A Projection-Oriented Mathematical Model for Second-Species Counterpoint

Octavio A. Agustín-Aquino[1](✉) and Guerino Mazzola[2]

[1] Instituto de Física y Matemáticas, Universidad Tecnológica de la Mixteca, Huajuapan de León, Oaxaca, Mexico
octavioalberto@mixteco.utm.mx

[2] School of Music, University of Minnesota, Minneapolis, MN, USA
mazzola@umn.edu

Abstract. Drawing inspiration from both the classical Guerino Mazzola's symmetry-based model for first-species counterpoint (one note against one note) and Johann Joseph Fux's *Gradus ad Parnassum*, we propose an extension for second-species (two notes against one note).

Keywords: Second-species · Counterpoint

1 Introduction

Guerino Mazzola's counterpoint model, founded on the concepts of

1. *strong dichotomy*, which encodes the notion of consonance and dissonance, and
2. *counterpoint symmetry*, which is the carrier of contrapuntal tension and allows to deduce the rules of counterpoint,

has been successful in explaining the necessity of regarding the fourth as a dissonance and obtaining the general prohibition of parallel fifths and tritone skips as a theorem. It also allows to define new understandings of consonance and dissonance, thereby leading to the concept of *counterpoint world*, i.e., paradigms for the handling of two-voice compositions represented as digraphs, whose vertices are consonant intervals and an arrow connects two of them whenever we have a valid progression. This, in turn, allows us to *morph* one world into another. See the monograph [2] and the treatise [4, Part VII] for a thorough account.

Despite these accomplishments, Mazzola's model is restricted to the case of *first-species* counterpoint, which means that only one note can be placed against another. Hence, in order to increase the potential of Mazzola's model for analysis and composition, it is indispensable to extend it to *second-species* counterpoint (i.e., two notes against one) and further. Our approach for a first step in this direction is to extend the notion of counterpoint interval to a 2-interval, i.e., one

This work was partially supported by a grant from the *Niels Hendrik Abel Board*.

M. Montiel et al. (Eds.): MCM 2022, LNAI 13267, pp. 75–85, 2022.
https://doi.org/10.1007/978-3-031-07015-0_7

such that two intervals are attached to a cantus firmus, the first one coming in the downbeat and the second one in the upbeat.

For our extension, the main idea is that the counterpoint symmetries in this case do not determine another 2-interval successor, but a first-species interval in the downbeat. The idea behind this is to blend the species of counterpoint more easily.

2 General Overview of Mazzola's Counterpoint Model

Here we quickly survey the key aspects of Mazzola's counterpoint model (we refer the reader to [2] and [4, Part VII] for a complete account). We consider the action of the group

$$\overrightarrow{GL}(\mathbb{Z}_{2k}) := \mathbb{Z}_{2k} \rtimes \mathbb{Z}_{2k}^{\times}$$

(which we call the group of *general affine symmetries*) on $\mathbb{Z}_{2k}$, which can be described in the following manner:

$$T^u.v(x) = vx + u;$$

here T^u is the *transposition* by u, and v is the *linear part* of the transformation.

We know [1,2] that, for any $k > 4$, there is at least one dichotomy $\Delta = (X/Y)$ of $\mathbb{Z}_{2k}$ such that there is a unique $p \in \overrightarrow{GL}(\mathbb{Z}_{2k})$ and

$$p(X) = Y \quad \text{and} \quad p \circ p = \mathrm{id}_{\mathbb{Z}_{2k}},$$

which is called the *polarity* of the dichotomy. The dichotomies with this property are called *strong*, and represent the division of intervals into generalized *consonances* X and *dissonances* Y.

