 47], [15, Proposition 1.7, p. 41]), (8.384), (8.357), (8.358) (with $q := \infty$), and (8.377), we may now estimate

$$\begin{aligned}
\big(Tf\big)^*_{\widetilde{\mu}}(t) &\le C\big(|T\widetilde{f}_t| + |T\widetilde{f}^t|\big)^*_{\widetilde{\mu}}(t) \le C\big(T\widetilde{f}_t\big)^*_{\widetilde{\mu}}(t/2) + C\big(T\widetilde{f}^t\big)^*_{\widetilde{\mu}}(t/2)\\
&\le C\big(T\widetilde{f}_t\big)^*_{\widetilde{\mu}}(t/2) + C\left(\frac{2}{t}\right)^{1/p} \sup_{t/2\le s<\tau} s^{1/p}\big(T\widetilde{f}^t\big)^*_{\widetilde{\mu}}(s)\\
&\le C\|T\widetilde{f}_t\|_{L^\infty(\Sigma,\widetilde{\mu})} + C\left(\frac{2}{t}\right)^{1/p} \|T\widetilde{f}^t\|_{L^{p,\infty}(\Sigma,\widetilde{\mu})}\\
&\le C\,M_\infty\|\widetilde{f}_t\|_{L^\infty(\Sigma,\widetilde{\mu})} + C\left(\frac{2}{t}\right)^{1/p} M_p\|\widetilde{f}^t\|_{L^{p,1}(\Sigma,\widetilde{\mu})}\\
&\le C\,M_\infty\, f^*_{\widetilde{\mu}}(t) + C\,M_p\, P_{1/p}\big(f^*_{\widetilde{\mu}}\big)(t),
\end{aligned} \tag{8.386}$$

for every $t \in (0,\tau)$. Collectively, (8.386), the monotonicity, homogeneity, and triangle inequality satisfied by the function norm $\|\cdot\|_{\overline{\mathbb{X}}}$ (cf. Definition 8.1), as well as the boundedness of $P_{1/p}$ on $\overline{\mathbb{X}}$ (since $0 < p < p_{\mathbb{X}}$; cf. (8.366)) allow us to we conclude that

$$\Big\|\big(Tf\big)^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} \le C(M_p + M_\infty)\Big\|f^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}}, \tag{8.387}$$

for some $C \in (0,\infty)$ depending only on p and the quasi-triangle inequality constant for T. With this in hand, we now make use of (8.355) to write

$$\|Tf\|_{\widetilde{\mathbb{X}}} = \Big\|\big(Tf\big)^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} \le C(M_p + M_\infty)\Big\|f^*_{\widetilde{\mu}}\Big\|_{\overline{\mathbb{X}}} = C(M_p + M_\infty)\|f\|_{\widetilde{\mathbb{X}}} \tag{8.388}$$

for each $f \in \widetilde{\mathbb{X}}$. Hence, the estimate claimed in (8.360) holds in the case described in the second line in (8.354). This completes the proof of Theorem 8.24. □

Moving on, assume that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Let $w \in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ be function satisfying $w > 0$ at σ-a.e. point in Σ; in particular, w is a weight function on (Σ,σ). As usual, w is identified with the weighted measure

$$\mathrm{d}w := w\,\mathrm{d}\sigma. \tag{8.389}$$

As such, w and σ have the same sigma-algebra of measurable sets, and the same null-sets; in particular, $\mathbb{M}_w = \mathbb{M}_\sigma$. Also, (Σ, w) is a sigma-finite measure space since the function w is locally integrable and the set Σ is sigma-compact. In addition, from (8.2) we know that, as a measure, w is non-atomic. Henceforth, make the assumption that w normalized, i.e.,

$$\int_\Sigma w\,\mathrm{d}\sigma = \sigma(\Sigma). \tag{8.390}$$

To proceed, let $\mathbb{X}$ be a RIBFS over (Σ, σ) and define

$$\|f\|_{\mathbb{X}(w)} := \big\|f_w^*\big\|_{\overline{\mathbb{X}}} \in [0, +\infty] \quad \text{for each} \quad f \in \mathbb{M}_w = \mathbb{M}_\sigma. \tag{8.391}$$

