SUM Classes and Quotient Generalized Interval Systems

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Abstract. The present paper develops algebraic properties of the SUM-class system first developed by Richard Cohn and explored by Robert Cook and Joseph Straus, in the context of David Lewin's Generalized Interval System (GIS) concept. Motivated by his observation that harmonic triads whose pitch classes sum to a given value modulo 12 share certain voice-leading properties, Cohn defined SUM classes for the 24 consonant (major and minor) triads, and defined transformations on these equivalence classes. We present the SUM-class system as a quotient GIS structure, and explore the dual quotient GIS implied by Lewin's theory for non-commutative GISs, and we generalize to other types of pitch-class sets (other set-classes).

Keywords: Generalized Interval System · Group homomorphism · Quotient group · SUM class

1 Introduction

In the context of consonant triads as pitch-class sets (i.e., major and minor triads, set-class 3-11), Cohn [1,2], and Cook [3] observed that the total pitch-class voice-leading interval between triads X and Y remains unchanged when either is transposed by a major third (4 semitones) in either direction (i.e., by T_4 or T_8). Cohn also noted that the pitch classes of major-third-related triads sum to the same value modulo 12. Generalizing to pitch-class sets of a given cardinality, the following definitions and proposition unify the voice-leading interval and SUM concepts and motivate the SUM-class definition. In this paper we assume the usual 12 pitch-class universe and all values are reduced modulo 12.

Definition 1. *Let* $X = \{x_1, \ldots, x_n\}$ *and* $Y = \{y_1, \ldots, y_n\}$ *be pitch-class sets of cardinality* $1 \leq n \leq 12$. *Let* $VL(X, Y) = \sum_{i=1}^{n}(y_i - x_i) \bmod 12$. *We call* $VL(X, Y)$ *the* voice-leading interval *from* X *to* Y.

While the sets in this definition are indexed, it turns out that the voice-leading interval does not depend on this ordering:

Definition 2. *Let* X *be a pitch-class set.* $SUM(X) = \sum_{x \in X}(x) \bmod 12$.

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Proposition 1. *Let $card(X) = card(Y)$. $VL(X,Y) = SUM(Y) - SUM(X)$.*

Proof. $VL(X,Y) = \sum_{i=1}^{n}(y_i - x_i) = (y_1 - x_1) + \cdots + (y_n - x_n) \bmod 12 = (y_1 + \cdots + y_n) - (x_1 + \cdots + x_n) = SUM(Y) - SUM(X)$ (adapted from [1], 286).

Since the proposition shows that the voice-leading interval depends only on the SUMs, defined in terms of commutative mod 12 addition, the voice-leading interval from X to Y is a property of the unordered sets. It is evident that $VL(X,Y) = -VL(Y,X)$.

Motivated by the above observation, we define SUM classes. For a given n, let S_n be the set of all pitch-class sets of cardinality n. By Definition 2, for all sets $X \in S_n$ we have a function SUM: $S_n \mapsto \mathbb{Z}_{12} : X \mapsto \mathrm{SUM}(X)$. The image of S_n under this function is a subset of $\mathbb{Z}_{12}$, $\mathrm{im}(S_n)$. Define the SUM classes of S_n to be the inverse images of elements in $\mathrm{im}(S_n)$. It follows that the SUM classes partition S_n and are the equivalence classes of the equivalence relation $\equiv_{S_n}$ where $X \equiv_{S_n} Y$ if and only if $\mathrm{SUM}(X) = \mathrm{SUM}(Y)$. This is the natural equivalence relation defined by the function SUM.

More often, one is interested in a refinement of this equivalence relation, where SUM is restricted to a given set-class (an orbit of the group T_n/I_n acting on pitch-class subsets). The set-classes partition the sets S_n, and SUM restricted to set-class α defines an equivalence $\equiv_\alpha$ on α. For example, if we take the canonical example of the 24 consonant triads, Forte set-class 3-11, the image of 3-11 under SUM is the set $\{1, 2, 4, 5, 7, 8, 10, 11\}$. The inverse images of these eight elements are the 3-11 SUM classes, which we label with square brackets, [1], [2], [4], [5], [7], [8], [10], [11]. For example, here SUM class [1] = {C-sharp minor, F minor, A minor}. E.g., SUM(F minor) $= 0 + 5 + 8 = 13 \equiv 1 \bmod 12$.

2 SUM-Class Transformation Groups and Quotient Generalized Interval Systems

We recall the definition of the T_n/I_n group: $T_n : \mathbb{Z}_{12} \mapsto \mathbb{Z}_{12} : z \mapsto n + z$; $I_n : \mathbb{Z}_{12} \mapsto \mathbb{Z}_{12} : z \mapsto n - z$. The group product is composition of functions, and $T_nT_m = T_{n+m}, T_nI_m = I_{n+m}, I_mT_n = I_{m-n}, I_mI_n = T_{m-n}$. It follows that the defining relations in terms of generators are $(T_1)^{12} = T_0$, $(I_j)^2 = T_0$, and $T_jI_0 = I_0T_{12-j}$; that is, the group is dihedral of order 24. Extending the definition of these operations on individual pitch classes to pitch-class sets, we have a group action on the power set $2^{\mathbb{Z}_{12}}$ of pitch-class subsets: for all subsets x, we have $T_0(x) = x$, and for all $f, g \in T_n/I_n, f(g(x)) = fg(x)$. The orbits are the set-classes, and therefore the group acts transitively on each set-class: if x, y are in a given set-class, there exists $f \in T_n/I_n$ such that $f(x) = y$; for set-classes with 24 elements (i.e., set-classes of sets of cardinality k, $2 < k < 10$, with only the trivial symmetry), the action is simply transitive: f mapping x to y is unique.

