Chapter 11
A Novel Solution of the Multi-Group Neutron Diffusion Equation by the Hankel Transform Formalism

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11.1 Introduction

The neutron multi-group equation is frequently used in applications for nuclear reactors. The division in energy groups has been used for a long time to develop more detailed solutions, since the separation by their speed or energy not only facilitates obtaining a better approximate model but also describes the diffusive process with more physical properties [Oz01, No21]. In addition, it is also common to use approaches with different types of geometry [Ma17, Ol19, Ma21], which provide some insight in the influences of the specific boundaries on neutronics. Nuclear reactor cores have different types of geometric approximations and one of the most used is one with axial symmetry in cylindrical coordinates. The choice of a specific coordinate system in general depends on the reactor type and the characteristics to be analysed.

In the course of time, several attempts were used to solve the neutron flux problem in reactor cores, among the most classic ones are procedures, which make use of integral transforms. This method has proven to be effective over many years of research and some representative works may be found in references [Du06, Vi08, Fe13]. Hence, in this work, we develop a methodology to solve the neutron diffusion equation analytically by a finite integral transform technique. In this line, recently Fernandes et al. [Fe11] solved the neutron diffusion equation in cylindrical geometry for a model with two energy groups using the Hankel transform in infinite space, and after constraining the solution to a finite domain, the Parseval identity was employed. In a similar solution procedure, the authors of reference [Gl03] solved the neutron transport equation in cylindrical geometry,

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while considering isotropic scattering and using the Hankel transform together with the Parseval identity. Thus, due to the promising results of these works and the fact that the approximation S_2 of the Boltzmann transport equation reduces to the diffusion equation, in the present work, we focus on the derivation of an analytical formulation for the fast and thermal neutron flux in the diffusion equation and exploring the finite Hankel transform. The derived solutions for different sources in cylindrical geometry are relevant for nuclear fuel element assembly calculations of reactor cores, as for example in pressurized water reactor core simulations.

11.2 Problem Formulation

We consider initially a steady-state problem with two energy groups in the neutron diffusion equation for a homogenized reactor core given by

$$-D_1 \triangle_r \phi_1 + \Sigma_{R1}\phi_1 = S_1 + \frac{1}{k_{eff}} \nu \chi_1 \Sigma_{f2}\phi_2 + \Sigma_{12}\phi_2$$

$$-D_2 \triangle_r \phi_2 + \Sigma_{R2}\phi_2 = \frac{1}{k_{eff}} \nu \chi_2 \Sigma_{f1}\phi_1 + S_2 + \Sigma_{21}\phi_1 \ ,$$

where ϕ_g is the neutron flux, D_g is the diffusion coefficient for the group g, $\triangle_r$ is the Laplacian operator in cylinder coordinates explicitly given by $\triangle_r = \partial_r^2 + \frac{1}{r}\partial_r$, Σ_{Rg} is the removal cross section of group g, k_{eff} is the effective multiplication factor from nuclear reactor theory, ν is the average number of neutrons emitted by fission, Σ_{fg} is the fission cross section, χ_g is the integrated spectrum for neutrons of group g and $\Sigma_{gg'}$ is the scattering cross section from g into group g'. The term S_g is the source term of group g which represents the term $\frac{1}{k_{eff}}\nu\chi_g\Sigma_{fg}\phi_g$ responsible for neutron multiplication, i.e. a manifestation of a chain reaction. The symmetry and boundary conditions for this problem defined individually for each energy group g are

$$\left.\frac{\partial \phi_g}{\partial r}\right|_{r=0} = 0 \ \text{ and } \ \left.\phi_g\right|_{r=R} = 0. \tag{11.1}$$

In order to apply the finite Hankel transform to the previous equations, where as an idealization R represents the extrapolated distance for the same problem as given in reference [La66]

$$H_0\{f(r)\} = \int_0^R r f(r) J_0(r\xi_i)\, dr \ ,$$

where ξ_i is the i-th root of $J_0(R\xi) = 0$, and the inversion of the finite Hankel transform is given by

$$H_0^{-1}\{f(\xi_i)\} = \frac{2}{R^2}\sum_{i=1}^{\infty} f(\xi_i)\frac{J_0(r\xi_i)}{J_1^2(R\xi_i)} \ .$$