Next we consider the *dual numbers*

$$\mathbb{Z}_{2k}[\epsilon] = \frac{\mathbb{Z}_{2k}[\mathcal{X}]}{\langle \mathcal{X}^2 \rangle} = \{x + \epsilon.y : x, y \in \mathbb{Z}_{2k}, \epsilon^2 = 0\}$$

in order to attach to each *cantus firmus* x the interval y that separates it from its *discantus*[1]. Thus for a strong dichotomy $\Delta = (X/Y)$ we have the consonant intervals

$$X[\epsilon] := \{c + \epsilon.x : c \in \mathbb{Z}_{2k}, x \in X\}$$

and the dissonant intervals $Y[\epsilon] = \mathbb{Z}_{2k} \setminus X[\epsilon]$. Considering the group

$$\overrightarrow{GL}(\mathbb{Z}_{2k}[\epsilon]) := \{T^{a+\epsilon.b}.(v + \epsilon.w) : a, b, w \in \mathbb{Z}_{2k}, v \in \mathbb{Z}_{2k}^{\times}\},$$

there is a canonical *autocomplementary* symmetry $p_\Delta^c \in \overrightarrow{GL}(\mathbb{Z}_{2k}[\epsilon])$ such that

$$p_\Delta^c(X[\epsilon]) = Y[\epsilon], \quad p_\Delta^c \circ p_\Delta^c = \mathrm{id}_{\mathbb{Z}_{2k}[\epsilon]},$$

and leaves the *tangent space* $c + \epsilon.\mathbb{Z}_{2k}$ invariant.

[1] The discantus can be understood in the *sweeping* $(x + y)$ or the *hanging* $(x - y)$ orientations, but we will only use the sweeping orientation from this point on.

With this preamble it is possible to state a classical paradox for first-species counterpoint theory: all the intervals $c + \epsilon.k$ used in a first-species counterpoint composition or improvisation are consonances. Hence, how can any tension between the voices arise, if at all? Mazzola's solution is inspired in the fact [6, pp. 33–35] that it is not that the point c which is to be confronted against $c + k$, but it is the consonant point $\xi = c_1 + \epsilon.k_1$ who will face a successor $\eta = c_2 + \epsilon.k_2$. The idea is to *deform* the dichotomy $(X[\epsilon]/Y[\epsilon])$ into $(gX[\epsilon], gY[\epsilon])$ through a symmetry $g \in \overrightarrow{GL}(\mathbb{Z}_{2k}[\epsilon])$, such that

1. the interval ξ becomes a deformed dissonance, i.e., $\xi \in gY[\epsilon]$,
2. the symmetry p^c_Δ is an autocomplementary function of

$$(gX[\epsilon], gY[\epsilon])$$

   which means that $p(gX[\epsilon]) = gY[\epsilon]$,

and thus we can transit from ξ to a consonance η which is also a deformed consonance, i.e., $\eta \in gX[\epsilon] \cap X[\epsilon]$. Since we wish to have the maximum amount of choices, we request also that

3. the set $gX[\epsilon] \cap X[\epsilon]$ is of maximum cardinality among the symmetries that satisfy conditions 1 and 2.

The elements of this latter set are the *admitted successors*.

3 Dichotomies of 2-Intervals

For the purposes of the second-species counterpoint, we need now an algebraic structure such that two intervals can be attached to a base tone. In the spirit of the model presented in the previous section, we take all the polynomials of the form[2]

$$c + \epsilon_1.x + \epsilon_2.y \in \frac{\mathbb{Z}_{2k}[\mathcal{X}, \mathcal{Y}]}{\langle \mathcal{X}^2, \mathcal{Y}^2, \mathcal{X}\mathcal{Y} \rangle} = \mathbb{Z}_{2k}[\epsilon_1, \epsilon_2]$$

where $\epsilon_1 \equiv \mathcal{X} \bmod \langle \mathcal{X}^2, \mathcal{Y}^2, \mathcal{X}\mathcal{Y} \rangle$, $\epsilon_2 \equiv \mathcal{Y} \bmod \langle \mathcal{X}^2, \mathcal{Y}^2, \mathcal{X}\mathcal{Y} \rangle$, c is the cantus firmus and x, y are the intervals (x is for the downbeat and y is for the upbeat). An element $\xi \in \mathbb{Z}_{2k}[\epsilon_1, \epsilon_2]$ is called a 2-*interval*. If $\Delta = (X/Y)$ is a strong dichotomy with polarity $p = T^u \circ v$, then

$$X[\epsilon_1, \epsilon_2] := \mathbb{Z}_{2k} + \epsilon_1.X + \epsilon_2.\mathbb{Z}_{2k}$$

is an dichotomy in $\mathbb{Z}_{2k}[\epsilon_1, \epsilon_2]$. We choose this dichotomy because the rules of counterpoint demand that the interval that comes on the downbeat to be a