From Lewin [4], 157–158, a simply transitive action of a group G on a set S is equivalent to a Generalized Interval System (GIS). The triadic GIS for $S =$ 3-11, $G = T_n/I_n$ is well known. We seek a GIS structure for the triadic

SUM-classes. Recalling Cohn's observation that voice-leading intervals between triads are invariant under pitch-class transpositions by 4 semitones, consider the subgroup $H = \{T_0, T_4, T_8\}$. H is a normal subgroup of $G = T_n/I_n$: since H commutes with all the transpositions, all that is needed to check is that for all k, $I_k(H) = (H)I_k$. $(H)I_k = (\{T_0, T_4, T_8\})I_k = \{I_k, I_{k+4}, I_{k+8}\}$ and $I_k(H) = I_k(\{T_0, T_4, T_8\}) = \{I_k, I_{k-4}, I_{k-8}\} = \{I_k, I_{k+8}, I_{k+4}\}$. Thus, left and right cosets coincide, and H is normal in G. By the principal theorems of group theory, the cosets form the quotient group $\frac{G}{H}$, isomorphic with the image of G under a homomorphism h with kernel H. The group product for $\frac{G}{H}$ is inherited from the parent group $G = T_n/I_n$ in the natural way (demonstrated below). Since for finite G $o(\frac{G}{H}) = o(G)/o(H)$, the order of the quotient group here is 24/3 = 8. The equivalence relation $\equiv_{3-11}$ partitioned the 24 harmonic triads into 8 SUM-classes, each with 3 members, and we assert that the quotient group and the set $\mathcal{S}$ of SUM classes form a GIS, a quotient GIS of the original (defined below).

We reduce the $\equiv_{3-11}$ equivalence classes to the respective sums x, symbolized $[x]$, but for present purposes in the triadic case it is useful to tabulate them (writing lower-case letters for minor, upper-case for major): [1] = {c♯, f, a}, [2] = {C♯, F, A}, [4] = {d, f♯, b♭}, [5] = {D, F♯, B♭}, [7] = {e♭, g, b}, [8] = {E♭, G, B}, [10] = {e, g♯, c}, [11] = {E, A♭, C}.

We similarly tabulate and name the elements of $\frac{G}{H}$, that is, the cosets of normal subgroup $\{T_0, T_4, T_8\}$ in T_n/I_n. To simplify the typography we name the cosets $\mathcal{T}_j = T_jH$, and $\mathcal{I}_j = I_jH, j = 0, 1, 2, 3$.

$$\mathcal{T}_0 = T_0H = \{T_0, T_4, T_8\}$$

$$\mathcal{T}_1 = T_1H = \{T_1, T_5, T_9\}$$

$$\mathcal{T}_2 = T_2H = \{T_2, T_6, T_{10}\}$$

$$\mathcal{T}_3 = T_3H = \{T_3, T_7, T_{11}\}$$

$$\mathcal{I}_0 = I_0H = \{I_0, I_4, I_8\}$$

$$\mathcal{I}_1 = I_1H = \{I_1, I_5, I_9\}$$

$$\mathcal{I}_2 = I_2H = \{I_2, I_6, I_{10}\}$$

$$\mathcal{I}_3 = I_3H = \{I_3, I_7, I_{11}\}$$

The general theory already tells us that $\frac{G}{H}$ is a group with product induced from the definition of operations in T_n/I_n, but let us explicitly state the composition rules. Given $\mathcal{T}_i, \mathcal{T}_j, \mathcal{I}_k, \mathcal{I}_l \in \frac{G}{H}$, we have $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_{(i+j) \bmod 4}, \mathcal{T}_i\mathcal{I}_k = \mathcal{I}_{(i+k) \bmod 4}, \mathcal{I}_k\mathcal{T}_i = \mathcal{I}_{(k-i) \bmod 4}, \mathcal{I}_k\mathcal{I}_l = \mathcal{T}_{(k-l) \bmod 4}$. Thus, we have relations $(\mathcal{T}_1)^4 = \mathcal{T}_0, (\mathcal{I}_j)^2 = \mathcal{T}_0$, and $\mathcal{T}_j\mathcal{I}_0 = \mathcal{I}_0\mathcal{T}_{4-j}$, that is, $\frac{G}{H}$ is dihedral of order 8 (the homomorphic image of a dihedral group).

Next we tabulate the group action of $\frac{G}{H}$ on the set $\mathcal{S}$ of $\equiv_{3-11}$ SUM classes. Each element defines a permutation of $\mathcal{S}$, expressed as a product of cycles:

$\mathcal{T}_0 : \mathcal{S} \to \mathcal{S} : (\,)$

$\mathcal{T}_1 : \mathcal{S} \to \mathcal{S}$: ([1] [4] [7] [10]) ([2] [5] [8] [11])

$\mathcal{T}_2 : \mathcal{S} \to \mathcal{S}$: ([1] [7]) ([4] [10]) ([2] [8]) ([5] [11])

$\mathcal{T}_3 : \mathcal{S} \to \mathcal{S}$: ([1] [10] [7] [4]) ([2] [11] [8] [5])

$\mathcal{I}_0 : \mathcal{S} \to \mathcal{S}$: ([1] [11]) ([2] [10]) ([4] [8]) ([5] [7])

$\mathcal{I}_1 : \mathcal{S} \to \mathcal{S}$: ([1] [2]) ([4] [11]) ([5] [10]) ([7] [8])

$\mathcal{I}_2 : \mathcal{S} \to \mathcal{S}$: ([1] [5]) ([2] [4]) ([7] [11]) ([8] [10])

$\mathcal{I}_3 : \mathcal{S} \to \mathcal{S}$: ([1] [8]) ([2] [7]) ([4] [5]) ([10] [11])

This tabulation shows that the group action of $\frac{G}{H}$ on $\mathcal{S}$ is simply transitive: for every ordered pair of elements (s_1, s_2) in $\mathcal{S}\times\mathcal{S}$, there exists exactly one $g \in \frac{G}{H}$ such that $g(s_1) = s_2$. As noted above, a simply transitive group action on a set S determines a GIS, so we have $\text{GIS}_1 = (\frac{G}{H}, \mathcal{S})$. It will later be significant to note that $\frac{G}{H}$ is non-commutative (non-abelian), so GIS_1 is a *non-commutative GIS*. This procedure holds generally, as demonstrated below, permitting the inclusion of GIS_1 in the class of *quotient GISs*.