Now, using the property

$$H_0\{-D_g \triangle_r \phi_g\} = -D_g \left(-\xi_i^2 \bar{\phi}_g(\xi_i) - R\xi\phi_g(R)J_1'(R\xi)\right)$$

and further applying the extrapolated distance boundary condition $\phi_g(R) = 0$, the finite Hankel transform of this operator term is

$$H_0\{-D_g \triangle_r \phi_g\} = D_g \xi_i^2 \bar{\phi}_g(\xi_i) \ .$$

The Hankel transform of the source terms is given by

$$H_0\{S_g\} = \bar{S}_g = \int_0^R r S_g J_0(r\xi_i)\, dr \ .$$

After application of the finite Hankel transform, one obtains a system of equations,

$$\begin{pmatrix} D_1\xi_i + \Sigma_{R1} & -\left(\frac{1}{k_{eff}}\chi_1\nu\Sigma_{f2} + \Sigma_{12}\right) \\ -\left(\frac{1}{k_{eff}}\chi_2\nu\Sigma_{f1} + \Sigma_{21}\right) & D_2\xi_i + \Sigma_{R2} \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix} = \begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \end{pmatrix} \ ,$$

which is a matrix equation which represents the multi-group problem and may be cast in compact form

$$M(\xi_i)\bar{\Phi} = \bar{S}(\xi_i) \ .$$

The solution of this equation system is formally given by

$$\bar{\Phi} = M^{-1}(\xi_i)\bar{S}(\xi_i) \ .$$

For convenience, we introduce now the shorthand notations

$$A_1(\xi_i) = D_1\xi_i^2 + \Sigma_{R1} \ , \tag{11.2}$$

$$A_2(\xi_i) = D_2\xi_i^2 + \Sigma_{R2} \ , \tag{11.3}$$

and

$$C = \underbrace{\left(\frac{1}{k_{eff}}\chi_1 \nu \Sigma_{f2} + \Sigma_{12}\right)}_{=p_1} \underbrace{\left(\frac{1}{k_{eff}}\chi_2 \nu \Sigma_{f1} + \Sigma_{21}\right)}_{=p_2} \tag{11.4}$$

so that the determinant of matrix M in compact form is

$$Det(M)(\xi_i) = A_1(\xi_i)A_2(\xi_i) - C \ .$$

With these conventions, one may write the transformed solution as

$$\bar{\phi}_1(\xi_i) = \frac{A_2(\xi_i)}{Det(M)(\xi_i)}\bar{S}_1 + \frac{p_1}{Det(M)(\xi_i)}\bar{S}_2 \ ,$$
$$\bar{\phi}_2(\xi_i) = \frac{A_1(\xi_i)}{Det(M)(\xi_i)}\bar{S}_2 + \frac{p_2}{Det(M)(\xi_i)}\bar{S}_1 \ .$$

These expressions are not the final solution yet, since they depend strongly on the choice for the sources terms. Nevertheless, using the definition of the inversion of the finite Hankel transform, we obtain for each energy group

$$\phi_1(r) = H_0^{-1}\{\bar{\phi}_1\} = \frac{2}{R^2}\sum_{i=1}^{\infty} \frac{A_2(\xi_i)\bar{S}_1}{Det(M)(\xi_i)} \frac{J_0(r\xi_i)}{J_1^2(R\xi_i)} + \frac{2}{R^2}p_1 \sum_{i=1}^{\infty} \frac{\bar{S}_2}{Det(M)(\xi_i)} \frac{J_0(r\xi_i)}{J_1^2(R\xi_i)} \ ,$$

$$\phi_2(r) = H_0^{-1}\{\bar{\phi}_2\} = \frac{2}{R^2}\sum_{i=1}^{\infty} \frac{A_1(\xi_i)\bar{S}_2}{Det(M)(\xi_i)} \frac{J_0(r\xi_i)}{J_1^2(R\xi_i)} + \frac{2}{R^2}p_2 \sum_{i=1}^{\infty} \frac{\bar{S}_1}{Det(M)(\xi_i)} \frac{J_0(r\xi_i)}{J_1^2(R\xi_i)} \ .$$

This couple of equations may be used to elaborate the solution of the multi-group neutron problem for a steady-state diffusion problem, here shown for two energy groups.