Since $\|\cdot\|_{\overline{\mathbb{X}}}$ is the norm in the rearrangement invariant Banach function space over $\Big([0, \sigma(\Sigma)), \mathcal{L}^1\Big)$ (cf. the statement in (8.301)) it follows from [15, Theorem 4.9, pp. 61–62] that $\|\cdot\|_{\mathbb{X}(w)}$ is a function norm over (Σ, w) (in the sense of Definition 8.1) which gives rise to the rearrangement invariant Banach function space

$$\mathbb{X}(w) := \{f \in \mathbb{M}_w : \big\|f_w^*\big\|_{\overline{\mathbb{X}}} < \infty\}. \tag{8.392}$$

Consider next the weighted space $\mathbb{X}'(w)$ constructed in relation to $\mathbb{X}'$, the associated space of $\mathbb{X}$ (which is itself a RIBFS over (Σ, σ); cf. (8.301)). Since (Σ, σ) and (Σ, w) are resonant (thanks to the current assumptions and [15, Theorem 2.7, p. 51]) it follows from the last part in [15, Theorem 4.10, p. 62] that

$$\mathbb{X}'(w) = (\mathbb{X}(w))', \tag{8.393}$$

where the latter space is the associated space of $\mathbb{X}(w)$, itself a RIBFS over (Σ, w) (cf. (8.301)). From [15, Corollary 4.3, p. 69], (8.7) written for $\overline{\mathbb{X}}$, and (8.392) we also obtain the following weighted version of (8.7):

$$\int_\Sigma |f(x)g(x)|\,\mathrm{d}w(x) \le \|f\|_{\mathbb{X}(w)}\|g\|_{\mathbb{X}'(w)} \tag{8.394}$$

for all $f, g \in \mathbb{M}_w = \mathbb{M}_\sigma$. Finally, from [15, Corollary 4.4, p. 69], [15, Definition 2.3, p. 45], and current assumptions we see that the following weighted version of (8.9) holds:

$$\|f\|_{\mathbb{X}(w)} = \sup\left\{\int_\Sigma |f(x)g(x)|\,\mathrm{d}w(x) : \; g \in \mathbb{X}'(w), \;\; \|g\|_{\mathbb{X}'(w)} \le 1\right\} \tag{8.395}$$

for each $f \in \mathbb{M}_w = \mathbb{M}_\sigma$. The reader is referred to [15, Chapter 2] for more details.

Corollary 8.2 *Assume that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Let $\mathbb{X}$ be a RIBFS over (Σ, σ) and denote by $p_{\mathbb{X}}$, $q_{\mathbb{X}}$ its lower and upper Boyd indices. Also, pick a weight $w \in A_{p_{\mathbb{X}}}(\Sigma, \sigma)$, which in the case when Σ is compact is normalized, i.e., satisfies*[8]

$$\fint_{\Sigma} w \, d\sigma = 1. \tag{8.396}$$

Then

1. *the Hardy–Littlewood maximal operator $\mathcal{M}$ on (Σ, σ) is bounded from the space $\mathbb{X}(w)$ into itself provided $1 < p_{\mathbb{X}} < \infty$;*
2. *the operator $f \mapsto \mathcal{M}' f := \mathcal{M}(f w)/w$ is sub-linear and bounded from the space $(\mathbb{X}(w))' = \mathbb{X}'(w)$ into itself whenever $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$.*

Moreover, in the above scenarios, $\|\mathcal{M}\|_{\mathbb{X}(w)\to\mathbb{X}(w)}$ and $\|\mathcal{M}'\|_{\mathbb{X}'(w)\to\mathbb{X}'(w)}$ are controlled by a constant which depends only on $\mathbb{X}$, $[w]_{A_{p_{\mathbb{X}}}}$, n and the Ahlfors regularity constant of Σ (and which, in fact, stays bounded as $[w]_{A_{p_{\mathbb{X}}}}$ stays bounded).