Proposition 2. *Let $(\mathcal{G}, S)$ be a GIS, with $\mathcal{H}$ a normal subgroup in $\mathcal{G}$. Then the quotient group $\frac{\mathcal{G}}{\mathcal{H}}$ induces a set S' of equivalence classes on S, such that $\frac{\mathcal{G}}{\mathcal{H}}$ acts simply transitively on S'. That is, $(\frac{\mathcal{G}}{\mathcal{H}}, S')$ is a GIS, and one may call it a* quotient GIS.

Lemma 1. *The quotient group $\frac{\mathcal{G}}{\mathcal{H}}$ induces a set S' of equivalence classes on S.*

Proof. Consider the restriction to $\mathcal{H}$ of the action of $\mathcal{G}$ on S, $\mathcal{H}(S) = \{h(s)|h \in \mathcal{H}, s \in S\}$. For all $s \in S$, let $\sigma_s = \mathcal{H}(s)$. Let $S' = \{\sigma_s | \forall s \in S\}$. S' defines a partition of S: All elements of S belong to some σ_s, since $e \in \mathcal{H}, e(s) = s \in \sigma_s$. Suppose we have $s_1 \in \sigma_{s_1}$ and $s_1 \in \sigma_{s_2}$. Then for some $h \in \mathcal{H}, s_1 = h(s_2)$. Let t be any element in σ_{s_1}, then for some $h_1 \in \mathcal{H}$, $t = h_1(s_1) = h_1 h(s_2)$, and $t \in \sigma_{s_2}$. Thus $\sigma_{s_1} = \sigma_{s_2}$. Suppose $s_1 \notin \sigma_{s_2}$, and $t \in \sigma_{s_1} \cap \sigma_{s_2}$. Then there exist $h_1, h_2 \in \mathcal{H}$ such that $h_1(s_1) = t = h_2(s_2)$. Then $h_1^{-1} h_2(s_2) = s_1$, so $s_1 \in \sigma_{s_2}$, contradiction, and σ_{s_1} and σ_{s_2} are disjoint. Thus S' partitions S, and the members of S' are equivalence classes of the equivalence relation defined by the partition.

Lemma 2. *$\frac{\mathcal{G}}{\mathcal{H}}$ acts simply transitively on S'.*

Proof. Let $x\mathcal{H}, y\mathcal{H}$ be distinct cosets $\in \frac{\mathcal{G}}{\mathcal{H}}$ (i.e., $y^{-1}x \notin \mathcal{H}$), and $\sigma_s \in S'$. For any $\sigma_s \in \mathcal{S}', \mathcal{H}(\sigma_s) = \mathcal{H}(\mathcal{H}(s)) = \mathcal{H}\mathcal{H}(s) = \mathcal{H}(s) = \sigma_s$, by definition, and

$x\mathcal{H}(y\mathcal{H}(\sigma_s)) = x\mathcal{H}y\mathcal{H}(\sigma_s)$, so $\frac{\mathcal{G}}{\mathcal{H}}$ acts transitively on S'. Suppose $x\mathcal{H}(\sigma_s) = y\mathcal{H}(\sigma_s)$. Then $(y\mathcal{H})^{-1}x\mathcal{H}(\sigma_s) = \sigma_s$, but also $(y\mathcal{H})^{-1}x\mathcal{H}(\sigma_s) = y^{-1}\mathcal{H}x\mathcal{H}(\sigma_s) = y^{-1}x\mathcal{H}\mathcal{H}(\sigma_s) = y^{-1}x\mathcal{H}(\sigma_s) = y^{-1}x(\sigma_s) = \sigma_s$, implies $y^{-1}x \in \mathcal{H}$, contrary to assumption. For every $s, t \in S'$, there exists $x\mathcal{H} \in \frac{\mathcal{G}}{\mathcal{H}}$ such that $x\mathcal{H}(s) = t$ (by definition of $\mathcal{S}'$), and $x\mathcal{H}$ is unique (by the above demonstration). Thus, $\frac{\mathcal{G}}{\mathcal{H}}$ acts simply transitively on S'.

$(\frac{\mathcal{G}}{\mathcal{H}}, S')$ is therefore a (quotient) GIS.

From Proposition 1 and the definition of the SUM classes, the corollary follows that the voice-leading interval from any member of $[x]$ to any member of $[y]$ is $y-x$ (modulo 12). Slightly abusing the notation, we have $VL([x],[y]) = y-x$. Recall that the voice-leading intervals are not the intervals of Generalized Interval System GIS_1—those are defined by the group action. From Lewin [4], 157, the interval i from $[x]$ to $[y]$, $i = \text{int}([x],[y])$, is identified with the unique $g \in \frac{G}{H}$ such that $g([x]) = [y]$, whereas by Definition 1 a voice-leading interval is an integer modulo 12. We are, however, concerned here with voice-leading intervals, the motivation for $\mathcal{S}$, the set of SUM classes.