11.3 Solution by the Infinite Hankel Transform

The initial problem in cylindrical coordinates is well defined inside the spatial domain with $r \in [0, R]$ and it was natural to choose the finite Hankel transform to solve the problem successfully. We now point out and discuss the consequences if one solves the same problem but using the infinite Hankel transform, which by definition is

$$H_0\{f(r); r \to \xi\} = \int_0^{\infty} rf(r)J_0(r\xi)\, dr,$$

and the inversion has the form

$$H_0^{-1}\{f(\xi);\xi\to r\}=\int_0^\infty \xi f(\xi)J_0(r\xi)\,d\xi\ . \tag{11.5}$$

This version of the transform is usually applied for half-open domains. As a matter of fact, the flux profile for both groups is well known as r goes to infinity, and we can define an extrapolated distance for both fluxes at R where these vanish and moreover redefine (11.1) for this kind of problems.

Since the property of the Hankel transform for the operator $\triangle_r$ is the same as for the finite Hankel transform, therefore,

$$H_0\{-D_g\triangle_r\phi_g\}=D_g\xi^2\bar{\phi}_g(\xi),$$

and the equation system after applying the Hankel transform can be written in matrix form as

$$\begin{pmatrix} D_1\xi+\Sigma_{R1} & -\left(\frac{1}{k_{eff}}\chi_1\nu\Sigma_{f2}+\Sigma_{12}\right)\\ -\left(\frac{1}{k_{eff}}\chi_2\nu\Sigma_{f1}+\Sigma_{21}\right) & D_2\xi+\Sigma_{R2}\end{pmatrix}\begin{pmatrix}\bar{\phi}_1\\ \bar{\phi}_2\end{pmatrix}=\begin{pmatrix}\bar{S}_1\\ \bar{S}_2\end{pmatrix}\ .$$

Using now the definitions (11.2), (11.3) and (11.4), the matrix equation reads

$$\begin{pmatrix} A_1(\xi) & -p_1\\ -p_2 & A_2(\xi)\end{pmatrix}\begin{pmatrix}\bar{\phi}_1\\ \bar{\phi}_2\end{pmatrix}=\begin{pmatrix}\bar{S}_1\\ \bar{S}_2\end{pmatrix}$$

with formal solution given by

$$\bar{\Phi}(\xi)=\frac{1}{Det(M)(\xi)}\begin{pmatrix}A_2(\xi)\bar{S}_1+p_1\bar{S}_2\\ A_1(\xi)\bar{S}_2+p_2\bar{S}_1\end{pmatrix}\ .$$

If we focus now our attention on the inversion problem for ϕ_1, we need to investigate the first term of the last equation using Eq. (11.5).

$$\bar{\phi}_1=\frac{1}{Det(M)(\xi)}\left(A_2(\xi)\bar{S}_1(\xi)+p_1\bar{S}_2(\xi)\right).$$

Using the inversion theorem, one gets

$$\phi_1(x)=\int_0^\infty \xi\left(\frac{A_2(\xi)J_0(r\xi)}{Det(M)(\xi)}\right)\bar{S}_1(\xi)\,d\xi+p_1\int_0^\infty \xi\left(\frac{\bar{S}_2(\xi)}{Det(M)(\xi)}\right)J_0(r\xi)\,d\xi\ .$$

The first term of the solution is clearly more complicated to solve, so that to this end we split the fast flux $\phi_1(x)=\phi_1^{(1)}(x)+\phi_1^{(2)}(x)$ and consider the following theorems.