[2] The original inspiration for using dual numbers in counterpoint was the Zariski tangent space, thus the definition of the tangent space of a morphism of schemes can be seen as a cue to use this kind of algebraic structure for second-species. See [7] for details.

consonance. A polarity for this dichotomy, which is analogous to the one for the first-species case, is

$$p^c = T^{c(1-v)+\epsilon_1.u+\epsilon_2.u} \circ v$$

because

$$\begin{aligned} p^c X[\epsilon_1,\epsilon_2] &= T^{c(1-v)} \circ v.\mathbb{Z}_{2k} + \epsilon_1.pX + \epsilon_2.p\mathbb{Z}_{2k} \\ &= \mathbb{Z}_{2k} + \epsilon_1.Y + \epsilon_2.\mathbb{Z}_{2k} \\ &= Y[\epsilon_1,\epsilon_2] \end{aligned}$$

and it is such that

$$p^c(c + \epsilon_1.\mathbb{Z}_{2k} + \epsilon_2.\mathbb{Z}_{2k}) = c + \epsilon_1.\mathbb{Z}_{2k} + \epsilon_2.\mathbb{Z}_{2k},$$

which means p^c fixes the tangent space to cantus firmus c as well.

We also check the following formula for future use:

$$\begin{aligned} p^{c_1+c_2} &= T^{(c_1+c_2)(1-v)+\epsilon_1.u+\epsilon_2.u} \circ v \\ &= T^{c_1(1-v)+c_2(1-v)+\epsilon_1.u+\epsilon_2.u} \circ v \\ &= T^{c_1} \circ T^{-vc_1} \circ T^{c_2(1-v)+\epsilon_1.u+\epsilon_2.u} \circ v \\ &= T^{c_1} \circ T^{c_2(1-v)+\epsilon_1.u+\epsilon_2.u} \circ v \circ T^{-c_1} \\ &= T^{c_1} \circ p^{c_2} \circ T^{-c_1}. \end{aligned} \tag{1}$$

4 Species Projections

If we represent the polynomial $c + \epsilon_1.x + \epsilon_2.y$ as a column vector, the candidates to (non-invertible) *species projections* are

$$\begin{aligned} g : \mathbb{Z}_{2k}[\epsilon_1,\epsilon_2] &\to \mathbb{Z}_{2k}[\epsilon_1], \\ \begin{pmatrix} c \\ x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} s & 0 & 0 \\ sw_1 & s & sw_2 \end{pmatrix} \begin{pmatrix} c \\ x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \\ &= [sc + t_1] + \epsilon_1.[s(w_1 c + x + w_2 y) + t_2] \end{aligned} \tag{2}$$

for we want to keep it as simple as possible and that the upbeat of the first interval to influence the downbeat of the successor, but not its upbeat one. We do not require the transformation to be bijective for we want it to be able to swap from second-species to first-species if necessary[3].

[3] For the converse swap the standard rules of counterpoint suffice: we can arbitrarily define the third component of the 2-interval. This is coherent with the local application of counterpoint rules in Fux's theory, and also with the particular idea of projection that stems from the fact that, in order to analyze a fragment, we "disregard" notes on the upbeat [3, pp. 41–43].

Definition 1. *A matrix of the form that appears in a species projection is called a* projection matrix.