For a given GIS (G, S), Lewin ([4], 48) defines an *interval-preserving transformation* to be a mapping $X : S \to S$ such that for all $s, t \in S$, $\text{int}(X(s), X(t)) = \text{int}(s,t)$. Here, *int* refers to the interval function of the GIS just discussed. Let's extend his definition to voice-leading intervals:

Definition 3. *If for all SUM classes $[x],[y] \in \mathcal{S}$ a transformation X on $\mathcal{S}$ has the property $VL(X([x]), X([y])) = VL([x],[y])$, then X is said to be a* voice-leading-interval-preserving transformation (VL-preserving transformation)*.*

Since $VL([x],[y]) = y - x$, X is a VL-preserving transformation if and only if $VL(X([x]), X([y])) = y - x$. The elements of $\frac{G}{H}$ are transformations of $\mathcal{S}$; are they VL-preserving?

Proposition 3. *The elements of the $\frac{G}{H}$ subgroup $\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ are VL-preserving.*

Proof. For $j = 0, 1, 2, 3$, $\mathcal{T}_j : \mathcal{S} \mapsto \mathcal{S} : [x] \mapsto [(x + 3j) \bmod 12]$. Then for all $[x],[y] \in \mathcal{S}$, $VL(\mathcal{T}_j([x]), \mathcal{T}_j([y])) = VL([(x+3j) \bmod 12], [(y+3j) \bmod 12]) = ((y+3j) - (x+3j)) \bmod 12 = (y-x) \bmod 12 = VL([x],[y])$. Therefore, the $\mathcal{T}_j$ are VL-preserving.

The elements of the coset of $\mathcal{T}$ as a (normal) subgroup of $\frac{G}{H}, \mathcal{I} = \{\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$, are not VL-preserving; rather, they are VL-reversing.

Definition 4. *If for all SUM classes $[x],[y] \in \mathcal{S}$ a transformation X on $\mathcal{S}$ has the property $VL(X([x]), X([y])) = VL([y],[x])$, then X is said to be a* voice-leading-interval-reversing transformation (VL-reversing transformation)*.*

Since $VL([x],[y]) = y - x, VL([y],[x]) = x - y$, and X is a VL-reversing transformation if and only $VL(X([x]), X([y])) = x - y$.

Proposition 4. *The elements of* $\mathcal{I}$ *are VL-reversing.*

Proof. For $j = 0, 1, 2, 3$, $\mathcal{I}_j : \mathcal{S} \mapsto \mathcal{S} : [x] \mapsto [(3j - x) \bmod 12]$. Then for all $[x], [y] \in \mathcal{S}$, $VL(\mathcal{I}_j([x]), \mathcal{I}_j([y])) = VL([(3j - x) \bmod 12], [(3j - y) \bmod 12]) = ((3j - y) - (3j - x)) \bmod 12 = (x - y) \bmod 12 = VL([y], [x])$. Therefore, the $\mathcal{I}_j$ are VL-reversing.

3 The Dual Quotient Generalized Interval System

Those familiar with Lewin's theory know that every non-commutative GIS (G, S) has its dual, with a different group G', isomorphic to G but with a distinct simply transitive action on S. The groups G and G' are subgroups of the group of all permutations of S, the symmetric group $Sym(S)$, so elements of the two groups compose with each other within $Sym(S)$. As Lewin demonstrates, the groups are each other's commuting groups; as subgroups of $Sym(S)$, each other's centralizer subgroup: every element of G commutes with every element of G'. It follows that they play the role of each other's group of interval-preserving transformations, as seen in the commutative diagram in Fig. 1. Let f belong to G and f' belong to G' and let s_1, s_2, s_3, s_4 be members of S. The diagram shows that f and f' commute: $f'f(s_1) = f'(s_2) = s_4$ and $ff'(s_1) = f(s_3) = s_4$. But this is equivalent to f' preserving the intervals in (G, S) and f preserving the intervals in (G', S): $\text{int}(s_1, s_2) = \text{int}(s_3, s_4)$ in (G, S) since $f(s_1) = s_2$ and $f(s_3) = s_4$, so in (G, S) $\text{int}(f'(s_1), f'(s_2)) = \text{int}(s_1, s_2)$, and thus f' satisfies Lewin's definition of an interval-preserving transformation. The same demonstration shows that in (G', S), $\text{int}'(f(s_1), f(s_3)) = \text{int}'(s_1, s_3)$; f is an interval-preserving transformation of (G', S).

$$\begin{array}{ccc} & f & \\ s_1 & \longrightarrow & s_2 \\ f' \downarrow & & \downarrow f' \\ s_3 & \longrightarrow & s_4 \\ & f & \end{array}$$

Fig. 1. Commutativity is equivalent to interval preservation

So much is known, but we are interested here also in voice-leading intervals, and in the dual of a (non-commutative) quotient GIS. It is known that the dual to the GIS $(T_n/I_n$, 3-11) is the GIS of the action of the full neo-Riemannian group on 3-11 ([5], 180–181). This group is often referred to as the *P*-*L*-*R* group, because it is generated by the three contextual inversions: P, which sends a triad to its Parallel, as C to c, c to C; L, which sends a triad to its *Leittonwechsel*, as C to e, e to C; and R, which sends a triad to its Relative, as C to a, a to C (see Cohn, [6]). One of these generators is redundant: composing RL and taking it as one generator, of order 12, and taking one of the involutions as the other generator, we have a presentation of the group that shows it to be isomorphic to the order 24 dihedral group, that is, to the T_n/I_n group: $(RL)^{12} = e, L^2 = e$ (where e is the identity operation), and $(RL)^j L = L(RL)^{12-j}$. We may therefore refer to this as the R/L group, and to the (R/L, 3-11) GIS. In this notation, the parallel structures with the T_n/I_n and its subgroups are made evident. Music theorists, who compute relatively easily with P, L, and R, conventionally express these transformations as compositions of them. In all but one case the minimal compositions are of length at most 4.