Theorem 11.1 (Hankel Inversion) *If $\sqrt{r'}f(r')$ is piecewise continuous and absolutely integrable along the real axis, then if $\gamma \geq -\frac{1}{2}$, $f_\gamma(\xi) = H_\gamma\{f(r')\}$, then*

$$\int_0^\infty \xi f_\gamma(\xi) J_\gamma(r'\xi)\, d\xi = \frac{1}{2}(f(r'+) + f(r'-)).$$

Theorem 11.2 (Parseval Relation) *If the functions $f(r')$ and $g(r')$ satisfy the conditions of Theorem 11.1 and if $f_\gamma(\xi)$ and $g_\gamma(\xi)$ are the Hankel transforms of order $\gamma \geq -\frac{1}{2}$, respectively, then*

$$\int_0^\infty r' f(r') g(r') dr' = \int_0^\infty \xi \bar{f}_\gamma(\xi) \bar{g}_\gamma(\xi)\, d\xi.$$

These two theorems are essential so that this alternative procedure may be applied. Upon substituting $\bar{f}_0(\xi)$ and $\bar{g}_0(\xi)$ with $\frac{A_2(\xi)J_0(r\xi)}{Det(M)(\xi)}$ and $\bar{S}_1$, respectively, and using Theorem 11.2, one obtains

$$\phi_1^{(1)}(x) = \int_0^\infty \xi \left(\frac{A_2(\xi)J_0(r\xi)}{Det(M)(\xi)}\right) \bar{S}_1\, d\xi = \int_0^\infty r' H_0^{-1}\left\{\frac{A_2(\xi)J_0(r\xi)}{Det(M)(\xi)}\right\} S_1(r')\, dr' .$$

In other words, we need to calculate $f(r')$.

$$f(r') = H_0^{-1}\left\{\frac{A_2(\xi)J_0(r\xi)}{Det(M)(\xi)}\right\} = \int_0^\infty \xi \frac{A_2(\xi)J_0(r\xi)}{Det(M)(\xi)} J_0(r'\xi)\, d\xi \; . \tag{11.6}$$

Recalling that $Det(M)(\xi) = A_1(\xi)A_2(\xi) - C$ and that the physically meaningful nuclear parameter set satisfies the following condition, $0 < \frac{C}{A_1(\xi)A_2(\xi)} < 1$ for all $\xi \in [0, \infty)$, we can expand the term $\frac{A_2}{Det(M)(\xi)}$

$$\begin{aligned}\frac{A_2(\xi)}{A_1(\xi)A_2(\xi) - C} &= \frac{1}{A_1(\xi)} \frac{1}{1 - \frac{C}{A_1(\xi)A_2(\xi)}} \\ &= \frac{1}{A_1(\xi)}\left(1 + \left(\frac{C}{A_1(\xi)A_2(\xi)}\right) + \left(\frac{C}{A_1(\xi)A_2(\xi)}\right)^2 + ...\right).\end{aligned}$$

Indeed, for all nuclear parameter sets known in the literature, they comply with $\frac{C}{A_1(\xi)A_2(\xi)} << 1$ for $\xi \geq 0$. After evaluating different kinds of parameter sets, one may obtain an estimate for the order of magnitude $\mathcal{O}\left(\frac{C}{A_1A_2}\right) = 10^{-3}$ and use this as a maximum for all values of ξ, so that one may safely take only the first term of the expansion.