Let $X[\epsilon_1, \epsilon_2.y] := \mathbb{Z}_{2k} + \epsilon_1.X + \epsilon_2.y$. We might define a counterpoint projection of a 2-interval $\xi = c + \epsilon_1.x + \epsilon_2.y$ as one such that

1. the condition $c + \epsilon_1.x \notin gX[\epsilon_1, \epsilon_2.y]$ holds,
2. the square

$$\begin{array}{ccc} \mathbb{Z}_{2k}[\epsilon_1, \epsilon_2] & \xrightarrow{g} & \mathbb{Z}_{2k}[\epsilon_1] \\ {\scriptstyle p^c}\downarrow & & \downarrow{\scriptstyle p^c_\Delta} \\ \mathbb{Z}_{2k}[\epsilon_1, \epsilon_2] & \xrightarrow[g]{} & \mathbb{Z}_{2k}[\epsilon_1] \end{array} \tag{3}$$

   commutes, where

$$p^c_\Delta := T^{c(1-v)+\epsilon_1.u} \circ v$$

   is the *canonical* polarity of $(X[\epsilon_1]/Y[\epsilon_1])$, and
3. the cardinality of $gX[\epsilon_1, \epsilon_2.y] \cap X[\epsilon_1]$ is maximal among the projections with the previous properties.

The reason for the second requirement is that when it is fulfilled then

$$p^c_\Delta(gX[\epsilon_1, \epsilon_2]) = g(p^c X[\epsilon_1, \epsilon_2]) = gY[\epsilon_1, \epsilon_2],$$

thus p^c_Δ is an autocomplementary function of $gX[\epsilon_1, \epsilon_2]$.

5 Algorithm for the Calculation of Projections

As with the first-species case, if for a projection of the form

$$g = T^{\epsilon_1.t_2} \circ M$$

where M is a projection matrix, we define

$$g^{(t_1)} = g \circ T^{\epsilon_1.s^{-1}w_1 t_1 + \epsilon_2.t_1}$$

then the relation

$$T^{t_1} \circ g = g^{(-t_1)} \circ T^{s^{-1}t_1 + \epsilon_2.t_1}, \tag{4}$$

holds, and hence contrapuntal projections can be calculated with cantus firmus 0 and successors can be suitably adjusted [2, Theorem 2.2].

Remark 1. The groups

$$T^{\mathbb{Z}_{2k}}, T^{\mathbb{Z}_{2k} + \epsilon_2 \mathbb{Z}_{2k}}$$

are subgroups of the group of automorphisms of $X[\epsilon_1]$ and $X[\epsilon_1, \epsilon_2]$, respectively.

The following identities are needed for the simplification of the calculation of contrapuntal symmetries.

Lemma 1. *For a species projection of the form $g = T^{\epsilon_1.t_2} \circ M$ the following holds:*

$$\begin{aligned}(g^{(t_1)})^{(t_2)} &= g^{(t_1+t_2)},\\ T^t \circ g(X[\epsilon_1,\epsilon_2]) &= g^{(-t)}(X[\epsilon_1,\epsilon_2]) \quad \textit{and}\\ T^t \circ g(Y[\epsilon_1,\epsilon_2]) &= g^{(-t)}(Y[\epsilon_1,\epsilon_2]).\end{aligned}$$

Proof. The first identity is straightforward:

$$\begin{aligned}(g^{(t_1)})^{(t_2)} &= (g \circ T^{\epsilon_1.s^{-1}w_1t_1+\epsilon_2.t_1})^{(t_2)}\\ &= g \circ T^{\epsilon_1.s^{-1}w_1t_1+\epsilon_2.t_1} \circ T^{\epsilon_1.s^{-1}w_1t_2+\epsilon_2.t_2}\\ &= g \circ T^{\epsilon_1.s^{-1}w_1(t_1+t_2)+\epsilon_2.(t_1+t_2)}\\ &= g^{(t_1+t_2)}.\end{aligned}$$