The distinguishing character of the three neo-Riemannian involutions is the parsimonious voice-leading they entail: all three preserve two common tones, the remaining tone moving by semitone under P and under L and by two semitones under R. One might imagine that this GIS is therefore well-suited to addressing voice-leading considerations, and indeed Cohn [1] treats SUM classes from this side of the duality (and moreover dives right into the quotient group). See also Cook [3], chapter 2. Since this is a true duality, one may logically begin from either side of it. The path chosen here is heuristic, considering that the T_n/I_n is perhaps better known and more convenient computationally.

For notational simplicity, set $Q = RL$: Q transposes major triads by T_7, minor triads by T_5. (NB: Music theorists often use right functional orthography in neo-Riemannian contexts. Since operations from the dual GISs will be composed, left orthography will be followed here.) The construction of the dual quotient group and dual quotient GIS_2 to $\text{GIS}_1 = (\frac{G}{H}, \mathcal{S})$ again begins with an order 3 cyclic subgroup $H' = \{Q^0, Q^4, Q^8\}$. H' is normal in $G' = R/L$, so there exists the quotient group $\frac{G'}{H'}$, of order 8. The group mirrors its dual, reflected in the notation for the cosets, which are expressed in terms of elements Q^j or $Q^j L$, for $j = 0, 1, 2, 3$, and as compositions of P, L, R. Following Cohn [1], we use $\mathcal{X}$ (for eXchange) for the sets of contextual inversions, $\mathcal{Y}$ for the others.

$$\mathcal{Y}_0 = \{Q^0, Q^4, Q^8\} = \{E, PL, LP\}$$

$$\mathcal{Y}_1 = \{Q^1, Q^5, Q^9\} = \{RL, RP, RPLP\}$$

$$\mathcal{Y}_2 = \{Q^2, Q^6, Q^{10}\} = \{RPRL, (RP)^2, LRPR\}$$

$$\mathcal{Y}_3 = \{Q^3, Q^7, Q^{11}\} = \{PLPR, LR, PR\}$$

$$\mathcal{X}_0 = \{L, Q^4 L, Q^8 L\} = \{L, P, PLP\}$$

$$\mathcal{X}_1 = \{QL, Q^5 L, Q^9 L\} = \{R, PLR, LPR\}$$

$$\mathcal{X}_2 = \{Q^2 L, Q^6 L, Q^{10} L\} = \{RLR, RPR, RPLPR\}$$

$$\mathcal{X}_3 = \{Q^3 L, Q^7 L, Q^{11} L\} = \{PRL, PRP, LRL\}$$

The composition rules are the same as in $\frac{G}{H}$, replacing $\mathcal{T}_j$ by $\mathcal{Y}_j$ and $\mathcal{I}_k$ by $\mathcal{X}_k$. The quotient groups are thus isomorphic as groups, but the action of $\frac{G'}{H'}$ on $\mathcal{S}$ is very different. It is tabulated below, again operations as products of cycles.

$\mathcal{Y}_0 : \mathcal{S} \mapsto \mathcal{S} : (\ )$

$\mathcal{Y}_1 : \mathcal{S} \mapsto \mathcal{S} : ([1]\ [4]\ [7]\ [10])\,([2]\ [11]\ [8]\ [5])$

$\mathcal{Y}_2 : \mathcal{S} \mapsto \mathcal{S} : ([1]\ [7])([2]\ [8])\,([4]\ [10])\,([5]\ [11])$

$\mathcal{Y}_3 : \mathcal{S} \mapsto \mathcal{S} : ([1]\ [10]\ [7]\ [4])\,([2]\ [5]\ [8]\ [11])$

$\mathcal{X}_0 : \mathcal{S} \mapsto \mathcal{S} : ([1]\ [2])\,([4]\ [5])\,([7]\ [8])\,([10]\ [11])$

$\mathcal{X}_1 : \mathcal{S} \mapsto \mathcal{S} : ([1]\ [11])\,([2]\ [4])\,([5]\ [7])\,([8]\ [10])$

$\mathcal{X}_2 : \mathcal{S} \mapsto \mathcal{S} : ([1])\ [8])\,([2]\ [7])\,([4]\ [11])\,([5]\ [10])$

$\mathcal{X}_3 : \mathcal{S} \mapsto \mathcal{S} : ([1]\ [5])\,([2]\ [10])\,([11]\ [7])\,([8]\ [4])$

Comparing the actions on $\mathcal{S}$ of the respective dual groups, one sees that, of course, the identity operations are the same—$\mathcal{Y}_0 = \mathcal{T}_0 = 1_{\mathcal{S}}$—and also $\mathcal{Y}_2 = \mathcal{T}_2$. These two operations commute with everything in both GIS_1 and GIS_2 (central in both quotient groups), so are interval-preserving transformations (moreover, operations) in both. $\mathcal{Y}_0$ and $\mathcal{Y}_2$ are also both obviously VL-preserving in GIS_2, but no other elements in $\frac{G'}{H'}$ are VL-preserving, as inspection of the tabulation makes clear. In GIS_1 the subgroup $\mathcal{T}$ was VL-preserving. In GIS_2, in the analogous subgroup $\mathcal{Y}$, the operations $\mathcal{Y}_1$ and $\mathcal{Y}_3$ are not VL-preserving. In the coset of $\mathcal{Y}$ the eXchange operations do interact cogently with the voice-leading intervals defined by VL, even though none are VL-preserving: each exchanges pairs $[x], [y]$ such that for a fixed n, $\mathrm{VL}([x], [y]) = \pm n \bmod 12$. The tabulation shows that $\mathcal{X}_0$ exchanges by ± 1, $\mathcal{X}_3$ by ± 4, $\mathcal{X}_2$ by ± 7, and $\mathcal{X}_1$ by ± 10. This fact motivated Cohn's notation for his sum-class transformation group, X_1, X_4, X_7, X_{10}. His Y/X group was not explicitly defined as a quotient structure, but is precisely the same group as the one derived here: $\mathcal{Y}_j = Y_{3j}, j = 0, 1, 2, 3, \mathcal{X}_0 = X_1, \mathcal{X}_1 = X_{10}, \mathcal{X}_2 = X_7, \mathcal{X}_3 = X_4$.