$$\frac{A_2(\xi)}{A_1(\xi)A_2(\xi) - C} \approx \frac{1}{A_1(\xi)}. \tag{11.7}$$

Consequently, Eq. (11.6) simplifies to

$$f(r') = \int_0^\infty \xi \frac{J_0(r\xi)}{A_1(\xi)} J_0(r'\xi)\, d\xi \ ,$$

so that by the definition of $A_1(\xi) = D_1\xi^2 + \Sigma_{R1} = D_1\left(\xi^2 + \sqrt{\frac{\Sigma_{R1}}{D_1}}^2\right)$, (11.3) may be explicitly written as

$$\begin{aligned} f(r') &= \frac{1}{D_1}\int_0^\infty \xi \frac{J_0(r\xi)}{\xi^2 + (\sqrt{\alpha_1})^2} J_0(r'\xi)\, d\xi \\ &= \begin{cases} \frac{1}{D_1} I_0(\sqrt{\alpha_1}r')K_0(\sqrt{\alpha_1}r)\ , & 0 < r' < r \\ \frac{1}{D_1} I_0(\sqrt{\alpha_1}r)K_0(\sqrt{\alpha_1}r')\ , & r' < r < \infty. \end{cases} \end{aligned}$$

Here, $\alpha_1 = \frac{\Sigma_{R1}}{D_1}$ and I_0 and K_0 are the modified Bessel functions of zero order. Then, we can write the final expression for $\phi_1^{(1)}$, i.e. the solution, using the fact that there is no source outside the cylinder

$$\begin{aligned} \phi_1^{(1)}(r) = &\frac{K_0(\sqrt{\alpha_1}r)}{D_1}\int_0^r r' I_0(\sqrt{\alpha_1}r')S_1(r')\, dr' \\ &+ \frac{I_0(\sqrt{\alpha_1}r)}{D_1}\int_r^R r' K_0(\sqrt{\alpha_1}r')S_1(r')dr'\ , \end{aligned}$$

and for $\phi_1^{(2)}$, we use only the definition of the Hankel transform to obtain

$$\phi_1^{(2)}(r) = p_1 S_2(r)\ .$$

By a similar procedure, we obtain the solution for $\phi_2(r)$ completing this way the entire solution of this problem using the infinite Hankel transform approach.

11.4 Results

We elaborated the general solutions in the previous sections, which for specific applications need the definitions of the parameter set and sources, respectively. Due to the fact that by virtue the specific source terms dominate the found solutions, in this section we present the influence of these source terms on the solution for a steady-state diffusion problem. The employed nuclear parameter sets are listed in Table 11.1, where for all cases we used $R = 5$, $k_{eff} = 0.95$ and $\nu = 2.5$ in the simulations.

Table 11.1 Nuclear parameter sets

	D_1	D_2	$S_0^{(1)}$	$S_0^{(2)}$	Σ_{R1}	Σ_{R2}	Σ_{12}	Σ_{21}	Σ_{f1}	Σ_{f2}
Set 1	1.43	0.39	4	0.0	0.029	0.104	0.015	0.00000	0.0041	0.0077
Set 2	1.43	0.39	4	0.1	0.029	0.104	0.015	0.00825	0.0041	0.0077
Set 3	1.43	0.51	4	0.1	0.052	0.081	0.015	0.00825	0.0041	0.0077
Set 4	1.13	0.39	4	0.1	0.052	0.081	0.015	0.00825	0.0051	0.0081

We consider cases with different sources and compare results from the application of the finite and infinite Hankel transform, respectively. To this end, we consider for all cases a dominant source with fast neutrons and one case with no thermal neutron source and three cases with a weak thermal neutron source. A further differentiation stems from different removal cross sections for the fast and the thermal neutron group. The last set is distinct in comparison to all other ones because of an increased fission cross section in the fast and the thermal neutron group. For the first case, only the fast neutron source contributes,

$$S_1(r) = S_0^{(1)} H(R-r)\ .$$

Upon applying the finite Hankel transform, the source term is

$$\begin{aligned}\bar{S}_1(\xi_i) = H_0\{S_1\} &= \int_0^R r S_0^{(1)} H(R-r) J_0(r\xi_i)\, dr \\ &= S_0^{(1)} \int_0^R r J_0(r\xi_i)\, dr \\ &= S_0^{(1)} \frac{R}{\xi_i} J_1(R\xi_i)\ ,\end{aligned}$$

and then using the final expression for the scalar neutron flux yields

$$\phi_1(r) = \frac{2}{R} S_0^{(1)} \sum_{i=1}^{\infty} \frac{A_2(\xi_i)}{\xi_i} \frac{1}{Det(M)(\xi_i)} \frac{J_0(r\xi_i)}{J_1(R\xi_i)} + \frac{2}{R} p_1 S_0^{(2)} \sum_{i=1}^{\infty} \frac{1}{\xi_i} \frac{1}{Det(M)(\xi_i)} \frac{J_0(r\xi_i)}{J_1(R\xi_i)}.$$