For the second identity, note that

$$\begin{aligned}(T^t \circ g)(X[\epsilon_1,\epsilon_2]) &= g^{(-t)} \circ T^{s^{-1}.t+\epsilon_2.t}(X[\epsilon_1,\epsilon_2])\\ &= g^{(-t)}X[\epsilon_1,\epsilon_2]\end{aligned}$$

using (4) and Remark 1. The case for $Y[\epsilon_1,\epsilon_2]$ is proved mutatis mutandis. □

Remark 2. If we have a species projection of the form $g = T^{z+\epsilon_1.t} \circ M$, then we define $f = T^{\epsilon_1.t} \circ M$ and thus $g = T^z \circ f$. Using Lemma 1, we have

$$g(X[\epsilon_1,\epsilon_2]) = (T^z \circ f)(X[\epsilon_1,\epsilon_2]) = f^{(-z)}(X[\epsilon_1,\epsilon_2]).$$

This means that in our discussion it suffices to consider projections whose translational part has zero non-dual component.

The following pair of results reduce the amount of computations required to obtain counterpoint projections.

Lemma 2. *Let $\xi = x + \epsilon_1.k$, g a species projection, and $z \in \mathbb{Z}_{2k}$. If*

$$\xi \notin g(X[\epsilon_1,\epsilon_2]) \quad \textit{and} \quad p_\Delta^x : g(X[\epsilon_1,\epsilon_2]) \xrightarrow{\cong} g(Y[\epsilon_1,\epsilon_2])$$

then

$$T^z(\xi) \notin (T^z \circ g)(X[\epsilon_1,\epsilon_2]) \quad \textit{and}$$
$$p_\Delta^{z+x} : (T^z \circ g)(X[\epsilon_1,\epsilon_2]) \xrightarrow{\cong} (T^z \circ g)(Y[\epsilon_1,\epsilon_2]).$$

Furthermore,

$$(T^z \circ g)(X[\epsilon_1,\epsilon_2]) \cap X[\epsilon_1] = T^z(g(X[\epsilon_1,\epsilon_2]) \cap X[\epsilon_1])$$

and, in particular,

$$|(T^z \circ g)(X[\epsilon_1,\epsilon_2]) \cap X[\epsilon_1,\epsilon_2]| = |g(X[\epsilon_1,\epsilon_2]) \cap X[\epsilon_1,\epsilon_2]|.$$

Proof. Since T^z is a symmetry of $g(X[\epsilon_1, \epsilon_2])$, it follows that $T^z(\xi) \notin T^z(g(X[\epsilon_1, \epsilon_2]))$. Now, using (1),

$$\begin{aligned}(p_\Delta^{x+z} \circ T^z \circ g)(X[\epsilon_1, \epsilon_2]) &= (T^z \circ p_\Delta^x \circ T^{-z} \circ T^z \circ g)(X[\epsilon_1, \epsilon_2]) \\ &= (T^z \circ p_\Delta^x \circ g)(X[\epsilon_1, \epsilon_2]) \\ &= (T^z \circ g)(Y[\epsilon_1, \epsilon_2]).\end{aligned}$$

From Remark 1 it follows that

$$\begin{aligned}(T^z \circ g)(X[\epsilon_1, \epsilon_2]) \cap X[\epsilon_1, \epsilon_2] &= (T^z \circ g)(X[\epsilon_1, \epsilon_2]) \cap T^z(X[\epsilon_1, \epsilon_2]) \\ &= T^z(g(X[\epsilon_1, \epsilon_2]) \cap X[\epsilon_1])\end{aligned}$$

since T^z is bijective. □

Theorem 1. *If $\xi = x + \epsilon_1.k + \epsilon_2.z \in X[\epsilon_1, \epsilon_2]$ and $g = T^{t_1+\epsilon_1.t_2} \circ M$ is any species projection that satisfies the counterpoint conditions, then there is a species projection $h = T^{\epsilon_1.t} \circ M$ such that it also satisfies the counterpoint conditions for ξ. Moreover: in order to verify that the conditions also hold for h, it suffices to check them for the 2-interval $\epsilon_1.k + \epsilon_2.z$, the projection $h^{(x)}$ and the polarity p_Δ^0.*