In [1], Cohn thematizes his X_1 and X_{10} operations because, just as the four-fold repetition of the alternation of R with P sends any given triad through an octatonic cycle, the four-fold repetition of the alternation of X_1 with X_{10} sends the SUM classes in a cycle matching an octatonic scale: $X_1([1]) = [2], X_{10}([2]) = [4], X_1([4]) = [5]$, etc. More significantly, the union of each pair of SUM classes related by X_1 forms a hexatonic region, the four pairs cover the four regions, and the subgroup formed by the union of the cosets associated with the identity Y_0 and with X_1 is the hexatonic subgroup, $\{E, PL, LP, P, L, PLP\}$. The union of each pair of SUM classes related by X_{10} forms a Weitzmann region, the four pairs cover the four regions, and the union of the cosets associated with the identity

Y_0 and with X_{10} is the Weitzmann subgroup $\{E, PL, LP, R, LPR, PLR\}$ (cf. [2,7]). In [2], Cohn refers to "voice-leading zones" in lieu of SUM classes, and presents many musical analyses employing them.

Applying Lewin's general theory, all the elements of $\frac{G'}{H'}$ commute with all of those of its dual $\frac{G}{H}$, all are interval-preserving transformations for GIS_1, and vice versa for elements of $\frac{G}{H}$ with respect to GIS_2. In [8], Orvek presents an analysis of passages from Charles Villiers Stanford's setting of Keats's poem *La belle dame sans merci*, employing a commutative diagram that relates two chromatic sequences. The sequences appear during a description of the protagonist's dream. The reduction to SUM classes employs a VL-preserving $\mathcal{T}$ transformation from GIS_1 commuting with an $\mathcal{X}$ transformation from GIS_2 that always exchanges triads at a fixed VL-interval distance. The reduction is shown in Fig. 2, adapted from ([8], fig. 2.4). Orvek presents a similar analysis, with set-class 3-3, of music from *Nacht* in Schoenberg's *Pierrot Lunaire*, op. 21. See also Cook's triadic analyses of music by César Franck [3].

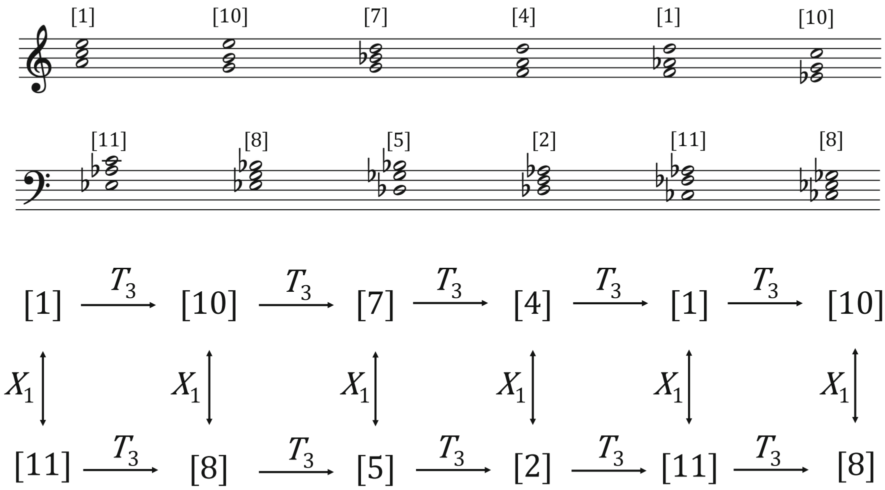

Fig. 2. A commutative transformational network relating mm. 98–108 (top) and mm. 122–128 (bottom) from Charles V. Stanford, *La belle dame sans merci*

4 Other Set-Classes

At the outset of this paper, the SUM-class equivalence relation was defined very generally, and then stated at the level of the asymmetric pitch-class set-classes, those with 24 members, and that is the level of generality at which we continue. The question is: is there always a quotient group that acts simply transitively on the SUM classes for a given asymmetric pitch-class set? The answer, sketched below, is yes, but it is not always a quotient of the T_n/I_n group. The problem is that, as we will see, some groups acting simply transitively on the SUM classes must be cyclic (generated by a single element). The quotient group of a dihedral group is the homomorphic image of that group, so is itself dihedral (except in the trivial case where the whole group is mapped to the group with one element, which only applies to the empty set-class and the aggregate $\mathbb{Z}_{12}$, where [0] or [6], respectively, is the only SUM class). We are also not interested in the other trivial case, the quotient by the identity subgroup, because there are at most 12 SUM classes. There are three normal subgroups of T_n/I_n of order 12, the cyclic subgroup of twelve transpositions, and two dihedral subgroups that contain the 6 even transpositions, one with the 6 inversions of even index, the other with the 6 inversions of odd index. In general, any subgroup H such that $o(H) = \frac{1}{2}o(G)$ is normal: you're either on the bus or off the bus, the bus here being H; there are just 2 cosets, H and $G \setminus H$, and $\forall g \in G, gH = Hg$, since $gH = H$ if and only if $g \in H$, (if and only if $g \notin H, gH = G \setminus H$). The quotient of T_n/I_n by any of the three subgroups is therefore a subgroup of order 2, potentially applicable to a set-class with just two SUM classes.