The procedure to obtain ϕ_2 works in close analogy to the one for ϕ_1. We obtained the following results for the selected parameter sets specified in Table 11.1 (Figs. 11.1, 11.2, 11.3 and 11.4).

By inspection of the obtained results, one observes qualitative agreement with what is expected from operational experience for processes inside a nuclear reactor core using this type of geometry. Quantitative properties are the flat current density at the origin, i.e. the flux with null derivative at $r = 0$ represents a symmetry condition. Furthermore, the vanishing flux at the outer boundary drags the flux from the maximum value at the center of the domain to decreasingly smaller values with

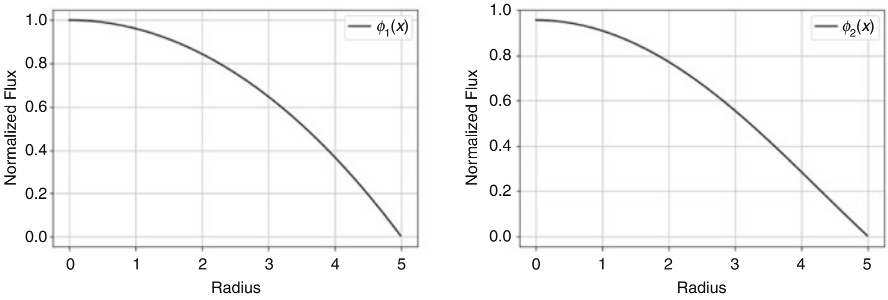

Fig. 11.1 The scalar neutron flux for the fast and thermal energy group Φ_1 and Φ_2 for parameter set 1

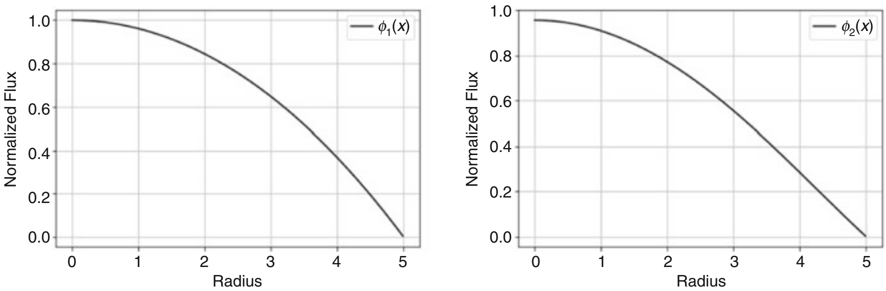

Fig. 11.2 The scalar neutron flux for the fast and thermal energy group Φ_1 and Φ_2 for parameter set 2

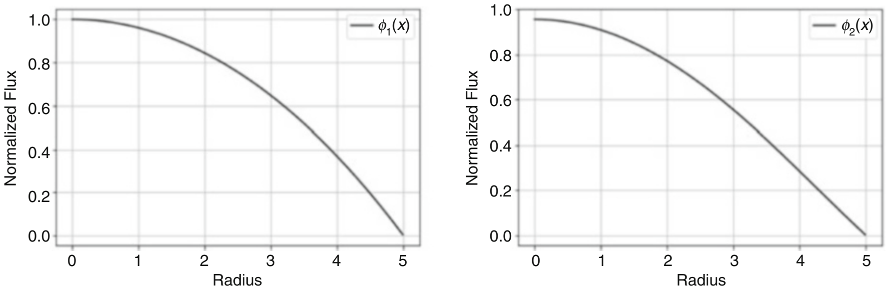

Fig. 11.3 The scalar neutron flux for the fast and thermal energy group Φ_1 and Φ_2 for parameter set 3

increasing radius. As a systematic feature for all parameter sets, the fast flux always shows a somewhat larger concavity than the thermal flux. In order to provide a quantitative comparison between the solutions from the finite and infinite Hankel transforms, a table with the numerical values for the normalized solutions ϕ_2 using both types of integral transforms is shown. Note that our findings agree fairly well with results in the literature [Da11].