Proof. The replacement of g follows from Remark 2. By Lemma 1, we have

$$(T^{-x} \circ h)(X[\epsilon_1, \epsilon_2]) = h^{(x)}(X[\epsilon_1, \epsilon_2]).$$

Using Lemma 2 with $z = -x$, we can verify that h is a counterpoint projection using $h^{(x)}$ with the interval $T^{-x}(\xi) = \epsilon_1.k + \epsilon_2.z$ and the polarity $p_\Delta^{-x+x} = p_\Delta^0$. From Lemma 2 it also follows that

$$\begin{aligned}(h^{(x)}(X[\epsilon_1, \epsilon_2]))) \cap (X[\epsilon_1] &= (T^{-x} \circ h)(X[\epsilon_1, \epsilon_2])) \cap X[\epsilon_1] \\ &= T^{-x}(h(X[\epsilon_1, \epsilon_2]) \cap X[\epsilon_1])\end{aligned}$$

which implies that any cardinalities computation we need to perform with h will be the same than doing them with $h^{(x)}$. □

Therefore, we can set $t_1 = 0$ and work with intervals of the form $\xi = \epsilon_1.y + \epsilon_2.z$. For (3) to commute, it is necessary and sufficient that

$$t_2 + su(1 + w_2) = u + vt_2. \tag{5}$$

For $\epsilon_1.y \notin gX[\epsilon_1, \epsilon_2.z]$ we need

$$y = sp(\ell) + t_2 + sw_2 z$$

for some $\ell \in X$. Hence, for some $\ell \in X$ we have

$$t_2 = y - s(p(\ell) + w_2 z). \tag{6}$$

Remark 3. Letting $w_2 = 0$ in (5) and (6), they reduce to the first-species case. Thus, taking $s = v$ and $\ell = y$ both are satisfied and hence we conclude that there exists at least one second-species counterpoint projection.

We only need to work with the following set

$$\begin{aligned} gX[\epsilon_1, \epsilon_2.z] &= \bigcup_{x \in \mathbb{Z}_k} g(x + \epsilon_1.X + \epsilon_2.z) \\ &= \bigcup_{x \in \mathbb{Z}_{2k}} (sx + \epsilon_1.(sw_1 x + sw_2 z + t_2 + sX)) \\ &= \bigcup_{r \in \mathbb{Z}_{2k}} (r + \epsilon_1.(w_1 r + sX + w_2 sz + t_2)) \\ &= \bigcup_{r \in \mathbb{Z}_{2k}} (r + \epsilon_1.T^{w_1 r + w_2 sz + t_2} \circ sX) \end{aligned}$$

to calculate the following cardinality

$$|gX[\epsilon_1, \epsilon_2.z] \cap X[\epsilon_1, \epsilon_2.z]| = \sum_{r \in \mathbb{Z}_{2k}} |T^{w_1 r + w_2 sz + t_2} \circ sX \cap X|.$$

When (6) holds, this reduces to

$$|gX[\epsilon_1, \epsilon_2.z] \cap X[\epsilon_1, \epsilon_2.z]| = \sum_{r \in \mathbb{Z}_{2k}} |T^{w_1 r + y - sp(\ell)} \circ sX \cap X|. \tag{7}$$

From now on we only need to adapt *mutatis mutandis* Hichert's algorithm [2, Algorithm 2.1] to search projections that maximize the intersection.

We must remark that (5) and (6) are perturbations of the conditions to find the counterpoint symmetries for the first-species case. These, together with (7), show that the conditions for deducing a counterpoint theorem [2, Theorem 2.3] hold again, which yields the following result.