The cyclic subgroups are normal, by the earlier argument for $\{T_0, T_4, T_8\}$. We omit the demonstration, but the only dihedral subgroups that are normal are the two of index 2 discussed above.

Orvek [8] has shown that in the cases of all but one asymmetric trichordal set-class, the situation is the same as for 3-11, and the order 3 cyclic subgroup $\{T_0, T_4, T_8\}$ effects the appropriate equivalence relations on the parent group and parent set-class. This is because transposing a trichord by any multiple n of 4 semitones adds $3 \times 4n = 12n \equiv 0 \bmod 12$ to the sum of the constituent pitch classes, thus leaves SUM classes fixed. The exception is the 3-4: (015) set class, which can be seen to admit only SUM classes [0], [3], [6], [9]. (Straus [9] refers to the exceptions as *maverick sets*.) The only order 4 quotient group is modulo the only order 6 normal subgroup, the transpositions of even index. This non-cyclic (dihedral) four-group has no element that holds SUM classes fixed, thus fails.

The appropriate simply transitive group on 3-4 from which to construct a quotient that succeeds is an abelian group T_n/J of order 24, isomorphic to $\mathbb{Z}_{12} \times \mathbb{Z}_2$, with the usual order 12 subgroup of transpositions and one contextual inversion that commutes with all the transpositions. The appropriate inversion J must hold all four SUM classes globally fixed, since it must be in the subgroup which is the identity element in the quotient subgroup. There are three possible choices for J; whichever choice is made, the other two appear as T_4J and T_8J. We may define the contextual inversion to be J such that it inverts about the pitch-class that is the endpoint of the semitone dyad that is 5 semitones away from the isolated pitch-class: thus J exchanges $\{0, 1, 5\} \longleftrightarrow \{7, 11, 0\}$. J is an involution by definition. It holds SUM classes fixed: a general element of 3-4 in prime form is $\{x, x+1, x+5\}, x \in \mathbb{Z}_{12}$, in SUM class $3x+6$; $J(\{x, x+1, x+5\}) = \{x, x-1, x-5\}$, and $x + (x-1) + (x-5) = x + (x+11) + (x+7) = 3x + 18 \equiv 3x + 6 \bmod 12$. By symmetry, involution J equally preserves the SUM class of inverted forms in 3-4.

For all n, T_n commutes with $J : T_nJ(\{z, z\pm 1, z\pm 5\}) = T_n(\{z, z\mp 1, z\mp 5\}) = \{n+z, n+z\mp 1, n+z\mp 5\})$, and $JT_n(\{z, z\pm 1, z\pm 5\}) = J(\{n+z, n+z\pm 1, n+z\pm 5\}) = \{n+z, n+z\mp 1, n+z\mp 5\}$. It follows that one may define $T_nJ := J_n$. Then $T_0J = J_0$, and $(J_0)^2 = T_0$. $T_nJ_m = T_n(T_mJ) = T_{m+n}J = J_{m+n}$; $J_mJ_n = T_mJT_nJ = T_mT_nJJ = T_mT_n = T_{m+n}$. Since T_n/J is abelian, all its subgroups are normal.

The quotient of Tn/J by normal subgroup $H = \{T_0, T_4, T_8, J_0, J_4, J_8\}$ is isomorphic to group $\mathbb{Z}_4$, gives rise to the 3-4 SUM classes, and acts simply transitively on them. $\frac{T_n/J}{H} = \{H, T_1H, T_2H, T_3H\}$ and the mapping $i : \frac{T_n/J}{H} \leftrightarrow \mathbb{Z}_4 : T_zH \leftrightarrow z, z = 0, 1, 2, 3$ is clearly an isomorphism.

The 3-4 SUM classes are:

$$[0] = \{\{2,3,7\},\{6,7,11\},\{10,11,3\},\{9,1,2\},\{1,5,6\},\{5,9,10\}\}$$
$$[3] = \{\{3,4,8\},\{7,8,0\},\{11,0,4\},\{10,2,3\},\{2,6,7\},\{6,10,11\}\}$$
$$[6] = \{\{0,1,5\},\{4,5,9\},\{8,9,1\},\{7,11,0\},\{11,3,4\},\{3,7,8\}\}$$
$$[9] = \{\{1,2,6\},\{5,6,10\},\{9,10,2\},\{8,0,1\},\{0,4,5\},\{4,8,9\}\}.$$

The simply transitive action of $\frac{T_n/J}{H}$ on $S = \{[0], [3], [6], [9]\}$ is:

$$H : S \mapsto S : (\quad)$$
$$T_1H : S \mapsto S : ([0][3][6][9])$$
$$T_2H : S \mapsto S : ([0][6])([3][9])$$
$$T_3H : S \mapsto S : ([0][9][6][3]).$$

Most (ten) asymmetric tetrachordal set-classes admit quotients by the cyclic subgroup $\{T_0, T_3, T_6, T_9\}$, because transposing a tetrachord by any multiple n of 3 semitones adds $4 \times 3n = 12n \equiv 0 \bmod 12$ to the sum of the constituent pitch classes, so leaves SUM classes fixed. But there are four examples (mavericks), identified in [8], where the quotient group must be cyclic, of order 3 (4-4, 4-14, 4-18: SUM classes $\{[0], [4], [8]\}$; 4-13: $\{[2], [6], [10]\}$). It is left as an exercise for the reader to carry out the case of the 4-4: (0125) set-class. In this case one may take the contextual inversion J to be inversion about the isolated pitch class, as in $J : \{0, 1, 2, 5\} \leftrightarrow \{5, 8, 9, 10\}$, and the normal subgroup is $\{T_0, T_3, T_6, T_9, J_0, J_3, J_6, J_9\}$.