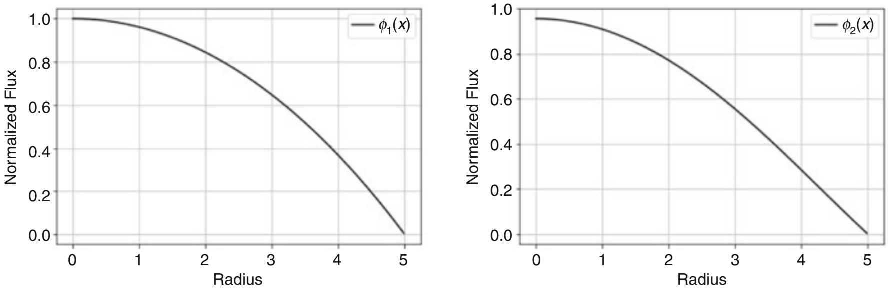

Fig. 11.4 The scalar neutron flux for the fast and thermal energy group Φ_1 and Φ_2 for parameter set 4

Table 11.2 Solution ϕ_2 using the finite Hankel (FHT) and the infinite Hankel Transform (IHT) for parameter set 1

r/R	FHT	IHT	r/R	FHT	IHT	r/R	FHT	IHT
0.0	1.000000	1.000000	0.0	1.000000	1.000000	0.0	1.000000	1.000000
0.1	0.987633	0.990253	0.1	0.987633	0.990253	0.1	0.987633	0.990253
0.2	0.950761	0.943312	0.2	0.950761	0.943312	0.2	0.950761	0.943312
0.3	0.890091	0.885213	0.3	0.890091	0.885213	0.3	0.890091	0.885213
0.4	0.806854	0.791032	0.4	0.806854	0.791032	0.4	0.806854	0.791032
0.5	0.702894	0.670122	0.5	0.702894	0.670122	0.5	0.702894	0.670122
0.6	0.580790	0.543311	0.6	0.580790	0.543311	0.6	0.580790	0.543311
0.7	0.444037	0.407630	0.7	0.444037	0.407630	0.7	0.444037	0.407630
0.8	0.297278	0.281963	0.8	0.297278	0.281963	0.8	0.297278	0.281963
0.9	0.146613	0.149313	0.9	0.146613	0.149313	0.9	0.146613	0.149313
1.0	0.000000	0.000000	1.0	0.000000	0.000000	1.0	0.000000	0.000000
$N = 50$			$N = 100$			$N = 500$		

In Table 11.2, results for ϕ_2 using the finite and the infinite Hankel transform are shown. Comparing the solutions for the finite Hankel transform for truncations at $N = 50$, $N = 100$ and $N = 500$ shows stability of the obtained solution, so that $N = 50$ already provides a solution with six significant digits. However, for solution by the finite Hankel transform, it is not obvious where to truncate the series in order to obtain an acceptable solution, which depend on the cumbersome task of determining the roots of the Bessel functions of order 0 and order 1. From the comparison of the solution ϕ_2 on the one hand by the finite and on the other hand by the infinite Hankel transform shows that the latter provides solutions fairly close to the ones by the finite integral transform. The advantage of the infinite Hankel transform over the finite case is that that there is no need to determine the lowest truncation of the series, which provides an acceptable solution. Besides having solved the stationary problem, where the found solution has value on its own right, the stationary case commonly provides the initial condition for a transient problem.