Theorem 2. *Given a marked strong dichotomy (X/Y) in $\mathbb{Z}_{2k}$, the 2-interval $\xi \in X[\epsilon_1, \epsilon_2]$ has at least k^2 and at most $2k^2 - k$ admitted successors given by a single counterpoint projection.*

Algorithm 3. *Here $\chi(x, y)$ is the function that returns the cardinality $T^x.yX \cap X$.*

Input: *A strong dichotomy $\Delta = (X/Y)$ and its polarity $T^u.v$.*
Output: *The set of counterpoint projections $\Sigma_{y,z} \subseteq H$ for each $\epsilon_1.y + \epsilon_2.z \in X[\epsilon_1, \epsilon_2]$.*

```
for all y ∈ X and z ∈ Z_12 do
    M ← 0, Σ_{y,z} ← ∅.
    for all s ∈ GL(Z_2k) do
        for all ℓ ∈ X do
            for all w_1, w_2 ∈ Z_2k do
                t_2 ← y − s((vℓ + u) + w_2 z).
```

if $t_2 + su(1+w_2) = u + vt_2$ **then**
  if $w_1 = 0$ **then**
    $S \leftarrow 2k\chi(t_2, s)$.
  else if $w_1 \in GL(\mathbb{Z}_{2k})$ **then**
    $S \leftarrow k^2$
  else
    $\rho \leftarrow \gcd(w_1, 2k)$
    $S \leftarrow \rho \sum_{j=0}^{\frac{2k}{\rho}-1} \chi(j\rho + t_2 + w_2 z, s)$.
  if $S > M$ **then**
    $\Sigma_{y,z} \leftarrow \left\{ T^{\epsilon_2 . t_2} \circ \begin{pmatrix} s & 0 & 0 \\ sw_1 & s & sw_2 \end{pmatrix} \right\}$.
    $S \leftarrow M$.
  else if $S = M$ **then**
    $\Sigma_{y,z} \leftarrow \Sigma_{y,z} \cup \left\{ T^{\epsilon . t_2} \circ \begin{pmatrix} s & 0 & 0 \\ sw_1 & s & sw_2 \end{pmatrix} \right\}$.
return $\Sigma_{y,z}$.

Example 1. The first (valid[4]) example of second-species counterpoint in the *Gradus ad Parnassum* [3, p. 45] is (see Fig. 1)

$$\xi_1 = 2 + \epsilon_1.7 + \epsilon_2.0,\ \xi_2 = 5 + \epsilon_1.4 + \epsilon_2.6,\ \xi_3 = 4 + \epsilon_1.8 + \epsilon_2.3,$$
$$\xi_4 = 2 + \epsilon_1.7 + \epsilon_2.0,\ \xi_5 = 7 + \epsilon_1.4 + \epsilon_2.5,\ \xi_6 = 5 + \epsilon_1.9 + \epsilon_2.4,$$
$$\xi_7 = 9 + \epsilon_1.3 + \epsilon_2.5,\ \xi_8 = 7 + \epsilon_1.9 + \epsilon_2.4,\ \xi_9 = 5 + \epsilon_1.9 + \epsilon_2.4,$$
$$\xi_{10} = 4 + \epsilon_1.7 + \epsilon_2.9,\ \xi_{11} = 2 + \epsilon_1.0.$$

Fig. 1. First (valid) example of second-species counterpoint in Fux's *Gradus ad Parnassum.*

[4] The first example is the student's attempt to write a second-species discantus by himself, but he makes two mistakes near the end of the exercise, namely the steps from the sequence $7 + \epsilon_1.7 + \epsilon_2.4,\ 5 + \epsilon_1.7 + \epsilon_2.4,\ 4 + \epsilon_1.7 + \epsilon_2.9$. They are also forbidden steps in the projection model!.