In the case of pentachords, all asymmetric pentachords admit all 12 possible sum-classes, (since 5 and 12 are coprime). An example, again left to the reader, is 5-4: (01236), the pentachord well known from the analysis by Lewin of Stockhausen's *Klavierstück III* in [11]. In this case a commutative GIS isomorphic to that employed by Lewin suffices, with a contextual inversion J about a cluster endpoint furthest from the isolated pitch class, analogous to that for the 3-4 case, and a quotient of order 12 by the subgroup $\{T_0, J\}$.

For the hexachords, transposing by any multiple of 2 semitones adds $6 \times 2n = 12n \equiv 0 \bmod 12$ to the sum of the constituent pitch classes, thus the subgroup $H = \{T_0, T_2, T_4, T_6, T_8, T_{10}\}$ leaves SUM classes fixed. The quotient four-group acts simply transitively on the SUM classes of all the asymmetric hexachords except for 6-9, 6-16 (SUM classes [0], [6]) and 6-14, 6-22 (SUM classes

[3], [9]). These exceptional cases may be realized as quotients of T_n/I_n by normal subgroups of order 12: for 6-9 (012357), 6-14 (013458), 6-22 (012468) we take $H = \{T_0, T_2, T_6, T_8, T_{10}, I_1, I_3, I_5, I_7, I_9, I_{11}\}$, and the quotient group is $\{H, T_1H\}$. For 6-9, the action of the quotient group on the SUM-classes S is $H : S \mapsto S : (\quad)$; $T_1H : S \mapsto S$: ([0][6]). For 6-14 and 6-22, the action is $H : S \mapsto S : (\quad)$; $T_1H : S \mapsto S$: ([3][9]). For 6-16 (014568), we take $H = \{T_0, T_2, T_4, T_6, T_8, T_{10}, I_0, I_2, I_4, I_6, I_8, I_{10}\}$, and the quotient group is $\{H, T_1H\}$, with action $H : S \mapsto S : (\quad); T_1H : S \mapsto S$: ([0][6]).

The treatment of the exceptional cases may be proven generally, including the commutativity of the ordinary transpositions with the appropriate contextual inversion, with recourse to Kochavi's study [10], in which he defines contextual inversions with respect to an indexing function for the members of the set class. Since both the parent GIS and the quotient GIS are commutative, there is no dual involved in the maverick cases.

The set-classes of larger cardinality than 6 may be treated in the same way as their complements with respect to SUM classes, quotient groups, and quotient GISs. In the case of a symmetrical set-class, the group acting on its SUM classes is a cyclic group, but the parent GIS already consists of a cyclic group (the T_n group or one of its subgroups) acting on the set-class. The quotient group falls out easily in such cases (see [8]). For example, the GIS for the augmented triad, set-class 3-12: (048), requires only the cyclic group $\{T_0, T_3, T_6, T_9\}$ acting simply transitively on the four members. There are just the four SUM classes, [0], [3], [6], [9], so the required quotient group for the quotient GIS is the trivial one by the identity subgroup $\{T_0\}$. Similarly, the GIS for the usual diatonic scale, 7-35: (013568t), requires the cyclic order 12 T_n group acting simply transitively on the 12 members of the inversionally symmetric set-class. The usual diatonic set admits the maximum 12 SUM classes, so again, the trivial quotient of T_n by its identity subgroup suffices for the quotient GIS.

The focus in this paper has been on mod 12 pitch-class sets, the groups that define their set-classes, and their quotients, because of the musical application. Most of these results could be extended to any dihedral group. It may be recalled that Proposition 2 is completely general: the Generalized Interval Systems and their quotients may be finite or infinite, and their groups may be of any structure. Many of Lewin's examples in [4] are effectively quotient GISs. Hook [12] defines a GIS homomorphism, equivalent to a quotient GIS, and proves its construction. His treatment is different in that he takes Lewin's initial intervallic GIS definition (perhaps more familiar to music theorists), rather than Lewin's equivalent formulation in terms of a simply transitive group action on a set, as has been done here.

References

1. Cohn, R.: Square dances with cubes. J. Music Theory **42**(2), 283–296 (1998)
2. Cohn, R.: Audacious Euphony: Chromaticism and the Triad's Second Nature. Oxford University Press, Oxford (2012)

3. Cook, R.: Transformational approaches to romantic harmony and the late works of César Franck. Ph.D. dissertation, University of Chicago (2001)
4. Lewin, D.: Generalized Musical Intervals and Transformations. Yale University Press New Haven (1987). Reprinted Oxford University Press, Oxford (2007)
5. Clampitt, D.: Pairwise well-formed scales: structural and transformational properties. Ph.D. dissertation, SUNY at Buffalo (1997)
6. Cohn, R.: Introduction to Neo-Riemannian theory: a survey and a historical perspective. J. Music Theory **42**(2), 167–180 (1998)
7. Cohn, R.: Weitzmann's regions, my cycles, and Douthett's dancing cubes. Music Theory Spectr. **22**(1), 89–103 (2000)
8. Orvek, D.: Generalized transformational voice-leading systems. M.A. thesis, Ohio State University (2019)
9. Straus, J.: Sum class. J. Music Theory **62**(2), 279–338 (2018)
10. Kochavi, J.: Some structural features of contextually-defined inversion operators. J. Music Theory **42**(2), 307–320 (1998)
11. Lewin, D.: Musical Form and Transformation: 4 Analytical Essays. Yale University Press, New Haven (1993). Reprinted Oxford University Press, Oxford (2007)
12. Hook, J.: Cross-type transformations and the path consistency condition. Music Theory Spectr. **29**(1), 1–39 (2007)