11.5 Conclusion

In the reported discussion, we presented and compared two integral methods to solve the stationary problem of two energy group neutron diffusion in cylinder geometry. Both methods, the finite and the infinite Hankel transform, generated comparable and acceptable results for the considered problems (parameter sets 1 to 4). While the finite Hankel transform seems to be the more natural tool to derive the solution due to the finite domain in consideration, the infinite sum of the analytical solution imposes the problem to determine truncation such that the approximate solution represents the exact solution to a prescribed accuracy. This task does not appear when the infinite Hankel transform is used, where it is the computation of the integrals that represents the challenge, and however numerical schemes for well-behaved integrands are usually no issue. All implemented simulations showed that both methods provide solutions with acceptable quality, but that the infinite integral transform is the simpler method especially due to the necessity to have a sufficiently large number N of terms in the series of the solution by the finite Hankel transform.

From the computational point of view, the source code for the implementation was written in Python 3.8 for both integral transforms and ran on a simple home computer, Intel(R) Core(TM) i3-4150 CPU @ 3.50 GHz (64-bit operating system) with Microsoft Windows 10 operational system. For the solution by the infinite Hankel transform the CPU, time amounted to a few seconds, while the finite Hankel transform provided also a solution in a small but larger computational time, however with increasing tendency for increasing N. Our findings allow to conject that the solution by the infinite Hankel transform in principle opens pathways to increase the problem setup, such as to include more energy groups and allow for heterogeneous domains, which designs these new cases closer to the ones of real reactor cores.

References

[BeGl70] Bell, G.I., Glasstone, S.: Nuclear Reactor Theory. Van Nostrand Reinhold Company, New York (1970)

[Da11] Dababneh, S.: An alternative solution of the neutron diffusion equation in cylindrical symmetry. Ann. Nuclear Energy **38**, 1040–1043 (2011)

[Du06] Dulla, S.: Space asymptotic method for the study of neutron propagation. Ann. Nuclear Energy **33**, 932–940 (2006)

[Fe11] Fernandes, J.C.L.: An analytical solution of multi-group neutron diffusion equation for a cylindrical symmetry using Hankel Transform, Thesis, Brazil (2011)

[Fe13] Fernandes, J.C.L.: On a comparative analysis of the solutions of the kinetic neutron diffusion equation by the Hankel transform formalism and the spectral method. Progr. Nuclear Energy **69**, 71–76 (2013)

[Gl03] Goncalves, G.A.: Analytical solution of neutron transport equation in cylindrical geometry, Thesis, Brazil (2003)

[La66] Lamarsh, J.R.: Introduction to Nuclear Reactor Theory. Addison Wesley Publishing, New York (1966)

[LeMi84] Lewis, E.E., Miller Jr., W.F.: Computational Methods of Neutron Transport. Wiley, New York (1984)
[Ma17] Manish, R.: Solution of neutron diffusion equation in 2d polar coordinates using nodal integral method. Ann. Nuclear Energy **105**, 69–78 (2017)
[Ma21] Manish, R.: Nodal Integral method for multi-group neutron diffusion equation in three dimensional cylindrical coordinate system. Ann. Nuclear Energy **151**, 107–114 (2021)
[No21] Di Nora, V.A.: Optimization of multi-group energy structures for diffusion analyses of sodium-cooled fast reactors assisted by simulated annealing – Part I: Methodology demonstration. Ann. Nuclear Energy**155**, 108–183 (2021)
[Ol19] Oliveira, F.R.: On an analytical solution for the two energy group neutron space-kinetic equation in heterogeneous cylindrical geometry. Ann. Nuclear Energy **133**, 216–220 (2019)
[Oz01] Ozgener, H.A.: A multi-region boundary element method for multigroup neutron diffusion calculations. Ann. Nuclear Energy **28**, 585–616 (2001)
[Oz73] Ozisik, M.N.: Radiative Transfer and Interaction with Conductions and Convection. Wiley, New York (1973)
[Vi08] Vilhena, M.T.B.: An analytical solution for the general perturbative diffusion equation by integral transform techniques. Ann. Nuclear Energy **35**, 2410–2413 (2008)