Proof To get started we note (8.2) guarantees that the measure w is non-atomic. In addition, (8.390) presently holds, thanks to (2.540) and the assumption made in (8.396). Bearing these observations in mind, it follows that

$$\begin{aligned}&\widetilde{\mathbb{X}}, \text{ constructed as in (8.355) for the measures } \mu := \sigma \text{ and } \widetilde{\mu} := w,\\ &\text{coincide with the space } \mathbb{X}(w) \text{ introduced in (8.391)–(8.392).}\end{aligned} \tag{8.397}$$

To deal with the claim made in item *(1)* of the statement, make the assumption that $< p_{\mathbb{X}} < \infty$. From item *(1)* in Proposition 2.20 we know that there exists some $\varepsilon \in (0, p_{\mathbb{X}} - 1)$ such that $w \in A_{p_{\mathbb{X}}-\varepsilon}(\Sigma, \sigma)$. Then Theorem 8.24 applied to the measures $\mu := \sigma$ and $\widetilde{\mu} := w$ on Σ, and the sub-linear operator $T := \mathcal{M}$, is applicable with $p := p_{\mathbb{X}} - \varepsilon \in (1, \infty)$ and $q := \infty$ since (8.357)–(8.358) hold for these choices, thanks to (2.530) and the fact that $w \in A_p(\Sigma, \sigma)$. In view of (8.397), from (8.360) we then conclude that $\mathcal{M}$ is bounded from $\mathbb{X}(w)$ into itself. The estimate in (8.360) together with (2.531) and the quantitative aspect of item *(1)* in Proposition 2.20 also show that $\|\mathcal{M}\|_{\mathbb{X}(w)\to\mathbb{X}(w)}$ is controlled solely in terms of $\mathbb{X}$, $[w]_{A_{p_{\mathbb{X}}}}$, n and the Ahlfors regularity constant of Σ (and stays bounded if $[w]_{A_{p_{\mathbb{X}}}}$ stays bounded).

To treat the claim made in item *(2)* of the statement, assume $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$. Once again, consider $\varepsilon \in (0, p_{\mathbb{X}} - 1)$ such that $w \in A_{p_{\mathbb{X}}-\varepsilon}(\Sigma, \sigma)$. If we now define $p := p_{\mathbb{X}} - \varepsilon$ and $q := q_{\mathbb{X}} + \varepsilon$, then $1 < p < p_{\mathbb{X}} \leq q_{\mathbb{X}} < q < \infty$, and we also have $w \in A_p(\Sigma, \sigma) \subseteq A_q(\Sigma, \sigma)$ (cf. item *(2)* in Proposition 2.20).

Given a weight w on Σ along with two integrability exponents $r, r' \in (1, \infty)$ satisfying $1/r + 1/r' = 1$, from the definition of operator $\mathcal{M}'$ it is immediate that

[8] This may always be achieved by multiplying w by a suitable constant, without affecting $[w]_{A_{p_{\mathbb{X}}}}$.

$\mathcal{M}'$ is bounded on $L^{r'}(\Sigma, w)$ if and only if the standard Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{r'}(\Sigma, w^{1-r'})$. In view of (2.530), the latter is equivalent to $w^{1-r'} \in A_{r'}(\Sigma, \sigma)$, which is further equivalent to $w \in A_r(\Sigma, \sigma)$ by item *(3)* in Proposition 2.20. This analysis shows that the operator $T := \mathcal{M}'$ is bounded both on $L^{p'}(\Sigma, w)$ and on $L^{q'}(\Sigma, w)$. Also, (8.309) shows that

$$1 < q' = (q_{\mathbb{X}} + \varepsilon)' < (q_{\mathbb{X}})' = p_{\mathbb{X}'} \leq q_{\mathbb{X}'} = (p_{\mathbb{X}})' < (p_{\mathbb{X}} - \varepsilon)' = p' < \infty. \tag{8.398}$$

Granted this, the desired conclusion follows by once again invoking Theorem 8.24, much as above. □

We may now prove a version of Theorem 8.1 for Muckenhoupt weighted RIBFS of the sort introduced in (8.392) as follows:

Theorem 8.25 *Assume that $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Suppose $\mathbb{X}$ is a RIBFS over (Σ, σ) with Boyd indices satisfying $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$. Fix some weight $w \in A_{p_{\mathbb{X}}}(\Sigma, \sigma)$ which, in the case when Σ is compact, is normalized so that $⨍_{\Sigma} w\, d\sigma = 1$. Recall from Corollary 8.2 that the Hardy–Littlewood maximal operator $\mathcal{M}$ on (Σ, σ) is bounded from the space $\mathbb{X}(w)$ into itself, and the operator $\mathcal{M}'$ acting on each σ-measurable function f on Σ according to $\mathcal{M}'f := \mathcal{M}(fw)/w$ is bounded from the space $(\mathbb{X}(w))' = \mathbb{X}'(w)$ into itself. Pick an integrability exponent $p_0 \in [1, \infty)$, denote by p_0' its Hölder conjugate exponent, and consider a non-decreasing function $\Phi : (0, \infty) \to (0, \infty)$. Finally, assume $\mathcal{F}$ is a family of pairs of σ-measurable functions on Σ.*

If for each $\omega \in A_{p_0}(\Sigma, \sigma)$ with $[\omega]_{A_{p_0}} \leq 2^{p_0} \|\mathcal{M}\|^{p_0-1}_{\mathbb{X}(w)\to\mathbb{X}(w)} \|\mathcal{M}'\|_{\mathbb{X}'(w)\to\mathbb{X}'(w)}$ one has

$$\|f\|_{L^{p_0}(\Sigma,\omega)} \leq \Phi([\omega]_{A_{p_0}})\|g\|_{L^{p_0}(\Sigma,\omega)} \ \textit{for every} \ \ (f, g) \in \mathcal{F}, \tag{8.399}$$

then one may conclude that for every pair $(f, g) \in \mathcal{F}$ one has

$$\|f\|_{\mathbb{X}(w)} \leq 2^{2+1/p_0'}\, \Phi\big(2^{p_0} \|\mathcal{M}\|^{p_0-1}_{\mathbb{X}(w)\to\mathbb{X}(w)} \|\mathcal{M}'\|_{\mathbb{X}'(w)\to\supsetneqq X(w)}\big)\|g\|_{\mathbb{X}(w)}. \tag{8.400}$$

Proof To place ourselves in the framework considered in previous sections, take $v := w$ and $\mathbb{X}_v := \mathbb{X}(w)$, which is a RIBFS over $(\Sigma, v\sigma)$. Corollary 8.2 then ensures that $\mathcal{M}$ is bounded on $\mathbb{X}_v = \mathbb{X}(w)$ while $\mathcal{M}'$ is bounded on $\mathbb{X}_v' = \mathbb{X}'(w)$, with operator norms controlled in terms of $\mathbb{X}$, $[w]_{A_{p_{\mathbb{X}}}}$, n, and the Ahlfors regularity constant of Σ. Granted this, Theorem 8.1 applies and yields the desired conclusion. □

Significantly, Theorem 8.2, Proposition 8.2, and Remark 8.2 hold in the setting considered in Theorem 8.25 for Ξ (large enough) depending only on $\mathbb{X}$, $[w]_{A_{p_{\mathbb{X}}}}$, n and the Ahlfors regularity constant of Σ.

Going further, we now take $\Omega \subseteq \mathbb{R}^n$ to be an open set such that $\partial\Omega$ is a UR set and define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Much as before, assume $\mathbb{X}$ is a RIBFS over $(\partial\Omega, \sigma)$ with $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$ and fix some weight $w \in A_{p_{\mathbb{X}}}(\partial\Omega, \sigma)$ which, in the case when Σ is compact, is normalized so that $\fint_\Sigma w \, d\sigma = 1$. We can pick then Ξ (large enough) depending only on $\mathbb{X}$, $[w]_{A_{p_\mathbb{X}}}$, n and the Ahlfors regularity constant of $\partial\Omega$ so that the versions of Propositions 8.3, 8.4, 8.5, 8.6, and 8.7, Theorems 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13, 8.14, 8.15, 8.16, and 8.17, Lemma 8.1, and Corollary 8.1 are all valid for the weighted Banach function space $\mathbb{X}(w)$ and with constants now also depending on $\mathbb{X}$ and $[w]_{A_{p_\mathbb{X}}}$.