Some counterpoint projections for the successors are

$$g_1 = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix}, g_2 = T^{\epsilon_1.6} \circ \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \end{pmatrix}, g_3 = T^{\epsilon_1.6} \circ \begin{pmatrix} 7 & 0 & 0 \\ 6 & 7 & 0 \end{pmatrix}$$

$$g_4 = g_1, g_5 = g_2, g_6 = T^{\epsilon_1.8} \circ \begin{pmatrix} 5 & 0 & 0 \\ 8 & 5 & 0 \end{pmatrix},$$

$$g_7 = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 8 \end{pmatrix}, g_8 = g_6, g_9 = g_6, g_{10} = g_1.$$

Let us examine in little bit more of detail the first transition. Note that $\eta = 11 + \epsilon_1.4 + \epsilon_2.11$ is a consonance, and that

$$g_1(\eta) = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ 4 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix},$$

which justifies the fact that the 2-interval $5+\epsilon_1.4+\epsilon_2.6$ is an admitted successor.

6 Comparison with Fux's Approach

Fux states the following in relation to second-species counterpoint (emphasis is our own) [3, p. 41]:

> The second species results when two half notes are set against a whole note. The first of them comes on the downbeat and must always be consonant; the second comes on the upbeat and *it may be dissonant if it moves from the preceding note and to the following note stepwise.* However, *if it moves by a skip, it must be consonant.*

We coded[5] in Octave the calculation of counterpoint projections for the Fuxian (K/D) dichotomy and some more to compare the performance between "restricted" Fux rules against the projection model. More explicitly, taking a second-species step

$$(0 + \epsilon_1.k_1 + \epsilon_2.t_1, c_2 + \epsilon_1.k_2)$$

such that we can proceed (in first-species) from $0 + \epsilon_1.k_1$ to $c_2 + \epsilon_1.k_2$, we verify the following cases:

1. the upbeat interval t_1 of the first 2-interval is allowed to be dissonant only when it connects a valid progression of consonances stepwise, i.e., $0 + t_1$ is between $0+k_1$ and c_2+k_2 and it is separated at most 2 semitones from them and
2. if t_1 is consonant, we duplicate the cantus firmus and check if $(0+\epsilon.k_1, 0+\epsilon.t_1)$, $(0 + \epsilon.t_1, c_2 + \epsilon.k_2)$ and $(0 + \epsilon.k_1, 0 + \epsilon.k_2)$ are valid first-species steps.

The results appear in Table 1 for cases 1 and 2.

[5] https://github.com/octavioalberto/counterpoint.

Table 1. Data for comparison of Fux's model with restrictions for second species against the projection model.

Number of steps	Case 1	Case 2
Total	192	2592
Valid only for Fux model	13	107
Valid only for the projection model	50	1227
Valid in both models	129	1137

We note that the number of cases the projection model cannot explain and only Fux can is relatively small: they amount to 6.8% and 4.1% for cases 1 and 2, respectively. Thus we can conclude that the vast majority of what is allowed by Fux's rules is allowed by the projection model, or that we have successfully extended Fux's handling of dissonance and consonance for second species. Even if this could be ascribed to the fact that the projection model admits 93.229% and 91.204% of the total of transitions in cases 1 and 2, respectively, it should be kept in mind that the original one-species model admits 89.671% of the possible steps between consonant intervals [5, p. 48].

Acknowledgements. We thank the anonymous reviewers whose suggestions significantly improved the exposition and clarity of this paper.

References

1. Agustín-Aquino, O.A.: Counterpoint in $2k$-tone equal temperament. J. Math. Music **3**(3), 153–164 (2009)
2. Agustín-Aquino, O.A., Junod, J., Mazzola, G.: Computational Counterpoint Worlds. Springer, Heidelberg (2015)
3. Mann, A.: The Study of Counterpoint. W. W. Norton & Company (1965)
4. Mazzola, G.: The Topos of Music, vol. I, 2nd edn. Springer, Heidelberg (2017)
5. Nieto, A.: Una aplicación del teorema de contrapunto. B.Sc. thesis (2010)
6. Sachs, K.J.: Der Contrapunctus im 14. und 15. Jahrhundert, Beihefte zum Archiv fü Musikwissenschaft, vol. 13. Franz Steiner Verlag (1974)
7. The Stacks project authors: The Stacks project, Section 0B28 (2022). https://stacks.math.columbia.edu/tag/0B28