Concerning the topic of boundary value problems, the versions of Theorems 8.18, 8.19, 8.20, 8.21, 8.22 in the aforementioned setting yield existence, estimates uniqueness, integral representation, uniqueness, and ultimately well-posedness for the Dirichlet problem, the Inhomogeneous Regularity Problem, the Homogeneous Regularity Problem, Neumann Problem, and the Transmission Problem for the weighted Banach function space $\mathbb{X}(w)$, where the implicit constants as well as the threshold $\delta \in (0, 1)$ now also depend on $\mathbb{X}$ and $[w]_{A_{p_\mathbb{X}}}$.

We mention some relevant examples of spaces to which the results in this chapter apply. Some of them have already been considered in the previous section, without using that $\mathbb{X}$ is rearrangement invariant. Having this extra feature allows us to consider the associated weighted spaces. Note, however, that variable Lebesgue spaces are not rearrangement invariant and, hence, the results in this section do not apply to this scale.

Example 8.4 In the case of the Lebesgue space $\mathbb{X} := L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ we have $p_\mathbb{X} = q_\mathbb{X} = p$, so we recover the results from previous chapters pertaining to the scale of (unweighted) Lebesgue spaces.

Example 8.5 If we take $\mathbb{X} := L^p(\partial\Omega, \sigma) + L^q(\partial\Omega, \sigma)$ with $p, q \in (1, \infty)$, then $p_\mathbb{X} = \min\{p, q\}$ and $q_\mathbb{X} = \max\{p, q\}$. In this case, we recover the results contained in Theorems 6.3, 6.14, and 6.17 for $w_0 = w_1$.

Example 8.6 If $\mathbb{X} := L^{p,q}(\partial\Omega, \sigma)$ with exponents $p \in (1, \infty)$ and $q \in [1, \infty]$ is a Lorentz space, then $p_\mathbb{X} = q_\mathbb{X} = p$. In this case, we recover the results in Remarks 4.8, 4.11, 4.16, 6.1, 6.2, 6.9, and 6.15. Moreover, the results in this chapter constitute an upgrade of the remarks just mentioned by allowing weighted Lorentz spaces.

Example 8.7 Given a Young function Φ and a closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, the Orlicz space $L^\Phi(\Sigma, \sigma)$ is given by the function norm

$$\|f\|_{L^\Phi(\Sigma,\sigma)} := \inf\left\{\lambda > 0 : \int_\Sigma \Phi\Big(\frac{|f(x)|}{\lambda}\Big) \, d\sigma(x) \leq 1\right\}. \tag{8.401}$$

Then $\mathbb{X} := L^\Phi(\Sigma, \sigma)$ is a RIBFS, and the associated weighted space $\mathbb{X}(w)$ is called a `weighted Orlicz space`. Clearly, Lebesgue spaces are Orlicz spaces corresponding to the family of functions $\Phi(t) := t^p$ indexed by $p \in (1, \infty)$. The spaces $L^p(\Sigma, \sigma) + L^q(\Sigma, \sigma)$ and $L^p(\Sigma, \sigma) \cap L^q(\Sigma, \sigma)$ may also be regarded as

Orlicz spaces with $\Phi(t) \approx \min\{t^p, t^q\}$ and $\Phi(t) \approx \max\{t^p, t^q\}$, respectively (see, e.g., [112, §5.3] for details). In both cases $p_{\mathbb{X}} = \min\{p, q\}$ and $q_{\mathbb{X}} = \max\{p, q\}$.

Other examples which have not been considered explicitly up to this point in this monograph include the Zygmund space $L^p(\log L)^\alpha(\Sigma, \sigma)$ with $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$, obtained by taking $\Phi(t) \approx t^p[\log(\mathrm{e} + t)]^\alpha$, in which case $p_{\mathbb{X}} = q_{\mathbb{X}} = p$ (once more see, e.g., [112, §5.3] for details). In practice, Boyd indices for Orlicz spaces may be computed from Φ using the dilation indices (cf. [15, Theorem 8.18, p. 277], [34, Remark 4.5 on p. 71, and Examples on p. 72]).

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J. J. Marín et al., *Singular Integral Operators, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics 344,
https://doi.org/10.1007/978-3-031-08234-4

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https://doi.org/10.1007/978-3-031-08234-4